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Nonabelian reciprocity laws and higher Brauer–Manin obstructions

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We reinterpret Kim's nonabelian reciprocity maps for algebraic varieties as obstruction towers of mapping spaces of étale homotopy types, removing technical hypotheses such as global basepoints and cohomological constraints. We then extend the theory by considering alternative natural series of extensions, one of which gives an obstruction tower whose first stage is the Brauer–Manin obstruction, allowing us to determine when Kim's maps recover the Brauer–Manin locus. A tower based on relative completions yields nontrivial reciprocity maps even for Shimura varieties; for the stacky modular curve, these take values in Galois cohomology of modular forms, and give obstructions to an adèlic elliptic curve with global Tate module underlying a global elliptic curve.

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Introduction

In [24], Minhyong Kim introduced a sequence of nonabelian reciprocity maps on the adèlic points $X(\mathbb{A}_F)$ of a variety X over a number field F equipped with a global point and satisfying certain cohomological conditions, with the global points contained within the kernel of all the maps. When $X = \mathbb{G}_m$, this sequence just consists of a single map, the Artin reciprocity law

$$\mathrm{rec}\colon \mathbb{A}_F^{\times} \to G_F^{\mathrm{ab}}$$

from the finite idèles of F to the abelianisation of its Galois group, with the property that $\mathrm{rec}(F^{\times}) = 0$.

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In this paper, we give a topological construction of the nonabelian reciprocity maps, based on homotopical obstruction theory. These are defined under more general hypotheses than those of [24]. In particular, we do not need to assume existence of a global point in order to define the maps, so our reciprocity laws can be used to test the Hasse principle. For arbitrary varieties, the reciprocity maps exist as a tower of spaces over $X(\mathbb{A}_F)$, with the cohomological conditions of [24] sufficing to ensure that the maps in the tower are injective.

Kim's nonabelian reciprocity laws are based on the lower central series of the geometric fundamental group, but other variants are possible with our approach. One variant produces a tower starting with the Brauer–Manin obstruction, allowing us to compare it with Kim's reciprocity laws. Another variant is based on relative completions, allowing us to study varieties whose geometric fundamental groups are perfect or nearly so.

For instance, the geometric fundamental group of the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves is the pro-finite completion $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ of $\mathrm{SL}_2(\mathbb{Z})$. This has finite abelianisation, so trivial pro-unipotent completion, which means the unipotent reciprocity maps of [24] are identically zero. However, the Maltsev completion of $\mathrm{SL}_2(\mathbb{Z})$ relative to $\mathrm{SL}_2(\hat{\mathbb{Z}})$ (resp. $\mathrm{SL}_2(\mathbb{Q}_\ell)$) is a pro-unipotent extension of $\mathrm{SL}_2(\hat{\mathbb{Z}})$ (resp. $\mathrm{SL}_2(\mathbb{Q}_\ell)$) by a pro-unipotent group freely generated by duals of spaces of weight 2 (resp. level 1) modular forms. Elements in Galois cohomology of these tensors then give nontrivial obstructions to an adèlic elliptic curve with global Tate module underlying a global elliptic curve.

Our point of view is that the reciprocity maps of [24] are obstruction towers in étale homotopy theory. The constructions of Artin and Mazur [1] and Friedlander [10] associate a pro-simplicial set $X_{\text{ét}}$ to any locally Noetherian simplicial scheme X. When X is smooth and quasiprojective over a field F, with separable closure $\bar{F}$, Theorem 11.5 of [10] shows that for $\overline{X} := X \otimes_F \bar{F}$, the geometric homotopy type $(\overline{X})_{\text{ét}}$ is the homotopy fibre of $X_{\text{ét}}$ over $(\operatorname{Spec} F)_{\text{ét}}$, because the space $(\operatorname{Spec} \bar{F})_{\text{ét}}$ is contractible. Moreover, $(\overline{X})_{\text{ét}}$ is equivalent to the pro-finite completion of the homotopy type of the complex manifold $X(\mathbb{C})$ for any embedding $F \hookrightarrow \mathbb{C}$, so $(\overline{X})_{\text{ét}}$ is a $K(\pi,1)$ whenever $X(\mathbb{C})$ is so.

We are interested in the simplicial set

$$\operatorname{map}_{(\operatorname{Spec} F)_{\text{ét}}}((\operatorname{Spec} F)_{\text{ét}}, X_{\text{ét}}),$$

ie the mapping space (or function complex) of pro-simplicial sets over $(\operatorname{Spec} F)_{\text{ét}}$ (see Definition 2.1). The space $(\operatorname{Spec} F)_{\text{ét}}$ is a $K(\pi,1)$, equivalent to the nerve BG_F of the Galois group G_F. Since morphisms of schemes give rise to morphisms of étale homotopy types, there is then a natural map

$$X(F) \to \mathrm{map}_{BG_F}(BG_F, X_{\text{ét}}).$$

When X is a $K(\pi,1)$ (such as any hyperbolic curve, surface of general type, or abelian variety) over F, we have (ignoring issues with basepoints)

$$\pi_i\,\mathrm{map}_{BG_F}(BG_F, X_{\text{ét}}) = \begin{cases} \mathrm{H}^{1-i}(F, \pi_1^{\text{ét}}(\overline{X})), & i \le 1, \\ 0, & i \ge 2. \end{cases}$$

For smooth varieties X, $\pi_1^{\text{ét}}(\overline{X})$ will always be of strictly negative weights, so we have $\mathrm{H}^0(F, \pi_1^{\text{ét}}(\overline{X})) = 0$, and

$$\mathrm{map}_{BG_F}(BG_F, X_{\text{ét}}) \simeq \mathrm{H}^1(F, \pi_1^{\text{ét}}(\overline{X})),$$

a discrete set of points. This nonabelian cohomology set is the main focus of [24], and for hyperbolic curves X, Grothendieck's section conjecture amounts to the prediction that the morphism

$$X(F) \to \mathrm{map}_{BG_F}(BG_F, X_{\text{ét}})$$

is an equivalence.

In this paper, we construct the reciprocity maps using obstruction theory analogous to Bousfield [2]. The idea is to identify towers $\{X_{\text{ét}}(n)\}_n$ of quotients of $X_{\text{ét}}$ over BG_F for which there exist nonabelian spectral sequences converging to $\mathrm{map}_{BG_F}(BG_F, X_{\text{ét}}(\infty))$, where $X_{\text{ét}}(\infty) := \underleftarrow{\mathrm{holim}}_n X_{\text{ét}}(n)$. The crucial property making these spectral sequences special is that they incorporate fibre sequences

$$\pi_0\,\mathrm{map}_{BG_F}(BG_F, X_{\text{ét}}(n)) \to \pi_0\,\mathrm{map}_{BG_F}(BG_F, X_{\text{ét}}(n-1)) \xrightarrow{\mathrm{ob}_n} \mathrm{Ob}_n,$$

giving obstructions to lifting homotopy classes of maps.

We can also take more general spaces as the source, considering a pro-finite homotopy type $BG_{\mathbb{A}_F^{\in\Sigma}}$ associated to the adèle ring $\mathbb{A}_F^{\in\Sigma} = \prod'_{v\in\Sigma} F_v$, for a (possibly infinite) nonempty set Σ of finite places. Reciprocity maps then arise in nonabelian spectral sequences converging to the homotopy groups of

$$X(\mathbb{A}_F^{\in\Sigma}) \times^h_{\mathrm{map}_{BG_F}(BG_{\mathbb{A}_F^{\in\Sigma}}, X_{\text{ét}}(\infty))} \mathrm{map}_{BG_F}(BG_F, X_{\text{ét}}(\infty)),$$

and the spaces in the spectral sequence are compactly supported cohomology groups $\mathrm{H}_c^*(\mathscr{O}_{F,\Sigma}, -)$, which can be rewritten as duals of Galois cohomology groups by Poitou–Tate duality. Defining the tower $\{X_{\text{ét}}(n)\}_n$ in terms of the lower central series of the geometric fundamental group $\pi_1^{\text{ét}}(\overline{X})$ recovers Kim's reciprocity maps [24]. Subtler towers based on relative completions give rise to reciprocity laws in more general situations.

Explicitly, for a modular curve Y_Γ we can consider the set $Y_\Gamma(\mathbb{A}_F^{\in\Sigma})_0$ of adèlic points x for which the Tate module $T_\ell E_{\bar{x}}$ of the associated elliptic curve lifts to a $G_{F,\Sigma}$–representation Λ. We then construct a sequence of subsets (glossing over subtleties related to potential higher automorphisms for now)

$$\cdots \to Y_\Gamma(\mathbb{A}_F^{\in\Sigma})_1 \to Y_\Gamma(\mathbb{A}_F^{\in\Sigma})_0$$

containing $Y_\Gamma(\mathscr{O}_{F,\Sigma})$. These are defined inductively by $Y_\Gamma(\mathbb{A}_F^{\in\Sigma})_n = \mathrm{ob}_n^{-1}(0)$ for reciprocity maps

$$\mathrm{ob}_n\colon Y_\Gamma(\mathbb{A}_F^{\in\Sigma})_{n-1} \to \mathrm{H}^2_c(G_{F,\Sigma}, T_n),$$

where the $\mathbb{Q}_\ell$–vector spaces T_n are given by homogeneous factors of a Lie algebra generated by

$$T_1 = \prod_m \mathrm{H}^1(\Gamma, V_m)^* \otimes V_m$$

for irreducible SL_2–representations V_m; via Eichler–Shimura, the groups $\mathrm{H}^1(\Gamma, V_m)$ can be interpreted as ℓ–adic realisations of motivic modular forms of weight $m+2$ and level Γ. If we instead assume that the Tate modules $T_p E_{\bar{x}}$ lift to $G_{F,\Sigma}$–representations ρ_p for all primes p, then we have a similar sequence, but with T_1 now defined in terms of modular forms of all levels. In this case, Helm and Voloch [18] show that whenever there is an adèlic elliptic curve compatible with the representations ρ_p, there must exist a rational elliptic curve giving rise to them, but our obstructions should measure the difference between these elliptic curves.

We can even incorporate higher homotopical information in constructing reciprocity laws for Deligne–Mumford stacks X, by looking at completions of étale homotopy types instead of their fundamental groups. The first obstruction map in the spectral sequence is then just the Brauer–Manin obstruction when we take the base $X_{\text{ét}}(0)$ of the tower to be BG_F, with refinements for (pro-)étale covers given by the subtler obstruction towers.

The structure of the paper is as follows. Section 1 lays the topological foundations for constructing reciprocity laws, developing generalisations of Bousfield's obstruction theory [2]. The most general statement is Proposition 1.5, giving obstruction spaces for homotopy limits of abelian extensions of simplicial groupoids.

Section 2 then applies this theory to give towers of obstructions to the existence of global points over a number field. The first such tower we consider is Example 2.5. Writing $\Pi_n := \pi_1^{\text{ét}}(X,\bar{x})/[\pi_1^{\text{ét}}(\overline{X},\bar{x})]_{n+1}$, $\bar{\pi} := \pi_1^{\text{ét}}(\overline{X},\bar{x})$ and $[\pi]_1 := \pi$, with $[\pi]_{k+1}$ the closure of $[\pi,[\pi]_k]$, this gives a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_{F,\Sigma},[\bar{\pi}]_s/[\bar{\pi}]_{s+1}) \Rightarrow \pi_{t-s}\,\mathrm{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma},B\Pi_\infty),$$

encoding Ellenberg's obstructions. There is a unipotent generalisation, Example 2.12, and further refinements for relative completion. Notably, Examples 2.16 and 2.17 give obstructions, in terms of modular forms, to lifting a G_F–representation Λ to an elliptic curve E over F with Tate module Λ.

In Section 3, this approach is refined to consider the difference between the obstruction towers for F and $\mathbb{A}_F^{\Sigma}$, yielding reciprocity laws in terms of Poitou–Tate duality. The main examples of resulting spectral sequences appear in Section 3.2.2, with Examples 3.16 and 3.18 recovering and generalising Kim's nonabelian reciprocity laws [24], while reciprocity laws for the stacky modular curve $\mathcal{M}_{1,1}$ appear in Example 3.19, giving obstructions to an adèlic elliptic curve being defined over F when its Tate module is known to be a G_F–representation.

Constructions in terms of higher homotopy types are then given in Example 3.21, with Section 3.2.3 showing how the spectral sequences for higher homotopy types start with the Brauer–Manin obstruction (or a pro-étale generalisation) as the first stage in the tower. Proposition 3.26 gives a sufficient condition for Kim's nonabelian reciprocity laws to recover the Brauer–Manin set. In Section 3.3, we then discuss more concrete ways to construct the reciprocity laws, with a fairly explicit description of the first obstruction for modular curves, and a discussion of the relation between higher Brauer–Manin obstructions and Massey products.

The appendix contains the technicalities needed to work with higher étale homotopical invariants of adèle rings, giving a morphism from $(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma})_{\text{ét}}$ to the homotopy type $BG_{\mathbb{A}_F^{\in\Sigma}}$ governing restricted products of local cohomology groups.

Readers unfamiliar with abstract homotopy theory are advised to skip Section 1 entirely, starting with Section 3.3 for an overview before reading the examples in Sections 2 and 3. We should warn at this stage that none of the examples exhibits explicit classes in Galois cohomology on which to evaluate the obstructions, but the weights of the Galois representations involved suggest they must exist in great generality.

Notation We will write $\cong$ for isomorphism and $\simeq$ for weak equivalence. Let $\mathbb{S}$ denote the category of simplicial sets with the Kan model structure, and $s\mathbb{S}$ the category of bisimplicial sets. We denote mapping spaces in model categories by map; in the case of simplicial model categories, these simplicial sets are just given by derived functors of the simplicially enriched Hom bifunctor, and in general they are given by the function complexes of Hovey [21, Section 5.4].

We fix a number field F, and a (possibly infinite) nonempty set Σ of finite places of F. Then G_F denotes the Galois group of F, and its quotient $G_{F,\Sigma}$ is the Galois group of the maximal extension of $\mathscr{O}_F$ unramified outside Σ. We write $\mathbb{A}_F^{\in\Sigma}$ for the adèle ring

$$\mathbb{A}_F^{\in\Sigma} := \prod_{v\in\Sigma}{}' F_v = \varinjlim_{\substack{T\subset\Sigma\\ T\text{ finite}}} \Big(\prod_{v\in T} F_v \times \prod_{v\in\Sigma-T} \mathscr{O}_{F,v} \Big).$$

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1 Obstruction theory from abelian extensions

Given a fibration $f\colon X \to Y$ of spaces with fibre Z, there is a long exact sequence

$$\cdots \to \pi_2 Z \to \pi_2 X \to \pi_2 Y \to \pi_1 Z \to \pi_1 X \to \pi_1 Y \to \pi_0 Z \to \pi_0 X \to \pi_0 Y$$

of homotopy groups and sets, where the final map need not be surjective (and at this stage we are being deliberately vague about basepoints).

Our primary goal in this section is to look for cases where this sequence extends one stage further, giving an obstruction map from $\pi_0 Y$ to some pointed set such that the fibre over the basepoint is the image of $\pi_0 X \to \pi_0 Y$. This will happen if there is some space B and a map $Y \to B$ in the homotopy category of spaces, with X the homotopy fibre over a point $b \in B$, and in this case Z above is automatically the loop space $\Omega(B, b)$.

An obvious example of this phenomenon is when X is a principal G–bundle over Y for a topological group G, so arises as the homotopy fibre of a map $Y \to BG$. We then have a long exact sequence

$$\cdots \to \pi_1 Y \to \pi_1 BG \to \pi_0 X \to \pi_0 Y \to \pi_0 BG,$$

noting that $\pi_n BG = \pi_{n-1} G$.

In this form, this statement is telling us nothing new, since $\pi_0 X \to \pi_0 Y$ is automatically surjective in such cases. However, the characterisation of X as a homotopy fibre also passes to homotopy limits of such diagrams. Given a small category I, together with I–diagrams Y and G in simplicial sets and simplicial groups, and a principal G–bundle X over Y, we can characterise X as the homotopy fibre of a map $Y \to BG$ in the homotopy category, and then

$$\underset{\substack{\longleftarrow \\ i \in I}}{\mathrm{holim}}\, X(i) \to \underset{\substack{\longleftarrow \\ i \in I}}{\mathrm{holim}}\, Y(i) \to \underset{\substack{\longleftarrow \\ i \in I}}{\mathrm{holim}}\, BG(i)$$

is a homotopy fibre sequence, so gives rise to a long exact sequence of homotopy groups and sets of the desired form; this is essentially the content of Corollary 1.11 below.

1.1 Central and abelian extensions of simplicial groups

1.1.1 Central extensions We now look at principal fibrations in the category of groups. First observe that an internal group object in the category of groups is an abelian group A by the Eckmann–Hilton argument, with multiplication $A \times A \xrightarrow{\cdot} A$ being a group homomorphism.

An A–space in groups is then a group G equipped with a group homomorphism $\mu\colon A \times G \to G$ such that the diagram

$$\begin{tikzcd} A \times A \times G \arrow[r, "(\mathrm{id}_A, \mu)"] \arrow[d, "(\cdot, \mathrm{id}_G)"'] & A \times G \arrow[d, "\mu"] \\ A \times G \arrow[r, "\mu"] & G \end{tikzcd}$$

commutes. In other words, $\mu(a, g) = \rho(a)g$ for the group homomorphism $\rho\colon A \to Z(G)$ to the centre of G given by $\rho(a) = \mu(a, 1)$. The A–action is faithful if ρ is injective, and then G is a principal A–space over G/A.

Applying the nerve functor, we have a simplicial abelian group BA (the group homomorphism $A \times A \xrightarrow{\cdot} A$ inducing a multiplication $BA \times BA \to BA$), and for every principal A–space G in groups over H, we get a principal BA–fibration BG over BH.

Definition 1.1 Define $\nabla\colon s\mathbb{S} \to \mathbb{S}$ to be the right adjoint to Illusie's total Dec functor given by $\mathrm{DEC}(X)_{mn} = X_{m+n+1}$. Explicitly,

$$\nabla_p(X) = \left\{ (x_0, x_1, \dots, x_p) \in \prod_{i=0}^{p} X_{i,p-i} \;\middle|\; \partial_0^v x_i = \partial_{i+1}^h x_{i+1} \text{ for all } 0 \le i < p \right\}$$

with operations

$$\begin{aligned}\partial_i(x_0, \dots, x_p) &= (\partial_i^v x_0, \partial_{i-1}^v x_1, \dots, \partial_1^v x_{i-1}, \partial_i^h x_{i+1}, \partial_i^h x_{i+2}, \dots, \partial_i^h x_p),\\ \sigma_i(x_0, \dots, x_p) &= (\sigma_i^v x_0, \sigma_{i-1}^v x_1, \dots, \sigma_0^v x_i, \sigma_i^h x_i, \sigma_i^h x_{i+1}, \dots, \sigma_i^h x_p).\end{aligned}$$

Given a simplicial diagram Γ of groupoids, the nerve $B\Gamma$ is a bisimplicial set, and we write $\overline{W}\Gamma := \nabla B\Gamma$, noting that this agrees with the definition of [11, Section V.7] when Γ has constant objects.

Note that the loop space $\Omega\overline{W}G$ of $\overline{W}G$ is weakly equivalent to G, so in particular $\pi_i\overline{W}G \cong \pi_{i-1}G$, with $\pi_0 G = *$.

In [4], it is established that the canonical natural transformation

$$\operatorname{diag} X \to \nabla X$$

from the diagonal is a weak equivalence for all X. Thus ∇X is a model for the homotopy colimit

$$\varinjlim_{n\in\Delta^{\mathrm{opp}}}{}^{\!\!\mathrm{holim}} X_n,$$

and, in particular, $\overline{W}\Gamma$ a model for $\underrightarrow{\mathrm{holim}}_{n\in\Delta^{\mathrm{opp}}} B(\Gamma_n)$.

Proposition 1.2 *Given a surjection $G \to H$ of simplicial groups with central kernel A, there is a simplicial set Y' weakly equivalent to $\overline{W}H$ and a map $f\colon Y' \to \overline{W}^2 A$ with fibre $\overline{W}G$, which is also the homotopy fibre. Moreover, the space Y' and weak equivalence $w\colon Y' \to \overline{W}H$ can be chosen functorially.*

Proof Writing $K = \overline{W}A$, the statement is essentially the well-known result [11, Theorem V.3.9] that $\overline{W}K$ classifies principal fibrations. The reasoning above applied to simplicial groups gives us a bisimplicial abelian group BA and a principal BA–fibration BG over BH. Applying the codiagonal functor ∇ then gives us a simplicial abelian group $\overline{W}A$ and a principal $\overline{W}A$–fibration $\overline{W}G$ over $\overline{W}H$. The map f then just comes by taking the homotopy quotient of $\overline{W}G \to \overline{W}H$ by the action of $\overline{W}A$.

Explicitly, we set $Y' = \overline{W}[\overline{W}G/\overline{W}A]$ for the simplicial groupoid $[\overline{W}G/\overline{W}A]$ with objects $\overline{W}G$ and morphisms given by $\overline{W}A$ acting on the right. Applying $\overline{W}$ twice to the map $[G/A] \to [H/1]$ of groupoids in groups gives the weak equivalence $Y' \to \overline{W}H$, since $\overline{W}[Y/1] = Y$ and the fibre $\overline{W}[A/A]$ is contractible. Similarly, the Kan fibration $Y' \to \overline{W}^2 A$ comes from the map $[G/A] \to [1/A]$ of groupoids in groups. □

1.1.2 Abelian extensions More generally, given a group H, a group object Γ in the comma category $\mathrm{Gp}\downarrow H$ of groups over H is of the form $\Gamma = H \ltimes A$ for an abelian group A equipped with an H–action.

Then a Γ–space in groups over H consists of a group G and a surjection $G \to H$, together with an associative action $\Gamma \times_H G \to G$ (all maps being group homomorphisms). Equivalently, for the group A above, we have a group homomorphism $G \ltimes A \to G$ over H, hence a G–equivariant map $A \to \ker(G \to H)$.

The condition for G to be a principal Γ–space is then just that the map $A \to \ker(G \to H)$ be an isomorphism. In other words, a pair (Γ, G) is the same as an abelian group A equipped with an H–action together with a surjective group homomorphism $G \to H$ with kernel A.

Given such a G, we can take the nerve, giving a surjective fibration $BG \to BH$ of simplicial sets with fibre BA over the unique vertex of BH. The simplicial set $B(H \ltimes A)$ is a group object in simplicial sets over BH, and BG is a principal $B(H \ltimes A)$–bundle.

Proposition 1.3 *Take a surjection $G \to H$ of simplicial groups with abelian kernel A. Then there exists a fibration*

$$Y' \to \overline{W}[H \ltimes \overline{W}A]$$

for which the projection $Y' \to \overline{W}H$ is a weak equivalence, with

$$Y' \times_{\overline{W}[H \ltimes \overline{W}A]} \overline{W}H \cong \overline{W}G.$$

Moreover, the space Y' and weak equivalence $w\colon Y' \to Y$ can be chosen functorially.

Proof We adapt the proof of Proposition 1.2. Set $Y' = \overline{W}[\overline{W}(G \ltimes A) \Rightarrow \overline{W}G]$ for the simplicial groupoid $[\overline{W}(G \ltimes A) \Rightarrow \overline{W}G]$ with objects $\overline{W}G$ and morphisms $\overline{W}(G \ltimes A)$. Applying $\overline{W}$ twice to the map $[(G \ltimes A) \Rightarrow G] \to [H \Rightarrow H]$ of groupoids in groups gives the weak equivalence $Y' \to \overline{W}H$, since $\overline{W}[Y \Rightarrow Y] = Y$ and the fibre $\overline{W}[A/A] = \overline{W}[A \ltimes A \Rightarrow A]$ is contractible. Similarly, the Kan fibration $Y' \to \overline{W}[H \ltimes \overline{W}A]$ comes from the map $[(G \ltimes A) \Rightarrow G] \to [(H \ltimes A) \Rightarrow H]$ of groupoids in groups. □

1.1.3 Groupoids The constructions above generalise to groupoids, and we will not concern ourselves with the full generality of internal groups in groupoids. We just observe that any abelian group is a fortiori an internal group in groupoids with one object, and that for any groupoid H, an H–representation A in abelian groups has associated groupoid $H \ltimes A$, which is a group object in groupoids over H.

Definition 1.4 We say that a morphism $f\colon G \to H$ is an abelian extension if it is an isomorphism on objects, surjective on morphisms, and the groups $A(x) := \ker(f\colon G(x,x) \to H(fx, fx))$ are abelian for all objects x of G.

Thus for any abelian extension $G \to H$ of groupoids with kernel A, we get a surjective fibration $BG \to BH$ of simplicial sets, and the fibre over $fx \in (BH)_0$ is just $A(x)$. Moreover, $B(H \ltimes A)$ is a group object in simplicial sets over BH, and BG is a principal $B(H \ltimes A)$–bundle.

Proposition 1.5 *Given an abelian extension $G \to H$ of simplicial groupoids with abelian kernel A, there is fibration $Y' \to \overline{W}[H \ltimes A]$ such that the projection $Y' \to \overline{W}H$ is a weak equivalence, with*

$$Y' \times_{\overline{W}[H \ltimes A]} \overline{W}H \cong \overline{W}G.$$

Moreover, the space Y' and weak equivalence $w\colon Y' \to Y$ can be chosen functorially.

Proof The proof of Proposition 1.3 carries over, replacing groupoids in groups with groupoids in groupoids. □

1.2 Passage to homotopy limits

For a small category I, we have a limit functor $\varprojlim_I\colon \mathbb{S}^I \to \mathbb{S}$ from I–diagrams of simplicial sets to simplicial sets. Recall from [11, Section VIII.2] or [19, Chapter 18] that $\underleftarrow{\mathrm{holim}}_I\colon \mathrm{Ho}(\mathbb{S}^I) \to \mathrm{Ho}(\mathbb{S})$ is the right-derived functor of $\varprojlim_I$; in other words, it is the universal functor under $\varprojlim_I$ preserving weak equivalences.

Definition 1.6 Given a small category I and two simplicial group-valued functors $G, H\colon I \to s\mathrm{Gp}$, we say that a natural transformation $G \to H$ is a central (resp. abelian) extension if the maps $G(i) \to H(i)$ are so for all $i \in I$.

Proposition 1.7 *Given a central extension $f\colon G \to H$ of I–diagrams with kernel A, there is a morphism*

$$\underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}H(i) \to \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}^2A(i)$$

in the homotopy category of simplicial sets with homotopy fibre $\operatorname{\underleftarrow{holim}}_{i\in I} \overline{W}G(i)$ over the distinguished point $*$.

Proof We just apply the derived functor $\operatorname{\underleftarrow{holim}}_{i\in I}$ to the diagrams from Proposition 1.2. □

Note that when $I = \Delta$, the simplex category, this recovers a fairly general case of Bousfield's obstruction maps from [2].

Corollary 1.8 *In the scenario of Proposition 1.7, there is a sequence*

$$\pi_0 \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}G(i) \xrightarrow{f_*} \pi_0 \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}H(i) \xrightarrow{\delta_*} \pi_0 \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}^2A(i)$$

of sets, exact in the sense that the fibre of δ_* over 0 is the image of f_*.

Moreover, there is a group action of $\pi_0 \operatorname{\underleftarrow{holim}}_{i\in I} \overline{W}A(i)$ on $\pi_0 \operatorname{\underleftarrow{holim}}_{i\in I} \overline{W}G(i)$ whose orbits are precisely the fibres of f_*.

For any $x \in \operatorname{\underleftarrow{holim}}_{i\in I} \overline{W}G(i)$, with $y = f_*x$, the homotopy fibre of f over y is weakly equivalent to $\operatorname{\underleftarrow{holim}}_{i\in I} \overline{W}A(i)$, and the sequence above extends to a long exact sequence

$$\cdots \xrightarrow{f_*} \pi_n\big(\underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}H(i), y\big) \xrightarrow{\delta} \pi_{n-1} \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}A(i) \to \pi_{n-1}\big(\underset{i\in I}{\operatorname{\underleftarrow{holim}}} X(i), x\big) \xrightarrow{f_*}$$
$$\cdots \xrightarrow{f_*} \pi_1\big(\underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}H(i), y\big) \xrightarrow{\delta} \pi_0 \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}A(i) \xrightarrow{-*x} \pi_0 \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}G(i).$$

Proof This is just the long exact sequence of a fibration [11, Lemma I.7.3] applied to $\delta\colon \operatorname{\underleftarrow{holim}}_{i\in I} Y'(i) \to \operatorname{\underleftarrow{holim}}_{i\in I} \overline{W}^2A(i)$, noting that

$$\Omega \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}^2A(i) \simeq \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \Omega\overline{W}^2A(i) \simeq \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}A(i),$$

so

$$\pi_n \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}^2A(i) \cong \pi_{n-1} \underset{i\in I}{\operatorname{\underleftarrow{holim}}} \overline{W}A(i)$$

for all $i > 0$. □

Remark 1.9 Were it not for the final term, Corollary 1.8 would just be the long exact sequence of homotopy for the map $\underleftarrow{\mathrm{holim}}_{i\in I}\overline{W}G(i)\to\underleftarrow{\mathrm{holim}}_{i\in I}\overline{W}H(i)$. The essential purpose of all our effort so far has thus been to incorporate the extra term $\pi_0\underleftarrow{\mathrm{holim}}_{i\in I}\overline{W}^2A(i)$, giving an obstruction to lifting connected components.

Proposition 1.10 *Given an abelian extension $G\to H$ of I–diagrams with kernel A, there is a morphism $\delta\colon\underleftarrow{\mathrm{holim}}_{i\in I}\overline{W}H(i)\to\underleftarrow{\mathrm{holim}}_{i\in I}\overline{W}(H\ltimes\overline{W}A(i))$ in the homotopy category of simplicial sets over $\underleftarrow{\mathrm{holim}}_{i\in I}\overline{W}H(i)$ with a homotopy pullback diagram*

$$\begin{array}{ccc}\underleftarrow{\mathrm{holim}}_i\overline{W}G(i) & \longrightarrow & \underleftarrow{\mathrm{holim}}_i\overline{W}H(i)\\ \downarrow & & \downarrow{\scriptstyle\delta}\\ \underleftarrow{\mathrm{holim}}_i\overline{W}H(i) & \xrightarrow{0} & \underleftarrow{\mathrm{holim}}_i\overline{W}(H(i)\ltimes\overline{W}A(i))\end{array}$$

In particular, if the adjoint action of H on A factors through some quotient $\overline{H}$, then for any $\bar{y}\in\underleftarrow{\mathrm{holim}}_i\overline{W}\overline{H}$, we have a fibration sequence

$$\big(\underleftarrow{\mathrm{holim}}_i\overline{W}G(i)\big)_{\bar{y}}\to\big(\underleftarrow{\mathrm{holim}}_i\overline{W}H(i)\big)_{\bar{y}}\to\underleftarrow{\mathrm{holim}}_i\overline{W}(\overline{H}(i)\ltimes\overline{W}A(i))_{\bar{y}}$$

on homotopy fibres over y.

Proof We just apply the derived functor $\underleftarrow{\mathrm{holim}}_{i\in I}$ to the diagrams from Proposition 1.3. □

Now, given an I–diagram X, write $\underleftarrow{X}:=\underleftarrow{\mathrm{holim}}_i X(i)$.

Corollary 1.11 *In the scenario of Proposition 1.10, an element y lies in the image of*

$$\pi_0\underleftarrow{\overline{W}G}\xrightarrow{f_*}\pi_0\underleftarrow{\overline{W}H}$$

if and only if $\delta_*(y)=0\in\pi_0(\underleftarrow{\overline{W}A}_{(y)})$, where $\underleftarrow{\overline{W}A}_{(y)}$ denotes the homotopy fibre of $\underleftarrow{\overline{W}(H\ltimes\overline{W}A)}\to\underleftarrow{\overline{W}H}$ over y.

Moreover, for each such y there is a transitive group action of $\pi_0(\underleftarrow{\overline{W}A}_{(y)})$ on the fibre of f_*.

For any $x\in\underleftarrow{\overline{W}G}$ with $y=f_*x$, the homotopy fibre of f over y is weakly equivalent to $\underleftarrow{\overline{W}A}_{(y)}$, and the sequence above extends to a long exact sequence

$$\cdots\xrightarrow{f_*}\pi_n(\underleftarrow{\overline{W}H},y)\xrightarrow{\delta}\pi_{n-1}(\underleftarrow{\overline{W}A}_{(y)})\to\pi_{n-1}(\underleftarrow{\overline{W}G},x)\xrightarrow{f_*}$$
$$\cdots\xrightarrow{f_*}\pi_1(\underleftarrow{\overline{W}H},y)\xrightarrow{\delta}\pi_0(\underleftarrow{\overline{W}A}_{(y)})\xrightarrow{-*x}\pi_0\underleftarrow{\overline{W}G}.$$

Proof The proof of Corollary 1.8 carries over. □

2 Towers of Diophantine obstructions

Recall that we are fixing a number field F, and a (possibly infinite) nonempty set Σ of finite places of F. When Σ consists of all finite places, we have a weak equivalence $BG_F \simeq (\mathrm{Spec}\, F)_{\text{ét}}$; in general, equation (1) of [42, Appendix A] combines with [10, Corollary 6.5] to show that the homotopy fibre of the surjective map $(\mathrm{Spec}\, \mathscr{O}_{F,\Sigma})_{\text{ét}} \to BG_{F,\Sigma}$ becomes contractible on derived pro-Σ completion, ie pro-L completion (see [1, Theorems 3.4 and 4.3]) for L the set of integer primes all of whose F–prime factors lie in Σ.

Given any pro-finite group Π and a pro-surjection $\Pi \to G_{F,\Sigma}$ (such as when Π is the arithmetic fundamental group of an $\mathscr{O}_{F,\Sigma}$–scheme), we have a fibration $B\Pi \to BG_{F,\Sigma}$ of pro-simplicial sets in the model structure of [22].

Thus for any pro-simplicial set Y over $BG_{F,\Sigma}$, we may consider the mapping space

$$\mathrm{map}_{BG_{F,\Sigma}}(Y, B\Pi)$$

for the same model structure; when $Y = BG_{F,\Sigma}$, this is the space of sections of $B\Pi \to BG_{F,\Sigma}$.

Explicitly, the proof of [22, Proposition 10.9] applied to the left function complex of [21, Remark 5.2.9, Section 5.4] allows us to describe mapping spaces of pro-simplicial sets in terms of the Edwards–Hastings strict model structure [7], reducing to the following description.

Definition 2.1 For pro-simplicial sets $X = \varprojlim_i X(i)$ and $Y = \varprojlim_j Y(j)$ such that each $Y(j)$ has finitely many nonzero homotopy groups, we may define the simplicial set $\mathrm{map}(X, Y)$ in terms of mapping spaces of simplicial sets as the homotopy limit

$$\mathrm{map}(X, Y) := \underset{\overleftarrow{j}}{\mathrm{holim}}\, \varinjlim_{i} \mathrm{map}(X(i), Y(j)).$$

For general pro-simplicial sets $X = \varprojlim_i X(i)$ and $Y = \varprojlim_j Y(j)$, we may define $\mathrm{map}(X, Y)$ as the homotopy limit

$$\mathrm{map}(X, Y) := \underset{\overleftarrow{j,k}}{\mathrm{holim}}\, \varinjlim_{i} \mathrm{map}(X(i), P_k Y(j)),$$

where P_k denotes a Postnikov tower.

For a diagram $X \xrightarrow{f} Z \leftarrow Y$ of pro-simplicial sets, the relative mapping space $\mathrm{map}_Z(X, Y)$ is the homotopy fibre of $\mathrm{map}(X, Y) \to \mathrm{map}(X, Z)$ over f.

2.1 Abelian extensions

Assume that we have abelian extension $\Pi'' \to \Pi'$ of pro-finite groups with kernel A, such that the conjugation action of Π' on A factors through some quotient G of Π'. When working with nilpotent completions of geometric fundamental groups, we may take $G = G_{F,\Sigma}$, but for relative completions (as needed for modular curves), G will be larger.

Writing $B(G \ltimes BA) := \overline{W}(G \ltimes BA)$, we have:

Proposition 2.2 *In the scenario above, and for any pro-simplicial set Y over BG, there is a natural fibration sequence*

$$\mathrm{map}_{BG}(Y, B\Pi'') \to \mathrm{map}_{BG}(Y, B\Pi') \to \mathrm{map}_{BG}(Y, B(G \ltimes BA))$$

of mapping spaces, the fibre being taken over the zero map $Y \to BG \to B(G \ltimes BA)$.

Proof The idea behind this statement is that the extension $\Pi'' \to \Pi'$ defines an element of $\mathrm{H}^1(\Pi', A)$, which we can write as a morphism ob: $\Pi' \to G \ltimes BA$ in the homotopy category of simplicial pro-finite groups over G. The proof of [33, Proposition 1.19] adapts to any Artinian category, and in particular to finite groups, allowing us to regard simplicial pro-finite groups as pro-objects in the category of (bounded) finite simplicial groups. We can then recover $B\Pi''$ as a homotopy fibre product

$$B\Pi' \times^h_{\mathrm{ob}, B(G \ltimes BA)} B\Pi',$$

leading to the fibration sequence above.

More formally, we write $\Pi'' = \varprojlim_{j \in J} \Pi''(j)$ as a filtered limit of finite quotient groups, inducing compatible expressions $A = \varprojlim_j A(j)$, $\Pi'(j) = \Pi''(j)/A(j)$ and $\Pi'(j) \twoheadrightarrow G(j)$ with $G = \varprojlim_j G(j)$.

The mapping spaces $\mathrm{map}(Y, B\Pi)$ are given by

$$\mathrm{map}_{BG}(Y, B\Pi) \simeq \operatorname*{holim}_{\substack{\longleftarrow \\ (n,j) \in \Delta \times J}} \mathrm{Hom}_{\mathrm{pro(Set)}}(Y_n, B\Pi(j)),$$

so we apply Proposition 1.10 to the abelian extension

$$\mathrm{Hom}_{\mathrm{pro(Set)}}(Y_n, \Pi''(j)) \to \mathrm{Hom}_{\mathrm{pro(Set)}}(Y_n, \Pi'(j))$$

of $(\Delta \times J)$–diagrams in groups, and then take homotopy fibres over the canonical basepoint of $\mathrm{map}(Y, BG)$. $\square$

We think of the base $\mathrm{map}_{BG}(Y, B(G \ltimes BA))$ of the fibration as an obstruction space; via the description of Definition 2.1, its homotopy groups are given by equivariant cohomology groups

$$\pi_i \,\mathrm{map}_{BG}(Y, B(G \ltimes BA)) \cong \mathrm{H}^{2-i}_G(Y, A),$$

so we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0_G(Y, A) &\to \pi_1 \,\mathrm{map}_{BG}(Y, B\Pi'') \to \pi_1 \,\mathrm{map}_{BG}(Y, B\Pi') \\ &\to \mathrm{H}^1_G(Y, A) \to \pi_0 \,\mathrm{map}_{BG}(Y, B\Pi'') \to \pi_0 \,\mathrm{map}_{BG}(Y, B\Pi') \to \mathrm{H}^2_G(Y, A). \end{aligned}$$

In particular, the obstruction to lifting a homotopy class of maps $Y \to B\Pi'$ to $B\Pi''$ lies in $\mathrm{H}^2_G(Y, A)$, and the ambiguity in this lift is given by an action of $\mathrm{H}^1_G(Y, A)$ on the fibres.

Remark 2.3 Given an abelian extension $\Pi'' \to \Pi'$ of pro-simplicial groups with kernel A, such that the conjugation action of Π' on A factors through some quotient G of Π', there is a natural fibration sequence

$$\mathrm{map}_{\overline{W}G}(Y, \overline{W}\Pi'') \to \mathrm{map}_{\overline{W}G}(Y, \overline{W}\Pi') \to \mathrm{map}_{\overline{W}G}(Y, \overline{W}(G \ltimes \overline{W}A))$$

of mapping spaces for any pro-space Y over $\overline{W}G$.

Example 2.4 In order to understand the first obstruction map

$$\mathrm{ob}\colon \pi_0 \,\mathrm{map}_{BG}(Y, B\Pi') \to \mathrm{H}^2_G(Y, A)$$

explicitly, consider the case when Y is reduced and connected, so $Y_0 = *$ and an element of $\pi_0 \,\mathrm{map}_{BG}(Y, B\Pi')$ is a conjugacy class of pro-group homomorphisms $\alpha\colon \pi_1(Y) \to \Pi'$ over G. Here, $\pi_1 Y$ is a pro-group with generators Y_1 and relations $\partial_1 y = \partial_0 y \cdot \partial_2 y$ for $y \in Y_2$. Since $\Pi'' \to \Pi'$ is surjective, we may lift α to a morphism $\tilde{\alpha}\colon Y_1 \to \Pi''$ of pro-sets. The obstruction $\mathrm{ob}(\alpha)$ then measures the failure of $\tilde{\alpha}$ to be a group homomorphism, in the form of the 2–cocycle

$$(y \in Y_2) \mapsto \tilde{\alpha}(\partial_2 y)\tilde{\alpha}(\partial_1 y)^{-1}\tilde{\alpha}(\partial_0 y).$$

2.2 Nilpotent obstruction towers

We can of course iterate the construction of Remark 2.3, by considering towers $\cdots \to \Pi_{n+1} \to \Pi_n \to \cdots \to \Pi_0 = G$ of surjections whose kernels are abelian G–representations. The motivating examples are given by the quotients of $\pi_1^{\text{ét}}(X)$ by the lower central series of $\pi_1^{\text{ét}}(\bar{X})$, and by their pro-$p$ completions relative to $G_{F,\Sigma}$ for $p \in \Sigma$.

Writing A_n for the kernel of $\Pi_n \to \Pi_{n-1}$ and $\Pi_\infty := \varprojlim_n \Pi_n$, we then have an exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_* \operatorname{map}_{BG}(Y, B\Pi_n) \arrow[r] & \pi_* \operatorname{map}_{BG}(Y, B\Pi_{n-1}) \arrow[r] \arrow[dl, dashed, "\delta"] & \cdots \arrow[r] \arrow[dl, dashed, "\delta"] & \pi_* \operatorname{map}_{BG}(Y, B\Pi_1) \\
& \mathrm{H}_G^{1-*}(Y, A_n) \arrow[u] & \mathrm{H}_G^{1-*}(Y, A_{n-1}) \arrow[u] & \cdots & \mathrm{H}_G^{1-*}(Y, A_1) \arrow[u, "\simeq"]
\end{tikzcd}$$

similar to that in [11, Section VI.2], but with the extra final terms $\mathrm{H}_G^2(Y, A_n)$. Here, the connecting homomorphism δ is of homological degree -1, so we have

$$\delta\colon \pi_i \operatorname{map}_{BG}(Y, B\Pi_{n-1}) \to \mathrm{H}_G^{2-i}(Y, A_n).$$

This induces a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}_G^{1+s-t}(Y, A_s) \Rightarrow \pi_{t-s} \operatorname{map}_{BG}(Y, B\Pi_\infty)$$

of groups and sets, where the terms $E_1^{s,t}$ are only defined for $t \geq \max(s-1, 0)$, and the indexing convention follows [11, Section VI.2], with $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. Unlike the fringed Bousfield–Kan spectral sequence of [11, Section VI.2], we have terms $E_r^{t+1,t}$ ensuring that we can recover the images of

$$\pi_0 \operatorname{map}_{BG}(Y, B\Pi_\infty) \to \pi_0 \operatorname{map}_{BG}(Y, B\Pi_s)$$

from our spectral sequence.

Explicitly, writing

$$\pi_i M_s^{(r)} := \operatorname{Im}(\pi_i \operatorname{map}_{BG}(Y, B\Pi_{s+r}) \to \pi_i \operatorname{map}_{BG}(Y, B\Pi_s)),$$

there are long exact sequences

$$\cdots \to E_r^{s-r+1,t-r+2} \to \pi_{t-s+1} M_{s-r+1}^{(r-1)} \to \pi_{t-s+1} M_{s-r}^{(r-1)} \\ \to E_r^{s,t} \to \pi_{t-s} M_s^{(r-1)} \to \pi_{t-s} M_{s-1}^{(r-1)} \to \cdots$$

(as in [11, Lemma VI.2.8], but with extra final terms $\pi_0 M_{t+1-r}^{(r-1)} \to E_r^{t+1,t}$).

The first page just corresponds to the exact sequences

$$0 \to \mathrm{H}_G^0(Y, A_s) \to \pi_1 M_s^{(0)} \to \pi_1 M_{s-1}^{(0)} \to \mathrm{H}_G^1(Y, A_s) \to \pi_0 M_s^{(0)} \to \pi_0 M_{s-1}^{(0)} \\ \to \mathrm{H}_G^2(Y, A_s).$$

Example 2.5 (nilpotent completion of $\pi_1^{\text{ét}}(\overline{X})$) If X is a scheme over F, and $\overline{X} := X \otimes_F \overline{F}$, with some geometric point $\bar{x}$, then the simplest examples are given by taking lower central series

$$\Pi_n := \pi_1^{\text{ét}}(X, \bar{x}) / [\pi_1^{\text{ét}}(\overline{X}, \bar{x})]_{n+1},$$

where for a pro-finite group π we define $[\pi]_{k+1}$ inductively to be the closure of $[\pi, [\pi]_k]$, with $[\pi]_1 := \pi$.

Thus $\Pi_0 = G_F$, and, taking $Y = BG_F$, we get the nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_F, [\bar{\pi}]_s / [\bar{\pi}]_{s+1}) \Rightarrow \pi_{t-s} \operatorname{map}_{BG_F}(BG_F, B\Pi_\infty)$$

of groups and sets, where we write $\bar{\pi} := \pi_1^{\text{ét}}(\overline{X}, \bar{x})$. If $\bar{x}$ lies over a point in $X(F)$, then Π_∞ is just the semidirect product of G_F and the pro-nilpotent completion of $\pi_1^{\text{ét}}(\overline{X}, \bar{x})$.

Since points in $X(F)$ map to elements in $\pi_0 \operatorname{map}_{BG_F}(BG_F, B\Pi_\infty)$, this spectral sequence gives obstructions to the existence of such rational points. The same constructions work when X is a Deligne–Mumford stack instead of a scheme, in which case we have a morphism from the groupoid $X(\mathscr{O}_F)$ to the fundamental groupoid $\pi_f \operatorname{map}_{BG_F}(BG_F, B\Pi_\infty)$.

The maps $d_r \colon E_r^{1,1} \to E_r^{r+1,r}$ are just Ellenberg's obstructions, which can be described in terms of Massey products as in Wickelgren's thesis [43].

Another variant is given by taking a smooth scheme X over $\mathscr{O}_{F,\Sigma}$ admitting a smooth relative compactification, and setting $\overline{X} := X \otimes_{\mathscr{O}_{F,\Sigma}} \mathscr{O}_{\overline{F},\Sigma}$, with some geometric point $\bar{x}$. For a prime ℓ all of whose F–prime factors lie in Σ, we can consider the relative pro-ℓ completion (or more generally pro-nilpotent pro-L for a set L of such primes) of $\pi_1^{\text{ét}}(X, \bar{x})$ over $G_{F,\Sigma}$ in the sense of [16], which will have the effect of replacing $[\bar{\pi}]_s / [\bar{\pi}]_{s+1}$ with ℓ–torsion groups — the corresponding maps are described in [44].

Alternatively, if we replaced BG_F with the étale homotopy type of an F–scheme Z, we would instead obtain topological obstructions to the existence of a map $Z \to X$ over F.

Example 2.6 (relative completion of $\pi_1^{\text{ét}}(\overline{X})$: descent obstructions) When the geometric fundamental group of X is perfect, its nilpotent completion is trivial, so the construction of Example 2.5 gives no information. However, we can remedy this by taking the completion relative to a larger group than G_F. We may take any quotient P of $\pi_1^{\text{ét}}(X,\bar{x})$ bigger than G_F, then write $K := \ker(\pi_1^{\text{ét}}(X,\bar{x}) \to P)$, and set $\Pi_n := \pi_1^{\text{ét}}(X,\bar{x})/[K]_{n+1}$.

This gives a nonabelian spectral sequence

$$E_1^{s,t} \Rightarrow \pi_{t-s}\,\mathrm{map}_{BG_F}(BG_F, B\Pi_\infty),$$

where

$$E_1^{s,t} = \begin{cases} \mathrm{H}^{1+s-t}(G_F, [K]_s/[K]_{s+1}), & s \geq 1, \\ \pi_t\,\mathrm{map}_{BG_F}(BG_F, BP), & s = 0, \end{cases}$$

of groups and sets. Here, the Galois action on $[K]_s/[K]_{s+1}$ depends on the relevant section $\sigma \in \pi_0\,\mathrm{map}_{BG_F}(BG_F, BP)$.

When P is a finite extension of G_F, each section σ as above gives a finite étale group scheme P^σ over F with $P^\sigma(\bar{F}) \cong \ker(P \to G_F)$, and hence BP^σ having étale homotopy type BP. Even when P is not a finite extension of G_F, we can write it as a filtered limit $\varprojlim_\alpha P_\alpha$ of such finite extensions, with each section σ giving a pro-(finite étale) group scheme $P^\sigma = \varprojlim_\alpha P_\alpha^\sigma$ over F. Maps $X_{\text{ét}} \to BP$ then correspond to P^σ–torsors $f^\sigma\colon Y^\sigma \to X$, and we may substitute $K \cong \pi_1(\bar{Y}^\sigma$ in the spectral sequence above.

Example 2.7 (relative completion of $\pi_1^{\text{ét}}(\bar{Y}_\Gamma)$) As a special case of Example 2.6, take a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$; we may then form a stacky modular curve Y_Γ over a number field F. The geometric fundamental group $\pi_1^{\text{ét}}(Y_\Gamma,\bar{x})$ is the pro-finite completion $\hat{\Gamma}$ of Γ, so a point $x \in Y_\Gamma(F)$ gives an isomorphism $\pi_1^{\text{ét}}(Y_\Gamma,\bar{x}) \cong \Gamma \rtimes G_F$. The Tate module of the universal elliptic curve over Y_Γ gives rise to a $\hat{\mathbb{Z}}$–local system of rank 2 on Y_Γ, and hence a map

$$\pi_1^{\text{ét}}(Y_\Gamma) \to \mathrm{GL}_2(\hat{\mathbb{Z}})$$

(for any choice of basepoint).

Since the local system has determinant $\hat{\mathbb{Z}}(1)$, this induces a map

$$\pi_1^{\text{ét}}(Y_\Gamma) \to \mathrm{GL}_2(\hat{\mathbb{Z}}) \times_{\mathbb{G}_m(\hat{\mathbb{Z}})} G_F,$$

and we may then take the relative pro-nilpotent completion over the image, or the relative pro-ℓ completion over the image in $\mathrm{GL}_2(\mathbb{Z}_\ell) \times_{\mathbb{G}_m(\mathbb{Z}_\ell)} G_F$. Since the maps $\mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{GL}_2(\mathbb{F}_\ell)$ are pro-ℓ extensions, completion relative to $\mathrm{GL}_2(\mathbb{F}_\ell)$ gives the same limit from a different tower.

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, with $\Gamma(N) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N))$, the spectral sequence resulting from the pro-nilpotent tower relative to $\mathrm{GL}_2(\mathbb{Z}/N) \times_{\mathbb{G}_m(\mathbb{Z}/N)} G_F$ is

$$\mathrm{H}^{1+s-t}(G_F, \widehat{[\Gamma(N)]_s/[\Gamma(N)]_{s+1}}) \Rightarrow \pi_{t-s}\,\mathrm{map}_{B(\mathrm{GL}_2(\mathbb{Z}/N)\times_{\mathbb{G}_m(\mathbb{Z}/N)}G_F)}(BG_F, B\Pi_\infty),$$

where $\Pi_\infty := \varprojlim_s \pi_1^{\text{ét}}(Y_\Gamma)/[\widehat{\Gamma}(N)]_s$.

The spectral sequence relative to $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \times_{\mathbb{G}_m(\widehat{\mathbb{Z}})} G_F$ instead has

$$\mathrm{H}^{1+s-t}\big(G_F, \varprojlim_N \widehat{[\Gamma(N!)]_s/[\Gamma(N!)]_{s+1}}\big) \Rightarrow \pi_{t-s}\,\mathrm{map}_{B(\mathrm{GL}_2(\widehat{\mathbb{Z}})\times_{\mathbb{G}_m(\widehat{\mathbb{Z}})}G_F)}(BG_F, B\Pi_\infty)$$

for $\Pi_\infty = \varprojlim_{s,N} \pi_1^{\text{ét}}(Y_\Gamma)/[\widehat{\Gamma}(N!)]_s$.

2.3 Unipotent extensions

We now look to consider towers $\cdots \to \Pi_{n+1} \to \Pi_n \to \cdots \to \Pi_0$ of unipotent extensions of Lie groups over $\mathbb{Q}_\ell$, where all F–prime factors of ℓ lie in of our set Σ of places of F.

Definition 2.8 Say that a simplicial group H is bounded if its Dold–Kan normalisation NH $\big($given by $N_nH = H_n \cap \bigcap_{i>0} \ker \partial_i\big)$ is so.

Lemma 2.9 *If U is a bounded simplicial unipotent algebraic group over $\mathbb{Q}_\ell$, equipped with a continuous action of a pro-finite group G, then $U(\mathbb{Q}_\ell)$ is the filtered colimit of its bounded simplicial pro-finite G–equivariant subgroups.*

Proof This is a slight generalisation of [34, Lemmas 3.10 and 3.14], which address the case where the G–action is semisimple. Standard arguments give a G–equivariant bounded simplicial $\mathbb{Z}_\ell$–submodule Λ of the Lie algebra $\mathfrak{u}$ of U, with Λ of finite rank and $\Lambda \otimes \mathbb{Q}_\ell \twoheadrightarrow \mathfrak{u}$. The closure $g(\Lambda)$ of Λ under monomial operations in the Campbell–Baker–Hausdorff product is still bounded and of finite rank, as $\mathfrak{u}$ is nilpotent, and the groups $g(\ell^{-n}\Lambda)$ realise $U(\mathbb{Q}_\ell)$ as a filtered colimit of the required form. □

Corollary 2.10 *Take an affine algebraic group T over $\mathbb{Q}_\ell$ and a surjection $\Pi \to T$ of simplicial affine group schemes, with $U := \ker(\Pi \to R)$ bounded unipotent. Then for any Zariski-dense pro-finite group $G \subset T(\mathbb{Q}_\ell)$, the simplicial topological group*

$$\Pi(\mathbb{Q}_\ell) \times_{T(\mathbb{Q}_\ell)} G$$

is a filtered colimit of those simplicial pro-finite subgroups which are bounded nilpotent extensions of G.

Proof Since $\Pi(\mathbb{Q}_\ell)\times_{T(\mathbb{Q}_\ell)}G$ is the fibre of $\Pi(\mathbb{Q}_\ell)\times_{T^{\mathrm{red}}(\mathbb{Q}_\ell)}G \to T(\mathbb{Q}_\ell)\times_{T^{\mathrm{red}}(\mathbb{Q}_\ell)}G$, it suffices to prove this for T reductive. As in [31], the simplicial unipotent extension $\Pi \to T$ then admits a section (ie a Levi decomposition), unique up to conjugation by $U(\mathbb{Q}_\ell)$; this gives an isomorphism $\Pi \cong T \ltimes U$. Since G is Zariski dense in the reductive group T, its action is semisimple so we may appeal to Lemma 2.9, writing

$$G \times_{T(\mathbb{Q}_\ell)} \Pi(\mathbb{Q}_\ell) \cong G \ltimes U \cong \varinjlim_{\alpha} G \ltimes N_\alpha$$

for bounded G–equivariant simplicial pro-finite subgroups N_α of U. □

The nerve $\overline{W}(\Pi(\mathbb{Q}_\ell)\times_{T(\mathbb{Q}_\ell)}G)$ is then an ind-pro-simplicial set, and defining mapping spaces for these by the usual convention

$$\mathrm{map}(Y, \{Z_\alpha\}) := \varinjlim_{\alpha} \mathrm{map}(Y, Z_\alpha)$$

for Y and Z_α pro-finite, we may apply Proposition 2.2 to unipotent extensions, by passing to filtered colimits:

Proposition 2.11 *Take a unipotent extension $\Pi'' \to \Pi'$ of algebraic groups over $\mathbb{Q}_\ell$ with commutative kernel A such that the conjugation action of Π' on A factors through some quotient Π of Π'. Then for any Zariski-dense map $G \to \Pi(\mathbb{Q}_\ell)$ with G pro-finite, and for any pro-simplicial set Y over BG, there is a natural fibration sequence*

$$\mathrm{map}_{BG}(Y, B(\Pi'' \times_\Pi G)) \to \mathrm{map}_{BG}(Y, B(\Pi' \times_\Pi G)) \to \mathrm{map}_{BG}(Y, B(G \ltimes BA))$$

of mapping spaces, the fibre being taken over the zero map $Y \to BG \to B(G \ltimes BA)$.

Example 2.12 (unipotent completion of $\pi_1^{\text{ét}}(\overline{X})$) If X is a smooth scheme over $\mathscr{O}_{F,\Sigma}$ admitting a smooth relative compactification, and $\overline{X} := X \otimes_{\mathscr{O}_{F,\Sigma}} \mathscr{O}_{\overline{F},\Sigma}$, with some geometric point $\bar{x}$, then we may use Proposition 2.11 to give a variant of Example 2.5. For simplicity, assume that we have a point $x \in X(\mathscr{O}_{F,\Sigma})$ under $\bar{x}$ (if not, we can recover analogues of the constructions below by taking a $G_{F,\Sigma}$–equivariant set $\mathcal{B} \subset X(\mathscr{O}_{\overline{F},\Sigma})$, then consider the $G_{F,\Sigma}$–equivariant surjection from the groupoid $\pi_1^{\text{ét}}(\overline{X}, \mathcal{B})$ to the contractible groupoid on objects $\mathcal{B}$).

Now, Friedlander [9] shows that $\overline{X}_{\text{ét}}$ has equivalent derived pro-Σ completion to the homotopy fibre of $X_{\text{ét}} \to (\mathrm{Spec}\, \mathscr{O}_{F,\Sigma})_{\text{ét}}$. Since the section x splits the long exact sequence of homotopy groups, this gives an isomorphism between the relative pro-Σ

completion of $\pi_1^{\text{ét}}(X, \bar{x})$ over $G_{F,\Sigma}$ and the semidirect product $G_{F,\Sigma} \ltimes \pi_1^{\text{ét}}(\overline{X}, \bar{x})^{\wedge\Sigma}$. We then consider the lower central series

$$\Pi_n := G_{F,\Sigma} \ltimes (\pi_1^{\text{ét}}(\overline{X}, \bar{x}) \otimes \mathbb{Q}_\ell)/[\pi_1^{\text{ét}}(\overline{X}, \bar{x}) \otimes \mathbb{Q}_\ell]_n$$

of the pro-unipotent Maltsev completion $\pi_1^{\text{ét}}(\overline{X}, \bar{x}) \otimes \mathbb{Q}_\ell$.

Thus $\Pi_0 = G_{F,\Sigma}$, and taking $Y = BG_{F,\Sigma}$ we get a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_{F,\Sigma}, [\bar{\pi} \otimes \mathbb{Q}_\ell]_s/[\bar{\pi} \otimes \mathbb{Q}_\ell]_{s+1}) \Rightarrow \pi_{t-s}\, \mathrm{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma}, B\Pi_\infty)$$

of groups and sets, where we write $\bar{\pi} := \pi_1^{\text{ét}}(\overline{X}, \bar{x})$.

Although this gives weaker obstructions than Example 2.5, the obstruction spaces are easier to calculate. The vector spaces $[\bar{\pi} \otimes \mathbb{Q}_\ell]_s/[\bar{\pi} \otimes \mathbb{Q}_\ell]_{s+1}$ are the graded pieces of a pro-nilpotent Lie algebra with generators $\mathrm{H}_1(\overline{X}, \mathbb{Q}_\ell)$ and relations non-canonically isomorphic to $\mathrm{H}_2(\overline{X}, \mathbb{Q}_\ell)$. Since points in $X(\mathscr{O}_{F,\Sigma})$ map to elements in $\pi_0\, \mathrm{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma}, B\Pi_\infty)$, this spectral sequence gives obstructions to the existence of such rational points.

2.4 Pro-unipotent extensions

Relative Maltsev completion was introduced by Hain in [12] for discrete groups, and as in [32] we consider the natural generalisation to pro-finite groups as follows:

Definition 2.13 Given a topological group Γ, a reductive pro-algebraic group R over $\mathbb{Q}_\ell$ and a Zariski-dense continuous representation $\rho\colon \Gamma \to R(\mathbb{Q}_\ell)$, define the Maltsev completion $(\Gamma)^{\rho,\mathrm{Mal}}$ to be the universal diagram

$$\Gamma \to \Gamma^{\rho,\mathrm{Mal}}(\mathbb{Q}_\ell) \xrightarrow{p} R(\mathbb{Q}_\ell),$$

with $p\colon \Gamma^{\rho,\mathrm{Mal}} \xrightarrow{p} R$ a pro-unipotent extension and the composition equal to ρ.

When the representation ρ is clear from the context, we will write $\Gamma^{R,\mathrm{Mal}} := \Gamma^{\rho,\mathrm{Mal}}$.

Remark 2.14 The pro-unipotent radical $\mathrm{R_u}(\Gamma, \rho)^{\mathrm{Mal}}$ is then given by $\exp(\mathfrak{u})$ for a pro-(finite-dimensional nilpotent) Lie algebra $\mathfrak{u}$. For $O(R)$ the ring of algebraic functions on R over $\mathbb{Q}_\ell$, equipped with its left R–action, the abelianisation of $\mathfrak{u}$ is dual to the continuous cohomology $\mathrm{H}^1(\Gamma, O(R))$, and there is a presentation of $\mathfrak{u}$ with relations dual to $\mathrm{H}^2(\Gamma, O(R))$. In particular, if $\mathrm{H}^2(\Gamma, O(R)) = 0$, then there are canonical isomorphisms

$$[\mathrm{R_u}(\Gamma, \rho)^{\mathrm{Mal}}]_n/[\mathrm{R_u}(\Gamma, \rho)^{\mathrm{Mal}}]_{n+1} \cong \big(\mathrm{CoLie}_n \mathrm{H}^1(\Gamma, O(R))\big)^*,$$

where $\mathrm{CoLie}_n(V) = \mathrm{Lie}(n)^* \otimes_{S_n} V^{\otimes n}$ for the Lie operad Lie. Explicitly, when V is finite-dimensional, $(\mathrm{CoLie}_n V)^*$ is the subspace of the free Lie algebra on generators V^* consisting of homogeneous terms of bracket length n.

Also note that if Γ is a discrete group and $\widehat{\Gamma}$ its pro-finite completion, then for any representation ρ of $\widehat{\Gamma}$, the map $\Gamma^{\rho,\mathrm{Mal}} \to \widehat{\Gamma}^{\rho,\mathrm{Mal}}$ is necessarily an isomorphism.

Examples 2.15 ($\widehat{\mathrm{SL}_2(\mathbb{Z})}$) Our main motivating example is to take $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and its pro-finite completion $\widehat{\Gamma}$, with $R = \mathrm{SL}_2$ (regarded as a group scheme over $\mathbb{Q}_\ell$) and $\widehat{\Gamma} \to \mathrm{SL}_2(\mathbb{Q}_\ell)$ the natural map.

Since the ring $O(\mathrm{SL}_2)$ of functions is given by $\bigoplus_m V_m \otimes (UV_m)^*$ for V_m the irreducible SL_2–representation of dimension $m+1$ over $\mathbb{Q}_\ell$ and UV_m the underlying vector space, we have

$$\mathrm{H}^*(\Gamma, O(\mathrm{SL}_2)) \cong \bigoplus_m \mathrm{H}^*(\Gamma, V_m) \otimes V_m^*.$$

Thus $\mathrm{H}^2(\Gamma, O(\mathrm{SL}_2)) = 0$, and Eichler–Shimura gives a description of the space $\mathrm{H}^1(\Gamma, O(\mathrm{SL}_2)) \otimes \mathbb{C}$ in terms of the decomposition of $\mathrm{H}^1(\Gamma, V_m) \otimes \mathbb{C}$ into modular forms and cusp forms of weight $m+2$ and level 1.

Our groups of interest are $\mathrm{H}^1(\Gamma, V_m)$ We may think of the spaces $\mathrm{H}^1(\Gamma, V_m)$ as ℓ–adic realisations of motives of modular forms, as in [5]. These $\mathbb{Q}_\ell$–vector spaces admit $G_{\mathbb{Q}}$–actions via the interpretation as summands of $\mathrm{H}^{m+1}_{\text{ét}}(\mathcal{M}_{1,m+1} \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)(m)$, interpreting $\mathcal{M}_{1,m+1}$ as the m–fold product of the universal elliptic curve $\mathcal{M}_{1,2}$ over the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves (the Tate twists arise because we wish to regard V_1 as a Tate module rather than its dual).

More generally, we can take Γ to be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, giving a similar expression involving modular forms of higher levels, but with relations coming from $\mathrm{H}^2(\Gamma, O(\mathrm{SL}_2))$ whenever it is nonzero.

Alternatively, we can look at the relative Maltsev completion of the canonical morphism

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\widehat{\mathbb{Z}}) \times \mathrm{SL}_2(\mathbb{Q}_\ell) =: R,$$

where we regard the pro-finite group $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ as an affine group scheme over $\mathbb{Q}_\ell$. Then we still have $\mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}), O(R)) \otimes \mathbb{Q} = 0$, and Leray–Serre gives

$$\mathrm{H}^*(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2(\widehat{\mathbb{Z}})) \otimes V) \cong \varinjlim_N \mathrm{H}^*(\Gamma(N!), V),$$

so

$$\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(R)) \cong \bigoplus_m \varinjlim_N \mathrm{H}^1(\Gamma(N!), V_m) \otimes V_m^*,$$

giving generators of $\mathrm{R_u}\widehat{\mathrm{SL}_2(\mathbb{Z})}^{(\mathrm{SL}_2(\hat{\mathbb{Z}})\times\mathrm{SL}_2),\mathrm{Mal}} = \mathrm{R_u}\mathrm{SL}_2(\mathbb{Z})^{(\mathrm{SL}_2(\hat{\mathbb{Z}})\times\mathrm{SL}_2),\mathrm{Mal}}$ in terms of modular and cusp forms of all weights and levels.

We can also just look at the relative Maltsev completion of the canonical morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\hat{\mathbb{Z}})$, again regarding $\mathrm{SL}_2(\hat{\mathbb{Z}})$ as an affine group scheme over $\mathbb{Q}_\ell$. We then have

$$\mathrm{H}^1\big(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2(\hat{\mathbb{Z}}))\big) \cong \varinjlim_N \mathrm{H}^1(\Gamma(N!), \mathbb{Q}_\ell)$$

with the corresponding H^2 vanishing, giving generators for $\mathrm{R_u}\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2(\hat{\mathbb{Z}}),\mathrm{Mal}}$ in terms of modular and cusp forms of weight 2 and all levels.

For our purposes, Proposition 2.2 is now not quite general enough, as our group schemes might not be of finite type. Consider an affine group scheme T over $\mathbb{Q}_\ell$ and a surjection $\Pi \to T$ of simplicial affine group schemes, with $U := \ker(\Pi \to R)$ bounded pro-unipotent, together with a Zariski-dense pro-finite group $G \subset T(\mathbb{Q}_\ell)$. We can then canonically write the morphism $\Pi \to T$ as a filtered limit of unipotent extensions $\Pi_a \to T_a$ of affine algebraic groups, with Corollary 2.10 giving that $\Pi_a(\mathbb{Q}_\ell) \times_{T_a(\mathbb{Q}_\ell)} G$ is an ind-pro-finite group, so $\Pi(\mathbb{Q}_\ell) \times_{T(\mathbb{Q}_\ell)} G$ is naturally a pro-ind-pro-finite group.

The nerve $\overline{W}(\Pi(\mathbb{Q}_\ell) \times_{T(\mathbb{Q}_\ell)} G)$ is then a pro-ind-pro-simplicial set, and, defining mapping spaces for these by the usual convention

$$\mathrm{map}(Y, \{Z_a\}) := \operatorname*{\underleftarrow{holim}}_a \mathrm{map}(Y, Z_a),$$

for Y pro-finite and Z_a ind-pro-finite, Proposition 2.11 extends verbatim to pro-unipotent extensions $\Pi'' \to \Pi'$.

Example 2.16 (modular forms of level 1) If $X = \mathcal{M}_{1,1}$ is the stacky modular curve over $\mathscr{O}_{F,\Sigma}$, and $x \in X(\mathscr{O}_{F,\Sigma})$, then the Tate module gives a surjective homomorphism $\pi_1^{\text{ét}}(\overline{X}, \bar{x}) \to \mathrm{SL}_2(\mathbb{Z}_\ell)$ whose relative pro-ℓ completion (or equivalently that over $\mathrm{SL}_2(\mathbb{F}_\ell)$) is the same as that of $\mathrm{SL}_2(\mathbb{Z})$. In particular, there is a natural action of $G_{F,\Sigma}$ on the relative Maltsev completion $\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2,\mathrm{Mal}} \cong \pi_1^{\text{ét}}(\overline{X}, \bar{x})^{\mathrm{SL}_2,\mathrm{Mal}}$, and we may consider the pro-unipotent extension

$$G_{F,\Sigma} \ltimes (\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2,\mathrm{Mal}} \times_{\mathrm{SL}_2(\mathbb{Q}_\ell)} \mathrm{SL}_2(\mathbb{Z}_\ell)) \to G_{F,\Sigma} \ltimes \mathrm{SL}_2(\mathbb{Z}_\ell),$$

setting

$$\Pi_n := G_{F,\Sigma} \ltimes \big((\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2,\mathrm{Mal}}/[\mathrm{R_u}]_{n+1}) \times_{\mathrm{SL}_2(\mathbb{Q}_\ell)} \mathrm{SL}_2(\mathbb{Z}_\ell)\big).$$

As in Example 2.7, for the representation $G_{F,\Sigma} \to \mathbb{Z}_\ell^*$ given by the Tate motive $\mathbb{Z}_\ell(1)$, we have $G_{F,\Sigma} \ltimes \mathrm{SL}_2(\mathbb{Z}_\ell) \cong G_{F,\Sigma} \times_{\mathbb{Z}_\ell^*} \mathrm{GL}_2(\mathbb{Z}_\ell)$, so a section of the projection $G_{F,\Sigma} \ltimes \mathrm{SL}_2(\mathbb{Z}_\ell) \to G_{F,\Sigma}$ is equivalent to giving a $G_{F,\Sigma}$–representation Λ of rank 2 over $\mathbb{Z}_\ell$, with determinant $\mathbb{Z}_\ell(1)$.

For the universal elliptic curve $f\colon E \to X$, we have the Tate module $\mathbb{T}_\ell := (\boldsymbol{R}^1 f_* \mathbb{Z}_\ell)^* \cong \boldsymbol{R}^1 f_* \mathbb{Z}_\ell(1)$, a lisse $\mathbb{Z}_\ell$–sheaf of rank 2 on X, giving a $G_{F,\Sigma}$–action on $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m)$ by identifying it with $\boldsymbol{R}^1 q_*(S^m \mathbb{T}_\ell) \otimes \mathbb{Q}$ for the structure morphism $q\colon X \to \operatorname{Spec} \mathscr{O}_{F,\Sigma}$.

Write

$$\begin{aligned} L_s &:= \mathrm{CoLie}_s\Big(\bigoplus_m \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m) \otimes S^m(\Lambda)^*\Big), \\ &\cong \mathrm{CoLie}_s\Big(\bigoplus_m \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m) \otimes S^m(\Lambda)(-m)\Big). \end{aligned}$$

Adapting Example 2.5, the pro-unipotent generalisation of Proposition 2.11 then combines with Examples 2.15 to give a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_{F,\Sigma}, L_s^*) \Rightarrow \pi_{t-s} \operatorname{map}_{B(G_{F,\Sigma} \times_{\mathbb{G}_m(\mathbb{Z}_\ell)} \mathrm{GL}_2(\mathbb{Z}_\ell))}(BG_{F,\Sigma}, B\Pi_\infty),$$

where the map $G_{F,\Sigma} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ is given by Λ. Note that $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m)(-m)$ is mixed of weights $m+1$ (cusp forms and their conjugates) and $2m+2$ (Eisenstein series), and that $S^m \Lambda_{\mathbb{Q}}$ is pure of weight $-m$. Thus $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m) \otimes S^m(\Lambda)(-m)$ is mixed of weights 1 and $m+2$, so L_s is of strictly positive weights, and $E^1_{s,s+1} = 0$.

Now set $X_{(n)} := \operatorname{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma}, B\Pi_n)$; thus $X_{(0)}$ consists of representations $G_{F,\Sigma} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ whose determinant is the Tate motive, conjugation by $\mathrm{SL}_2(\mathbb{Z}_\ell)$ giving equivalences, so $\pi_1(X_{(0)}, [\Lambda])$ consists of elements of $\mathrm{SL}_2(\mathbb{Z}_\ell)$ commuting with the action of $G_{F,\Sigma}$ on Λ. Since $\pi_i X_{(n)} = 0$ for $i > 1$, we then have exact sequences

$$0 \to \pi_1 X_{(n)} \to \pi_1 X_{(n-1)} \to \mathrm{H}^1(F, L_n^*) \to \pi_0 X_{(n)} \to \pi_0 X_{(n-1)} \to \mathrm{H}^2(F, L_n^*),$$

with a map $X(\mathscr{O}_{F,\Sigma}) \to X_{(\infty)}$. Here, $X(\mathscr{O}_{F,\Sigma})$ is the nerve of the groupoid of maps $\operatorname{Spec} \mathscr{O}_{F,\Sigma} \to X$, so $\pi_0 X(\mathscr{O}_{F,\Sigma})$ is the set of isomorphism classes of elliptic curves over $\mathscr{O}_{F,\Sigma}$, and $\pi_1(X(\mathscr{O}_{F,\Sigma}), x)$ the group of automorphisms of the elliptic curve E_x over $\mathscr{O}_{F,\Sigma}$; the higher homotopy groups all vanish.

In other words, given a $G_{F,\Sigma}$–representation Λ of rank 2 over $\mathbb{Z}_\ell$, with determinant $\mathbb{Z}_\ell(1)$, these sequences give a tower of obstructions to lifting Λ to an elliptic curve over $\mathscr{O}_{F,\Sigma}$ with Tate module Λ, and characterise the ambiguity of the lift at each stage. As in Examples 2.15, there is an entirely similar treatment for pro-finite completions of congruence subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, replacing $\mathcal{M}_{1,1}$ with the modular curve Y_Γ.

Example 2.17 (modular forms of all levels) Again taking $X = \mathcal{M}_{1,1}$ to be the stacky modular curve over a number field F and $x \in X(F)$, we may consider the pro-unipotent extension

$$G_F \ltimes (\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2 \times \mathrm{SL}_2(\hat{\mathbb{Z}}),\mathrm{Mal}} \times_{\mathrm{SL}_2(\mathbb{Q}_\ell)\times \mathrm{SL}_2(\hat{\mathbb{Z}})} \mathrm{SL}_2(\hat{\mathbb{Z}})) \to G_F \ltimes \mathrm{SL}_2(\hat{\mathbb{Z}}),$$

setting

$$\Pi_n := G_F \ltimes ((\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2 \times \mathrm{SL}_2(\hat{\mathbb{Z}}),\mathrm{Mal}}/[\mathrm{R_u}]_{n+1}) \times_{\mathrm{SL}_2(\mathbb{Q}_\ell)\times \mathrm{SL}_2(\hat{\mathbb{Z}})} \mathrm{SL}_2(\hat{\mathbb{Z}})).$$

Choose a section of the projection $G_F \ltimes \mathrm{SL}_2(\hat{\mathbb{Z}}) \to G_F$; this is equivalent to giving a G_F–representation Λ of rank 2 over $\hat{\mathbb{Z}}$, with determinant $\hat{\mathbb{Z}}(1)$. Write

$$M_s := \mathrm{CoLie}_s\Big(\bigoplus_m \varinjlim_N \mathrm{H}^1(\Gamma(N!), V_m) \otimes_{\hat{\mathbb{Z}}} S^m_{\hat{\mathbb{Z}}}(\Lambda)(-m)\Big).$$

As in Example 2.16, we then have a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_F, M_s^*) \Rightarrow \pi_{t-s}\,\mathrm{map}_{B(G_F \times_{\mathbb{G}_m(\hat{\mathbb{Z}})} \mathrm{GL}_2(\hat{\mathbb{Z}}))}(BG_F, B\Pi_\infty),$$

where the map $G_F \to \mathrm{GL}_2(\hat{\mathbb{Z}})$ is given by Λ. Since

$$O(\mathrm{SL}_2(\hat{\mathbb{Z}})) \otimes \mathrm{H}^1(\Gamma(N!), V_m) \otimes S^m(\Lambda)(-m)$$

is mixed of weights 1 (cusp forms of all levels and their conjugates) and $m+2$ (Eisenstein series of all levels), M_s is of strictly positive weights, and $E^1_{s,s+1} = 0$.

Now set $X_{(n)} := \mathrm{map}_{BG_F}(BG_F, B\Pi_n)$; thus $X_{(0)}$ consists of representations $G_F \to \mathrm{GL}_2(\hat{\mathbb{Z}})$ whose determinant is the Tate motive, conjugation by $\mathrm{SL}_2(\hat{\mathbb{Z}})$ giving equivalences. Since $\pi_i X_{(n)} = 0$ for $i > 1$, we then have exact sequences

$$0 \to \pi_1 X_{(n)} \to \pi_1 X_{(n-1)} \to \mathrm{H}^1(F, M_n^*) \to \pi_0 X_{(n)} \to \pi_0 X_{(n-1)} \to \mathrm{H}^2(F, M_n^*),$$

with a map $X(F) \to X_{(\infty)}$.

Example 2.18 (modular forms of weight 2) Again taking $X = \mathcal{M}_{1,1}$ to be the stacky modular curve over F, and $x \in X(F)$, we may consider the pro-unipotent extension

$$G_F \ltimes \mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2(\widehat{\mathbb{Z}}),\mathrm{Mal}} \to G_F \ltimes \mathrm{SL}_2(\widehat{\mathbb{Z}}),$$

setting

$$\Pi_n := G_F \ltimes (\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2(\widehat{\mathbb{Z}}),\mathrm{Mal}}/[\mathrm{R_u}]_{n+1}).$$

As in Example 2.18, choose a G_F–representation Λ of rank 2 over $\widehat{\mathbb{Z}}$, with determinant $\widehat{\mathbb{Z}}(1)$. Write

$$M_s := \mathrm{CoLie}_s\big(\varinjlim_N \mathrm{H}^1(\Gamma(N!), \mathbb{Q}_\ell)\big);$$

thus M_1 is related to weight 2 modular forms; as a Galois representation it is mixed of weights 1 and 2. We then have a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_F, M_s^*) \Rightarrow \pi_{t-s}\, \mathrm{map}_{B(G_F \times_{\mathbb{G}_m(\widehat{\mathbb{Z}})} \mathrm{GL}_2(\widehat{\mathbb{Z}}))}(BG_F, B\Pi_\infty),$$

where the map $G_F \to \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is given by Λ.

Set $X_{(n)} := \mathrm{map}_{BG_F}(BG_F, B\Pi_n)$; since $\pi_i X_{(n)} = 0$ for $i > 1$, we then have exact sequences

$$0 \to \pi_1 X_{(n)} \to \pi_1 X_{(n-1)} \to \mathrm{H}^1(F, M_n^*) \to \pi_0 X_{(n)} \to \pi_0 X_{(n-1)} \to \mathrm{H}^2(F, M_n^*),$$

with a map $X(F) \to X_{(\infty)}$.

Example 2.19 (étale fundamental groups) For any smooth Deligne–Mumford stack X over $\mathscr{O}_{F,\Sigma}$ admitting a smooth relative compactification, with $x \in X(\mathscr{O}_{F,\Sigma})$, we can generalise the examples above by considering any $G_{F,\Sigma}$–equivariant Zariski-dense representation ρ: $\pi_1^{\text{ét}}(\overline{X}, \bar{x})^{\wedge\Sigma} \to R(\mathbb{Q}_\ell)$ to a pro-reductive affine group scheme R over $\mathbb{Q}_\ell$. If there is no rational basepoint, we can instead take a $G_{F,\Sigma}$–equivariant set $\mathcal{B} \subset X(\mathscr{O}_{\overline{F},\Sigma})$ of basepoints, then consider the $G_{F,\Sigma}$–equivariant surjection from the groupoid $\pi_1^{\text{ét}}(\overline{X}, \mathcal{B})$ to the contractible groupoid on objects $\mathcal{B}$, with relative Maltsev completions as in [31, Section 3.2]).

We may then set

$$\Pi_n := G_{F,\Sigma} \ltimes \big((\pi_1^{\text{ét}}(\overline{X}, \bar{x})^{R,\mathrm{Mal}}/[\mathrm{R_u}]_{n+1}) \times_{R(\mathbb{Q}_\ell)} \rho(\pi_1^{\text{ét}}(\overline{X}, \bar{x})^{\wedge\Sigma})\big),$$

with $P_n := \ker(\Pi_n \to \Pi_{n-1})$ a quotient of $\big(\mathrm{CoLie}_n\big(\mathrm{H}^1(\overline{X}, O(R))\big)\big)^*$ as described in Remark 2.14.

For any section σ of the projection $G_{F,\Sigma} \ltimes \rho(\pi_1^{\text{ét}}(\overline{X}, \bar{x})^{\wedge\Sigma}) \to G_{F,\Sigma}$, we then have a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_{F,\Sigma}, P_s) \Rightarrow \pi_{t-s} \operatorname{map}_{B(G_{F,\Sigma} \ltimes \rho(\pi_1^{\text{ét}}(\overline{X},\bar{x})^{\wedge\Sigma}))}(BG_{F,\Sigma}, B\Pi_\infty),$$

where the map $G_{F,\Sigma} \to G_{F,\Sigma} \ltimes \rho(\pi_1^{\text{ét}}(\overline{X}, \bar{x})^{\wedge\Sigma})$ is given by σ.

Example 2.20 (étale homotopy types) We may refine the previous example by considering étale homotopy types in place of fundamental groups. Take a locally Noetherian simplicial scheme X, and a geometric point $\bar{x}$. We can then form the étale topological type $X_{\text{ét}} \in \mathrm{pro}(\mathbb{S})$ as defined in [10, Definition 4.4]. In particular, we can apply this to a simplicial scheme resolving a locally Noetherian algebraic stack (by [36, Theorem 4.7], such resolutions exist even for higher Artin stacks, and the description of [36, Theorems 4.10 and Remark 4.11] ensures that the choice does not affect the homotopy type).

Note that $(X_{\text{ét}})_0$ is the set of geometric points of X_0 (with some bound imposed on the cardinalities of the associated fields). Consider the reduced pro-simplicial set $(X_{\text{ét}}, \bar{x}) \subset X_{\text{ét}}$ given by setting $(X_{\text{ét}}, \bar{x})_n$ to consist of n–simplices with fixed vertex $\bar{x}$. We may then apply the simplicial loop groupoid functor of [6] to get a pro-simplicial groupoid $GX_{\text{ét}}$, and restricting to the vertex $\bar{x}$ gives a pro-simplicial groupoid $G(X_{\text{ét}}, \bar{x})$ with $\pi_0 G(X_{\text{ét}}, \bar{x}) \cong \pi_1^{\text{ét}}(X, \bar{x})$.

If X is defined over $\mathscr{O}_{F,\Sigma}$, with each X_n admitting a smooth relative compactification, set $\overline{X} := X \otimes_{\mathscr{O}_{F,\Sigma}} \mathscr{O}_{\overline{F},\Sigma}$. Now fix a Zariski-dense representation $\rho\colon \pi_1^{\text{ét}}(X, \bar{x}) \to S(\mathbb{Q}_\ell)$ to a pro-reductive pro-algebraic group S, and let R be the Zariski closure of $\rho(\pi_1^{\text{ét}}(\overline{X}, \bar{x}))$, and set $T := S/R$. We now need to consider fibre sequences, because $G_{F,\Sigma}$ does not explicitly act on our model for $\overline{X}_{\text{ét}}$. If the $G_{F,\Sigma}$–representation $\mathrm{H}^*(\overline{X}, V)$ is an extension of T–representations for all R–representations V, then [34, Theorem 3.32] combines with [9] to give a fibre sequence

$$\overline{W}G(\overline{X}_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}} \to \overline{W}G(X_{\text{ét}}, \bar{x})^{S,\mathrm{Mal}} \to BG(\mathscr{O}_{F,\Sigma,\text{ét}})^{T,\mathrm{Mal}}$$

of pro-algebraic homotopy types over $\mathbb{Q}_\ell$. By [42, Appendix A, equation (1)], $BG_{F,\Sigma}$ and $\mathscr{O}_{F,\Sigma,\text{ét}}$ have isomorphic cohomology for ℓ–torsion coefficients, so $G(BG_{F,\Sigma})^{T,\mathrm{Mal}} \simeq G(\mathscr{O}_{F,\Sigma,\text{ét}})^{T,\mathrm{Mal}}$ and we have a long exact sequence

$$\cdots \to \varpi_n(\overline{X}, \bar{x})^{R,\mathrm{Mal}} \to \varpi_n(X, \bar{x})^{S,\mathrm{Mal}} \to \varpi_n(BG_{F,\Sigma})^{T,\mathrm{Mal}} \to \varpi_{n-1}(\overline{X}, \bar{x})^{R,\mathrm{Mal}} \to$$
$$\cdots \to \pi_1^{\text{ét}}(\overline{X}, \bar{x})^{R,\mathrm{Mal}} \to \pi_1^{\text{ét}}(X, \bar{x})^{S,\mathrm{Mal}} \to G_{F,\Sigma}^{T,\mathrm{Mal}} \to 1$$

of pro-algebraic homotopy groups; in particular we will have an exact sequence of completed fundamental groups whenever $\varpi_2(BG_{F,\Sigma})^{T,\mathrm{Mal}} = 0$, ie if $G_{F,\Sigma}$ is 2–good relative to T in the sense of [34, Definition 3.35; 35, Section 1.2.3].

We may then set $\widetilde{\Pi}_n$ to be the simplicial topological group given by the homotopy fibre product

$$\widetilde{\Pi}_n := (G(X_{\text{ét}}, \bar{x})^{S,\mathrm{Mal}}/[U]_{n+1}) \times^h_{G(BG_{F,\Sigma})^{T,\mathrm{Mal}}} G_{F,\Sigma},$$

where $U = \mathrm{R_u} G(\overline{X}_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}}$; in particular, $\widetilde{\Pi}_0 = S \times_T G_{F,\Sigma}$. Note that since $B\widetilde{\Pi}_\infty$ is equipped with a map from $\overline{W}G(\overline{X}_{\text{ét}}, \bar{x})$, there is a canonical morphism

$$X(\mathscr{O}_{F,\Sigma}) \to \mathrm{map}_{B(G_{F,\Sigma})}(BG_{F,\Sigma}, B\widetilde{\Pi}_\infty)$$

in the homotopy category of pro-ind-pro-simplicial sets.

We will then have a nonabelian spectral sequence

$$E_1^{s,t} \Rightarrow \pi_{t-s}\, \mathrm{map}_{B(G_{F,\Sigma})}(BG_{F,\Sigma}, B\widetilde{\Pi}_\infty),$$

with

$$E_1^{s,t} = \begin{cases} \mathbb{H}^{1+s-t}(G_{F,\Sigma}, [U]_s/[U]_{s+1}), & s \geq 1, \\ \pi_t\, \mathrm{map}_{BT}(BG_{F,\Sigma}, BS), & s = 0, \end{cases}$$

where $[U]_s/[U]_{s+1}$ is dual to $\mathrm{CoLie}_n((\boldsymbol{R}\Gamma(\overline{X}, O(R))/\mathbb{Q}_\ell)[1])$ (associating $O(R)$ with an ind-lisse $\mathbb{Q}_\ell$ sheaf via ρ and Remark 2.14) for CoLie now the cofree graded Lie coalgebra, and $\widetilde{\mathrm{H}}^\bullet = \mathrm{H}^\bullet/\mathrm{H}^0$. Beware that the terms $E_1^{s,t}$ depend on an element of $E_1^{0,0}$ to determine the Galois action on U.

The filtration $[U]_n$ corresponds to the good truncation filtration on $\boldsymbol{R}\Gamma(\overline{X}, O(R))$, but there are variants for other filtrations, replacing $\mathrm{H}^{1+\bullet}(\overline{X}, O(R))$ with the E_1 page of the associated spectral sequence. For the case of the weight filtration on a quasiprojective variety, with representations tamely ramified around the divisor, see [34, Corollary 6.16].

Note that taking path components of simplicial groups $\widetilde{\Pi}_n$ gives morphisms

$$\mathrm{map}_{B(G_{F,\Sigma})}(BG_{F,\Sigma}, B\widetilde{\Pi}_n) \to \mathrm{map}_{B(G_{F,\Sigma})}(BG_{F,\Sigma}, B\Pi_n)$$

for the groups Π_n of Example 2.19. When $\mathrm{H}^{\geq 2}(\overline{X}, O(R)) = 0$ (such as for the stacky modular curve), the filtration $[U]_n$ is just equivalent to the lower central series filtration of Example 2.19, so the morphisms are weak equivalences. For general X, the towers will be different, but whenever the higher relative Maltsev homotopy groups of X vanish, the towers will converge to the same limit.

Remark 2.21 To recover Example 2.16 from Example 2.20, we take S to be the Zariski closure of the image of the representation

$$\pi_1^{\text{ét}}(\mathcal{M}_{1,1}, \bar{x}) \to \mathrm{GL}(\mathbb{T}_{\ell,\bar{x}} \otimes \mathbb{Q}) \times \prod_m \mathrm{GL}(\mathrm{H}^1(\Gamma, V_m))$$

given by combining the monodromy representation on $\mathbb{T}_{\ell,\bar{x}} \otimes \mathbb{Q}$ with the pullbacks of the $G_{F,\Sigma}$–representations $\mathrm{H}^1(\Gamma, V_m)$. Then the Zariski closure R of the image of $\pi_1^{\text{ét}}(\bar{X}, \bar{x})$ is just $\mathrm{SL}_2 \times \{1\}$, and the quotient $T := S/R$ is the Zariski closure of the representation $G_{F,\Sigma} \to \mathbb{G}_m(\mathbb{Q}_\ell) \times \prod_m \mathrm{GL}(\mathrm{H}^1(\Gamma, V_m))$. The conditions of [34, Theorem 3.32] are then satisfied by construction.

Remark 2.22 (algebraic monoids and weighted completion) A variant of Example 2.20 is given by taking a Zariski-dense representation $\rho\colon \pi_1^{\text{ét}}(X, \bar{x}) \to S(\mathbb{Q}_\ell)$ to a pro-reductive pro-algebraic monoid S, with R the Zariski closure of $\rho(\pi_1^{\text{ét}}(\bar{X}, \bar{x}))$ and $T := S/R$. Then the theory of relative Maltsev completion still works to give a homotopy fibre sequence

$$G(\bar{X}_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}} \to G(X_{\text{ét}}, \bar{x})^{S,\mathrm{Mal}} \to G(\mathscr{O}_{F,\Sigma,\text{ét}})^{T,\mathrm{Mal}}$$

of simplicial pro-algebraic monoids, and we may restrict to invertible elements (that is, $G(\bar{X}_{\text{ét}}, \bar{x})^{R,\mathrm{Mal}} \times_R R^\times$ etc) and proceed as before.

For Example 2.16, that would mean adapting Remark 2.21 by taking S to be the Zariski closure of the image of the representation

$$\pi_1^{\text{ét}}(\mathcal{M}_{1,1}, \bar{x}) \to \mathrm{End}(\mathbb{T}_{\ell,\bar{x}} \otimes \mathbb{Q}) \times \prod_m \mathrm{End}(\mathrm{H}^1(\Gamma, V_m)^*).$$

The group R would still be SL_2, and the obstruction spaces would be the same, but this gives a smaller sequence deriving them by ignoring data from irrelevant representations.

As noted in [35, Remark 1.24], the weighted completions of [15] relative to a pro-reductive group S with central cocharacter $\chi\colon \mathbb{G}_m \to S$ can also be regarded as completions relative to a monoid, namely $S \times_{\chi,\mathbb{G}_m} \mathbb{A}^1$. This has the effect of excluding some, but not all, irrelevant representations in Example 2.16 (analogous to the distinction between effective motives and motives of nonnegative weight).

Again, weighted completions generate the same obstructions as unweighted completion. In particular, for a relative curve $C \to T$ in characteristic 0 and a generic point η of T, we may consider the fibre sequence $C_{\bar{\eta}} \to C_\eta \to \operatorname{Spec} k(T)$ as in [13]. If we let S be the Zariski closure of the natural representation from $\pi_1^{\text{ét}}(C, \bar{\eta}_C)$ to $\mathrm{GL}(\mathrm{H}^1_{\text{ét}}(C_{\bar{\eta}}, \mathbb{Q}_\ell)^*)$

(or to $\mathrm{End}(\mathrm{H}^1_{\text{ét}}(C_{\bar{\eta}}, \mathbb{Q}_\ell)^*)$, or to any algebraic monoid in between), then we have a fibre sequence

$$G((C_{\bar{\eta}})_{\text{ét}})^{1,\mathrm{Mal}} \to G((C_\eta)_{\text{ét}})^{S,\mathrm{Mal}} \to G(\operatorname{Spec} k(T)_{\text{ét}})^{S,\mathrm{Mal}}.$$

The associated obstruction and lifting data of the same type as the unipotent obstructions we encountered in Example 2.12, with relative completion in this case just providing an alternative description. These obstructions (and particularly the second stage of the tower) are the main technical ingredient of [13].

3 Nonabelian reciprocity laws as obstruction maps

3.1 Adèlic mapping spaces and compact supports

Definition 3.1 In the category of pro-simplicial sets, we set

$$BG_{\mathbb{A}_F^{\in\Sigma}} := \varprojlim_{\substack{T\subset\Sigma \\ \text{finite}}} \biggl(\coprod_{v\in T} BG_v \sqcup \coprod_{v\in\Sigma-T} B(G_v/I_v)\biggr),$$

where $G_v = G_{F_v} \subset G_F$ and $I_v \lhd G_v$ is the inertia subgroup; beware that both the coproduct and the limit are taken in the category of pro-simplicial sets.

Note that there is a natural map $BG_{\mathbb{A}_F^{\in\Sigma}} \to BG_{F,\Sigma}$.

Definition 3.2 Given a finite abelian group U equipped with a continuous $G_{F,\Sigma}$–action, define

$$\boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, U) := \prod_v{}' \boldsymbol{R}\Gamma(G_v, U) = \varinjlim_T \biggl(\prod_{v\in T} \boldsymbol{R}\Gamma(G_v, U) \times \prod_{v\in\Sigma-T} \boldsymbol{R}\Gamma(G_v/I_v, U)\biggr),$$

where $\boldsymbol{R}\Gamma(G,-)$ denotes the continuous cohomology complex and T ranges over all finite subsets of Σ containing the places at which the action on U is ramified.

Definition 3.3 Given a continuous pro-finite $G_{F,\Sigma}$–representation U, define

$$\boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, U) := \boldsymbol{R}\varprojlim_i \boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, U_i),$$

where the U_i range over the finite Galois-equivariant quotients of U.

Similarly, given a continuous discrete torsion $G_{F,\Sigma}$–representation U, define

$$\boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, U) := \varinjlim_i \boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, U_i),$$

where the U_i range over the finite Galois-equivariant subgroups of U.

Definition 3.4 Given a continuous $G_{F,\Sigma}$–representation V in finite-dimensional vector spaces over $\mathbb{Q}_\ell$, define

$$\boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, V) := \varinjlim_j \boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, V_j),$$

where the V_j range over the filtered direct system of all pro-finite subrepresentations of V.

Given a continuous $G_{F,\Sigma}$–representation $V = \varprojlim_\alpha V_\alpha$ in pro-finite-dimensional vector spaces over $\mathbb{Q}_\ell$, define

$$\boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, V) := \boldsymbol{R}\varprojlim_\alpha \boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, V_\alpha).$$

For any $G_{F,\Sigma}$–equivariant lattice Λ in a finite-dimensional $G_{F,\Sigma}$–representation V over $\mathbb{Q}_\ell$, the system $\{\ell^{-n}\Lambda\}_n$ of pro-finite subrepresentations is cofinal, so

$$\boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, V) \simeq \boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}}, \Lambda) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Remark 3.5 Given a finite abelian group U equipped with a continuous $G_{F,\Sigma}$–action, observe that for all r,

$$\begin{aligned}&\operatorname{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B(G_{F,\Sigma} \ltimes B^r U))\\ &\quad\simeq \varinjlim_{\substack{T\subset\Sigma\\ \text{finite}}} \biggl(\prod_{v\in T} \operatorname{map}_{BG_{F,\Sigma}}(BG_v, B(G_{F,\Sigma} \ltimes B^r U))\\ &\qquad\qquad\qquad \times \prod_{v\in\Sigma-T} \operatorname{map}_{BG_{F,\Sigma}}(B(G_v/I_v), B(G_{F,\Sigma} \ltimes B^r U))\biggr),\end{aligned}$$

so $\pi_i \operatorname{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B(G_{F,\Sigma} \ltimes B^r U)) \cong \mathrm{H}^{r+1-i}(G_{\mathbb{A}_F^{\in\Sigma}}, U)$.

The convention of Definition 3.4 ensures that this equivalence extends to pro-finite groups or (pro-)finite-dimensional $\mathbb{Q}_\ell$–vector spaces U (regarding BU as a pro-simplicial set or a (pro-)ind-pro-simplicial set).

Beware that $BG_{\mathbb{A}_F^{\in\Sigma}}$ is not necessarily the same as the étale homotopy type $(\operatorname{Spec}\mathbb{A}_F^{\in\Sigma})_{\text{ét}}$. However, there is a map from the former to the pro-finite completion of the latter (see Corollary A.5); on the level of fundamental groups this is just the observation that a finite lisse étale sheaf on $\mathbb{A}_F^{\in\Sigma}$ is only ramified at F_v for finitely many places in $v \in \Sigma$.

We may now adapt all the examples from Section 2 to consider adèlic points instead of rational points. In particular:

Example 3.6 (nilpotent completion of $\pi_1^{\text{ét}}(\overline{X})$) Using the pro-simplicial set $BG_{\mathbb{A}_F}$, we may adapt Example 2.5. If X is a Deligne–Mumford stack over F, and $\overline{X} := X \otimes_F \overline{F}$, with some geometric point $\bar{x}$, again consider the lower central series

$$\Pi_n := \pi_1^{\text{ét}}(X, \bar{x})/[\pi_1^{\text{ét}}(\overline{X}, \bar{x})]_{n+1},$$

where we write $[\pi]_1 := \pi$ and $[\pi]_{k+1} := [\pi, [\pi]_k]$. Thus $\Pi_0 = G_F$, and taking $Y = BG_{\mathbb{A}_F}$ in the tower of Section 2.2, we get the nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(\mathbb{A}_F, [\bar{\pi}]_s/[\bar{\pi}]_{s+1}) \Rightarrow \pi_{t-s} \operatorname{map}_{BG_F}(BG_{\mathbb{A}_F}, B\Pi_\infty)$$

of groups and sets, where we write $\bar{\pi} := \pi_1^{\text{ét}}(\overline{X}, \bar{x})$.

The reasoning above (without recourse to Corollary A.5) gives a morphism of groupoids from $X(\mathbb{A}_F)$ to the fundamental groupoid $\pi_f \operatorname{map}_{BG_F}(BG_{\mathbb{A}_F}, B\Pi_\infty)$, so the spectral sequence gives obstructions to the existence of such adèlic points.

A variant of this construction is given by taking X smooth over $\mathscr{O}_{F,\Sigma}$ admitting a smooth relative compactification. For $\overline{X} := X \otimes_{\mathscr{O}_{F,\Sigma}} \mathscr{O}_{\overline{F},\Sigma}$, we can then take Π_∞ to be the relative pro-ℓ completion of $\pi_1^{\text{ét}}(X, \bar{x})$ over $G_{F,\Sigma}$ (with our convention that all F–prime factors of ℓ lie in Σ), giving a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(\mathbb{A}_F^{\in\Sigma}, ([\bar{\pi}]_s/[\bar{\pi}]_{s+1}) \otimes_{\hat{\mathbb{Z}}} \mathbb{Z}_\ell) \Rightarrow \pi_{t-s} \operatorname{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_\infty)$$

with a morphism $X(\mathbb{A}_F^{\in\Sigma}) \to \pi_f \operatorname{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_\infty)$.

Example 3.7 (unipotent completion of $\pi_1^{\text{ét}}(\overline{X})$) For unipotent adèlic obstructions, we can adapt Example 2.12, taking a smooth scheme X over $\mathscr{O}_{F,\Sigma}$ admitting a smooth relative compactification, with $\overline{X} := X \otimes_{\mathscr{O}_{F,\Sigma}} \mathscr{O}_{\overline{F},\Sigma}$ and $\bar{x}$ a geometric point. Assume that we have a point $x \in X(\mathscr{O}_{F,\Sigma})$ under $\bar{x}$ (if not, there are analogous statements using a $G_{F,\Sigma}$–equivariant set $\mathcal{B} \subset X(\mathscr{O}_{\overline{F},\Sigma})$ of basepoints instead), and consider the lower central series

$$\Pi_n := G_{F,\Sigma} \ltimes \left(\pi_1^{\text{ét}}(\overline{X}, \bar{x}) \otimes \mathbb{Q}_\ell/[\pi_1^{\text{ét}}(\overline{X}, \bar{x}) \otimes \mathbb{Q}_\ell]_n\right)$$

of the pro-unipotent Maltsev completion $\pi_1^{\text{ét}}(\overline{X}, \bar{x}) \otimes \mathbb{Q}_\ell$.

Thus $\Pi_0 = G_{F,\Sigma}$, and taking $Y = BG_{\mathbb{A}_F^{\in\Sigma}}$, we get a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_{\mathbb{A}_F^{\in\Sigma}}, [\bar{\pi} \otimes \mathbb{Q}_\ell]_s/[\bar{\pi} \otimes \mathbb{Q}_\ell]_{s+1}) \Rightarrow \pi_{t-s} \operatorname{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_\infty)$$

of groups and sets, where we write $\bar{\pi} := \pi_1^{\text{ét}}(\bar{X}, \bar{x})$. As in Example 3.6, there is a natural morphism $X(\mathbb{A}_F^{\in\Sigma}) \to \pi_f \operatorname{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma}, B\Pi_\infty)$ of groupoids, but the obstruction spaces are easier to calculate in this setting.

Example 3.8 (modular forms of level 1) As in Example 2.16, let $X = \mathcal{M}_{1,1}$ be the stacky modular curve, take $x \in X(\mathscr{O}_{F,\Sigma})$, and consider the resulting pro-unipotent extension

$$G_{F,\Sigma} \ltimes (\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2,\mathrm{Mal}} \times_{\mathrm{SL}_2(\mathbb{Q}_\ell)} \mathrm{SL}_2(\mathbb{Z}_\ell)) \to G_{F,\Sigma} \ltimes \mathrm{SL}_2(\mathbb{Z}_\ell),$$

then set

$$\Pi_n := G_{F,\Sigma} \ltimes \big((\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2,\mathrm{Mal}}/[\mathrm{R_u}]_{n+1}) \times_{\mathrm{SL}_2(\mathbb{Q}_\ell)} \mathrm{SL}_2(\mathbb{Z}_\ell)\big).$$

Using Example 2.7, a lift $BG_{\mathbb{A}_F^{\in\Sigma}} \to G_{F,\Sigma} \ltimes \mathrm{SL}_2(\mathbb{Z}_\ell)$ of the homomorphism $BG_{\mathbb{A}_F^{\in\Sigma}} \to G_{F,\Sigma}$ is equivalent to giving G_v–representations Λ_v of rank 2 over $\mathbb{Z}_\ell$ for $v \in \Sigma$, with determinant $\mathbb{Z}_\ell(1)$, such that for each n, there are only finitely many $v \in \Sigma$ with Λ_v/ℓ^n ramified. Write Λ for the system $\{\Lambda_v\}_v$.

As in Example 2.7, write $L_s := \mathrm{CoLie}_s\big(\bigoplus_m \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m) \otimes S^m(\Lambda)(-m)\big)$. The pro-unipotent generalisation of Proposition 2.11 then combines with Examples 2.15 to give a nonabelian spectral sequence

$$E_1^{s,t} = \mathrm{H}^{1+s-t}(G_{\mathbb{A}_F^{\in\Sigma}}, L_s^*) \Rightarrow \pi_{t-s} \operatorname{map}_{B(G_{F,\Sigma} \times_{\mathbb{G}_m(\mathbb{Z}_\ell)} \mathrm{GL}_2(\mathbb{Z}_\ell))}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_\infty),$$

where the map $BG_{\mathbb{A}_F^{\in\Sigma}} \to \mathrm{GL}_2(\mathbb{Z}_\ell)$ is given by Λ. Note that $E^1_{s,s+1} = 0$ as L_s is of nonzero weights.

Now set $X_{(n)} := \operatorname{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_n)$; thus $X_{(0)}$ consists of sets $\{\Lambda_v\}_v$ as above, conjugation by $\mathrm{SL}_2(\mathbb{Z}_\ell)$ giving equivalences, so $\pi_1(X_{(0)}, [\Lambda])$ consists of elements of $\mathrm{SL}_2(\mathbb{Z}_\ell)$ commuting with the actions of the G_v on Λ. Since $\pi_i X_{(n)} = 0$ for $i > 1$, we then have exact sequences

$$0 \to \pi_1 X_{(n)} \to \pi_1 X_{(n-1)} \to \mathrm{H}^1(G_{\mathbb{A}_F^{\in\Sigma}}, L_n^*) \to \pi_0 X_{(n)} \to \pi_0 X_{(n-1)} \to \mathrm{H}^2(G_{\mathbb{A}_F^{\in\Sigma}}, L_n^*),$$

with a map $X(\mathbb{A}_F^{\in\Sigma}) \to X_{(\infty)}$. Here, $\pi_0 X(\mathbb{A}_F^{\in\Sigma})$ is the set of isomorphism classes of elliptic curves over $\mathbb{A}_F^{\in\Sigma}$, and $\pi_1(X(\mathbb{A}_F^{\in\Sigma}), x)$ the group of automorphisms of the elliptic curve E_x over $\mathbb{A}_F^{\in\Sigma}$.

In other words, given a system $\Lambda = \{\Lambda_v\}_{v\in\Sigma}$ of rank 2 local Galois representations over $\mathbb{Z}_\ell$ as above, these sequences give a tower of obstructions to lifting Λ to an elliptic curve over $\mathbb{A}_F^{\in\Sigma}$ with Tate module Λ, and characterise the ambiguity of the lift at each stage. As in Examples 2.15, there is an entirely similar treatment for pro-finite completions of congruence subgroups $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$, replacing $\mathcal{M}_{1,1}$ with the modular curve Y_Γ.

Example 3.9 (étale homotopy types) We now consider étale homotopy types in place of fundamental groups, as in Example 2.20. Take a smooth Deligne–Mumford stack X over $\mathscr{O}_{F,\Sigma}$ admitting a smooth relative compactification, and set $\overline{X} := X \otimes_{\mathscr{O}_{F,\Sigma}} \mathscr{O}_{\overline{F},\Sigma}$. For a geometric point $\bar{x}$ and a Zariski-dense representation $\rho\colon \pi_1^{\text{ét}}(X,\bar{x}) \to S(\mathbb{Q}_\ell)$ to a pro-reductive pro-algebraic group S, let R be the Zariski closure of $\rho(\pi_1^{\text{ét}}(\overline{X},\bar{x}))$, and set $T := S/R$.

We then look at the pro-simplicial group $G(X_{\text{ét}},\bar{x})$ associated to the étale topological type $X_{\text{ét}} \in \mathrm{pro}(\mathbb{S})$. If the $G_{F,\Sigma}$–representation $\mathrm{H}^*(\overline{X},V)$ is an extension of T–representations for all R–representations V, then we may again set Π_n to be the simplicial topological group given by the homotopy fibre product

$$\Pi_n := (G(X_{\text{ét}},\bar{x})^{S,\mathrm{Mal}}/[U]_{n+1}) \times^h_{G(BG_{F,\Sigma})^{T,\mathrm{Mal}}} G_{F,\Sigma},$$

where $U = \mathrm{R_u} G(\overline{X}_{\text{ét}},\bar{x})^{R,\mathrm{Mal}}$. Since $B\Pi_\infty$ is equipped with a map from $\overline{W}G(\overline{X}_{\text{ét}},\bar{x})$, Corollary A.5 gives a canonical morphism

$$X(\mathbb{A}_F^{\in\Sigma}) \to \mathrm{map}_{B(G_{F,\Sigma})}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_\infty)$$

in the homotopy category of pro-ind-pro-simplicial sets.

We then have a nonabelian spectral sequence

$$E_1^{s,t} \Rightarrow \pi_{t-s}\,\mathrm{map}_{B(G_{F,\Sigma})}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_\infty),$$

with

$$E_1^{s,t} = \begin{cases} \mathbb{H}^{1+s-t}(\mathbb{A}_F^{\in\Sigma}, [U]_s/[U]_{s+1}), & s \ge 1, \\ \pi_t\,\mathrm{map}_{BT}(BG_{\mathbb{A}_F^{\in\Sigma}}, BS), & s = 0, \end{cases}$$

where $[U]_s/[U]_{s+1}$ is dual to $\mathrm{CoLie}_n\big((\boldsymbol{R}\Gamma(\overline{X}, O(R))/\mathbb{Q}_\ell)[1]\big)$.

3.2 Reciprocity laws

The idea behind nonabelian reciprocity laws is to compare the towers of obstructions for rational and adèlic points, giving a relative obstruction tower for rational points over adèlic points.

Definition 3.10 Given a continuous $G_{F,\Sigma}$–representation U, we set

$$\boldsymbol{R}\Gamma_c(G_{F,\Sigma},U) := \mathrm{cocone}(\boldsymbol{R}\Gamma(G_{F,\Sigma},U) \to \boldsymbol{R}\Gamma(G_{\mathbb{A}_F^{\in\Sigma}},U)),$$

where U can be any of the types of representation considered in Definitions 3.2–3.4.

3.2.1 Abelian Poitou–Tate duality

Definition 3.11 Define a contravariant functor $(-)^\vee$ on the category of abelian groups by

$$A^\vee := \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}).$$

Definition 3.12 Define a contravariant functor $(-)^\vee(1)$ on the category of continuous $G_{F,\Sigma}$–representations in locally compact topological torsion abelian groups (in the sense of [20]) by

$$A^\vee(1) := \mathrm{Hom}_{\mathbb{Z},\mathrm{cts}}(A,\mu_\infty).$$

Note that $(-)^\vee$ preserves the subcategory of finite representations, and interchanges pro-finite and discrete representations.

Lemma 3.13 *If Σ is a finite set of finite places containing all primes dividing ℓ, and U a continuous pro-ℓ $G_{F,\Sigma}$–representation, then there is a canonical equivalence*

$$\boldsymbol{R}\Gamma_c(G_{F,\Sigma},U) \simeq \boldsymbol{R}\Gamma(G_{F,\Sigma},U^\vee(1))^\vee[-3].$$

If V is a continuous $G_{F,\Sigma}$–representation in finite-dimensional vector spaces over $\mathbb{Q}_\ell$, then we also have

$$\boldsymbol{R}\Gamma_c(G_{F,\Sigma},V) \simeq \boldsymbol{R}\Gamma(G_{F,\Sigma},V^*(1))^*[-3].$$

Proof The first statement is the formulation of Poitou–Tate duality given in [26], refining a homological isomorphism from [30]. For the second statement, take a $G_{F,\Sigma}$–equivariant lattice $\Lambda \subset V$, and then (writing $\Lambda^* := \mathrm{Hom}_{\mathbb{Z}_\ell}(\Lambda,\mathbb{Z}_\ell)$)

$$\begin{aligned}
\boldsymbol{R}\Gamma_c(G_{F,\Sigma},V) &\simeq \boldsymbol{R}\Gamma_c(G_{F,\Sigma},\Lambda)\otimes\mathbb{Q}\\
&\simeq \boldsymbol{R}\Gamma(G_{F,\Sigma},\Lambda^\vee(1))^\vee[-3]\otimes\mathbb{Q}\\
&\simeq \boldsymbol{R}\underline{\mathrm{Hom}}_{\mathbb{Z}_\ell}\big(\boldsymbol{R}\Gamma(G_{F,\Sigma},\Lambda^*(1))\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell,\mathbb{Q}_\ell/\mathbb{Z}_\ell\big)[-3]\otimes\mathbb{Q}\\
&\simeq \boldsymbol{R}\underline{\mathrm{Hom}}_{\mathbb{Z}_\ell}\big(\boldsymbol{R}\Gamma(G_{F,\Sigma},\Lambda^*(1)),\mathbb{Z}_\ell\big)[-3]\otimes\mathbb{Q}\\
&\simeq \boldsymbol{R}\underline{\mathrm{Hom}}_{\mathbb{Z}_\ell}\big(\boldsymbol{R}\Gamma(G_{F,\Sigma},\Lambda^*(1)),\mathbb{Q}_\ell\big)[-3],
\end{aligned}$$

the last isomorphism following because $\mathrm{H}^*(G_{F,\Sigma},\Lambda^*(1))$ has finite rank, Σ being finite. The result now follows because $V^* \cong \Lambda^*\otimes\mathbb{Q}_\ell$. $\square$

Lemma 3.14 *If Σ is a possibly infinite set of finite places, and U a continuous $G_{F,\Sigma}$–representation in pro-finite abelian groups whose order is a unit outside Σ, then there is a canonical equivalence*

$$\boldsymbol{R}\Gamma_c(G_{F,\Sigma},U)\simeq \boldsymbol{R}\Gamma(G_{F,\Sigma},U^\vee(1))^\vee[-3],$$

following the continuous cohomology conventions of Definition 3.3.

Proof When U is finite, this is essentially the Poitou–Tate duality of Theorem 1.4.10 of [29]. In general, writing $U=\varprojlim_\alpha U_\alpha$ for U_α finite, we have

$$\begin{aligned}\boldsymbol{R}\Gamma_c(G_{F,\Sigma},U)&\simeq \boldsymbol{R}\varprojlim_\alpha \boldsymbol{R}\Gamma_c(G_{F,\Sigma},U_\alpha)\\ &\simeq \boldsymbol{R}\varprojlim_\alpha \boldsymbol{R}\Gamma(G_{F,\Sigma},U_\alpha^\vee(1))^\vee[-3]\\ &\simeq \big(\varinjlim_\alpha \boldsymbol{R}\Gamma(G_{F,\Sigma},U_\alpha^\vee(1))\big)^\vee[-3]\\ &= \boldsymbol{R}\Gamma(G_{F,\Sigma},U^\vee(1))^\vee[-3].\end{aligned}$$ □

Remark 3.15 If we wanted to extend Lemma 3.14 to more general coefficients, we would have to pass to a larger category than the category $\mathcal{TT}$ of locally compact topological torsion groups. The category $\mathcal{TT}$ precisely consists of the Tate objects over the category of finite abelian groups in the sense of [3]. Since $\boldsymbol{R}\Gamma(G_{F,\Sigma},-)$ and $\boldsymbol{R}\Gamma_c(G_{F,\Sigma},-)$ are functors from finite groups to complexes of Tate objects, their natural extension to coefficients in $\mathcal{TT}$ will take values in complexes of 2–Tate objects over finite abelian groups (or equivalently Tate objects over $\mathcal{TT}$), and Poitou–Tate duality will extend formally to that category.

3.2.2 Nonabelian reciprocity laws We may now adapt all the examples from Section 2 to obtain obstructions to adèlic points being rational points, with terms in the spectral sequence given by Galois cohomology $\mathrm{H}^*_c(G_{F,\Sigma},-)$ with compact supports. Since the coefficients we consider have negative weights, the lower cohomology groups with compact supports tend to be small; when they vanish, the obstruction towers have no ambiguity in the lift at each stage.

Example 3.16 (nilpotent completion of $\pi_1^{\text{ét}}(\overline{X})$) If X is a Deligne–Mumford stack over F, and $\overline{X}=X\otimes_F\overline{F}$, with some geometric point $\bar{x}$, then as in Examples 2.5 and 3.6 we may consider the lower central series

$$\Pi_n:=\pi_1^{\text{ét}}(X,\bar{x})/[\pi_1^{\text{ét}}(\overline{X},\bar{x})]_{n+1},$$

where we write $[\pi]_1:=\pi$ and $[\pi]_{k+1}$ for the closure of $[\pi,[\pi]_k]$. Write $\Pi_\infty=\varprojlim_n \Pi_n$.

We then define the tower $\cdots \to X(\mathbb{A}_F)_n \to X(\mathbb{A}_F)_0 = X(\mathbb{A}_F)$ by the homotopy fibre products

$$X(\mathbb{A}_F)_n := X(\mathbb{A}_F) \times^h_{\mathrm{map}_{BG_F}(BG_{\mathbb{A}_F}, B\Pi_n)} \mathrm{map}_{BG_F}(BG_F, B\Pi_n),$$

defined using the morphism $X(\mathbb{A}_F) \to \mathrm{map}_{BG_F}(BG_{\mathbb{A}_F}, B\Pi_\infty)$ from Section 3.1.

Taking homotopy fibres of the fibration sequences in Section 2.2, we then get a nonabelian spectral sequence

$$E_1^{s,t} \Rightarrow \pi_{t-s}X(\mathbb{A}_F)_\infty, \quad \text{where } E_1^{s,t} = \begin{cases} \mathrm{H}_c^{1+s-t}(G_F, [\bar{\pi}]_s/[\bar{\pi}]_{s+1}), & s \geq 1, \\ \pi_t X(\mathbb{A}_F), & s = 0, \end{cases}$$

of groups and sets, where we write $\bar{\pi} := \pi_1^{\text{ét}}(\bar{X}, \bar{x})$. This comes from the exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_* X(\mathbb{A}_F)_s \arrow[r] & \cdots \arrow[r] \arrow[dl, dashed, "\delta"] & \pi_* X(\mathbb{A}_F)_1 \arrow[r] & \pi_* X(\mathbb{A}_F)_0 \arrow[dl, dashed, "\delta"] \arrow[d, equal] \\
& \mathrm{H}_c^{1-*}(G_F, [\bar{\pi}]_s/[\bar{\pi}]_{s+1}) \arrow[u] & \cdots & \mathrm{H}_c^{1-*}(G_F, \bar{\pi}/[\bar{\pi}]_2) \arrow[u] & \pi_* X(\mathbb{A}_F)
\end{tikzcd}$$

with δ of cohomological degree $+1$.

As in Section 2, we have a map $X(F) \to X(\mathbb{A}_F)_\infty$, so the spectral sequence gives obstructions to an adèlic point being rational. When X is a scheme (or algebraic space), $\pi_0 X(\mathbb{A}_F) = \pi_* X(\mathbb{A}_F)$ and $\pi_i(\mathbb{A}_F) = 0$ for $i > 0$.

By Lemma 3.14, $\mathrm{H}_c^{1+s-t}(G_F, [\bar{\pi}]_s/[\bar{\pi}]_{s+1})$ is isomorphic to

$$\mathrm{H}^{2+t-s}\big(G_F, ([\bar{\pi}]_s/[\bar{\pi}]_{s+1})^\vee(1)\big)^\vee.$$

Thus elements of $\mathrm{H}^1\big(G_F, ([\bar{\pi}]_s/[\bar{\pi}]_{s+1})^\vee(1)\big)$ give obstructions to lifting points in $\pi_0 X(\mathbb{A}_F)$ to $X(F)$, and the ambiguities of the lifts at each stage are dual to the groups $\mathrm{H}^2\big(G_F, ([\bar{\pi}]_s/[\bar{\pi}]_{s+1})^\vee(1)\big)$, which are often finite for weight reasons as in [23]. The higher homotopy groups $\pi_{\geq 2} X(\mathbb{A}_F)_n$ are necessarily 0, by vanishing of $\mathrm{H}_c^{\leq 0}$.

Remark 3.17 Since $[\bar{\pi}]_n/[\bar{\pi}]_{n+1}$ is contained in the centre of $\bar{\pi}/[\bar{\pi}]_{n+1}$, it seems that the spectral sequence in Example 3.16 can alternatively be obtained as an inverse limit of the nonabelian Poitou–Tate exact sequence of [40, Theorem 168].

Example 3.18 (unipotent completion of $\pi_1^{\text{ét}}(\bar{X})$) Examples 2.12 and 3.7 adapt along the lines of Example 3.16. Take a smooth Deligne–Mumford X over $\mathscr{O}_{F,\Sigma}$ admitting a smooth relative compactification, with $\bar{X} := X \otimes_{\mathscr{O}_{F,\Sigma}} \mathscr{O}_{\bar{F},\Sigma}$ and $\bar{x}$ a geometric point. Assume that we have a point $x \in X(\mathscr{O}_{F,\Sigma})$ under $\bar{x}$ (if not, there are analogous statements using a $G_{F,\Sigma}$–equivariant set $\mathcal{B}$ of basepoints instead).

Now set

$$\Pi_n := G_{F,\Sigma} \ltimes (\pi_1^{\text{ét}}(\bar{X},\bar{x}) \otimes \mathbb{Q}_\ell)/[\pi_1^{\text{ét}}(\bar{X},\bar{x}) \otimes \mathbb{Q}_\ell]_n,$$

and

$$X(\mathbb{A}_F^{\in\Sigma})_n := X(\mathbb{A}_F^{\in\Sigma}) \times^h_{\mathrm{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_n)} \mathrm{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma}, B\Pi_n),$$

to give a nonabelian spectral sequence

$$E_1^{s,t} \Rightarrow \pi_{t-s} X(\mathbb{A}_F^{\in\Sigma})_\infty,$$

where

$$E_1^{s,t} = \begin{cases} \mathrm{H}_c^{1+s-t}(G_{F,\Sigma}, [\bar{\pi} \otimes \mathbb{Q}_\ell]_s/[\bar{\pi} \otimes \mathbb{Q}_\ell]_{s+1}), & s \geq 1, \\ \pi_t X(\mathbb{A}_F^{\in\Sigma}), & s = 0, \end{cases}$$

of groups and sets, where we write $\bar{\pi} := \pi_1^{\text{ét}}(\bar{X},\bar{x})$.

Lemma 3.14 shows that $\mathrm{H}_c^{1+s-t}(G_{F,\Sigma}, [\bar{\pi} \otimes \mathbb{Q}_\ell]_s/[\bar{\pi} \otimes \mathbb{Q}_\ell]_{s+1})$ is isomorphic to $\mathrm{H}^{2+t-s}(G_{F,\Sigma}, ([\bar{\pi}]_s/[\bar{\pi}]_{s+1})^\vee(1))^\vee \otimes \mathbb{Q}_\ell$. Since X is smooth, $[\bar{\pi} \otimes \mathbb{Q}_\ell]_s/[\bar{\pi} \otimes \mathbb{Q}_\ell]$ is a pro-finite-dimensional Galois $\mathbb{Q}_\ell$–representation of negative weights, so the local monodromy weight conjectures (as in the Poitou–Tate dual form of [23, Conjecture 6.3]) would imply $E^1_{s,s} = 0$ for $s > 0$, with the exact couple yielding the spectral sequence then degenerating to exact sequences

$$0 \to \pi_0 X(\mathbb{A}_F^{\in\Sigma})_{(n)} \to \pi_0 X(\mathbb{A}_F^{\in\Sigma})_{(n-1)} \to \mathrm{H}_c^2(G_{F,\Sigma}, [\bar{\pi} \otimes \mathbb{Q}_\ell]_s/[\bar{\pi} \otimes \mathbb{Q}_\ell]_{s+1}),$$

so the tower becomes a sequence of subsets.

Example 3.19 (modular forms of level 1) As in Examples 2.16 and 3.8, let $X = \mathcal{M}_{1,1}$ be the stacky modular curve, take $x \in X(\mathscr{O}_{F,\Sigma})$, and set

$$\Pi_n := G_{F,\Sigma} \ltimes \big((\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2,\mathrm{Mal}}/[\mathrm{R_u}]_{n+1}) \times_{\mathrm{SL}_2(\mathbb{Q}_\ell)} \mathrm{SL}_2(\mathbb{Z}_\ell)\big),$$

where SL_2 is here regarded as an algebraic group over $\mathbb{Q}_\ell$.

We now write

$$X(\mathbb{A}_F^{\in\Sigma})_n := X(\mathbb{A}_F^{\in\Sigma}) \times^h_{\mathrm{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_n)} \mathrm{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma}, B\Pi_n).$$

Since $\Pi_0 = G_{F,\Sigma} \ltimes \mathrm{SL}_2(\mathbb{Z}_\ell)$, the space $X(\mathbb{A}_F^{\in\Sigma})_0$ consists of pairs (x, Λ) with x an adèlic point and Λ a $G_{F,\Sigma}$–representation of rank 2 over $\mathbb{Z}_\ell$ with determinant $\mathbb{Z}_\ell(1)$, together with an isomorphism $T_\ell E_{\bar{x}} \cong \Lambda$ of $BG_{\mathbb{A}_F^{\in\Sigma}}$–representations.

Writing $L_s := \mathrm{CoLie}_s\big(\bigoplus_m \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m) \otimes S^m(\Lambda)(-m)\big)$, Proposition 2.11 and Examples 2.15 then give a nonabelian spectral sequence

$$E_1^{s,t} \Rightarrow \pi_{t-s} X(\mathbb{A}_F^{\in\Sigma})_\infty, \quad \text{where } E_1^{s,t} = \begin{cases} \mathrm{H}_c^{1+s-t}(G_{F,\Sigma}, L_s^*), & s \geq 1, \\ \pi_t X(\mathbb{A}_F^{\in\Sigma})_0, & s = 0. \end{cases}$$

In other words, given a global Galois representation Λ and, for each $v \in \Sigma$, a local elliptic curve E_v lifting each underlying G_v–representation, with constraints on ramification, these sequences give a tower of obstructions to lifting $(\Lambda, \{E_v\}_{v\in\Sigma})$ to an elliptic curve E over $\mathscr{O}_{F,\Sigma}$ with Tate module $T_\ell E(\bar{F}) \otimes \mathbb{Q} \cong \Lambda \otimes \mathbb{Q}$ and localisations E_v; the sequences also characterise the ambiguity of the lift at each stage.

As in Examples 2.15, the group $\mathrm{H}^1(\Gamma, V_m)(-m)$ consists of modular forms and cusp forms of weight $m+2$ and level 1. Thus L_s is a Galois $\mathbb{Q}_\ell$–representation of weights $\geq s$, so it follows that L_s^* is a pro-finite-dimensional Galois $\mathbb{Q}_\ell$–representation of weights $\leq -s$. As in Example 3.18, the local monodromy weight conjectures would cause the exact couple yielding the spectral sequence to degenerate to the exact sequences

$$0 \to \pi_0 X(\mathbb{A}_F^{\in\Sigma})_{(n)} \to \pi_0 X(\mathbb{A}_F^{\in\Sigma})_{(n-1)} \to \mathrm{H}_c^2(G_{F,\Sigma}, L_s^*)$$

equipped with a map $X(\mathscr{O}_{F,\Sigma}) \to X(\mathbb{A}_F^{\in\Sigma})_{(\infty)}$.

As in Examples 2.15, there is an entirely similar treatment for congruence subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, replacing $\mathcal{M}_{1,1}$ with the modular curve Y_Γ. If we instead started from a representation Λ over $\hat{\mathbb{Z}}$, relative Maltsev completion of $\mathrm{SL}_2(\mathbb{Z})$ over $\mathrm{SL}_2 \times \mathrm{SL}_2(\hat{\mathbb{Z}})$ as in Example 2.17 would give rise to reciprocity laws associated to modular forms of all levels. Meanwhile, relative Maltsev completion of $\mathrm{SL}_2(\mathbb{Z})$ over $\mathrm{SL}_2(\hat{\mathbb{Z}})$ as in Example 2.18 gives rise to reciprocity laws associated to weight 2 modular forms of all levels.

Remark 3.20 We may write

$$\begin{aligned} &\mathrm{H}_c^i(G_{F,\Sigma}, L_s^*) \\ &\qquad \cong \mathrm{Lie}(n) \otimes^{S_n} \mathrm{H}_c^i\bigg(G_{F,\Sigma}, \bigg(\bigg(\bigoplus_m \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m) \otimes S^m(\Lambda)(-m)\bigg)^{\otimes n}\bigg)^*\bigg). \end{aligned}$$

As in Example 2.16, we may then consider the sheaf $\mathbb{T}_\ell$ of relative Tate modules on Y_Γ, with $\boldsymbol{R}q_*\mathbb{T}_\ell \otimes \mathbb{Q}[1] \simeq \boldsymbol{R}^1 q_*\mathbb{T}_\ell \otimes \mathbb{Q} \cong \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m)$, for the structure map

$q\colon Y_\Gamma \to \operatorname{Spec} \mathscr{O}_{F,\Sigma}$. Applying Poitou–Tate duality in the form of Lemma 3.14 to this $\mathbb{Z}_\ell$–lattice then gives

$$\begin{aligned}
&\boldsymbol{R}\Gamma_c\biggl(G_{F,\Sigma}, \biggl(\biggl(\bigoplus_m \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), V_m)\otimes S^m(\Lambda)(-m)\biggr)^{\otimes s}\biggr)^*\biggr)\\
&\qquad\simeq \boldsymbol{R}\Gamma\biggl(G_{F,\Sigma}, \biggl(\bigoplus_{m\ge 1} \boldsymbol{R}q_*S^m\mathbb{T}_\ell \otimes S^m(\Lambda)(-m)\biggr)^{\otimes s}\otimes \mu_{\ell^\infty}\biggr)^\vee \otimes \mathbb{Q}[3+s]\\
&\qquad\simeq \boldsymbol{R}\Gamma\biggl(X^s, \bigotimes_{i=1}^s\biggl(\bigoplus_{m\ge 1} \mathrm{pr}_i^* S^m\mathbb{T}_\ell\otimes q^*S^m(\Lambda)(-m)\biggr)\otimes \mu_{\ell^\infty}\biggr)^\vee \otimes \mathbb{Q}[3+s],
\end{aligned}$$

providing an expression for $E_1^{s,t}$ as a summand of $\mathrm{H}^{2+t}(X^s, \cdots\otimes \mu_{\ell^\infty})^\vee\otimes\mathbb{Q}$. As we will see in Example 3.24, the $s=1$ case is a part of the Brauer–Manin obstruction, divisible elements in cohomology giving rise to obstructions.

By [18], for Σ cofinite and $\rho\colon G_{F,\Sigma}\to \mathrm{SL}_2(\widehat{\mathbb{Z}})$, nonemptiness of $\mathcal{M}_{1,1}(\mathbb{A}_{\mathbb{Q}}^{\in\Sigma})_\rho$ implies nonemptiness of $X(\mathbb{Z}_\Sigma)_\rho$. The variants of Example 3.19 for relative Maltsev completions of $\mathrm{SL}_2(\mathbb{Z})$ over $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ or over $\mathrm{SL}_2(\mathbb{Q}_\ell)\times\mathrm{SL}_2(\widehat{\mathbb{Z}})$ should then help to identify $X(\mathbb{Z}_\Sigma)_\rho\subset \mathcal{M}_{1,1}(\mathbb{A}_{\mathbb{Q}}^{\in\Sigma})_\rho$.

Example 3.21 (relative Maltsev étale homotopy types) As in Examples 2.20 and 3.9, we may consider étale homotopy types in place of fundamental groups. Take a smooth Deligne–Mumford stack X over $\mathscr{O}_{F,\Sigma}$ admitting a smooth relative compactification, a geometric point $\bar{x}$ and a Zariski-dense representation $\rho\colon \pi_1^{\text{ét}}(X,\bar{x})\to S(\mathbb{Q}_\ell)$ to a pro-reductive pro-algebraic group S, let R be the Zariski closure of $\rho(\pi_1^{\text{ét}}(\overline{X},\bar{x}))$, and set $T:=S/R$.

Now set Π_n to be the simplicial topological group given by the homotopy fibre product

$$\Pi_n := (G(X_{\text{ét}},\bar{x})^{S,\mathrm{Mal}}/[U]_{n+1})\times^h_{G(BG_{F,\Sigma})^{T,\mathrm{Mal}}} G_{F,\Sigma},$$

where $U=\mathrm{R_u}G(\overline{X}_{\text{ét}},\bar{x})^{R,\mathrm{Mal}}$.

The formula of Example 3.19 then gives a tower of spaces $\{X(\mathbb{A}_F^{\in\Sigma})_n\}_n$ and an associated nonabelian spectral sequence

$$E_1^{s,t}\Rightarrow \pi_{t-s}(X(\mathbb{A}_F^{\in\Sigma})\times^h_{\mathrm{map}_{B(G_{F,\Sigma})}(BG_{\mathbb{A}_F^{\in\Sigma}},\overline{W}\Pi_\infty)}\mathrm{map}_{B(G_{F,\Sigma})}(BG_{F,\Sigma},\overline{W}\Pi_\infty)),$$

with

$$E_1^{s,t}=\begin{cases}\mathbb{H}_c^{1+s-t}(G_{F,\Sigma},[U]_s/[U]_{s+1}), & s\ge 1,\\ \pi_t X(\mathbb{A}_F^{\in\Sigma})_0, & s=0,\end{cases}$$

where $[U]_s/[U]_{s+1}$ is dual to $\mathrm{CoLie}_s((\boldsymbol{R}\Gamma(\overline{X}, O(R))/\mathbb{Q}_\ell)[1])$ and

$$X(\mathbb{A}_F^{\in\Sigma})_0 = X(\mathbb{A}_F^{\in\Sigma}) \times^h_{\mathrm{map}_{BT}(BG_{\mathbb{A}_F^{\in\Sigma}}, BS)} \mathrm{map}_{BT}(B(G_{F,\Sigma}), BS).$$

Remark 3.22 As in [34, Theorem 6.4], Lafforgue's theorem [25, Theorem VII.6 and Corollary VII.8] and Esnault and Kerz [8] imply that the ind-lisse sheaf $\rho^{-1}O(R)$ on $\overline{X}$ is pure of weight 0. If $\overline{X}$ is smooth and proper, Corollary 6.7 of [34] then implies that the group $\mathrm{H}^{-i}([U]_s/[U]_{s+1})$ in Example 3.21 is pure of weight $-i-s$.

The obstruction spaces for étale homotopy sections $\pi_0 X(\mathbb{A}_F^{\in\Sigma})_{(\infty)}$ are given in the spectral sequence by the terms $E_1^{s,s-1}$. Assuming that ρ is of geometric origin, the local monodromy weight conjectures (as in [23, Conjecture 6.3]) would imply that the groups H^1_c vanish, so the only nontrivial contributions to $E_1^{s,s-1}$ come from

$$\mathrm{H}^2_c\big(G_{F,\Sigma}, \big(\mathrm{CoLie}_s \mathrm{H}^1(\overline{X}, O(R))\big)^*\big)$$

as in Example 3.18, and from

$$\mathrm{H}^3_c\big(G_{F,\Sigma}, \big(\mathrm{H}^2(\overline{X}, O(R)) \otimes \mathrm{CoLie}_{s-1}\mathrm{H}^1(\overline{X}, O(R))\big)^*\big).$$

The latter group can only be only nonzero for $s=1$, when $\mathrm{H}^2(\overline{X}, O(R))$ contains copies of the Tate motive, in which case the reciprocity map is detecting the Brauer–Manin obstruction of a pro-étale covering whose geometric fibres are $\rho(\pi_1^{\text{ét}}(\overline{X}, \bar{x}))$–torsors as in Example 3.27 below. These copies of the Tate motive then generate a large contribution $\mathrm{H}^2_c\big(G_{F,\Sigma}, \mathrm{H}^2(\overline{X}, O(R))^*\big)$ to the $E_1^{1,1}$ term, producing an ambiguity in the lift much larger than the new obstruction, meaning the map $X(\mathbb{A}_F^{\in\Sigma})_1 \to X(\mathbb{A}_F^{\in\Sigma})$ would then be far from injective.

3.2.3 Brauer–Manin obstructions We now look at Example 3.21 and analogous completions of étale homotopy types, giving rise to obstruction towers refining the nonabelian reciprocity laws by incorporating higher homotopical information. A common feature is that the first obstruction map in the tower is just the Brauer–Manin obstruction, or related (pro-)étale refinements in the case of relative completion. Because the higher obstructions induce nonabelian reciprocity laws, they will be nontrivial in any case where the higher reciprocity maps of [24] are nonzero on the relevant Brauer–Manin set.

If $O(R)_{\mathbb{Z}_\ell}$ is a $\pi_1^{\text{ét}}(\overline{X})$–equivariant $\mathbb{Z}_\ell$–form for the ring $O(R)$ of functions on the reductive group featuring in Example 3.21, then we may use Poitou–Tate duality to rewrite the term $E_1^{1,t}$ as

$$\begin{aligned}\mathbb{H}_c^{2-t}(G_{F,\Sigma},[U]_1/[U]_2) &\cong \mathbb{H}^{2+t}(G_{F,\Sigma},\boldsymbol{R}\Gamma(\overline{X},O(R)_{\mathbb{Z}_\ell}\otimes\mu_{\ell^\infty})/\mu_{\ell^\infty})^\vee\otimes\mathbb{Q}\\ &\cong \begin{cases}\mathrm{H}^2_{\text{ét}}(X,O(R)_{\mathbb{Z}_\ell}\otimes\mu_{\ell^\infty})/\mathrm{H}^2(G_{F,\Sigma},\mu_{\ell^\infty})^\vee, & t=0,\\ \mathrm{H}^{2+t}_{\text{ét}}(X,O(R)_{\mathbb{Z}_\ell}\otimes\mu_{\ell^\infty})^\vee, & t>0;\end{cases}\end{aligned}$$

when $R=1$ (unipotent completion of the geometric fibre), we have $O(R)_{\mathbb{Z}_\ell}=\mathbb{Z}_\ell$, and the first obstruction map $d_1\colon E_1^{0,0}\to E_1^{1,0}$ is the rationalised Brauer–Manin obstruction

$$\pi_0 X(\mathbb{A}_F^{\in\Sigma})\to(\mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^\infty})/\mathrm{H}^2(G_{F,\Sigma},\mu_{\ell^\infty}))^\vee\otimes\mathbb{Q}.$$

Remark 3.23 We may write $E_1^{s,t}$ as cohomology of a complex defined in terms of the Lie operad and the complexes $\boldsymbol{R}\Gamma(X^n,\mu_{\ell^\infty})^\vee\otimes\mathbb{Q}$ for $n\le s$. In particular,

$$E_1^{2,t}\cong\mathrm{H}^{2+t}\,\mathrm{Tot}\big(\boldsymbol{R}\Gamma(X^2,\mu_{\ell^\infty})^\vee\otimes\mathbb{Q}/S_2\xrightarrow{\mathrm{pr}_{1*}-\mathrm{pr}_{2*}}\boldsymbol{R}\Gamma(X,\mu_{\ell^\infty})^\vee\otimes\mathbb{Q}\big),$$

where S_2 acts by switching the factors in X^2. For $R\neq 1$, the expression in Remark 3.20 for modular curves generalises whenever $\mathrm{H}^{>0}(\overline{X},\mathbb{Q}_\ell)=0$, but usually there are extra factors reflecting the difference between reduced and nonreduced cohomology.

Taking nilpotent completion instead of unipotent completion gives the following:

Example 3.24 (étale homotopy types and the Brauer–Manin obstruction) Take a smooth Deligne–Mumford stack X over $\mathcal{O}_{F,\Sigma}$ admitting a smooth relative compactification, and a geometric point $\bar{x}$. Applying relative pro-Σ completion over $G_{F,\Sigma}$ levelwise (see [34, Section 1]) to the pro-simplicial group $G(X_{\text{ét}},\bar{x})$ of Example 2.20 gives a pro-(finite simplicial group) $\widehat{G}(X_{\text{ét}},\bar{x})$ as in the proof of Proposition 2.2; up to homotopy, this is independent of the choices made, by [34, Proposition 1.32]. When Σ is the set of all primes (corresponding to $\mathcal{O}_{F,\Sigma}=F$), note that $\widehat{G}(X_{\text{ét}},\bar{x})$ is just the pro-finite completion of $G(X_{\text{ét}},\bar{x})$.

We now refine Example 3.16 by considering relative pro-nilpotent completions of the whole pro-finite homotopy type $\widehat{G}(X_{\text{ét}},\bar{x})$ instead of the fundamental group. For completions relative to $G_{F,\Sigma}$, we set $K:=\ker(\widehat{G}(X_{\text{ét}},\bar{x})\to G_{F,\Sigma})$ and

$$\Pi_n:=\widehat{G}(X_{\text{ét}},\bar{x})/[K]_{n+1},$$

which is a pro-(finite simplicial group).

We then construct a tower $\cdots\to X(\mathbb{A}_F^{\in\Sigma})_1\to X(\mathbb{A}_F^{\in\Sigma})_0=X(\mathbb{A}_F^{\in\Sigma})$ of homotopy fibre products

$$X(\mathbb{A}_F^{\in\Sigma})_n:=X(\mathbb{A}_F^{\in\Sigma})\times^h_{\mathrm{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}},\overline{W}\Pi_n)}\mathrm{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma},\overline{W}\Pi_n),$$

defined using the morphism $X(\mathbb{A}_F^{\in\Sigma}) \to \mathrm{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, B\Pi_\infty)$ from Section 3.1 and Corollary A.5.

This gives a nonabelian spectral sequence

$$E_1^{s,t} \Rightarrow \pi_{t-s} X(\mathbb{A}_F^{\in\Sigma})_\infty, \quad \text{where } E_1^{s,t} = \begin{cases} \mathbb{H}_c^{1+s-t}(G_{F,\Sigma}, [K]_s/[K]_{s+1}), & s \geq 1, \\ \pi_t X(\mathbb{A}_F^{\in\Sigma}), & s = 0, \end{cases}$$

where we regard the simplicial abelian groups $[K]_s/[K]_{s+1}$ as chain complexes.

Nielsen–Schreier implies that the simplicial group K is given levelwise by pro-finite completions of free groups, so the $s=1$ term is given by $[K]_1/[K]_2 \simeq (G(\overline{X}_{\text{ét}}, \overline{x})^{\wedge\Sigma})^{\text{ab}}$, which is just the reduced homology complex of $\overline{X}$ with $\prod_\ell \mathbb{Z}_\ell$ coefficients, where the product runs over those primes ℓ which are units in $\mathscr{O}_{F,\Sigma}$. Poitou–Tate duality in the form of Lemma 3.14 applied to the complexes $\widehat{G}(\overline{X}_{\text{ét}}, \overline{x})^{\text{ab}}$ thus gives

$$\mathbb{H}_c^{2-t}(G_{F,\Sigma}, [K]_1/[K]_2) \cong \begin{cases} \prod_\ell (\mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^\infty})/\mathrm{H}^2(G_{F,\Sigma}, \mu_{\ell^\infty}))^\vee, & t = 0, \\ \prod_\ell \mathrm{H}^{2+t}_{\text{ét}}(X, \mu_{\ell^\infty})^\vee, & t > 0, \end{cases}$$

where ℓ runs over all primes which are units in $\mathscr{O}_{F,\Sigma}$ and we follow the usual convention for continuous cohomology, regarding μ_{ℓ^∞} as the ind-sheaf $\varinjlim_{n\in\mathbb{N}} \mu_{\ell^n}$.

If we set $\mathrm{Br}_\Sigma(X) := \mathrm{Im}\big(\bigoplus_\ell \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^\infty}) \to \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)\big)$ to be the Σ–torsion cohomological Brauer group, then the first obstruction map $d_1\colon E_1^{0,0} \to E_1^{1,0}$ is thus the map

$$\pi_0 X(\mathbb{A}_F^{\in\Sigma}) \to \prod_\ell (\mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^\infty})/\mathrm{H}^2(G_{F,\Sigma}, \mu_{\ell^\infty}))^\vee,$$

induced by the natural map $\mathrm{BM}_\Sigma\colon \pi_0 X(\mathbb{A}_F^{\in\Sigma}) \to \mathrm{Br}_\Sigma(X)^\vee$, which is just the Brauer–Manin obstruction of [28] when $\mathscr{O}_{F,\Sigma} = F$.

Writing $\pi_0 X(\mathbb{A}_F^{\in\Sigma})^{\mathrm{Br}_\Sigma}$ for the kernel of BM_Σ, we thus have

$$\pi_0 X(\mathbb{A}_F^{\in\Sigma})^{\mathrm{Br}_\Sigma} = \mathrm{Im}(\pi_0 X(\mathbb{A}_F^{\in\Sigma})_1 \to \pi_0 X(\mathbb{A}_F^{\in\Sigma}))$$

for the tower above, and the later pages of the spectral sequence give obstructions to lifting further up the tower. Beware, however, that when $E_1^{s,s} \neq 0$, the lifts are not unique at each stage; in particular, if a point lies in the kernel of BM_Σ, we have a $\prod_\ell \mathrm{H}^3_{\text{ét}}(X, \mu_{\ell^\infty})^\vee$–torsor of possible choices on which to apply the secondary obstruction.

When X is an algebraic space rather than a stack, we have $\pi_0 X(\mathbb{A}_F^{\in\Sigma}) = X(\mathbb{A}_F^{\in\Sigma})$, and may simply write $X(\mathbb{A}_F^{\in\Sigma})^{\mathrm{Br}_\Sigma}$ for the image of $\pi_0 X(\mathbb{A}_F^{\in\Sigma})_1$.

Remark 3.25 Because the simplicial pro-group K of Example 3.24 is given levelwise by pro- Σ completions of free groups, the Magnus embedding (applied to pro-finite groups as in [43]) gives an isomorphism $[K]_s/[K]_{s+1} \cong \widehat{\mathrm{Lie}}_s(K^{\mathrm{ab}})$, where $\bigoplus_{s\geq 1}\mathrm{Lie}_s$ is the free Lie algebra functor, graded by bracket length, and $\widehat{\mathrm{Lie}}_s$ the pro-finite completion of Lie_s, applied levelwise to the simplicial abelian group. These functors are homotopy invariant when applied to chain complexes of projective modules via the Dold–Kan correspondence, but are not easy to calculate; they give the terms arising in the unstable Adams spectral sequence.

Over $\mathbb{Q}$, the functor $\bigoplus_s \mathrm{Lie}_s$ corresponds via the Dold–Kan correspondence to the free Lie algebra functor on chain complexes. Thus the spaces $E_1^{s,t}\otimes\mathbb{Q}$ are much simpler to describe in terms of free Lie algebras, but they correspond to the obstructions for the unipotent completion of Example 3.21 (with $R=1$).

We are now in a position to compare Kim's nonabelian reciprocity laws with the Brauer–Manin obstruction. Restricting to a single prime ℓ would give a similar statement for the ℓ–torsion part of the Brauer–Manin obstruction.

Proposition 3.26 *If the natural maps*

$$\mathrm{H}^2_{\mathrm{cts}}(\pi_1^{\text{ét}}(\overline{X})/[\pi_1^{\text{ét}}(\overline{X})]_{n+1}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

is surjective for all primes ℓ which are units in $\mathscr{O}_{F,\Sigma}$, then the image of the map $X(\mathbb{A}_F^{\in\Sigma})_n \to X(\mathbb{A}_F^{\in\Sigma})$ from Example 3.16 is contained in the Brauer–Manin set $X(\mathbb{A}_F^{\in\Sigma})^{\mathrm{Br}_\Sigma}$.

Proof Take a free pro-simplicial resolution $\widetilde{P}$ of $P := \pi_1^{\text{ét}}(\overline{X})^{\wedge\Sigma}/[\pi_1^{\text{ét}}(\overline{X})^{\wedge\Sigma}]_{n+1}$, and observe that the cofibrancy of $G(\overline{X}_{\text{ét}})$ ensures that the natural map $G(\overline{X}_{\text{ét}}) \to P$ lifts to a map $G(\overline{X}_{\text{ét}}) \to \widetilde{P}$, unique up to homotopy.

Since a point of $X(\mathbb{A})_n$ incorporates the datum of a P–valued Galois representation, the composite map

$$X(\mathbb{A})_n \to X(\mathbb{A}) \to \mathbb{H}^2_c\big(G_{F,\Sigma}, (G(\overline{X}_{\text{ét}})^{\wedge\Sigma})^{\mathrm{ab}}\big) \to \mathrm{H}^2_c(G_{F,\Sigma}, P^{\mathrm{ab}})$$

is necessarily 0. The kernel of the middle map is the Σ–torsion Brauer–Manin set as in Example 3.24, and via Poitou–Tate duality we can rewrite the final map as

$$\prod_\ell \mathbb{H}^2(G_{F,\Sigma}, \boldsymbol{R}\widetilde{\Gamma}_{\text{ét}}(\overline{X}, \mu_{\ell^\infty}))^\vee \to \prod_\ell \mathbb{H}^2(G_{F,\Sigma}, \boldsymbol{R}\widetilde{\Gamma}_{\mathrm{cts}}(P, \mu_{\ell^\infty}))^\vee,$$

where $\boldsymbol{R}\widetilde{\Gamma}$ denotes the reduced cohomology complex.

It suffices to show that this map is injective, or equivalently that its dual is surjective. This will follow from the Leray spectral sequences provided the maps

$$\mathrm{H}^i_{\mathrm{cts}}(P, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{H}^i_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

are isomorphisms for $i = 1$ and surjective for $i = 2$. The first condition is automatic and the second is our hypothesis. □

Considering the relative merits of the higher Brauer–Manin obstructions of Example 3.24 and the nonabelian reciprocity laws of Example 3.16, the latter generally avoid ambiguity of lifts to the higher stages of the tower, but converge more slowly.

Example 3.27 (étale Brauer–Manin obstructions) While Example 3.24 considered completions of the étale homotopy type $\widehat{G}(X_{\text{ét}}, \bar{x})$ relative to $G_{F,\Sigma}$, it also makes sense to consider completions with respect to larger quotients P of $\pi_0\widehat{G}(X_{\text{ét}}, \bar{x})$ over $G_{F,\Sigma}$ (ie relative pro-Σ quotients P of $\pi_1^{\text{ét}}(X, \bar{x})$ over $G_{F,\Sigma}$). We can write $K := \ker(\widehat{G}(X_{\text{ét}}, \bar{x}) \to P)$, and set $\Pi_n := \widehat{G}(X_{\text{ét}}, \bar{x})/[K]_{n+1}$.

As before, we define a tower $\{X(\mathbb{A}_F^{\in\Sigma})_n\}_n$ by

$$X(\mathbb{A}_F^{\in\Sigma})_n := X(\mathbb{A}_F^{\in\Sigma}) \times^h_{\mathrm{map}_{BG_{F,\Sigma}}(BG_{\mathbb{A}_F^{\in\Sigma}}, \overline{W}\Pi_n)} \mathrm{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma}, \overline{W}\Pi_n);$$

note that points in $X(\mathbb{A}_F^{\in\Sigma})_0$ now include the data of sections of $P \to G_{F,\Sigma}$, because $\Pi_0 = P$. The reasoning of Example 3.16 again gives a nonabelian spectral sequence

$$E_1^{s,t} \Rightarrow \pi_{t-s}X(\mathbb{A}_F^{\in\Sigma})_\infty, \quad \text{where } E_1^{s,t} = \begin{cases} \mathbb{H}_c^{1+s-t}(G_{F,\Sigma}, [K]_s/[K]_{s+1}), & s \geq 1, \\ \pi_t X(\mathbb{A}_F^{\in\Sigma})_0, & s = 0, \end{cases}$$

of groups and sets. The terms $\mathbb{H}_c^{1+s-t}(G_{F,\Sigma}, [K]_s/[K]_{s+1})$ depend on the section σ of $P \to G_{F,\Sigma}$ induced by the relevant element of $\pi_0 X(\mathbb{A}_F^{\in\Sigma})_0$, the Galois action then coming from the natural P–action on $[K]_s/[K]_{s+1}$.

As in Example 2.6, each section σ above gives a pro-(finite étale Σ–torsion) group scheme P^σ over $\mathscr{O}_{F,\Sigma}$ with BP^σ having étale homotopy type BP, and maps $X_{\text{ét}} \to BP$ correspond to P^σ–torsors $f^\sigma\colon Y^\sigma \to X$. The first obstruction map $d_1\colon E_1^{0,0} \to E_1^{1,0}$ in the spectral sequence above is the disjoint union, over inner automorphism classes of sections σ, of the Brauer–Manin obstructions

$$\pi_0 Y^\sigma(\mathbb{A}_F^{\in\Sigma})/P^\sigma(\mathscr{O}_{F,\Sigma}) \to \prod_\ell (\mathrm{H}^2_{\mathrm{pro(ét)}}(Y^\sigma, \mu_{\ell^\infty})/\mathrm{H}^2(G_{F,\Sigma}, \mu_{\ell^\infty}))^\vee$$

of the Y^σ (defined as derived limits unless $\ker(P \to G_{F,\Sigma})$ is finite), so we have

$$\mathrm{Im}(\pi_0 X(\mathbb{A}_F^{\in\Sigma})_1 \to \pi_0 X(\mathbb{A}_F^{\in\Sigma})) = \bigcup_{\substack{\sigma\colon G_{F,\Sigma}\to P \\ \text{a section}}} f^\sigma(\pi_0 Y^\sigma(\mathbb{A}_F^{\in\Sigma})^{\mathrm{Br}_\Sigma})$$

(when X is an algebraic space, we can drop the π_0's). When $\mathscr{O}_{F,\Sigma} = F$, combining these for all finite extensions P of G_F will thus give Skorobogatov's étale Brauer–Manin obstruction [39].

For smooth proper varieties, the space of adèlic points is compact, and by Tychonoff's theorem the inverse limit of nonempty compact spaces is nonempty, so considering pro-étale covers in this way will just recover the étale Brauer–Manin obstruction in this case.

When $F = \mathscr{O}_{F,\Sigma}$, the universal case to consider would take $P = \pi_1^{\text{ét}}(X, \bar{x})$, with the spectral sequence then detecting exclusively higher homotopical information, and $\overline{Y}^\sigma$ being a universal cover $\widetilde{X}$ of X. For this choice of P, we may therefore set

$$\begin{aligned} \pi_0 X(\mathbb{A}_F)^{\mathrm{pro(ét),Br}} &= \mathrm{Im}(\pi_0 X(\mathbb{A}_F)_1 \to \pi_0 X(\mathbb{A}_F)) \\ &= \bigcup_{\substack{\sigma\colon G_F\to \pi_1^{\text{ét}}(X,\bar{x}) \\ \text{a section}}} f^\sigma(\pi_0 Y^\sigma(\mathbb{A}_F)^{\mathrm{Br}}) \end{aligned}$$

(again, we can drop the π_0's when X is an algebraic space).

Since in this case G_F has cohomological dimension 2, the higher homotopy groups $\pi_{\geq 2}([K]_s/[K]_{s+1})$ never contribute to the obstruction spaces $E_1^{s,s-1}$ for $\pi_0 X(\mathbb{A}_F)$ in the nonabelian spectral sequence above. For the universal case $P = \pi_1^{\text{ét}}(X, \bar{x})$, we have $\pi_i K = \pi_{i+1}^{\text{ét}}(\widetilde{X})$, and $\pi_1[K]_2 = 0$ (the Hurewicz map for π_2 being an isomorphism). Thus $E_1^{s,s-1} = 0$ for $s > 1$, meaning all higher obstructions vanish and

$$\pi_0 X(\mathbb{A}_F)^{\mathrm{pro(ét),Br}} = \mathrm{Im}(\pi_0 X(\mathbb{A}_F)_\infty \to \pi_0 X(\mathbb{A}_F)).$$

Moreover the sequence $[K]_n$ is increasingly connected, so $\Pi_\infty \simeq \widehat{G}(X_{\text{ét}}, \bar{x})$. Together, these phenomena imply that vanishing of the pro-étale Brauer–Manin obstruction alone implies the existence of a compatible section of the map $X_{\text{ét}}^\wedge \to (\operatorname{Spec} \mathscr{O}_F)_{\text{ét}}^\wedge$ of profinite étale homotopy types when X is geometrically connected. This is not nearly as impressive as it might seem, since the construction of the pro-étale Brauer–Manin obstruction assumes a compatible section of $\pi_1^{\text{ét}}(X) \to G_F$.

Remark 3.28 (relation to Harpaz–Schlank) Our spaces $X(\mathbb{A}_F)_n$ in this section are closely related to those of [17], which (after including Archimedean places) considers spaces $X(\mathbb{A}_F)^h$ broadly of the form

$$X(\mathbb{A}_F) \times^h_{\mathrm{map}_{BG_F}((\mathrm{Spec}\,\mathbb{A}_F)_{\text{ét}}, X_{\text{ét}})} \mathrm{map}_{BG_F}(BG_F, X_{\text{ét}}),$$

as well as variants $X(\mathbb{A}_F)^{\mathbb{Z}h}$, $X(\mathbb{A}_F)^{h,n}$ and $X(\mathbb{A}_F)^{\mathbb{Z}h,n}$. In our terms, $X(\mathbb{A}_F)^{\mathbb{Z}h}$ corresponds to replacing $X_{\text{ét}}$ with $\overline{W}(G(X_{\text{ét}})^{\mathrm{ab}})$ above; the others are given by taking Postnikov towers.

Rather than imposing smoothness hypotheses and appealing to [9] as we have done, Harpaz and Schlank [17] construct a G_F–equivariant homotopy type $\acute{\mathrm{E}}\mathrm{t}_{/K}(X)$, and effectively works with the homotopy quotient $\acute{\mathrm{E}}\mathrm{t}_{/K}(X)/^h G_F$ in place of $X_{\text{ét}}$ above. In [17, Theorem 11.1], the étale Brauer set is shown to correspond to the set $X(\mathbb{A}_F)^h$, which is a somewhat stronger statement than our final observation in Example 3.27.

The main new ingredient in our constructions and comparisons is that by modelling pro-finite homotopy types as simplicial pro-finite groups and groupoids following [34, Section 1; 33, Proposition 1.19], we are able to work systematically with much more general towers than the Postnikov tower.

3.3 Alternative characterisations of the reciprocity laws

We now give a more pedestrian interpretation of the obstruction maps from Section 1, and show how this can give rise to a more explicit description of the first obstruction map in cases of interest. This first obstruction map seems to be well known to experts, but we are not aware of a reference.

3.3.1 Cohomological obstruction classes Extensions $e\colon 0 \to A \to \Pi'' \to \Pi' \to 1$ of a group Π' by an abelian Π'–representation A are classified by

$$\mathrm{H}^2(\Pi', A),$$

by which we mean continuous cohomology when considering extensions of topological groups.

Given a group homomorphism $\psi\colon G \to \Pi'$, the obstruction to lifting ψ to a homomorphism $\widetilde{\psi}\colon G \to \Pi''$ is then given by

$$\psi^*[e] \in \mathrm{H}^2(G, A).$$

If $\psi^*[e] = 0$, then the difference between two choices for $\widetilde{\psi}$ is a derivation, so the set of choices is a torsor for the group

$$\mathrm{H}^1(G, A).$$

Taking Π' and Π'' to be suitable quotients of the arithmetic fundamental group of a scheme X over $\mathscr{O}_{F,\Sigma}$, the Diophantine obstruction maps on spaces of sections

$$\pi_0 \operatorname{map}_{BG_{F,\Sigma}}(BG_{F,\Sigma}, B\Pi') \to \mathrm{H}^2(G_{F,\Sigma}, A)$$

of Section 2 are all of this form. The adèlic obstruction maps of Section 3.1 are a slight variant coming from looking at restricted products

$$\prod_{v\in\Sigma}{}' \pi_0 \operatorname{map}_{BG_{F,\Sigma}}(BG_v, B\Pi') \to \prod_{v\in\Sigma}{}' \mathrm{H}^2(G_v, A).$$

The reciprocity maps associated to an $\mathbb{A}_F^{\in\Sigma}$–point in Section 3.2 then effectively look at the difference between these obstructions, yielding an obstruction in $\mathrm{H}^2_c(G_{F,\Sigma}, A)$ via the exact sequence

$$\prod_{v\in\Sigma}{}' \mathrm{H}^1(G_v, A) \xrightarrow{\partial} \mathrm{H}^2_c(G_{F,\Sigma}, A) \to \mathrm{H}^2(G_{F,\Sigma}, A) \to \prod_{v\in\Sigma}{}' \mathrm{H}^2(G_v, A).$$

In general, this is not very easy to work with, but when the extension e splits, so $\Pi'' = \Pi' \ltimes A$, the adèlic point defines a derivation in $\alpha \in \prod'_{v\in\Sigma} \mathrm{H}^1(G_v, A)$, with associated abelian obstruction $\partial(\alpha) \in \mathrm{H}^2_c(G_{F,\Sigma}, A)$ to lifting the adèlic point to a rational point.

Example 3.29 In nilpotent or unipotent settings such as Example 3.18, the first stage in the tower is a split extension

$$G_F \ltimes \pi_1^{\text{ét}}(\overline{X}, \bar{x})^{\text{ab}} \to G_F, \quad G_{F,\Sigma} \ltimes (\pi_1^{\text{ét}}(\overline{X}, \bar{x}) \otimes \mathbb{Q}_\ell)^{\text{ab}} \cong G_{F,\Sigma} \ltimes \mathrm{H}^1(\overline{X}, \mathbb{Q}_\ell)^* \to G_{F,\Sigma}.$$

Then an $\mathbb{A}_F^{\in\Sigma}$–point y defines a class in $\mathrm{H}^1(\mathbb{A}_F^{\in\Sigma}, \mathrm{H}^1(\overline{X}, \mathbb{Q}_\ell)^*)$ whose image in $\mathrm{H}^2_c(G_{F,\Sigma}, \mathrm{H}^1(\overline{X}, \mathbb{Q}_\ell)^*)$ is the first unipotent obstruction to y being a rational point.

Example 3.30 Relative Maltsev completions as in Example 3.19 are a little more complicated. For $X = \mathcal{M}_{1,1}$ the stacky modular curve, take $x \in X(\mathscr{O}_{F,\Sigma})$, giving rise to a $G_{F,\Sigma}$–representation V of dimension 2 over $\mathbb{Q}_\ell$. We then set $P_0 = G_{F,\Sigma} \ltimes \mathrm{SL}_2(\mathbb{Q}_\ell)$, and

$$P_1 := G_{F,\Sigma} \ltimes (\mathrm{SL}_2(\mathbb{Z})^{\mathrm{SL}_2,\mathrm{Mal}}/[\mathrm{R_u}]_2), = G_{F,\Sigma} \ltimes (\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2))^* \rtimes \mathrm{SL}_2(\mathbb{Q}_\ell)),$$

with $\Pi_i = P_i \times_{\mathrm{SL}_2(\mathbb{Q}_\ell)} \mathrm{SL}_2(\mathbb{Z}_\ell)$, where we are writing $O(\mathrm{SL}_2)$ for the ring of algebraic functions on the scheme SL_2 over $\mathbb{Q}_\ell$.

Now, P_1 is an extension of P_0 by $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2) \otimes \mathbb{Q}_\ell)^*$, so is given by a class in $\mathrm{H}^2\big(P_0, \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2))^*\big)$, where we may regard $\mathrm{SL}_2(\mathbb{Q}_\ell)$ as an algebraic group. Since SL_2 is reductive, the Leray–Serre spectral sequence then gives

$$\mathrm{H}^2\big(P_0, \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2))^*\big) \cong \mathrm{H}^2\big(G_{F,\Sigma}, \big(\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2))^*\big)^{\mathrm{SL}_2}\big),$$

which vanishes because $\mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q}_\ell) = 0$.

We therefore have a split extension $\Pi_1 \cong \Pi_0 \ltimes \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2))^*$. (For more general relative Maltsev completions, a similar conclusion will still hold by combining Leray–Serre with the splitting of the extension $\Pi_1 \to G_{F,\Sigma}$.)

Thus an adèlic elliptic curve E defines a class in $\mathrm{H}^1\big(\mathbb{A}_F^{\in\Sigma}, \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2))^*\big)$, whose image in $\mathrm{H}^2_c\big(G_{F,\Sigma}, \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), O(\mathrm{SL}_2))^*\big)$ is the first obstruction to E being defined over $\mathscr{O}_{F,\Sigma}$ with Tate module $T_\ell(E(\bar{F})) \otimes \mathbb{Q} \simeq V$.

3.3.2 The first obstruction for modular curves We now give an explicit description of the abelian obstruction of Example 3.30, seeking elliptic curves with given Tate module.

On the modular curve $q\colon Y_\Gamma \to \operatorname{Spec} \mathscr{O}_{F,\Sigma}$, the Tate module of the universal elliptic curve $f\colon E \to Y_\Gamma$ gives a lisse $\mathbb{Z}_\ell$–sheaf $\mathbb{T}_\ell$ of rank 2, and we write $\mathbb{T}_{\mathbb{Q}_\ell} := \mathbb{T}_\ell \otimes \mathbb{Q}$. On pulling back to $\bar{Y}_\Gamma$, the sheaves $S^m\mathbb{T}_{\mathbb{Q}_\ell}$ correspond to the irreducible representations V_m of SL_2, and we consider the Galois representations $\mathrm{H}^1(\Gamma, V_m) := \boldsymbol{R}^1 q_* S^m \mathbb{T}_{\mathbb{Q}_\ell}$. For each m, the adjunction $q^* \dashv \boldsymbol{R}q_*$ defines a class

$$\eta_m \in \mathrm{Ext}^1_{Y_\Gamma,\mathbb{Q}_\ell}(q^*\mathrm{H}^1(\Gamma, V_m), S^m\mathbb{T}_{\mathbb{Q}_\ell}).$$

Now take an adèlic point $x \in Y_\Gamma(\mathbb{A}_F^{\in\Sigma})$, and assume that there is a $G_{F,\Sigma}$–representation Λ with $\det \Lambda = \mathbb{Z}_\ell(1)$ and an isomorphism $\alpha\colon \Lambda \otimes \mathbb{Q} \cong \mathbb{T}_{\mathbb{Q}_\ell,x}$ which is G_v–equivariant for all $v \in \Sigma$. A necessary condition for x to lie in $Y_\Gamma(\mathscr{O}_{F,\Sigma})$ compatibly with α is that the class $x^*\eta_m \in \prod_{v\in\Sigma} \mathrm{Ext}^1_{G_v}(\mathrm{H}^1(\Gamma, V_m), S^m\Lambda \otimes \mathbb{Q})$ lie in the image of $\mathrm{Ext}^1_{G_{F,\Sigma}}(\mathrm{H}^1(\Gamma, V_m), S^m\Lambda \otimes \mathbb{Q})$. Following the conventions of Section 3.2.1 to replace the product with a suitable restricted product, we get an obstruction

$$\partial(x^*\eta_m) \in \mathrm{H}^2_c(G_{F,\Sigma}, \mathrm{H}^1(\Gamma, V_m)^* \otimes S^m\Lambda).$$

Combining these gives a map

$$\mathrm{H}^1(G_{F,\Sigma}, \mathrm{GL}_2(\mathbb{Q}_\ell)) \times_{\mathrm{H}^1(\mathbb{A}_F^{\in\Sigma}, \mathrm{GL}_2(\mathbb{Q}_\ell))} Y_\Gamma(\mathbb{A}_F^{\in\Sigma}) \to \prod_{m\geq 1} \mathrm{H}^2_c(G_{F,\Sigma}, \mathrm{H}^1(\Gamma, V_m)^* \otimes S^m\Lambda),$$

which is the first reciprocity map associated to the relative completion of $\Gamma \to \mathrm{SL}_2(\mathbb{Q}_\ell)$ in Example 3.19, via the isomorphism $O(\mathrm{SL}_2) \otimes \mathbb{Q}_\ell \cong \bigoplus_m V_m \otimes V_m^*$. We may then use Poitou–Tate duality as in Example 3.24 to rewrite the target of the map as

$$\prod_{m\geq 1} (\mathrm{H}^2_{\text{ét}}(Y_\Gamma, S^m\mathbb{T}_\ell \otimes q^*S^m\Lambda^* \otimes \mu_{\ell^\infty})^\vee \otimes \mathbb{Q});$$

adapting Example 3.27, this can be recovered from the Brauer–Manin obstruction of an inverse system of finite étale covers of Y_Γ, which in this case correspond to twisted level structures associated to the $G_{F,\Sigma}$–representations Λ/ℓ^n.

Remark 3.31 An intermediate step in the construction above associates to each elliptic curve E over F a class in

$$\mathrm{Ext}^1_{G_{F,\Sigma}}\big(\mathrm{H}^1(\Gamma, V_m), S^m T_\ell(E(\bar{F}) \otimes \mathbb{Q})\big).$$

The corresponding construction for complex elliptic curves and mixed Hodge structures is given in [14, Remark 13.3] (evaluating the section at the point $[E]$). The extension arises geometrically as the relative cohomology group $\mathrm{H}^1(\overline{Y}_\Gamma, [E]; S^m\mathbb{T}_{\mathbb{Q}_\ell})$.

3.3.3 Higher Brauer–Main obstructions via cochain algebras The unipotent obstructions which we have considered were formulated in terms of morphisms of simplicial pro-unipotent groups, so could be thought of as a form of Quillen homotopy type [38]. An equivalent alternative formulation would be to look at morphisms of Sullivan homotopy types [41], which are just algebras of cochains.

Taking a Deligne–Mumford stack X over $\mathscr{O}_{F,\Sigma}$ and writing $\overline{X} := X \otimes \mathscr{O}_{\overline{F},\Sigma}$, the cochain complex $\boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell)$ carries a natural cup product, and is in fact naturally quasi-isomorphic to a commutative differential graded algebra over $\mathbb{Q}_\ell$. Equivalently this means that $\boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell)$ carries the structure of a unital Com_∞–algebra (or strongly homotopy commutative algebra): it has a symmetric bilinear multiplication m_2, which is associative up to a homotopy m_3, and there is a hierarchy of higher homotopies m_n formulated in terms of the Lie operad. In the $R = 1$ case, Example 3.21 looks at the morphism

$$\boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$$

defined by an adèlic point, and studies obstructions to lifting it to a Com_∞–morphism $\{f_n\}_{n\geq 1}$ which is equivariant for the global Galois group $G_{F,\Sigma}$, rather than just the pro-groupoid

$$G_{\mathbb{A}_F^{\in\Sigma}} := \varprojlim_{\substack{T\subset\Sigma \\ T\,\text{finite}}} \Big(\coprod_{v\in T} G_v \sqcup \coprod_{v\in\Sigma-T} G_v/I_v \Big)$$

formed from local Galois groups.

(1) The first reciprocity law seeks just to lift this as a morphism of complexes, fixing $\mathbb{Q}_\ell \subset \boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell)$, so the first obstruction lies in

$$\mathbb{E}\mathrm{xt}^1_{G_{F,\Sigma},c}(\boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell)/\mathbb{Q}_\ell, \mathbb{Q}_\ell) \cong (\mathrm{H}^2(X, \mu_{\ell^\infty})/\mathrm{H}^2(G_{F,\Sigma}, \mu_{\ell^\infty}))^\vee \otimes \mathbb{Q};$$

this is just the rational Brauer–Manin obstruction.

(2) The secondary obstruction of Section 3.2.3 depends on a choice $f_1\colon \boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ of $G_{F,\Sigma}$–equivariant chain map, together with a homotopy h_1 of $G_{\mathbb{A}_F^{\in\Sigma}}$–representations making f_1 compatible with our chosen adèlic point. Such a lift exists whenever the rational ℓ–torsion Brauer–Manin obstruction vanishes, and we now need to look at whether it respects the cup product. We thus ask whether the diagram

$$\begin{tikzcd}
\boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell) \otimes \boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell) \arrow[r, "m_2"] \arrow[dr, "f_1 \otimes f_1"'] & \boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell) \arrow[d, "f_1"] \\
& \mathbb{Q}_\ell
\end{tikzcd}$$

commutes, up to a homotopy f_2, in the derived category of $G_{F,\Sigma}$–representations, with a further $G_{\mathbb{A}_F^{\in\Sigma}}$–equivariant homotopy h_2 between f_2 and the homotopy $f_1 \otimes h_1 + h_1 \otimes f_1 + (h_1 d) \otimes h_1 - h_1 \circ m_2$ providing the known $G_{\mathbb{A}_F^{\in\Sigma}}$–equivariant commutativity of f_1. The resulting obstruction lies in

$$\mathbb{E}\mathrm{xt}^0_{G_{F,\Sigma},c}(\boldsymbol{R}\Gamma(\overline{X}^2, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \cong \mathrm{H}^3(X^2, \mu_{\ell^\infty})^\vee \otimes \mathbb{Q},$$

but this restricts to the finer obstruction described in Remark 3.23 when we take symmetry and the unit into account.

(3) The third obstruction is more complicated, measuring obstructions to choosing the next component (f_3, h_3) of a Com_∞–morphism. If we choose a model A of $\boldsymbol{R}\Gamma(\overline{X}, \mathbb{Q}_\ell)$ which is strictly (graded-)commutative, this means we seek a map $f_3\colon A^{\otimes 3} \to A[-1]$ satisfying

$$(d \circ f_3 \mp f_3 \circ d)(a, b, c) = f_2(ab, c) \pm f_2(a, bc) \mp f_1(a) f_2(b, c) \mp f_2(a, b) f_2(c),$$

which must vanish on the unit $1 \in A$ and on shuffle products. The right-hand side and associated $G_{\mathbb{A}_F^{\in\Sigma}}$–equivariant homotopy in terms of h_2 give rise to an obstruction class in

$$\mathbb{E}\mathrm{xt}^{-1}_{G_{F,\Sigma},c}(\boldsymbol{R}\Gamma(\overline{X}^3, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \cong \mathrm{H}^4(X^3, \mu_{\ell^\infty})^\vee \otimes \mathbb{Q},$$

which is closely related to Massey triple products.

(4) Explicit descriptions for the higher obstructions follow from the formulas for Com_∞–morphisms as in [27, Sections 10.2.2 and 13.1.13] (take the expression for A_∞–morphisms in [27, Proposition 10.2.12] and replace $\mathcal{A}s$ with $\mathcal{L}ie$ by taking invariants under shuffle permutations). These are related to higher Massey products.

To express Example 3.21 in these terms beyond the $R = 1$ case, we may reformulate via [31, Proposition 3.15 and Corollary 4.41] to seek $(G_{F,\Sigma} \ltimes R)$–equivariant morphisms

$$\boldsymbol{R}\Gamma(\overline{X}, O(R)) \to O(R)$$

for a pro-reductive algebraic groupoid R over $\mathbb{Q}_\ell$ and Zariski-dense Galois-equivariant homomorphism $\pi_1(\overline{X}, \mathcal{B}) \to R(\mathbb{Q}_\ell)$ with a Galois-equivariant set of basepoints $\mathcal{B}$. The descriptions above adapt, with the sheaf $O(R)_{\mathbb{Z}_\ell} \otimes \mu_{\ell^\infty}$ (regarded as a $(\pi_1(\overline{X}, \mathcal{B}) \times R)$–representation via the left and right actions) replacing μ_{ℓ^∞}.

Remark 3.32 If we wished to construct obstructions in the nilpotent, rather than unipotent, setting, we should seek Galois-equivariant morphisms $\boldsymbol{R}\Gamma(\overline{X}, \hat{\mathbb{Z}}) \to \hat{\mathbb{Z}}$ of cosimplicial commutative rings. The first obstruction is just Brauer–Manin, but the torsion in the higher obstructions is very difficult to describe, as discussed in Remark 3.25.

Remark 3.33 The description in terms of cochain algebras will readily adapt to more general cohomology theories with cup product. For instance, a motivic analogue of Section 2 would be given by seeking Com_∞–morphisms $M(Y_\Gamma) \to M(F)$ of cohomological F–motives, assuming existence of a suitable Com_∞–structure enriching the cup product on motivic cohomology. The obstruction tower just depends on a filtration on the Com_∞–operad, whereas a Postnikov-type filtration in terms of motivic homotopy groups [37, Section 3.5] would require a suitable t–structure. This approach could also be used to construct motivic obstructions to adèlic points being global, along the lines of this section, but it is not obvious what the motivic analogue of Poitou–Tate duality should be.

Appendix Pro-finite homotopy types for adèles

Definition A.1 Write $s^\flat\mathrm{Gpd}$ for the category consisting of simplicial groupoids G for which

(1) the simplicial set $\mathrm{Ob}G$ of G is constant and finite;

(2) each $G_i(x,y)$ is finite;

(3) the group $N_iG(x,x) := G_i(x,x) \cap \bigcap_{j>0} \ker \partial_j$ is trivial for all but finitely many i.

Note that the second condition is equivalent to saying that the map $G \to \mathrm{cosk}_n G$ to the n–coskeleton is an isomorphism for sufficiently large n.

Lemma A.2 *The functor U from $\mathrm{pro}(s^{\flat}\mathrm{Gpd})$ to simplicial pro-finite groupoids given by $(U\{G(\alpha)\}_\alpha)_n := \{G(\alpha)_n\}_\alpha$ is an equivalence of categories; moreover, we may restrict to inverse systems in which all morphisms are surjective.*

Proof Since $s^{\flat}\mathrm{Gp}$ is an Artinian category, the proofs of [33, Proposition 1.19] (which dealt with Artinian local rings rather than finite groupoids) and of [34, Lemma 1.17] carry over to this generality. □

Definition A.3 Given a simplicial scheme Y, define $\Gamma^{\mathbb{S}}(Y,-)$ to be the global sections functor from simplicial étale presheaves on Y to simplicial sets. Write $\boldsymbol{R}\Gamma^{\mathbb{S}}_{\text{ét}}(Y,-)$ for its right-derived functor with respect to the model structure for étale hypersheaves. Explicitly,

$$\boldsymbol{R}\Gamma^{\mathbb{S}}_{\text{ét}}(Y,\mathscr{F}) \simeq \varinjlim_{Y'_\bullet}{}^{\!\!\mathrm{holim}} \; \underset{\overleftarrow{n\in\Delta}}{\mathrm{holim}}\, \Gamma(Y'_n,\mathscr{F}),$$

where $Y'_\bullet$ runs over simplicial étale hypercovers of Y.

Given an inverse system $\mathscr{F} = \{\mathscr{F}_i\}_i$, set

$$\boldsymbol{R}\Gamma^{\mathbb{S}}_{\text{ét}}(Y,\mathscr{F}) := \underset{\overleftarrow{i}}{\mathrm{holim}}\, \boldsymbol{R}\Gamma^{\mathbb{S}}_{\text{ét}}(Y,\mathscr{F}_i).$$

Lemma A.4 *There is a canonical morphism*

$$\boldsymbol{R}\Gamma^{\mathbb{S}}_{\text{ét,cts}}(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma}, \overline{W}G) \to \mathrm{map}(BG_{\mathbb{A}_F^{\in\Sigma}}, \overline{W}G)$$

in $\mathrm{Ho}(\mathbb{S})$, functorial in simplicial pro-finite groupoids G.

Proof Because $\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma}$ is quasicompact, the category of quasicompact hypercovers of $\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma}$ is left filtering in the category of all hypercovers, by the argument of [10, Proposition 7.1]. Thus, for all simplicial presheaves $\mathscr{F}$,

$$\begin{aligned}\boldsymbol{R}\Gamma^{\mathbb{S}}_{\text{ét}}(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma},\mathscr{F}) \\ \simeq \underset{\overrightarrow{Y'_\bullet\in\mathrm{HR}(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma})}}{\mathrm{holim}} \; \underset{\overleftarrow{n\in\Delta}}{\mathrm{holim}}\, \Gamma(Y'_n,\mathscr{F}) \leftarrow \underset{\overrightarrow{Y'_\bullet\in\mathrm{qcHR}(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma})}}{\mathrm{holim}} \; \underset{\overleftarrow{n\in\Delta}}{\mathrm{holim}}\, \Gamma(Y'_n,\mathscr{F})\end{aligned}$$

is an equivalence, where $\mathrm{HR}(Y)$ is the category of simplicial hypercovers $Y'_\bullet \to Y$ and $\mathrm{qcHR}(Y)$ the full subcategory of simplicial hypercovers $Y'_\bullet \to Y$ with each Y'_n quasicompact.

Given a simplicial presheaf $\mathscr{F}$ for which the map $\mathscr{F} \to \mathrm{cosk}_m \mathscr{F}$ is an isomorphism, the map

$$\varinjlim_{Y'_\bullet \in \mathrm{qcHR}(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma})}{}^{\!\!\mathrm{holim}} \ \varprojlim_{n\in\Delta}{}^{\!\!\mathrm{holim}}\ \Gamma(Y'_n, \mathscr{F}) \leftarrow \varinjlim_{Y'_\bullet \in \mathrm{qcHR}^\flat(\mathrm{Spec}\,\mathbb{A}_F)}{}^{\!\!\mathrm{holim}} \ \varprojlim_{n\in\Delta}{}^{\!\!\mathrm{holim}}\ \Gamma(Y'_n, \mathscr{F})$$

is an equivalence, where $\mathrm{qcHR}^\flat(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma})$ consists of quasicompact hypercovers Y' which are truncated in the sense that $Y' = \mathrm{cosk}_r(Y'/\mathbb{A}_F^{\in\Sigma})$ for some r (in fact $r = m$ suffices for the case in hand).

Given a quasicompact hypercover $Y'_\bullet \to \mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma}$, write $Y'_{\bullet,v}$ for its pullback along $\mathrm{Spec}\,F_v \to \mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma}$. Thus each $Y'_{i,v}$ is the spectrum of a finite product of finite field extensions of F_v. Because Y'_i is of finite type over $\mathbb{A}_F^{\in\Sigma}$, it is defined over $\left(\prod_{v\in\Sigma}\mathscr{O}_v\right)\otimes_{\mathbb{Z}}\mathbb{Z}[S_i^{-1}]$ for some finite set $S_i \subset \Sigma$ of primes. For $v \in S_i$, it then follows that $Y'_{i,v}$ is the spectrum of a finite product of finite unramified field extensions of F_v. When the hypercover $Y'_\bullet$ is r–truncated, we can set $S = \bigcup_{i\le r} S_i$, and then see that

$$\{Y'_{\bullet,v}\}_v \in \left(\prod_{v\in\Sigma-S}\mathrm{qcHR}^{\mathrm{nr}}(\mathrm{Spec}\,F_v)\right)\times\left(\prod_{v\in S}\mathrm{qcHR}(\mathrm{Spec}\,F_v)\right) \subset \prod_v \mathrm{HR}(\mathrm{Spec}\,F_v),$$

where $\mathrm{qcHR}^{\mathrm{nr}}$ consists of quasicompact hypercovers built from unramified field extensions.

Writing

$$\prod_v{}' \mathrm{qcHR}(\mathrm{Spec}\,F_v) := \bigcup_{S\subset\Sigma \text{ finite}}\left(\prod_{v\in\Sigma-S}\mathrm{qcHR}^{\mathrm{nr}}(\mathrm{Spec}\,F_v)\right)\times\left(\prod_{v\in S}\mathrm{qcHR}(\mathrm{Spec}\,F_v)\right),$$

we then get a map

$$\varinjlim_{Y'_\bullet \in \mathrm{qcHR}^\flat(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma})}{}^{\!\!\mathrm{holim}} \ \varprojlim_{n\in\Delta}{}^{\!\!\mathrm{holim}}\ \Gamma(Y'_n, \mathscr{F}) \to \varinjlim_{Y'_\bullet \in \prod'_v \mathrm{qcHR}^\flat(\mathrm{Spec}\,F_v)}{}^{\!\!\mathrm{holim}} \ \varprojlim_{n\in\Delta}{}^{\!\!\mathrm{holim}}\ \Gamma(Y'_n, \mathscr{F}).$$

Returning to the statement of the lemma, since both functors send filtered inverse limits to homotopy limits, Lemma A.2 allows us to restrict to the case where $G \in s^\flat\mathrm{Gpd}$. Thus the map $G \to \mathrm{cosk}_{m-1} G$ is an isomorphism for some m, so $\overline{W}G \cong \mathrm{cosk}_m \overline{W}G$ and satisfies the conditions for $\mathscr{F}$ above. Then we have

$$\boldsymbol{R}\Gamma^{\mathbb{S}}_{\text{ét}}(\mathrm{Spec}\,\mathbb{A}_F^{\in\Sigma}, \overline{W}G) \to \varinjlim_{Y'_\bullet \in \prod'_v \mathrm{qcHR}^\flat(\mathrm{Spec}\,F_v)}{}^{\!\!\mathrm{holim}} \ \prod_v \varprojlim_{n\in\Delta}{}^{\!\!\mathrm{holim}}\ \Gamma((Y'_v)_n, \overline{W}G).$$

Now, we can rewrite the right-hand side as

$$\varinjlim_{S\subset\Sigma \text{ finite}} \operatorname*{holim}_{\substack{\longleftarrow \\ n\in\Delta}}\Bigg(\prod_{v\in\Sigma-S}\ \operatorname*{holim}_{\substack{\longrightarrow \\ (Y'_v)_\bullet\in\mathrm{qcHR}^{\mathrm{nr}}(\operatorname{Spec} F_v)}} \Gamma((Y'_v)_n,\overline{W}G)\Bigg) \times\Bigg(\prod_{v\in S}\ \operatorname*{holim}_{\substack{\longrightarrow \\ (Y'_v)_\bullet\in\mathrm{qcHR}(\operatorname{Spec} F_v)}} \Gamma((Y'_v)_n,\overline{W}G)\Bigg).$$

Since $(\operatorname{Spec} F_v)^\wedge_{\text{ét}} \simeq BG_v$ and $(\operatorname{Spec} \mathscr{O}_{F,v})^\wedge_{\text{ét}} \simeq B(G_v/I_v)$, this is weakly equivalent to

$$\varinjlim_{S\subset\Sigma \text{ finite}} \prod_{v\in\Sigma-S} \operatorname{map}(B(G_v/I_v),\overline{W}G)\times\prod_{v\in S}\operatorname{map}(BG_v,\overline{W}G),$$

which is just $\operatorname{map}(BG_{\mathbb{A}_F^{\in\Sigma}},\overline{W}G)$, as required. □

Corollary A.5 *There is a canonical morphism*

$$BG_{\mathbb{A}_F^{\in\Sigma}} \to (\operatorname{Spec}\mathbb{A}_F^{\in\Sigma})^\wedge_{\text{ét}}$$

in the homotopy category of pro-simplicial sets, where $\wedge$ *denotes pro-finite completion, and* $X_{\text{ét}}$ *the étale topological type as in* [10, Definition 4.4].

Proof Since simplicial pro-finite groupoids model pro-finite homotopy types by [34, Proposition 1.29], it suffices to show that we have natural morphisms

$$\operatorname{map}((\operatorname{Spec}\mathbb{A}_F^{\in\Sigma})_{\text{ét}},\overline{W}G)\to\operatorname{map}(BG_{\mathbb{A}_F^{\in\Sigma}},\overline{W}G)$$

for simplicial pro-finite groupoids G, and this is precisely the content of Lemma A.4. □

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