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Algebraic & Geometric Topology 20 (2020) 1073–1126

Rational homology cobordisms of plumbed manifolds

PAOLO ACETO

We investigate rational homology cobordisms of 3–manifolds with nonzero first Betti number. This is motivated by the natural generalization of the *slice-ribbon conjecture* to multicomponent links. In particular we consider the problem of which rational homology $S^1 \times S^2$'s bound rational homology $S^1 \times D^3$'s. We give a simple procedure to construct rational homology cobordisms between plumbed 3–manifolds. We introduce a family of plumbed 3–manifolds with $b_1 = 1$. By adapting an obstruction based on Donaldson's diagonalization theorem we characterize all manifolds in our family that bound rational homology $S^1 \times D^3$'s. For all these manifolds a rational homology cobordism to $S^1 \times S^2$ can be constructed via our procedure. Our family is large enough to include all Seifert fibered spaces over the 2–sphere with vanishing Euler invariant. In a subsequent paper we describe applications to arborescent link concordance.

57M27; 57M12, 57M25

1 Introduction

The study of concordance properties of classical knots and links in the 3–sphere is a highly active field of research in low-dimensional topology. Problems in this area involve

Published: 27 May 2020 DOI: 10.2140/agt.2020.20.1073

a wide range of techniques, from the use of sophisticated combinatorial invariants derived from knot homology theories to the interplay with 3– and 4–manifold topology.

One of the most famous unsolved problems in this field is the so-called *slice-ribbon conjecture*. A knot $K \subset S^3$ is *smoothly slice* if it bounds a properly embedded smooth disk in the 4–ball. A smoothly slice knot is *ribbon* if the spanning disk $D^2 \subset D^4$ can be chosen so that there are no local maxima of the radial function $\rho\colon D^4 \to [0,1]$ restricted to the image of D^2. The slice ribbon conjecture states that every slice knot is ribbon. Since it was first formulated by Fox in 1962 (as a question rather than a conjecture) there have been many efforts towards understanding slice and ribbon knots. One stimulating aspect of this topic is that it naturally leads to several related questions on 3–manifold topology.

In [7], Lisca proved that the slice ribbon conjecture holds true for 2–bridge knots. He used an obstruction based on Donaldson's diagonalization theorem to determine which lens spaces bound rational homology balls. This technique has been used by Lecuona [5] to prove that the slice ribbon conjecture holds true for an infinite family of Montesinos knots. In [2], Donald refined the obstruction used by Lisca to determine which connected sums of lens spaces embed smoothly in S^4. The starting point of this work is an adaption of these ideas to the study of slice links with more than one component.

The basic idea of [7] can be described as follows. If a knot K is slice its branched double cover $\Sigma(K)$ is a rational homology sphere that bounds a rational homology ball W. If K is a 2–bridge knot then $\Sigma(K)$ is a lens space, say $L(p,q)$. Each lens space is the boundary of a canonical plumbed 4–manifold $X(p,q)$ with negative definite intersection form. By taking the union $X' = X(p,q) \cup -W$ we obtain a smooth closed oriented 4–manifold with unimodular, negative definite intersection form, and by Donaldson's diagonalization theorem this intersection form is diagonalizable over the integers. The inclusion $X(p,q) \hookrightarrow X'$ induces an embedding of intersection lattices $(H_2(X(p,q);\mathbb{Z}), Q_{X(p,q)}) \hookrightarrow (\mathbb{Z}^N, -I_N)$. This fact turns out to be a powerful obstruction which eventually leads to a complete list of lens spaces that bound rational homology balls.

A link $L \subset S^3$ is (smoothly) slice if it bounds a disjoint union of properly embedded disks in the 4–ball, one for each component of L. Let L be a slice link with n components ($n > 1$). The first observation is that $\Sigma(L)$ is a 3–manifold with $b_1 = n-1$ which bounds a smooth 4–manifold W with the rational homology of a boundary connected sum of $n-1$ copies of $S^1 \times D^3$ (see Proposition 3.1). Motivated by this fact and focusing on the case $n=2$ we are led to the following general problem:

Question 1.1 Which rational homology $S^1 \times S^2$'s bound rational homology $S^1 \times D^3$'s?

In Section 4 we introduce a general procedure which allows one to construct rational homology cobordisms between plumbed 3–manifolds. For any plumbed 3–manifold Y our procedure gives infinitely many plumbed 3–manifolds which are rational homology cobordant to Y. We then introduce a family of plumbed 3–manifolds with $b_1 = 1$. This family includes, up to orientation reversal, all Seifert fibered spaces over the 2–sphere with vanishing Euler invariant. We prove that if a given Y in our family bounds a rational homology $S^1 \times D^3$ then Y can be constructed with our procedure (see Theorem 5.1). This gives us a complete list of the 3–manifolds in our family that bound a rational $S^1 \times D^3$. By specializing Theorem 5.1 to star-shaped plumbing graphs, we obtain the following characterization for the Seifert fibered spaces over the 2–sphere which bound rational homology $S^1 \times D^3$'s:

Theorem 1.2 *A Seifert fibered manifold* $Y = (0; b; (\alpha_1, \beta_1), \dots, (\alpha_h, \beta_h))$ *bounds a* $\mathbb{Q}H - S^1 \times D^3$ *if and only if the Seifert invariants occur in complementary pairs and* $e(Y) = 0$.

Two pairs of Seifert invariants (α_i, β_i) and (α_j, β_j) are *complementary* if they can be chosen so that $\beta_i/\alpha_i + \beta_j/\alpha_j = -1$, ie if $\alpha_i = \alpha_j$ and $\beta_i + \beta_j = -\alpha_i$ (see Section 2.4 for precise definitions).

This result (as well as Theorem 5.1) is obtained by using an obstruction based on Donaldson's theorem. Roughly speaking we proceed as follows. Each Y in our family bounds a negative *semidefinite* plumbed 4–manifold X. If Y bounds a rational homology $S^1 \times D^3$, say W, we can form the closed 4–manifold $X' = X \cup -W$. The intersection form $Q_{X'}$ will again be negative definite and this fact provides the constraints we need for our analysis.

In a subsequent paper [1], we will describe the applications of our work on arborescent link concordance. To each Y we can associate the family $L(Y)$ of arborescent links whose branched double cover is Y. In general, the family $L(Y)$ contains many nonisotopic links. However, these links are all related to each other by Conway mutation. In [1] we will prove the following:

Theorem 1.3 *Let L be a link in $L(Y)$ for some Y described by a plumbing graph satisfying the hypothesis of Theorem 5.1* (*eg any Montesinos link*). *The following conditions are equivalent*:

- *Y bounds a rational homology $S^1 \times D^3$.*
- *There exists $L' \in L(Y)$ that bounds a properly embedded smooth surface S in D^4 with $\chi(S) = 2$ without local maxima.*

In particular, *if L is a 2–component slice link then it has a ribbon mutant.*

This paper is organized as follows. In Section 2 we provide an introduction to plumbed manifolds following Neumann and Raymond [9; 10; 11]. We also introduce some new terminology that will be useful later on. In Section 3 we give some motivation for our work relating rational homology cobordism of 3–manifolds and link concordance. We also state our lattice-theoretical obstruction. In Section 4 we introduce a method that allows one to construct rational homology cobordisms between plumbed 3–manifolds. In Section 5 we state our main theorem (Theorem 5.1) and give a proof modulo a technical result (Theorem 7.1). Sections 6–10 are dedicated to the technical analysis needed to prove Theorem 7.1.

Acknowledgements

I would like to thank my supervisor Paolo Lisca for his support and for suggesting this topic, and Giulia Cervia for her constant encouragement and her help in drawing pictures. I acknowledge partial support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 674978).

2 Plumbed manifolds

In this section, following [9; 10; 11], we review the basic definitions and properties of plumbed 3–manifolds. We recall Neumann's normal form of a plumbing graph, and the generalized continued fraction associated to a plumbing graph. We show how these data behave with respect to orientation reversal. We briefly recall the definitions of lens spaces and Seifert manifolds viewed as special plumbed manifolds.

Definition 2.1 A *plumbing graph* Γ is a finite tree where every vertex has an integral weight assigned to it.

To every plumbing graph Γ we can associate a smooth oriented 4–manifold $P\Gamma$ with boundary $\partial P\Gamma$ in the following way. For each vertex take a disc bundle over the 2–sphere with Euler number prescribed by the weight of the vertex. Whenever two vertices are connected by an edge we identify the trivial bundles over two small discs (one in each sphere) by exchanging the role of the fiber and the base coordinates. We call $P\Gamma$ (resp. $\partial P\Gamma$) a *plumbed* 4*–manifold* (resp. *plumbed* 3*–manifold*).

This definition can be extended to reducible 3–manifolds; if the graph is a finite forest (ie a disjoint union of trees), we take the boundary connected sum of the plumbed 4–manifolds associated to each connected component of Γ. Unless otherwise stated, by a plumbing graph we will always mean a connected one, as in Definition 2.1.

Every plumbed 4–manifold has a nice surgery description which can be obtained directly from the plumbing graph. To every vertex we associate an unknotted circle framed according to the weight of the vertex. Whenever two vertices are connected by an edge, the corresponding circles are linked in the simplest possible way, ie like the Hopf link. The framed link obtained in this way also gives an integral surgery presentation for the corresponding plumbed 3–manifold. The group $H_2(P(\Gamma);\mathbb{Z})$ is a free abelian group generated by the zero sections of the sphere bundles (ie by vertices of the graph). Moreover, with respect to this basis, the intersection form of $P(\Gamma)$, which we indicate by Q_Γ, is described by the matrix M_Γ whose entries (a_{ij}) are defined as follows:

- $a_{i,i}$ equals the Euler number of the corresponding disc bundle;
- $a_{i,j}=1$ if the corresponding vertices are connected;
- $a_{i,j}=0$ otherwise.

Finally note that M_Γ is also a presentation matrix for the group $H_1(\partial P\Gamma;\mathbb{Z})$.

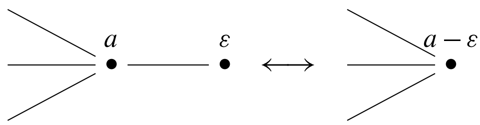
2.1 The normal form of a plumbing graph

We will be mainly interested in plumbed 3–manifolds. There are some elementary operations on the plumbing graph which alter the 4–manifold but not its boundary. Following [9], we will state a theorem which establishes the existence of a unique *normal form* for the graph of a plumbed 3–manifold. In [9] these results are stated in a more general context. Here we extrapolate only what we need in order to deal with plumbed manifolds.

First consider the *blowdown* operation. It can be performed in any of three situations depicted below:

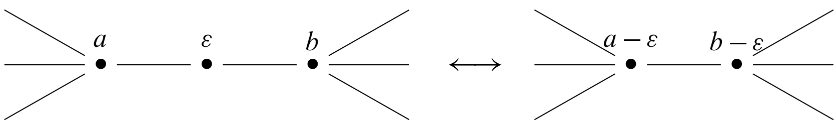
(1) We can add or remove an isolated vertex with weight $\varepsilon \in \{\pm 1\}$ from any plumbing graph.
$$\Gamma \sqcup \overset{\varepsilon}{\bullet} \longleftrightarrow \Gamma$$

(2) A vertex with weight $\varepsilon \in \{\pm 1\}$ linked to a single vertex of a plumbing graph can be removed as shown below:

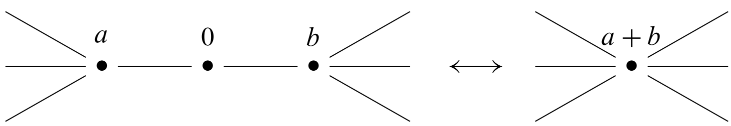


From now on we use three edges coming out of a vertex to indicate that any number of edges may be linked to that vertex.

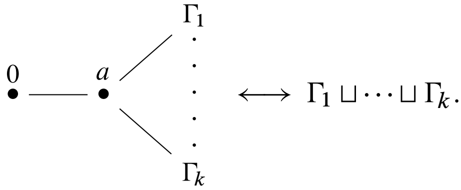
(3) Finally, if a ± 1–weighted vertex is linked to exactly two vertices it can be removed, as shown below:



Next we have the 0–*chain absorption* move. A 0–weighted vertex linked to two vertices can be removed and the plumbing graph changes as shown:



The *splitting* move can be applied in the following situation. Given a plumbing graph with a 0–weighted vertex which is linked to a single vertex v, we may remove both vertices (and all the corresponding edges) obtaining a disjoint union of plumbing trees. We may depict this move by



Proposition 2.2 [9] *Applying any of the above operations and their inverses to a plumbing graph does not change the oriented diffeomorphism type of the corresponding plumbed* 3*–manifold.*

Before discussing the normal form of a plumbing graph we need some terminology. A *linear chain* of a plumbing graph is a portion of the graph consisting of some vertices $v_1, \dots, v_k$ $(k \geq 1)$ such that

- each v_i with $1 < i < k$ is linked only to v_{i-1} and v_{i+1};
- v_1 and v_k are linked to at most two vertices.

A linear chain is *maximal* if it is not contained in any larger linear chain. A vertex of a plumbing graph is said to be

(1) *isolated* if it is not linked to any other vertex;

(2) *final* if it is linked exactly to one vertex;

(3) *internal* otherwise.

Note that isolated and final vertices always belong to some linear chain, while an internal vertex belongs to some linear chain if and only if it is linked to exactly two vertices.

Definition 2.3 A plumbing graph Γ is said to be in *normal form* if one of the following holds:

(1) $\Gamma = \varnothing$ or $\Gamma = \overset{0}{\bullet}$.

(2) Every vertex of a linear chain has weight less than or equal to -2.

Theorem 2.4 [9] *Every plumbing graph can be reduced to a unique normal form via a sequence of blowdowns*, 0*–chain absorptions*, *splittings and their inverses. Moreover*, *two oriented plumbed* 3*–manifolds are diffeomorphic* (*preserving the orientation*) *if and only if their plumbing graphs have the same normal form.*

Remark 2.5 Using this theorem, one can specify a certain class of plumbed 3–manifolds simply by describing the shape of the plumbing graph in its normal form. In particular we will see at the end of this section that lens spaces and some Seifert manifolds admit such a description.

2.2 The continued fraction of a plumbing graph

In this section, following [10] we introduce some additional data associated to a plumbing graph. As we have seen to any plumbing graph Γ we can associate an integral symmetric bilinear form Q_Γ. All the usual invariants of Q_Γ will be denoted referring only to the graph. In particular, rank, signature and determinant will be denoted respectively by $\operatorname{rk}\Gamma$, $(b_+\Gamma, b_-\Gamma, b_0\Gamma)$ and $\det\Gamma$.

Let (Γ, v) be a connected *rooted plumbing graph*, ie a plumbing graph together with the choice of a particular vertex. If we remove from Γ the vertex v and all the corresponding edges, we obtain a plumbing graph Γ_v which is the disjoint union of some trees $\Gamma_1, \dots, \Gamma_k$ (k is the valency of v). Every such tree has a distinguished vertex v_j, which is the one adjacent to v.

Definition 2.6 With the notation above we define the *continued fraction* of Γ as

$$\operatorname{cf}(\Gamma) := \frac{\det\Gamma}{\det\Gamma_v} \in \mathbb{Q} \cup \{\infty\}.$$

We put $\alpha/0 = \infty$ for each $\alpha \in \mathbb{Q}$.

Remark 2.7 This value $\operatorname{cf}(\Gamma)$ depends on the *rooted* plumbing graph (Γ, v). By abusing notation we do not indicate this dependence explicitly. In the sequel, it will always be clear from the context which vertex has been chosen.

Proposition 2.8 [10] *If the weight of the distinguished vertex is $b \in \mathbb{Z}$ then*

$$\det\Gamma = b \cdot \det\Gamma_v - \sum_{i=1}^{k} \Big(\det(\Gamma_i)_{v_i} \prod_{j \neq i} \det\Gamma_j \Big)$$

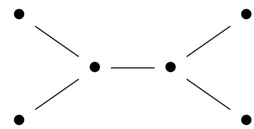
and

$$\operatorname{cf}(\Gamma) = b - \sum_{i=1}^{k} \frac{1}{\operatorname{cf}(\Gamma_i)}.$$

2.3 Reversing the orientation

Let Γ be a plumbing graph in normal form. In this section, following [9], we explain how to compute the normal form for the plumbed manifold $-\partial P\Gamma$, ie $\partial P\Gamma$ with reversed orientation. We call this plumbing graph the *dual graph* of Γ and we denote it by Γ^*.

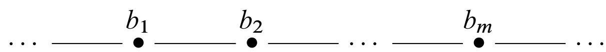
For a vertex v of a plumbing graph which is not on a linear chain we define the quantity $c(v)$ to be the number of linear chains adjacent to v, ie the number of vertices belonging to a linear chain that are linked to v. For instance, in the graph



both the trivalent vertices have $c = 2$. We indicate with $(\dots, -2^{[a]}, \dots)$ a portion of a string with a -2–chain of length $a > 0$, ie a linear chain consisting of a vertices each with weight -2.

Theorem 2.9 [9] *Let Γ be a plumbing graph in normal form. Its dual graph Γ^* can be obtained as follows. The weight $w(v)$ of every vertex which is not on a linear chain is replaced with $-w(v) - c(v)$, and every maximal linear chain of the form*

is replaced with



where the weights are determined as follows. If

$$(a_1, \dots, a_n) = (-2^{[n_0]}, -m_1 - 3, -2^{[n_1]}, -m_2 - 3, \dots, -m_s - 3, -2^{[n_s]})$$

with $n_i \geq 0$, $m_i \geq 0$ and $s > 0$, then

$$(b_1, \dots, b_m) = (-n_0 - 2, -2^{[m_1]}, -n_1 - 3, \dots, -n_{s-1} - 3, -2^{[m_s]}, -n_s - 2).$$

If $(a_1, \dots, a_{n_0}) = (-2^{[n_0]})$ then $(b_1) = (-n_0 - 1)$.

The reason why we are interested in this construction of the dual graph of a plumbing graph in normal form will be clear in Section 5. Essentially we are trying to detect nullcobordant 3–manifolds using obstructions based on Donaldson's diagonalization theorem. Since the property we want to detect does not depend on the orientation of a given 3–manifold, it is natural to examine both a plumbing graph Γ and its dual Γ^*. Moreover, the normal form is specifically defined to give a plumbing graph that minimizes the quantity $b_+(\Gamma)$ among all plumbing graphs representing $\partial P\Gamma$ (see [10, Theorem 1.2]).

We now introduce a quantity that will play an important role in the analysis developed in Sections 6–9.

Definition 2.10 Let Γ be a plumbing graph in normal form, and let $v_1, \dots, v_n$ be its vertices. We define

$$I(\Gamma) := \sum_{i=1}^{n} -3 - w(v_i).$$

The following proposition is proved in [7]. It can also be proved directly using Theorem 2.9.

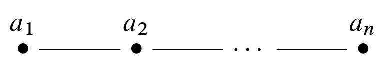
Proposition 2.11 *Let Γ be a linear plumbing graph in normal form. We have*

$$I(\Gamma) + I(\Gamma^*) = -2.$$

2.4 Lens spaces and Seifert manifolds

We briefly recall the plumbing description for lens spaces and Seifert manifolds.

In this context it is convenient to define a lens space as a closed 3–manifold whose Heegaard genus is ≤ 1. The difference with the usual definition is that we are including S^3 and $S^1 \times S^2$. It is well known that every lens space has a plumbing graph which is either empty (S^3) or a linear plumbing graph and that every linear plumbing graph represents a lens space. It follows from Theorem 2.4 that the normal form of a plumbing graph representing a lens space other than S^3 or $S^1 \times S^2$ is a linear plumbing graph



where $a_i \leq -2$ for each i. It is easy to check that given a linear plumbing graph as above we have

$$\mathrm{cf}(\Gamma) = a_1 - \cfrac{1}{a_2 - \frac{1}{a_3 - \cdots}} =: [a_1, \dots, a_n]^-.$$

This fact justifies the name continued fraction. Note that $\mathrm{cf}(\Gamma) < -1$. The usual notation for a lens space $L(p,q)$, defined as $-\frac{p}{q}$–surgery on the unknot, can recovered from the continued fraction as follows. Write $\mathrm{cf}(\Gamma) = \frac{p}{-q}$, so that $p > q \geq 1$ and $(p,q) = 1$. Then $\partial P\Gamma = L(p,q)$.

Remark 2.12 If Γ is a nonempty linear plumbing graph in normal form which is not a 0–weighted single vertex, then $\det \Gamma \neq 0$. We will make extensive use of this fact throughout this work without further reference.

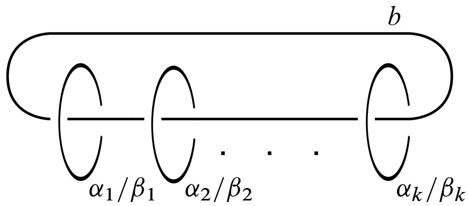


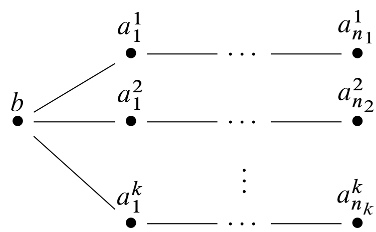
Figure 1: A surgery description for the Seifert fibered manifold $(0; b; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k))$.

A closed Seifert fibered manifold (see [11]) can be described by its *unnormalized Seifert invariants*

$$(g; b; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)),$$

where $g \geq 0$ is the genus of the base surface, $b \in \mathbb{Z}$, $\alpha_i > 1$ and $(\alpha_i, \beta_i) = 1$. This data (which is not unique) uniquely determines the manifold. When $g = 0$ a surgery description for such a manifold is depicted in Figure 1. The following theorem is proved in [11]:

Theorem 2.13 *Let Γ be the following star-shaped plumbing graph in normal form*:



Then $\partial P\Gamma$ is a Seifert manifold with unnormalized Seifert invariants

$$(0, b; (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)),$$

where

$$\frac{\alpha_i}{\beta_i} = [a_1^i, \dots, a_{n_i}^i]^-.$$

The quantity

$$e(Y) := b - \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i}$$

is called the *Euler number* of Y. It is easy to check that

$$e(Y) = \mathrm{cf}(\Gamma), \tag{1}$$

where Γ is the plumbing graph in normal form associated to Y.

Definition 2.14 Let Γ_1 and Γ_2 be two linear plumbing graphs in normal form:

$$\Gamma_1 := \overset{a_1}{\bullet}\!-\!\!\!-\,\overset{a_2}{\bullet}\!-\!\!\!-\,\cdots\,-\!\!\!-\!\overset{a_n}{\bullet} \qquad \Gamma_2 := \overset{b_1}{\bullet}\!-\!\!\!-\,\overset{b_2}{\bullet}\!-\!\!\!-\,\cdots\,-\!\!\!-\!\overset{b_m}{\bullet}$$

Γ_1 and Γ_2 are said to be *complementary* if $\Gamma_2 = \Gamma_1^*$.

Proposition 2.15 *With the notation of Definition 2.14 the following conditions are equivalent*:

1. Γ_1 *and* Γ_2 *are complementary.*
2. $\partial P\big(\overset{b_m}{\bullet}-\cdots-\overset{b_1}{\bullet}-\overset{-1}{\bullet}-\overset{a_1}{\bullet}-\cdots-\overset{a_n}{\bullet}\big) \cong S^1 \times S^2$.
3. $1/\mathrm{cf}(\Gamma_1) + 1/\mathrm{cf}(\Gamma_2) = -1$.

Proof (1) $\Longrightarrow$ (2) This can be checked directly using Theorem 2.9. A series of -1–blowdowns will turn the linear graph above into a 0–weighted single vertex.

(2) $\Longrightarrow$ (3) Consider the continued fraction of the graph representing $S^1 \times S^2$ with respect to the only -1–weighted vertex. We have

$$0 = \frac{\det\big(\overset{b_m}{\bullet}-\cdots-\overset{b_1}{\bullet}-\overset{-1}{\bullet}-\overset{a_1}{\bullet}-\cdots-\overset{a_n}{\bullet}\big)}{\det(\Gamma_1)\det(\Gamma_2)} = -1 - \frac{1}{\mathrm{cf}(\Gamma_1)} - \frac{1}{\mathrm{cf}(\Gamma_2)}.$$

The first equality above holds because for any plumbing graph we have $b_1(\partial P\Gamma) = b_0(\Gamma)$.

(3) $\Longrightarrow$ (1) By the same formula used above we obtain

$$\det\big(\overset{b_m}{\bullet}-\cdots-\overset{b_1}{\bullet}-\overset{-1}{\bullet}-\overset{a_1}{\bullet}-\cdots-\overset{a_n}{\bullet}\big) = 0.$$

After a -1–blowdown we obtain

$$\det\big(\overset{b_m}{\bullet}-\!\!\!-\cdots-\!\!\!-\overset{b_1+1}{\bullet}-\!\!\!-\overset{a_1+1}{\bullet}-\!\!\!-\cdots-\!\!\!-\overset{a_n}{\bullet}\big) = 0;$$

therefore, this plumbing graph is not in normal form, which means that at least one weight among a_1 and b_1 is -2. Suppose, for instance, that $a_1 = -2$.

If $n = 1$, it is easy to see that $m = 1$ as well, and $b_1 = -2$, from which the conclusion follows. Therefore we may assume that $n > 1$.

If $m = 1$, by blowing down the vertex whose weight is $a_1 + 1$ we obtain

$$\overset{b_1+2}{\bullet}-\!\!\!-\!\!\!-\overset{a_2+1}{\bullet}-\!\!\!-\!\!\!-\cdots-\!\!\!-\!\!\!-\overset{a_n}{\bullet}$$

Again, this graph has vanishing determinant and therefore is not in normal form. If $b_1 = -3$, we blow down the vertex whose weight is b_1+2. It follows easily that $a_2 = -2$ and that $n = 2$. If $b_1 < -3$, then $a_2 = -2$; we blow down the vertex whose weight is $a_2 + 1$ and we iterate the argument. This shows that $(a_1, \dots, a_n) = (-2, \dots, -2)$ and that $n = -b_1 - 1$.

If $m > 1$, we claim that $b_1 \le -3$. To see this, assume by contradiction that $b_1 = -2$. By blowing down the vertex whose weight is $a_1 + 1$ we obtain

$$\overset{b_m}{\bullet} \text{---} \cdots \text{---} \overset{b_2}{\bullet} \text{---} \overset{0}{\bullet} \text{---} \overset{a_2+1}{\bullet} \text{---} \cdots \text{---} \overset{a_n}{\bullet}$$

which, by 0–chain absorption, becomes

$$\overset{b_m}{\bullet} \text{---} \cdots \text{---} \overset{b_2+a_2+1}{\bullet} \text{---} \overset{a_3}{\bullet} \text{---} \cdots \text{---} \overset{a_n}{\bullet}$$

This last graph is in normal form, which contradicts the fact that its determinant is zero. This proves the claim.

Now the argument can be iterated. Each time we blow down a -1–vertex we obtain a new linear graph which has exactly one -1–vertex. By repeatedly blowing down -1–vertices we will eventually obtain the graph

$$\overset{b_m}{\bullet} \text{---} \overset{-1}{\bullet} \text{---} \overset{a_n}{\bullet}$$

Since the determinant must vanish, it is easy to verify that $a_n = b_m = -2$ and that

$$\partial P\big(\overset{b_m}{\bullet} \text{---} \overset{-1}{\bullet} \text{---} \overset{a_n}{\bullet} \big) = S^1 \times S^2.$$

This proves (2) and, by Theorem 2.9, also (1). In fact, by induction on the number of blowup operations one can verify that each linear graph corresponds to a pair of complementary strings. This can be done by starting with the last graph we obtained above and then going backwards via blowups. □

Remark 2.16 Strictly speaking, the definition of complementary linear graphs should involve an extra bit of data. In Definition 2.14 we implicitly fixed an initial vertex and a final one on each graph (as suggested by the indexing of the weights). Only in this way does the condition $\Gamma_2 = \Gamma_1^*$ make sense.

It is useful to extend in the obvious way the notion of complementary linear graphs to that of *complementary legs* in a star-shaped plumbing graph. We also say that a pair of Seifert invariants are complementary if they correspond to complementary legs in the associated star-shaped plumbing graph in normal form. It follows by Proposition 2.15 that pairs of complementary legs correspond to pairs of Seifert invariants (α_i, β_i) and (α_j, β_j) that satisfy

$$\frac{\beta_i}{\alpha_i} + \frac{\beta_j}{\alpha_j} = -1.$$

Note that, in general, this formula does not hold if we do not compute the Seifert invariants from the weights of a star-shaped plumbing graph in normal form as in Theorem 2.13.

2.5 The linear complexity of a tree

Let Γ be a plumbing graph in normal form. Let $\mathrm{lc}(\Gamma)$ be the cardinality of the smallest subset of vertices we need to remove from Γ in order to obtain a linear graph. We call $\mathrm{lc}(\Gamma)$ the *linear complexity* of Γ and we set $\mathrm{lc}(\varnothing) = -1$. We stress the fact that because of the uniqueness of the normal form of a plumbing graph it makes sense to talk about the linear complexity of a plumbed 3–manifold. Note that:

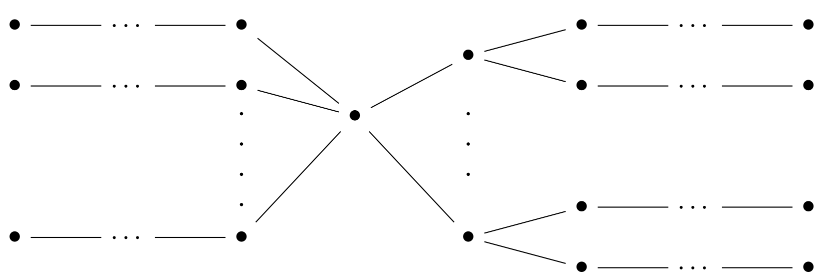
- $\mathrm{lc}(\Gamma) = 0$ if and only if $\partial P\Gamma$ is a lens space.
- If $\partial P\Gamma$ is a Seifert manifold then $\mathrm{lc}(\Gamma) = 1$.
- $\mathrm{lc}(\Gamma_1 \sqcup \Gamma_2) = \mathrm{lc}(\Gamma_1) + \mathrm{lc}(\Gamma_2)$.

Proposition 2.17 *Let Γ be a plumbing graph in normal form such that $\mathrm{lc}(\Gamma) = 1$ and for at least one choice of a vertex $v \in \Gamma$ the graph Γ_v is linear and negative definite. Then*

$$\det \Gamma = 0 \iff \mathrm{cf}\,\Gamma = 0.$$

Proof The proof follows directly from Definition 2.6 and we omit the details. □

In Section 5 we will deal mainly with plumbed 3–manifolds with $\mathrm{lc}(\Gamma) = 1$. A generic plumbing graph Γ with $\mathrm{lc}(\Gamma) = 1$ looks like the one shown below:



Such a graph is made of a distinguished vertex v and several linear components. These linear components are joined to v via a final vertex (on the left-hand side of the picture above) or via an internal vertex (right-hand side).

3 Motivations and obstructions

In this section we start dealing with rational homology cobordisms. As a motivation, we first explain in Proposition 3.1 how rational homology cobordisms of 3–manifolds are relevant for link concordance problems. Then, in Proposition 3.3, we state our lattice-theoretical obstruction, which will be used in the proof of Theorem 5.1.

Two closed, oriented 3–manifolds Y_1 and Y_2 are *rational homology cobordant* (or $\mathbb{Q}H$–cobordant) if there exists a smooth compact 4–manifold W such that

- $\partial W = Y_1 \cup -Y_2$;
- both inclusions $Y_i \to W$ induce isomorphisms $H_*(Y_i;\mathbb{Q}) \cong H_*(W;\mathbb{Q})$.

It is well known that if a rational homology sphere is obtained as the branched double cover along a slice knot then it bounds a rational homology ball. In the next proposition we make an analogous observation concerning branched double covers along slice links with more than one component.

Proposition 3.1 *Let $L \subset S^3$ be a link. Let $S \subset D^4$ be a properly embedded smooth surface without closed components such that $\partial S = L$. Let W be the double cover of D^4 branched along S. Assume that*

$$b_1(\partial W) \le \chi(S) - 1.$$

Then $b_1(W) = \chi(S) - 1$ and $b_2(W) = b_3(W) = 0$. In particular, if $b_1(\partial W) > 0$, we have an isomorphism

$$H_*(W;\mathbb{Q}) \cong H_*\Bigl(\natural_{i=1}^{\chi(S)-1} S^1 \times D^3;\mathbb{Q}\Bigr).$$

Proof As shown in [6], we have a long exact sequence

$$\cdots \to H_i(D^4, S \cup S^3) \to H_i(W, \partial W) \to H_i(D^4, S^3) \to H_{i-1}(D^4, S \cup S^3) \to \cdots,$$

from which we obtain an isomorphism $H_1(D^4, S \cup S^3) \cong H_1(W, \partial W)$. It follows from the exact sequence of the pair that $H_1(D^4, S \cup S^3) = 0$. We conclude that

$0 = H_1(W, \partial W) = H^3(W)$. From the exact sequence of the pair $(W, \partial W)$ with rational coefficients we get

$$\cdots \to H_1(\partial W) \to H_1(W) \to 0.$$

We obtain

$$b_1(W) \leq b_1(\partial W) \leq \chi(S) - 1.$$

Since

$$\chi(W) = 2\chi(B^4) - \chi(S) = 2 - \chi(S) \implies 1 - b_1(W) + b_2(W) = 2 - \chi(S),$$

we see that $b_1(W) = \chi(S) - 1$ and $b_2(W) = 0$. □

Corollary 3.2 *Let L be a slice link with n components ($n > 1$). Let W be the branched double cover of the four-ball branched along a collection of slicing discs for L. We have an isomorphism*

$$H_*(W; \mathbb{Q}) \cong H_*\Big(\natural_{i=1}^{n-1} S^1 \times D^3; \mathbb{Q}\Big)$$

Proof It is well known that $b_1(\partial W) = |L| - 1$ (see for instance [4]). Here $|L|$ denotes the number of components of the link L. Then we may apply Proposition 3.1. □

Motivated by Proposition 3.1, we investigate $\mathbb{Q}H$–cobordisms of plumbed 3–manifolds with $b_1 \geq 1$. Note that if a 3–manifold Y bounds a $\mathbb{Q}H - \natural_n S^1 \times D^3$, then $b_1(Y)$ equals the number of $S^1 \times D^3$ summands.

Proposition 3.3 *Let Y be a connected 3–manifold with $b_1(Y) = n$. Suppose that Y bounds smooth 4–manifolds X and W with the following properties:*

- *X is simply connected, negative semidefinite and $\operatorname{rk} Q_X = b_2(X) - n$.*
- *$H_*(W; \mathbb{Q}) = H_*\big(\natural_{i=1}^{n} S^1 \times D^3; \mathbb{Q}\big)$.*

Then there exists a morphism of integral lattices

$$((H_2(X); \mathbb{Z}), Q_X) \to (\mathbb{Z}^{b_2(X)-n}, -\mathrm{Id}).$$

In particular, for every definite sublattice $(G, Q_G) \subset (H_2(X), Q_X)$ whose rank is $b_2(X) - n$, we obtain an embedding of integral lattices

$$(G, Q_G) \to (\mathbb{Z}^{b_2(X)-n}, -\mathrm{Id}).$$

Proof Consider the smooth 4–manifold $X' := X \cup_Y -W$. The Mayer–Vietoris exact sequence with integral coefficients reads

$$\to H_3(X')\to H_2(Y)\to H_2(X)\oplus H_2(W)\to H_2(X')\to H_1(Y)\to H_1(W)\to H_1(X')\to 0.$$

Note that $b_1(Y)=b_1(W)$; moreover, the map $H_1(Y;\mathbb{Q})\to H_1(W;\mathbb{Q})$ is an isomorphism. It follows that $b_1(X')=0$. The group $H_2(W)$ is finite. Note that $b_3(X')=0$. If we consider the above exact sequence with rational coefficients, we obtain

$$0\to H_2(Y;\mathbb{Q})\to H_2(X;\mathbb{Q})\to H_2(X';\mathbb{Q})\to 0;$$

therefore, $b_2(X')=b_2(X)-b_2(Y)=b_2(X)-n$. Now note that, by Novikov additivity, $\sigma(X')=\sigma(X)$. This shows that X' is a smooth, closed, negative definite 4–manifold. By Donaldson's diagonalization theorem its intersection form is equivalent to the standard negative definite form on $\mathbb{Z}^{b_2(X')}$. The inclusion $X\to X'$ induces the desired morphism of integral lattices.

The last assertion follows easily. The map

$$\varphi\colon (G,Q_G)\to(\mathbb{Z}^{b_2(X)-n},-\mathrm{Id})$$

preserves the intersection form. Since Q_G is negative definite, φ must be injective and is therefore an embedding of integral lattices. □

4 Constructing $\mathbb{Q}H$–cobordisms

In this section we introduce a procedure for constructing rational homology cobordisms between plumbed 3–manifolds; our method is explained in Proposition 4.5. We then introduce some *elementary building blocks* which are sufficient to produce all manifolds satisfying the hypotheses of Theorem 5.1 which bound rational homology $S^1\times D^3$'s.

Recall that a *rooted* plumbing graph (Γ,v) is a plumbing graph with a distinguished vertex. In particular, a rooted plumbing graph is necessarily nonempty.

Definition 4.1 Let (Γ_1,v_1) and (Γ_2,v_2) be two rooted plumbing graphs. Let Γ be the plumbing graph obtained from $\Gamma_1\sqcup\Gamma_2$ by identifying the two distinguished vertices and taking the sum of the corresponding weights. We say that Γ is obtained by *joining* together Γ_1 and Γ_2 along v_1 and v_2 and we write

$$\Gamma := \Gamma_1 \vee_{v_1,v_2} \Gamma_2.$$

The following proposition follows immediately from Proposition 2.8:

Proposition 4.2 *With the above notation we have*

$$\mathrm{cf}(\Gamma_1 \vee_{v_1,v_2} \Gamma_2) = \mathrm{cf}(\Gamma_1) + \mathrm{cf}(\Gamma_2)$$

provided that the continued fractions on the right are computed with respect to the vertices v_1 and v_2, and the continued fraction on the left is computed with respect to the vertex resulting from joining v_1 and v_2.

Lemma 4.3 *Let W be a connected 4–dimensional handlebody without 3–handles. If $H_*(\partial W;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$ then $H_1(W;\mathbb{Q}) = 0$.*

In particular, if W is built using a single 1–handle h^1 and a single 2–handle h^2, then the algebraic intersection of these handles does not vanish.

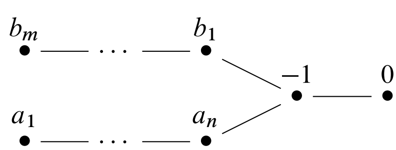
Proof The homology exact sequence of the pair $(W, \partial W)$ with rational coefficients reads

$$\cdots \to H_1(\partial W) \to H_1(W) \to H_1(W, \partial W) \to 0.$$

Since $H_1(\partial W) = 0$ and by Lefschetz duality $H_1(W, \partial W) = H^3(W) = 0$ the conclusion follows. If there are only two handles h^1 and h^2, the attaching sphere of h^2 must have nonzero intersection number with the belt sphere of h^1, otherwise h^1 would represent a nontrivial element in $H_1(W)$. □

The following lemma is an immediate consequence of the splitting move:

Lemma 4.4 *Let $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$ be strings (where each coefficient is ≤ -2). The 3–manifold described by the plumbing graph*



is a rational homology sphere.

Proposition 4.5 *Let (Γ, v) be a rooted plumbing graph such that $\partial P(\Gamma) = S^1 \times S^2$ and $\partial P(\Gamma \setminus \{v\})$ is a rational homology sphere. Let (Γ', v') be any rooted plumbing graph.*

Then $b_1(\partial P(\Gamma')) = b_1(\partial P(\Gamma' \vee_{v',v} \Gamma))$ and these manifolds are $\mathbb{Q}H$–cobordant.

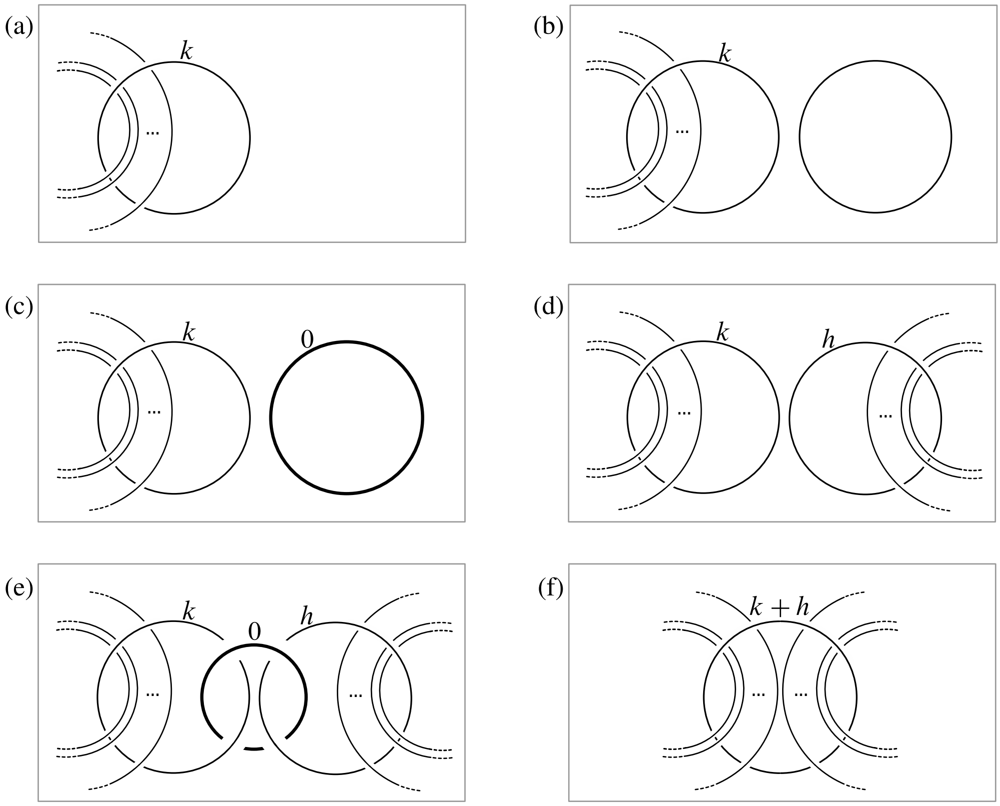


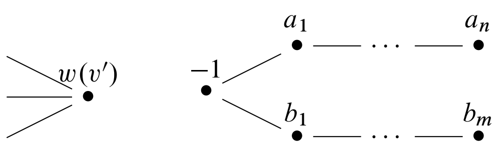
Figure 2: A rational homology cobordism between $\partial P\Gamma'$ and $\partial P(\Gamma' \vee_{v',v} \Gamma)$.

Proof In Figure 2(a) we have a surgery description for $\partial P\Gamma'$. First we attach a 4–dimensional 1–handle to $\partial P\Gamma' \times I$ as shown in Figure 2(b). In Figure 2(c) we draw the boundary of the four manifold obtained after the 1–handle attachment. This is just $\partial P\Gamma' \sharp S^1 \times S^2$. In Figure 2(d) we draw the same manifold replacing the 0–framed circle with the surgery diagram associated to the graph Γ. Now we attach a 4–dimensional 2–handle as shown in Figure 2(e). Via a zero-absorption move the result of this 2–handle attachment is a 4–manifold whose bottom boundary is $\partial P(\Gamma' \vee_{v',v} \Gamma)$. This is shown in Figure 2(f). We have constructed a cobordism W between $\partial P\Gamma'$ and $\partial P(\Gamma' \vee_{v',v} \Gamma)$ which consists of one 1–handle and one 2–handle. In order to prove that W is in fact a $\mathbb{Q}H$–cobordism it suffices to check that the algebraic intersection between the attaching sphere of the 2–handle and the belt sphere of the 1–handle does not vanish.

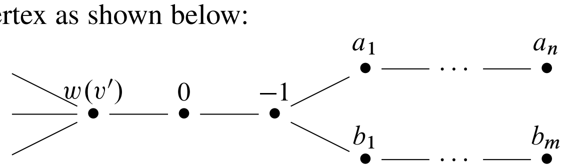
Let us write α for the attaching sphere of the 2–handle. The first homology group of $\partial P\Gamma' \sharp S^1 \times S^2$ is $\mathbb{Q}^{b_1(\partial P\Gamma')} \oplus \mathbb{Q}$. Our algebraic intersection number is nonzero if and only if α represents a nontrivial element when projected into $H_1(S^1 \times S^2)$. Note that in $H_1(\partial P\Gamma' \sharp S^1 \times S^2)$ the curve α is homologous to the pair of curves α_1 and α_2 shown in Figure 3. This means that the projection of α in $H_1(S^1 \times S^2)$ is equivalent to α_2. The fact that α_2 is a nontrivial element in $H_1(S^1 \times S^2)$ follows immediately from our hypotheses on (Γ, v). To see this, let $\widetilde{L}$ be the link that gives a surgery description for $S^1 \times S^2$ in Figure 2(d). Applying the splitting move on the link $\alpha_2 \cup \widetilde{L}$ we see that the 3–manifold described by this link is precisely $\partial P(\Gamma \setminus \{v\})$, which by our assumption is a rational homology sphere. This fact ensures that α_2 represents a nontrivial element in $H_1(S^1 \times S^2; \mathbb{Q})$.

It follows that $b_1(\partial P\Gamma') = b_1(\partial P(\Gamma' \vee_{v',v} \Gamma))$ and that W is a $\mathbb{Q}H$–cobordism. $\square$

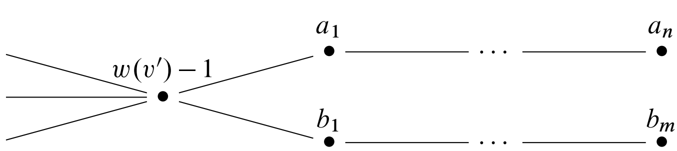
Remark 4.6 The 2–handle attachment used in Proposition 4.5 can also be described in terms of plumbing graphs as follows. We start with $\partial P(\Gamma' \sqcup \Gamma)$, which has the description



where, for simplicity, we have chosen Γ as in Lemma 4.4. The 2–handle then appears as an additional vertex as shown below:



This last level of the cobordism can be described also by the following plumbing graph, using the 0–chain absorption move:



Example 4.7 Let $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$ be two complementary strings. The plumbing graph associated to the string $(a_n, \dots, a_1, -1, b_1, \dots, b_m)$ represents $S^1 \times S^2$. By the previous proposition all lens spaces associated to strings of the form

$$(a_n, \dots, a_1, -2, b_1, \dots, b_m)$$

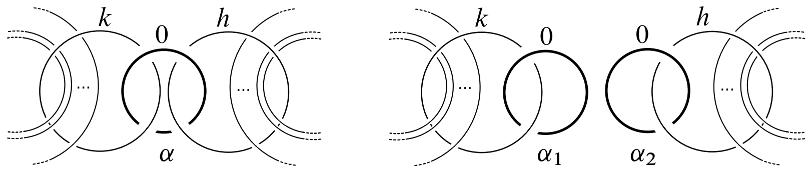
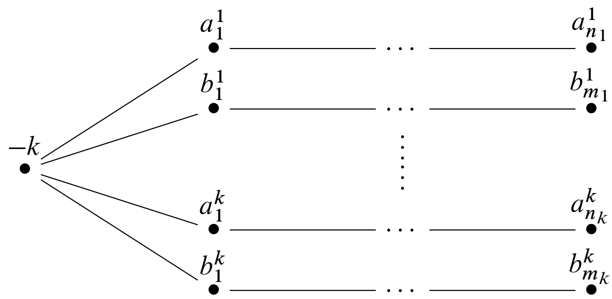


Figure 3: The thick curve on the leftmost diagram is homologous to the sum of the two thick curves on the rightmost diagram.

are $\mathbb{Q}H$–cobordant to S^3. In fact, the corresponding plumbing graph is obtained by joining together a -1–weighted vertex and a graph as in Lemma 4.4.

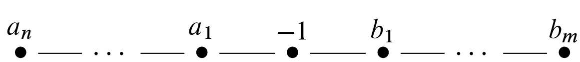
Example 4.8 Choose strings $(a^i_{n_i}, \dots, a^i_1, -1, b^i_1, \dots, b^i_{m_i})$, where $i = 1, \dots, k$, as in the previous example. Consider the plumbed 3–manifold described by the star-shaped plumbing graph



By Proposition 4.5 such a manifold is $\mathbb{Q}H$–cobordant to $S^1 \times S^2$ and thus it bounds a $\mathbb{Q}H - S^1 \times D^3$. In Section 5 we will see that these are the only Seifert manifolds over the 2–sphere with this property.

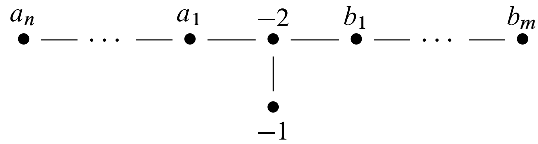
4.1 Elementary building blocks

In the previous example we have used the graph

$\Gamma_1 :=$ 

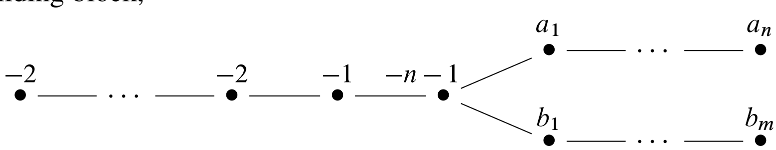
as a building block for constructing rational homology cobordisms of 3–manifolds. This is somehow the simplest way to use Proposition 4.5. The process can be iterated by constructing more complicated pieces to be used as building blocks.

Keeping in mind that we are interested in plumbed manifolds with $\mathrm{lc} = 1$, we may introduce three more building blocks. The graph Γ_1 can be slightly modified, obtaining

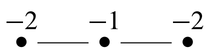
$\Gamma_2 :=$


Another building block can be obtained starting with

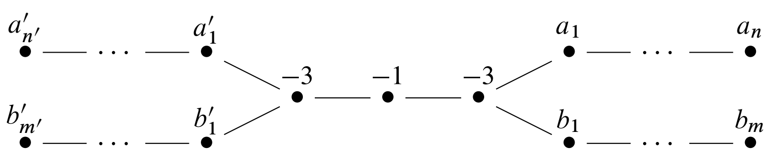
where $n-1$ is the length of the -2–chain ($n \geq 2$). This is just a special case of Γ_1. Now we join this graph with Γ_1 along the vertices of weight $-n$ and -1. We obtain our third building block,

$\Gamma_3 :=$


Note that $\partial P\Gamma_3 = S^1 \times S^2$. A fourth building block can be constructed as follows. We start with



and then we attach to the final vertices of this graph two linear graphs like Γ_1. We obtain

$\Gamma_4 :=$


Note that this last graph does not represent $S^1 \times S^2$ since its normal form can be obtained by blowing down the -1–vertex. Each of the four building blocks we have introduced has a distinguished -1–weighted vertex. From now on we will implicitly consider each of these graphs as a rooted plumbing graph where the preferred vertex is the one whose weight is -1.

Definition 4.9 The four families of rooted plumbing graphs introduced above will be called *building blocks* of the first, second, third and fourth type, respectively.

The following proposition is an immediate consequence of Proposition 4.5:

Proposition 4.10 *Let Γ be a plumbing graph obtained by joining together two or more building blocks of any type along their -1–vertices. Then*:

(1) Γ *is in normal form.*

(2) $\mathrm{lc}(\Gamma) = 1$.

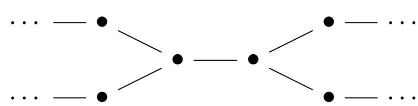
(3) $\partial P\Gamma$ *bounds a* $\mathbb{Q}H - S^1 \times D^3$.

Our main result, Theorem 5.1, should be thought of as a converse of this last proposition.

5 Main results

In this section we state our main result, Theorem 5.1. We give a proof modulo a technical result, Theorem 7.1, whose statement and proof are postponed to the next sections. We explain how to specialize our result to Seifert fibered spaces over the 2–sphere in Theorem 5.2.

First we introduce some terminology. Let Γ be a plumbing graph in normal form such that $\mathrm{lc}(\Gamma) = 1$. Choose $v \in \Gamma$ such that $\widetilde{\Gamma} := \Gamma \setminus \{v\}$ is linear. The linear graph $\widetilde{\Gamma}$ is a disjoint union of connected linear graphs $\Gamma_1, \dots, \Gamma_k$. We call Γ_i a *final leg* or an *internal leg* according to whether v is linked to a final vertex of Γ_i or an internal one. We indicate with $i(\Gamma, v)$ and $f(\Gamma, v)$ the number of internal and final legs of Γ. Finally, each internal leg of Γ has a distinguished vertex which is 3–valent in Γ. We call these vertices the *nodes* of Γ, and we indicate with $N(\Gamma)$ the set of all the nodes. Note that, in some cases, these definitions depend on the choice of the vertex v. This is the case for three-legged star-shaped plumbing graphs (there are four choices for the vertex v) and plumbing graphs like



where there are two possible choices for the vertex v.

Theorem 5.1 *Let Γ be a plumbing graph in normal form with* $\mathrm{lc}(\Gamma) = 1$. *Choose a vertex $v \in \Gamma$ such that $\widetilde{\Gamma} := \Gamma \setminus \{v\}$ is linear. Suppose that each node of Γ has weight less than or equal to -2 and that*

$$I(\widetilde{\Gamma}) \leq -f(\Gamma, v) - 2i(\Gamma, v) - \sum_{u \in N(\Gamma)} \max\{0, w(u) + 3\}. \tag{2}$$

The following conditions are equivalent:

- *the* 3*–manifold* $\partial P\Gamma$ *bounds a* $\mathbb{Q}H - S^1 \times D^3$;
- *equality holds in* (2) *and* Γ *is obtained by joining together building blocks along* -1*–vertices.*

Now we present a proof of this theorem that relies on a technical result, Theorem 7.1, which will be proved in the following sections.

Proof If $\partial P\Gamma$ is obtained by joining together building blocks along -1–vertices then the conclusion follows from Proposition 4.10.

Let Γ be a plumbing graph in normal form satisfying the hypotheses of the theorem and let W be a $\mathbb{Q}H - S^1 \times D^3$ such that $\partial W = \partial P\Gamma$. Let N be the number of vertices of Γ. Note that $b_0(\Gamma) = b_1(\partial P\Gamma) = 1$; moreover, $H_2(P\Gamma; \mathbb{Z})$ contains a free subgroup of rank $N-1$ on which Q_Γ is negative definite (it is the subgroup $\mathbb{Z}\widetilde{\Gamma}$ spanned by all vertices in $\widetilde{\Gamma}$). It follows that Q_Γ is negative semidefinite; more precisely,

$$(b_0(\Gamma), b_-(\Gamma), b_+(\Gamma)) = (1, N-1, 0).$$

Therefore we are in the situation described in Proposition 3.3. There exists a morphism of integral lattices

$$\Phi\colon \big(H_2(X(\Gamma); \mathbb{Z}), Q_\Gamma\big) \to (\mathbb{Z}^{N-1}, -\mathrm{Id}).$$

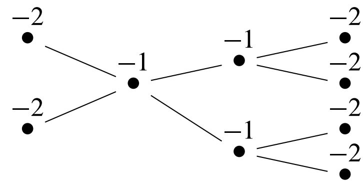
Precomposing this map with the inclusion $(\mathbb{Z}\widetilde{\Gamma}, Q_{\widetilde{\Gamma}}) \hookrightarrow \big(H_2(X(\Gamma); \mathbb{Z}), Q_\Gamma\big)$, we obtain an embedding of integral lattices

$$\widetilde{\Phi}\colon (\mathbb{Z}\widetilde{\Gamma}, Q_{\widetilde{\Gamma}}) \to (\mathbb{Z}^{N-1}, -\mathrm{Id}).$$

Let us write $\{v_1, \dots, v_{N-1}\}$ for the set of vertices of $\widetilde{\Gamma}$. Now consider the subset $S := \{\Phi(v_1), \dots, \Phi(v_{N-1})\} \subset \mathbb{Z}^{N-1}$. The extra vector $\Phi(v)$ is linked once to each connected component of $\widetilde{\Gamma}$ and is orthogonal to every other vector. The subset S satisfies all the hypotheses of Theorem 7.1 and the conclusion follows. □

Even though the class of plumbed manifolds that satisfy the hypotheses of Theorem 5.1 is quite large (it includes, up to orientation reversal, all Seifert fibered spaces over the 2–sphere with vanishing Euler invariant) some of the assumptions on the plumbing graph are rather technical and unnatural. The need for these hypotheses can be explained as follows.

The fact that every vertex in $\widetilde{\Gamma}$ has weight less or equal to -2 allows us to avoid indefinite plumbing graphs. Consider, for instance, the following plumbing graph:

$\Gamma :=$ 

Note that Γ is in normal form. We have

$$(b_0\Gamma, b_+\Gamma, b_-\Gamma) = (1, 1, 7).$$

Moreover, this plumbing graph is *selfdual*, meaning that $\Gamma^* = \Gamma$, therefore reversing the orientation does not help. Theorem 5.1 does not say if $\partial P\Gamma$ bounds a $\mathbb{Q}H - S^1 \times D^3$. However, in this particular case $\partial P\Gamma$ does bound a $\mathbb{Q}H - S^1 \times D^3$. This can be checked easily using Proposition 4.5. By splitting off three building blocks of the first type and then applying the splitting move, we obtain a 0–weighted single vertex. It follows that $\partial P\Gamma$ is $\mathbb{Q}H$–cobordant to $S^1 \times S^2$.

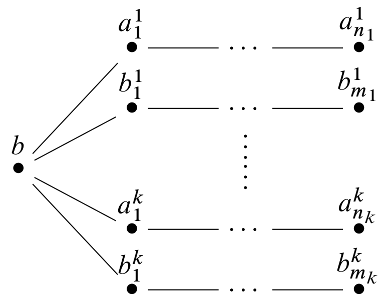
The reason why we need the condition (2) can be explained as follows. In the proof of Theorem 5.1 we have shown that $\widetilde{\Gamma}$ gives rise to a subset $S = \{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ with certain properties. The starting point of our analysis is that these subsets are well understood provided that $I(S) < 0$. We use the known results on such subsets, as developed in [7; 8], to show that the possible graphs of $S \cup \{v\}$, where v is the vector that corresponds to the extra vertex in Γ, are obtained by joining together building blocks along -1–vertices.

5.1 Seifert manifolds

As we show in the next theorem, the assumption $I(\widetilde{\Gamma}) < 0$ in Theorem 5.1 can be avoided when both Γ and Γ^* are negative semidefinite. This is not true for every graph with $\mathrm{lc}(\Gamma) = 1$ and $b_0(\Gamma) = 1$. It is true, however, if we restrict ourselves to star-shaped plumbing graphs. The following theorem should be compared with Theorem 1.3 in [2] (in fact the same technique is used for the proof).

Theorem 5.2 *A Seifert fibered manifold $Y = (0; b; (\alpha_1, \beta_1), \dots, (\alpha_h, \beta_h))$ bounds a $\mathbb{Q}H - S^1 \times D^3$ if and only if the Seifert invariants occur in complementary pairs and $e(Y) = 0$.*

Proof Assume that the Seifert invariants occur in complementary pairs and that $e(Y) = 0$. By Theorem 2.13, we may write $Y = \partial P\Gamma$, where Γ is the following plumbing graph in normal form:



Here the legs are pairwise complementary. Call $\Gamma_1^a, \Gamma_1^b, \dots, \Gamma_k^a, \Gamma_k^b$ the legs of Γ. The condition $e(Y)=0$ implies that $b=-k$. Indeed,

$$0 = e(Y) = \mathrm{cf}(\Gamma) = b - \sum_{i=1}^{k}\left(\frac{1}{\mathrm{cf}(\Gamma_i^a)} + \frac{1}{\mathrm{cf}(\Gamma_i^b)}\right) = b + \sum_{i=1}^{k} 1.$$

The conclusion follows from Proposition 4.5, as explained in Example 4.8.

Now assume that Y bounds a $\mathbb{Q}H - S^1 \times D^3$. Then, so does $-Y$. Let Γ and Γ^* be their plumbing graphs in normal form, and let $\widetilde{\Gamma}$ and $\widetilde{\Gamma}^*$ be the graphs obtained from Γ and Γ^* by removing the central vertices. Note that $\widetilde{\Gamma}^*$ is in fact the dual of $\widetilde{\Gamma}$, so there is no ambiguity with this notation. By Proposition 2.11 we have

$$I(\widetilde{\Gamma}) + I(\widetilde{\Gamma}^*) = -k,$$

where k is the number of pairs of legs in Γ. In particular we may assume, for instance, that $I(\widetilde{\Gamma}) \le -k$. Since for a star-shaped graph we have no nodes and no internal legs, condition (2) becomes $I(\widetilde{\Gamma}) \le -f(\widetilde{\Gamma}) = -k$. Therefore we may apply Theorem 5.1. Γ is obtained by joining together building blocks along their -1–vertices. Since Γ is star-shaped, only building blocks of the first type may occur, which means that Y belongs to the family described in Example 4.8. □

6 The language of linear subsets

In this section we start our technical analysis needed to complete the proof of Theorem 5.1. We begin providing a brief introduction to the language of good subsets and we prove Lemma 6.5, which will be used extensively throughout the rest of the paper. In Section 7 we state the main technical results, Theorems 7.1 and 7.2, and explain the strategy of the proofs. In Section 8 we carry out a detailed analysis of certain good subsets and we conclude by proving Theorem 7.2. In Section 9 we prove what we need to fill the gap between Theorem 7.1 and Theorem 7.2. Finally, in Section 10 we give the proof of Theorem 7.1.

An *intersection lattice* is a pair (G, Q_G) of a free abelian group G together with a $\mathbb{Z}$–valued symmetric bilinear form on it. We indicate with $(\mathbb{Z}^N, -\mathrm{Id})$ the intersection lattice with the standard negative definite form defined by

$$e_i \cdot e_j = -\delta_{ij}.$$

We will always work with $\mathbb{Z}^N$ with the above form on it, so in most cases we will omit the form and indicate the intersection lattice simply by $\mathbb{Z}^N$. Let $S = \{v_1, \dots, v_N\} \subset \mathbb{Z}^N$ be such that

- $v_i \cdot v_i \leq -2$;
- $v_i \cdot v_j \in \{0, 1\}$ if $i \neq j$.

Define the *intersection graph* of S as the graph having a vertex for each element of S and an edge for every pair (v_i, v_j) such that $v_i \cdot v_j = 1$. We indicate this graph with Γ_S. The graph Γ_S can be given integral weights on its vertices: the weight of the vertex corresponding to v_i is $v_i \cdot v_i$.

Definition 6.1 A subset $S \subset \mathbb{Z}^N$ satisfying the above properties is said to be a *linear subset* whenever Γ_S is a linear graph. We will also say that S is *treelike* whenever its graph is a tree. In this case we require that $v_i \cdot v_i \leq -2$ only when v_i corresponds to a vertex on a linear chain.

Note that the graph of a treelike subset is a plumbing graph in normal form. We will use all the terminology we have introduced for plumbing graphs and intersection forms in this new context without stating the obvious definitions. For example, given a linear subset S, a vector $v \in S$ can be isolated, internal or final just like the vertex of a plumbing graph.

Given $v \in \mathbb{Z}^N$ and some basis vector e_i, we say that e_i *hits* v (or that v hits e_i) if $v \cdot e_i \neq 0$. Two vectors $v, w \in \mathbb{Z}^N$ are *linked* if there exists a basis vector that hits both of them. A subset $S \subset \mathbb{Z}^N$ is *irreducible* if for every pair of vectors $v, w \in S$ there exists a sequence of vectors in S,

$$v_0 = v, v_1, \dots, v_n = w,$$

such that v_i and v_{i+1} are linked for $i = 0, \dots, n-1$. A subset which is not irreducible is said to be *reducible*. A linear irreducible subset is called a *good subset*. A good subset whose graph is connected is a *standard subset*. We indicate with $c(S)$ the number of connected components of Γ_S. This should not be confused with the number of irreducible components, for which we do not introduce any symbol. In general an irreducible component may have a graph consisting of several connected components.

There are some elementary operations that, under certain assumptions, can be performed on a linear subset in order to obtain a smaller linear subset. Here we restrict ourselves to -2*–final expansions* and -2*–final contractions* because these are the only operations that we need. In [7] a more general notion of expansions and contractions is used. We indicate with $\pi_h\colon \mathbb{Z}^N \to \mathbb{Z}^{N-1}$ the projection onto the subgroup $\langle e_1, \dots, e_{h-1}, e_{h+1}, \dots, e_N \rangle$.

Definition 6.2 Let $S = \{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ be a linear subset. Suppose that there exists e_i such that

- e_i only hits two vectors v_h and v_k;
- both of these vectors are final;
- $v_h \cdot v_h = -2$ and $v_k \cdot v_k < -2$.

We say that the subset $S' := \pi_h(S \setminus \{v_h\})$ is obtained from S by -2*–final contraction* and we write $S \searrow S'$. We also say that S is obtained from S' by -2*–final expansion* and we write $S' \nearrow S$.

If we think of a subset $S \subset \mathbb{Z}^{n-1}$ as a square matrix whose columns are the vectors $v_1, \dots, v_n$, then a -2–final contraction consists in removing one column and one row provided that the above conditions are satisfied. Note that a -2–final contraction (or expansion) of a linear subset S is again a linear subset S' whose graph $\Gamma_{S'}$ has the same number of components as Γ_S provided that the vector v_h in Definition 6.2 is not isolated.

Definition 6.3 Let $S' = \{v_1, \dots, v_N\} \subset \mathbb{Z}^N$ with $N \geq 3$ be a good subset. Let $C' = \{v_{s-1}, v_s, v_{s+1}\} \subset S'$ be such that $\Gamma_{C'}$ is a connected component of $\Gamma_{S'}$ with $v_{s-1} \cdot v_{s-1} = v_{s+1} \cdot v_{s+1} = -2$ and $v_s \cdot v_s < -2$. Suppose that there exists e_j which hits all the vectors in C' and no other vector of S'. Let S be a subset obtained from S' via a sequence of -2–final expansions performed on C'. The component $C \subset S$ corresponding to $C' \subset S'$ is called a *bad component* of the good subset S.

We indicate the number of bad components of a good subset with $b(S)$. Given elements $v_1, \dots, v_j$ of a linear subset we also define

$$E(v_1, \dots, v_j) := |\{k \mid e_k \cdot v_1 \neq 0, \dots, e_k \cdot v_j \neq 0\}|.$$

The situation we need to study is that of a linear subset together with an extra vector v which is orthogonal to all but one vector, say w_i, of each connected component S_i of S and $v\cdot w_i = 1$. This last condition is expressed by saying that v *is linked once to* w_i.

The following lemmas will be used several times in the next sections.

Lemma 6.4 *Let Γ be a linear plumbing graph in normal form with connected components $\Gamma_1,\dots,\Gamma_k$. Choose vertices $v_i\in\Gamma_i$ for $1\le i\le k$. Let Γ' be the graph obtained from Γ by adding a new vertex v with weight $w(v)\le -1$ and new edges for the pairs (v,v_i). If $\det\Gamma'=0$, then one of the following holds*:

- $w(v) > -k$.
- $w(v_j) = -2$ *for some* $j\in\{1,\dots,k\}$.

Proof Since $\det\Gamma'=0$ and $\det\Gamma\neq 0$, by Proposition 2.17 we must have $\operatorname{cf}\Gamma'=0$. Computing $\operatorname{cf}\Gamma'$ with respect to the vertex v, using Proposition 2.8, we obtain

$$w(v)-\sum_{i=1}^{k}\frac{1}{w(v_i)-1/\alpha_i-1/\beta_i}=0,$$

where α_i and β_i are the continued fractions of the two components of $\Gamma_i\setminus\{v_i\}$, rooted at the vertices adjacent to v_i. Note that, if v_i is final, there is only one component. In this case we set $1/\beta_i=0$. Suppose that for each $1\le j\le k$ we have $w(v_j)\le -3$. We need to prove that $w(v)>-k$. Each α_i (and β_i if v_i is internal) is the continued fraction of a linear connected plumbing graph in normal form rooted at a final vertex. Therefore $\alpha_i,\beta_i<-1$ and, since $w(v_j)\le -3$, we have

$$w(v_i)-\frac{1}{\alpha_i}-\frac{1}{\beta_i}<-1 \implies \sum_{i=1}^{k}\frac{1}{w(v_i)-1/\alpha_i-1/\beta_i}>-k.$$

Combining this fact with the expression for $\operatorname{cf}\Gamma$ we obtain $w(v)>-k$ and we are done. □

Lemma 6.5 *Let $S\subset\mathbb{Z}^N$ be a linear subset. Let $S_1,\dots,S_n$ be the connected components of S. Suppose there is a vector $v\in\mathbb{Z}^N$ which is* ***linked once*** *to a vector of each S_i, say v_i (ie $v\cdot v_i=1$), and is orthogonal to every other vector of $S_i\setminus v_i$. Then*

$$v\cdot v>\sum_{i=1}^{n}v_i\cdot v_i.$$

Proof Let M be the $N \times N$ matrix whose columns are the elements of S. The conditions on the extra vector v can be expressed as a linear system of equations, namely

$$ {}^{t}M v = \sum_{i=1}^{n} e_{k_i}, \tag{3} $$

where the $(k_i)^{\text{th}}$ column of M is v_i. Multiplying both sides of (3) by M, we get

$$ M\,{}^{t}M v = M \sum_{i=1}^{n} e_{k_i} = \sum_{i=1}^{n} v_i. \tag{4} $$

The matrix $M\,{}^{t}M$ is conjugated to ${}^{t}MM$; in particular, they have the same eigenvalues. The matrix $-{}^{t}MM$ represents the intersection form of $P\Gamma_S$. It consists of n blocks, one for each connected component of S. Each block can be diagonalized as shown in Chapter V of [3]; the k^{th} eigenvalue is given by the negative continued fractions corresponding to the first k diagonal entries. In particular, it is easy to prove by induction that, for each eigenvalue λ, we have $\lambda < -1$. It follows that

$$ \|v\|^2 < \|M\,{}^{t}M v\|^2 = \Big\| \sum_{i=1}^{n} v_i \Big\|^2 = \sum_{i=1}^{n} \|v_i\|^2, $$

where $\|\cdot\|$ denotes the usual Euclidean norm. Rewriting the above inequality using the standard negative definite product in $\mathbb{Z}^N$, we obtain

$$ v \cdot v > (M\,{}^{t}M v) \cdot (M\,{}^{t}M v) = \Big(\sum_{i=1}^{n} v_i \Big) \cdot \Big(\sum_{i=1}^{n} v_i \Big) = \sum_{i=1}^{n} v_i \cdot v_i. \qquad \square $$

7 Main technical results and strategy of the proof

The key technical result that will complete the proof of Theorem 5.1 is the following:

Theorem 7.1 *Let $S \subset \mathbb{Z}^N$ be a linear subset. Suppose that there exists $v \in \mathbb{Z}^N$ which is linked once to a vector of each connected component of S and is orthogonal to any other vector of S. Assume also that*

$$ I(\Gamma_S) \le -f(\Gamma_{S\cup\{v\}}, v) - 2i(\Gamma_{S\cup\{v\}}, v) - \sum_{u \in N(\Gamma_{S\cup\{v\}})} \max\{0, u \cdot u + 3\}. \tag{5} $$

Then $\Gamma_{S\cup\{v\}}$ can be obtained by joining together two or more building blocks along their -1–vertices.

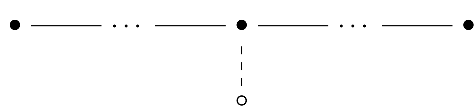
The main ingredient for the proof of Theorem 7.1 is the following result, which explains that the irreducible components of the given subset together with the corresponding extra vector give rise to building blocks:

Theorem 7.2 *Let $S \subset \mathbb{Z}^N$ be a good subset such that $I(S) + c(S) \leq 0$ and $I(S) + b(S) < 0$. Suppose there exists $v \in \mathbb{Z}^N$ which is linked once to a vector of each connected component of S and is orthogonal to all the other vectors of S. Then $v \cdot v = -1$ and $\Gamma_{S \cup \{v\}}$ is a building block.*

The idea of the proof of Theorem 7.2 is the following. The assumptions on S are chosen so that, by the results of [8] the subset S falls in one of the following classes:

(1) $c(S) = 1$, so that the graph of S is a single linear component

In this case we will prove that the extra vector v is linked to a internal vector of S and that the graph of $S \cup \{v\}$, which is of the form

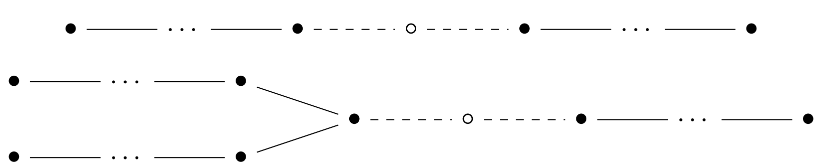


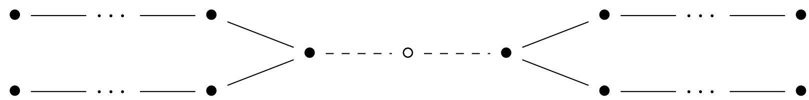
is a building block of the second type. Here the extra vector v has been depicted with a white dot and the edges coming out of it are dashed.

(2) $c(S) = 2$. In this case the graph of S consists of two linear components. There are three possible graphs for $S \cup \{v\}$ according to whether v is linked to a pair of final vectors, to a final vector and an internal one or to two internal vectors. We will prove that

- in the first case $b(S) = 0$ and $\Gamma_{S \cup \{v\}}$ is a building block of the first type;
- in the second case $b(S) = 1$ and $\Gamma_{S \cup \{v\}}$ is a building block of the third type;
- in the third case $b(S) = 2$ and $\Gamma_{S \cup \{v\}}$ is a building block of the fourth type.

The graphs corresponding to these three possibilities are the following:





The analysis required by the above four cases may be sketched as follows. We may think of S as a square matrix where each column is an element of S. The condition on the extra vector v may be translated into a matrix equation, namely

$$ {}^{t}Sv = e_i \quad \text{for some } i \leq N $$

for the first case, and

$$ {}^{t}Sv = e_i + e_j \quad \text{for some } i, j \leq N $$

for the other cases. In each case there is an obvious solution to the above equations, which gives rise to a subset whose graph is a building block. Using this language, the content of Theorem 7.2 amounts to saying that the only integral solutions to the above systems of equations are the obvious ones. This fact will be proved by assuming that there is a nonobvious solution and then finding a contradiction with the constraints provided by Lemma 6.5.

8 Irreducible subsets

In this section we collect all the results we need to prove Theorem 7.2. As explained at the end of the previous section, we will need to examine several cases.

Proposition 8.1 *Let $S = \{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ be a standard subset. Suppose there exists $v \in \mathbb{Z}^n$ which is linked once to a vector, say v_k, of S and is orthogonal to every other vector of S. Then*:

- *v_k is internal and $v_k \cdot v_k = -2$.*
- *$v \cdot v = -1$.*
- *$\Gamma_{S \cup \{v\}}$ is a building block of the second type.*
- *$I(S) = -3$.*

Proof Assume by contradiction that v_k is final. Then, $\Gamma_{S \cup \{v\}}$ is a linear plumbing graph consisting of $n+1$ linearly dependent vectors and, as in Proposition 2.17, it is easy to see that $\mathrm{cf}(\Gamma_{S \cup \{v\}}) = 0$, which means that

$$ \mathrm{cf}(\Gamma_{S \cup \{v\}}) = v \cdot v - \frac{1}{\mathrm{cf}(\Gamma_S)} = 0. $$

This is impossible because $\mathrm{cf}(\Gamma_S) < -1$ and $v \cdot v \leq -1$. It follows that v_k is internal. By Proposition 2.17 the continued fraction associated to $S \cup \{v\}$ must vanish and it can be written as

$$v_k \cdot v_k - \frac{q_1}{p_1} - \frac{q_2}{p_2} - \frac{1}{v \cdot v} = 0,$$

where the p_i/q_i are the continued fractions associated to the linear graphs obtained from S by deleting v_k. Since $0 < -q_i/p_i < 1$, it follows that $v_k \cdot v_k \in \{-1, -2\}$. The case $v_k \cdot v_k = -1$ cannot occur because S is standard; therefore, $v_k \cdot v_k = -2$.

By Lemma 6.5, we have $v \cdot v > v_k \cdot v_k = -2$; therefore, $v \cdot v = -1$. We may write $v = e_t$ for some $t \in \{1, \dots, n\}$. Since v is orthogonal to every vector of $S \setminus \{v_k\}$, we can perform the transformation

$$S \cup \{v\} \mapsto S' := \pi_t(S).$$

At the level of graphs this is just a blowdown move. Since $n = |S'| \subset \mathbb{Z}^{n-1}$ we see that $\det \Gamma_{S'} = 0$. It follows that $\mathrm{cf}(\Gamma_{S'}) = 0$, which means that condition (3) of Proposition 2.15 holds, where Γ_1 and Γ_2 are the connected components of $S' \setminus \{\pi_s(v_k)\}$. This shows that $\Gamma_{S'}$ is a building block of the first type and $\Gamma_{S \cup \{v\}}$ is a building block of the second type.

Since $S \setminus \{v_k\}$ consists of two complementary legs, we have $I(S \setminus \{v_k\}) = -2$ and so $I(S) = -3$. □

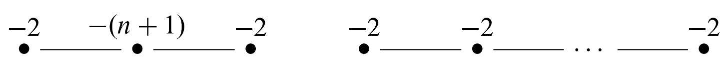
In the next proposition we make explicit a characterization of certain good subsets which is contained in [8]. Following the proof of the main theorem in [8] in the first case (S irreducible), one can see that each subcase corresponds to one of the items listed in the following proposition. We also note, for future reference, that given a linear graph the operation of -2–final expansion commutes with taking the dual.

Proposition 8.2 *Let S be a good subset such that $I(S)+c(S) \leq 0$ and $I(S)+b(S) < 0$. Then $c(S) \leq 2$. Assume $c(S) = 2$.*

(1) *If $b(S) = 0$ then Γ_S consists of two complementary legs.*

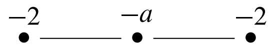
(2) *If $b(S) = 1$ then one of the following holds*:

- Γ_S *is obtained from the graph*



(the -2–chain has length $n-1$ and $n \geq 2$) *via a finite number of -2–final expansions performed on the leftmost component.*

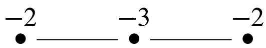
- $\Gamma_S = \Gamma_1 \sqcup \Gamma_2$, *where* Γ_1 *is obtained from the graph*



with $a \geq 3$

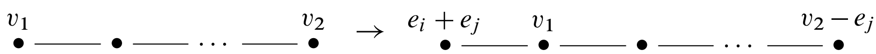
via a finite number of -2*–final expansions and* Γ_2 *is dual to a graph obtained from the one above via a finite number of* -2*–final expansions.*

(3) *If* $b(S) = 2$ *then* $\Gamma_S = \Gamma_1 \sqcup \Gamma_2$, *where each* Γ_i *is obtained from*



via a finite sequence of -2*–final expansions.*

Remark 8.3 It may be useful to explain how the graph of a linear subset changes via -2–final expansions. Suppose that S is a linear subset and that, for some index i, e_i hits only two final vectors v_1 and v_2. If v_1 and v_2 belong to the same connected component of Γ_S, then a -2–final expansion changes the graph as follows:



where we are assuming that $v_1 = -e_i + \cdots$ and $v_2 = e_i + \cdots$. An analogous operation can be performed when v_1 and v_2 belong to different connected components.

Proposition 8.4 *Let* $S = S_1 \cup S_2$ *be a good subset with no bad components such that* $I(S) < 0$ *and* $c(S) = 2$. *Let* v *be an element of, say,* S_1.

(1) *If* v *is internal and* $v \cdot v \geq -3$, *there exists a vector* $v' \in S_2$ *such that* $E(v, v') = 2$.

(2) *If* v *is internal and* $v \cdot v = -k < -3$, *there exists a* -2*–chain in* S_2 *of the form*

$$(\dots, e_1 - e_2, e_2 - e_3, \dots, e_{k-3} - e_{k-2}, \dots)$$

and $|e_i \cdot v| = 1$ *for each* $i \leq k-2$.

(3) *If* v *is final and* $v \cdot v = -k < -2$, *there exists a* -2*–chain in* S_2 *of the form*

$$(e_1 - e_2, e_2 - e_3, \dots, e_{k-2} - e_{k-1}, \dots)$$

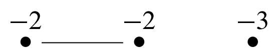
and $|e_i \cdot v| = 1$ *for each* $1 \leq i \leq k-2$.

Proof It is shown in [8] (in the proof of Theorem 1.1, first case and first subcase) that a subset S satisfying our hypothesis is obtained via a sequence of -2–final expansions as described in Lemma 4.7 in [8] from a subset of the form $\{e_1 - e_2, e_1 + e_2\}$. In particular, $|e_i \cdot v| \in \{0,1\}$ for each i and every $v \in S$. This means that we can always write

$$v = \sum_{i=1}^{|v \cdot v|} \varepsilon_i e_i, \quad \text{where } \varepsilon_i \in \{\pm 1\}.$$

If v is internal and $v \cdot v = -2$, write $v = e_1 + e_2$. Again by Lemma 4.7 in [8], every basis vector that hits an internal vector hits exactly three vectors of S. It follows that e_1 hits two more vectors, say v' and v''. Suppose that e_2 does not hit any of these vectors. Then we must have $v' \cdot v = v'' \cdot v = 1$. Now e_2 must hit some vector, say v'''. Since e_1 does not hit v''', we would have $v''' \cdot v = 1$. But then v would be adjacent to three vectors, which is impossible. The same argument works if $v \cdot v = -3$; we omit the details.

If $v \cdot v = k \leq -4$, write $v = \sum_{i=1}^{k} e_i$. It is clear from the proof of the main theorem in [8] (again first case and first subcase) that the subset S is obtained by -2–final expansions from a subset S' whose associated graph is



Then the assertion is easily proved by induction on the number of expansions needed to obtain S from S'; we omit the details.

The third assertion is proved similarly. If $v \cdot v = -k < -2$ then S originates from a subset S' via $k-2$ -2–final expansions. Similarly, v originates from a final vector $v' \in S'$, with $v' \cdot v' = -2$. Each -2–final expansion creates a new -2–final vector in S_2 linked to the one resulting from the previous expansion. □

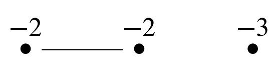
8.1 First case: $b(S) = 0$

In this subsection we examine the subset in Proposition 8.2 with no bad components. We will need the following lemma:

Lemma 8.5 *Let S be a good subset such that $I(S) < 0$, $c(S) = 2$ and $b(S) = 0$. Let v_i and v_j be two vectors in S. We have*:

- *If $v_i \cdot v_j = 1$ then $E(v_i, v_j) = 1$.*
- *If $v_i \cdot v_j = 0$ then $E(v_i, v_j) \in \{0, 2\}$.*

Proof The lemma clearly holds for the subset S_3 whose graph is



By [8] we know that the subset S is obtained by -2–final expansions from S_3 via a sequence of -2–final expansions

$$S_3 \nearrow S_4 \nearrow \cdots \nearrow S_n = S.$$

Suppose the lemma holds for S_{n-1}. The conclusion follows easily from the fact that the new vector which has been introduced has square -2. $\square$

Proposition 8.6 *Let $S \subset \mathbb{Z}^N$ be a good subset such that $c(S) = 2$ and $b(S) = 0$. Suppose that there exists a vector $v \in \mathbb{Z}^N$ that is linked once to a vector of each connected component of S and is orthogonal to all the remaining vectors of S. Then*:

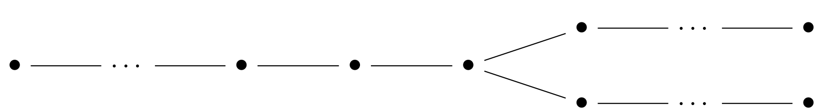
- *v is linked to a pair of final vectors.*
- $v \cdot v = -1$.
- *The graph of $S \cup \{v\}$ is a building block of the first type.*
- $I(S) = -2$.

Proof Write $S = S_1 \cup S_2$ and w_1 and w_2 for the two vectors linked once with v. First note that if both w_1 and w_2 are final vectors then the graph associated to $S \cup \{v\}$ is linear and, since $\det \Gamma_{S\cup\{v\}} = 0$, the corresponding plumbed manifold is diffeomorphic to $S^1 \times S^2$. This means that $\Gamma_{S\cup\{v\}}$ cannot be in normal form, which is only possible if $v \cdot v = -1$. By Proposition 2.15, the graph $\Gamma_{S\cup\{v\}}$ is a building block of the first type. Also by Proposition 2.15, the two components of S are complementary and so $I(S) = -2$. Therefore it is enough to show that both w_1 and w_2 are final.

Assume by contradiction that w_1 is an internal vector. Then we have $v \cdot v < -1$. To see this note that if $v \cdot v = -1$ then, by Lemma 4.7 in [8], the vector v can only hit final vectors. By Lemma 6.4, at least one vector among w_1 and w_2 has square -2.

We have two possibilities:

First case (the vector w_2 is final) The graph $\Gamma_{S\cup\{v\}}$ has the form



It is a star-shaped plumbing graph in normal form with three legs. Since $\det \Gamma_{S\cup\{v\}} = 0$, the weight of the central vertex, which is w_1, can only be -1 or -2. Since S is a good subset, we have $w_1 \cdot w_1 = -2$. We may write $w_1 = e_1 + e_2$. Recall that by Lemma 6.5 we must have

$$\|v\|^2 < 2 + \|w_2\|^2. \tag{6}$$

Moreover, we claim that

$$E(w_1, w_2) = 0. \tag{7}$$

To see this note that since $w_1 \cdot w_2 = 0$ and $w_1 \cdot w_1 = -2$ we have $E(w_1, w_2) \in \{0, 2\}$. If both e_1 and e_2 hit w_2 then, by Lemma 4.7(3) in [8], at least one of them hits exactly two vectors in S. But then, again by Lemma 4.7(2) in [8], these two vectors are not internal. This contradicts the fact that w_1 is internal.

Now we proceed by distinguishing several cases according to the weight of w_2.

First subcase $(w_2 \cdot w_2 = -2)$ By (7) we may write

$$w_1 = e_1 + e_2, \quad w_2 = e_3 + e_4.$$

Note that (6) tells us that $\|v\|^2 < 4$; in particular, $|v \cdot e_i| \le 1$ for each e_i. Therefore, since $1 = v \cdot w_1 = v \cdot e_1 + v \cdot e_2$, either $v \cdot e_1 = 0$ or $v \cdot e_2 = 0$. Similarly, either $v \cdot e_3 = 0$ or $v \cdot e_4 = 0$. Without loss of generality we may write $v = -e_1 - e_3 + v'$, where $v' \cdot e_i = 0$ for $i \le 4$. By (6), we have $\|v'\|^2 \le 1$. Since w_1 is internal, by Lemma 4.7 in [8] we know that e_1 hits exactly three vectors in S, say w_1, u_1 and u_2. The condition $v \cdot u_1 = v \cdot u_2 = 0$ shows that $v' \neq 0$; say $v' = e_5$. We obtain the expression $v = -e_1 - e_3 + e_5$. We have $v \cdot u_i = -e_1 \cdot u_i + e_5 \cdot u_i = 0$ for $i = 1, 2$. Therefore we may write $u_i = \varepsilon_i(e_1 + e_5) + u_i'$ with $u_i' \cdot e_1 = u_i' \cdot e_5 = 0$ and $\varepsilon_i = \pm 1$ for $i = 1, 2$. This fact together with $|u_1 \cdot u_2| \le 1$ implies that $E(u_1, u_2) > 2$, which contradicts Lemma 8.5.

Second subcase $(w_2 \cdot w_2 = -3)$ By (7) we may write

$$w_1 = e_1 + e_2, \quad w_2 = e_3 + e_4 + e_5.$$

By Lemma 4.7 in [8], there exists a final vector w_3 which, without loss of generality, we can write as $w_3 = e_3 - e_4$. Now let us write

$$v = v' + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5,$$

where $v' \cdot e_i = 0$ for each $i \le 5$. Since at least two α_i are nonzero, it follows by (6) that $|\alpha_i| \le 1$ for each $i \le 5$ and that $\sum_{i=1}^{5} |\alpha_i| < 5$. In particular, at least one coefficient is zero. The conditions $v \cdot w_1 = v \cdot w_2 = 1$ and $v \cdot w_3 = 0$ quickly imply:

- $(\alpha_1, \alpha_2) \in \{(-1,0),(0,-1)\}$.
- $(\alpha_3, \alpha_4) \in \{(0,0),(1,1),(-1,-1)\}$.
- $\alpha_5 \in \{-1,1\}$.

If $(\alpha_3, \alpha_4) = (1,1)$ then $\|v\|^2 = 4 + \|v'\|^2$ and therefore $v' = 0$. We can write

$$v = \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5.$$

Let w_4 be the vector of S_1 such that $w_3 \cdot w_4 = 1$. We may write this vector as $w_4 = w_4' + e_4$, and since $w_4 \cdot w_2 = 0$ we may write $w_4 = w_4'' + e_4 - e_5$. Clearly $e_i \cdot w_4'' = 0$ when $3 \leq i \leq 5$. Assume that $e_2 \cdot w_4'' = 0$. Then, since $v \cdot w_4 = 0$, we would have $\alpha_4 = \alpha_5$, which does not match with the previous conditions we obtained for these coefficients. If $|e_2 \cdot w_4''| = 1$ then $E(v, w_4) = 3$, which contradicts the fact that $v \cdot w_4 = 0$.

If $(\alpha_3, \alpha_4) = (-1,-1)$, the argument is analogous.

Therefore we may assume that $(\alpha_3, \alpha_4) = (0,0)$. In this situation we may perform a -2–final contraction on S that has the effect of deleting the vector w_3 and decreasing the norm of w_2 by 1. The extra vector v is not affected by this operation and all the hypothesis that we need remain valid. In this situation v is linked to a final vector whose weight is -2 and therefore we may repeat the argument given in the first subcase.

Third subcase $(w_2 \cdot w_2 < -3)$ We may write $w_2 = \sum_{i=1}^{k} e_i$, with $k \geq 4$. By Proposition 8.4 there is a -2–chain of the form

$$(e_1 - e_2, e_2 - e_3, \dots, e_{k-2} - e_{k-1}, \dots).$$

By (7) we know that w_1 does not belong to this chain. Therefore v must be orthogonal to every vector in this chain. It follows that either v hits all of the vectors in the set $\{e_1, \dots, e_{k-1}\}$ or it does not hit any of them.

If v hits all of the vectors in the set $\{e_1, \dots, e_{k-1}\}$, we can write, without loss of generality,

$$v = v' + \alpha \sum_{i=1}^{k-1} e_i,$$

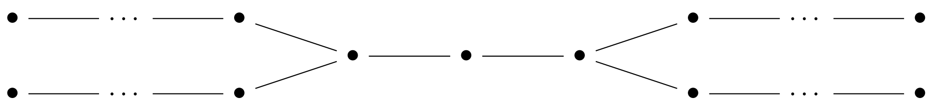
where $v' \cdot e_i = 0$ for $i \leq k-1$ and $\alpha \in \mathbb{Z} \setminus \{0\}$. But then the condition $v \cdot w_2 = 1$ implies $v \cdot e_k = \alpha(k-1) + 1$ and therefore

$$\|v\|^2 \geq \alpha^2(k-1) + (\alpha(k-1)+1)^2 \geq k - 1 + k^2 \geq k + 2,$$

and this contradicts (6).

If v does not hit any of the vectors in the set $\{e_1, \dots, e_{k-2}\}$, we can perform a series of -2–final contractions that will eliminate these vectors. These contractions do not alter the vector v. Let w_2' be the image of w_2 after these contractions are performed. Since $w_2' \cdot w_2' = -2$, we can apply the argument given in the first subcase.

Second case (the vector w_2 is internal) The graph $\Gamma_{S \cup \{v\}}$ has the form



Recall that we have shown that $v \cdot v < -1$. By Lemma 6.4, we may assume, as in the first case, that one of the vectors w_1 and w_2 — say w_1 — has square -2. As a consequence, (6) holds. Note that if $w_2 \cdot w_2 = -2$, the argument given in the first case works as well in this situation. Therefore we may assume that $w_2 \cdot w_2 \leq -3$.

Let e_s be a base vector that hits two final vectors of S. It is easy to see that if $e_s \cdot v = 0$ then the -2–final contraction $S \searrow S'$ associated to e_s does not affect the vector v. In this situation the subset S' satisfies all the hypotheses in the statement and the conclusions hold for S' if and only if they hold for S. This process may be iterated, via a sequence of -2–final contractions $S \searrow \cdots \searrow \bar{S}$, until one of the following holds:

(1) the image in $\bar{S}$ of one vector among w_1 and w_2 is a final vector;

(2) no more contractions can be performed on S without affecting the vector v.

If the first condition holds, we may apply the argument given in the first case. Assume the second condition holds. The subset $\bar{S}$ has two -2–final vectors of the form $e_{j_1} - e_{j_2}$ and $e_{j_3} - e_{j_4}$. By our assumption,

$$v \cdot e_{j_i} \neq 0 \quad \text{for each } 1 \leq i \leq 4. \tag{8}$$

Now we distinguish two cases:

First subcase $(w_2 \cdot w_2 = -3)$ In this case (8) contradicts (6).

Second subcase $(w_2 \cdot w_2 < -3)$ By Proposition 8.4, there is a -2–chain of the form

$$(\dots, -e_1 + \cdots, e_1 - e_2, e_2 - e_3, \dots, e_{k-3} - e_{k-2}, e_{k-2} + \cdots, \dots)$$

and $w_2 = \sum_{i=1}^{k} e_i$. Since v is orthogonal to every vector in the -2–chain, either $v \cdot e_i \neq 0$ for each $i \leq k-2$ or $v \cdot e_i = 0$ for each $i \leq k-2$. In the first case we quickly obtain a contradiction with (6) (by taking into account (8)). In the second case we may remove the whole -2–chain performing the transformation

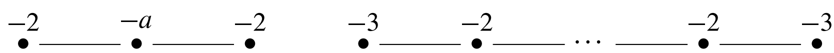
$$(\dots,-e_1+\cdots,e_1-e_2,\dots,e_{k-3}-e_{k-2},e_{k-2}+\cdots,\dots)$$
$$\to(\dots,-e_1+\cdots,e_1+\cdots,\dots).$$

The image of the vector w_2 under this transformation is $\overline{w}_2=e_1+e_{k-1}+e_k$. Since $\overline{w}_2\cdot\overline{w}_2=-3$, we may argue as in the first subcase, and we are done. □

8.2 Second case: $b(S)=1$

In this section we deal with the subsets of Theorem 7.2 having a single bad component. As stated in Proposition 8.2, there are two different classes of such subsets. First we show that for one of these classes it is not possible to find an extra vector v satisfying the hypothesis of Theorem 7.2. Then we deal with the other class of subsets, which will give rise to building block of the third type.

Proposition 8.7 *Let S' be a good subset such that $b(S')=1$ and its graph $\Gamma_{S'}$ is of the form*



where $a\geq 3$ and the -2–chain has length $a-3$. Let S be a good subset which is obtained via -2–final expansions from S' as explained in Proposition 8.2. Then there exists no vector $v\in\mathbb{Z}^N$ linked once to a vector of each connected component of S and orthogonal to all the other vectors of S.

Proof Assume by contradiction that there exists $v\in\mathbb{Z}^N$ linked once to a vector of each connected component of S and orthogonal to all the other vectors of S. We write $S=S_1\cup S_2$, where S_1 is obtained from the bad component of S' via -2–final expansions and S_2 is obtained from the nonbad component of S' in a similar way. Note that the only vector of S_1 which is linked to a vector of S_2 is the central one. Call this vector w. More precisely, we may choose base vectors of $\mathbb{Z}^N$ $\{e_1,\dots,e_k,e_{k+1},\dots,e_N\}$ so that

- if $i\leq k+1$ we have $e_i\cdot u=0$ for each $u\in S_2$;
- if $i\geq k+2$ we have $e_i\cdot u=0$ for each $u\in S_1\setminus\{w\}$;
- $e_{k+1}\cdot w\neq 0$ and for some $j\geq k+2$ we have $e_j\cdot w\neq 0$.

Note that $|S_1| = k+2$ and $|S_2| = N-k-2$. Now we proceed by distinguishing several cases:

First case $(w \cdot v = 0)$ We can write $v = v_1 + v_2$ such that v_1 is spanned by $\{e_1, \dots, e_{k+1}\}$ and v_2 by $\{e_{k+2}, \dots, e_N\}$. In particular, v_1 (resp. v_2) is orthogonal to every element of S_2 (resp. S_1), and moreover both v_1 and v_2 are nonzero. The subset $\widetilde{S}_1 := (S_1 \setminus \{w\}) \subset \mathbb{Z}^{k+1}$ consists of two complementary linear components, T_1 and T_2. Since $w \cdot v = 0$, the vector v_1 is linked once to a vector of, say, T_1 and is orthogonal to the other vectors of $\widetilde{S}_1$. The graph $\Gamma_{\widetilde{S}_1 \cup \{v_1\}}$ is given by the disjoint union $\Gamma_{T_1 \cup \{v_1\}} \sqcup \Gamma_{T_2}$, where $\Gamma_{T_1 \cup \{v_1\}}$ is either star-shaped with three legs or linear and Γ_{T_2} is linear. It is easy to see that $\Gamma_{T_1 \cup \{v_1\}}$ cannot be linear. Indeed, since $\det \Gamma_{T_1 \cup \{v_1\}} = 0$, we would have $v_1 \cdot v_1 = -1$, which is easily seen to be impossible. Therefore we may assume that $\Gamma_{T_1 \cup \{v_1\}}$ is star-shaped with three legs. Since $\det \Gamma_{\widetilde{S}_1 \cup \{v_1\}} = 0$, we have

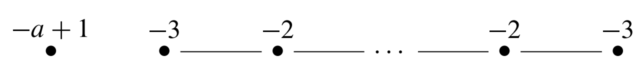
$$0 = \det \Gamma_{\widetilde{S}_1 \cup \{v_1\}} = \det(\Gamma_{T_1 \cup \{v_1\}} \sqcup \Gamma_{T_2}) = \det \Gamma_{T_1 \cup \{v_1\}} \det \Gamma_{T_2}.$$

Since $\det \Gamma_{T_2} \neq 0$, we must have $\det \Gamma_{T_1 \cup \{v\}} = 0$. It follows that, as in the proof of Proposition 8.1, v is linked once to a vector of T_1 with -2 square. This quickly leads to a contradiction with Lemma 6.5.

Second case $(w \cdot v = 1)$ We may write $v = v_1 + v_2$ as in the first case. Since v_1 is orthogonal to the vectors of $S_1 \setminus \{w\}$, we must have $v_1 = 0$ (because v_1 is orthogonal to $k+1$ linearly independent vectors in $\mathbb{Z}^{k+1}$). Consider the good subset

$$\widetilde{S} := (S \setminus S_1) \cup \{\pi_{k+1}(w)\}.$$

The vector $v = v_2$ is linked once to a vector of each connected component of $\widetilde{S}$ and is orthogonal to the other vectors of $\widetilde{S}$. The graph $\Gamma_{\widetilde{S} \cup \{v\}}$ is either star-shaped with three legs (if v is linked once to an internal vector of S_2) or linear (if v is linked once to a final vector of S_2). The latter possibility cannot occur. To see this, suppose that $\Gamma_{\widetilde{S} \cup \{v\}}$ is linear. Since $\det \Gamma_{\widetilde{S} \cup \{v\}} = 0$, we must have $v \cdot v = -1$. Moreover, by Proposition 2.15 the two components of $\widetilde{S}$ are complementary. Since one of these components consists of a single vertex, the other one must be a -2–chain, which is not the case. Therefore we may assume that the graph $\Gamma_{\widetilde{S} \cup \{v\}}$ is star-shaped with three legs. The subset $\widetilde{S}$ is obtained via -2–final expansions (performed on the rightmost component) from a subset whose graph is



where $a \geq 3$ and the -2–chain has length $a-3$. Up to automorphisms of the integral lattice $\mathbb{Z}^a$ this subset may be written as

$$\tilde{\tilde{S}} := \left\{ \sum_{i=1}^{a-1} e_i \right\} \cup \{e_1 - e_2 + e_a, e_2 - e_3, \dots, e_{a-2} - e_{a-1}, e_{a-1} + e_a - e_1\}. \tag{9}$$

Note that, as in the proof of Proposition 8.1, the vector v must be linked to a -2–vector — say u — of $\tilde{S} \setminus \{\pi_{k+1}(w)\}$. We have two possibilities, which we examine separately.

First subcase (the vector u is not affected by the series of -2–final contractions from $\tilde{S}$ to $\tilde{\tilde{S}}$) In this case the vector u belongs to the -2–chain that appears in (9). By Lemma 6.5, we must have $v \cdot v < -a-2$. Write $u = e_k - e_{k+1}$ with $2 \leq k \leq a-1$. It is easy to see that v can be written as

$$v = v' + \alpha \sum_{i=2}^{k} e_i + (1+\alpha) \sum_{i=k+1}^{a-1} e_i,$$

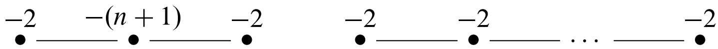
where $\alpha \in \mathbb{Z} \setminus \{0, -1\}$. This expression quickly leads to a contradiction with the inequality $v \cdot v < -a-2$.

Second subcase (the vector u is the result of one of the -2–final expansions from $\tilde{\tilde{S}}$ to $\tilde{S}$) Write $u = e_s + e_t$. We have either $e_s \cdot v \neq 0$ or $e_t \cdot v \neq 0$, and it is easy to see that v must hit at least another base vector which is not in $\{e_1, \dots, e_{a-1}\}$. Moreover, since $w \cdot v = 1$, the vector v hits at least one vector among $\{e_1, \dots, e_{a-1}\}$. Since v is orthogonal to all the vectors in the -2–chain in (9), we see that $e_2 \cdot v = \dots = e_{a-1} \cdot v$. If $e_2 \cdot v \neq 0$ then we quickly obtain a contradiction with Lemma 6.5 by computing $e_1 \cdot v$. If $e_2 \cdot v = 0$, we may write $v = v' - e_1 + e_a$, where $e_j \cdot v' = 0$ for each $j \leq a$. In this situation we can change the subset $\tilde{S}$ by removing the coordinate vectors appearing in the -2–chain of $\tilde{\tilde{S}}$ and the vector w. We call this new subset T; it is obtained from the subset

$$\{e_1 - e_a, e_1 + e_a\}$$

via -2–final expansions. The vector v is not affected by this transformation. Note that T is a good subset with two complementary connected components and that v is linked once to a vector of one connected component and is orthogonal to any other vector. The graph $\Gamma_{T \cup \{v\}}$ is the disjoint union of a three-legged star-shaped graph and a linear one. Now we can argue as in the first case. Since $\det \Gamma_{T \cup \{v\}} = 0$, the vector v must be linked to a -2–weighted vertex, which quickly leads to a contradiction with Lemma 6.5. □

Proposition 8.8 *Let $S = S_1 \cup S_2$ be a good subset such that $c(S) = 2$, $b(S) = 1$ and $I(S) < 0$. Suppose that Γ_S is obtained from*



(where the -2–chain has length $n-1$ and $n \geq 2$) *via a finite number of -2–final expansions performed on the leftmost component. Assume that there exists $v \in \mathbb{Z}^N$ linked once to a vector of each connected component of S and orthogonal to any other vector of S. Then*:

- *v is linked to the central vector of the bad component of S and to a final vector of the -2–chain.*
- $v \cdot v = -1$.
- *The graph $\Gamma_{S \cup \{v\}}$ is a building block of the third type.*
- $I(S) = -3$.

Proof The vectors corresponding to the -2–chain can be written as

$$(e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n).$$

The vectors corresponding to the bad component (before the -2–final expansions are performed) can be written as

$$S_3 = \left\{ -e_{n+1} - e_{n+2}, \sum_{j=1}^{n+1} e_j, -e_{n+1} + e_{n+2} \right\}.$$

Note that the central vector is not altered by -2–final expansions and the same holds for one of the two final vectors.

Claim *The extra vector v is linked to a final vector of the -2–chain.*

To see this, suppose v is linked to an internal vector — say $e_i - e_{i+1}$ — where $1 < i < n-1$. Then we can write

$$v = v' + \alpha \sum_{j=1}^{i} e_j + (1+\alpha) \sum_{j=i+1}^{n} e_j, \quad \text{where } \alpha \in \mathbb{Z} \setminus \{0, -1\}, \tag{10}$$

and $v' \cdot e_i = 0$ for $1 \leq i \leq n$. Now v must be linked to some vector of the bad component; first assume v is linked to the central vector whose weight is $n+1$. In this case, Lemma 6.5 implies that $\|v\|^2 < n+3$. Using the expression for v in (10) we obtain

$$\|v'\|^2 + i\alpha^2 + (n-i)(1+\alpha)^2 < n+3,$$

which is impossible when $\alpha \notin \{0,-1\}$. If $\alpha \in \{0,-1\}$, it is easy to see that the vector v cannot be orthogonal nor linked once to the central vector of the bad component.

Now assume v is linked to some vector — say w — of the bad component other than the central one. If $n \le 3$ the claim is satisfied so we may assume that $n > 3$. It follows by Lemma 6.5 that

$$\|v'\|^2 + i\alpha^2 + (n-i)(1+\alpha)^2 < 2 + \|w\|^2.$$

In particular,

$$\|w\|^2 > 3 \quad \text{and} \quad \|w\|^2 - \|v'\|^2 > 2.$$

We can write $w = \sum_{h=1}^{k} e_{j_h}$, where $k \ge 4$. The relevant portion of the bad component can be written as

$$\left(\dots, u + e_{j_1} - e_{j_2}, e_{j_2} - e_{j_3}, \dots, e_{j_{k-2}} - e_{j_{k-1}} + u', \dots, \sum_{h=1}^{k} e_{j_h}, \dots\right).$$

In particular, there is a -2–chain of length $k-3$. If v' hits one of the basis vectors in this chain then it hits them all, and this would contradict the inequality $\|w\|^2 - \|v'\|^2 > 2$. Therefore we may assume that $e_{j_2} \cdot v = \dots = e_{j_{k-2}} \cdot v = 0$. In this situation we can change the bad component by removing the vectors $e_{j_2}, \dots, e_{j_{k-2}}$. The relevant portion of this new component can be written as

$$(\dots, u + e_{j_1} - e_{j_2}, e_{j_2} - e_{j_{k-1}} + u', \dots, e_{j_1} + e_{j_2} + e_{j_{k-1}} + e_{j_k}, \dots).$$

Everything we said so far holds for this new component; in particular, the inequality $\|w\|^2 - \|v'\|^2 > 2$ now implies $\|v'\|^2 = 1$, which is easily seen to be impossible and the claim is proved.

We can write $v = -e_1 + v'$, where v' does not hit any vector in the -2–chain. Note that if v is linked to the central vector of the bad component then we must have $v' = 0$. This is because $-e_1$ is linked once to a final vector of the -2–chain and once to the central vector of the bad component and there is at most one vector in $\mathbb{Z}^N$ with this property (the conditions on v can be expressed as a nonsingular $n \times n$ system of equations).

In this case the plumbing graph corresponding to $S \cup \{v\}$ is a building block of the third type.

Therefore, in order to conclude we need to show that $v' = 0$. Assume $v' \neq 0$; then v must be linked to some vector of the bad component — say w — other than the central one. By Lemma 6.5 we have $\|v\|^2 = 1 + \|v'\|^2 < 2 + \|w\|^2$; therefore,

$$\|v'\|^2 \leq \|w\|^2. \tag{11}$$

We can write $w = \sum_{h=1}^{k} e_{j_h}$; again the relevant portion of the bad component can be written as

$$\left(\ldots, u + e_{j_1} - e_{j_2}, e_{j_2} - e_{j_3}, \ldots, e_{j_{k-2}} - e_{j_{k-1}} + u', \ldots, \sum_{h=1}^{k} e_{j_h}\right).$$

If $k = 2$ then $w = e_{j_1} - e_{j_2}$ and v' can be written as

$$v' = \alpha e_{j_1} + (1+\alpha) e_{j_2} \quad \text{with } \alpha \in \mathbb{Z} \setminus \{0, -1\},$$

but then $\|v'\|^2 \geq 5$, which contradicts (11). If $k = 3$, write $w = e_{j_1} + e_{j_2} + e_{j_3}$. It is easy to show that again the possible expressions for v' contradict (11) (one needs to distinguish the three possibilities where v' hits one, two or all of the vectors among $\{e_{j_1}, e_{j_2}, e_{j_3}\}$). If $k \geq 4$, there is a -2–chain associated to w whose length is $k-3$ and either v' hits every vector in this chain or it does not hit any of them. If v' hits every vector in the -2–chain, it is easy to see that this would contradict again (11). If v' does not hit any vector in the -2–chain, the chain can be contracted as we did before, and we are back to the case $k = 3$. The fact that $I(S) = -3$ follows from [8]. It can also be checked directly by observing that -2–final expansions do not alter the quantity $I(S)$. □

8.3 Third case: $b(S) = 2$

In this subsection we examine the good subsets with two bad components satisfying the hypothesis of Theorem 7.2 and we show that they give rise to building blocks of the fourth type.

Proposition 8.9 *Let S be a good subset such that $c(S) = b(S) = 2$ and $I(S) < 0$. Suppose that there exists $v \in \mathbb{Z}^N$ which is linked once to a vector of each connected component of S and is orthogonal to the other vectors of S. Then*:

- *v is linked to the central vectors of each bad component of S.*
- *$v \cdot v = -1$.*
- *The graph $\Gamma_{S \cup \{v\}}$ is a building block of the fourth type.*
- *$I(S) = -4$.*

Proof Write $S = S_1 \cup S_2$. By Proposition 8.2, the string associated to S is of the form $s_1 \cup s_2$, where each s_i is obtained from $(2,3,2)$ via -2–final expansions.

First let us assume that the extra vector v is linked to both the central vectors of the two bad components. Then note that $\det(\Gamma_{S\cup\{v\}}) = 0$. By Proposition 2.17, this is equivalent to $\mathrm{cf}(\Gamma_{S\cup\{v\}}) = 0$; therefore,

$$0 = \mathrm{cf}(\Gamma_{S\cup\{v\}}) = v \cdot v - \frac{1}{\mathrm{cf}(\Gamma_{S_1})} - \frac{1}{\mathrm{cf}(\Gamma_{S_2})},$$

where each Γ_{S_i} is rooted at its central vector. The graph obtained from Γ_{S_i} by removing the central vector consists of two components which are dual of each other. Therefore, $\mathrm{cf}(\Gamma_{S_i}) = -2$, which implies $v \cdot v = -1$.

It is clear that the graph $\Gamma_{S\cup\{v\}}$ is a building block of the fourth type. To see this first blow down the extra vector and split the graph along one of its trivalent vertices. The fact that $I(S) = -4$ is a straightforward computation.

In order to conclude we need to rule out the possibility of v being linked to a noncentral vector. Let $w_1 \in S_1$ and $w_2 \in S_2$ be the two vectors of S which are linked to v. Suppose w_1 is noncentral.

Claim *Possibly after a sequence of contractions which do not alter the extra vector v, we may assume that $\|v\|^2 \geq \|w_1\|^2$.*

We prove the claim in three steps, which correspond to the three cases $w_1 \cdot w_1 = -2$, $w_1 \cdot w_1 = -3$ and $w_1 \cdot w_1 \leq -4$. If $w_1 \cdot w_1 = -2$, we can write $w_1 = e_1 + e_2$ and assume $e_1 \cdot v \neq 0$. If $e_2 \cdot v \neq 0$, we are done. If $e_2 \cdot v = 0$, note that e_1 must hit some other vector $u \in S_1$. Since $u \cdot v = 0$, we see that v must hit some basis vector other than e_1 and therefore $\|v\|^2 \geq 2 = \|w_1\|^2$. If $w_1 \cdot w_1 = -3$, we may write $w_1 = e_1 + e_2 + e_3$ and assume $e_1 \cdot v \neq 0$. If $e_2 \cdot v \neq 0$ and $e_3 \cdot v \neq 0$ we are done. If this is not the case, it is easy to find two more basis vectors that hit v arguing just like above. If $w_1 \cdot w_1 \leq -4$, we may write $w_1 = \sum_{i=1}^{k} e_i$; in this case there is a -2–chain associated to w_1. The relevant portion of S can be written as

$$\Bigl(\dots, u + e_1 - e_2, e_2 - e_3, e_3 - e_4, \dots, e_{k-2} - e_{k-1}, e_{k-1} - e_k + u', \dots, \sum_{i=1}^{k} e_i, \dots\Bigr).$$

Note that either v hits every vector in the -2–chain or it does not hit any of them. If v hits every vector in the -2–chain, the inequality $\|v\|^2 \geq \|w_1\|^2$ follows easily. If v

does not hit any vector in the -2–chain, we remove from S the -2–chain. We obtain a new subset $\widetilde{S} \subset \mathbb{Z}^{N-k+3}$. The relevant portion of $\widetilde{S}$ can be written as

$$(\dots, u + e_1 - e_{k-1}, e_{k-1} - e_k + u', \dots, e_1 + e_{k-1} + e_k, \dots).$$

Now we can repeat the argument we used for the case $w_1 \cdot w_1 = -2$ and the claim is proved. There are two possibilities, according as whether w_2 is central or not. If w_2 is not central, we may repeat the argument used in the claim; we obtain the inequality $\|v\|^2 \geq \|w_1\|^2 + \|w_2\|^2$, which contradicts Lemma 6.5. If w_2 is central, it is easy again to contradict Lemma 6.5. □

8.4 Conclusion

Now we are ready to prove Theorem 7.2.

Proof of Theorem 7.2 By Proposition 4.10 in [8] we have $c(S) \leq 2$. If $c(S) = 1$ then S is standard and the conclusion follows from Proposition 8.1. If $c(S) = 2$, there are four possibilities as explained in Proposition 8.2. If $b(S) = 0$, the conclusion follows from Proposition 8.6. If $b(S) = 1$, the two different cases are settled by Propositions 8.7 and 8.8. When $b(S) = 2$ we can apply Proposition 8.9. □

9 Orthogonal subsets

In this section we basically fill the gap between Theorems 7.2 and 7.1. Roughly speaking, we need to remove the technical assumption $I(S) + b(S) < 0$, since this is not a property of the plumbing graph. The main result of this section is Proposition 9.5, which shows that the subsets that are of interest for us have at most two components. Given a linear subset $S = \{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ we define, following [8], $p_k(S)$ as the number of e_i which hit exactly k vectors in S. Thinking of S as a matrix $p_k(S)$ is the number of rows with k nonzero entries. Note that

$$\sum_{i=1}^{n} p_i(S) = n, \tag{12}$$

$$\sum_{i=1}^{n} i\, p_i(S) \leq -\sum_{i=1}^{n} v_i \cdot v_i. \tag{13}$$

A linear subset $S = \{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ is said to be *orthogonal* if $v_i \cdot v_j = 0$ whenever $i \neq j$.

Lemma 9.1 *Let $S = \{v_1, \dots, v_n\}$ be a good orthogonal subset such that $n \geq 3$ and $I(S) = 0$. The following conditions are satisfied*:

(1) *Either there exists $v \in S$ such that $v \cdot v = -2$ or $v \cdot v = -3$ for each $v \in S$.*

(2) *$p_3(S) = n$ and $p_i(S) = 0$ for each $i \neq 3$.*

Proof Assume that there is no vector $v \in S$ such that $v \cdot v = -2$, ie that $v_i \cdot v_i \leq -3$ for each $1 \leq i \leq n$. Since $\sum_{i=1}^n v_i \cdot v_i = -I(S) - 3n$, we see that $v_i \cdot v_i = -3$ for each $1 \leq i \leq n$.

Now we prove that $p_1(S) = 0$. Assume by contradiction that $v_j = \alpha e_1 + \pi_1(v_j)$ for some $v_j \in S$ and that no other vector in S hits e_1. Since S is irreducible, we have $\pi_1(v_j) \neq 0$. Moreover, $\pi_1(v_j) \cdot v_i = 0$ for each $i \neq j$ and, since the vectors $v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n$ are independent in $\mathbb{Z}^{n-1}$, we must have $\pi_1(v_j) = 0$, which is a contradiction; therefore, $p_1(S) = 0$.

Now we show that $p_2(S) = 0$. Assume by contradiction that $p_2(S) \neq 0$. Let e_i, v_j and v_h be such that e_i only hits v_j and v_h among the elements of S. We may assume that, say, v_h is such that $v_h \cdot v_h \leq -3$ (otherwise, the set $\{v_h, v_j\}$ would be an irreducible component of S, which is impossible because S is irreducible and $|S| \geq 3$). Either $v_j \cdot v_j \leq -3$ or $v_j \cdot v_j = -2$. If $v_j \cdot v_j = -2$ then we may write $v_j = e_i + e_s$ and, since e_i only hits v_h and v_j, the same conclusion holds for e_s. Write $v_h = \alpha e_i - \alpha e_s + v_h'$ with $\alpha \neq 0$. Since v_h' is orthogonal to any vector in $S \setminus \{v_j, v_h\}$, it must vanish. Therefore the subset $\{v_j, v_h\}$ is an irreducible component of S. But this is impossible because S is irreducible and $|S| \geq 3$. Therefore we may assume that $v_j \cdot v_j \leq -3$. Consider the subset

$$S' = S \setminus \{v_h, v_j\} \cup \{\pi_i(v_j)\}.$$

It is easy to check that S' is an orthogonal subset; moreover, the same argument used to show that $v_j \cdot v_j \leq -3$ shows that $\pi_i(v_j) \cdot \pi_i(v_j) \leq -2$ and therefore S' is good. We have

$$\begin{aligned} I(S') &= I(S) + v_h \cdot v_h + 3 + v_j \cdot v_j + 3 - \pi_i(v_j) \cdot \pi_i(v_j) - 3 \\ &= I(S) + v_h \cdot v_h + 3 + v_j \cdot v_j - \pi_i(v_j) \cdot \pi_i(v_j) \\ &\leq I(S) + v_j \cdot v_j - \pi_i(v_j) \cdot \pi_i(v_j) \\ &< I(S). \end{aligned}$$

In particular, $I(S') < 0$. By Lemma 4.9 in [8], we must have $c(S') \leq 2$. Since $|S| = c(S) \geq 3$, we have $c(S') = 2$. It is easy to check that S' must be of the form

$$S' = \{e_1 + e_2, e_1 - e_2\}.$$

Now it is easy to see that S' cannot be expanded to a good orthogonal subset S such that $I(S) = 0$. In fact there are no good orthogonal subset such that $(c(S), I(S)) = (3, 0)$. This is a contradiction and we conclude that $p_2(S) = 0$.

Finally, note that by (12) we have

$$\sum_{i=1}^{k} (i-3) p_i(S) \leq 0,$$

which means that $p_i(S) = 0$ for each $i \geq 4$. □

Proposition 9.2 *Let S be a good orthogonal subset such that $I(S) = 0$. Then $c(S) = 4$. If, moreover, there exists $v \in S$ such that $v \cdot v = -2$ then, up to automorphisms of the integral lattice $\mathbb{Z}^4$, S has the matrix*

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}.$$

Proof It is easy to check that $|S| > 2$. By Lemma 9.1 we have two possibilities:

First case (there exists $v \in S \subset \mathbb{Z}^N$ which can be written as $v = e_1 + e_2$) Since $p_3(S) = n$, e_1 hits two more vectors, say v' and v''. Since $v' \cdot v = v'' \cdot v = 0$, we see that e_2 hits v' and v'' as well. Writing S as a matrix whose first three columns are v, v' and v'', we have

$$\begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & -1 & 0 & \cdots & 0 \\ 0 & * & * & & & \\ \vdots & \vdots & \vdots & & & \\ 0 & * & * & & & \end{pmatrix},$$

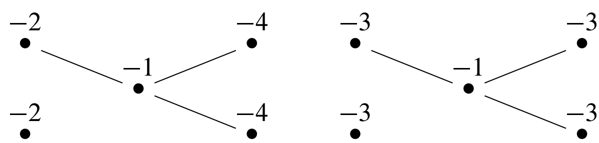
where the fact that $|v' \cdot e_i| = |v'' \cdot e_i| = 1$ for $i = 1, 2$ follows from the fact that each row of the matrix above has exactly three nonzero entries and therefore $0 = I(S) = \sum_{i,j} a_{i,j}^2 - 3n \geq 0$, and equality holds if and only if $|a_{i,j}| \leq 1$. Consider the subset

$$S' = S \setminus \{v, v', v''\} \cup \{\pi_1(v'), \pi_1(v'')\} \subset \mathbb{Z}^{N-1}.$$

Note that $\pi_1(v') \cdot \pi_1(v'') = 1$. It is easy to see that S' is a good subset. Moreover, $(c(S'), I(S')) = (N-2, -1)$ and $b(S') = 0$. By Proposition 4.10 in [8] we have $c(S') \leq 2$, which implies $N \leq 4$. It is easy to verify that $N \geq 4$. We conclude that $N = 4$. The matrix description for S follows easily by filling the remaining entries in the above matrix.

Second case ($v \cdot v = -3$ for each $v \in S$) In this case we only need to show that $c(S) = 4$. Choose a vector — say v_1 — of S. We may write $v_1 = e_1 + e_2 + e_3$. Since $p_1(S) = 0$, e_1 must hit some other vector — say v_2 — of S. Since $v_1 \cdot v_2 = 0$, up to exchanging the role of e_2 and e_3 we may write $v_2 = e_1 - e_2 + e_4$. Since $p_2(S) = 0$, e_1 must hit some other vector of S. Call this vector v_3. It is easy to see that e_2 cannot hit v_3. Since $v_1 \cdot v_3 = v_2 \cdot v_3 = 0$, we may write $v_3 = e_1 - e_3 - e_4$. Since $p_2(S) = 0$, e_2 must hit some other vector — say v_4 — of S. Now the orthogonality condition implies $v_4 = e_2 - e_3 + e_4$. Since $p_3(S) = n$, we see that the subset $\{v_1, \dots, v_4\}$ is irreducible. Since S is irreducible too, we conclude that $S = \{v_1, \dots, v_4\}$. □

Lemma 9.3 *Let $S = \{v_1, v_2, v_3, v_4\} \subset \mathbb{Z}^4$ be a subset as in Proposition 9.2. Let $v \in \mathbb{Z}^4 \setminus \{0\}$ be such that, for each $i = 1, \dots, 4$, we have $v \cdot v_i \in \{0, 1\}$. Then the graph of $S \cup \{v\}$ is one of the following*:



Proof Let M be the matrix of S. For each $J \subset \{1, 2, 3, 4\}$ consider the linear system of equations

$$^{t}M v = -\sum_{j \in J} e_j .$$

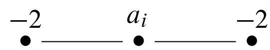
The lemma is equivalent to the fact that among these linear systems the only ones which are solvable in $\mathbb{Z}^4$ correspond to the above graphs. We omit the details. □

Lemma 9.4 *Let $S \subset \mathbb{Z}^N$ be a good subset such that $-I(S) = b(S) = c(S) = 4$. There exists no vector $v \in \mathbb{Z}^N$ linked once to a vector of each connected component of S and orthogonal to the vectors of S.*

Proof Let us write $S = B_1 \cup \dots \cup B_4$, where each B_i is a bad component. By definition of bad component there is a sequence of -2–final contractions

$$S \searrow \cdots \searrow \widetilde{S}$$

such that $\widetilde{S} = \widetilde{B}_1 \cup \cdots \cup \widetilde{B}_4$ and each $\widetilde{B}_i$ is a bad component whose graph is of the form



for some $a_i \leq -3$. For each $i = 1, \ldots, 4$, let $v_i \in B_i$ be the only vector of B_i that is linked once to v, and let u_i be the central vector of B_i.

Claim *$v_i = u_i$ for each $i \leq 4$.*

To see this we may argue exactly as in the proof of Proposition 8.8. Indeed, assume by contradiction that $v_i \neq u_i$. Let v' be the projection of v onto the subspace generated by the basis vectors that span the subset $S_i' := S_i \setminus u_i$. Note that S_i' is a good subset consisting of two complementary components. The vector v' is linked once to a vector of a connected component and is orthogonal to all the other vectors of S_i'. We have already observed in the proof of Proposition 8.8 that such a vector does not exist. This proves the claim.

It is easy to see that $E(v, w) = 0$ for each $w \in S \setminus \{u_1, \ldots, u_4\}$. Let $\overline{S} := \{\overline{u}_1, \ldots, \overline{u}_4\}$ be the subset obtained by projecting each u_i onto the subspace orthogonal to the one generated by the basis vectors that span the subset $S_i' := S_i \setminus u_i$.

We have

$$-4 = I(S) = I(\widetilde{S}) = -8 + I(\{u_1, \ldots, u_4\}) = -8 + 4 + I(\overline{S}) \implies I(\overline{S}) = 0.$$

Therefore, $\overline{S}$ is of the form described in Proposition 9.2 and $v \cdot u_i = v \cdot \overline{u}_i$ for each $i = 1, \ldots, 4$. The fact that $v \cdot \overline{u}_i = 1$ for each $i \leq 4$ contradicts Lemma 9.3. □

Proposition 9.5 *Let $S \subset \mathbb{Z}^N$ be a good subset such that $I(S) + c(S) \leq 0$. Suppose that there exists $v \in \mathbb{Z}^N$ which is linked once to a vector of each connected component of S and is orthogonal to all the vectors. Then $c(S) \leq 2$.*

Proof By Proposition 4.10 in [8], if $I(S) < -b(S)$ then $c(S) \leq 2$. Assume by contradiction that $c(S) \geq 3$. Then, $I(S) \geq -b(S)$ and we have

$$-b(S) \leq I(S) \leq -c(S) \leq -b(S);$$

therefore, $I(S) = -c(S) = -b(S)$. Write $S = B_1 \cup \cdots \cup B_k$, where each B_i is a bad component. Let S' be the subset obtained from S via a sequence of -2–final contractions such that each bad component has been reduced to its minimal configuration consisting of three vectors as in Definition 6.3. The graph of S' has the form

$$\overset{-2}{\bullet}\text{———}\overset{a_1}{\bullet}\text{———}\overset{-2}{\bullet}\qquad\overset{-2}{\bullet}\text{———}\overset{a_2}{\bullet}\text{———}\overset{-2}{\bullet}\qquad\cdots\qquad\overset{-2}{\bullet}\text{———}\overset{a_k}{\bullet}\text{———}\overset{-2}{\bullet}$$

where $a_i \le -3$ for each $1 \le i \le k$. Note that S' is a good subset and $(c(S'), I(S')) = (c(S), I(S))$. Since $I(S') = -k$, we have

$$\sum_{i=1}^{k} a_i = -4k. \tag{14}$$

Each bad component can be written as

$$\overset{e_1+e_2}{\bullet}\text{————}\overset{-e_2+w_i}{\bullet}\text{————}\overset{e_2-e_1}{\bullet}$$

where $w_i \cdot e_1 = w_i \cdot e_2 = 0$ and $w_i \cdot w_i \le -2$. Consider the subset $S'' = \{w_1, \dots, w_k\}$. Its graph is

$$\overset{a_1+1}{\bullet}\qquad\overset{a_2+1}{\bullet}\qquad\cdots\qquad\overset{a_k+1}{\bullet}$$

Note that this is a good orthogonal subset and by (14) we have

$$\sum_{i=1}^{k} w_i \cdot w_i = \sum_{i=1}^{k}(a_i + 1) = -3k.$$

Therefore, the subset S'' satisfies the hypotheses of Lemma 9.1 and Proposition 9.2. In particular, $k = 4$.

The proof is concluded by using Lemma 9.4, which shows that there exist no subset S and vector v with the above properties. □

10 Conclusion of the proof

Putting together Theorem 7.2 and Proposition 9.5, we can finally prove Theorem 7.1.

Proof of Theorem 7.1 Let $S = S_1 \cup \dots \cup S_k$ be the decomposition of S into its irreducible components. We may write $v = v_1 + \dots + v_k$ so that each v_i is the projection of v onto the subspace that corresponds to S_i. From (5) we obtain

$$I(S) + c(S) = \sum_{i=1}^{k} I(S_i) + c(S_i) \le 0.$$

We may choose an irreducible component S_j such that $I(S_j) + c(S_j) \le 0$. By Proposition 9.5, we have $c(S_j) \le 2$. Moreover, $I(S_j) + b(S_j) \le I(S_j) + c(S_j) \le 0$.

We claim that $I(S_j) + b(S_j) < 0$. Assume by contradiction that $I(S_j) = -b(S_j) = -c(S) = -2$ and write $S_j = B_1 \cup B_2$. Since it is easy to check that for every bad component B we have $I(B) \geq -2$, we may assume that one of the following holds:

- $I(B_1) = I(B_2) = -1$.
- $I(B_1) = -2$ and $I(B_2) = 0$.

Arguing as in the proof of Proposition 9.5 we would get orthogonal subsets whose associated graph is either

or

It is easy to check that none of these configurations are realizable, and the claim is proved.

We can now apply Theorem 7.2. The graph $\Gamma_{S_j \cup \{v_j\}}$ is a building block. Moreover, (5) holds for the subset $S \setminus S_j$. To see this, one needs to compare the value $I(S_j)$ and the contribution of Γ_{S_j} to the right-hand side of (5). For example, if Γ_{S_j} is a building block of the first type then $I(S_j) = -2$ and Γ_{S_j} contributes to the right-hand side of (5) with two final legs (ie with a -2). The other cases can be checked similarly. Therefore we may iterate the argument above with all the irreducible components of S, and we are done. □

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Mathematical Institute, University of Oxford
Oxford, United Kingdom

paolo.aceto@maths.ox.ac.uk

Received: 24 March 2015 Revised: 30 April 2019