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Rational homology 3–spheres and simply connected definite bounding

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For each rational homology 3–sphere Y which bounds simply connected definite 4–manifolds of both signs, we construct an infinite family of irreducible rational homology 3–spheres which are homology cobordant to Y but cannot bound any simply connected definite 4–manifold. As a corollary, for any coprime integers p and q, we obtain an infinite family of irreducible rational homology 3–spheres which are homology cobordant to the lens space $L(p,q)$ but cannot be obtained by a knot surgery.

57M25, 57M27

1 Introduction

Throughout this paper, all manifolds are assumed to be smooth, compact, orientable and oriented, and diffeomorphisms are orientation-preserving unless otherwise stated.

The intersection form of a $4n$–dimensional manifold has been used to study the topology of its boundary. For instance, the first exotic 7–spheres discovered by Milnor [12] were distinguished by using the intersection form of 8–manifolds whose boundaries are the exotic 7–spheres. In the case of dimension 4, Donaldson's diagonalization theorem [6] implies that if a homology 3–sphere bounds a 4–manifold with nondiagonalizable definite intersection form, then it cannot bound any rational homology 4–ball.

In light of the above results, for any 3–manifold Y, it seems natural to ask which bilinear forms are realized by the intersection form of a 4–manifold with boundary Y. In the case where Y is a rational homology 3–sphere, Choe and Park [3] define $\mathcal{I}(Y)$ (resp. $\mathcal{I}^{\mathrm{TOP}}(Y)$) as the set of all negative definite bilinear forms realized by the intersection form of a 4–manifold (resp. topological 4–manifold) with boundary Y, up to stable-equivalence. They prove in [3] that $|\mathcal{I}^{\mathrm{TOP}}(Y)| = \infty$ for any Y, while $|\mathcal{I}(Y)| < \infty$ if $\mathcal{I}(-Y)$ is not empty. Moreover, it is well known that either $\mathcal{I}(Y) \neq \varnothing$ or $\mathcal{I}(-Y) \neq \varnothing$ holds for any Seifert rational homology sphere Y. (It is also known that many Seifert rational homology spheres bound definite 4–manifolds of both signs. For instance,

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see [3] and Owens and Strle [13].) Here we note that all 4–manifolds constructed in the proof of the above results are simply connected. Hence, if we define $\mathcal{I}_s(Y)$ (resp. $\mathcal{I}_s^{\mathrm{TOP}}(Y)$) by replacing "4–manifolds" in the definition of $\mathcal{I}(Y)$ (resp. $\mathcal{I}^{\mathrm{TOP}}(Y)$) with "simply connected 4–manifolds", then we can prove that $|\mathcal{I}_s^{\mathrm{TOP}}(Y)| = \infty$ and if $\mathcal{I}_s(-Y) \neq \varnothing$ then $|\mathcal{I}_s(Y)| < \infty$ for any Y, and either $\mathcal{I}_s(Y) \neq \varnothing$ or $\mathcal{I}_s(-Y) \neq \varnothing$ holds if Y is Seifert fibered. Then, how different are they? The aim of this paper is to prove the following theorem, which shows a big gap between $\mathcal{I}(Y)$ and $\mathcal{I}_s(Y)$:

Theorem 1.1 *For any rational homology 3–sphere Y satisfying $\mathcal{I}_s(Y) \neq \varnothing$ and $\mathcal{I}_s(-Y) \neq \varnothing$, there exist infinitely many rational homology 3–spheres $\{Y_k\}_{k=1}^{\infty}$ which satisfy the following properties*:

(1) *Y_k is homology cobordant to Y.*

(2) *$\mathcal{I}(Y_k) = \mathcal{I}(Y) \neq \varnothing$ and $\mathcal{I}(-Y_k) = \mathcal{I}(-Y) \neq \varnothing$.*

(3) *$\mathcal{I}_s(Y_k) = \varnothing$ and $\mathcal{I}_s(-Y_k) = \varnothing$.*

(4) *If $k \neq k'$ then Y_k is diffeomorphic to neither $Y_{k'}$ nor $-Y_{k'}$.*

(5) *Each Y_k is irreducible and toroidal.*

Remark 1.2 If we do not require the irreducibility of Y_k, then Theorem 1.1 can be proved easily. (In that case, we can take $Y_k := Y \# \Sigma(2,3,6k-1) \# (-\Sigma(2,3,6k-1))$, where $\Sigma(2,3,6k-1)$ denotes the $(2,3,6k-1)$–Brieskorn sphere.) We will give the proof in the end of Section 3. The irreducibility of Y_k is also meaningful in terms of knot surgery obstructions, since it is known — see Gordon and Luecke [10] — that a reducible 3–manifold Y obtained by a knot surgery must have a lens space as a connected-sum component. On the other hand, the fact that all Y_k are toroidal is caused by a technical reason. This fact also implies that all our examples are not hyperbolic.

Here, rational homology 3–spheres Y_0 and Y_1 are *homology cobordant* if there exists a cobordism W from Y_0 to Y_1 (ie $\partial W = (-Y_0) \amalg Y_1$) such that the inclusion $Y_i \hookrightarrow W$ induces an isomorphism between $H_*(Y_i;\mathbb{Z})$ and $H_*(W;\mathbb{Z})$ for each $i \in \{0,1\}$. (Then we call W a *homology cobordism*.) We note that since $\mathcal{I}(Y)$ is an invariant under homology cobordism (more generally, rational homology cobordism), the first property implies the second property. Moreover, the third property implies that any Y_k is non-Seifert. We also note that there exist infinitely many rational homology 3–spheres satisfying $\mathcal{I}_s(Y) \neq \varnothing$ and $\mathcal{I}_s(-Y) \neq \varnothing$. For instance, any p/q surgery of S^3 over any 0–negative knot (defined in Cochran, Harvey and Horn [5]) with $p/q > 0$

satisfies this condition. (In this case, there is a negative definite cobordism W from the lens space $L(p,q)$ to such a p/q surgery such that $i_*(\pi_1(L(p,q)))$ normally generates $\pi_1(W)$, and $\mathcal{I}_s(L(p,q)) \neq \varnothing$.)

In order to prove Theorem 1.1, we first prove the following proposition, which is obtained by generalizing Auckly's construction in [1]:

Proposition 1.3 *For any rational homology* 3*–spheres* Y *and* M*, there exist a rational homology* 3*–sphere* Y_M *and a homology cobordism* $W_{Y,M}$ *from* $Y \# M \# (-M)$ *to* Y_M *which satisfy*

(1) $i_*\colon \pi_1(Y_M) \to \pi_1(W_{Y,M})$ *is surjective,*

(2) $i_*\colon \pi_1(Y \# M \# (-M)) \to \pi_1(W_{Y,M})$ *is bijective, and*

(3) Y_M *is irreducible and toroidal,*

where i_* *denotes the induced homomorphism from the inclusion.*

Then, by assuming that $\mathcal{I}_s(Y) \neq \varnothing$, $\mathcal{I}_s(-Y) \neq \varnothing$ and $|\mathcal{I}_s(M)| > 1$, and combining the first property of $W_{Y,M}$ in Proposition 1.3 with Taubes's theorem in [14] (stated as Theorem 3.2), we prove that $\mathcal{I}_s(Y_M) = \varnothing$ and $\mathcal{I}_s(-Y_M) = \varnothing$. Finally, we combine the second property of $W_{Y,M}$ in Proposition 1.3 with the Chern–Simons invariants for 3–manifolds to find an infinite family $\{M_k\}_{k=1}^{\infty}$ of integer homology 3–spheres such that the 3–manifolds $\{Y_{M_k}\}_{k=1}^{\infty}$ are mutually distinct. (Note that if M is an integer homology 3–sphere, then $Y \# M \# (-M)$ is homology cobordant to Y.)

As an application of Theorem 1.1, we provide a huge number of irreducible rational homology 3–spheres that are not obtained by a knot surgery. Here we note that if Y is obtained by a knot surgery, then either $\mathcal{I}_s(Y) \neq \varnothing$ or $\mathcal{I}_s(-Y) \neq \varnothing$ holds (see [13]). Hence the 3–manifolds $\{Y_k\}_{k=1}^{\infty}$ in Theorem 1.1 are not obtained by a knot surgery. Therefore, for instance, we have the following corollary:

Corollary 1.4 *For any nonzero coprime integers* p *and* q*, there are infinitely many irreducible rational homology* 3*–spheres which are homology cobordant to* $L(p,q)$ *but not obtained by a knot surgery.*

It is proved in Gilmer and Livingston [9] that any two lens spaces $L(p,q)$ and $L(p',q')$ are homology cobordant if and only if they are homeomorphic. This seems to deduce that for rational homology 3–spheres, the gap between homomorphism and (integer) homology cobordism is not so big. However, Corollary 1.4 tells us that the gap is big enough that the homology cobordism class of any lens space contains infinitely many irreducible rational homology 3–spheres which are not obtained by a knot surgery.

In addition, it is worth comparing Corollary 1.4 with Boyer and Lines's result [2] in terms of surgery obstruction. In [2], Boyer and Lines give infinitely many rational homology 3–spheres $\{M_k\}_{k=1}^{\infty}$ such that $H_1(M_k;\mathbb{Z}) \cong \mathbb{Z}_5$ and $\pi_1(M_k)$ has weight 1 but M_k is not obtained by a knot surgery for any k. On the other hand, for each positive integer p, we give infinitely many irreducible rational homology 3–spheres $\{M_k^p\}_{k=1}^{\infty}$ such that $H_1(M_k^p;\mathbb{Z}) \cong \mathbb{Z}_p$ but M_k^p is not obtained by a knot surgery for any k, while it remains open whether the M_k^p can be taken so that $\pi_1(M_k^p)$ has weight 1.

Next we consider homology cobordism on integer homology 3–spheres. In this case, we can work more systematically by using the homology cobordism group $\Theta_{\mathbb{Z}}^3$. We define a subgroup $\mathcal{B}_s$ of $\Theta_{\mathbb{Z}}^3$ by

$$\mathcal{B}_s := \{x \in \Theta_{\mathbb{Z}}^3 \mid \mathcal{I}_s(Y) \neq \varnothing \text{ and } \mathcal{I}_s(-Y) \neq \varnothing \text{ for some } Y \in x\},$$

and then we have the following corollary of Theorem 1.1:

Corollary 1.5 *Any element of $\mathcal{B}_s$ contains infinitely many irreducible homology 3–spheres which are not obtained by a knot surgery.*

As concrete examples, for any coprime integers $p, q > 0$ and any $k \in \mathbb{Z}_{>0}$, the homology cobordism class of the $(p, q, pqk+1)$–Brieskorn sphere belongs to $\mathcal{B}_s$. (In fact, it is known that the $(p, q, pqk+1)$–Brieskorn sphere is the $1/k$ surgery of S^3 over the $(p, -q)$–torus knot, and the $(p, -q)$–torus knot is 0–negative.) Therefore, any linear combination of them contains infinitely many irreducible homology 3–spheres which are not obtained by a knot surgery.

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2 Proof of Proposition 1.3

Proof of Proposition 1.3 We first describe the construction of Y_M in the proposition. Let Y and M be rational homology 3–spheres and $Y'_M := Y \# M \# (-M)$. By

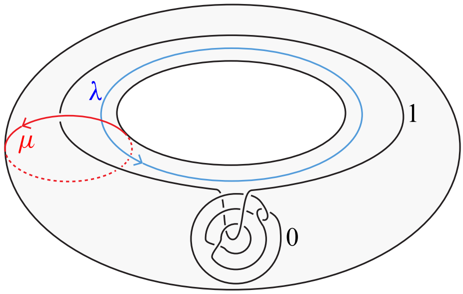

Figure 1: A 3–manifold C.

the result of [15], there exists a null-homologous knot K in Y'_M whose complement $Y'_M \setminus K$ has a hyperbolic structure. We denote the exterior of K in Y'_M by E_K. Let C be a 3–manifold with torus boundary as shown in Figure 1, where μ and λ in the figure are simple closed curves on the boundary of C. (More precisely, the 3–manifold C is defined as follows. First, we regard the shaded annulus in Figure 1 as in $\mathbb{R}^2$. We denote the annulus by A. Then, the 2–component framed link diagram in A represents a framed link in $\mathbb{R}^3$, where A is lying in $\mathbb{R}^2 \times \{0\}$. We denote the framed link in $\mathbb{R}^3$ by L. Then, we can take a closed interval $[s,t] \subset \mathbb{R}$ so that the interior of the solid torus $A \times [s,t]$ contains L. By performing the surgery of $A \times [s,t]$ along L, we obtain the manifold C.) Define Y_M as $C \cup_{\text{torus}} E_K$ by the identifications

$$\mu \mapsto \text{the meridian of } K, \qquad \lambda \mapsto \text{the preferred longitude of } K.$$

It is easy to see that Y_M is a rational homology 3–sphere.

Claim 1 *Y_M is irreducible and toroidal.*

Proof We make a similar argument to [1]. More precisely, we use the following lemmas:

Lemma 2.1 [1, page 13] *Let N be a 3–manifold and F a properly embedded incompressible surface in N. If every component of $N - F$ is irreducible, then N is also irreducible.*

Lemma 2.2 (Papakyriakopoulos's loop theorem) *If the boundary of a 3–manifold N is incompressible in N, then $i_*\colon \pi_1(\partial N) \to \pi_1(N)$ is injective.*

Lemma 2.3 (see Theorem 2.6 in [11], for example) *Let*

$$\begin{tikzcd} C \arrow[r, "i_A"] \arrow[d, "i_B"'] & A \arrow[d, "j_A"] \\ B \arrow[r, "j_B"'] & A *_C B \end{tikzcd}$$

be the defining diagram of $A *_C B$. If i_A and i_B are injective, then j_A and j_B are injective.

By Lemma 2.1, it suffices to prove

(1) ∂C is incompressible in Y_M, and

(2) both C and E_K are irreducible

for proving Claim 1. (Note that the first condition implies that Y_M is a toroidal 3–manifold with essential torus ∂C.) Moreover, Lemmas 2.2 and 2.3 imply that ∂C is incompressible in Y_M if ∂C is incompressible both in C and in E_K. To prove it, suppose that ∂C is incompressible both in C and in E_K. Then it follows from Lemma 2.2 that both of the induced homomorphisms $(i_C)_*\colon \pi_1(\partial C) \to \pi_1(C)$ and $(i_{E_K})_*\colon \pi_1(\partial C) \to \pi_1(E_K)$ are injective. In addition, since $\pi_1(Y_M) = \pi_1(C) *_{\pi_1(\partial C)} \pi_1(E_K)$, Lemma 2.3 implies that both of the induced homomorphisms $(j_C)_*\colon \pi_1(C) \to \pi_1(Y_M)$ and $(j_{E_K})_*\colon \pi_1(E_K) \to \pi_1(Y_M)$ are injective. Now, assume that there exists a compressing disk for ∂C in Y_M, and then $i_*\colon \pi_1(\partial C) \to \pi_1(Y_M)$ is not injective. However, since $i_* = (j_C)_* \circ (i_C)_*$ and the right-hand side is injective, it leads to a contradiction.

Here, we note that the 3–manifold C is exactly the same as the manifold C appearing in [1], and it is proved that ∂C is incompressible in C, and C is irreducible. Now let us prove that $\partial C = \partial E_K$ is incompressible in E_K, and E_K is irreducible. The irreducibility of E_K immediately follows from the fact that E_K has a hyperbolic structure. Assume that there exists a compressing disk D for ∂E_K in E_K. Then it follows from elementary arguments that ∂D is a preferred longitude for K, and hence K bounds a disk in Y'_M. This implies that E_K is homeomorphic to $Y'_M \# (S^1 \times D^2)$, and E_K does not have any hyperbolic structure. This leads to a contradiction, and hence ∂E_K is incompressible in E_K. □

Next, let $W_{Y,M}$ denote a cobordism described by the relative Kirby diagram shown in Figure 2. Here, the tangle diagram $\langle D \rangle$ in Figure 2 is obtained as follows. We first take a

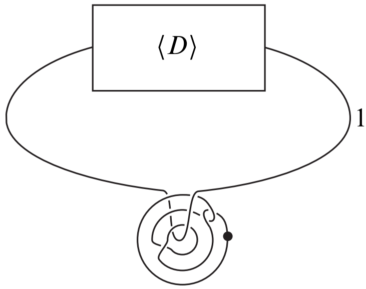

Figure 2: A cobordism $W_{Y,M}$.

diagram D' of K in Y'_M (ie a knot diagram of K in a surgery diagram of Y'_M) such that the linking number between K and each component of the surgery link for Y'_M is zero.

To obtain such a diagram D', we first fix an l–component framed link $L = \coprod_{i=1}^{l} L_i$ in S^3 with integer framings $\{n_i\}_{i=1}^{l}$ which represents Y'_M. Then Y'_M is diffeomorphic to $(S^3 \setminus \nu(L)) \cup \left(\coprod_l S^1 \times D^2\right)$, where $\nu(L)$ is an open tubular neighborhood of L in S^3 and $\coprod_l S^1 \times D^2$ corresponds to Dehn fillings. Through the diffeomorphism, we can regard K as lying in $(S^3 \setminus \nu(L)) \cup \left(\coprod_l S^1 \times D^2\right)$. Moreover, by an isotopy, we can deform K into K' such that K' lies in $S^3 \setminus \nu(L)$. In particular, we can take a link diagram of the link $L \amalg K'$ in S^3, and by adding the framing n_i to each L_i, we have a diagram D' of K' in Y'_M. Since K' is isotopic to K, this D' is also a diagram of K. To satisfy $\operatorname{lk}(L_i, K') = 0$ for each i, it suffices to take the band sum of K' and several parallel copies of the n_i–framed longitude λ_i of each L_i. This follows from the fact that K' is null-homologous in Y'_M and $H_1(Y'_M; \mathbb{Z})$ has an abelian group presentation whose generators are the meridians of the L_i and relations are the λ_i.

Next, we derive a tangle diagram D from D' by regarding D' as in S^2 and removing a small open disk in S^2 whose intersection with D' is a small arc in the component corresponding to K. Finally, by putting brackets around each surgery coefficient in D, we have the diagram $\langle D \rangle$. Then, it follows from elementary handle theory that $W_{Y,M}$ is a homology cobordism from Y'_M to Y_M, and it admits a handle decomposition consisting of a single 1–handle h^1 and single 2–handle h^2.

Claim 2 $i_*\colon \pi_1(Y_M) \to \pi_1(W_{Y,M})$ *is surjective.*

Proof By considering the dual decomposition, we have a handle decomposition of $W_{Y,M}$ consisting of a single 2–handle and single 3–handle. This implies that $i_*\colon \pi_1(Y_M) \to \pi_1(W_{Y,M})$ is surjective. □

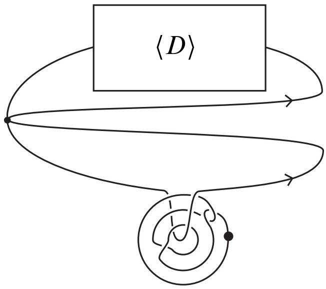

Figure 3: A loop l in $(Y'_M \times [0,1]) \cup h^1$.

Claim 3 *$i_*\colon \pi_1(Y'_M) \to \pi_1(W_{Y,M})$ is bijective.*

Proof Let $\langle S \mid R\rangle$ be a presentation for $\pi_1(Y'_M) = \pi_1(Y'_M \times [0,1])$ and l a loop as shown in Figure 3. Then $\pi_1((Y'_M \times [0,1]) \cup h^1)$ is presented by $\langle S \cup \{x\} \mid R\rangle$, where x corresponds to h^1, and ∂h^2 is homotopic to l in $(Y'_M \times [0,1]) \cup h^1$. This implies that the homotopy class of ∂h^2 is a word of the form xw, where w is a word on S, and hence $\pi_1(W_{Y,M}) = \pi_1((Y'_M \times [0,1]) \cup h^1 \cup h^2)$ is represented by $\langle S \cup \{x\} \mid R \cup \{xw\}\rangle$. Moreover, the diagram

$$\begin{tikzcd} \langle S \mid R\rangle \arrow[r, "f"] \arrow[d, "\cong"'] & \langle S \cup \{x\} \mid R \cup \{xw\}\rangle \arrow[d, "\cong"] \\ \pi_1(Y'_M) \arrow[r, "i_*"'] & \pi_1(W_{Y,M}) \end{tikzcd}$$

is commutative, where f maps $y \in S$ to y. We construct the inverse of f. Define a map $g\colon \langle S \cup \{x\} \mid R \cup \{xw\}\rangle \to \langle S \mid R\rangle$ by

$$y \mapsto \begin{cases} y & \text{if } y \in S, \\ w^{-1} & \text{if } y = x. \end{cases}$$

Then it is easy to see that g is well defined and both $f \circ g$ and $g \circ f$ are the identity maps. (Note that $w^{-1} = x$ in $\langle S \cup \{x\} \mid R \cup \{xw\}\rangle$.) This completes the proof. □

The above arguments complete the proof of Proposition 1.3. □

3 Definite bounding and Taubes's theorem

In this section, we prove the following proposition by using a theorem of Taubes:

Proposition 3.1 *Let Y and M be rational homology 3–spheres, and Y_M a rational homology 3–sphere as given by Proposition 1.3. If $\mathcal{I}_s(Y) \neq \varnothing$, $\mathcal{I}_s(-Y) \neq \varnothing$ and $|\mathcal{I}_s(M)| > 1$, then we have $\mathcal{I}_s(Y_M) = \mathcal{I}_s(-Y_M) = \varnothing$.*

First we state Taubes's theorem. This is an end-periodic version of Donaldson's diagonalization theorem.

Theorem 3.2 [14] *Let Y be a rational homology 3–sphere and*

$$W := K \cup_Y W_0 \cup_Y W_1 \cup_Y \cdots$$

a connected noncompact 4–manifold satisfying the following conditions:

- *K is a simply connected, negative definite 4–manifold with $\partial K = Y$.*
- *W_0 is a negative definite cobordism from Y to itself.*
- *W_i are copies of W_0 for any $i \in \{1, 2, \dots\}$.*

We also assume that there is no nontrivial representation from $\pi_1(W_0)$ to $\mathrm{SU}(2)$. Then the intersection form of K is diagonalizable.

The assumption about $\mathrm{SU}(2)$–representation is essential. If the assumption is removed, then one can easily find a counterexample to the theorem. (For instance, take $Y = \Sigma(2,3,5)$ and $W_0 = Y \times I$.) This is an essential reason why we can claim the nonexistence of simply connected definite bounding.

As a corollary of Theorem 3.2, we have the following lemma:

Lemma 3.3 *Let M be a rational homology 3–sphere. If there exists a simply connected, negative definite cobordism W from M to itself, then $|\mathcal{I}_s(M)| \leq 1$.*

Proof Suppose that $|\mathcal{I}_s(M)| > 1$. Then there exists a simply connected, negative definite 4–manifold K with $\partial K = M$ whose intersection form is not diagonalizable. However, the end-periodic manifold $K \cup_M \cup W \cup_M W \cup_M \cdots$ satisfies all assumptions of Theorem 3.2. This leads to a contradiction. □

Proof of Proposition 3.1 We first assume that $\mathcal{I}_s(Y_M) \neq \varnothing$. Then there exists a simply connected, negative definite 4–manifold X with $\partial X = Y_M$. Moreover, since $\mathcal{I}_s(-Y) \neq \varnothing$, $-Y$ also bounds a simply connected, negative definite 4–manifold U. We use X, U and $-W_{Y,M}$ to construct a simply connected, negative definite cobordism from M to itself.

We first glue X with $-W_{Y,M}$ along Y_M, and denote it by X'. (Note that $\partial(-W_{Y,M}) = -Y_M \amalg (Y \# M \# (-M))$.) Then X' is negative definite and $\partial X' = Y \# M \# (-M)$.

Furthermore, since $\pi_1(X) = 1$ and $i_*\colon \pi_1(-Y_M) \to \pi_1(-W_{Y,M})$ is surjective, we have $\pi_1(X') = 1$. Next, by attaching two 3–handles to X', we obtain a 4–manifold X'' with $\partial X'' = Y \amalg M \amalg (-M)$. Finally, by gluing X'' with U along Y, we have a 4–manifold W with boundary $M \amalg (-M)$. By the construction, it is easy to check that $\pi_1(W) = 1$ and W is negative definite.

Now, by applying Lemma 3.3 to W, we conclude that $|\mathcal{I}_s(M)| \leq 1$. However, since $|\mathcal{I}_s(M)| > 1$ is assumed, this leads to a contradiction. As a consequence, we have $\mathcal{I}_s(Y_M) = \varnothing$.

If we assume $\mathcal{I}_s(-Y_M) \neq \varnothing$, then a similar argument gives $|\mathcal{I}_s(M)| \leq 1$, which contradicts the assumption $|\mathcal{I}_s(M)| > 1$. (In this case, use $\mathcal{I}_s(Y) \neq \varnothing$ and $W_{Y,M}$ instead of $\mathcal{I}_s(-Y) \neq \varnothing$ and $-W_{Y,M}$.) This completes the proof. □

Here we show the existence of a class of integer homology 3–spheres with $|\mathcal{I}_s(M)| > 1$. To show that, we first give several lemmas which are useful for studying $\mathcal{I}_s(M)$ concretely.

Lemma 3.4 *Let M_0 and M_1 be integer homology 3–spheres. If there exists a cobordism W from M_0 to M_1 which admits a handle decomposition consisting of only 2–handles and whose intersection form is isomorphic to $\bigoplus_m \langle -1 \rangle$ for some $m \in \mathbb{Z}_{>0}$, then we have $\mathcal{I}_s(M_0) \subset \mathcal{I}_s(M_1)$.*

Proof For an element $x \in \mathcal{I}_s(M_0)$, let X be a simply connected 4–manifold with boundary M_0 whose intersection form Q_X is a representative of x. Then $X \cup_{M_0} W$ is obtained by attaching finitely many 2–handles to X, and hence it is simply connected. Moreover, its boundary is M_1 and intersection form is isomorphic to $Q_X \oplus \big(\bigoplus_m \langle -1 \rangle\big)$. This proves $x \in \mathcal{I}_s(M_1)$. □

Lemma 3.5 *For a knot K in S^3 and $p/q \in \mathbb{Q}$, let $S^3_{p/q}(K)$ denote the p/q surgery of S^3 over K. Then we have $\mathcal{I}_s(-S^3_{1/n}(K)) \subset \mathcal{I}_s(-S^3_{1/(n+1)}(K))$ for any $n \in \mathbb{Z}_{>0}$.*

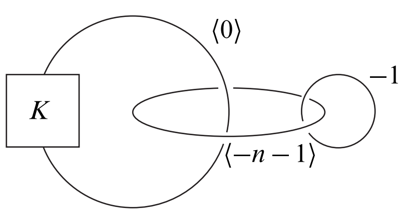

Figure 4: The cobordism W_n.

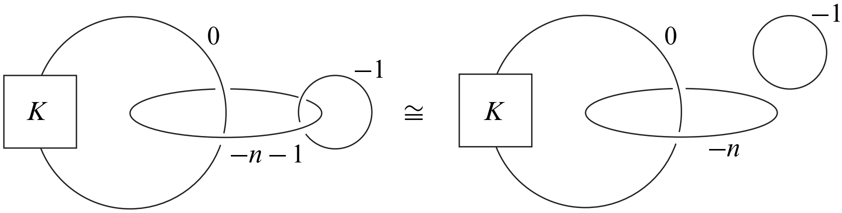

Figure 5: The 4–manifold X_n.

Proof For any $n \in \mathbb{Z}_{>0}$, let W_n be the cobordism given by the relative Kirby diagram in Figure 4. Note that W_n consists of a single 2–handle, and has boundary $-S^3_{1/(n+1)}(K) \amalg S^3_{1/n}(K)$. (In particular, it can be regarded as a cobordism from $-S^3_{1/n}(K)$ to $-S^3_{1/(n+1)}(K)$.) We will see that the intersection form of W_n is $\langle -1 \rangle$, and then applying Lemma 3.4 to W_n will prove Lemma 3.5.

Let X_n be a 4–manifold as given by the Kirby diagrams in Figure 5, and X'_n a 4–dimensional submanifold of X_n obtained by attaching 2–handles along the 2–component sublink in the left diagram of Figure 5 whose framing is $(0, -n-1)$. Then we see the diffeomorphism $X_n \cong X'_n \cup_{S^3_{1/n}(K)} W_n$. For a general 4–manifold X, let $b_2^+(X)$ (resp. $b_2^-(X)$) denote the number of positive (resp. negative) eigenvalues of the intersection form of X. Then it is easy to check that $b_2^+(X_n) = 1$, $b_2^-(X_n) = 2$ and $b_2^+(X'_n) = b_2^-(X'_n) = 1$. These imply that $b_2^+(W_n) = 0$ and $b_2^-(W_n) = 1$. Now, we can conclude that the intersection form of W_n is negative definite and unimodular, and hence it is isomorphic to $\langle -1 \rangle$. □

Next, a *positive crossing* is a crossing as shown in the left of Figure 6, and a *positive crossing change* is a deformation of a knot diagram as shown in Figure 6.

Lemma 3.6 *Suppose that a diagram D_0 of a knot K_0 is deformed into a diagram of a knot K_1 by performing positive crossing changes at some crossings $\{c_1, \dots, c_m\}$ on D_0. Then we have $\mathcal{I}_s(-S^3_{1/n}(K_1)) \subset \mathcal{I}_s(-S^3_{1/n}(K_0))$ for any $n \in \mathbb{Z}_{>0}$.*

Proof For any $n \in \mathbb{Z}_{>0}$, we make a relative Kirby diagram $\langle D_0 \rangle$ from D_0 in the following way:

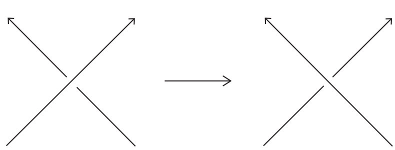

Figure 6: A positive crossing change.

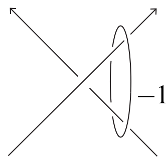

Figure 7: A 2–handle attached near each c_k.

- Replace a neighborhood of each crossing c_k with the picture shown in Figure 7. (Then, each component except for the original one has a framing.)
- Associate the framing $\langle 1/n \rangle$ to the original component.

Then $\langle D_0 \rangle$ represents a cobordism W_n consisting of m 2–handles. Moreover, we can verify that

- $\partial W_n = -S^3_{1/n}(K_0) \amalg S^3_{1/n}(K_1)$, and
- the intersection form of W_n is isomorphic to $\bigoplus_{k=1}^m \langle -1 \rangle$.

Therefore, we can apply Lemma 3.4 to W_n, and it gives

$$\mathcal{I}_s(-S^3_{1/n}(K_1)) \subset \mathcal{I}_s(-S^3_{1/n}(K_0)). \qquad \square$$

Now, we present a class of integer homology 3–spheres with $|\mathcal{I}_s(M)| > 1$. Here, a knot in S^3 is called a *positive knot* if it admits a diagram containing only positive crossings. (Such a diagram is called a *positive diagram*.)

Proposition 3.7 *For any nontrivial positive knot K and positive integer n, we have $|\mathcal{I}_s(-S^3_{1/n}(K))| > 1$.*

To prove the proposition, we combine the following theorem of Cochran and Gompf with the above lemmas:

Theorem 3.8 [4, Theorem 3.1] *Any positive diagram D of a nontrivial positive knot can be deformed into a diagram of the right-hand trefoil $T_{2,3}$ by performing positive crossing changes at finitely many crossings on D.*

Proof of Proposition 3.7 First note that $-S^3_1(T_{2,3})$ is the Poincaré sphere $\Sigma(2,3,5)$, and hence $\mathcal{I}_s(-S^3_1(T_{2,3}))$ contains the stable-equivalence classes of $\langle -1 \rangle$ and $-E_8$. In particular, we have $|\mathcal{I}_s(-S^3_1(T_{2,3}))| > 1$. Therefore, it follows from Lemma 3.5

that $|\mathcal{I}_s(-S^3_{1/n}(T_{2,3}))| \geq |\mathcal{I}_s(-S^3_1(T_{2,3}))| > 1$. Now, by combining Theorem 3.8 with Lemma 3.6, we have

$$|\mathcal{I}_s(-S^3_{1/n}(K))| \geq |\mathcal{I}_s(-S^3_{1/n}(T_{2,3}))| > 1. \qquad \square$$

For coprime positive integers p and q, let $T_{p,q}$ denote the (p,q)–torus knot. It is well known that $T_{p,q}$ is a nontrivial positive knot and $-S^3_{1/n}(T_{p,q})$ is the $(p,q,pqn-1)$–Brieskorn sphere $\Sigma(p,q,pqn-1)$. Therefore, as a corollary of Proposition 3.7, we have the following:

Corollary 3.9 *For any coprime positive integers p and q, we have*

$$|\mathcal{I}_s(\Sigma(p,q,pqn-1))| > 1.$$

In the last of this section, we prove the following proposition as mentioned in Remark 1.2:

Proposition 3.10 *For any rational homology 3–sphere Y satisfying $\mathcal{I}_s(Y) \neq \varnothing$ and $\mathcal{I}_s(-Y) \neq \varnothing$ and $n \in \mathbb{Z}_{>0}$, set $Y_n := Y \# \Sigma(2,3,6n-1) \# (-\Sigma(2,3,6n-1))$. Then $\{Y_n\}_{n=1}^{\infty}$ satisfy the following properties*:

(1) *Y_n is homology cobordant to Y.*

(2) *$\mathcal{I}(Y_n) = \mathcal{I}(Y) \neq \varnothing$ and $\mathcal{I}(-Y_n) = \mathcal{I}(-Y) \neq \varnothing$.*

(3) *$\mathcal{I}_s(Y_n) = \varnothing$ and $\mathcal{I}_s(-Y_n) = \varnothing$.*

(4) *If $n \neq n'$ then Y_n is diffeomorphic to neither $Y_{n'}$ nor $-Y_{n'}$.*

Proof The first property is obviously satisfied, and hence the second property immediately follows from the assumption.

Next, we prove that the third property is satisfied. Actually, the proof is obtained by replacing Y_M and $W_{Y,M}$ in the proof of Proposition 3.1 with Y_n and $Y_n \times I$, respectively. Then, for the present case, the assumption $\mathcal{I}_s(Y_n) \neq \varnothing$ gives a simply connected, negative definite cobordism W from $\Sigma(2,3,6n-1)$ to itself. Therefore, Lemma 3.3 gives the inequality $|\Sigma(2,3,6n-1)| \leq 1$, which contradicts Corollary 3.9. Hence we have $\mathcal{I}_s(Y_n) = \varnothing$. Similarly, we also have $\mathcal{I}_s(-Y_n) = \varnothing$.

Finally, we prove that the fourth property is satisfied. Indeed, this immediately follows from the prime decomposition theorem for 3–manifolds and the fact that the $\Sigma(2,3,6n-1)$ are mutually nondiffeomorphic. This completes the proof. $\square$

4 Chern–Simons invariants

In this section, we give a method for finding an infinite family $\{M_k\}$ such that $\{Y_{M_k}\}$ are mutually nondiffeomorphic. The goal of this section is to prove the following proposition. Here we denote the (p,q,r)–Brieskorn sphere by $\Sigma(p,q,r)$.

Proposition 4.1 *Let Y be a rational homology 3–sphere, p and q coprime integers, $M_n := \Sigma(p,q,pqn-1)$ and $Y_n := Y_{M_n}$ a rational homology 3–sphere as given by Proposition 1.3. Then there exists a numerical sequence $\{n_k\}_{k=1}^{\infty}$ such that if $k \neq k'$, then Y_{n_k} is diffeomorphic to neither $Y_{n_{k'}}$ nor $-Y_{n_{k'}}$.*

It is shown in Corollary 3.9 that $|\mathcal{I}_s(\Sigma(p,q,pqn-1))| > 1$. Hence we can apply Proposition 3.1 to Y_{M_n} whenever Y satisfies $\mathcal{I}_s(Y) \neq \varnothing$ and $\mathcal{I}_s(-Y) \neq \varnothing$. In order to prove Proposition 4.1, we use the Chern–Simons invariants for 3–manifolds. Here we recall the Chern–Simons invariants. For a given 3–manifold Y, let P_Y be the product SO(3) bundle. First we introduce several definitions which are used for gauge theory. We denote by $\mathrm{Map}(Y, \mathrm{SO}(3))$ the set of smooth maps from Y to SO(3). The group structure on SO(3) induces a group structure on $\mathrm{Map}(Y, \mathrm{SO}(3))$. Let $\mathcal{A}_Y^f$ be the set of SO(3) flat connections on P_Y. Since $\mathrm{Map}(Y, \mathrm{SO}(3))$ can be identified with the set of automorphisms on P_Y, $\mathrm{Map}(Y, \mathrm{SO}(3))$ acts on $\mathcal{A}_Y^f$ by the pullback of connections. The set of SO(3)–connections on P_Y can be identified with the set of $\mathfrak{so}(3)$–valued 1–forms on Y. Therefore we regard any element of $\mathcal{A}_Y^f$ as an element of $\Omega^1(Y) \otimes \mathfrak{so}(3)$. Under these identifications, the action of $\mathrm{Map}(Y, \mathrm{SO}(3))$ on $\mathcal{A}_Y^f$ is

$$g^*a = g^{-1}dg + g^{-1}ag$$

for $a \in \Omega^1(Y) \otimes \mathfrak{so}(3)$. This action defines the quotient

$$R(Y) := \mathcal{A}_Y^f / \mathrm{Map}(Y, \mathrm{SO}(3)).$$

Then the Chern–Simons functional

$$\widetilde{\mathrm{cs}}\colon \mathcal{A}_Y^f \to \mathbb{R} \tag{1}$$

is defined by

$$\widetilde{\mathrm{cs}}(a) = \frac{1}{8\pi^2} \int_Y \mathrm{Tr}\big(a \wedge da + \tfrac{2}{3} a \wedge a \wedge a\big)$$

for $a \in \Omega^1(Y) \otimes \mathfrak{so}(3)$. It is known that

$$\widetilde{\mathrm{cs}}(g^*a) = \widetilde{\mathrm{cs}}(a) + \deg(g),$$

where $g \in \mathrm{Map}(Y, \mathrm{SO}(3))$, $a \in \Omega^1(Y) \otimes \mathfrak{so}(3)$ and $\deg(g)$ is the mapping degree of g. Therefore the map (1) descends to the map

$$\mathrm{cs}\colon R(Y) \to \mathbb{R}/\mathbb{Z}.$$

Since the space $R(Y)$ is compact and the map cs is locally constant, one can show the set $\mathrm{Im}\,\mathrm{cs} \subset \mathbb{R}/\mathbb{Z}$ is a finite set.

By using the Chern–Simons functional, Furuta [8] defines a numerical invariant ϵ as follows. (In [7], Fintushel and Stern also consider such an invariant.) Here we identify $(0,1]$ with $\mathbb{R}/\mathbb{Z}$ via the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and regard cs as a map from $R(Y)$ to $(0,1]$.

Definition 4.2 For a 3–manifold Y, we define

$$\epsilon(Y) := \begin{cases} \min_{a \in \mathrm{cs}^{-1}(0,1)} \mathrm{cs}(a) & \text{if } \mathrm{cs}^{-1}(0,1) \neq \varnothing, \\ 1 & \text{if } \mathrm{cs}^{-1}(0,1) = \varnothing. \end{cases}$$

There is a connected sum inequality for ϵ stated as follows:

Lemma 4.3 *For any two 3–manifolds Y_1 and Y_2, we have*

$$\epsilon(Y_1 \# Y_2) \leq \min\{\epsilon(Y_1), \epsilon(Y_2)\}.$$

Proof For proving the lemma, it suffices to prove that $\epsilon(Y_1 \# Y_2) \leq \epsilon(Y_1)$. Let ρ be an SO(3) flat connection on Y_1 satisfying $\mathrm{cs}(\rho) = \epsilon(Y_1)$ and θ the product connection on Y_2. By taking the connected sum of ρ_M and θ, we get an SO(3) flat connection $\rho \# \theta$ over $Y_1 \# Y_2$. Then it follows from the definitions of cs and ϵ that

$$\epsilon(Y_1 \# Y_2) \leq \mathrm{cs}(\rho \# \theta) = \mathrm{cs}(\rho) = \epsilon(Y_1). \qquad \square$$

Next, we prove the following lemma. This lemma says that if we have a cobordism such that the inclusion of one side of its boundary induces an isomorphism on the fundamental groups, then we can estimate the value of ϵ.

Lemma 4.4 *Let Y_1 and Y_2 be 3–manifolds. Suppose that there is a cobordism W from Y_1 to Y_2 such that $i_*\colon \pi_1(Y_1) \to \pi_1(W)$ is bijective. Then the inequality*

$$\epsilon(Y_2) \leq \epsilon(Y_1)$$

holds.

Proof Suppose that ρ is an SO(3) flat connection satisfying $\mathrm{cs}(\rho) = \epsilon(Y_1)$. Since $\pi_1(Y_1) \to \pi_1(W)$ is bijective, we can extend ρ over W using the holonomy correspondence. We denote the extended connection by $\widetilde{\rho}$. Then the equalities

$$0 = \frac{1}{8\pi^2} \int_W \mathrm{Tr}(F(\rho) \wedge F(\rho)) = \mathrm{cs}(\rho) - \mathrm{cs}(\widetilde{\rho}|_{Y_2})$$

hold. Therefore, we have

$$\epsilon(Y_2) \leq \mathrm{cs}(\widetilde{\rho}|_{Y_2}) = \mathrm{cs}(\rho) = \epsilon(Y_1). \qquad \square$$

In our situation, we have the following estimate for $\epsilon(Y_M)$:

Corollary 4.5 *For any Y and M, we have*

$$\epsilon(Y_M) \leq \epsilon(M).$$

Proof By applying Lemma 4.4 to $W_{Y,M}$, we have

$$\epsilon(Y_M) \leq \epsilon(Y \# M \# (-M)) \leq \epsilon(M),$$

where the second inequality follows from Lemma 4.3. $\square$

Proof of Proposition 4.1 It is proved by Furuta [8] and Fintushel and Stern [7] that

$$\epsilon(M_n) = \frac{1}{pq(pqn-1)}.$$

Therefore, it follows from Corollary 4.5 that for any n, we have

$$\epsilon(Y_n) \leq \epsilon(M_n) = \frac{1}{pq(pqn-1)}.$$

We construct a numerical sequence $\{n_k\}_{k=1}^{\infty}$ by induction. First, we define $n_1 := 1$. Next, suppose that $\{n_k\}_{k=1}^{m}$ is defined for some m. Since $1/pq(pqn-1) \to 0$ as $n \to \infty$, there exists an integer n such that

$$\frac{1}{pq(pqn-1)} < \min_{1 \leq k \leq m} \{\epsilon(Y_{n_k}), \epsilon(-Y_{n_k})\}.$$

Then we define $n_{m+1} := n$.

Now, let us prove that $\{n_k\}_{k=1}^{\infty}$ is the desired sequence. Suppose that $k \neq k'$. Without loss of generality, we may assume that $k > k'$. Then, by the definition of $\{n_k\}_{k=1}^{\infty}$, the inequalities

$$\epsilon(Y_{n_k}) \leq \frac{1}{pq(pqn_k-1)} < \min_{l<k} \{\epsilon(Y_{n_l}), \epsilon(-Y_{n_l})\}$$

hold. In particular, since $k' < k$, we have

$$\epsilon(Y_{n_k}) < \min\{\epsilon(Y_{n_{k'}}), \epsilon(-Y_{n_{k'}})\}.$$

This proves that Y_{n_k} is diffeomorphic to neither $Y_{n_{k'}}$ nor $-Y_{n_{k'}}$. □

5 Proof of the main theorem

In this section, we prove Theorem 1.1, which is stated as follows:

Theorem 1.1 *For any rational homology 3–sphere Y satisfying $\mathcal{I}_s(Y) \neq \varnothing$ and $\mathcal{I}_s(-Y) \neq \varnothing$, there exist infinitely many rational homology 3–spheres $\{Y_k\}_{k=1}^{\infty}$ which satisfy the following properties*:

(1) *Y_k is homology cobordant to Y.*

(2) *$\mathcal{I}(Y_k) = \mathcal{I}(Y) \neq \varnothing$ and $\mathcal{I}(-Y_k) = \mathcal{I}(-Y) \neq \varnothing$.*

(3) *$\mathcal{I}_s(Y_k) = \varnothing$ and $\mathcal{I}_s(-Y_k) = \varnothing$.*

(4) *If $k \neq k'$ then Y_k is diffeomorphic to neither $Y_{k'}$ nor $-Y_{k'}$.*

(5) *Each Y_k is irreducible and toroidal.*

Proof Let $M_n := \Sigma(2,3,6n-1)$, Y_{M_n} be a rational homology 3–sphere as given by Proposition 1.3 and $\{n_k\}_{k=1}^{\infty}$ a numerical sequence as given by Proposition 4.1. Then we define $Y_k := Y_{M_{n_k}}$. Let us prove that $\{Y_k\}_{k=1}^{\infty}$ is the desired family.

First, since Y_k is homology cobordant to $Y \# M_{n_k} \# (-M_{n_k})$ and M_{n_k} is an integer homology 3–sphere, Y_k is homology cobordant to Y for any k.

Second, since $\mathcal{I}$ is a homology cobordism invariant, $\mathcal{I}(Y) \supset \mathcal{I}_s(Y)$, and we assume that $\mathcal{I}_s(Y) \neq \varnothing$ and $\mathcal{I}_s(-Y) \neq \varnothing$, both $\mathcal{I}(Y_k) = \mathcal{I}(Y) \neq \varnothing$ and $\mathcal{I}(-Y_k) = \mathcal{I}(-Y) \neq \varnothing$ hold for any k.

Third, since $|\mathcal{I}_s(M_{n_k})| > 1$ holds for any k, it follows from Proposition 3.1 that $\mathcal{I}_s(Y_k) = \varnothing$ and $\mathcal{I}_s(-Y_k) = \varnothing$.

Fourth, it follows from Proposition 4.1 that if $k \neq k'$ then Y_k is diffeomorphic to neither $Y_{k'}$ nor $-Y_{k'}$.

Finally, it follows from Proposition 1.3 that each Y_k is irreducible and toroidal. This completes the proof. □

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