

On the Brun spectral sequence for topological Hochschild homology

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We generalize a spectral sequence of Brun for the computation of topological Hochschild homology. The generalized version computes the E -homology of $\mathrm{THH}(A; B)$, where E is a ring spectrum, A is a commutative S -algebra and B is a connective commutative A -algebra. The input of the spectral sequence are the topological Hochschild homology groups of B with coefficients in the E -homology groups of $B \wedge_A B$. The mod p and v_1 topological Hochschild homology of connective complex K -theory has been computed by Ausoni and later again by Rognes, Sagave and Schlichtkrull. We present an alternative, short computation using the generalized Brun spectral sequence.

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1 Introduction

Topological Hochschild homology assists in computations of algebraic K -theory. Bökstedt refined the Dennis trace map to obtain a map $\mathbf{K}(-) \rightarrow \mathrm{THH}(-)$, from algebraic K -theory to topological Hochschild homology, that factors over topological cyclic homology. Topological cyclic homology can be calculated from topological Hochschild homology using homotopy fixed points and Tate spectral sequences. In particular, the recent treatment of topological cyclic homology by Nikolaus and Scholze [24] simplifies computations and opens broad perspectives, calling for new examples. In a number of cases topological cyclic homology is a good approximation of algebraic K -theory. For example, by Hesselholt and Madsen [14], $\mathbf{K}(\mathbb{Z}_p)_p$ is equivalent to the connective cover of $\mathrm{TC}(\mathbb{Z}_p)_p$ and by Dundas, Goodwillie and McCarthy [12] the same follows for every connective S -algebra B with $\pi_0(B) = \mathbb{Z}_p$.

The goal of this paper is to provide an additional tool for the computation of topological Hochschild homology by generalizing a spectral sequence of Brun [9]. As explained below, it does not rely on flatness assumptions, as does for example the Bökstedt spectral sequence. As a first example, we apply this spectral sequence to compute the mod p and v_1 topological Hochschild homology of p -completed connective complex

K -theory, $V(1)_* \text{THH}(\text{ku})$, for primes $p \geq 5$. The $V(1)$ -homotopy of $\text{THH}(\text{ku})$ has been first computed by Ausoni in [2] (using the Bökstedt spectral sequence), leading via topological cyclic homology to a computation of the $V(1)$ -homotopy of $\text{K}(\text{ku})$ [3].

Topological Hochschild homology computations also play an important role in the development of the algebraic geometry of S -algebras, and in particular as a test case for new concepts, such as useful variants of the notion of étale extensions. This is illustrated for example by the work of Rognes, Sagave and Schlichtkrull [32; 29] on log étale extensions, which has led to a computation of $V(1)_* \text{THH}(\text{ku})$ using logarithmic topological Hochschild homology [30], much in the spirit of computations by Hesselholt and Madsen for discrete valuation rings [15].

Our main motivation for constructing the generalized Brun spectral sequence, and for simplifying the computation for ku , is that the previous methods do not seem sufficient for the computation of $\text{THH}(\text{K}(\mathbb{F}_q))$, where $\text{K}(\mathbb{F}_q)$ is the algebraic K -theory spectrum of the finite field with q elements. By work of Quillen [28], $\text{K}(\mathbb{F}_q)$ is closely linked to ku ; in particular, if p and q are coprime, we have $\text{ku} \simeq \text{K}(\overline{\mathbb{F}}_q)_p$, where $\overline{\mathbb{F}}_q$ is the algebraic closure of \mathbb{F}_q , and there is a fiber sequence relating ku and $\text{K}(\mathbb{F}_q)_p$. The strategy of computing $V(1)_* \text{THH}(\text{ku})$ presented in this paper is applied in Höning [16] to compute $V(1)_* \text{THH}(\text{K}(\mathbb{F}_q))$ in certain cases.

The spectral sequence

Recall that for a commutative S -algebra B the Bökstedt spectral sequence has the form

$$E_{*,*}^2 = \text{HH}_{*,*}^{\mathbb{F}_p}((H\mathbb{F}_p)_* B) \Rightarrow (H\mathbb{F}_p)_* \text{THH}(B).$$

Here, $H\mathbb{F}_p$ is the Eilenberg–Mac Lane spectrum of \mathbb{F}_p and $\text{HH}_{*,*}^{\mathbb{F}_p}(-)$ denotes ordinary Hochschild homology over the ground ring \mathbb{F}_p . The spectral sequence is particularly useful if the pages $E_{*,*}^r$ are flat over the mod p homology of B . This is for example the case for $B = H\mathbb{F}_p$, $B = H\mathbb{Z}$ and for the Adams summand $B = \ell \subset \text{ku}$ (see Bökstedt [8] and McClure and Staffeldt [23]). If the flatness condition is satisfied, the Bökstedt spectral sequence is an $(H\mathbb{F}_p)_* H\mathbb{F}_p$ -comodule $(H\mathbb{F}_p)_* B$ -bialgebra spectral sequence and this structure can be very helpful to compute the differentials (see Angeltveit and Rognes [1]). However, if the flatness condition is not satisfied, computations with the Bökstedt spectral sequence can be harder. For example, if $B = \text{ku}$, the flatness condition is not satisfied, and if $B = \text{K}(\mathbb{F}_q)$, it is not always satisfied. The Bökstedt spectral can be generalized from $H\mathbb{F}_p$ to a more general ring spectrum E , if

we have a Künneth isomorphism for E . But, for example for the mod p Moore spectrum $V(0)$ and for $V(1)$ one does not have a Künneth isomorphism, so that the Bökstedt spectral sequence does not allow a direct computation of $V(0)$ – or $V(1)$ –homotopy.

The main goal of this paper is to prove the following theorem:

Theorem 1.1 *Let A be a cofibrant commutative S –algebra and let B be a connective cofibrant commutative A –algebra. Let E be a ring spectrum. Then there is a strongly convergent, multiplicative spectral sequence of the form*

$$E_{n,m}^2 = \pi_n \text{THH}(B; HE_m(B \wedge_A B)) \Rightarrow E_{n+m} \text{THH}(A; B).$$

If the $E_m(B \wedge_A B)$ happen to be \mathbb{F}_p –vector spaces and $\pi_0(B)/p\pi_0(B) = \mathbb{F}_p$ as rings, we have

$$E_{n,m}^2 = E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \pi_n \text{THH}(B; H\mathbb{F}_p).$$

If $E = S$ is the sphere spectrum and if $A \rightarrow B$ is given by applying the Eilenberg–Mac Lane spectrum functor $H(-)$ to a morphism of commutative rings, the spectral sequence above has the same form as the one by Brun [9, Theorem 6.2.10]. Our proof is inspired by Brun’s proof, but it is not a direct generalization. Brun works with functors with smash product and functors with tensor product and his spectral sequence is based on a bisimplicial abelian group. We work with S –algebras in the sense of Elmendorf, Kriz, Mandell and May [13] and our spectral sequence is an instance of the Atiyah–Hirzebruch spectral sequence.

Application to ku

We apply the generalized Brun spectral sequence to ku . The computation is organized in the following three steps:

- Computation of

$$V(0)_*(H\mathbb{Z}_p \wedge_{\text{ku}} H\mathbb{Z}_p) = \pi_*(H\mathbb{F}_p \wedge_{\text{ku}} H\mathbb{Z}_p)$$

via the Künneth spectral sequence. Here, \mathbb{Z}_p is the ring of p –adic integers.

- Computation of $V(0)_* \text{THH}(\text{ku}; H\mathbb{Z}_p)$ via the generalized Brun spectral sequence

$$\begin{aligned} E_{*,*}^2 &= V(0)_*(H\mathbb{Z}_p \wedge_{\text{ku}} H\mathbb{Z}_p) \otimes_{\mathbb{F}_p} \pi_* \text{THH}(H\mathbb{Z}_p; H\mathbb{F}_p) \\ &\Rightarrow V(0)_* \text{THH}(\text{ku}; H\mathbb{Z}_p). \end{aligned}$$

Here, note that $\pi_* \text{THH}(H\mathbb{Z}_p; H\mathbb{F}_p)$ has been computed by Bökstedt [8].

- Computation of $V(1)_* \text{THH}(\text{ku})$ via the generalized Brun spectral sequence

$$E_{*,*}^2 = V(1)_* \text{ku} \otimes_{\mathbb{F}_p} \pi_* \text{THH}(\text{ku}; H\mathbb{F}_p) \Rightarrow V(1)_* \text{THH}(\text{ku}).$$

Here, note that $\pi_* \text{THH}(\text{ku}; H\mathbb{F}_p) = V(0)_* \text{THH}(\text{ku}; H\mathbb{Z}_p)$.

The first generalized Brun spectral sequence mentioned above has to collapse at the E^2 -page and there cannot be any multiplicative extensions. In the second generalized Brun spectral sequence mentioned above there is one possible differential. Its existence follows from the relation $u^{p-1} = 0$ in $(H\mathbb{F}_p)_* \text{ku}$, where $u \in (H\mathbb{F}_p)_2 \text{ku}$ is the image of the Bott element under the Hurewicz map $\pi_*(\text{ku}) \rightarrow (H\mathbb{F}_p)_* \text{ku}$. In contrast to the Bökstedt spectral sequence for ku there are no multiplicative extensions in the Brun spectral sequence and we obtain the following concise description of $V(1)_* \text{THH}(\text{ku})$:

Theorem 1.2 *The $V(1)$ -homotopy of $\text{THH}(\text{ku})$ is the homology of the differential graded algebra*

$$P_{p-1}(u) \otimes_{\mathbb{F}_p} E(\sigma u, \lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1), \quad d(\mu_1) = u^{p-2} \sigma u.$$

Here, $P(-)$ denotes the polynomial algebra over \mathbb{F}_p , $E(-)$ denotes the exterior algebra over \mathbb{F}_p and $P_{p-1}(-)$ denotes the truncated polynomial algebra over \mathbb{F}_p . The degrees are given by $|u| = 2$, $|\sigma u| = 3$, $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

Structure of the paper

We work in the setting of Elmendorf, Kriz, Mandell and May [13]. In Section 2 we collect some of the properties of the categories we are working with. The main goal of Section 3 is to give a detailed proof of the multiplicativity of the Atiyah–Hirzebruch spectral sequence. In Section 3.1 we transfer some methods of [11] by Dugger about multiplicative structures on spectral sequences to the EKMM setting. In Section 3.2 we use this to prove the multiplicativity of the Atiyah–Hirzebruch spectral sequence and some additional properties. The content of Sections 2 and 3 consist essentially of recollections, and are included here to provide details that we did not find in the literature. In Section 4 we apply the Atiyah–Hirzebruch spectral sequence to derive the existence of the generalized Brun spectral sequence. Finally, in Section 5 we compute $V(1)_* \text{THH}(\text{ku})$ using the Brun spectral sequence.

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2 Preliminaries

We assume familiarity with the theory of Elmendorf, Kriz, Mandell and May [13]. We denote the category of (compactly generated weak Hausdorff) spaces by \mathcal{U} and the category of based spaces by \mathcal{T} . Let \mathcal{S} , $\mathcal{S}[\mathbb{L}]$ and \mathcal{M}_S be the categories of spectra, of \mathbb{L} -spectra and of S -modules, respectively. For a commutative S -algebra R let \mathcal{M}_R be the category of R -modules. We denote the tensor products of the closed symmetric monoidal categories \mathcal{M}_S and \mathcal{M}_R by \wedge_S and \wedge_R .

Recall from [13, Chapter VII] that the categories \mathcal{S} , $\mathcal{S}[\mathbb{L}]$, \mathcal{M}_S and \mathcal{M}_R are topological model categories. In particular, these categories are topologically complete and cocomplete. The tensor with a based space X is denoted by $- \wedge X$. Colimits and tensors in \mathcal{M}_R , \mathcal{M}_S and $\mathcal{S}[\mathbb{L}]$ are created in \mathcal{S} . By [13, page 132] the functors considered in [13] are enriched functors. We point out that for a \mathcal{T} -enriched functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between \mathcal{T} -tensored categories we have a natural morphism

$$F(c) \wedge X \rightarrow F(c \wedge X),$$

which is an isomorphism if F is the left adjoint of an adjoint pair of \mathcal{T} -enriched functors. A homotopy is a map of the form $E \wedge I_+ \rightarrow F$ in the respective category, where $I = [0, 1]$ is the unit interval, and the homotopy categories are denoted by $h\mathcal{S}$, $h\mathcal{S}[\mathbb{L}]$, $h\mathcal{M}_S$ and $h\mathcal{M}_R$.

We denote the homotopy category of the model category \mathcal{M}_R by \mathcal{D}_R . Since all objects in \mathcal{M}_R are fibrant, cell R -modules are cofibrant and since $M \wedge I_+$ is a good cylinder object if M is a cofibrant R -module, we get that

$$\mathcal{D}_R(M, N) \cong h\mathcal{M}_R(\Gamma^R M, \Gamma^R N) \cong h\mathcal{M}_R(\Gamma^R M, N),$$

where $\Gamma^R M$ and $\Gamma^R N$ are cell approximations of M and N . The category \mathcal{D}_R is a tensor triangulated category in the sense of [6, Definition 1.1]. The tensor product \wedge_R^L is given by

$$M \wedge_R^L N := \Gamma^R M \wedge_R \Gamma^R N.$$

If M is a cofibrant R -module, then $M \wedge_R -$ preserves weak equivalences. Thus, for any $M, N \in \mathcal{D}_R$ we have that $M \wedge_R^L N$ is isomorphic in \mathcal{D}_R to $\Gamma^R M \wedge_R N$ and $M \wedge_R \Gamma^R N$.

To deal with the signs in the spectral sequences we follow Dugger’s treatment in [11] and fix orientations on various manifolds:

Remark 2.1 Equip \mathbb{R} with the usual orientation and \mathbb{R}^n with the product orientation. These orientations induce orientations on D^n and I . We equip I with the basepoint 0 , and D^n and the sphere S^{n-1} with the basepoint $(-1, 0, \dots, 0)$. The orientations on D^n and I induce orientations on the boundaries S^{n-1} and $\{0, 1\}$. Here, we use the same convention as in [11, Remark 2.2] for the boundary orientation, so that the basepoint in S^0 has orientation -1 and the nonbasepoint in S^0 has orientation $+1$ and so that we have the following formula for two manifolds M and N with boundary:

$$(1) \quad \partial(M \times N) = \partial M \times N \cup (-1)^{\dim(M)} M \times \partial N.$$

The orientations on the interiors of D^n and I determine orientations on D^n/S^{n-1} and on $I/\partial I$. For $n \geq 1$ or $m \geq 1$ the product orientation on the right-hand side of

$$S^n \wedge S^m \setminus \{*\} \cong S^n \setminus \{*\} \times S^m \setminus \{*\}$$

defines an orientation on $S^n \wedge S^m$. In $S^0 \wedge S^0$ we equip the nonbasepoint with the orientation 1 and the basepoint with the orientation -1 . For the rest of this paper we fix basepoint-preserving, orientation-preserving homeomorphisms $D^n/S^{n-1} \cong S^n$, $S^n \wedge S^m \cong S^{n+m}$ and $I/\partial I \cong S^1$.

To deal with the signs in the spectral sequences we give a precise definition of suspension isomorphisms and boundary maps in long exact sequences: Recall that we have left adjoint functors

$$\mathcal{S} \xrightarrow{\mathbb{L}(-)} \mathcal{S}[\mathbb{L}] \xrightarrow{S \wedge_{\mathcal{L}}(-)} \mathcal{M}_S \xrightarrow{R \wedge_S(-)} \mathcal{M}_R,$$

whose composition is denoted by \mathbb{F}_R . Spheres in \mathcal{M}_R are denoted by S_R^n and are defined by applying $\mathbb{F}_R(-)$ to the spheres S^n in \mathcal{S} . Note that the S_R^n are CW R -modules and that S_R^0 is a cell approximation of R . The homotopy groups of an R -module M are given by

$$M_n = \pi_n(M) = h.\mathcal{M}_R(S_R^n, M).$$

For $n \geq 0$ we define an isomorphism $S^n \wedge S^1 \cong S^{n+1}$ in \mathcal{S} by

$$\Sigma^\infty S^n \wedge S^1 \cong \Sigma^\infty (S^n \wedge S^1) \cong \Sigma^\infty S^{n+1}.$$

Here, Σ^∞ is the left adjoint to the zeroth space functor Ω^∞ and the second map is given by the fixed orientation-preserving homeomorphism. For $n < 0$ we define an isomorphism $S^n \wedge S^1 \cong S^{n+1}$ using [18, Proposition I.4.2]. We define isomorphisms $S^n_R \wedge S^1 \cong S^{n+1}_R$ by

$$\mathbb{F}_R S^n \wedge S^1 \cong \mathbb{F}_R (S^n \wedge S^1) \cong \mathbb{F}_R S^{n+1}.$$

These isomorphisms define suspension isomorphisms $\pi_n(M) \cong \pi_{n+1}(M \wedge S^1)$ for $M \in \mathcal{M}_R$. Let $f: M \rightarrow N$ be a morphism in \mathcal{M}_R and let $C(f) := N \cup_M (M \wedge I)$ be its cofiber. We use the fixed homeomorphism $I/\partial I \cong S^1$ and the suspension isomorphism to define the boundary map in the long exact sequence

$$\cdots \rightarrow \pi_n(M) \rightarrow \pi_n(N) \rightarrow \pi_n(C(f)) \xrightarrow{d} \pi_{n-1}(M) \rightarrow \cdots .$$

If f satisfies the homotopy extension property, then the canonical map $C(f) \rightarrow N/M := * \cup_M N$ is a homotopy equivalence. Following [13] we call a map in \mathcal{M}_R satisfying the homotopy extension property a cofibration. Note that a cofibration is not necessary a cofibration in the model categorical sense.

In the proof of the existence of the Brun spectral sequence we will need that different functors behave well with respect to algebras and ring spectra. For this observe the following: The canonical functor $i_R: \mathcal{M}_R \rightarrow \mathcal{D}_R$ is lax symmetric monoidal. The structure maps are given by applying i_R to the maps

$$\Gamma^R M \wedge_R \Gamma^R N \rightarrow M \wedge_R N.$$

Moreover, for a morphism of commutative S -algebras $\phi: R \rightarrow R'$ we have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{R'} & \xrightarrow{\phi^*} & \mathcal{M}_R \\ \downarrow i_{R'} & & \downarrow i_R \\ \mathcal{D}_{R'} & \xrightarrow{\phi^*} & \mathcal{D}_R \end{array}$$

All the functors in this diagram are lax symmetric monoidal and the structure maps of the two lax monoidal functors $i_R \circ \phi^*$ and $\phi^* \circ i_{R'}$ agree. Now, let $\phi: R \rightarrow R'$ and $\psi: R' \rightarrow R''$ be maps of commutative S -algebras. Then, the two functors $(\psi \circ \phi)^*$, $\phi^* \circ \psi^*: \mathcal{D}_{R''} \rightarrow \mathcal{D}_R$ are the same and the structure maps of these two

lax monoidal functors agree. Recall that an R -ring spectrum is an object $A \in \mathcal{D}_R$ with maps $A \wedge_R^L A \rightarrow A$ and $R \rightarrow A$ in \mathcal{D}_R satisfying the left and right unit laws. As a consequence of the above we get that for R -algebras M and N the map

$$M \wedge_R^L N \rightarrow M \wedge_R N$$

in \mathcal{D}_R is a map of R -ring spectra. Moreover, for maps of commutative S -algebras $R \xrightarrow{\phi} R' \xrightarrow{\psi} R''$ we get commutative diagrams

$$\begin{array}{ccc} \mathcal{A}_{R'} & \longrightarrow & \mathcal{A}_R \\ \downarrow & & \downarrow \\ \mathcal{R}_{R'} & \longrightarrow & \mathcal{R}_R \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{R}_{R''} & \xrightarrow{\psi^*} & \mathcal{R}_{R'} \\ & \searrow (\psi \circ \phi)^* & \downarrow \phi^* \\ & & \mathcal{R}_R \end{array}$$

Here, \mathcal{A}_R and $\mathcal{A}_{R'}$ denote the categories of algebras over R and R' , and \mathcal{R}_R , $\mathcal{R}_{R'}$ and $\mathcal{R}_{R''}$ denote the categories of ring spectra over R , R' and R'' , respectively.

We now recall some details about the lax monoidality of the functor $\pi_*(-)$, because we need these in the proof of the multiplicativity of the spectral sequences. By [13, page 92] there are homotopy equivalences

$$S_R^k \wedge_R S_R^l \simeq S_R^{k+l}$$

that are unital, associative and commutative up to the sign $(-1)^{kl}$. They define maps

$$\mathcal{D}_R(S_R^k, M) \otimes_{\mathbb{Z}} \mathcal{D}_R(S_R^l, N) \rightarrow \mathcal{D}_R(S_R^{k+l}, M \wedge_R^L N)$$

for $M, N \in \mathcal{D}_R$ which make $\pi_*(-)$ into a lax symmetric monoidal functor from \mathcal{D}_R to the category of \mathbb{Z} -graded abelian groups $\mathcal{G}\mathcal{A}$. Here, the symmetry in $\mathcal{G}\mathcal{A}$ is given by

$$A_* \otimes_{\mathbb{Z}} B_* \rightarrow B_* \otimes_{\mathbb{Z}} A_*, \quad a \otimes b \mapsto (-1)^{|a||b|} b \otimes a,$$

where $|a|$ and $|b|$ denote the degrees of a and b . The equivalences $S_R^k \wedge_R S_R^l \simeq S_R^{k+l}$ are compatible with suspension in the sense that the diagrams

$$(2) \quad \begin{array}{ccc} S_R^k \wedge_R S_R^l & \xrightarrow{\quad\quad\quad} & S_R^{k+l} \\ \downarrow & & \downarrow \\ (S_R^{k-1} \wedge S^1) \wedge_R S_R^l & \xrightarrow{\quad\quad\quad} & (S_R^{k-1} \wedge_R S_R^l) \wedge S^1 \xrightarrow{\quad\quad\quad} S_R^{k+l-1} \wedge S^1 \end{array}$$

$$(3) \quad \begin{array}{ccc} S_R^k \wedge_R S_R^l & \xrightarrow{\quad\quad\quad} & S_R^{k+l} \\ \downarrow & & \downarrow \\ S_R^k \wedge_R (S_R^{l-1} \wedge S^1) & \longrightarrow (S_R^k \wedge_R S_R^{l-1}) \wedge S^1 \longrightarrow & S_R^{k+l-1} \wedge S^1 \end{array}$$

commute in \mathcal{D}_R up to the indicated sign. We will later compare the product on usual homotopy groups to a product on relative homotopy groups. For this we need that for $k, l \geq 0$ the equivalence $S_R^k \wedge_R S_R^l \simeq S_R^{k+l}$ can be identified with the following map: By [13, Corollary III.3.7; 18, Propositions II.3.6 and II.1.4] there is a natural isomorphism

$$(4) \quad \mathbb{F}_R \Sigma^\infty X \wedge_R \mathbb{F}_R \Sigma^\infty Y \cong \mathbb{F}_R \Sigma^\infty (X \wedge Y)$$

for $X, Y \in \mathcal{T}$. For $k, l \geq 0$ the equivalence $S_R^k \wedge_R S_R^l \simeq S_R^{k+l}$ is then homotopic to the map that is given by taking $X = S^k$ and $Y = S^l$ in (4) and by using the orientation-preserving homeomorphism $S^k \wedge S^l \cong S^{k+l}$. Let $R \rightarrow R'$ be a morphism of commutative S -algebras. Then, for $M \in \mathcal{D}_{R'}$ one has natural isomorphisms

$$\mathcal{D}_{R'}(S_{R'}^n, M) \cong \mathcal{D}_R(S_R^n, M)$$

compatible with the suspension isomorphisms. One can show that they define a monoidal natural isomorphism between the functors

$$\pi_*: \mathcal{D}_{R'} \rightarrow \mathcal{G}\mathcal{A} \quad \text{and} \quad \mathcal{D}_{R'} \rightarrow \mathcal{D}_R \xrightarrow{\pi_*} \mathcal{G}\mathcal{A}.$$

Let us denote the category of commutative S -algebras by $\mathcal{C}\mathcal{A}_S$. Note that the pushout of a diagram $G \leftarrow H \rightarrow J$ in $\mathcal{C}\mathcal{A}_S$ is given by $G \wedge_H J$. By [13, Chapter VII] the category $\mathcal{C}\mathcal{A}_S$ is a \mathcal{U} -enriched topological model category. We denote the homotopy category of this model category by $\overline{h}\mathcal{C}\mathcal{A}_S$. The tensor of a commutative S -algebra A and a space X is denoted by $A \otimes X$. If R is a cofibrant commutative S -algebra, the tensor $R \otimes I$ is a good cylinder object. Since all objects are fibrant, it follows that for commutative S -algebras R and R' the set $\overline{h}\mathcal{C}\mathcal{A}_S(R, R')$ is the set of homotopy classes of maps of commutative S -algebras $QR \rightarrow QR'$, where QR and QR' are cofibrant replacements of R and R' in $\mathcal{C}\mathcal{A}_S$ and where homotopies are given by maps $QR \otimes I \rightarrow QR'$. One has a map $R \wedge I_+ \rightarrow R \otimes I$ for every commutative S -algebra R , so that maps that are homotopic as maps of commutative S -algebras are homotopic as maps of S -modules.

Let R be a commutative S -algebra. We denote the category of commutative R -algebras by $\mathcal{C}\mathcal{A}_R$. As $\mathcal{C}\mathcal{A}_R$ can be identified with the category of commutative

S -algebras under R , it inherits a model category structure from $\mathcal{C}\mathcal{A}_S$. This model category structure is the same as the one defined in [13, Chapter VII], because in both cases weak equivalences are created in \mathcal{S} and fibrations are created in \mathcal{M}_S .

Lemma 2.2 *Let A be a cofibrant commutative S -algebra, let $A \rightarrow B$ be a cofibration in $\mathcal{C}\mathcal{A}_S$ and let $A \rightarrow C$ be a map in $\mathcal{C}\mathcal{A}_S$, where C is a cofibrant commutative S -algebra. Then the map*

$$B \wedge_A^L C \rightarrow B \wedge_A C$$

in \mathcal{D}_A is an isomorphism of A -ring spectra.

Proof We only have to show that the map $\Gamma B \wedge_A \Gamma C \rightarrow B \wedge_A C$ is a weak equivalence. To see this, factor the map $A \rightarrow C$ in $\mathcal{C}\mathcal{A}_S$ as

$$A \twoheadrightarrow \tilde{C} \xrightarrow{\sim} C$$

and consider the homotopy commutative diagram

$$\begin{array}{ccc} \Gamma B \wedge_A \Gamma C & \longrightarrow & B \wedge_A C \\ \uparrow & & \uparrow \\ \Gamma B \wedge_A \Gamma \tilde{C} & \longrightarrow & B \wedge_A \tilde{C} \end{array}$$

Obviously, the left vertical map is a weak equivalence. The lower horizontal map is a weak equivalence by [13, Theorems VII.6.5 and VII.6.7] and the right vertical map is a weak equivalence by [13, Theorem VII.7.4] □

3 The multiplicativity of the Atiyah–Hirzebruch spectral sequence

3.1 Multiplicative spectral sequences associated to towers of cofibrations

The goal of this subsection is to transfer some of the methods of [11] to the EKMM setting. We fix a commutative S -algebra R .

Lemma 3.1 *Let $\cdots \rightarrow X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow \cdots$ be a sequence of cofibrations of R -modules. Then there is a spectral sequence $E_{*,*}^*(X_\bullet)$ of standard homological type with*

$$(5) \quad E_{n,m}^1 = \pi_{n+m}(X_n/X_{n-1}).$$

If $X_n = *$ for $n < 0$, then the spectral sequence converges strongly:

$$E_{n,m}^1 \Rightarrow \pi_{n+m}(\operatorname{colim}_i X_i).$$

Proof We have the unrolled exact couple

$$\begin{array}{ccccccc} \cdots & \xrightarrow{i} & \pi_*(X_{n-1}) & \xrightarrow{i} & \pi_*(X_n) & \xrightarrow{i} & \pi_*(X_{n+1}) \xrightarrow{i} \cdots \\ & & \downarrow j & \swarrow \partial & \downarrow j & \swarrow \partial & \downarrow j \\ & & \pi_*(X_{n-1}/X_{n-2}) & & \pi_*(X_n/X_{n-1}) & & \pi_*(X_{n+1}/X_n) \end{array}$$

Here, ∂ is defined by $(-1)^m$ times the map

$$\pi_m(X_n/X_{n-1}) \cong \pi_m(C(X_{n-1} \rightarrow X_n)) \xrightarrow{d} \pi_{m-1}(X_{n-1}).$$

The reason for the sign will become clear in Lemma 3.2. This defines the spectral sequence. Clearly, if $X_n = *$ for $n < 0$, one has $\pi_*(X_n/X_{n-1}) = 0$ for $n < 0$ and $\lim_n \pi_*(X_n) = 0$. So, by [7, Theorem 6.1] the spectral sequence converges strongly to $\operatorname{colim}_n \pi_*(X_n)$.

We show that the canonical map

$$\alpha: \operatorname{colim}_n \pi_m(X_n) \rightarrow \pi_m(\operatorname{colim}_n X_n)$$

is a bijection: Each map $X_{n-1} \rightarrow X_n$ is a cofibration of R -modules and thus a cofibration of spectra by the retraction of mapping cylinders criterion. By [18, Lemma I.8.1] each map $X_{n-1} \rightarrow X_n$ is a spacewise closed inclusion. Thus, by [13, Proposition III.1.7], every map $L \rightarrow \operatorname{colim}_n X_n$, where L is a compact R -module in the sense of [13, Definition III.1.6], factors through some X_j . Applying this for $L = S_R^m$ and $L = S_R^m \wedge I_+$ one gets the surjectivity and injectivity of α . For the injectivity also note that the maps $X_j \rightarrow \operatorname{colim}_n X_n$ are cofibrations and thus monomorphisms in the category of R -modules. □

To deal with multiplicative structures, we follow Dugger’s treatment and work with homotopy groups of morphisms. For a map $M \rightarrow N$ of R -modules and $n \geq 1$ we define $\pi_n(N, M)$ as the set of homotopy classes of diagrams

$$(6) \quad \begin{array}{ccc} \mathbb{F}_R \Sigma^\infty S^{n-1} & \xrightarrow{\beta} & M \\ \downarrow & & \downarrow \\ \mathbb{F}_R \Sigma^\infty D^n & \xrightarrow{\alpha} & N \end{array}$$

Recall that the right adjoint of $\mathbb{F}_R \Sigma^\infty(-)$ is given by the composition

$$\mathcal{M}_R \rightarrow \mathcal{M}_S \xrightarrow{F_{\mathcal{L}}(S, -)} \mathcal{S}[\mathbb{L}] \rightarrow \mathcal{S} \xrightarrow{\Omega^\infty} \mathcal{T},$$

where $F_{\mathcal{L}}(-, -)$ is the function \mathbb{L} -spectrum. Thus, we have a bijection

$$\pi_n(N, M) \cong \pi_n(\Omega^\infty F_{\mathcal{L}}(S, N), \Omega^\infty F_{\mathcal{L}}(S, M)),$$

where the right-hand side is the homotopy group of a morphism of based spaces defined in [11]. Using [11] one gets that $\pi_n(N, M)$ has a natural group structure for $n = 2$ and a natural abelian group structure for $n \geq 3$, that $\pi_n(M, *) \cong \pi_n(M)$ and that the sequence

$$\pi_n(M) \rightarrow \pi_n(N) \rightarrow \pi_n(N, M) \xrightarrow{\kappa} \pi_{n-1}(M) \rightarrow \pi_{n-1}(N)$$

is exact. Here, κ is the map that sends the class of the diagram (6) to the class of β .

Lemma 3.2 *Let $M \rightarrow N$ be a cofibration of R -modules. Then the diagram*

$$\begin{array}{ccccccccc} \pi_n(M) & \longrightarrow & \pi_n(N) & \longrightarrow & \pi_n(N, M) & \xrightarrow{\kappa} & \pi_{n-1}(M) & \longrightarrow & \pi_{n-1}(M) \\ \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \phi & & \downarrow \text{id} & & \downarrow \text{id} \\ \pi_n(M) & \longrightarrow & \pi_n(N) & \longrightarrow & \pi_n(N/M) & \xrightarrow{(-1)^n d} & \pi_{n-1}(M) & \longrightarrow & \pi_{n-1}(M) \end{array}$$

is commutative and ϕ is an isomorphism.

Proof For the commutativity of the third square note that by naturality we only have to consider the case where $N \rightarrow M$ is the cofibration $\mathbb{F}_R \Sigma^\infty S^{n-1} \rightarrow \mathbb{F}_R \Sigma^\infty D^n$ and here we only have to show the equality of the maps κ and $(-1)^n d \circ \phi$ on the class in $\pi_n(\mathbb{F}_R \Sigma^\infty D^n, \mathbb{F}_R \Sigma^\infty S^{n-1})$ that is represented by the diagram

$$\begin{array}{ccc} \mathbb{F}_R \Sigma^\infty S^{n-1} & \xrightarrow{\text{id}} & \mathbb{F}_R \Sigma^\infty S^{n-1} \\ \downarrow & & \downarrow \\ \mathbb{F}_R \Sigma^\infty D^n & \xrightarrow{\text{id}} & \mathbb{F}_R \Sigma^\infty D^n \end{array}$$

For the equality on this element one has to show that

$$\begin{array}{ccc} C(\mathbb{F}_R \Sigma^\infty S^{n-1} \rightarrow \mathbb{F}_R \Sigma^\infty D^n) & \longrightarrow & \mathbb{F}_R \Sigma^\infty S^{n-1} \wedge S^1 \\ \cong \downarrow & & \downarrow \\ \mathbb{F}_R \Sigma^\infty D^n / \mathbb{F}_R \Sigma^\infty S^{n-1} \cong \mathbb{F}_R \Sigma^\infty D^n / S^{n-1} & & \mathbb{F}_R \Sigma^\infty (S^{n-1} \wedge S^1) \\ \cong \downarrow & & \downarrow \\ \mathbb{F}_R \Sigma^\infty S^n & \xrightarrow{(-1)^n \text{id}} & \mathbb{F}_R \Sigma^\infty S^n \end{array}$$

is commutative in the homotopy category of R -modules. Roughly speaking, this follows by applying the formula (1) for the boundary orientation of a product to $D^n \times I$.

By the five lemma ϕ is an isomorphism for $n \geq 2$. For $n = 1$ note that we can endow $\pi_1(N, M)$ with an abelian group structure by identifying it with the set of homotopy classes of commutative diagrams

$$\begin{array}{ccc} S_R^{-2} \wedge S^2 & \longrightarrow & M \\ \downarrow & & \downarrow \\ (S_R^{-2} \wedge I) \wedge S^2 & \longrightarrow & N \end{array}$$

and by using the comultiplication of S^2 . One easily sees that $\pi_1(N) \rightarrow \pi_1(N, M)$, κ and ϕ are group homomorphisms with respect to this group structure. \square

Following Dugger we now consider products on $\pi_*(-, -)$: Let $D^{n+m} \rightarrow D^n \times D^m$ be an orientation-preserving, pointed homeomorphism. It induces an orientation-preserving homeomorphism of the boundaries

$$S^{n+m-1} \xrightarrow{\cong} (S^{n-1} \times D^m) \amalg_{(S^{n-1} \times S^{m-1})} (D^n \times S^{m-1})$$

and we get the commutative diagram

$$(7) \quad \begin{array}{ccc} S^{n+m-1} & \longrightarrow & (S^{n-1} \wedge D^m) \amalg_{(S^{n-1} \wedge S^{m-1})} (D^n \wedge S^{m-1}) \\ \downarrow & & \downarrow \\ D^{n+m-1} & \longrightarrow & D^n \wedge D^m \end{array}$$

Given two maps of R -modules $M \rightarrow N$ and $O \rightarrow L$ we define a product

$$(-) \cdot (-): \pi_n(N, M) \times \pi_m(L, O) \rightarrow \pi_{n+m}(N \wedge_R L, (M \wedge_R L) \amalg_{(M \wedge_R O)} (N \wedge_R O))$$

by applying $\mathbb{F}_R \Sigma^\infty(-)$ to (7) and by using the isomorphism (4).

Lemma 3.3 *Let $M \rightarrow N$ and $O \rightarrow L$ be maps of R -modules. Then the following diagram commutes:*

$$\begin{array}{ccc} \pi_k(N, M) \times \pi_l(L, O) & \longrightarrow & \pi_{k+l}(N \wedge_R L, (M \wedge_R L) \amalg_{(M \wedge_R O)} (N \wedge_R O)) \\ \downarrow & & \downarrow \\ \pi_k(N/M) \times \pi_l(L/O) & \longrightarrow & \pi_{k+l}(N \wedge_R L / (M \wedge_R L) \amalg_{(M \wedge_R O)} N \wedge_R O) \\ & \searrow & \downarrow \\ & & \pi_{k+l}(N/M \wedge_R L/O) \end{array}$$

Proof Because of naturality we only have to prove commutativity in the universal case. This can be reduced to the corresponding statement for spaces by using that we

have natural maps

$$(8) \quad \pi_*(B, A) \rightarrow \pi_*(\mathbb{F}_R \Sigma^\infty B, \mathbb{F}_R \Sigma^\infty A)$$

and

$$(9) \quad \pi_*(B) \rightarrow \pi_*(\mathbb{F}_R \Sigma^\infty B)$$

that identify for $A = *$ and that are compatible with products. Here, the product on the left-hand side of (9) is defined by means of the fixed orientation-preserving homeomorphisms $S^k \wedge S^l \cong S^{k+l}$. The statement for spaces follows from the fact that two pointed, orientation-preserving homeomorphisms $S^{k+l} \rightarrow D^k/S^{k-1} \wedge D^l/S^{l-1}$ are homotopic. □

Lemma 3.4 *Let $M \rightarrow N$ and $O \rightarrow L$ be morphisms of R -modules. Let j_* , $(i_1)_*$ and $(i_2)_*$ be the maps*

$$\begin{aligned} \pi_*((M \wedge_R L) \amalg_{(M \wedge_R O)} (N \wedge_R O)) &\rightarrow \pi_*((M \wedge_R L) \amalg_{(M \wedge_R O)} (N \wedge_R O), M \wedge_R O), \\ \pi_*(M \wedge_R L, M \wedge_R O) &\rightarrow \pi_*((M \wedge_R L) \amalg_{(M \wedge_R O)} (N \wedge_R O), M \wedge_R O), \\ \pi_*(N \wedge_R O, M \wedge_R O) &\rightarrow \pi_*((M \wedge_R L) \amalg_{(M \wedge_R O)} (N \wedge_R O), M \wedge_R O), \end{aligned}$$

respectively. Let $x \in \pi_k(N, M)$ and $y \in \pi_l(L, O)$ with $k, l \geq 3$. Then we have

$$j_*\kappa(x \cdot y) = (i_1)_*(\kappa x \cdot y) + (-1)^k (i_2)_*(x \cdot \kappa y)$$

in $\pi_{k+l-1}((M \wedge_R L) \amalg_{(M \wedge_R O)} (N \wedge_R O), M \wedge_R O)$.

Proof By naturality we only have to check the equality in the universal case. By using the maps (8) and (9) we can reduce the statement to the corresponding statement for spaces. The statement for spaces is proven in [11, Proposition 4.1]. Roughly speaking, it follows by applying the formula (1) for the boundary orientation of a product to $D^k \times D^l$. □

We now study the multiplicative properties of the spectral sequence (5). We assume that we have towers of cofibrations of R -modules X_\bullet , Y_\bullet and W_\bullet . Furthermore, we assume that we have a pairing of towers $X_\bullet \wedge_R Y_\bullet \rightarrow W_\bullet$, by which we mean maps $X_n \wedge_R Y_k \rightarrow W_{n+k}$ such that the following diagrams commute in \mathcal{M}_R :

$$\begin{array}{ccccc} X_{n-1} \wedge_R Y_k & \longrightarrow & X_n \wedge_R Y_k & \longleftarrow & X_n \wedge_R Y_{k-1} \\ \downarrow & & \downarrow & & \downarrow \\ W_{n+k-1} & \longrightarrow & W_{n+k} & \longleftarrow & W_{n+k-1} \end{array}$$

We get unique maps $X_n/X_{n-1} \wedge_R Y_k/Y_{k-1} \rightarrow W_{n+k}/W_{n+k-1}$ making the diagrams

$$\begin{CD} X_n \wedge_R Y_k @>>> W_{n+k} \\ @VVV @VVV \\ X_n/X_{n-1} \wedge_R Y_k/Y_{k-1} @>>> W_{n+k}/W_{n+k-1} \end{CD}$$

commutative. Let $E_{*,*}^*(X_\bullet)$, $E_{*,*}^*(Y_\bullet)$ and $E_{*,*}^*(W_\bullet)$ be the spectral sequences associated to the towers (see Lemma 3.1). We get maps

$$\star: E_{n,m}^1(X_\bullet) \otimes_{\mathbb{Z}} E_{k,l}^1(Y_\bullet) \rightarrow E_{n+k,m+l}^1(W_\bullet).$$

In the following lemma we prove the compatibility of the product with the differentials for classes in total degrees ≥ 3 :

Lemma 3.5 *Let $x \in E_{n,m}^1(X_\bullet) \cap Z_{n,m}^r$ and $y \in E_{k,l}^1(Y_\bullet) \cap Z_{k,l}^r$ be r^{th} cycles with $n+m \geq 3$ and $k+l \geq 3$. Then there are classes $\bar{x} \in \pi_{n+m-1}(X_{n-r})$, $\bar{y} \in \pi_{k+l-1}(Y_{k-r})$ and $\bar{w} \in \pi_{n+m+k+l-1}(W_{n+k-r})$ such that the following holds:*

- Under $\pi_{n+m-1}(X_{n-r}) \rightarrow \pi_{n+m-1}(X_{n-1})$, $\pi_{k+l-1}(Y_{k-r}) \rightarrow \pi_{k+l-1}(Y_{k-1})$ and $\pi_{n+m+k+l-1}(W_{n+k-r}) \rightarrow \pi_{n+m+k+l-1}(W_{n+k-1})$ the classes \bar{x} , \bar{y} and \bar{w} get mapped to $\partial(x)$, $\partial(y)$ and $\partial(x \star y)$.
- Let \tilde{x} , \tilde{y} and \tilde{w} be the images of the classes \bar{x} , \bar{y} and \bar{w} under the maps

$$\begin{aligned} \pi_{n+m-1}(X_{n-r}) &\rightarrow \pi_{n+m-1}(X_{n-r}/X_{n-r-1}), \\ \pi_{k+l-1}(Y_{k-r}) &\rightarrow \pi_{k+l-1}(Y_{k-r}/Y_{k-r-1}), \\ \pi_{n+m+k+l-1}(W_{n+k-r}) &\rightarrow \pi_{n+m+k+l-1}(W_{n+k-r}/W_{n+k-r-1}). \end{aligned}$$

Then

$$\tilde{w} = \tilde{x} \star y + (-1)^{n+m} x \star \tilde{y}.$$

Proof Using Lemma 3.2 and the fact that the maps $\mathbb{F}_R \Sigma^\infty S^{s-1} \rightarrow \mathbb{F}_R \Sigma^\infty D^s$ satisfy the homotopy extension property, one gets classes $x' \in \pi_{n+m}(X_n, X_{n-r})$ and $y' \in \pi_{k+l}(Y_k, Y_{k-r})$ that are mapped to x and y under

$$\pi_{n+m}(X_n, X_{n-r}) \rightarrow \pi_{n+m}(X_n, X_{n-1}) \xrightarrow{\phi} \pi_{n+m}(X_n/X_{n-1})$$

and

$$\pi_{k+l}(Y_k, Y_{k-r}) \rightarrow \pi_{k+l}(Y_k, Y_{k-1}) \xrightarrow{\phi} \pi_{k+l}(Y_k/Y_{k-1}).$$

We define $\bar{x} := \kappa(x') \in \pi_{n+m-1}(X_{n-r})$ and $\bar{y} := \kappa(y') \in \pi_{k+l-1}(Y_{k-r})$. By Lemma 3.3 the element $x \star y$ is the image of $x' \cdot y'$ under

$$\begin{aligned} \pi_{n+m+k+l}(X_n \wedge_R Y_k, P') &\rightarrow \pi_{n+m+k+l}(W_{n+k}, W_{n+k-r}) \\ &\rightarrow \pi_{n+m+k+l}(W_{n+k}, W_{n+k-1}) \rightarrow \pi_{n+m+k+l}(W_{n+k}/W_{n+k-1}), \end{aligned}$$

where $P' := (X_{n-r} \wedge_R Y_k) \amalg_{(X_{n-r} \wedge_R Y_{k-r})} (X_n \wedge_R Y_{k-r})$. We define \bar{w} to be the image of $x' \cdot y'$ under

$$\begin{aligned} \pi_{n+m+k+l}(X_n \wedge_R Y_k, P') &\rightarrow \pi_{n+m+k+l}(W_{n+k}, W_{n+k-r}) \\ &\xrightarrow{\kappa} \pi_{n+m+k+l-1}(W_{n+k-r}). \end{aligned}$$

The claim then follows from Lemmas 3.2, 3.3 and 3.4. □

In order to show that Lemma 3.5 holds for classes in arbitrary total degree, we next prove that suspension isomorphisms give isomorphisms of spectral sequences. Let M be an R -module, and let X and Y be pointed spaces. Analogously to $M \wedge X$, we define R -modules $X \wedge M$ and $X \wedge M \wedge Y$ by applying the spectrification functor [18, Appendix] to the prespectra given by $(X \wedge M)(V) = X \wedge M(V)$ and $(X \wedge M \wedge Y)(V) = X \wedge M(V) \wedge Y$. Recall that we fixed isomorphisms

$$S_R^{m+1} \cong S_R^m \wedge S^1.$$

For $m \in \mathbb{Z}$ we define $S_R^{m+1} \cong S^1 \wedge S_R^m$ as the composition

$$S_R^{m+1} \cong S_R^m \wedge S^1 \cong S^1 \wedge S_R^m,$$

and we define $S_R^{m+2} \cong S^1 \wedge S_R^m \wedge S^1$ as the composition

$$S_R^{m+2} \cong S_R^{m+1} \wedge S^1 \cong (S^1 \wedge S_R^m) \wedge S^1 \cong S^1 \wedge S_R^m \wedge S^1.$$

These isomorphisms define group isomorphisms

$$\begin{aligned} \sigma_r: \pi_m(M) &\rightarrow \pi_{m+1}(M \wedge S^1), \\ \sigma_l: \pi_m(M) &\rightarrow \pi_{m+1}(S^1 \wedge M), \\ \sigma_b: \pi_m(M) &\rightarrow \pi_{m+2}(S^1 \wedge M \wedge S^1) \end{aligned}$$

such that

$$(10) \quad \begin{array}{ccc} \pi_m(M) & \xrightarrow{\sigma_r} & \pi_{m+1}(M \wedge S^1) \\ & \searrow \sigma_l & \downarrow \\ & & \pi_{m+1}(S^1 \wedge M) \end{array}$$

and

$$(11) \quad \begin{array}{ccc} \pi_m(M) & \xrightarrow{\sigma_b} & \pi_{m+2}(S^1 \wedge M \wedge S^1) \\ \sigma_l \downarrow & & \downarrow \cong \\ \pi_{m+1}(S^1 \wedge M) & \xrightarrow{\sigma_r} & \pi_{m+2}((S^1 \wedge M) \wedge S^1) \end{array}$$

are commutative. By commutativity of (10) and since the map

$$\pi_*(M \wedge (S^1 \wedge S^1)) \rightarrow *(M \wedge (S^1 \wedge S^1)),$$

interchanging the two S^1 's, is the negative of the identity map, we get that $\sigma_r \circ \sigma_l = -\sigma_l \circ \sigma_r$. Using this and the commutativity of (11), one gets $\sigma_b \circ \sigma_r = -\sigma_r \circ \sigma_b$.

Lemma 3.6 *Let $\cdots \rightarrow X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow \cdots$ be a sequence of cofibrations of R -modules. The maps σ_r , σ_l and σ_b define isomorphisms of unrolled exact couples and thus isomorphisms of spectral sequences*

$$\begin{aligned} E_{*,*}^*(X_\bullet) &\rightarrow E_{*,*+1}^*(X_\bullet \wedge S^1), \\ E_{*,*}^*(X_\bullet) &\rightarrow E_{*,*+1}^*(S^1 \wedge X_\bullet), \\ E_{*,*}^*(X_\bullet) &\rightarrow E_{*,*+2}^*(S^1 \wedge X_\bullet \wedge S^1). \end{aligned}$$

Proof Consider for example the third case. We only need to check that σ_b and ∂ commute. Recall from the proof of Lemma 3.1 that there was a sign in the definition of ∂ . The claim follows from the fact that for a cofibration of R -modules $f: M \rightarrow N$ and for pointed spaces X and Y the diagram

$$\begin{array}{ccccc} (X \wedge N \wedge Y)/(X \wedge M \wedge Y) & \xleftarrow{\cong} & C(\text{id}_X \wedge f \wedge \text{id}_Y) & \longrightarrow & (X \wedge M \wedge Y) \wedge S^1 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ X \wedge (N/M) \wedge Y & \xleftarrow{\cong} & X \wedge C(f) \wedge Y & \longrightarrow & X \wedge (M \wedge S^1) \wedge Y \end{array}$$

commutes, from $\sigma_b \circ \sigma_r = -\sigma_r \circ \sigma_b$ and from the fact that interchanging two S^1 's induces the negative of the identity in homotopy. □

We want to define pairings of suspensions of towers. Note that for R -modules M and N and for pointed spaces X and Y we can define a natural isomorphism

$$X \wedge (M \wedge_R N) \wedge Y \cong (X \wedge M) \wedge_R (N \wedge Y).$$

Let X_\bullet, Y_\bullet and W_\bullet be towers of cofibrations of R -modules. Suppose that we have a pairing of towers $X_\bullet \wedge_R Y_\bullet \rightarrow W_\bullet$. Then the maps

$$(X \wedge X_n) \wedge_R (Y_k \wedge Y) \xrightarrow{\cong} X \wedge (X_n \wedge_R Y_k) \wedge Y \rightarrow X \wedge W_{n+k} \wedge Y$$

define a pairing of towers $(X \wedge X_\bullet) \wedge_R (Y_\bullet \wedge Y) \rightarrow X \wedge W_\bullet \wedge Y$.

Lemma 3.7 *The diagram*

$$\begin{array}{ccc} E_{n,m}^1(X_\bullet) \otimes E_{k,l}^1(Y_\bullet) & \xrightarrow{\quad \star \quad} & E_{n+k,m+l}^1(W_\bullet) \\ \sigma_l \otimes \sigma_r \downarrow & & \downarrow \sigma_b \\ E_{n,m+1}^1(S^1 \wedge X_\bullet) \otimes E_{k,l+1}^1(Y_\bullet \wedge S^1) & \xrightarrow{\quad \star \quad} & E_{n+k,m+l+2}^1(S^1 \wedge W_\bullet \wedge S^1) \end{array}$$

commutes up to the sign $(-1)^{k+l}$.

Proof Because of (11) and (10) we only have to check that the diagram

$$\begin{array}{ccc} \pi_s(X_n/X_{n-1}) \otimes \pi_t(Y_k/Y_{k-1}) & \longrightarrow & \pi_{s+t}(X_n/X_{n-1} \wedge_R Y_k/Y_{k-1}) \\ \downarrow \sigma_r \otimes \text{id} & & \downarrow \sigma_r \\ \pi_{s+1}(X_n/X_{n-1} \wedge S^1) \otimes \pi_t(Y_k/Y_{k-1}) & \longrightarrow & (-1)^t \pi_{s+t+1}((X_n/X_{n-1} \wedge_R Y_k/Y_{k-1}) \wedge S^1) \\ \downarrow \cong & & \downarrow \cong \\ \pi_{s+1}(S^1 \wedge X_n/X_{n-1}) \otimes \pi_t(Y_k/Y_{k-1}) & \longrightarrow & \pi_{s+t+1}((X_n/X_{n-1} \wedge S^1) \wedge_R Y_k/Y_{k-1}) \\ \downarrow \text{id} \otimes \sigma_r & & \downarrow \sigma_r \\ \pi_{s+1}(S^1 \wedge X_n/X_{n-1}) \otimes \pi_{t+1}(Y_k/Y_{k-1} \wedge S^1) & \longrightarrow & \pi_{s+t+2}(((S^1 \wedge X_n/X_{n-1}) \wedge_R Y_k/Y_{k-1}) \wedge S^1) \\ & & \downarrow \cong \\ \pi_{s+1}(S^1 \wedge X_n/X_{n-1}) \otimes \pi_{t+1}(Y_k/Y_{k-1} \wedge S^1) & \longrightarrow & \pi_{s+t+2}((S^1 \wedge X_n/X_{n-1}) \wedge_R (Y_k/Y_{k-1} \wedge S^1)) \end{array}$$

commutes up to the indicated sign. This follows from (2) and (3). □

Using the above, one easily gets:

Lemma 3.8 *Lemma 3.5 is true without the assumptions $n + m \geq 3$ and $k + l \geq 3$.*

We can conclude:

Corollary 3.9 *In the situation of Lemma 3.5 the following properties hold: The map*

$$\star: E_{n,m}^r(X_\bullet) \otimes_{\mathbb{Z}} E_{k,l}^r(Y_\bullet) \rightarrow E_{n+k,m+l}^r(W_\bullet), \quad [\alpha] \otimes [\beta] \mapsto [\alpha \star \beta],$$

is well-defined. The Leibniz rule holds:

$$d^r(x \star y) = d^r(x) \star y + (-1)^{n+m} x \star d^r(y).$$

For $r \geq 2$ the following diagram commutes:

$$\begin{array}{ccc} E_{*,*}^r(X_\bullet) \otimes_{\mathbb{Z}} E_{*,*}^r(Y_\bullet) & \xrightarrow{\quad \star \quad} & E_{*,*}^r(W_\bullet) \\ \cong \downarrow & & \downarrow \cong \\ H_*(E_{*,*}^{r-1}) \otimes_{\mathbb{Z}} H_*(E_{*,*}^{r-1}) & \longrightarrow & H_*(E_{*,*}^{r-1} \otimes_{\mathbb{Z}} E_{*,*}^{r-1}) \xrightarrow{H_*(\star)} H_*(E_{*,*}^{r-1}) \end{array}$$

Let $X_\bullet \wedge_R Y_\bullet \rightarrow W_\bullet$ be pairing of towers of cofibrations. We study the convergence properties of the associated pairing of spectral sequences: We have a product

$$\begin{aligned} \star: \pi_n(\operatorname{colim}_i X_i) \otimes_{\mathbb{Z}} \pi_m(\operatorname{colim}_j Y_j) &\rightarrow \pi_{n+m}(\operatorname{colim}_i X_i \wedge_R \operatorname{colim}_j Y_j) \\ &\xrightarrow{\cong} \pi_{n+m}(\operatorname{colim}_{i,j} X_i \wedge_R Y_j) \rightarrow \pi_{n+m}(\operatorname{colim}_k W_k). \end{aligned}$$

We define $F_s^n(X_\bullet)$ by the image of $\pi_n(X_s) \rightarrow \pi_n(\operatorname{colim}_i X_i)$. We get a filtration

$$\dots \subseteq F_{s-1}^n(X_\bullet) \subseteq F_s^n(X_\bullet) \subseteq \dots$$

of $\pi_n(\operatorname{colim}_i X_i)$. This is the filtration we used implicitly in the proof of the convergence statement of Lemma 3.1. Analogously, we define filtrations of the groups $\pi_m(\operatorname{colim}_j Y_j)$ and $\pi_{n+m}(\operatorname{colim}_k W_k)$.

The product \star respects the filtrations, so we get a product

$$F_s^n(X_\bullet)/F_{s-1}^n(X_\bullet) \otimes_{\mathbb{Z}} F_t^m(Y_\bullet)/F_{t-1}^m(Y_\bullet) \rightarrow F_{s+t}^{n+m}(W_\bullet)/F_{s+t-1}^{n+m}(W_\bullet).$$

By Corollary 3.9 the map

$$\star: E_{*,*}^1(X_\bullet) \otimes_{\mathbb{Z}} E_{*,*}^1(Y_\bullet) \rightarrow E_{*,*}^1(W_\bullet)$$

induces a product

$$\star: E_{*,*}^\infty(X_\bullet) \otimes_{\mathbb{Z}} E_{*,*}^\infty(Y_\bullet) \rightarrow E_{*,*}^\infty(W_\bullet).$$

These two products are compatible:

Lemma 3.10 Suppose that we have $X_n = Y_n = W_n = *$ for $n < 0$, so that the spectral sequences converge. Then the following diagram commutes:

$$\begin{array}{ccc}
 F_s^n(X_\bullet)/F_{s-1}^n(X_\bullet) \otimes F_t^m(Y_\bullet)/F_{t-1}^m(Y_\bullet) & \xrightarrow{\star} & F_{s+t}^{n+m}(W_\bullet)/F_{s+t-1}^{n+m}(W_\bullet) \\
 \cong \downarrow & & \downarrow \cong \\
 E_{s,n-s}^\infty(X_\bullet) \otimes E_{t,m-t}^\infty(Y_\bullet) & \xrightarrow{\star} & E_{s+t,n+m-(s+t)}^\infty(W_\bullet)
 \end{array}$$

Proof This easily follows from the definitions of the vertical isomorphisms (see [7, Lemma 5.6]). □

3.2 The Atiyah–Hirzebruch spectral sequence

In this subsection we use the results of the previous subsection to derive the existence of the Atiyah–Hirzebruch spectral sequence and some of its properties, especially its multiplicativity. It is stated in [13, Section IV.3] that the Atiyah–Hirzebruch spectral sequence has the expected multiplicative properties. We work this out in more detail.

Remark 3.11 Recall that the Eilenberg–Mac Lane spectrum HA of a commutative ring A can be realized as a commutative S –algebra: Multiplicative infinite loop space theory provides a functor from commutative rings to E_∞ ring spectra [22; 20; 19] and the functor $S \wedge_{\mathcal{L}} -$ induces a functor from E_∞ ring spectra to commutative S –algebras [13, Corollary II.3.6]. Since we can functorially replace commutative S –algebras by cofibrant ones, we can assume that HA is cofibrant.

If M is an A –module, then by [13, Section IV.2] the Eilenberg–Mac Lane spectrum HM can be realized as an HA –module. Eilenberg–Mac Lane spectra are unique: if R is a connective ((-1) –connected) commutative S –algebra and if X and Y are R –modules with $\pi_*(X) = 0 = \pi_*(Y)$ for $* \neq s$, then, by [17, page 507], we have an isomorphism

$$(12) \quad \mathcal{D}_R(X, Y) \cong \text{Hom}_{\pi_0(R)}(\pi_s(X), \pi_s(Y)).$$

Remark 3.12 Let R be a connective commutative S –algebra. By Proposition IV.3.1 of [13] there is a morphism

$$R \rightarrow H\pi_0(R)$$

in $\bar{h}\mathcal{C}\mathcal{A}_S$ that induces the identity on π_0 . The proof shows that it is natural: if $R \rightarrow R'$ is a morphism of commutative S -algebras, then

$$(13) \quad \begin{array}{ccc} R & \longrightarrow & H\pi_0(R) \\ \downarrow & & \downarrow \\ R' & \longrightarrow & H\pi_0(R') \end{array}$$

is commutative in $\bar{h}\mathcal{C}\mathcal{A}_S$.

Let R be a connective cofibrant commutative S -algebra, so that we have a map of commutative S -algebras $R \rightarrow H\pi_0(R)$ realizing the identity on π_0 . Let G be a connective R -module. Then we have a map $G \rightarrow H\pi_0(G)$ in \mathcal{D}_R realizing the identity map on π_0 (compare [31, Theorem II.4.13]). Here, $H\pi_0(G)$ is an R -module by pulling back the $H\pi_0(R)$ -action along the map $R \rightarrow H\pi_0(R)$.

We now prove the existence of the Atiyah–Hirzebruch spectral sequence.

Theorem 3.13 *Let R be a connective cofibrant commutative S -algebra. Let M be a connective R -module and let G be an arbitrary R -module. Then we have a strongly convergent spectral sequence of the form*

$$E_{n,m}^2 = \pi_n(HG_m \wedge_R^L M) \Rightarrow \pi_{n+m}(G \wedge_R^L M).$$

Here, HG_m is an R -module by pulling back the $H\pi_0(R)$ -action along $R \rightarrow H\pi_0(R)$.

Proof This is [13, Theorem IV.3.7]. Since we will need it later, we explain the proof in more detail. We replace G by a cell approximation and denote this approximation again by G . Since R is connective there is a CW R -module ΓM and a weak equivalence of R -modules $\Gamma M \rightarrow M$ [13, Theorem III.2.10]. Because M is connective we can assume that the n -skeleton ΓM^n is $*$ for $n < 0$ [13, proof of Theorem IV.3.6]. The $(n+1)$ -skeleton ΓM^{n+1} is built from the n -skeleton by attaching $(n+1)$ -cells. Since the coproduct of cofibrations is a cofibration and since cofibrations are stable under cobase change, $\Gamma M^n \rightarrow \Gamma M^{n+1}$ is a cofibration. Because $G \wedge_R (-)$ is left adjoint, it preserves cofibrations. Hence,

$$G \wedge_R \Gamma M^n \rightarrow G \wedge_R \Gamma M^{n+1}$$

is a cofibration, too. Therefore, Lemma 3.1 gives a strongly convergent spectral sequence of the form

$$E_{n,m}^1 = \pi_{n+m}(G \wedge_R \Gamma M^n / \Gamma M^{n-1}) \Rightarrow \pi_{n+m}(\operatorname{colim}_n G \wedge_R \Gamma M^n).$$

We have $\operatorname{colim}_n G \wedge_R \Gamma M^n = G \wedge_R \operatorname{colim}_n \Gamma M^n = G \wedge_R \Gamma M$. Thus, it remains to identify the E^2 -page. The quotient $\Gamma M^n / \Gamma M^{n-1}$ is a wedge of n -spheres S^n_R , so by [13, Proposition III.3.9] the Künneth map induces an isomorphism

$$G_m \otimes_{R_0} \pi_n(\Gamma M^n / \Gamma M^{n-1}) \rightarrow \pi_{n+m}(G \wedge_R \Gamma M^n / \Gamma M^{n-1}).$$

Since the Künneth map is compatible with the suspension, the differential $d^1: E^1_{n,m} \rightarrow E^1_{n-1,m}$ identifies with $(-1)^{n+m}$ times the map

$$(14) \quad G_m \otimes_{R_0} \pi_n(\Gamma M^n / \Gamma M^{n-1}) \xrightarrow{\cong} G_m \otimes_{R_0} \pi_n(C) \xrightarrow{\operatorname{id} \otimes d} G_m \otimes_{R_0} \pi_{n-1}(\Gamma M^{n-1}) \rightarrow G_m \otimes_{R_0} \pi_{n-1}(\Gamma M^{n-1} / \Gamma M^{n-2}),$$

where $C = C(\Gamma M^{n-1} \rightarrow \Gamma M^n)$. We now fix m . The above argument shows that there is a strongly convergent spectral sequence

$${}_m E^1_{n,k} \Rightarrow \pi_{n+k}(HG_m \wedge^L_R M)$$

whose E^1 -page is given by

$${}_m E^1_{n,k} \cong \begin{cases} 0 & \text{if } k \neq 0, \\ G_m \otimes_{R_0} \pi_n(\Gamma M^n / \Gamma M^{n-1}) & \text{if } k = 0, \end{cases}$$

and whose differential $d^1: {}_m E^1_{n,0} \rightarrow {}_m E^1_{n-1,0}$ is $(-1)^n$ times the map (14). Since this spectral sequence has to collapse at the E^2 -page we get

$$E^2_{n,m} \cong {}_m E^2_{n,0} \cong \pi_n(HG_m \wedge^L_R M). \quad \square$$

Lemma 3.14 *Let R be a connective cofibrant commutative S -algebra. Let $M \rightarrow \bar{M}$ and $G \rightarrow \bar{G}$ be morphisms in \mathcal{D}_R , where M and \bar{M} are connective. Then we have a morphism of Atiyah–Hirzebruch spectral sequences between the spectral sequence converging to $\pi_*(G \wedge^L_R M)$ and the spectral sequence converging to $\pi_*(\bar{G} \wedge^L_R \bar{M})$. The map $\pi_*(G \wedge^L_R M) \rightarrow \pi_*(\bar{G} \wedge^L_R \bar{M})$ respects the filtrations and the induced map on the associated graded identifies with the induced map on E^∞ -pages. On the E^2 -pages the map is the obvious one.*

Proof This follows easily from the fact that maps between CW modules over a connective S -algebra are homotopic to cellular maps [13, Theorem III.2.9]. \square

Lemma 3.15 *In the situation of Theorem 3.13 we have the following: if G is connective, the edge homomorphism*

$$\pi_n(G \wedge^L_R M) = F^n_n \rightarrow F^n_n / F^{n-1}_n \xrightarrow{\cong} E^\infty_{n,0} \hookrightarrow E^2_{n,0} \cong \pi_n(HG_0 \wedge^L_R M)$$

is the map induced by $G \rightarrow HG_0$.

Proof This follows by applying Lemma 3.14 to the morphisms $M \xrightarrow{\text{id}} M$ and $G \rightarrow HG_0$ and by observing that the edge homomorphism of the Atiyah–Hirzebruch spectral sequence converging to $\pi_*(HG_0 \wedge_R^L M)$ is the identity. \square

Lemma 3.16 *Let $R \rightarrow R'$ be a morphism of connective cofibrant commutative S -algebras. Let M be a connective R' -module and let G be an R' -module. Then we get a morphism of Atiyah–Hirzebruch spectral sequences converging to the map $\pi_*(G \wedge_R^L M) \rightarrow \pi_*(G \wedge_{R'}^L M)$. On the E^2 -pages the map is given by the canonical maps $\pi_*(HG_m \wedge_R^L M) \rightarrow \pi_*(HG_m \wedge_{R'}^L M)$. Here, note that we can realize HG_m as an $H\pi_0(R')$ -module and consider it as an R -module via the map $R \rightarrow R' \rightarrow H\pi_0(R')$, and we can realize HG_m as an $H\pi_0(R)$ -module and consider it as an R -module via $R \rightarrow H\pi_0(R)$. These two R -modules are isomorphic by (12).*

Proof Let $\Gamma^R G$ and $\Gamma^{R'} G$ be the cell approximations of G used in the construction of the spectral sequences and let $\Gamma^R M$ and $\Gamma^{R'} M$ be the CW approximation of M used in the construction of the spectral sequences. Let $\Gamma^R G \rightarrow \Gamma^{R'} G$ and $\Gamma^R M \rightarrow \Gamma^{R'} M$ be maps in \mathcal{M}_R lifting the identity maps up to homotopy. Because of $(\Gamma^R M)^n = * = (\Gamma^{R'} M)^n$ for $n < 0$, one can show, as in the proof of the existence of cellular approximations of maps between CW complexes (see [10, Theorem 8.5.4; 34, Lemma II.3.2]), that $\Gamma^R M \rightarrow \Gamma^{R'} M$ is homotopic in \mathcal{M}_R to a map that respects the skeleton filtrations of $\Gamma^R M$ and $\Gamma^{R'} M$. For this note that the only properties of the sequence of cofibrations of R -modules

$$\dots \rightarrow (\Gamma^{R'} M)^{n-1} \rightarrow (\Gamma^{R'} M)^n \rightarrow (\Gamma^{R'} M)^{n+1} \rightarrow \dots \rightarrow \Gamma^{R'} M$$

that are needed in the proof are that the n^{th} homotopy group of the R -module homomorphism $(\Gamma^{R'} M)^n \rightarrow \Gamma^{R'} M$ is zero for $n \geq 1$ and that $\pi_0((\Gamma^{R'} M)^0) \rightarrow \pi_0(\Gamma^{R'} M)$ is surjective. Both follow from the fact that $\pi_n(\Gamma^{R'} M / (\Gamma^{R'} M)^n) = 0$ by [13, page 57].

It is clear that the maps

$$\Gamma^R G \wedge_R (\Gamma^R M)^n \rightarrow \Gamma^{R'} G \wedge_R (\Gamma^{R'} M)^n \rightarrow \Gamma^{R'} G \wedge_{R'} (\Gamma^{R'} M)^n$$

induce a morphism of unrolled exact couples and therefore a map of spectral sequences which converges to the map $\pi_*(G \wedge_R^L M) \rightarrow \pi_*(G \wedge_{R'}^L M)$. Note that the map

$$\pi_{n+m}(\Gamma^R G \wedge_R (\Gamma^R M)^n / (\Gamma^R M)^{n-1}) \rightarrow \pi_{n+m}(\Gamma^{R'} G \wedge_{R'} (\Gamma^{R'} M)^n / (\Gamma^{R'} M)^{n-1})$$

identifies with the map

$$G_m \otimes_{R_0} \pi_n((\Gamma^R M)^n / (\Gamma^R M)^{n-1}) \rightarrow G_m \otimes_{R'_0} \pi_n((\Gamma^{R'} M)^n / (\Gamma^{R'} M)^{n-1}).$$

Applying the proven statements as well to the R' -modules HG_m , one gets the identification of the map of spectral sequences on E^2 -pages. For this also note that the isomorphism between the two realizations of HG_m induces a morphism of spectral sequences by Lemma 3.14. \square

We now prove the multiplicativity of the Atiyah–Hirzebruch spectral sequence:

Proposition 3.17 *Let R be a connective cofibrant commutative S -algebra. Let M, N and L be connective R -modules and let G be an R -module. Let ${}^M E_{*,*}^*, {}^N E_{*,*}^*$ and ${}^L E_{*,*}^*$ be the spectral sequences defined in Theorem 3.13. Then maps $G \wedge_R^L G \rightarrow G$ and $M \wedge_R^L N \rightarrow L$ in \mathcal{D}_R induce a pairing of spectral sequences*

$${}^M E_{*,*}^* \otimes_{\mathbb{Z}} {}^N E_{*,*}^* \rightarrow {}^L E_{*,*}^*$$

that converges to the product

$$\pi_*(G \wedge_R^L M) \otimes_{\mathbb{Z}} \pi_*(G \wedge_R^L N) \rightarrow \pi_*(G \wedge_R^L L).$$

Proof Since the spectral sequences were constructed using a cell approximation of G , we can assume that the product $G \wedge_R^L G \rightarrow G$ in \mathcal{D}_R can be represented by a map $G \wedge_R G \rightarrow G$ in \mathcal{M}_R . Let $\Gamma M, \Gamma N$ and ΓL be the CW approximations of M, N and L used in the definition of the spectral sequences. Then the map $M \wedge_R^L N \rightarrow L$ can be represented by a map

$$\Gamma M \wedge_R \Gamma N \rightarrow \Gamma L$$

in \mathcal{M}_R that is cellular. We define maps $(G \wedge_R \Gamma M^n) \wedge_R (G \wedge_R \Gamma N^k) \rightarrow G \wedge \Gamma L^{n+k}$ by $(G \wedge_R \Gamma M^n) \wedge_R (G \wedge_R \Gamma N^k) \xrightarrow{\cong} (G \wedge_R G) \wedge_R (\Gamma M^n \wedge_R \Gamma N^k) \rightarrow G \wedge (\Gamma M \wedge_R \Gamma N)^{n+k} \rightarrow G \wedge \Gamma L^{n+k}$.

Since the diagrams

$$\begin{array}{ccccc} \Gamma M^{n-1} \wedge_R \Gamma N^k & \longrightarrow & \Gamma M^n \wedge_R \Gamma N^k & \longleftarrow & \Gamma M^n \wedge_R \Gamma N^{k-1} \\ \downarrow & & \downarrow & & \downarrow \\ (\Gamma M \wedge_R \Gamma N)^{n+k-1} & \longrightarrow & (\Gamma M \wedge_R \Gamma N)^{n+k} & \longleftarrow & (\Gamma M \wedge_R \Gamma N)^{n+k-1} \end{array}$$

and

$$\begin{array}{ccc} (\Gamma M \wedge_R \Gamma N)^{n+k-1} & \longrightarrow & (\Gamma M \wedge_R \Gamma N)^{n+k} \\ \downarrow & & \downarrow \\ \Gamma L^{n+k-1} & \longrightarrow & \Gamma L^{n+k} \end{array}$$

commute in \mathcal{M}_R , the proposition follows from Corollary 3.9 and Lemma 3.10. \square

4 The Brun spectral sequence

The goal of this section is to apply the results of the previous section to construct a generalization of the following spectral sequence of Brun (see [9, Theorem 6.2.10]):

Theorem 4.1 (Brun) *Let $R \rightarrow R_1$ be a ring homomorphism of commutative rings. There is a multiplicative spectral sequence of the form*

$$E_{n,m}^2 = \pi_n \text{THH}(HR_1, H \text{Tor}_m^R(R_1, R_1)) \Rightarrow \pi_{n+m} \text{THH}(HR, HR_1).$$

We first recall two definitions of topological Hochschild homology. For this, recall the definition of geometric realization.

Remark 4.2 Let X_\bullet be a simplicial object in \mathcal{M}_S or \mathcal{S} . Then its geometric realization $|X_\bullet|$ is defined as the coend

$$\int^\Delta X_q \wedge (\Delta_q)_+,$$

where Δ_q denotes the standard topological q -simplex. Since tensors and colimits in \mathcal{M}_S are created in \mathcal{S} geometric realization in \mathcal{M}_S is given by geometric realization in \mathcal{S} . Now, let X_\bullet be a simplicial commutative S -algebra. Then, by Proposition X.1.5 of [13] its geometric realization can be endowed with the structure of a commutative S -algebra. We can also form the internal geometric realization

$$\int^\Delta X_q \otimes \Delta_q \in \mathcal{C}\mathcal{A}_S.$$

By [13, Proposition VII.3.3] these two commutative S -algebras are isomorphic.

Definition 4.3 (THH, first definition) Let A be a cofibrant commutative S -algebra. Let B be an (A, A) -bimodule. The simplicial S -module $\text{THH}(A; B)_\bullet$ is defined by

$$\text{THH}(A; B)_n := B \wedge_S \underbrace{A \wedge_S \cdots \wedge_S A}_{n \text{ times}}$$

together with the usual face and degeneracy maps (see [13, Definition IX.2.1]). We define

$$\text{THH}(A; B) := |\text{THH}(A; B)_\bullet|.$$

We write $\text{THH}(A)$ for $\text{THH}(A; A)$. We denote the homotopy groups of $\text{THH}(A; B)$ and $\text{THH}(A)$ by $\text{THH}_*(A; B)$ and $\text{THH}_*(A)$.

Definition 4.4 (THH, second definition) Let A be a cofibrant commutative S -algebra and let B be an (A, A) -bimodule. Let $A^e = A \wedge_S A^{\text{op}}$ be the enveloping algebra of A . We define

$$\text{THH}(A; B) = B \wedge_{A^e}^L A \in \mathcal{D}_{A^e}.$$

By [13, Proposition IX.2.5] the image of $B \wedge_{A^e}^L A$ under $\mathcal{D}_{A^e} \rightarrow \mathcal{D}_S$ is isomorphic to the image of $|n \mapsto B \wedge_S A^{\wedge_S n}|$ under $\mathcal{M}_S \rightarrow \mathcal{D}_S$ if B is a cell A^e -module.

We now study algebra structures for both definitions of THH in the case where B is a commutative A -algebra. In Lemma 4.6 we will see that the isomorphism of [13, Proposition IX.2.5] preserves multiplicative structures.

Remark 4.5 Let A be a cofibrant commutative S -algebra. Let B be a commutative A -algebra. Then $[n] \mapsto B \wedge_S A^{\wedge_S n}$ is a simplicial commutative S -algebra. Thus, by Remark 4.2, its geometric realization is an object in $\mathcal{C}\mathcal{A}_S$. Moreover, the maps $B \rightarrow B \wedge_S A^{\wedge_S n}$ and $B \wedge_S A^{\wedge_S n} \rightarrow B$ endow $|[n] \mapsto B \wedge_S A^{\wedge_S n}|$ with the structure of an augmented commutative B -algebra. Here, note that by the universal property of coends we have $|\underline{B}| \cong B$, where \underline{B} denotes the constant simplicial object.

Note that the (A, A) -bimodule structures on A and B come from the A^e -algebra structures given by the maps of commutative S -algebras $A^e = A \wedge_S A \rightarrow A$ and

$$(A^e \rightarrow A \rightarrow B) = (A^e \rightarrow B^e \rightarrow B).$$

Since the smash product of two ring spectra is a ring spectrum we get that $B \wedge_{A^e}^L A$ is an A^e -ring spectrum.

Lemma 4.6 Let $A \rightarrow B$ be a morphism of cofibrant commutative S -algebras. Then the image of $B \wedge_{A^e}^L A$ under $\mathcal{D}_{A^e} \rightarrow \mathcal{D}_S$ and the image of $|[n] \mapsto B \wedge_S A^{\wedge_S n}|$ under $\mathcal{M}_S \rightarrow \mathcal{D}_S$ are isomorphic as S -ring spectra.

Proof Let $B^S(A, A, A)_\bullet$ be the bar construction (see [13, Definition IV.7.2]). It is a simplicial commutative S -algebra with $B^S(A, A, A)_n = A \wedge_S A^{\wedge_S n} \wedge_S A$. The inclusions of the first and the last smash factor give a map of commutative S -algebras

$$A^e \cong |\underline{A^e}| \rightarrow |B^S(A, A, A)_\bullet|.$$

We have:

- By [13, Lemma VII.7.3] the map of commutative A^e -algebras

$$|B^S(A, A, A)_\bullet| \rightarrow |\underline{A}| \cong A$$

is a weak equivalence.

- The map $A^e \rightarrow |B_\bullet^S(A, A, A)|$ is a cofibration of commutative S -algebras. To see this we consider the simplicial 1-simplex $[n] \mapsto \Delta_n^1$. We interpret Δ^1 as a discrete simplicial space. Since the functor $A \otimes -: \mathcal{U} \rightarrow \mathcal{C}\mathcal{A}_S$ commutes with coproducts, we can identify $|B_\bullet^S(A, A, A)|$ with the geometric realization of the simplicial commutative S -algebra

$$[n] \mapsto A \otimes (\Delta^1)_n.$$

By [13, Proposition VII.3.2] this is isomorphic in $\mathcal{C}\mathcal{A}_S$ to $A \otimes |\Delta^1|$. Likewise, $|\underline{A}^e|$ is isomorphic to $A \otimes |\partial\Delta^1|$. The map $|\underline{A}^e| \rightarrow |B^S(A, A, A)_\bullet|$ can be identified with

$$A \otimes |\partial\Delta^1| = (A \otimes |\partial\Delta^1| \wedge_{S \otimes |\partial\Delta^1|} S \otimes |\Delta^1|) \rightarrow A \otimes |\Delta^1|,$$

which is a cofibration in $\mathcal{C}\mathcal{A}_S$ because the model category structure is topological.

We are now ready to prove the lemma. We have an isomorphism of commutative S -algebras

$$[[n] \mapsto B \wedge_S A^{\wedge_S n}] \cong |\underline{B} \wedge_{\underline{A}^e} B^S(A, A, A)_\bullet|.$$

By the universal properties of coends and pushouts, and the fact that the functor $-\otimes X: \mathcal{C}\mathcal{A}_S \rightarrow \mathcal{C}\mathcal{A}_S$ commutes with colimits, we get that this is isomorphic to

$$|\underline{B}| \wedge_{|\underline{A}^e|} |B^S(A, A, A)_\bullet| \cong B \wedge_{A^e} |B^S(A, A, A)_\bullet|.$$

Since A^e is a cofibrant commutative S -algebra (as coproduct of cofibrant objects), we get by Lemma 2.2 that the map

$$B \wedge_{A^e}^L |B^S(A, A, A)_\bullet| \rightarrow B \wedge_{A^e} |B^S(A, A, A)_\bullet|$$

in \mathcal{D}_{A^e} is an isomorphism of A^e -ring spectra. Since

$$B \wedge_{A^e}^L |B^S(A, A, A)_\bullet| \cong B \wedge_{A^e}^L A$$

as A^e -ring spectra, this finishes the proof. □

Remark 4.7 Let

$$(15) \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be a commutative diagram of cofibrant commutative S -algebras. We then have a commutative diagram in \mathcal{D}_S

$$\begin{array}{ccc} B \wedge_{A^e}^L A & \xrightarrow{\cong} & |[n] \mapsto B \wedge_S A^{\wedge_S n}| \\ \downarrow & & \downarrow \\ D \wedge_{C^e}^L C & \xrightarrow{\cong} & |[n] \mapsto D \wedge_S C^{\wedge_S n}| \end{array}$$

where the left vertical map is given by $B \wedge_{A^e}^L A \rightarrow D \wedge_{A^e}^L C \rightarrow D \wedge_{C^e}^L C$.

Until the end of the section, we work with Definition 4.4 for THH (second definition). In order to proof the generalization of Brun’s spectral sequence, we first identify THH with another ring spectrum. After that, we apply the Atiyah–Hirzebruch spectral sequence.

Lemma 4.8 *Let $S \rightarrow A \rightarrow B$ be cofibrations of commutative S -algebras. Then we have an isomorphism of A^e -ring spectra*

$$\mathrm{THH}(A; B) \cong (B \wedge_A B) \wedge_{B^e}^L B.$$

Proof We factor the map $B^e \rightarrow B$ in commutative S -algebras as

$$B^e \twoheadrightarrow \tilde{B} \xrightarrow{\sim} B.$$

We then have an isomorphism of A^e -ring spectra

$$\mathrm{THH}(A; B) \cong \tilde{B} \wedge_{A^e}^L A.$$

We claim that the map

$$\tilde{B} \wedge_{A^e}^L A \rightarrow \tilde{B} \wedge_{A^e} A$$

in \mathcal{D}_{A^e} is an isomorphism of A^e -ring spectra. By Lemma 2.2 we only have to show that $A^e \rightarrow B^e$ is a cofibration of commutative S -algebras. This is true because $A \rightarrow B$ is a cofibration, and, by denoting the coproduct inclusions by i_1 and i_2 , we have pushout diagrams

$$\begin{array}{ccc} A & \xrightarrow{i_2} & A \wedge_S A \\ \downarrow & & \downarrow \\ B & \xrightarrow{i_2} & A \wedge_S B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{i_1} & A \wedge_S B \\ \downarrow & & \downarrow \\ B & \xrightarrow{i_1} & B \wedge_S B \end{array}$$

Since $\tilde{B} \cong B^e \wedge_{B^e} \tilde{B}$ as B^e -algebras, we have an isomorphism of A^e -algebras

$$\tilde{B} \wedge_{A^e} A \cong A \wedge_{A^e} \tilde{B} \cong A \wedge_{A^e} (B^e \wedge_{B^e} \tilde{B}).$$

By using the universal property of pushouts in the category of commutative S -algebras, one sees that

$$A \wedge_{A^e} (B^e \wedge_{B^e} \tilde{B}) \cong (A \wedge_{A^e} B^e) \wedge_{B^e} \tilde{B}.$$

Obviously, the isomorphism is an isomorphism of A^e -algebras. Again, by using the universal property of pushouts one sees that we have an isomorphism of commutative S -algebras

$$A \wedge_{A^e} B^e \cong B \wedge_A B.$$

The isomorphism is an isomorphism of B^e -algebras. Therefore, we get

$$(A \wedge_{A^e} B^e) \wedge_{B^e} \tilde{B} \cong (B \wedge_A B) \wedge_{B^e} \tilde{B}$$

as B^e -algebras. We claim that the map

$$(B \wedge_A B) \wedge_{B^e}^L \tilde{B} \rightarrow (B \wedge_A B) \wedge_{B^e} \tilde{B}$$

in \mathcal{D}_{B^e} is an isomorphism of B^e -ring spectra. This follows from Lemma 2.2 if we can show that $B \wedge_A B$ is a cofibrant commutative S -algebra. But this is clear, because $A \rightarrow B \wedge_A B$ is a cofibration in $\mathcal{C}\mathcal{A}_S$, since cofibrations are stable under cobase change. As we have

$$(B \wedge_A B) \wedge_{B^e}^L \tilde{B} \cong (B \wedge_A B) \wedge_{B^e}^L B$$

as B^e -ring spectra, this finishes the proof. □

Remark 4.9 Given a diagram of the form (15), where the horizontal maps are cofibrations in $\mathcal{C}\mathcal{A}_S$, one gets a commutative diagram in \mathcal{D}_S

$$\begin{CD} \mathrm{THH}(A; B) @>\cong>> (B \wedge_A B) \wedge_{B^e}^L B \\ @VVV @VVV \\ \mathrm{THH}(C; D) @>\cong>> (D \wedge_C D) \wedge_{D^e}^L D \end{CD}$$

Here, the right vertical map is given by

$$(B \wedge_A B) \wedge_{B^e}^L B \rightarrow (D \wedge_C D) \wedge_{B^e}^L D \rightarrow (D \wedge_C D) \wedge_{D^e}^L D.$$

One uses that maps in $\mathcal{C}\mathcal{A}_S$ can be factored functorially into cofibrations followed by weak equivalences.

Before we can prove the generalization of Brun’s spectral sequence we need another remark:

Remark 4.10 Let A be a commutative S –algebra and let E be an S –ring spectrum that is a cell S –module. Then, by [13, Proposition III.4.1], $E \wedge_S A$ is a cell A –module. We can equip $E \wedge_S A$ with the structure of an A –ring spectrum. Its image under $\mathcal{D}_A \rightarrow \mathcal{D}_S$ is isomorphic to the S –ring spectrum $E \wedge_S^L A$. If A is a commutative R –algebra and B is an R –ring spectrum, we have

$$(E \wedge_S A) \wedge_R^L B \cong E \wedge_S^L (A \wedge_R^L B)$$

as S –ring spectra. The isomorphism is natural in E, A and B . If $R \rightarrow R'$ is a morphism of commutative S –algebras, B is an R' –ring spectrum and A is a commutative R' –algebra, the following diagram commutes in \mathcal{D}_S :

$$\begin{CD} (E \wedge_S A) \wedge_R^L B @>\cong>> E \wedge_S^L (A \wedge_R^L B) \\ @VVV @VVV \\ (E \wedge_S A) \wedge_{R'}^L B @>\cong>> E \wedge_S^L (A \wedge_{R'}^L B) \end{CD}$$

For an S –module E we write $E_*(-)$ for $\pi_*(E \wedge_S^L -)$.

Theorem 4.11 Let A be a cofibrant commutative S –algebra and let B be a connective cofibrant commutative A –algebra. Let E be an S –ring spectrum. Then there is a multiplicative spectral sequence of the form

$$E_{n,m}^2 = \mathrm{THH}_n(B; HE_m(B \wedge_A B)) \Rightarrow E_{n+m}(\mathrm{THH}(A; B)).$$

Proof Since B is a cofibrant commutative S –algebra, the map

$$\Gamma^S B \wedge_S \Gamma^S B \rightarrow B \wedge_S B$$

is a weak equivalence by [13, Theorems VII.6.5 and VII.6.7]. By [17, page 506] we therefore have a strongly convergent spectral sequence of the form

$$E_{n,m}^2 = \mathrm{Tor}_{n,m}^{S_*}(B_*, B_*) \Rightarrow \pi_{n+m}(B \wedge_S B).$$

This shows that $B \wedge_S B$ is connective. Since B^e is a cofibrant in $\mathcal{C}\mathcal{A}_S$, Theorem 3.13 yields a strongly convergent spectral sequence of the form

$$E_{n,m}^2 = \pi_n(H\pi_m(\Gamma^S E \wedge_S (B \wedge_A B)) \wedge_{B^e}^L B) \Rightarrow \pi_{n+m}((\Gamma^S E \wedge_S (B \wedge_A B)) \wedge_{B^e}^L B).$$

By Remark 4.10 and Proposition 3.17 the spectral sequence is multiplicative. We have

$$E_{n,m}^2 = \pi_n(HE_m(B \wedge_A B) \wedge_{B^e}^L B) = \mathrm{THH}_n(B; HE_m(B \wedge_A B)).$$

By Remark 4.10 and by Lemma 4.8 we have isomorphisms

$$\begin{aligned} \pi_*((\Gamma^S E \wedge_S (B \wedge_A B)) \wedge_{B^e}^L B) &\cong \pi_*(E \wedge_S^L ((B \wedge_A B) \wedge_{B^e}^L B)) \\ &\cong \pi_*(E \wedge_S^L \mathrm{THH}(A; B)) \end{aligned}$$

that are compatible with the multiplications. □

We want to find a more computable description of the E^2 -page in special cases. To see that different ring spectra are isomorphic, the following remark will be useful:

Remark 4.12 Let R be a connective commutative S -algebra. Let A and B be R -ring spectra with $\pi_*(A) = 0 = \pi_*(B)$ for $* \neq 0$. Suppose that there is an isomorphism $A_0 \cong B_0$ that is R_0 -linear and compatible with the unit and multiplication. Then, by using [17, page 507], one gets $A \cong B$ as R -ring spectra.

Now let A be a connective cofibrant commutative S -algebra. Then we have a map of commutative S -algebras $A \rightarrow H\pi_0(A)$ realizing the identity on π_0 . We consider $\mathrm{THH}(A; H\pi_0(A)) = H\pi_0(A) \wedge_{A^e}^L A$. By Remark 4.12 this does not depend on the choice of the representative of the homotopy class $A \rightarrow H\pi_0(A)$ in $\overline{h\mathcal{C}A}_S$. By using the commutative diagram (13) we also see that $\mathrm{THH}(A; H\pi_0(A))$ is isomorphic to $H\pi_0(A) \wedge_{A^e}^L A$ built via $A^e \rightarrow H\pi_0(A^e) \rightarrow H\pi_0(A)$.

Lemma 4.13 *In the situation of Theorem 4.11 we assume that $E_m(B \wedge_A B)$ is an \mathbb{F}_p -vector space for all m and that $\pi_0(B)/p\pi_0(B) = \mathbb{F}_p$ as rings. Then we have an isomorphism*

$$E_{n,m}^2 \cong E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \mathrm{THH}_n(B; H\mathbb{F}_p)$$

that is compatible with the multiplication. Here, the multiplication on the right side is given by $a \otimes b \cdot c \otimes d = (-1)^{|b||c|} ac \otimes bd$. The (B, B) -bimodule structure on $H\mathbb{F}_p$ is given by the morphism of commutative S -algebras $B \rightarrow H\pi_0(B) \rightarrow H\mathbb{F}_p$.

Proof We have

$$E_{n,m}^2 \cong \pi_n(HE_m(B \wedge_A B) \wedge_{B^e} \Gamma^{B^e} B).$$

The B^e -action on $HE_m(B \wedge_A B)$ is given by pulling back the $H\pi_0(B^e)$ -action along the map $B^e \rightarrow H\pi_0(B^e)$. The Künneth map

$$\pi_0(B) \otimes_{\mathbb{Z}} \pi_0(B) \rightarrow \pi_0(B \wedge_S B)$$

is a ring map, and it is an isomorphism because it is the edge homomorphism in the Tor spectral sequence

$$E_{n,m}^2 = \text{Tor}_{n,m}^{S^*}(B_*, B_*) \Rightarrow \pi_{n+m}(B \wedge_S B)$$

(see [17, page 506]) and because B is connective. The $\pi_0(B) \otimes \pi_0(B) = \pi_0(B \wedge_S B)$ -action on $E_m(B \wedge_A B)$ is given by pulling back the \mathbb{F}_p -action along

$$\pi_0(B) \otimes \pi_0(B) \rightarrow \pi_0(B) \rightarrow \pi_0(B)/p\pi_0(B) = \mathbb{F}_p.$$

This is because the two actions of $\pi_0(B)$ induced by the two inclusions

$$\pi_0(B) \rightarrow \pi_0(B) \otimes \pi_0(B)$$

have to factor over $\pi_0(B)/p\pi_0(B)$ -actions because $E_m(B \wedge_A B)$ is an \mathbb{F}_p -vector space, and because there is only one $\pi_0(B)/p\pi_0(B) = \mathbb{F}_p$ -action on a given abelian group. Because of the uniqueness of Eilenberg–Mac Lane spectra we can write

$$E_{n,m}^2 \cong \pi_n(HE_m(B \wedge_A B) \wedge_{B^e} \Gamma^{B^e} B),$$

where we realize $HE_m(B \wedge_A B)$ as an $H\mathbb{F}_p$ -module and equip it with a B^e -module structure by pullback along

$$B^e \rightarrow H\pi_0(B^e) \rightarrow H\pi_0(B) \rightarrow H\mathbb{F}_p.$$

We can also assume that $HE_m(B \wedge_A B)$ is a cell $H\mathbb{F}_p$ -module. We have

$$HE_m(B \wedge_A B) \wedge_{B^e} \Gamma^{B^e} B \cong HE_m(B \wedge_A B) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B).$$

With the Tor spectral sequence we get

$$\begin{aligned} E_{n,m}^2 &= E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B) \\ &= E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e}^L B). \end{aligned}$$

By Remark 4.12 this is isomorphic to

$$E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \text{THH}_n(B; H\mathbb{F}_p).$$

It remains to prove that the identification is compatible with the multiplication.

Our spectral sequence is given as an Atiyah–Hirzebruch spectral sequence with E^1 –page

$$E_{n,m}^1 = \pi_{n+m}(F \wedge_{B^e} (\Gamma^{B^e} B)^n / (\Gamma^{B^e} B)^{n-1}),$$

where $F = \Gamma^{B^e}(\Gamma^S E \wedge_S (B \wedge_A B))$. The multiplication on the E^1 –page is given by the maps

$$(16) \quad F \wedge_{B^e} F \rightarrow F$$

and

$$(17) \quad (\Gamma^{B^e} B)^n / (\Gamma^{B^e} B)^{n-1} \wedge_{B^e} (\Gamma^{B^e} B)^{n'} / (\Gamma^{B^e} B)^{n'-1} \rightarrow (\Gamma^{B^e} B)^{n+n'} / (\Gamma^{B^e} B)^{n+n'-1}.$$

Here, the map (17) comes from a cellular representative

$$(18) \quad \Gamma^{B^e} B \wedge_{B^e} \Gamma^{B^e} B \rightarrow \Gamma^{B^e} B$$

of the product $B \wedge_{B^e}^L B \rightarrow B$. One easily sees that the isomorphism between

$$E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \mathrm{THH}_n(B; H\mathbb{F}_p) \cong F_m \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B)$$

and the homology of the E^1 –page is given as follows: For

$$x \in F_m \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B)$$

choose a preimage y under

$$F_m \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e} (\Gamma^{B^e} B)^n) \twoheadrightarrow F_m \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B).$$

Then x is mapped to the homology class of the image of y under the composition of

$$F_m \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e} (\Gamma^{B^e} B)^n) \rightarrow F_m \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e} (\Gamma^{B^e} B)^n / (\Gamma^{B^e} B)^{n-1})$$

and

$$\begin{aligned} F_m \otimes_{\mathbb{F}_p} \pi_n(H\mathbb{F}_p \wedge_{B^e} (\Gamma^{B^e} B)^n / (\Gamma^{B^e} B)^{n-1}) & \\ & \cong F_m \otimes_{\mathbb{F}_p} \mathbb{F}_p \otimes_{(B^e)_0} \pi_n((\Gamma^{B^e} B)^n / (\Gamma^{B^e} B)^{n-1}) \\ & \cong F_m \otimes_{(B^e)_0} \pi_n((\Gamma^{B^e} B)^n / (\Gamma^{B^e} B)^{n-1}) \\ & \cong \pi_{n+m}(F \wedge_{B^e} (\Gamma^{B^e} B)^n / (\Gamma^{B^e} B)^{n-1}). \end{aligned}$$

Every intermediate graded abelian group appearing in this identification can be endowed with a multiplication by using the maps $H\mathbb{F}_p \wedge_{B^e} H\mathbb{F}_p \rightarrow H\mathbb{F}_p$, (16), (17), (18) and

$$(\Gamma^{B^e} B)^n \wedge_{B^e} (\Gamma^{B^e} B)^{n'} \rightarrow (\Gamma^{B^e} B)^{n+n'}.$$

For multiplications on tensor products we use the sign convention given in the statement of the lemma. Then every map in the identification is compatible with the multiplication. This proves the lemma. \square

Lemma 4.14 *Suppose that we have a commutative diagram in $\mathcal{C}\mathcal{A}_S$*

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{\quad} & D \end{array}$$

where A and C are cofibrant commutative S -algebras, B and D are connective and where the horizontal maps are cofibrations in $\mathcal{C}\mathcal{A}_S$. Furthermore, suppose that we have a morphism of S -ring spectra $E \rightarrow F$. Then we get a morphism of Brun spectral sequences converging to the map $E_* \text{THH}(A; B) \rightarrow F_* \text{THH}(C; D)$. If additionally $E_m(B \wedge_A B)$ and $F_m(D \wedge_C D)$ are \mathbb{F}_p -vector spaces and $\pi_0(B)/p\pi_0(B) = \mathbb{F}_p = \pi_0(D)/p\pi_0(D)$ as rings, the map on the E^2 -pages identifies with the map

$$E_*(B \wedge_A B) \otimes_{\mathbb{F}_p} \text{THH}_*(B; H\mathbb{F}_p) \rightarrow F_*(D \wedge_C D) \otimes_{\mathbb{F}_p} \text{THH}_*(D; H\mathbb{F}_p)$$

that is given by the obvious map on the first tensor factor and that is induced by the map

$$\text{THH}(B; H\mathbb{F}_p) \xrightarrow{\cong} H\mathbb{F}_p \wedge_{B^e}^L B \rightarrow H\mathbb{F}_p \wedge_{B^e}^L D \rightarrow H\mathbb{F}_p \wedge_{D^e}^L D$$

on the second tensor factor. Here, the B^e -module action on $H\mathbb{F}_p$ in the second smash product is given by

$$B^e \rightarrow D^e \rightarrow D \rightarrow H\pi_0(D) \rightarrow H\mathbb{F}_p$$

and the isomorphism is defined using Remark 4.12.

Proof By Remarks 4.7, 4.9 and 4.10 the morphism $E_* \text{THH}(A; B) \rightarrow F_* \text{THH}(C; D)$ corresponds to the composition of

$$(\Gamma^S E \wedge_S (B \wedge_A B)) \wedge_{B^e}^L B \rightarrow (\Gamma^S F \wedge_S (D \wedge_C D)) \wedge_{B^e}^L D$$

and

$$(\Gamma^S F \wedge_S (D \wedge_C D)) \wedge_{B^e}^L D \rightarrow (\Gamma^S F \wedge_S (D \wedge_C D)) \wedge_{D^e}^L D.$$

By Lemmas 3.14 and 3.16, we have Atiyah–Hirzebruch spectral sequences converging to these maps.

Now, assume that the additional conditions are satisfied. We identify the map on E^2 -pages. Recall from the proof of Lemma 4.13 that we identified

$$HE_m(B \wedge_A B) \wedge_{B^e} \Gamma^{B^e} B$$

with

$$HE_m(B \wedge_A B) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B,$$

where in the second smash product $HE_m(B \wedge_A B)$ is realized as a cell $H\mathbb{F}_p$ -module and the B^e -action on $H\mathbb{F}_p$ is given by

$$(19) \quad B^e \rightarrow H\pi_0(B^e) \rightarrow H\pi_0(B) \rightarrow H\mathbb{F}_p.$$

The analogous statement holds for E, A and B replaced by F, C and D . We have a cell approximation $\Gamma^1 \xrightarrow{\sim} H\mathbb{F}_p$ of $H\mathbb{F}_p$ as $H\mathbb{F}_p \wedge_S B^e$ -module, where the B^e -module action is given by (19). We also have a cell approximation $\Gamma^2 \xrightarrow{\sim} H\mathbb{F}_p$ of $H\mathbb{F}_p$ as an $H\mathbb{F}_p \wedge_S B^e$ -module, where the B^e -module action is given by

$$B^e \rightarrow D^e \rightarrow H\pi_0(D^e) \rightarrow H\pi_0(D) \rightarrow H\mathbb{F}_p.$$

By fact (12), $H\mathbb{F}_p$ equipped with the first $H\mathbb{F}_p \wedge_S B^e$ -module structure is isomorphic in $\mathcal{D}_{H\mathbb{F}_p \wedge_S B^e}$ to $H\mathbb{F}_p$ equipped with the second module structure. The isomorphism is represented by a map $\Gamma^1 \rightarrow \Gamma^2$. By (12) we have a map between the cell $H\mathbb{F}_p$ -modules $HE_m(B \wedge_A B)$ and $HF_m(D \wedge_C D)$ inducing the obvious map in homotopy. The maps $H\mathbb{F}_p \leftarrow \Gamma^1 \rightarrow \Gamma^2 \rightarrow H\mathbb{F}_p$, $HE_m(B \wedge_A B) \rightarrow HF_m(D \wedge_C D)$ and $\Gamma^{B^e} B \rightarrow \Gamma^{B^e} D$ induce a map $(f_1^m)_*$,

$$\begin{aligned} \pi_*(HE_m(B \wedge_A B) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B) \\ \rightarrow \pi_*(HF_m(D \wedge_C D) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} D). \end{aligned}$$

We also have a map $(f_2^m)_*$,

$$\begin{aligned} \pi_*(HF_m(D \wedge_C D) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} D) \\ \rightarrow \pi_*(HF_m(D \wedge_C D) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge_{D^e} \Gamma^{D^e} D), \end{aligned}$$

induced by a map $\Gamma^{B^e} D \rightarrow \Gamma^{D^e} D$ in \mathcal{M}_{B^e} lifting the identity up to homotopy. Using (12) it follows that the map on E^2 -pages is given by the maps $(f_2^m)_* \circ (f_1^m)_*$. In the proof of Lemma 4.13 we also did the identifications

$$\begin{aligned} \pi_*(HE_m(B \wedge_A B) \wedge_{H\mathbb{F}_p} H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B) &\cong E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge_{B^e} \Gamma^{B^e} B) \\ &\cong E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \pi_*(H\mathbb{F}_p \wedge_{B^e}^L B) \\ &\cong E_m(B \wedge_A B) \otimes_{\mathbb{F}_p} \text{THH}_*(B; H\mathbb{F}_p) \end{aligned}$$

and analogously for F , C and D . Again by using (12), one sees that under these isomorphisms the maps $(f_2^m) \circ (f_1^m)_*$ correspond to the map in the statement of the lemma. \square

Remark 4.15 One easily sees that the Brun spectral sequence is functorial with respect to morphisms of S -ring spectra and the vertical maps in diagrams as in Lemma 4.14.

5 The topological Hochschild homology of ku

In [2] Ausoni computed the mod (p, v_1) homotopy of the topological Hochschild homology of p -completed connective complex K -theory for an odd prime p . Ausoni used the Bökstedt spectral sequence. In this chapter we present a shorter computation for $p \geq 5$ using the Brun spectral sequence.

We take for the ring spectrum E in the Brun spectral sequence the mod p Moore spectrum and the mod (p, v_1) Toda-Smith complex. Recall the following:

Remark 5.1 Let p be a prime with $p \geq 5$. Let $V(0)$ be the mod p Moore spectrum and let $V(1)$ be the mod (p, v_1) Toda-Smith complex. We can suppose that $V(0)$ and $V(1)$ are cell S -modules. Because of $p \geq 5$ we have the following properties (see [26; 27; 25]):

- The S -modules $V(0)$ and $V(1)$ are associative and commutative S -ring spectra.
- We have distinguished triangles in \mathcal{D}_S

$$(20) \quad S \xrightarrow{p\text{-id}} S \rightarrow V(0) \rightarrow \Sigma S$$

and

$$\Sigma^{2p-2}V(0) \rightarrow V(0) \rightarrow V(1) \rightarrow \Sigma^{2p-1}V(0).$$

The maps $S \rightarrow V(0)$ and $V(0) \rightarrow V(1)$ are maps of S -ring spectra.

Let p be a prime with $p \geq 5$. Let ku be p -completed connective complex K -theory. Following Ausoni we take as a commutative S -algebra model for ku $K(\bigcup_l \mathbb{F}_{l^{p^i(p-1)}})_p$, the p -completion of the algebraic K -theory of $\bigcup_{i \geq 0} \mathbb{F}_{l^{p^i(p-1)}}$, where l is a prime that generates $(\mathbb{Z}/p^2)^*$. By [19, Example VI.5.2; 21; 22; 13, Corollary II.3.6], algebraic K -theory can be realized as a functor from commutative rings to commutative S -algebras. By p -completion we mean Bousfield localization with respect to $V(0)$ as defined in [13, Section VIII.1]. By the proofs of [13, Lemmas VII.5.8 and VII.5.2 and

Theorem VIII.2.2] the p -completion of a commutative S -algebra can be constructed as a commutative S -algebra in such a way that we get a functor $(-)_p: \mathcal{C}\mathcal{A}_S \rightarrow \mathcal{C}\mathcal{A}_S$ with values in cofibrant commutative S -algebras.

We have

$$\pi_*(\mathbf{ku}) = \mathbb{Z}_p[u]$$

as rings, where $|u| = 2$. Using the cofiber sequence (20) one gets

$$V(0)_* \mathbf{ku} = P(u),$$

where $P(-)$ denotes the polynomial algebra over \mathbb{F}_p . By [2] we have

$$V(1)_* \mathbf{ku} = P_{p-1}(u).$$

Here, $P_{p-1}(-) = P(u)/(u^{p-1})$ is the truncated polynomial algebra over \mathbb{F}_p . The map $V(0)_* \mathbf{ku} \rightarrow V(1)_* \mathbf{ku}$ is the quotient map. By [2, Theorem 2.5] we have

$$H\mathbb{F}_p \wedge_* \mathbf{ku} = P_{p-1}(u) \otimes P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_2, \dots),$$

where the tensor product is taken over \mathbb{F}_p and where $E(-)$ denotes the exterior algebra over \mathbb{F}_p . We have $|\bar{\xi}_i| = 2p^i - 2$ and $|\bar{\tau}_i| = 2p^i - 1$, and u is the image of $u \in \pi_2(\mathbf{ku})$ under the Hurewicz map

$$\pi_*(\mathbf{ku}) \xrightarrow{\cong} \pi_*(S \wedge_S^L \mathbf{ku}) \rightarrow \pi_*(H\mathbb{F}_p \wedge_S^L \mathbf{ku}).$$

Recall that $\ell := \mathbf{K}(\bigcup \mathbb{F}_{1/p^i})_p$ is a commutative S -algebra model for the p -completed connective Adams summand [23, Proposition 9.2; 5, Section 2]. The map $\pi_*(\ell) \rightarrow \pi_*(\mathbf{ku})$ induced by the inclusion of fields identifies $\pi_*(\ell)$ with the subring $\mathbb{Z}_p[u^{p-1}]$ of $\pi_*(\mathbf{ku})$. We have $V(0)_* \ell = P(u^{p-1})$ and $V(1)_* \ell = \mathbb{F}_p$.

In order to calculate $V(1)_* \text{THH}(\mathbf{ku})$ we proceed as follows:

We factor the map $\mathbf{ku} \rightarrow H\mathbb{Z}_p$ in the category of commutative S -algebras as

$$\mathbf{ku} \twoheadrightarrow \hat{H}\mathbb{Z}_p \xrightarrow{\sim} H\mathbb{Z}_p.$$

To simplify the notation we will write $H\mathbb{Z}_p$ for $\hat{H}\mathbb{Z}_p$ most of the time.

We have a Brun spectral sequence

$$(21) \quad E_{*,*}^2 = \text{THH}_*(H\mathbb{Z}_p; HV(0)_*(H\mathbb{Z}_p \wedge_{\mathbf{ku}} H\mathbb{Z}_p)) \Rightarrow V(0)_* \text{THH}(\mathbf{ku}; H\mathbb{Z}_p)$$

and by Lemma 4.13 the E^2 -page can be identified with

$$E_{*,*}^2 = V(0)_*(H\mathbb{Z}_p \wedge_{\mathbf{ku}} H\mathbb{Z}_p) \otimes \text{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p).$$

We compute this spectral sequence in Section 5.1. In Section 5.2 we compute the Brun spectral sequence

$$(22) \quad E_{*,*}^2 = \mathrm{THH}_*(\mathrm{ku}; H(V(1)_* \mathrm{ku})) \Rightarrow V(1)_* \mathrm{THH}(\mathrm{ku}).$$

Again by Lemma 4.13, the E^2 -page can be identified with

$$E_{*,*}^2 = V(1)_* \mathrm{ku} \otimes \mathrm{THH}_*(\mathrm{ku}; H\mathbb{F}_p) \cong P_{p-1}(u) \otimes \mathrm{THH}_*(\mathrm{ku}; H\mathbb{F}_p).$$

Note that by Remark 4.10 we have an isomorphism of S -ring spectra

$$V(0) \wedge_S^L \mathrm{THH}(\mathrm{ku}; H\mathbb{Z}_p) = V(0) \wedge_S^L (H\mathbb{Z}_p \wedge_{\mathrm{ku}}^L \mathrm{ku}) \cong (V(0) \wedge_S H\mathbb{Z}_p) \wedge_{\mathrm{ku}}^L \mathrm{ku}.$$

Since the $H\mathbb{Z}_p$ -ring spectra $V(0) \wedge_S H\mathbb{Z}_p$ and $H\mathbb{F}_p$ are isomorphic by Remark 4.12, this is isomorphic to $\mathrm{THH}(\mathrm{ku}; H\mathbb{F}_p)$. Thus, the abutment of the spectral sequence (21) computes the input of the spectral sequence (22).

Notation 5.2 An infinite cycle in a spectral sequence is a class b such that $d^s(b) = 0$ for all s . A permanent cycle is an infinite cycle that is not in the image of d^s for any s . We write $b \doteq c$ if b and c are equal up to multiplication by a unit in \mathbb{F}_p .

5.1 The mod p homotopy of $\mathrm{THH}(\mathrm{ku}; H\mathbb{Z}_p)$

In this subsection we calculate the \mathbb{F}_p -algebra $V(0)_* \mathrm{THH}_*(\mathrm{ku}, H\mathbb{Z}_p)$ using the spectral sequence (21). By Bökstedt’s computations [8], we have

$$\mathrm{THH}_*(H\mathbb{Z}_p; H\mathbb{F}_p) = E(\lambda_1) \otimes P(\mu_1)$$

with $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

Lemma 5.3 We have $V(0)_*(H\mathbb{Z}_p \wedge_{\mathrm{ku}} H\mathbb{Z}_p) = E(\sigma u)$ with $|\sigma u| = 3$.

Proof Since $H\mathbb{Z}_p$ is a cofibrant commutative ku -algebra, we have

$$V(0) \wedge_S^L (H\mathbb{Z}_p \wedge_{\mathrm{ku}} H\mathbb{Z}_p) \cong V(0) \wedge_S^L (H\mathbb{Z}_p \wedge_{\mathrm{ku}}^L H\mathbb{Z}_p)$$

as S -ring spectra by Lemma 2.2. Using Remark 4.10 one gets that this is isomorphic to the S -ring spectrum $H\mathbb{F}_p \wedge_{\mathrm{ku}}^L H\mathbb{Z}_p$.

By [17] we have a strongly convergent spectral sequence of the form

$$E_{n,m}^2 = \mathrm{Tor}_{n,m}^{\mathrm{ku}_*}(\mathbb{F}_p, \mathbb{Z}_p) \Rightarrow \pi_{n+m}(H\mathbb{F}_p \wedge_{\mathrm{ku}}^L H\mathbb{Z}_p).$$

We have the free resolution of \mathbb{Z}_p as a ku_* -module

$$0 \rightarrow \Sigma^2 ku_* \xrightarrow{u} ku_* \rightarrow \mathbb{Z}_p \rightarrow 0.$$

Thus, the E^2 -page is \mathbb{F}_p in the bidegrees $(0, 0)$ and $(1, 2)$ and zero in all other bidegrees. There cannot be any differentials. Since there is only one graded-commutative \mathbb{F}_p -algebra that has $\mathbb{F}_p \oplus \Sigma^3 \mathbb{F}_p$ as an underlying \mathbb{F}_p -vector space, this proves the lemma. \square

Thus, the E^2 -page of the spectral sequence (21) is

$$E(\sigma u) \otimes E(\lambda_1) \otimes P(\mu_1),$$

where σu , λ_1 and μ_1 have the bidegrees $(0, 3)$, $(2p - 1, 0)$ and $(2p, 0)$.

Lemma 5.4 *The spectral sequence (21) collapses at the E^2 -page.*

Proof The class σu is an infinite cycle, because it lies in column zero. The class λ_1 is an infinite cycle, because the E^2 -page is zero in total degree $2p - 2$. The class μ_1 is also an infinite cycle: In total degree $2p - 1$ the E^2 -page is given by $\mathbb{F}_p\{\lambda_1\}$. The differentials of μ_1 lies in columns $\leq 2p - 2$ and λ_1 lies in column $2p - 1$. Thus, the differentials of μ_1 cannot hit λ_1 . \square

Lemma 5.5 *We have*

$$V(0)_* \text{THH}(ku; H\mathbb{Z}_p) = E(\sigma u, \lambda_1) \otimes P(\mu_1)$$

with $|\sigma u| = 3$, $|\lambda_1| = 2p - 1$ and $|\mu_1| = 2p$.

Proof Since the $E^\infty = E^2$ -page is a free graded-commutative \mathbb{F}_p -algebra, there cannot be any multiplicative extensions. \square

5.2 The $V(1)$ -homotopy of $\text{THH}(ku)$

In this subsection we calculate $V(1)_* \text{THH}(ku)$ via the spectral sequence (22). By Lemma 5.5 we have

$$E_{*,*}^2 = P_{p-1}(u) \otimes E(\sigma u, \lambda_1) \otimes P(\mu_1),$$

where u , σu , λ_1 and μ_1 have bidegrees $(0, 2)$, $(3, 0)$, $(2p - 1, 0)$ and $(2p, 0)$, respectively.

Lemma 5.6 *There are nontrivial elements*

$$u' \in V(1)_2 \text{ THH}(\text{ku}) \quad \text{and} \quad \sigma u' \in V(1)_3 \text{ THH}(\text{ku})$$

satisfying

$$(u')^{p-2} \sigma u' = 0.$$

Proof To see this, we pass to homology. We have $u^{p-1} = 0$ in $(H\mathbb{F}_p)_* \text{ku}$. Recall from [1, page 1238, Proposition 5.10] that we have a map $\text{ku} \wedge S^1 \rightarrow \text{THH}(\text{ku})$ such that the induced map in homology $\sigma_*: (H\mathbb{F}_p)_* \text{ku} \rightarrow (H\mathbb{F}_p)_{*+1} \text{THH}(\text{ku})$ is a derivation. We apply σ_* to the equation $u^{p-1} = 0$ and get

$$u^{p-2} \sigma_*(u) = 0$$

in $(H\mathbb{F}_p)_* \text{THH}(\text{ku})$, where u is the image of u under the unit $(H\mathbb{F}_p)_* \text{ku} \rightarrow (H\mathbb{F}_p)_* \text{THH}(\text{ku})$. Since the unit has a left inverse, $u \in (H\mathbb{F}_p)_2 \text{THH}(\text{ku})$ is not zero. We claim that the class $\sigma_*(u)$ is also not zero. To see this we consider the Bökstedt spectral sequence [1, Section 4]

$$E_{*,*}^2(\text{ku}) = \mathbb{H}_{*,*}^{\mathbb{F}_p}((H\mathbb{F}_p)_* \text{ku}) \Rightarrow (H\mathbb{F}_p)_* \text{THH}(\text{ku}).$$

Here, $\mathbb{H}_{*,*}^{\mathbb{F}_p}((H\mathbb{F}_p)_* \text{ku})$ denotes the Hochschild homology of $(H\mathbb{F}_p)_* \text{ku}$ over the ground ring \mathbb{F}_p . By [23, Proposition 3.2] the class $\sigma_* u$ is a representative of the class $\sigma(u) \in E_{1,2}^\infty(\text{ku})$ that is given in the Hochschild complex by $1 \otimes u$. The class $\sigma(u) \in E_{1,2}^\infty(\text{ku})$ is not zero, because by standard computations of Hochschild homology (see [2, Propositions 3.2 and 3.3]) the bigraded \mathbb{F}_p -vector space $E_{*,*}^2(\text{ku})$ does not contain any class of total degree 4 in a filtration degree ≥ 3 , and therefore $\sigma(u) \in E_{1,2}^2(\text{ku})$ cannot be hit by any differential.

Recall that the mod p homology of an S -module admits a natural comodule structure over the dual Steenrod algebra $A_* = (H\mathbb{F}_p)_* H\mathbb{F}_p$ [4, Theorem 1.1]. By [33, Corollary 17.4] the image of the Hurewicz map is contained in the \mathbb{F}_p -subspace of A_* -comodule primitives. Thus, the class $u \in (H\mathbb{F}_p)_* \text{ku}$ as well as its image under the unit $(H\mathbb{F}_p)_* \text{ku} \rightarrow (H\mathbb{F}_p)_* \text{THH}(\text{ku})$ are A_* -comodule primitives. Since the suspension isomorphism $(H\mathbb{F}_p)_*(-) \rightarrow (H\mathbb{F}_p)_{*+1}(- \wedge S^1)$ is compatible with the comodule action, it follows that the class $\sigma_*(u)$ is also primitive.

We consider the diagram

$$(H\mathbb{F}_p)_* \text{THH}(\text{ku}) \xrightarrow{f} (H\mathbb{F}_p)_*(V(1) \wedge_S^L \text{THH}(\text{ku})) \xleftarrow{h} V(1)_* \text{THH}(\text{ku}),$$

where the right map is the Hurewicz morphism. The map f is injective because composed with the Künneth isomorphism

$$(H\mathbb{F}_p)_*(V(1) \wedge_S^L \text{THH}(\text{ku})) \cong (H\mathbb{F}_p)_*V(1) \otimes (H\mathbb{F}_p)_* \text{THH}(\text{ku})$$

it is the inclusion

$$(H\mathbb{F}_p)_* \text{THH}(\text{ku}) \rightarrow (H\mathbb{F}_p)_*V(1) \otimes (H\mathbb{F}_p)_* \text{THH}(\text{ku}), \quad a \mapsto 1 \otimes a.$$

The S -module $V(1) \wedge_S^L \text{THH}(\text{ku})$ is a module (in \mathcal{D}_S) over the S -ring spectrum $V(1) \wedge_S^L \ell \cong H\mathbb{F}_p$. Therefore, it is isomorphic in \mathcal{D}_S to a coproduct of S -modules of the form $H\mathbb{F}_p \wedge_S^L S_S^n$ and the Hurewicz morphism h induces an isomorphism onto the A_* -comodule primitives of $(H\mathbb{F}_p)_*(V(1) \wedge_S^L \text{THH}(\text{ku}))$. We obtain two nontrivial elements u' and $\sigma_*(u)'$ in $V(1)_* \text{THH}(\text{ku})$ satisfying the equation

$$(u')^{p-2} \sigma_*(u)' = 0. \quad \square$$

Lemma 5.7 *The classes u , σu and λ_1 in the Brun spectral sequence (22) are infinite cycles. We have $d^s(\mu_1) = 0$ for $2 \leq s \leq 2p - 4$ and there is a differential*

$$d^{2p-3}(\mu_1) \doteq u^{p-2} \sigma u.$$

The spectral sequence collapses at the E^{2p-2} -page.

Proof The class u is an infinite cycle because it lies in column zero. It is also a permanent cycle because the unit

$$V(1)_* \text{ku} \rightarrow V(1)_* \text{THH}(\text{ku})$$

is injective and so $V(1)_2 \text{THH}(\text{ku})$ is not zero. This implies that σu is an infinite cycle, because the only possible nontrivial differential on σu is $d^3(\sigma u) \doteq u$. The class λ_1 is an infinite cycle because the E^2 -page is zero in total degree $2p - 2$.

There is a differential $d^{2p-3}(\mu_1) \doteq u^{p-2} \sigma u$: By Lemma 5.6 we have two nontrivial elements u' and $\sigma_*(u)'$ in $V(1)_* \text{THH}(\text{ku})$ satisfying the equation

$$(u')^{p-2} \sigma_*(u)' = 0.$$

Since the E^∞ -page of the Brun spectral sequence is $\mathbb{F}_p\{u\}$ in total degree two and $\mathbb{F}_p\{\sigma u\}$ in total degree three, up to a unit, u' has to be a representative of u and $\sigma u'$ has to be a representative of σu . Therefore, we have $u^{p-2} \sigma u = 0$ in $E_{*,*}^\infty$ and the class $u^{p-2} \sigma u \in E_{*,*}^2$ has to be hit by a differential. The only class in total degree $2p$ is μ_1 . Thus, we have $d^s(\mu_1) = 0$ for $2 \leq s \leq 2p - 4$ and $d^{2p-3}(\mu_1) \doteq u^{p-2} \sigma u$.

Since we have $E_{*,m}^2 = 0$ if $m > 2p - 4$ or $m < 0$, the spectral sequence has to collapse at the E^{2p-2} -page. □

Lemma 5.8 We have $E_{*,*}^\infty = E_{*,*}^{2p-2} = \Omega_{*,*}^\infty \otimes E(\lambda_1)$, where $\Omega_{*,*}^\infty$ is the bigraded commutative \mathbb{F}_p -algebra with generators

$$u, \mu_2, a_0, \dots, a_{p-1}, b_1, \dots, b_{p-1}$$

in bidegrees $|u| = (0, 2)$, $|\mu_2| = (2p^2, 0)$, $|a_i| = (2pi + 3, 0)$ and $|b_i| = (2pi, 2)$ and relations

- (23) $u^{p-1} = 0,$
- (24) $u^{p-2}a_i = 0$ for $0 \leq i \leq p - 2,$
- (25) $u^{p-2}b_i = 0$ for $1 \leq i \leq p - 1,$
- (26) $b_i b_j = ub_{i+j}$ for $i + j \leq p - 1,$
- (27) $a_i b_j = ua_{i+j}$ for $i + j \leq p - 1,$
- (28) $b_i b_j = ub_{i+j-p}\mu_2$ for $i + j \geq p,$
- (29) $a_i b_j = ua_{i+j-p}\mu_2$ for $i + j \geq p,$
- (30) $a_i a_j = 0$ for $0 \leq i, j \leq p - 1.$

Here, we use the convention $b_0 = u$. By bigraded commutative \mathbb{F}_p -algebra we mean a bigraded \mathbb{F}_p -algebra which is commutative in the following sense: if a and b are elements of total degree n and m , we have $ab = (-1)^{nm}ba$.

Proof We have $E_{*,*}^{2p-2} = H_*((\Omega_{*,*}, d')) \otimes E(\lambda_1)$, where

$$\Omega_{*,*} = P_{p-1}(u) \otimes E(\sigma u) \otimes P(\mu_1)$$

and d' is the restriction of the differential of E^{2p-3} . Computing homology, we find that, as an \mathbb{F}_p -vector space,

$$(31) \quad H_*((\Omega_{*,*}, d')) = \bigoplus_{n \geq 0} \mathbb{F}_p\{\mu_1^{pn}\} \oplus \bigoplus_{\substack{n \geq 0 \\ i=1, \dots, p-2}} \mathbb{F}_p\{u^i \mu_1^n\} \\ \oplus \bigoplus_{\substack{n \geq 0 \\ i=0, \dots, p-3}} \mathbb{F}_p\{u^i \sigma u \mu_1^n\} \oplus \bigoplus_{n \geq 0} \mathbb{F}_p\{u^{p-2} \sigma u \mu_1^{pn+p-1}\}.$$

We define classes in $H_*((\Omega_{*,*}, d'))$ by $\mu_2 := \mu_1^p$, $a_i := \sigma u \mu_1^i$ for $i = 0, \dots, p - 1$ and $b_i := u \mu_1^i$ for $i = 1, \dots, p - 1$. Obviously, these classes satisfy the relations (23)–(30).

Note that $u^{p-2}\sigma u\mu_1^{p-1}$ is not in the image of d' , so that $u^{p-2}a_i = 0$ only holds for $i = 0, \dots, p-2$. We get a map of bigraded \mathbb{F}_p -algebras

$$\Omega_{*,*}^\infty \xrightarrow{f} H_*((\Omega_{*,*}, d')).$$

Using the relations (23)–(30) one easily sees that f maps a generating set bijectively onto the basis of $H_*((\Omega_{*,*}, d'))$ given in (31). It follows that f is an isomorphism. \square

Theorem 5.9 *We have*

$$V(1)_* \text{THH}(ku) = E(\lambda_1) \otimes \Omega_{*,*}^\infty,$$

where we consider $\Omega_{*,*}^\infty$ as a graded \mathbb{F}_p -algebra via the total grading.

Proof We use a method from [2]: Recall that we have chosen $K(\bigcup_{l^{p^i(p-1)}} \mathbb{F}_l)_{p-1}$ as a model for ku , where l is a prime that generates $(\mathbb{Z}/p^2)^*$. The Galois group of $\bigcup_{l^{p^i(p-1)}} \mathbb{F}_l$ over \mathbb{F}_l is given by $\mathbb{Z}_p \times \mathbb{Z}/p-1$. Let δ be a generator of the subgroup $\mathbb{Z}/p-1$. By Lemma 4.14 the diagram

$$\begin{array}{ccc} ku & \xrightarrow{\text{id}} & ku \\ \delta \downarrow & & \downarrow \delta \\ ku & \xrightarrow{\text{id}} & ku \end{array}$$

induces a morphism of Brun spectral sequences $\delta_{*,*}^*$ converging to

$$\delta_*: V(1)_* \text{THH}(ku) \rightarrow V(1)_* \text{THH}(ku).$$

By linear algebra every finite-dimensional vector space over \mathbb{F}_p admitting an automorphism f with $f^{p-1} = \text{id}$ splits into eigenspaces with respect to f . We get that the Brun spectral sequence splits into a direct sum of $p-1$ spectral sequences (corresponding to the $p-1$ possible eigenvalues). Furthermore, $V(1)_* \text{THH}(ku)$ splits into eigenspaces and the subspectral sequence corresponding to an eigenvalue β converges to the eigenspace corresponding to β .

For the class $u \in V(0)_* ku$ we have $\delta_*(u) = \alpha u$ for a generator α of \mathbb{F}_p^* [2, page 1267]. It follows that the same is true for the classes $u \in V(1)_* ku$ and $u \in (H\mathbb{F}_p)_* ku$. We say that a class has δ -weight $i \in \mathbb{Z}/p-1$ if it lies in the eigenspace corresponding to α^i .

The class $\sigma u \in E_{3,0}^2$ has δ -weight 1, because $u \in (H\mathbb{F}_p)_* ku$ has δ -weight 1 and because the map $\sigma_*: (H\mathbb{F}_p)_* ku \rightarrow (H\mathbb{F}_p)_{*+1} \text{THH}(ku)$ is natural. We claim that

the classes $\lambda_1, \mu_1 \in E_{*,*}^2$ have δ -weight 0: Note that $\bigcup \mathbb{F}_{1p^i}$ is the subfield of $\bigcup \mathbb{F}_{1p^i(p-1)}$ fixed under δ . It follows that $\delta: \mathbf{ku} \rightarrow \mathbf{ku}$ restricts to the identity map on the Adams summand ℓ . We consider the ℓ^e -module structure on $H\mathbb{F}_p$ given by the morphism of commutative S -algebras

$$\ell \rightarrow \mathbf{ku} \rightarrow H\mathbb{Z}_p \rightarrow H\mathbb{F}_p.$$

We then have commutative diagram in \mathcal{D}_S

$$\begin{array}{ccc} H\mathbb{F}_p \wedge_{\mathbf{ku}^e}^L \mathbf{ku} & \xrightarrow{\cong} & H\mathbb{F}_p \wedge_{\mathbf{ku}^e}^L \mathbf{ku} \\ \uparrow & & \uparrow \\ H\mathbb{F}_p \wedge_{\ell^e}^L \ell & \xrightarrow{\text{id}} & H\mathbb{F}_p \wedge_{\ell^e}^L \ell \end{array}$$

where the upper horizontal map is the one inducing $\delta_{*,*}^2$ on the second tensor factor in

$$E_{*,*}^2 = V(1)_* \mathbf{ku} \otimes \text{THH}_*(\mathbf{ku}; H\mathbb{F}_p).$$

It thus suffices to show that λ_1 and μ_1^p are in the image of $\pi_*(H\mathbb{F}_p \wedge_{\ell^e}^L \ell) \rightarrow \pi_*(H\mathbb{F}_p \wedge_{\mathbf{ku}^e}^L \mathbf{ku})$. The diagram

$$\begin{array}{ccc} \mathbf{ku} & \longrightarrow & H\mathbb{Z}_p \\ \uparrow & & \uparrow \text{id} \\ \ell & \longrightarrow & H\mathbb{Z}_p \end{array}$$

factors as

$$\begin{array}{ccccc} \mathbf{ku} & \twoheadrightarrow & \widehat{H}\mathbb{Z}_p & \xrightarrow{\sim} & H\mathbb{Z}_p \\ \uparrow & & \uparrow & & \uparrow \text{id} \\ \ell & \twoheadrightarrow & \widetilde{H}\mathbb{Z}_p & \xrightarrow{\sim} & H\mathbb{Z}_p \end{array}$$

The map $\pi_*(H\mathbb{F}_p \wedge_{\ell^e}^L \ell) \rightarrow \pi_*(H\mathbb{F}_p \wedge_{\mathbf{ku}^e}^L \mathbf{ku})$ identifies with

$$V(0)_* \text{THH}(\ell; \widetilde{H}\mathbb{Z}_p) \rightarrow V(0)_* \text{THH}(\mathbf{ku}; \widehat{H}\mathbb{Z}_p)$$

and we have a morphism of Brun spectral sequences converging to this map. The E^2 -page of the first Brun spectral sequence is

$$E_{*,*}^2 = E(\sigma v_1) \otimes E(\lambda'_1) \otimes P(\mu'_1),$$

where $|\sigma v_1| = (0, 2p - 1)$, $|\lambda'_1| = (2p - 1, 0)$ and $|\mu'_1| = (2p, 0)$. By [13, Theorem X.2.4, Proposition IX.2.8 and Theorems VII.6.5 and VII.6.7], commutative diagrams of cofibrant commutative S -algebras, where the vertical maps are weak equivalences, induces a weak equivalence in $\text{THH}(-; -)$. Therefore, the map of spectral sequences

has to send λ'_1 to λ_1 and μ'_1 to μ_1 up to a unit. Since λ'_1 and μ'^p_1 have to survive to the E^∞ -page and since $V(0)_{2p^2} \text{THH}(\text{ku}, \widehat{H}\mathbb{Z}_p)$ and $V(0)_{2p-1} \text{THH}(\text{ku}, \widehat{H}\mathbb{Z}_p)$ are one-dimensional this proves that λ_1 and μ^p_1 are in the image of $\pi_*(H\mathbb{F}_p \wedge^L_{\ell} \ell) \rightarrow \pi_*(H\mathbb{F}_p \wedge^L_{\text{ku}^e} \text{ku})$.

We get that the classes u, λ_1, μ_2, a_i for $i = 0, \dots, p-1$ and b_i for $i = 1, \dots, p-1$ in the E^∞ -page of the Brun spectral sequence converging to $V(1)_* \text{THH}(\text{ku})$ have the δ -weights 1, 0, 0, 1 and 1, respectively. One easily checks that these classes have unique representatives in $V(1)_* \text{THH}(\text{ku})$ with the same δ -weight. We use the same names for the representatives.

The relations $u^{p-1} = 0, u^{p-2}a_i = 0$ for $i = 0, \dots, p-2, u^{p-2}b_i = 0$ for $i = 1, \dots, p-1, b_i b_j = ub_{i+j}$ for $i + j \leq p-1$ and $b_i b_j = ub_{i+j-p}\mu_2$ for $i + j \geq p$ hold in $V(1)_* \text{THH}(\text{ku})$, because they hold in the E^∞ -page and because the E^∞ -page is zero in lower filtration in the respective total degrees.

We claim that the relations $a_i b_j = ua_{i+j}$ for $i + j \leq p-1$ and $a_i b_j = ua_{i+j-p}\mu_2$ for $i + j \geq p$ hold in $V(1)_* \text{THH}(\text{ku})$. For this note that the only classes with the same total degree and lower filtration degree in the E^∞ -page are

$$\lambda_1 u^2 b_{i+j-1} \quad \text{if } 1 \leq i + j \leq p, \quad \lambda_1 u^2 b_{i+j-1-p}\mu_2 \quad \text{if } p + 1 \leq i + j \leq 2p - 2,$$

where we again use the convention $b_0 = u$. Since these classes have δ -weight 3, whereas $a_i b_j$ has δ -weight 2, this proves the claim. We claim that $a_i a_j = 0$ in $V(1)_* \text{THH}(\text{ku})$. For this note that the only classes with the same total degree and lower filtration degree in the E^∞ -page are

$$\begin{aligned} u^2 b_{i+j} & \quad \text{if } 0 \leq i + j \leq p - 1, \\ u^2 b_{i+j-p}\mu_2 & \quad \text{if } p \leq i + j \leq 2p - 2, \\ \lambda_1 u^2 a_{i+j-1} & \quad \text{if } 1 \leq i + j \leq p, \\ \lambda_1 \mu_2 u^2 a_{i+j-1-p} & \quad \text{if } p + 1 \leq i + j \leq 2p - 2. \end{aligned}$$

These classes have δ -weight 3, whereas $a_i a_j$ has δ -weight 2. □

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