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An explicit model for the homotopy theory of finite-type Lie n–algebras

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Lie n–algebras are the L_∞ analogs of chain Lie algebras from rational homotopy theory. Henriques showed that finite-type Lie n–algebras can be integrated to produce certain simplicial Banach manifolds, known as Lie ∞–groups, via a smooth analog of Sullivan's realization functor. We provide an explicit proof that the category of finite-type Lie n–algebras and (weak) L_∞–morphisms admits the structure of a category of fibrant objects (CFO) for a homotopy theory. Roughly speaking, this CFO structure can be thought of as the transfer of the classical projective CFO structure on nonnegatively graded chain complexes via the tangent functor. In particular, the weak equivalences are precisely the L_∞–quasi-isomorphisms. Along the way, we give explicit constructions for pullbacks and factorizations of L_∞–morphisms between finite-type Lie n–algebras. We also analyze Postnikov towers and Maurer–Cartan/deformation functors associated to such Lie n–algebras. The main application of this work is our joint paper with C Zhu (1127–1219), which characterizes the compatibility of Henriques' integration functor with the homotopy theory of Lie n–algebras and that of Lie ∞–groups.

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1 Introduction

The motivation for this paper lies within the various algebraic formalisms developed decades ago for classifying rational homotopy types. In [15], Quillen established the

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relationship between the rational homotopy theory of simply connected spaces and the homotopy theory of connected chain Lie algebras over $\mathbb{Q}$. By a chain Lie algebra, we mean a chain complex L of vector spaces concentrated in nonnegative degrees equipped with a differential graded Lie algebra (dgla) structure. A chain Lie algebra L is connected if it is strictly positively graded, ie $L_0 = 0$. Quillen formalized the homotopy theory of connected chain Lie algebras using a model structure in which a weak equivalence is defined to be a dgla morphism whose underlying chain map induces an isomorphism in homology, and in which a fibration is defined to be a dgla morphism whose underlying chain map is surjective in all degrees.

From here, we can make a quick leap to the approach developed by Sullivan in [17], provided we restrict our attention to "finite-type" chain Lie algebras, ie those algebras whose underlying chain complex is finite-dimensional in each degree. First, recall that the Chevalley–Eilenberg algebra $\mathrm{CE}(L)$ associated to a chain Lie algebra L is the commutative dg algebra (cdga) obtained by taking the linear dual of the bar construction of L. If L is of finite type and connected, then $\mathrm{CE}(L)$ is simply connected, and admits the structure of a Sullivan algebra. Roughly, a cdga is a Sullivan algebra if, as a graded commutative algebra, it is freely generated by a filtered graded vector space that satisfies certain compatibility conditions with the differential. Sullivan algebras are the cofibrant cdgas in the model structure introduced by Bousfield and Gugenheim [3]. In particular, the Sullivan algebra $\mathrm{CE}(L)$ is a model for the rational homotopy type of its realization: the Kan simplicial set $\langle \mathrm{CE}(L) \rangle := \hom_{\mathsf{cdga}}(\mathrm{CE}(L), \Omega^*_{\mathrm{poly}}(\Delta^\bullet))$, where $\Omega^*_{\mathrm{poly}}(\Delta^n)$ is the cdga of polynomial de Rham forms on the geometric n–simplex.

A more direct path from chain Lie algebras to Kan complexes is via deformation theory. Associated to any $\mathbb{Z}$–graded dgla $(L, d, [\cdot,\cdot])$ is its set of Maurer–Cartan elements $\mathrm{MC}(L) := \bigl\{a \in L_{-1} \mid da + \frac{1}{2}[a,a] = 0\bigr\}$. Furthermore, the dgla structure on L induces a natural simplicial dgla structure on the tensor product $L \otimes_{\mathbb{Q}} \Omega^*_{\mathrm{poly}}(\Delta^\bullet)$. As noted by Getzler [8], if L is of finite type, then there is a natural isomorphism of simplicial sets

$$\langle \mathrm{CE}(L) \rangle \cong \textstyle\int L,$$

where $\bigl(\int L\bigr)_n := \mathrm{MC}(L \otimes \Omega^*_{\mathrm{poly}}(\Delta^n))$. Hence, if L is finite-type connected, then L is a Lie model for the rational homotopy type of the space $\int L$. Moreover, the "simplicial Maurer–Cartan functor" $\int$ is compatible with the respective homotopy theories. Indeed, by combining Quillen's work [15] with the results of Bousfield and Gugenheim [3], it follows that $\int$ preserves both weak equivalences and fibrations. In particular, $\int L$ is a Kan complex for every finite-type, connected chain Lie algebra L.

This leads to an obvious question: What is the analog of this story for arbitrary finite-type chain Lie algebras over a field of characteristic zero? That is, suppose there are no constraints imposed on degree zero elements and no assumptions involving nilpotency or completeness. To begin with, Quillen's model for the homotopy theory of connected rational chain Lie algebras extends in a straightforward way to the more general case over any field of characteristic zero. It follows from the work of Getzler and Jones [9, Theorem 4.4] that the category of chain Lie algebras over such a field admits a model structure induced by the projective model structure on nonnegatively graded chain complexes. Hence, a weak equivalence is, as before, a dgla morphism whose underlying chain map is a quasi-isomorphism, and a fibration is defined to be a dgla morphism whose underlying chain map is surjective in positive degrees. This is a natural generalization of Quillen's model structure for the rational connected case.

However, the spatial realization of finite-type chain Lie algebras with no constraints on connectivity is a more subtle endeavor. Indeed, all finite-dimensional nonnilpotent Lie algebras are examples of such chain Lie algebras. Consequently, as demonstrated by Sullivan [17, Theorem 8.1], the realization of chain Lie algebras over $\mathbb{R}$ necessarily involves the diffeogeometric integration of Lie algebras to Lie groups (ie Lie's Third Theorem).

The existence of a smooth realization functor for such chain Lie algebras is a special case of the more general problem addressed by Henriques in his work [10] on the integration of Lie n–algebras. Lie n–algebras are L_∞–algebras (or "strong homotopy Lie algebras") whose underlying chain complexes are nonnegatively graded. Thus they include chain Lie algebras as a special case. However, the "correct" notion of morphism between Lie n–algebras is significantly weaker than just a linear map which preserves the L_∞–structure on the nose. This implies that the category of Lie n–algebras and weak L_∞–morphisms is not a category of algebras over the L_∞ operad. Hence, a model for the homotopy theory of Lie n–algebras does not follow from the aforementioned result of Getzler and Jones. The main result (Theorem 5.2) of this paper resolves this issue by explicitly providing such a model.

Henriques' integration procedure for finite-type Lie n–algebras involves replacing the polynomial de Rham forms in Sullivan's realization functor with the dg Banach algebra of C^r–differential forms. The output of this procedure is a grouplike simplicial Banach manifold, or "Lie ∞–group", which satisfies a diffeogeometric analog of the horn filling condition for Kan simplicial sets. Simplicial manifolds of this kind have been used as geometric models for the higher stages of the Whitehead tower of the orthogonal group. The most famous example of such a model is called the "string Lie 2–group", which Henriques showed can be obtained by integrating its infinitesimal analog, the "string Lie 2–algebra".

The results of this paper, when combined with our joint work with Zhu in the companion paper [16], address the compatibility of Henriques' integration functor with the homotopy theories of Lie n–algebras and Lie ∞–groups. This can be understood as the smooth analog of the aforementioned results of Quillen and Bousfield and Gugenheim which characterize the homotopical properties of the realization functor for connected Lie models for rational homotopy types.

Overview and main results

After reviewing standard facts concerning L_∞–algebras in the sections leading up to Section 3.2, we consider in Definition 3.6 two classes of morphisms in the category $\mathsf{Lie}_n\mathsf{Alg}$ of Lie n–algebras. We say an L_∞–morphism

$$(f_1, f_2, \dots)\colon (L, \ell_1, \ell_2, \ell_3, \dots) \to (L', \ell'_1, \ell'_2, \ell'_3, \dots)$$

is a *weak equivalence* if and only if the chain map $f_1\colon (L, \ell_1) \to (L', \ell'_1)$ is a quasi-isomorphism, and we say it is a *fibration* if and only if the chain map f_1 is a surjection in all positive degrees. Thus weak equivalences coincide with L_∞–quasi-isomorphisms. Although every weak equivalence between Lie n–algebra induces a quasi-isomorphism between their associated Chevalley–Eilenberg (co)algebras, the converse is not true, in contrast with the simply connected case mentioned in the above introduction. (See Remarks 3.7 for an explicit counterexample.)

In Section 3.3, we prove several useful technical results concerning morphisms in $\mathsf{Lie}_n\mathsf{Alg}$. We show in Proposition 3.8 that every strict L_∞–morphism (in the sense of Section 3.1.1) can be factored into a fibration followed by a weak equivalence. In particular, the diagonal map $L \to L \oplus L$ in $\mathsf{Lie}_n\mathsf{Alg}$ admits such a factorization. Therefore, it follows from our main theorem (see below) and Brown's factorization lemma (see Lemma 5.4) that every weak L_∞–morphism in $\mathsf{Lie}_n\mathsf{Alg}$ admits such a factorization.

Next, in Lemma 3.11, we show that every fibration can be factored into an isomorphism followed by a strict fibration. The proof is a simple modification of a result of Vallette [19] concerning the factorization of "∞–epimorphisms". (See, for example, Definition 3.3.) This "strictification of fibrations" is a very useful tool which we apply repeatedly throughout this paper and in the companion paper [16].

In Proposition 4.1 and Corollary 4.4, we explicitly construct pullbacks of fibrations and acyclic fibrations along arbitrary morphisms in $\mathsf{Lie}_n\mathsf{Alg}$. Moreover, the pullback of an (acyclic) fibration is again an (acyclic) fibration. The proof of these facts involves a clever use of certain coalgebra endomorphisms, which we learned from studying Vallette's proof of his Theorem 4.1 in [19].

Since our main application is integration, in Section 5 we restrict our attention to the full subcategory $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ of finite-type Lie n–algebras. The category $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ does not admit a model structure, since it does not have all limits and colimits. So instead, we work within Brown's framework [4] of a category of fibrant objects, or "CFO", for a homotopy theory (Definition 5.1). The main result (Theorem 5.2) of the paper is:

Theorem *Let $n \in \mathbb{N} \cup \{\infty\}$. The category $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ of finite-type Lie n–algebras over a field of characteristic zero and weak L_∞–morphisms has the structure of a category of fibrant objects, in which a morphism*

$$(f_1, f_2, \dots)\colon (L, \ell_1, \ell_2, \ell_3, \dots) \to (L', \ell'_1, \ell'_2, \ell'_3, \dots)$$

is

- *a weak equivalence if and only if the chain map $f_1\colon (L, \ell_1) \to (L', \ell'_1)$ is a quasi-isomorphism of chain complexes, and*
- *a fibration if and only if the chain map $f_1\colon (L, \ell_1) \to (L', \ell'_1)$ is a surjection in all positive degrees* (ie in all degrees ≥ 1).

Hence, the homotopy theory that we consider for Lie n–algebras is inherited from the projective model structure for nonnegatively graded chain complexes (Section 2), via the tangent functor which assigns to an L_∞–morphism $(f_1, f_2, \dots)$ as above, the chain map $f_1\colon (L, \ell_1) \to (L', \ell'_1)$. In particular, our results imply that the tangent functor is an exact functor between categories of fibrant objects (Corollary 5.8). We also note that this CFO structure is compatible with the one induced on the category of chain Lie algebras by the aforementioned Getzler–Jones/Quillen model structure. (Recall that chain Lie algebras form a nonfull subcategory of $\mathsf{Lie}_\infty\mathsf{Alg}$.)

Also in Section 5, we compare the category of fibrant objects structure on finite-type Lie n–algebras with Vallette's CFO structure (Theorem 5.9) on $\mathbb{Z}$–graded L_∞–algebras.

In Section 6, we analyze Maurer–Cartan (MC) sets of certain $\mathbb{Z}$–graded L_∞–algebras which are constructed by tensoring Lie n–algebras with bounded commutative dg algebras. This is a familiar procedure used in deformation theory for constructing deformation functors out of pronilpotent L_∞–algebras. Maurer–Cartan sets are also used to define Henriques' integration functor. We prove that this construction sends pullback diagrams of fibrations in $\mathsf{Lie}_n\mathsf{Alg}$ to pullback diagrams of MC sets. In Corollary 6.7, we show that an analogous statement holds in the smooth category when the MC sets are equipped with a Banach manifold structure. The proofs crucially depend on the explicit description of pullbacks given in Section 4.

Finally, in Section 7, we analyze a very useful Postnikov tower construction for Lie n–algebras which was introduced by Henriques in [10]. They play a key role in his proof that Lie n–algebras integrate to fibrant simplicial manifolds. We also introduce an important class of fibrations in $\mathsf{Lie}_n\mathsf{Alg}$ called "quasisplit fibrations" (Definition 7.1). Such fibrations naturally arise in applications, eg string extensions. We show that a morphism of Postnikov towers induced by a quasisplit fibration admits a convenient functorial decomposition (Propositions 7.5 and 7.6). We use this result and the aforementioned results concerning MC sets in [16] to show that Henriques' integration functor is an exact functor with respect to the class of quasifibrations. (See Theorem 9.16 in [16].)

In this paper, we work with L_∞–algebras within the context of dg cocommutative coalgebras, rather than commutative dg algebras, even though we are ultimately interested in finite-type objects. This is because our main result, Theorem 5.2, as well as many of the auxiliary results also hold for infinite-dimensional Lie n–algebras. (See Remark 5.3.) Furthermore, many of the technical tools we develop in order to prove our main results are the "nonnegatively graded" variations of Vallette's work [19] on the homotopy theory of $\mathbb{Z}$–graded dg $\mathcal{P}^{¡}$ coalgebras (where $\mathcal{P}^{¡}$ is the Koszul dual cooperad of an operad $\mathcal{P}$). For the convenience of the reader, we quickly recall in the next section various facts concerning cocommutative coalgebras, as well as establish the notation and conventions that we use throughout the paper.

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2 Preliminaries and notation

2.1 Graded linear algebra

Throughout, $\Bbbk$ denotes a field of characteristic zero. For a $\mathbb{Z}$–graded vector space V we denote by $\boldsymbol{s}V$ (resp. by $\boldsymbol{s}^{-1}V$) the suspension (resp. the desuspension) of V. In other words,

$$\boldsymbol{s}V_i := V[-1]_i = V_{i-1}, \quad \boldsymbol{s}^{-1}V_i := V[1]_i = V_{i+1}.$$

We denote by $|x|$ the degree of a homogeneous element $x \in V$. Similarly, $|f|$ denotes the degree of a linear map $f\colon V \to V'$ between graded vector spaces V and V'.

Let $x_1, \dots, x_n$ be elements of V and $\sigma \in \mathbb{S}_n$ a permutation. The notation $\epsilon(\sigma) = \epsilon(\sigma; x_1, \dots, x_n)$ is reserved for the Koszul sign, which is defined by the equality

$$x_1 \vee \cdots \vee x_n := \epsilon(\sigma) x_{\sigma(1)} \vee \cdots \vee x_{\sigma(n)} \in S(V),$$

which holds in the free graded commutative algebra $S(V)$ generated by V. Note that $\epsilon(\sigma)$ does not include the sign $(-1)^\sigma$ of the permutation σ.

The notation $\sigma \cdot (x_1 \otimes x_2 \otimes \cdots \otimes x_n)$ is reserved for the left action of $\mathbb{S}_n$ on $V^{\otimes n}$, ie

$$\sigma \cdot x_1 \otimes x_2 \otimes \cdots \otimes x_n := \epsilon(\sigma) x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}. \tag{2-1}$$

We denote by $\mathrm{Sh}_{p_1,\dots,p_k} \subseteq \mathbb{S}_n$ the subset of $(p_1, \dots, p_k)$–shuffles in $\mathbb{S}_n$, ie $\mathrm{Sh}_{p_1,\dots,p_k}$ consists of elements $\sigma \in \mathbb{S}_n$, $n = p_1 + p_2 + \cdots + p_k$ such that

$$\begin{gathered}
\sigma(1) < \sigma(2) < \cdots < \sigma(p_1), \\
\sigma(p_1+1) < \sigma(p_1+2) < \cdots < \sigma(p_1+p_2), \\
\vdots \\
\sigma(n-p_k+1) < \sigma(n-p_k+2) < \cdots < \sigma(n).
\end{gathered}$$

We use *homological* conventions for all differential graded (dg) structures except for the cochain algebras that appear in Section 6.

2.2 Notation

2.2.1 The model category $\mathsf{Ch}_{\geq 0}^{\mathrm{proj}}$ We refer the reader to the introductory article [6] for basic facts concerning model categories. Throughout our paper, $\mathsf{Ch}_{\geq 0}$ denotes the category of chain complexes over $\Bbbk$ concentrated in nonnegative degrees. The notation $\mathsf{Ch}_{\geq 0}^{\mathrm{proj}}$ is reserved for the category $\mathsf{Ch}_{\geq 0}$ equipped with the projective model structure

[6, Theorem 7.2]. Weak equivalences are the quasi-isomorphisms, and fibrations are those chain maps which are surjective in all positive degrees. Since we work over a field, the cofibrations are those chain maps which are injective in all degrees.

Since all objects in $\mathsf{Ch}_{\geq 0}^{\mathrm{proj}}$ are (bi)fibrant, we will also consider $\mathsf{Ch}_{\geq 0}^{\mathrm{proj}}$ as a category of fibrant objects (Definition 5.1) whenever it is convenient to do so.

2.2.2 Notation for categories We will consider several closely related categories. We list them here as a convenient reference for the reader as they traverse through the paper:

- Ch denotes the category of $\mathbb{Z}$–graded (ie unbounded) chain complexes over $\Bbbk$.
- CoCom (resp. $\mathsf{dgCoCom}$) denotes the category of $\mathbb{Z}$–graded (resp. dg) conilpotent coaugmented cocommutative coalgebras (Section 2.3).
- $\mathsf{L_\infty Alg}$ is the category whose objects are $\mathbb{Z}$–graded L_∞–algebras and whose morphisms are weak L_∞–morphisms (Section 3.1). We will tacitly identify $\mathsf{L_\infty Alg}$ as the full subcategory of $\mathsf{dgCoCom}$ consisting of those dg coalgebras whose underlying graded coalgebras are cofree (Section 2.4).
- $\mathsf{CoCom}_{\geq 0}$ (resp. $\mathsf{dgCoCom}_{\geq 0}$) denotes the full subcategory of CoCom (resp. $\mathsf{dgCoCom}$) consisting of those graded (resp. dg) coalgebras whose underlying graded vector spaces are concentrated in nonnegative degrees.
- Fix $n \in \mathbb{N} \cup \{\infty\}$. We denote by $\mathsf{Lie}_n\mathsf{Alg}$ the category of Lie n–algebras: the full subcategory of $\mathsf{L_\infty Alg}$ consisting of those L_∞–algebras whose underlying chain complex is concentrated in degrees $0, 1, \ldots, n-1$ (Section 3.2). Hence, the morphisms in $\mathsf{Lie}_n\mathsf{Alg}$ are always taken to be *weak* L_∞–morphisms. Again we will usually identify $\mathsf{Lie}_n\mathsf{Alg}$ as a full subcategory of $\mathsf{dgCoCom}_{\geq 0}$.

  The notation $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ is reserved for the full subcategory of $\mathsf{Lie}_n\mathsf{Alg}$ consisting of finite-type Lie n–algebra (Section 3.2).
- We denote by $\mathsf{cdga}_{\geq 0}^{\mathrm{bnd}}$ the category of bounded cochain algebras (Section 6). Its objects are *cohomologically* graded unital commutative dg $\Bbbk$–algebras whose underlying graded vector spaces are concentrated in nonnegative degrees and bounded from above. The morphisms in $\mathsf{cdga}_{\geq 0}^{\mathrm{bnd}}$ are unit-preserving cdga morphisms.

2.3 Conilpotent cocommutative coalgebras

The following facts and notational conventions concerning dg coalgebras are standard, and the reader already familiar with treatments of L_∞–algebras as dg coalgebras, eg [12] or [14, Section 10.1.6], can likely skip this section.

For a basic introduction to dg coalgebras and their morphisms, we suggest [7, Sections 3d and 22a; 15, Appendix B; 11, Section 2]. We recall that a graded counital cocommutative coalgebra (C, Δ, ϵ) is *coaugmented* if and only if it is equipped with a distinguished degree zero element $\mathbf{1} \in C_0$ satisfying $\Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ and $\epsilon(\mathbf{1}) = 1$. For such a coalgebra, we have a decomposition of vector spaces

$$C \cong \Bbbk \cdot \mathbf{1} \oplus \overline{C}, \quad \overline{C} := \ker \epsilon.$$

Let $\overline{\Delta}\colon \overline{C} \to \overline{C} \otimes \overline{C}$ denote the *reduced comultiplication* defined as $\overline{\Delta}(c) := \overline{\Delta}c = \Delta c - c \otimes \mathbf{1} - \mathbf{1} \otimes c$. We call the noncounital cocommutative coalgebra $(\overline{C}, \overline{\Delta})$ the associated *reduced coalgebra*. The reduced diagonal $\overline{\Delta}^{(n)}$ is recursively defined by the formulas

$$\overline{\Delta}^{(0)} := \mathrm{id}, \quad \overline{\Delta}^{(1)} := \overline{\Delta}, \quad \overline{\Delta}^{(n)} := (\overline{\Delta} \otimes \mathrm{id}^{\otimes(n-1)}) \circ \overline{\Delta}^{(n-1)}\colon \overline{C} \to \overline{C}^{\otimes(n+1)}.$$

A coaugmented counital cocommutative coalgebra $(C, \Delta, \epsilon, \mathbf{1})$ is *conilpotent* if and only if $\overline{C} = \bigcup_n \ker \overline{\Delta}^{(n)}$. We define CoCom to be the category whose objects are $\mathbb{Z}$–graded conilpotent coaugmented counital cocommutative coalgebras. Similarly, we define $\mathsf{dgCoCom}$ to be the category whose objects are coalgebras $(C, \Delta, \epsilon, \mathbf{1})$ in CoCom equipped with a degree -1 codifferential δ satisfying $\delta(\mathbf{1}) = 0$.

The full subcategories $\mathsf{CoCom}_{\geq 0} \subseteq \mathsf{CoCom}$ and $\mathsf{dgCoCom}_{\geq 0} \subseteq \mathsf{dgCoCom}$ are defined analogously.

Reduced noncounital dg coalgebras $(\overline{C}, \overline{\Delta}, \delta)$ and their morphisms The codifferential δ of any conilpotent dg coalgebra $(C, \Delta, \epsilon, \mathbf{1}, \delta) \in \mathsf{dgCoCom}$ is uniquely determined by its restriction to the corresponding reduced coalgebra $(\overline{C}, \overline{\Delta})$. We will use the same notation for δ and its restriction to $\overline{C}$. Similarly, since we are over a field, morphisms in $\mathsf{dgCoCom}$ are uniquely determined by their restriction to the associated reduced dg coalgebras [11, Section 2.1]. From now on, when dealing with the categories $\mathsf{dgCoCom}_{\geq 0}$, $\mathsf{dgCoCom}$, etc, we will tacitly work with the associated reduced dg coalgebras and their morphisms, and make no mention of counits or coaugmentations.

2.4 Cofree conilpotent coalgebras and their morphisms

Let V be a graded vector space. The symmetric algebra generated by V,

$$S(V) = \Bbbk \oplus \overline{S}(V), \quad \overline{S}(V) := V \oplus S^2(V) \oplus S^3(V) \oplus \cdots,$$

is naturally a graded conilpotent cocommutative coalgebra with comultiplication Δ defined as the unique morphism of algebras such that $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$. The comultiplication for the corresponding reduced coalgebra $(\bar{S}(V), \bar{\Delta})$ is explicitly

$$\bar{\Delta}(v_1, v_2, \dots, v_m) = \sum_{1 \le p \le m-1} \sum_{\sigma \in \mathrm{Sh}(p,m-p)} \epsilon(\sigma)(v_{\sigma(1)} \vee v_{\sigma(2)} \vee \cdots \vee v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \vee v_{\sigma(p+2)} \vee \cdots \vee v_{\sigma(m)}).$$

Let V and V' be graded vector spaces. Let $\Phi\colon \bar{S}(V) \to \bar{S}(V')$ be a linear map. For $p, m \ge 1$ the notation Φ^p_n is reserved for the restriction–projections

$$\Phi^p_m\colon \bar{S}^m(V) \to \bar{S}^p(V') \quad \Phi^p_m := \mathrm{pr}_{\bar{S}^p(V')} \circ \Phi|_{\bar{S}^m(V)};$$

Φ is obviously completely determined by its restriction–projection maps $\{\Phi^r_m\}$. Furthermore, we denote by $\Phi^1\colon \bar{S}(V) \to V'$ the linear map

$$\Phi^1 := \Phi^1_1 + \Phi^1_2 + \cdots .$$

We recall that $(\bar{S}(V), \bar{\Delta})$ is cofree over V in the category CoCom (see Lemma 22.1 in [7]). In particular, a degree zero linear map $F^1\colon \bar{S}(V) \to V'$ uniquely determines a coalgebra morphism $F\colon \bar{S}(V) \to \bar{S}(V')$ via the formula

$$\begin{aligned} F(v_1, \dots, v_m) &= F^1_m(v_1, \dots, v_m) \\ &\quad + \sum_{p=1}^{m-1} \sum_{\substack{k_1, k_2, \dots, k_{p+1} \ge 1 \\ k_1 + k_2 + \cdots + k_{p+1} = m}} \sum_{\sigma \in \mathrm{Sh}(k_1, k_2, \dots, k_{p+1})} \frac{\epsilon(\sigma)}{(p+1)!} \\ &\qquad \times F^1_{k_1}(v_{\sigma(1)}, \dots, v_{\sigma(k_1)}) \vee F^1_{k_2}(v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)}) \vee \\ &\qquad\qquad \cdots \vee F^1_{k_{p+1}}(v_{\sigma(m-k_{p+1}+1)}, \dots, v_{\sigma(m)}). \end{aligned} \tag{2-2}$$

This gives explicit formulas for the restriction–projections F^p_m,

$$\begin{aligned} F^p_m(v_1, \dots, v_m) &= \sum_{\substack{k_1, k_2, \dots, k_p \ge 1 \\ k_1 + k_2 + \cdots + k_p = m}} \sum_{\sigma \in \mathrm{Sh}(k_1, k_2, \dots, k_p)} \frac{\epsilon(\sigma)}{p!} F^1_{k_1}(v_{\sigma(1)}, \dots, v_{\sigma(k_1)}) \\ &\qquad \vee F^1_{k_2}(v_{\sigma(k_1+1)}, \dots, v_{\sigma(k_1+k_2)}) \vee \cdots \vee F^1_{k_p}(v_{\sigma(m-k_p+1)}, \dots, v_{\sigma(m)}). \end{aligned} \tag{2-3}$$

In particular, we have

$$
\begin{aligned}
&F_m^m(v_1,\dots,v_m) = F_1^1(v_1)\vee F_1^1(v_2)\vee\cdots\vee F_1^1(v_m),\\
&F_m^p(v_1,\dots,v_m) = 0 \quad \text{for all } p>m.
\end{aligned}
\tag{2-4}
$$

Hence, a coalgebra morphism between cofree conilpotent coalgebras $F\colon \bar S(V)\to\bar S(V')$ is uniquely determined by its *structure maps* $F_k^1\colon \bar S^k(V)\to V'$.

Composition Let $F\colon \bar S(V)\to\bar S(V')$ and $G\colon \bar S(V')\to\bar S(V'')$ be coalgebra morphisms. It follows from (2-2)–(2-4) that the composition $GF\colon \bar S(V)\to\bar S(V'')$ is the unique coalgebra morphism whose structure maps $(GF)_m^1\colon \bar S^m(V)\to V''$ are

$$
(GF)_m^1(v_1,\dots,v_m) = \sum_{p=1}^m G_p^1 F_m^p(v_1,\dots,v_m).
\tag{2-5}
$$

Coderivations Analogously, we recall that a degree -1 linear map $\delta^1\colon \bar S(V)\to V$ uniquely determines a degree -1 coderivation $\delta\colon \bar S(V)\to\bar S(V)$ via the formula

$$
\begin{aligned}
\delta_m(v_1,\dots,v_m)
&= \delta_m^1(v_1,\dots,v_m) + \sum_{i=1}^{m-1}\sum_{\sigma\in\mathrm{Sh}(i,m-i)} \epsilon(\sigma)\delta_i^1(v_{\sigma(1)},\dots,v_{\sigma(i)})\\
&\qquad\qquad \vee v_{\sigma(i+1)}\vee\cdots\vee v_{\sigma(m)}.
\end{aligned}
\tag{2-6}
$$

(See, for example, Lemma 2.4 in [12]). This gives explicit formulas for the restriction–projections,

$$
\begin{aligned}
\delta_m^p(v_1,\dots,v_m) = \sum_{\sigma\in\mathrm{Sh}(m-p+1,p-1)} &\epsilon(\sigma)\delta_{m-p+1}^1(v_{\sigma(1)},\dots,v_{\sigma(m-p+1)})\\
&\vee v_{\sigma(m-p+2)}\vee\cdots\vee v_{\sigma(m)}.
\end{aligned}
\tag{2-7}
$$

In particular, we have

$$
\begin{aligned}
&\delta_m^m(v_1,\dots,v_m) = (\delta_1^1)^{\otimes}(v_1\vee v_2\vee\cdots\vee v_m),\\
&\delta_m^p(v_1,\dots,v_m) = 0 \quad \text{for all } p>m,
\end{aligned}
\tag{2-8}
$$

where $(\delta_1^1)^{\otimes}$ denotes the usual derivation on $S(V)$ induced by the linear map $\delta_1^1\colon V\to V$. Hence, a coderivation on a cofree conilpotent coalgebra is uniquely determined by its *structure maps* $\delta_m^1\colon \bar S^m(V)\to V$.

It follows from (2-6)–(2-8) that a degree -1 coderivation δ on $\bar S(V)$ is a *codifferential*, ie $\delta^2=0$, if and only if

$$
\sum_{k=1}^m \delta_k^1\delta_m^k(v_1,\dots,v_m) = 0 \quad \text{for all } m\geq 1.
\tag{2-9}
$$

Analogously, a coalgebra morphism of the form $F\colon \bar{S}(V) \to \bar{S}(V')$ lifts to a dg coalgebra morphism $F\colon (\bar{S}(V), \delta) \to (\bar{S}(V'), \delta')$ if and only if

$$\sum_{k=1}^{m} \delta'^{1}_{k} F^{k}_{m}(v_1, \dots, v_m) = \sum_{k=1}^{m} F^{1}_{k} \delta^{k}_{m}(v_1, \dots, v_m) \quad \text{for all } m \geq 1. \tag{2-10}$$

3 Lie n–algebras

3.1 L_∞–algebras and L_∞–morphisms

An L_∞*–algebra* (L, ℓ) is a $\mathbb{Z}$–graded vector space L equipped with a collection $\ell = \{\ell_1, \ell_2, \ell_3, \dots\}$ of graded skew-symmetric linear map, or "brackets",

$$\ell_k\colon \Lambda^k L \to L, \quad 1 \leq k < \infty,$$

with $|\ell_k| = k-2$, satisfying an infinite sequence of Jacobi-like identities of the form

$$\sum_{\substack{i+j=m+1\\ \sigma\in \mathrm{Sh}(i,m-i)}} (-1)^{\sigma} \epsilon(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(m)}) = 0 \tag{3-1}$$

for all $m \geq 1$ [12, Definition 2.1]. In particular, equation (3-1) implies that $(L, \ell_1) \in \mathsf{Ch}$. Equivalently, an L_∞–structure on a graded vector space L is a degree -1 codifferential δ on the coalgebra $\bar{S}(sL)$. (See, for example, Theorem 2.4 in [12].) The correspondence between the structure maps

$$\delta^1_m\colon \bar{S}^m(sL) \to sL$$

and the brackets is given by the formula

$$\delta^1_m = (-1)^{m(m-1)/2} s \circ \ell_m \circ (s^{-1})^{\otimes m}. \tag{3-2}$$

Let L and L' be graded vector spaces. We recall that there is a one-to-one correspondence between collections $f = \{f_1, f_2, \dots\}$ of skew-symmetric linear maps

$$f_k\colon \Lambda^k L \to L', \quad 1 \leq k < \infty, \tag{3-3}$$

with $|f_k| = k-1$, and coalgebra morphisms

$$F\colon \bar{S}(sL) \to \bar{S}(sL')$$

whose degree 0 structure maps $F^1_k\colon \bar{S}^k(sL) \to sL$ are given by the formula

$$F^1_k = (-1)^{k(k-1)/2} s \circ f_k \circ (s^{-1})^{\otimes k}. \tag{3-4}$$

A *morphism* (ie a weak L_∞–morphism) of L_∞–algebras

$$f\colon (L,\ell)\to(L',\ell')$$

is a collection $f=\{f_1,f_2,\dots\}$ of skew-symmetric linear maps as in (3-3) whose corresponding coalgebra morphism (3-4) satisfies (2-10) and therefore lifts to a morphism of dg coalgebras

$$F\colon (\bar S(sL),\delta)\to(\bar S(sL'),\delta').$$

We treat the category $\mathsf{L_\infty Alg}$ of L_∞–algebras as a full subcategory of $\mathsf{dgCoCom}$. Hence, composition of morphisms in $\mathsf{L_\infty Alg}$ is given by (2-5).

A morphism $f\colon (L,\ell)\to(L',\ell')$ is an L_∞*–isomorphism* if and only if the linear map $f_1\colon L\to L'$ is an isomorphism in Ch. It is easy to show that this condition implies that f corresponds to an actual isomorphism in the category $\mathsf{dgCoCom}$.

Notation 3.1 In contrast with some other conventions found in the literature, we will write weak L_∞–morphisms in $\mathsf{L_\infty Alg}$ using a single lowercase (Latin or Greek) letter, eg

$$f\colon (L,\ell)\to(L',\ell'),$$

and the k–ary map in the collection f will always be denoted by f_k. The dg coalgebra morphism encoded by the collection f will always be written using the corresponding uppercase letter, eg $F\colon \bar S(sL)\to\bar S(sL')$.

Remark 3.2 By definition, if $f\colon (L,\ell)\to(L',\ell')$ is an L_∞–morphism, then the coalgebra morphism $F\colon \bar S(sL)\to\bar S(sL')$ satisfies $F\delta=\delta' F$. Therefore, by setting $m=1$ in (2-10), we observe that the degree 0 map f_1 is necessarily a chain map

$$f_1\colon (L,\ell_1)\to(L',\ell'_1).$$

Furthermore, if $x,y\in L$, then by setting $m=2$ in (2-10) we see that

$$f_1(\ell_2(x,y))-\ell'_2(f_1(x),f_1(y))=\ell'_1 f_2(x,y). \tag{3-5}$$

Since the bilinear bracket ℓ_2 on an L_∞–algebra (L,ℓ) induces a Lie algebra structure on $H_0(L)$, it follows from (3-5) that $H_0(f_1)\colon H_0(L)\to H_0(L')$ is a morphism of Lie algebras.

Next, we recall three useful classes of L_∞–morphisms.

Definition 3.3 Let $f\colon (L,\ell)\to(L',\ell')$ be a morphism of L_∞–algebras. We say f is

(1) an L_∞*–quasi-isomorphism* if the chain map $f_1\colon (L,\ell_1)\to(L',\ell'_1)$ is a quasi-isomorphism, ie the induced map on homology
$$H(f_1)\colon H(L)\to H(L')$$
is an isomorphism;

(2) an L_∞*–epimorphism* [19, Definition 4.1] if the chain map $f_1\colon (L,\ell_1)\to(L',\ell'_1)$ is a surjection in degree n for all $n\in\mathbb{Z}$.

3.1.1 Strict L_∞–morphisms Finally, recall that a morphism $f\colon (L,\ell)\to(L',\ell')$ in $\mathsf{L_\infty Alg}$ is a *strict L_∞–morphism* if and only if $f_k=0$ for all $k\geq 2$. If
$$f=f_1\colon (L,\ell)\to(L',\ell')$$
is a strict morphism, then it follows from the definition that the restriction–projections (2-3) of the coalgebra morphism F satisfy
$$F^k_m=0 \quad \text{if } k\neq m.$$
Combining this with (2-10), it follows that every k–ary bracket ℓ_k is preserved by the chain map f_1:
$$\ell'_k\circ f_1^{\otimes k}=f_1\circ\ell_k \quad \text{for all } k\geq 1.$$

3.2 The category of Lie n–algebras

Let (L,ℓ) be an L_∞–algebra. If the underlying graded vector space L is concentrated in the first $n-1$ nonnegative degrees, ie
$$L=\bigoplus_{i\geq 0}^{n-1}L_i,$$
then L is called a *Lie n–algebra* [1, Definition 4.3.2].

For a fixed $n\in\mathbb{N}\cup\{\infty\}$, we denote by $\mathsf{Lie}_n\mathsf{Alg}$ the full subcategory of $\mathsf{L_\infty Alg}$ whose objects are Lie n–algebras. Note that if $n<\infty$ then, for degree reasons, $\ell_k=0$ for all $k>n+1$. Similarly, if $f\colon (L,\ell)\to(L',\ell')$ is a morphism in $\mathsf{Lie}_n\mathsf{Alg}$, then $f_k=0$ for all $k>n$.

Example 3.4 We recall a few elementary, but important, examples:

(1) A Lie 1–algebra is just a Lie algebra. This gives a full and faithful embedding of the category of Lie algebras over $\Bbbk$ into $\mathsf{Lie}_n\mathsf{Alg}$ for any n.

(2) We say a Lie n–algebra (L,ℓ) is *abelian* if and only if $\ell_k = 0$ for all $k \geq 2$. Hence, an abelian Lie n–algebra is the same thing as a chain complex concentrated in degrees $0,\dots,n-1$.

(3) Let $\mathsf{dgla}_{\geq 0}^{<n}$ denote the category of chain Lie algebras whose underlying chain complex is concentrated in degrees $0,\dots,n-1$. There is a functor $\mathsf{dgla}_{\geq 0}^{<n} \to \mathsf{Lie}_n\mathsf{Alg}$ which sends a chain Lie algebra $(L,d,[\cdot,\cdot])$ to the Lie n–algebra (L,ℓ), where $\ell_1 = d$, $\ell_2 = [\cdot,\cdot]$ and $\ell_k = 0$ for all $k > 2$. Under this embedding, dgla morphisms are mapped to strict Lie n–algebra morphisms.

A simple but nontrivial example of a Lie 2–algebra is the "string Lie 2–algebra" [10, Definition 8.1]. See Section 7 for further discussion.

Proposition 3.5 *The category* $\mathsf{Lie}_n\mathsf{Alg}$ *is closed under finite products. Moreover, the forgetful functor* $\mathsf{Lie}_n\mathsf{Alg} \to \mathsf{dgCoCom}_{\geq 0}$ *creates products.*

Proof Indeed, the categorical product of any two Lie n–algebras (L,ℓ) and (L',ℓ') is the Lie n–algebra $(L \oplus L', \ell \oplus \ell')$ where, for all $k \geq 1$,

$$\ell_k \oplus \ell'_k((x_1,x'_1),\dots,(x_k,x'_k)) := (\ell_k(x_1,\dots,x_k), \ell'_k(x'_1,\dots,x'_k)).$$

Furthermore, the usual projections $\mathrm{pr}\colon L \oplus L' \to L$ and $\mathrm{pr}'\colon L \oplus L' \to L'$ lift to strict L_∞–epimorphisms

$$(L,\ell) \xleftarrow{\mathrm{pr}} (L \oplus L', \ell \oplus \ell') \xrightarrow{\mathrm{pr}'} (L',\ell')$$

The product of Lie n–algebras then coincides with the product in $\mathsf{dgCoCom}_{\geq 0}$ via the natural isomorphism of graded vector spaces $S(V \oplus V') \cong S(V) \otimes S(V')$. □

Definition 3.6 Let $f\colon (L,\ell) \to (L',\ell')$ be a morphism of Lie n–algebras.

(1) We say f is a *weak equivalence* if the chain map $f_1\colon (L,\ell_1) \to (L',\ell'_1)$ is a quasi-isomorphism.

(2) We say f is a *fibration* if the chain map $f_1\colon (L,\ell_1) \to (L',\ell'_1)$ is surjective in all positive degrees.

(3) We say f is an *acyclic fibration* if f is a weak equivalence and a fibration.

Following the standard terminology from deformation theory, let

$$\mathsf{tan}_{\geq 0}\colon \mathsf{Lie}_n\mathsf{Alg} \to \mathsf{Ch}_{\geq 0}^{\mathrm{proj}}$$

denote the *tangent functor*, which is defined by the assignments

$$(L,\ell)\mapsto(L,\ell_1),\quad (L,\ell)\xrightarrow{f}(L',\ell')\longmapsto(L,\ell_1)\xrightarrow{f_1}(L'_1,\ell'_1). \tag{3-6}$$

Then it follows from Section 2.2.1 that $f\colon(L,\ell)\to(L',\ell')$ is a weak equivalence (resp. fibration) of Lie n–algebras if and only if $\tan_{\geq 0} f$ is a weak equivalence (resp. fibration) in $\mathsf{Ch}^{\mathrm{proj}}_{\geq 0}$. We show later in Corollary 5.8 that the tangent functor is an exact functor.

Before proceeding further, we record below some basic facts about weak equivalences and fibrations in $\mathsf{Lie}_n\mathsf{Alg}$.

Remarks 3.7 (1) Clearly, a morphism f in $\mathsf{Lie}_n\mathsf{Alg}$ is a weak equivalence if and only if it is an L_∞–quasi-isomorphism. In particular, a morphism $f\colon\mathfrak{g}\to\mathfrak{h}$ between Lie algebras is a weak equivalence in $\mathsf{Lie}_1\mathsf{Alg}$ if and only if f is an isomorphism.

(2) A morphism f in $\mathsf{Lie}_n\mathsf{Alg}$ is an acyclic fibration if and only if it is both an L_∞–quasi-isomorphism and an L_∞–epimorphism. Indeed, if (L,ℓ) is a Lie n–algebra, then $H_0(L)=L_0/\operatorname{im}\ell_1$.

(3) If $f\colon(L,\ell)\to(L',\ell')$ is a weak equivalence, then its corresponding dg coalgebra map $F\colon(\bar{S}(sL),\delta)\to(\bar{S}(sL'),\delta')$ is a quasi-isomorphism, ie $H(F)\colon H(\bar{S}(sL))\xrightarrow{\cong} H(\bar{S}(sL'))$. One can verify this by equipping the chain complexes $(\bar{S}(sL),\delta)$ and $(\bar{S}(sL'),\delta')$ with their canonical ascending filtrations induced by tensor word-length, and then analyzing the associated spectral sequence.

(4) The converse of the above statement is false: there exist morphisms in $\mathsf{Lie}_n\mathsf{Alg}$ which are not weak equivalences, whose corresponding coalgebra maps are nevertheless quasi-isomorphisms in $\mathsf{dgCoCom}$. The following example of such a morphism is probably well known.

Let $\mathfrak{g}:=\Bbbk e_1\oplus\Bbbk e_2$ be the 2–dimensional solvable Lie algebra with bracket $[e_1,e_2]=e_1$, and let $\mathfrak{h}:=\Bbbk\tilde{e}$ be the 1–dimensional abelian Lie algebra. Consider the Chevalley–Eilenberg homology of $\mathfrak{g}$ and $\mathfrak{h}$. We have

$$\bar{S}(s\mathfrak{g})=(\Bbbk se_1\oplus\Bbbk se_2)\oplus\Bbbk se_1\vee se_2,\quad \bar{S}(s\mathfrak{h})=\Bbbk s\tilde{e}.$$

Obviously, $H(\bar{S}(s\mathfrak{h}))\cong\Bbbk s\tilde{e}$. Let $\delta_{\mathfrak{g}}$ denote the codifferential encoding the Lie bracket of $\mathfrak{g}$. Then (2-6) and (3-2) imply that

$$\delta_{\mathfrak{g}}(se_1)=\delta_{\mathfrak{g}}(se_2)=0,\quad \delta_{\mathfrak{g}}(se_1\vee se_2)=s[e_1,e_2]=se_1.$$

Hence, $H(\bar{S}(s\mathfrak{g}))\cong\Bbbk se_2$.

Now consider the map $f\colon \mathfrak{g} \to \mathfrak{h}$, defined by $f(e_1) := 0$, $f(e_2) := \tilde{e}$. Clearly f is a Lie algebra morphism, but it is not a weak equivalence (since it is not an isomorphism). Let $F\colon (\bar{S}(s\mathfrak{g}), \delta) \to (\bar{S}(s\mathfrak{h}), 0)$ denote the dg coalgebra morphism corresponding to f. Since f is strict, it follows from (2-4) and (3-4) that $F(se_1) = 0$, $F(se_1 \vee se_2) = F(se_1) \vee F(se_2) = 0$ and

$$F(se_2) = s\tilde{e}.$$

This last equality, combined with the above homology calculations, implies that F is a quasi-isomorphism in $\mathsf{dgCoCom}$.

3.3 Factoring Lie n–algebra morphisms

We now give an explicit factorization for morphisms in $\mathsf{Lie}_n\mathsf{Alg}$, which we will use to show the existence of path objects.

Factoring chain maps in $\mathsf{Ch}^{\mathrm{proj}}_{\geq 0}$ Suppose $f\colon (V, d_V) \to (W, d_W)$ is a morphism of chain complexes. We will factor f explicitly as $f = p_f \jmath$, where $\jmath$ is an acyclic cofibration and p_f is a fibration as defined in Section 2.2.1. Let $(P(W), d_{P(W)}) \in \mathsf{Ch}_{\geq 0}$ denote the chain complex with underlying graded vector space

$$P(W)_0 := \{0\} \oplus W_1, \qquad P(W)_i := W_i \oplus W_{i+1} \quad \text{for } i \geq 1, \tag{3-7}$$

with differential

$$d_{P(W)}(x, y) := (0, x).$$

Note that the complex $(P(W), d_{P(W)})$ is acyclic. In particular, the linear map

$$h\colon P(W) \to P(W)[1], \quad h(x, y) := (y, 0), \tag{3-8}$$

is a contracting chain homotopy. Let $\pi\colon P(W) \to W$ denote the degree zero linear map

$$\pi(x, y) := x + d_W y.$$

It is easy to verify that π is a chain map and that it is surjective in all positive degrees. We can therefore factor f into an acyclic cofibration followed by a fibration:

$$\begin{gathered} V \xrightarrow{\jmath} V \oplus P(W) \xrightarrow{p_f} W, \\ \jmath(v) := (v, (0,0)), \quad p_f(v, (x, y)) := f(v) + \pi(x, y). \end{gathered} \tag{3-9}$$

Factoring strict maps in $\mathsf{Lie}_n\mathsf{Alg}$ Next we extend the above factorization in $\mathsf{Ch}^{\mathrm{proj}}$ to strict morphisms in $\mathsf{Lie}_n\mathsf{Alg}$.

Proposition 3.8 *Let $f = f_1\colon (L,\ell) \to (L',\ell')$ be a strict morphism of Lie n–algebras. Then f can be factored in the category $\mathsf{Lie}_n\mathsf{Alg}$ as*

$$(L,\ell) \xrightarrow{\jmath} (\widetilde{L},\widetilde{\ell}) \xrightarrow{\phi} (L',\ell'),$$

where $\jmath$ is a weak equivalence and ϕ is a fibration in $\mathsf{Lie}_n\mathsf{Alg}$.

For the proof, we'll use the following lemma:

Lemma 3.9 [19, Theorem A.1] *Let $(\widetilde{L},\widetilde{\ell})$ and (L',ℓ') be Lie n–algebras. Let $m > 1$ and suppose $\{\Phi_i^1\colon \bar{S}^i(s\widetilde{L}) \to sL'\}_{1\le i\le m-1}$ is a collection of linear maps satisfying*

$$\sum_{i=1}^{k} \delta_i'^1 \Phi_k^i = \sum_{i=1}^{k} \Phi_i^1 \widetilde{\delta}_k^i \quad \text{for all } k \le m-1,$$

where $\widetilde{\delta}$ and δ' are the codifferentials encoding the L_∞–structures on $\widetilde{L}$ and L', respectively. Then:

(1) *The linear map $c_m\colon \bar{S}^m(s\widetilde{L}) \to sL'$ defined as*

$$c_m := \sum_{k=1}^{m-1} \Phi_k^1 \widetilde{\delta}_m^k - \sum_{k=2}^{m} \delta_k'^1 \Phi_m^k$$

is a degree -1 cycle in the chain complex $(\mathrm{Hom}(\bar{S}^m(s\widetilde{L}), sL'), \partial)$, where

$$\partial c_m = \delta_1'^1 \circ c_m - (-1)^{|c_m|} c_m \circ \widetilde{\delta}_m^m.$$

(2) *There exists a linear map $\Phi_m^1\colon \bar{S}^m(s\widetilde{L}) \to sL'$ such that $\{\Phi_1^1,\dots,\Phi_{m-1}^1,\Phi_m^1\}$ satisfy*

$$\sum_{i=1}^{m} \delta_i'^1 \Phi_m^i = \sum_{i=1}^{m} \Phi_i^1 \widetilde{\delta}_m^i$$

if and only the homology class $[c_m]$ is trivial.

The lemma can be verified by direct computation, which we leave to the reader. Alternatively, both the lemma and Proposition 3.8 follow from applying the obstruction theory developed in [19] to weak L_∞–morphisms between algebras over the Lie_∞ operad.

Proof of Proposition 3.8 Let $f = f_1\colon (L,\ell) \to (L',\ell')$ be a strict morphism of Lie n–algebras. Via (3-9), we factor the chain map $f\colon (L,\ell_1) \to (L',\ell'_1)$ in $\mathsf{Ch}^{\mathrm{proj}}_{\geq 0}$ as

$$L \xrightarrow{\jmath} \widetilde{L} \xrightarrow{p_f} L',$$

where, for brevity, we denote by $\widetilde{L}$ the chain complex

$$(\widetilde{L},\widetilde{\ell}_1) := (L \oplus P(L'), \ell_1 \oplus d_{P(L')}).$$

We then extend the differential $\widetilde{\ell}_1$ to the L_∞–structure

$$\widetilde{\ell}_k((v_1,\vec{v}'_1),\dots,(v_k,\vec{v}'_k)) := (\ell_k(v_1,\dots,v_k),(0,0)) \quad \text{for all } k \geq 2$$

for all $v_i \in L$ and $\vec{v}'_i = (x'_i, y'_i) \in P(L')$. Note that, by construction, if L and L' are concentrated in degrees $0,\dots,n-1$, then $\widetilde{L}$ is as well. Hence, $(\widetilde{L},\widetilde{\ell}) \in \mathsf{Lie}_n\mathsf{Alg}$.

It is easy to see that the inclusion of complexes $\jmath\colon L \to \widetilde{L}$ lifts to a strict L_∞–morphism $\jmath\colon (L,\ell) \to (\widetilde{L},\widetilde{\ell})$ which is, by construction, a weak equivalence in $\mathsf{Lie}_n\mathsf{Alg}$.

Switching to the coalgebra picture, let $\widetilde{\delta}$ be the codifferential on

$$S(s\widetilde{L}) \cong S(sL) \otimes S(sP(L'))$$

that encodes the L_∞–structure on $\widetilde{L}$. We have the equality

$$\widetilde{\delta} = \delta \otimes \mathrm{id} + \mathrm{id} \otimes \widehat{\delta},$$

where $\widehat{\delta}$ denotes the extension of the differential $d_{P(L')}$ on $P(L')$ to $\bar{S}(sP(L'))$, ie $\widehat{\delta}^1_1 = s d_{P(L')} s^{-1}$. Let J denote the coalgebra map corresponding to the strict L_∞–morphism $\jmath$. Our goal is to construct a morphism in $\mathsf{dgCoCom}_{\geq 0}$,

$$\Phi\colon (\bar{S}(s\widetilde{L}),\widetilde{\delta}) \to (\bar{S}(sL'),\delta'),$$

that has the following properties:

(1) $\Phi^1_1 = s \circ p_f \circ s^{-1}$.

(2) $\Phi J = F$, and hence $\Phi^1_k J^k_k = 0$ for all $k \geq 2$, since f is strict.

The above implies that the morphism Φ, combined with J, will give us the desired factorization of f in $\mathsf{L}_\infty\mathsf{Alg}$.

We construct Φ by induction. Let $\Phi^1_1 := s \circ p_f \circ s^{-1}$. Let $m > 1$ and suppose $\{\Phi^1_i\colon \bar{S}^i(s\widetilde{L}) \to sL' \mid 1 \leq i \leq m-1\}$ is a collection of linear maps satisfying

$$\sum_{i=1}^{k} \delta'^1_i \Phi^i_k = \sum_{i=1}^{k} \Phi^1_i \widetilde{\delta}^i_k \quad \text{for all } k \leq m-1$$

and

$$\Phi^1_k J^k_k = 0 \quad \text{for } 2 \le k \le m-1. \tag{3-10}$$

Lemma 3.9(1) implies that the degree -1 linear map

$$c_m\colon \bar{S}^m(\boldsymbol{s}\widetilde{L}) \to \boldsymbol{s}L', \quad c_m := \sum_{k=1}^{m-1} \Phi^1_k \widetilde{\delta}^k_m - \sum_{k=2}^{m} \delta'^1_k \Phi^k_m,$$

as an element of the chain complex $\mathrm{Hom}_{\Bbbk}(\bar{S}^m(\boldsymbol{s}\widetilde{L}), \boldsymbol{s}L')$, satisfies

$$\partial c_m = \delta'^1_1 \circ c_m - (-1)^{|c_m|} c_m \circ \widetilde{\delta}^m_m = 0.$$

Recalling (2-5), which describes the composition of morphisms in $\mathsf{Lie}_n\mathsf{Alg}$, we observe that (3-10) and the hypothesis that f is a strict L_∞–morphism imply that c_m vanishes when restricted to the subspace $\operatorname{im} J^m_m$. Hence, c_m descends to cocycle $\widetilde{c}_m$ in the subcomplex $(\mathrm{Hom}_{\Bbbk}(\operatorname{coker} J^m_m, \boldsymbol{s}L'), \partial)$, where

$$\operatorname{coker} J^m_m \cong \bigoplus_{i=0}^{m-1} S^i(\boldsymbol{s}L) \otimes S^{m-i}(\boldsymbol{s}P(L')) = \bigoplus_{i+j=m} S^i(\boldsymbol{s}L) \otimes \bar{S}^j(\boldsymbol{s}P(L')),$$

and where ∂ denotes, by slight abuse of notation, the restriction of the differential on the ambient complex $\mathrm{Hom}_{\Bbbk}(\bar{S}^m(\boldsymbol{s}\widetilde{L}), \boldsymbol{s}L')$.

Now let $h\colon P(L') \to P(L')[1]$ be the contracting chain homotopy defined in (3-8). By the symmetric version of the "tensor trick" (eg [14, Section 10.3.6]), we extend h to a contracting chain homotopy H on the complex $\big(\bar{S}(\boldsymbol{s}P(L')), \widehat{\delta}\big)$. Explicitly, the restriction of H to length k tensors $H_k\colon \bar{S}^k(\boldsymbol{s}P(L')) \to \bar{S}^k(\boldsymbol{s}P(L'))[1]$ is defined as

$$H_k(\vec{v}'_1, \dots, \vec{v}'_k) := \sum_{\sigma \in \mathbb{S}_k} \sigma^{-1} \cdot (\mathrm{id}^{\otimes k-1} \otimes \boldsymbol{s}h\boldsymbol{s}^{-1}) \sigma \cdot (\vec{v}'_1, \dots, \vec{v}'_k) \quad \text{for all } \vec{v}'_i \in \boldsymbol{s}P(L'),$$

where $\sigma\cdot$ denotes the left action defined in (2-1). A direct calculation shows that indeed $\mathrm{id} = \widehat{\delta}H + H\widehat{\delta}$. Since $\widetilde{\delta}^m_m = \sum_{i+j=m} (\delta^i_i \otimes \mathrm{id} + \mathrm{id} \otimes \widehat{\delta}^j_j)$, it follows that the map

$$\mathrm{id}_{S(\boldsymbol{s}L)} \otimes H\colon \operatorname{coker} J^m_m \to \operatorname{coker} J^m_m[1]$$

is a contracting chain homotopy for the complex $(\operatorname{coker} J^m_m, \widetilde{\delta}^m_m)$. Therefore, the linear map $\widetilde{c}_m \circ (\mathrm{id} \otimes H)\colon \operatorname{coker} J^m_m \to \boldsymbol{s}L'$ satisfies

$$\widetilde{c}_m = -\partial(\widetilde{c}_m \circ (\mathrm{id} \otimes H)). \tag{3-11}$$

Finally, we extend H to all of $S(\boldsymbol{s}P(L'))$ by declaring $H(1_{\Bbbk}) := 0$, and we let $\Phi^1_m\colon \bar{S}(\boldsymbol{s}\widetilde{L}) \to \boldsymbol{s}L'$ be the linear map

$$\Phi^1_m((v_1,\vec{v}_1'),\dots,(v_k,\vec{v}_m')) := -c_m \circ (\mathrm{id}\otimes H)((v_1,\vec{v}_1'),\dots,(v_k,\vec{v}_m')).$$

Hence, (3-11) implies that $c_m = \partial\Phi^1_k$. It then follows from Lemma 3.9(2) that the collection $\{\Phi^1_1,\dots,\Phi^1_{m-1},\Phi^1_m\}$ satisfies

$$\sum_{i=1}^{k}\delta_i'^{1}\Phi^i_k = \sum_{i=1}^{k}\Phi^1_i\widetilde{\delta}^i_k \quad \text{for all } k\le m,$$

and $\Phi^1_k J^k_k = 0$ for $2\le k\le m$. This completes the inductive step, and therefore the proof of the proposition. $\square$

Remark 3.10 The factorization recalled in the beginning of this section of a chain map $f\colon (V,d_V)\to(W,d_W)$ in $\mathsf{Ch}^{\mathrm{proj}}_{\ge 0}$ is functorial. Indeed, a commutative diagram in $\mathsf{Ch}_{\ge 0}$ of the form

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\alpha"'] & W \arrow[d, "\beta"] \\ V' \arrow[r, "f'"] & W' \end{tikzcd}$$

factors as

$$\begin{tikzcd} V \arrow[r, "\jmath"] \arrow[d, "\alpha"'] & V\oplus P(W) \arrow[r, "p_f"] \arrow[d, "\gamma"] & W \arrow[d, "\beta"] \\ V' \arrow[r, "\jmath'"] & V'\oplus P(W') \arrow[r, "p_{f'}"] & W' \end{tikzcd}$$

where $\gamma\colon V\oplus P(W)\to V'\oplus P(W')$ is the chain map

$$\gamma(v,(x,y)) := \bigl(\alpha(v),(\beta(x),\beta(y))\bigr).$$

It would be convenient if the factorization in Proposition 3.8 could be made functorial in a similar way, perhaps using the symmetrized homotopy $H\colon \bar{S}(\boldsymbol{s}P(L'))\to \bar{S}(\boldsymbol{s}P(L'))[1]$. Then, Theorem 5.2 would imply that $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ is equipped with a functorial path object. We leave this as an open question.

Factoring arbitrary maps in $\mathsf{Lie}_n\mathsf{Alg}$ Let $f\colon (L,\ell)\to(L',\ell')$ be an arbitrary (not necessarily strict) L_∞–morphism in $\mathsf{Lie}_n\mathsf{Alg}$. Then f can be factored into a weak equivalence composed with a fibration in the following way. First, we observe that the diagonal map diag: $(L',\ell')\to(L'\oplus L',\ell'\oplus\ell')$ is a strict L_∞–morphism. Hence, Proposition 3.8 gives us an explicit factorization of the diagonal into a weak equivalence followed by a fibration

$$(L',\ell') \xrightarrow{J} (L'^{I},\ell'^{I}) \xrightarrow{\phi} (L'\oplus L',\ell'\oplus\ell').$$

The Lie n–algebra (L'^{I},ℓ'^{I}) is a path object for (L',ℓ'), in the sense of Definition 5.1. When combined with Corollary 4.4, which concerns the existence of pullbacks of (acyclic) fibrations, the existence of a path object for (L',ℓ') implies the existence of a factorization[1] of f. We provide more details later, in Corollary 5.5, for morphisms in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$, our main category of interest.

3.3.1 Strictification of fibrations In the last part of this section, we show that every fibration in $\mathsf{Lie}_n\mathsf{Alg}$ can be factored into an isomorphism followed by a strict fibration. We will use this fact repeatedly throughout the paper. The proof is a modification of a result of Vallette [19] concerning the factorization of "∞–epimorphisms" between $\mathbb{Z}$–graded homotopy algebras.

Lemma 3.11 *Let $f\colon (L,\ell)\to(L',\ell')$ be a fibration between Lie n–algebras. Then there exists a Lie n–algebra $(L,\widetilde{\ell})$ and an isomorphism $\phi\colon (L,\widetilde{\ell}) \xrightarrow{\cong} (L,\ell)$ such that*

$$f\phi\colon (L,\widetilde{\ell})\to(L',\ell')$$

is a strict fibration with $f\phi=(f\phi)_1=f_1$.

Proof Let $F\colon \overline{S}(sL)\to\overline{S}(sL')$ be the coalgebra morphism corresponding to the fibration f. Then the linear map $F^1_1\colon \bigoplus_{i\geq 1} sL_i \to \bigoplus_{i\geq 1} sL'_i$ is surjective in all degrees $i\geq 2$. Let $\sigma\colon \bigoplus_{i\geq 1} sL'_i\to\bigoplus_{i\geq 1} sL_i$ be a map of graded vector space whose restrictions satisfy the equalities

$$\sigma|_{sL'_1}=0,\qquad F^1_1\circ\sigma|_{\bigoplus_{i\geq 2} sL'_i}=\mathrm{id}.$$

Let $\Phi^1_1:=\mathrm{id}\colon sL\to sL$. Let $m\geq 2$ and suppose we have defined a sequence of degree 0 linear maps $\{\Phi^1_k\colon \overline{S}^k(sL)\to sL\}_{k\geq 1}^{m-1}$. It follows from (2-3) that this sequence gives well-defined linear maps $\Phi^k_m\colon \overline{S}^m(sL)\to\overline{S}^k(sL)$ for $2\leq k\leq m$. Now define $\Phi^1_m\colon \overline{S}^m(sL)\to sL$ as

$$\Phi^1_m=-\sum_{k\geq 2}^{m}\sigma F^1_k\Phi^k_m.$$

This construction inductively yields a coalgebra isomorphism $\Phi\colon \overline{S}(sL)\to\overline{S}(sL)$. Now let δ be the codifferential on $\overline{S}(sL)$ corresponding to the Lie n–algebra structure

[1] This is Brown's factorization lemma (see Lemma 5.4).

on L. We define a new codifferential $\widetilde{\delta} := \Phi^{-1}\delta\Phi$. Clearly, we can promote Φ to an isomorphism of dg coalgebras $\Phi\colon (\bar{S}(sL), \widetilde{\delta}) \to (\bar{S}(sL), \delta)$. And since $\Phi^1_1 = \mathrm{id}$, it follows from (2-5) that we have

$$(F\Phi)^1_1 = F^1_1\colon sL \to sL'.$$

It remains to show that $F\Phi$ is strict, ie $(F\Phi)^1_m = 0$ for all $m \geq 2$. Equation (2-5) and the construction of Φ imply that

$$(F\Phi)^1_m = -F^1_1\left(\sum_{k\geq 2}^{m} \sigma F^1_k \Phi^k_m\right) + \sum_{k\geq 2}^{m} F^1_k \Phi^k_m.$$

If $m \geq 2$, then for any homogeneous element $y \in \bar{S}^m(sL)$, we have $|y| > 1$. Hence, $|F^1_k \Phi^k_m(y)| > 1$ for any $k \geq 1$. It then follows from the definition of σ that $F^1_1(\sigma F^1_k \Phi^k_m(y)) = F^1_k \Phi^k_m(y)$. Therefore, $(F\Phi)^1_m = 0$ for all $m \geq 2$. □

4 Pullbacks in $\mathsf{Lie}_n\mathsf{Alg}$

In this section, we give an explicit description of pullbacks of fibrations and acyclic fibrations in the category $\mathsf{Lie}_n\mathsf{Alg}$. After recalling the construction of pullbacks in $\mathsf{dgCoCom}_{\geq 0}$, we focus on the special case of pulling back strict fibrations in $\mathsf{Lie}_n\mathsf{Alg}$. We then use Lemma 3.11 to address the more general nonstrict case.

Consider a diagram of Lie n–algebras of the form $(L', \ell') \xrightarrow{g} (L'', \ell'') \xleftarrow{f} (L, \ell)$. The category $\mathsf{dgCoCom}_{\geq 0}$ is complete, so the pullback of the corresponding diagram of dg coalgebras exists,

(4-1)
$$\begin{tikzcd}
(\bar{P}, \delta_P) \arrow[r, "\mathrm{Pr}"] \arrow[d, "\mathrm{Pr}'"'] \arrow[dr, phantom, "\lrcorner"] & (\bar{S}(sL), \delta) \arrow[d, "F"] \\
(\bar{S}(sL'), \delta') \arrow[r, "G"] & (\bar{S}(sL''), \delta'')
\end{tikzcd}$$

The graded coalgebra $\bar{P}$ is the equalizer of the diagram

$$\bar{S}(sL \oplus sL') \underset{G\,\mathrm{Pr}'}{\overset{F\,\mathrm{Pr}}{\rightrightarrows}} \bar{S}(sL''),$$

which can be characterized as the largest subcoalgebra of $\bar{S}(sL \oplus sL')$ contained in the vector space $\ker(F\,\mathrm{Pr} - G\,\mathrm{Pr}')$. By generalizing a construction of Sweedler

[18, Section 16.1], we have an explicit description of $\overline{P}$

$$\overline{P} = \big\{ v \in \ker(F\,\mathrm{Pr} - G\,\mathrm{Pr}') \mid (\mathrm{id} \otimes F\,\mathrm{Pr})\overline{\Delta}(v) = (\mathrm{id} \otimes G\,\mathrm{Pr}')\overline{\Delta}(v) \in \overline{S}(sL \oplus sL') \otimes \overline{S}(sL'') \big\}. \tag{4-2}$$

Above $\overline{\Delta}$ is the reduced comultiplication on $\overline{S}(sL \oplus sL')$. Using arguments analogous to those in the proof of Lemma 16.1.1 in [18], it is not difficult to show that the restriction $\overline{\Delta}|_P$ gives $\overline{P}$ the structure of a cocommutative coalgebra. The codifferential δ_P is, of course, the restriction of the codifferential $\delta_\oplus$ on $\overline{S}(sL \oplus sL')$.

4.1 Strict fibrations

The pullback of a strict fibration in $\mathsf{Lie}_n\mathsf{Alg}$ has a convenient explicit description.

Proposition 4.1 *Suppose $f = f_1\colon (L, \ell) \to (L'', \ell'')$ is a strict fibration in $\mathsf{Lie}_n\mathsf{Alg}$ and $g\colon (L', \ell') \to (L'', \ell'')$ is an arbitrary morphism between Lie n–algebras. Let $(\widetilde{L}, \widetilde{\ell}_1) \in \mathsf{Ch}_{\geq 0}$ denote the pullback of the diagram of chain maps $(L'_1, \ell'_1) \xrightarrow{g_1} (L''_1, \ell''_1) \xleftarrow{f_1} (L, \ell_1)$.*

(1) *The pullback square in $\mathsf{Ch}_{\geq 0}$ containing f_1 and g_1 lifts to a commutative diagram in $\mathsf{Lie}_n\mathsf{Alg}$,*

$$\begin{tikzcd} (\widetilde{L}, \widetilde{\ell}) \arrow[r] \arrow[d] & (L, \ell) \arrow[d, "f"] \\ (L', \ell') \arrow[r, "g"] & (L'', \ell'') \end{tikzcd} \tag{4-3}$$

(2) *Let $(\overline{P}, \delta_P)$ be the pullback of the diagram $(\overline{S}(sL'), \delta') \xrightarrow{G} (\overline{S}(sL''), \delta'') \xleftarrow{F} (\overline{S}(sL), \delta)$, where F and G are the dg coalgebra morphisms corresponding to f and g, respectively. Then there exists an isomorphism of dg coalgebras*

$$(\overline{P}, \delta_P) \xrightarrow{\cong} (\overline{S}(s\widetilde{L}), \widetilde{\delta})$$

which identifies $(\overline{S}(s\widetilde{L}), \widetilde{\delta})$ as the pullback of F and G in $\mathsf{dgCoCom}_{\geq 0}$. Hence, the diagram (4-3) *is a pullback diagram in $\mathsf{Lie}_n\mathsf{Alg}$.*

Before we prove Proposition 4.1, we will need to discuss a few technical constructions.

Useful endomorphisms of the product coalgebra As above, consider a diagram of the form $(L', \ell') \xrightarrow{g} (L'', \ell'') \xleftarrow{f = f_1} (L, \ell)$ in $\mathsf{Lie}_n\mathsf{Alg}$, in which f is a strict fibration.

In order to give an explicit description of the L_∞–structure on the pullback, we first construct[2] two auxiliary endomorphisms of the graded coalgebra $\overline{S}(sL' \oplus sL)$.

Throughout this section, we denote elements of the direct sum $sL' \oplus sL$ as vectors $\vec{v} := (v', v)$. We first give a convenient description of $s\widetilde{L}$, the suspension of the pullback of f_1, as a graded vector space. Since f is a fibration, the linear map $F_1^1\colon \bigoplus_{i\geq 1} sL_i \to \bigoplus_{i\geq 1} sL''_i$ is surjective in all degrees $i \geq 2$. Let $\sigma\colon \bigoplus_{i\geq 1} sL''_i \to \bigoplus_{i\geq 1} sL_i$ be a map of graded vector spaces whose restrictions satisfy the equalities

$$\sigma|_{sL''_1} = 0, \quad F_1^1 \circ \sigma|_{\bigoplus_{i\geq 2} sL''_i} = \mathrm{id}. \tag{4-4}$$

Then we have

$$s\widetilde{L}_1 = sL'_1 \times_{sL''_1} sL_1, \quad s\widetilde{L}_{i\geq 2} = sL'_{i\geq 2} \oplus \ker F_1 \tag{4-5}$$

and a pullback diagram of graded vector spaces

$$\begin{tikzcd} s\widetilde{L} \arrow[r, "\mathrm{pr} + \sigma G_1 \mathrm{pr}'"] \arrow[d, "\mathrm{pr}'"'] \arrow[dr, phantom, "\lrcorner"] & sL \arrow[d, "F_1"] \\ sL' \arrow[r, "G_1"] & sL'' \end{tikzcd} \tag{4-6}$$

Clearly, $\overline{S}(s\widetilde{L}) \subseteq \overline{S}(sL' \oplus sL)$ as graded coalgebras. We define the linear maps $H_k^1\colon \overline{S}^k(sL' \oplus sL) \to sL' \oplus sL$ to be

$$H_1^1(\vec{v}) := (v', \sigma G_1(v') + v), \quad H_k^1(\vec{v}_1, \ldots, \vec{v}_k) := (0, \sigma G_k^1(v'_1, \ldots, v'_k)). \tag{4-7}$$

Similarly, let $J_k^1\colon \overline{S}^k(sL' \oplus sL) \to sL' \oplus sL$ denote the linear maps

$$J_1^1(\vec{v}) := (v', -\sigma G_1(v') + v), \quad J_k^1(\vec{v}_1, \ldots, \vec{v}_k) := (0, -\sigma G_k^1(v'_1, \ldots, v'_k)). \tag{4-8}$$

Claim 4.2 *We have the equalities $HJ = JH = \mathrm{id}_{\overline{S}(sL' \oplus sL)}$.*

Proof Indeed, using the definition of σ, a direct computation verifies that $(HJ)_1^1 = H_1^1 J_1^1 = \mathrm{id}_{sL' \oplus sL}$. Now suppose $m \geq 2$. It remains to show $(HJ)_m^1 = 0$. It follows from (2-5) that $(HJ)_m^1 = \sum_{k=1}^m H_k^1 J_m^k$. From (2-3), we see that the formula for J_m^k involves a summation of tensor products of linear maps of the form

$$\sum_{j_1 + j_2 + \cdots + j_k = m} J_{j_1}^1 \otimes J_{j_2}^1 \otimes \cdots \otimes J_{j_k}^1. \tag{4-9}$$

[2] To the best of our knowledge, this construction is due to Bruno Vallette. It is implicit in his proof of Theorem 4.1 in [19].

Hence, if $k < m$, then in each term of above sum, there exists a $j_i > 1$, and so it follows from the definition (4-8) of J that $\mathrm{pr}' J^1_{j_i} = 0$. Combining this observation with the definition (4-7) of H, we deduce that if $2 \le k < m$, then $H^1_k J^k_m = 0$. Therefore,

$$
\begin{aligned}
&(HJ)^1_m(\vec{v}_1, \dots, \vec{v}_m) \\
&\quad = H^1_1 J^1_m(\vec{v}_1, \dots, \vec{v}_m) + H^1_m J^m_m(\vec{v}_1, \dots, \vec{v}_m) \\
&\quad = H^1_1(0, -\sigma G^1_m(v'_1, \dots, v'_m)) \\
&\qquad\quad + H^1_m\big((v'_1, -\sigma G_1(v'_1) + v_1), (v'_2, -\sigma G_1(v'_2) + v_2), \dots, (v'_m, -\sigma G_1(v'_m) + v_m)\big) \\
&\quad = (0, -\sigma G^1_m(v'_1, \dots, v'_m)) + (0, \sigma G^1_m(v'_1, \dots, v'_m)) = 0.
\end{aligned}
$$

Hence, $HJ = \mathrm{id}_{\bar{S}(sL' \oplus sL)}$. The same proof (mutatis mutandis) shows that $JH = \mathrm{id}_{\bar{S}(sL' \oplus sL)}$, and so the claim has been verified. □

Now let $\delta_\oplus$ denote the codifferential on the product $\bar{S}(sL' \oplus sL)$, and define

$$\tilde{\delta} := J \circ \delta_\oplus \circ H|_{\bar{S}(s\tilde{L})}.$$

Claim 4.3 *$\tilde{\delta}$ induces a well-defined codifferential on $\bar{S}(s\tilde{L})$.*

Proof Note that if $\operatorname{im} \tilde{\delta} \subseteq \bar{S}(s\tilde{L})$, then Claim 4.2 implies that $\tilde{\delta}^2 = 0$, and hence $\tilde{\delta}$ is a codifferential. Therefore, all we need to show is that $\operatorname{im} \tilde{\delta}^1_m \subseteq s\tilde{L}$ for $m \ge 1$. We proceed by considering a few cases.

Case *m* = 1 If $\vec{v} \in s\tilde{L}_{i \ge 2}$, then it follows from (2-5) and the definitions of H and J that

$$\tilde{\delta}^1_1(\vec{v}) = J^1_1(\delta_\oplus)^1_1 H^1_1(\vec{v}) = \big(\delta'^1_1(v'), -\sigma G_1(\delta'^1_1(v')) + \delta^1_1 \sigma G_1(v') + \delta^1_1(v)\big).$$

If $|\vec{v}| = 2$, then $(\delta_\oplus)^1_1 \vec{v} \in sL'_1 \oplus sL_1$. Therefore, $\sigma G_1(\delta'^1_1(v')) = 0$ and so

$$F_1\big(-\sigma G_1(\delta'^1_1(v')) + \delta^1_1 \sigma G_1(v') + \delta^1_1(v)\big) = \delta''^1_1(F_1 \sigma G_1(v')) = G_1(\delta'^1_1(v')).$$

Hence, $\tilde{\delta}^1_1(\vec{v}) \in s\tilde{L}_1$. If $|\vec{v}| > 2$, then $\delta^1_1 v \in \ker F_1$ and so

$$
\begin{aligned}
F_1\big(-\sigma G_1(\delta'^1_1(v')) + \delta^1_1 \sigma G_1(v') + \delta^1_1(v)\big) &= -G_1(\delta'^1_1(v')) + \delta''^1_1(F \sigma G_1(v')) \\
&= -G_1(\delta'^1_1(v')) + \delta''^1_1 G_1(v') = 0.
\end{aligned}
$$

Hence, $\tilde{\delta}^1_1(\vec{v}) \in s\tilde{L}_{i > 1}$.

Case $m \geq 2$ Let $\vec{v}_1, \dots, \vec{v}_m \in s\widetilde{L}$. It follows from (2-5) that

$$\widetilde{\delta}^1_m(\vec{v}_1, \dots, \vec{v}_m) = \sum_{k \geq 1}^{m} \sum_{l \geq 1}^{k} J^1_l (\delta_\oplus)^l_k H^k_m(\vec{v}_1, \dots, \vec{v}_m). \tag{4-10}$$

First suppose that $|(\delta_\oplus)^l_k H^k_m(\vec{v}_1, \dots, \vec{v}_m)| = 1$. Then, for degree reasons it must be the case that $m = 2$, and $\vec{v}_1, \vec{v}_2 \in s\widetilde{L}_1$ are in lowest degree. This implies that $(\delta_\oplus)^2_2 H^2_2(\vec{v}_1, \vec{v}_2) = (\delta_\oplus)^2_2(H^1_1(\vec{v}_1) \vee H^1_1(\vec{v}_2)) = 0$. Therefore, by expanding (4-10) we obtain

$$\begin{aligned} \widetilde{\delta}^1_2(\vec{v}_1, \vec{v}_2) &= J^1_1((\delta_\oplus)^1_1 H^1_2(\vec{v}_1, \vec{v}_2) + (\delta_\oplus)^1_2 H^2_2(\vec{v}_1, \vec{v}_2)) \\ &= (0, \delta^1_1 \sigma G^1_2(v'_1, v'_2)) + (\delta'^1_2(v'_1, v'_2), \delta^1_2(v_1, v_2)). \end{aligned}$$

Since F corresponds to a strict L_∞–morphism, $F_1(\delta^1_2(v_1, v_2)) = \delta''^1_2(F_1(v_1), F_1(v_2))$. Furthermore, since $\vec{v}_1, \vec{v}_2 \in s\widetilde{L}_1$, we have $(F_1(v_1), F_1(v_2)) = (G_1(v'_1), G_1(v'_2))$. From these two equalities, we deduce that

$$F_1(\delta^1_1 \sigma G^1_2(v'_1, v'_2) + \delta^1_2(v_1, v_2)) = (\delta''^1_2 G^2_2 + \delta''^1_1 G^1_2)(v'_1, v'_2). \tag{4-11}$$

Since G is a morphism of dg coalgebras, we have

$$(\delta''^1_2 G^2_2 + \delta''^1_1 G^1_2)(v'_1, v'_2) = (G^1_1 \delta'^1_2 + G^1_2 \delta'^2_2)(v'_1, v'_2) = G^1_1 \delta'^1_2(v'_1, v'_2).$$

By substituting this last equality into (4-11), we conclude that

$$G_1(\delta'^1_2(v'_1, v'_2)) = F_1(\delta^1_1 \sigma G^1_2(v'_1, v'_2) + \delta^1_2(v_1, v_2)),$$

and hence $\widetilde{\delta}^1_2(\vec{v}_1, \vec{v}_2) \in s\widetilde{L}_1$.

Now we consider the remaining subcase and suppose $|\vec{v}_1 + \cdots + \vec{v}_m| > 2$. From (2-3), we see that the formula for H^k_m consists of a sum of tensor products of linear maps $H^1_{j_1} \otimes H^1_{j_2} \otimes \cdots \otimes H^1_{j_k}$, just as J^k_m involved the summation (4-9). Hence, we can apply the argument used in the verification of Claim 4.2 to the present case and deduce that if $2 \leq l \leq k < m$, then

$$J^1_l (\delta_\oplus)^l_k H^k_m(\vec{v}_1, \dots, \vec{v}_m) = 0.$$

Consequently,

$$\begin{aligned} \widetilde{\delta}^1_m(\vec{v}_1, \dots, \vec{v}_m) \\ = J^1_1\Bigg(\sum_{k=1}^{m-1} (\delta_\oplus)^1_k H^k_m(\vec{v}_1, \dots, \vec{v}_m)\Bigg) + \sum_{l=1}^{m} J^1_l (\delta_\oplus)^l_m H^m_m(\vec{v}_1, \dots, \vec{v}_m). \end{aligned} \tag{4-12}$$

Note that in order to show $\widetilde{\delta}^1_m(\vec{v}_1, \dots, \vec{v}_m) \in s\widetilde{L}$, it suffices to verify that

$$\mathrm{pr}\widetilde{\delta}^1_m(\vec{v}_1, \dots, \vec{v}_m) \in \ker F_1,$$

since $|\vec{v}_1 + \cdots + \vec{v}_m| > 2$. Let us focus on the first summation on the right-hand side of (4-12). It follows from the definition of $\delta_\oplus$ that we have $(\delta_\oplus)^1_k H^k_m = (\delta'^1_k \operatorname{pr}'^{\otimes k} H^k_m, \delta^1_k \operatorname{pr}^{\otimes k} H^k_m)$. Since $k < m$, we have $\operatorname{pr}'^{\otimes k} H^k_m = 0$, and hence

$$\operatorname{pr} J^1_1\Bigl(\sum_{k=1}^{m-1} (\delta_\oplus)^1_k H^k_m(\vec{v}_1,\dots,\vec{v}_m)\Bigr) = \sum_{k=1}^{m-1} \delta^1_k \operatorname{pr}^{\otimes k} H^k_m(\vec{v}_1,\dots,\vec{v}_m).$$

Since F corresponds to a strict L_∞–morphism, applying F_1 to the right-hand side of the above equality gives

$$\sum_{k=1}^{m-1} F_1\delta^1_k \operatorname{pr}^{\otimes k} H^k_m(\vec{v}_1,\dots,\vec{v}_m) = \sum_{k=1}^{m-1} \delta''^1_k (F_1 \operatorname{pr})^{\otimes k} H^k_m(\vec{v}_1,\dots,\vec{v}_m). \tag{4-13}$$

The expansion of $(F_1 \operatorname{pr})^{\otimes k} H^k_m$ using (2-3) involves sums of the form

$$\sum_{j_1+j_2+\cdots+j_k=m} (F_1 \operatorname{pr}) H^1_{j_1} \otimes (F_1 \operatorname{pr}) H^1_{j_2} \otimes \cdots \otimes (F_1 \operatorname{pr}) H^1_{j_k}.$$

If $j_r > 1$ and $\vec{w}_1,\dots,\vec{w}_{j_r} \in \{\vec{v}_1,\dots,\vec{v}_m\} \subseteq \boldsymbol{s}\widetilde{L}$, then

$$(F_1 \operatorname{pr}) H^1_{j_r}(\vec{w}_1,\dots,\vec{w}_{j_r}) = F_1 \sigma G^1_{j_r}(w'_1,\dots,w'_{j_r}) = G^1_{j_r}(w'_1,\dots,w'_{j_r}),$$

where the last equality above follows from the fact that $|\vec{w}_1,\dots,\vec{w}_{j_r}| > 1$. For the $j_r = 1$ case, if $\vec{v}_i \in \boldsymbol{s}\widetilde{L}_1$, then the definition of H^1_1 implies that

$$F_1 \operatorname{pr} H^1_1(\vec{v}_i) = F_1(\sigma G_1(v'_i) + v_i) = F_1(v_i) = G_1(v_i).$$

And if $\vec{v}_i \in \boldsymbol{s}\widetilde{L}_{j\geq 2}$, then $v_i \in \ker F$, which implies that

$$F_1 \operatorname{pr} H^1_1(\vec{v}_i) = F_1(\sigma G_1(v'_i)) = G_1(v'_i). \tag{4-14}$$

Therefore, by combining equations (4-13)–(4-14), we conclude that

$$F_1 \operatorname{pr} J^1_1\Bigl(\sum_{k=1}^{m-1} (\delta_\oplus)^1_k H^k_m(\vec{v}_1,\dots,\vec{v}_m)\Bigr) = \sum_{k=1}^{m-1} \delta''^1_k G^k_m(v'_1,\dots,v'_m). \tag{4-15}$$

Now consider the second summation on the right-hand side of (4-12). Equation (2-7) and the definitions of $\delta_\oplus$ and $J^1_{l\geq 2}$ imply that

$$\operatorname{pr} \sum_{l=2}^{m} J^1_l (\delta_\oplus)^l_m H^m_m = -\sum_{l=2}^{m} \sigma G^1_l \delta'^l_m \operatorname{pr}'^{\otimes m} H^m_m.$$

Hence,

$$\operatorname{pr}\sum_{l=2}^{m} J_l^1(\delta_\oplus)_m^l H_m^m(\vec{v}_1,\dots,\vec{v}_m) = -\sum_{l=2}^{m}\sigma G_l^1 \delta'^l_m(v'_1,\dots,v'_m).$$

Furthermore, it follows directly from the definition of J_1^1 that

$$\begin{aligned}\operatorname{pr} J_1^1(\delta_\oplus)_m^1 H_m^m(\vec{v}_1,\dots,\vec{v}_m)&\\ &= (-\sigma G_1 \delta'^1_m \operatorname{pr}'^{\otimes m} H_m^m + \delta_m^1 \operatorname{pr}^{\otimes m} H_m^m)(\vec{v}_1,\dots,\vec{v}_m)\\ &= -\sigma G_1^1 \delta'^1_m(v'_1,\dots,v'_m) + \delta_m^1(\sigma G_1(v'_1)+v_1,\dots,\sigma G_1(v'_m)+v_m).\end{aligned}$$

We apply F_1, and by using the above equalities, we obtain

$$\begin{aligned}F_1 \operatorname{pr}\Bigl(\sum_{l=1}^{m} J_l^1(\delta_\oplus)_m^l H_m^m(\vec{v}_1,\dots,\vec{v}_m)\Bigr)&\\ = -\sum_{l=2}^{m} F_1\sigma G_l^1\delta'^l_m(v'_1,\dots,v'_m) &- F_1\sigma G_1^1\delta'^1_m(v'_1,\dots,v'_m)\\ &+ F_1\delta_m^1(\sigma G_1(v'_1)+v_1,\dots,\sigma G_1(v'_m)+v_m)\\ = -\sum_{l=2}^{m} G_l^1\delta'^l_m(v'_1,\dots,v'_m) &- G_1^1\delta'^1_m(v'_1,\dots,v'_m)\\ &+ \delta''^1_m(F_1)^{\otimes m}(\sigma G_1(v'_1)+v_1,\dots,G_1(v'_m)+v_m)\\ = -\sum_{l=1}^{m} G_l^1\delta'^l_m(v'_1,\dots,v'_m) &+ \delta''^1_m G_m^m(v'_1,\dots,v'_m).\end{aligned}\tag{4-16}$$

To obtain the last two lines above, we used the fact that F is a strict L_∞–morphism, as well as the definition of σ, and the fact that either $v_i \in \ker F_1$ or $G_1(v'_i) = F_1(v_i)$ for all $i = 1,\dots,m$. Finally, we combine (4-15) with (4-16) to obtain

$$F_1 \operatorname{pr}\widetilde{\delta}_m^1(\vec{v}_1,\dots,\vec{v}_m) = \sum_{k=1}^{m}\delta''^1_k G_m^k(v'_1,\dots,v'_m) - \sum_{l=1}^{m} G_l^1\delta'^l_m(v'_1,\dots,v'_m).$$

Therefore, since G is a dg coalgebra morphism, we conclude that

$$F_1 \operatorname{pr}\widetilde{\delta}_m^1(\vec{v}_1,\dots,\vec{v}_m) = 0.$$

This completes the verification of the claim. □

The proof of Proposition 4.1

Proof of (1) Claims 4.2 and 4.3 above imply that $(\bar{S}(s\widetilde{L}),\widetilde{\delta})$ is a dg coalgebra, and that $H\colon (\bar{S}(s\widetilde{L}),\widetilde{\delta}) \to (\bar{S}(sL'\oplus sL),\delta_\oplus)$ is a dg coalgebra morphism. We wish to show that the following diagram in $\mathsf{Lie}_n\mathsf{Alg} \subseteq \mathsf{dgCoCom}_{\geq 0}$ commutes:

$$\begin{tikzcd}
(\bar{S}(s\widetilde{L}), \widetilde{\delta}) \arrow[r, "\Pr H"] \arrow[d, "\Pr' H"'] & (\bar{S}(sL), \delta) \arrow[d, "F"] \\
(\bar{S}(sL'), \delta') \arrow[r, "G"] & (\bar{S}(sL''), \delta'')
\end{tikzcd} \tag{4-17}$$

It suffices to show that for all $m \geq 1$, the linear maps

$$(G \Pr' H)^1_m\colon \bar{S}^m(s\widetilde{L}) \to sL'', \quad (F \Pr H)^1_m\colon \bar{S}^m(s\widetilde{L}) \to sL''$$

are equal. It follows from (2-5) that

$$\begin{aligned}
(G \Pr' H)^1_m &= \sum_{k=1}^m \sum_{i=1}^k G^1_i \Pr'^i_k H^k_m = \sum_{k=1}^m G^1_k \operatorname{pr}'^{\otimes k} H^k_m, \\
(F \Pr H)^1_m &= \sum_{k=1}^m \sum_{i=1}^k F^1_i \Pr^i_k H^k_m = \sum_{k=1}^m F^1_k \operatorname{pr}^{\otimes k} H^k_m = F^1_1 \operatorname{pr} H^1_m.
\end{aligned} \tag{4-18}$$

Note that the last equality above follows from the hypothesis that F is strict.

We first consider the $m = 1$ case. Let $\vec{v} \in s\widetilde{L}$. From the definition (4-7) of H, we have

$$F^1_1 \operatorname{pr} H^1_1(\vec{v}) = F^1_1(\sigma G_1(v') + v).$$

Hence, the commutativity of diagram (4-6) implies that $F^1_1 \operatorname{pr} H^1_1(\vec{v}) = G^1_1 \operatorname{pr}' H^1_1(\vec{v})$.

Now suppose $m \geq 2$. From (2-3), we see that the formula for H^k_m involves a summation of tensor products of linear maps of the form

$$\sum_{i_1+i_2+\cdots+i_k=m} H^1_{i_1} \otimes H^1_{i_2} \otimes \cdots \otimes H^1_{i_k}.$$

Hence, if $k < m$, then in each term of above sum, there exists an $i_r > 1$, and then it follows from the definition of H that $\operatorname{pr}' H^1_{i_r} = 0$. So we deduce that

$$(G \Pr' H)^1_m = G^1_m \operatorname{pr}'^{\otimes m} H^m_m.$$

Therefore, for any $\vec{v}_1, \dots, \vec{v}_m \in s\widetilde{L}$, we obtain the equalities

$$\begin{aligned}
(G \Pr' H)^1_m(\vec{v}_1, \dots, \vec{v}_m) &= G^1_m \operatorname{pr}'^{\otimes m} H^m_m(\vec{v}_1, \dots, \vec{v}_m) \\
&= G^1_m(\operatorname{pr}' H^1_1(\vec{v}_1), \operatorname{pr}' H^1_1(\vec{v}_2), \dots, \operatorname{pr}' H^1_1(\vec{v}_m)) \\
&= G^1_m(v'_1, v'_2, \dots, v'_m).
\end{aligned}$$

On the other hand, it follows from (4-18) and the definition of H that

$$\begin{aligned}(F\,\mathrm{Pr}\,H)^1_m(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m) &= F^1_1 \mathrm{pr}\,H^1_m(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m)\\ &= F^1_1\sigma G^1_m(v'_1,v'_2,\dots,v'_m)\\ &= G^1_m(v'_1,v'_2,\dots,v'_m).\end{aligned}$$

Hence, we conclude that (4-17) indeed commutes, and, by construction, it lifts the pullback square in $\mathsf{Ch}_{\geq 0}$ to the category $\mathsf{Lie}_n\mathsf{Alg}$. □

Proof of (2) Let $(\overline{P},\delta_P)$ denote the pullback of diagram (4-1) in $\mathsf{dgCoCom}_{\geq 0}$ with $F\colon(\overline{S}(sL),\delta)\to(\overline{S}(sL''),\delta'')$ corresponding to a strict fibration in $\mathsf{Lie}_n\mathsf{Alg}$.

Recall from (4-2) that $\overline{P}$ is a subcoalgebra of $\overline{S}(sL'\oplus sL)$. Let

$$J|_{\overline{P}}\colon \overline{P}\to\overline{S}(sL'\oplus sL)$$

denote the restriction of the coalgebra morphism J defined in (4-8). We claim that $\operatorname{im} J|_{\overline{P}}\subseteq\overline{S}(s\widetilde{L})$. Since $\overline{S}(sL'\oplus sL)$ is cofree in $\mathsf{CoCom}_{\geq 0}$, it suffices to check that the image of the linear map $J^1|_P\colon \overline{P}\to sL'\oplus sL$ is contained in $s\widetilde{L}$.

Let $\vec{y}\in\overline{P}$ be an element of homogeneous degree. Write $\vec{y}$ as sum of elements of increasing word length, ie $\vec{y}=\vec{y}_1+\vec{y}_2+\dots+\vec{y}_r$, with $\vec{y}_i\in\overline{S}^{n_i}(sL'\oplus sL)$ and $1=n_1<n_2<\dots<n_r$. Suppose $n_r=1$. Then $\vec{y}=\vec{y}_1=(v',v)\in sL'\oplus sL$, and $\vec{y}\in\ker(F\,\mathrm{Pr}-G\,\mathrm{Pr}')$ implies that $(v',v)\in s\widetilde{L}$. Therefore $J(\vec{y})=(v',-\sigma G_1(v')+v)\in s\widetilde{L}$.

For the higher arity case, suppose $n_r>1$. The equality $F\,\mathrm{Pr}(\vec{y})=G\,\mathrm{Pr}'(\vec{y})$ implies that $F^1\,\mathrm{Pr}(\vec{y})=G^1\,\mathrm{Pr}'(\vec{y})$. Since F is a strict L_∞–morphism, $F^1_{m\geq 2}=0$. Therefore, we deduce that

$$F_1\,\mathrm{pr}(\vec{y}_1)=\sum_{i=1}^{r}G^1_{n_i}\,\mathrm{pr}'^{\otimes n_i}(\vec{y}_i). \tag{4-19}$$

Note that $|\vec{y}|>1$, since $n_r>1$. Hence, it suffices to show that $\mathrm{pr}\,J^1(\vec{y})\in\ker F_1$. It follows directly from the definition of J that

$$\mathrm{pr}\,J^1(\vec{y})=\sum_{i=1}^{r}\mathrm{pr}\,J^1_{n_i}(\vec{y}_i)=\mathrm{pr}\,\vec{y}_1-\sum_{i=1}^{r}\sigma G^1_{n_i}\mathrm{pr}'^{\otimes n_i}(\vec{y}_i).$$

Applying F_1 to the above gives

$$F_1\,\mathrm{pr}\,J^1(\vec{y})=F_1\,\mathrm{pr}\,\vec{y}_1-\sum_{i=1}^{r}G^1_{n_i}\,\mathrm{pr}'^{\otimes n_i}(\vec{y}_i).$$

It then follows from (4-19) that $F_1\,\mathrm{pr}\,J^1(\vec{y})=0$.

Thus, $J|_{\overline{P}}\colon \overline{P} \to \overline{S}(s\widetilde{L})$ is a well-defined coalgebra morphism. Since $\delta_P = \delta_{\oplus}|_P$ and $HJ = \mathrm{id}$, the morphism $J|_P$ is compatible with the differentials. Furthermore, the following diagram in $\mathsf{dgCoCom}_{\geq 0}$ commutes:

$$\begin{tikzcd}
(\overline{P},\delta_P) \arrow[dr, "J"] \arrow[drr, bend left, "\mathrm{Pr}"] \arrow[ddr, bend right, "\mathrm{Pr}'"'] & & \\
& (\overline{S}(s\widetilde{L}),\widetilde{\delta}) \arrow[r, "\mathrm{Pr}\,H"] \arrow[d, "\mathrm{Pr}'\,H"'] & (\overline{S}(sL),\delta) \arrow[d, "F"] \\
& (\overline{S}(sL'),\delta') \arrow[r, "G"] & (\overline{S}(sL''),\delta'')
\end{tikzcd}$$

To see that $J|_{\overline{P}}$ is an isomorphism, note that the commutative diagram (4-17) implies that the subcoalgebra $H(\overline{S}(s\widetilde{L})) \subseteq \overline{S}(sL' \oplus sL)$ is contained in the vector space $\ker(F\,\mathrm{Pr} - G\,\mathrm{Pr}')$. Therefore, the universal property of the coalgebra $\overline{P}$ implies that $H(\overline{S}(s\widetilde{L})) \subseteq \overline{P}$, and it then follows by Claim 4.2 that $H|_{\overline{S}(s\widetilde{L})}$ is the inverse of $J|_{\overline{P}}$.

Hence, we conclude that $(\overline{S}(s\widetilde{L}),\widetilde{\delta})$ is a pullback in $\mathsf{dgCoCom}_{\geq 0}$ and therefore a pullback in $\mathsf{Lie}_n\mathsf{Alg}$. This completes the proof of statement (2) of the proposition. □

4.2 Pullbacks of fibrations and acyclic fibrations

Corollary 4.4 *Suppose $f\colon (L,\ell) \to (L'',\ell'')$ is an (acyclic) fibration in $\mathsf{Lie}_n\mathsf{Alg}$ and $g\colon (L',\ell) \to (L'',\ell'')$ is an arbitrary morphism between Lie n–algebras. Then the pullback of the diagram*

$$(L',\ell') \xrightarrow{g} (L'',\ell'') \xleftarrow{f} (L,\ell)$$

exists in $\mathsf{Lie}_n\mathsf{Alg}$, and the morphism induced by the pullback of f along g is an (acyclic) fibration.

Proof Suppose f is a (acyclic) fibration. Lemma 3.11 implies that there exist a Lie n–algebra $(L,\widehat{\ell})$ and an isomorphism $\phi\colon (L,\widehat{\ell}) \xrightarrow{\cong} (L,\ell)$ such that $f\phi\colon (L,\widehat{\ell}) \to (L'',\ell'')$ is a strict (acyclic) fibration with $f\phi = (f\phi)_1 = f_1$. It follows from Proposition 4.1, that there exists a pullback diagram in $\mathsf{Lie}_n\mathsf{Alg}$ of the form

$$\begin{tikzcd}
(\widetilde{L},\widetilde{\ell}) \arrow[r] \arrow[d, "\widetilde{f\phi}"'] \arrow[dr, phantom, "\lrcorner"] & (L,\widehat{\ell}) \arrow[d, "f\phi"] \\
(L',\ell') \arrow[r, "g"] & (L'',\ell'')
\end{tikzcd}$$

Moreover, Proposition 4.1 implies that the image of above diagram under the tangent functor (3-6) is the pullback diagram of $(f\phi)_1$ along g_1 in $\mathsf{Ch}_{\geq 0}$. Hence, $\tan_{\geq 0}(\widetilde{f\phi})$ is a (acyclic) fibration in $\mathsf{Ch}^{\mathrm{proj}}_{\geq 0}$, and therefore $\widetilde{f\phi}$ is a (acyclic) fibration in $\mathsf{Lie}_n\mathsf{Alg}$.

Now let F, G and Φ denote the dg coalgebra morphisms corresponding to f, g and ϕ, respectively. Let $(\overline{C}, \delta_C)$ denote the dg coalgebra witnessing the pullback of F along G. The second statement of Proposition 4.1, combined with the pasting lemma for pullbacks, implies that the following diagram in $\mathsf{dgCoCom}_{\geq 0}$ commutes:

$$\begin{tikzcd}
(\bar{S}(s\widetilde{L}), \widetilde{\delta}) \arrow[r] \arrow[d, "\cong"', "\widetilde{\Phi}"] \arrow[dr, phantom, "\lrcorner"] & (\bar{S}(sL), \widehat{\delta}) \arrow[d, "\cong"', "\Phi"] \\
(\overline{C}, \delta_C) \arrow[r] \arrow[d, "\widetilde{F}"'] \arrow[dr, phantom, "\lrcorner"] & (\bar{S}(sL), \delta) \arrow[d, "F"] \\
(\bar{S}(sL'), \delta') \arrow[r, "G"] & (\bar{S}(sL''), \delta'')
\end{tikzcd}$$

Since $\widetilde{\Phi}$ is an isomorphism, $\overline{C}$ is the cofree coalgebra in $\mathsf{CoCom}_{\geq 0}$ cogenerated by $\widetilde{\Phi}(s\widetilde{L})$. Hence, $(\overline{C}, \delta_C)$ is also a Lie n–algebra and $\widetilde{F}$ is an (acyclic) fibration. □

We end this section with the corollary below, whose proof follows immediately from the construction of the pullbacks in Proposition 4.1 and Corollary 4.4.

Corollary 4.5 *Let $f\colon (L,\ell) \to (L'',\ell'')$ be a fibration in $\mathsf{Lie}_n\mathsf{Alg}$. Then the tangent functor* (3-6) *maps pullback squares in $\mathsf{Lie}_n\mathsf{Alg}$ of the form*

$$\begin{tikzcd}
(\widetilde{L}, \widetilde{\ell}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (L, \ell) \arrow[d, "f"] \\
(L', \ell') \arrow[r, "g"] & (L'', \ell'')
\end{tikzcd}$$

to pullback squares in $\mathsf{Ch}_{\geq 0}$.

5 $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ as a category of fibrant objects

Let (L,ℓ) be a Lie n–algebra with underlying graded vector space $L = \bigoplus_{i\geq 0}^{n-1} L_i$. If each L_i is finite-dimensional, we say (L,ℓ) is of *finite type*. For a fixed $n \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ the full subcategory of $\mathsf{L}_\infty\mathsf{Alg}$ whose objects are finite-type Lie n–algebras.

Definition 5.1 [4, Section 1] Let C be a category with finite products, with terminal object $* \in \mathsf{C}$, and equipped with two classes of morphisms called *weak equivalences* and *fibrations*. A morphism which is both a weak equivalence and a fibration is called an *acyclic fibration*. Then C is a *category of fibrant objects* (*CFO*) for a homotopy theory if and only if:

(1) Every isomorphism in C is an acyclic fibration.

(2) The class of weak equivalences satisfies "2 out of 3". That is, if f and g are composable morphisms in C and any two of f, g and $g \circ f$ are weak equivalences, then so is the third.

(3) The composition of two fibrations is a fibration.

(4) The pullback of a fibration exists, and is a fibration. That is, if $Y \xrightarrow{g} Z \xleftarrow{f} X$ is a diagram in C with f a fibration, then the pullback $X \times_Z Y$ exists, and the induced projection $X \times_Z Y \to Y$ is a fibration.

(5) The pullback of an acyclic fibration exists, and is an acyclic fibration.

(6) For any object $X \in \mathsf{C}$ there exists a (not necessarily functorial) *path object*, that is, an object X^I equipped with morphisms

$$X \xrightarrow{s} X^I \xrightarrow{(d_0,d_1)} X \times X$$

such that s is a weak equivalence, (d_0, d_1) is a fibration and their composite is the diagonal map.

(7) All objects of C are *fibrant*. That is, for any $X \in \mathsf{C}$ the unique map $X \to *$ is a fibration.

We now prove the main result of the paper.

Theorem 5.2 *Let $n \in \mathbb{N} \cup \{\infty\}$. The category $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ of finite-type Lie n–algebras and weak L_∞–morphisms has the structure of a category of fibrant objects, in which the weak equivalences and fibrations are those morphisms that satisfy the defining criteria given in Definition 3.6. That is, a morphism $f\colon (L,\ell) \to (L',\ell')$ is*

- *a weak equivalence if and only if f is an L_∞–quasi-isomorphism,*
- *a fibration if and only if the associated chain map $f_1\colon (L,\ell_1) \to (L',\ell'_1)$ is a surjection in all positive degrees.*

Proof We begin by noting that Proposition 3.5 implies that $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ has finite products. Next, it follows immediately from the definition of weak equivalences and fibrations that: every isomorphism is an acyclic fibration, the weak equivalences satisfy "2 out of 3", the composition of two fibrations is again a fibration, and that the trivial map $(L,\ell)\to 0$ is a fibration for any $(L,\ell)\in\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$. Hence axioms (1)–(3) and (7) in Definition 5.1 for a CFO are satisfied.

To verify axiom (4), suppose $(L',\ell')\xrightarrow{g}(L'',\ell'')\xleftarrow{f}(L,\ell)$ is a diagram in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ and f is a fibration. It follows from Corollary 4.4 that the pullback $(\widetilde{L},\widetilde{\ell})$ of the diagram exists in $\mathsf{Lie}_n\mathsf{Alg}$ and that the morphism induced by the pullback is a fibration. Corollary 4.4 also implies that the underlying complex of $(\widetilde{L},\widetilde{\ell})$ is the pullback of the diagram $\tan_{\geq 0}((L',\ell')\xrightarrow{g}(L'',\ell'')\xleftarrow{f}(L,\ell))$ in $\mathsf{Ch}_{\geq 0}$. Hence, $(\widetilde{L},\widetilde{\ell})$ is clearly of finite type, and axiom (4) is satisfied. The same argument also verifies axiom (5).

Finally, recall that any diagonal map $\mathrm{diag}\colon(L,\ell)\to(L\oplus L,\ell\oplus\ell)$ is a strict morphism in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$. Hence, Proposition 3.8 implies that diag has a factorization $(L,\ell)\xrightarrow{\jmath}(\widetilde{L},\widetilde{\ell})\xrightarrow{\phi}(L\oplus L,\ell\oplus\ell)$ in the category $\mathsf{Lie}_n\mathsf{Alg}$, in which $\jmath$ is a weak equivalence and ϕ is a fibration. Recall from the proof of Proposition 3.8 that $\widetilde{L}=(L\oplus P(L\oplus L))$, where $P(L\oplus L)$ is the graded vector space defined in (3-7). Therefore, $\widetilde{L}$ is of finite type, and $(\widetilde{L},\widetilde{\ell})$ is a path object for (L,ℓ) in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$. Hence, axiom (6) in Definition 5.1 is satisfied, and this completes the proof. □

Remark 5.3 Clearly, the same proof shows that the category $\mathsf{Lie}_n\mathsf{Alg}$ is also a category of fibrant objects with the same weak equivalences, fibrations and path objects as $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$.

We can now complete the discussion that we started in Section 3.3 concerning the factorization of arbitrary weak L_∞–morphisms in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$. Let us recall Brown's factorization lemma:

Lemma 5.4 [4, Section 1] *Let C be a category of fibrant objects. Let $f\colon X\to Y$ be a morphism in C, and let Y^I be a path object for Y. Then f can be factored as*

$$X\xrightarrow{\jmath}X\times_Y Y^I\xrightarrow{\phi_f}Y,$$

where ϕ_f is a fibration, and $\jmath$ is a weak equivalence which is a section (right inverse) of an acyclic fibration.

The morphisms $\jmath$ and ϕ_f in the lemma can easily be expressed in terms of the maps $Y \xrightarrow{s} Y^I \xrightarrow{(d_0,d_1)} Y \times Y$ that appear in factorization of the diagonal. See, for example, Section 2.1 of [16] for details.

Hence, Theorem 5.2 and the factorization lemma imply the following:

Corollary 5.5 *Let $f\colon (L,\ell) \to (L',\ell')$ be a weak L_∞–morphism between finite-type Lie n–algebras. Then f can be factored in the category $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ as*

$$(L,\ell) \xrightarrow{\jmath} (\widetilde{L},\widetilde{\ell}) \xrightarrow{p_f} (L',\ell'),$$

where $\jmath$ is a weak equivalence and p_f is a fibration in $\mathsf{Lie}_n\mathsf{Alg}$.

The CFO structure on Lie n–algebras can be thought of as a lift of the projective CFO structure (Section 2.2.1) on $\mathsf{Ch}_{\geq 0}$ via the tangent functor (3-6) in the following sense:

Definition 5.6 [2, Definition 2.3.3] A functor $F\colon \mathsf{C} \to \mathsf{D}$ between categories of fibrant objects is a *(left) exact functor* if and only if:

(1) F preserves the terminal object, fibrations, and acyclic fibrations.

(2) Any pullback square in C of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

in which $f\colon X \to Y$ is a fibration in C is mapped by F to a pullback square in D.

Remark 5.7 Axiom (1) in the above definition combined with Lemma 5.4 implies that an exact functor $F\colon \mathsf{C} \to \mathsf{D}$ between CFOs sends weak equivalences to weak equivalences.

Corollary 5.8 *The tangent functor $\tan_{\geq 0}\colon \mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}} \to \mathsf{Ch}^{\mathrm{proj}}_{\geq 0}$ is an exact functor between categories of fibrant objects.*

Proof Combine Theorem 5.2 and Corollary 4.5. $\square$

Comparison with the Vallette CFO structure on $\mathsf{L}_\infty\mathsf{Alg}$

In [19], Vallette showed that the category $\mathsf{L}_\infty\mathsf{Alg}$ of $\mathbb{Z}$–graded L_∞–algebras is exactly the subcategory of bifibrant objects in the Hinich model structure [11] on $\mathsf{dgCoCom}$. This result implies the following theorem:

Theorem 5.9 [19, Theorem 2.1] (1) *The category $\mathsf{L}_\infty\mathsf{Alg}$ of $\mathbb{Z}$–graded L_∞–algebras and weak L_∞–morphisms has the structure of a category of fibrant objects in which a morphism*

$$f\colon (L,\ell)\to(L',\ell')$$

is a weak equivalence if and only if it is an L_∞–quasi-isomorphism, and a fibration if and only if it is an L_∞–epimorphism (Definition 3.3).

(2) *Every L_∞–algebra (L,ℓ) has a functorial path object whose underlying graded vector space is*

$$(L\otimes\Omega^*_{\mathrm{poly}}(\Delta^1))_m\cong L_m[t]\oplus L_{m+1}[t]\,dt,$$

where $\Omega^*_{\mathrm{poly}}(\Delta^1)$ denotes the commutative dg algebra of polynomial de Rham forms on the geometric 1–simplex.

Let us make a few comparisons between the CFO structure on $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ given in Theorem 5.2 and Vallette's CFO structure on $\mathsf{L}_\infty\mathsf{Alg}$.

First, the weak equivalences in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ and $\mathsf{L}_\infty\mathsf{Alg}$ obviously coincide. Furthermore, it follows from Remark 3.7(2) that the acyclic fibrations also coincide. If $(L,\ell)\in\mathsf{Lie}_n\mathsf{Alg}$, then the path object $(L\otimes\Omega^*_{\mathrm{poly}}(\Delta^1),\ell^\Omega)$ for $(L,\ell)\in\mathsf{L}_\infty\mathsf{Alg}$ is not a Lie n–algebra. And the degree zero truncation of $L\otimes\Omega^*_{\mathrm{poly}}(\Delta^1)$ is certainly not of finite type. However, Vallette's Proposition 3.3 in [19] implies that $\mathsf{L}_\infty\mathsf{Alg}$ is also equipped with a functorial cylinder object. If (L,ℓ) is a finite-type Lie n–algebra, then the degree zero truncation of the cylinder object $(L\otimes J,\widetilde{\ell})$ is also of finite type. It would be interesting to work out further details and show that the CFO structure on $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ satisfies Brown's additional axioms (F) and (G) in [4, Section 6]. This would imply that $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ is "almost" a model category, in the sense of Vallette [19, Section 4.1].

6 Maurer–Cartan sets

In this section, we analyze Maurer–Cartan (MC) sets of $\mathbb{Z}$–graded L_∞–algebras constructed by tensoring Lie n–algebras with bounded commutative dg algebras.

This construction naturally arises when studying formal deformation problems in characteristic zero. It also appears, as mentioned in the introduction, in the definition of the spatial realization functor for chain Lie algebras. The smooth analog of the Maurer–Cartan set is featured in the definition of Henriques' integration functor [10] for Lie n–algebras.

We begin by recalling some basic facts about Maurer–Cartan elements from Section 2 of [5]. We note that the L_∞–algebras in [5] are assumed to be complete and filtered, which is not the case in this paper. However, the particular results that we recall below will still hold since all L_∞–algebras involved are sufficiently "tame" in the following sense:

Definition 6.1 A $\mathbb{Z}$–graded L_∞–algebra (L,ℓ) is *tame* if there exists an $N \geq 1$ such that, for all $k \geq N$,

$$\ell_k(x_1,\dots,x_k) = 0 \quad \text{for all } x_1,\dots,x_k \in L_{-1}.$$

A (weak) L_∞–morphism $f\colon (L,\ell) \to (L',\ell')$ between tame L_∞–algebra is a *tame morphism* if there exists an $N \geq 1$ such that, for all $k \geq N$,

$$f_k(x_1,\dots,x_k) = 0 \quad \text{for all } x_1,\dots,x_k \in L_{-1}.$$

For trivial reasons, every Lie n–algebra is tame and every morphism between Lie n–algebra is tame for any $n \in \mathbb{N} \cup \{\infty\}$. Given a tame $\mathbb{Z}$–graded L_∞–algebra (L,ℓ), the *curvature* curv: $L_{-1} \to L_{-2}$ is the function

$$\begin{aligned} \text{(6-1)} \qquad \operatorname{curv}(a) &:= \boldsymbol{s}^{-1} \sum_{k\geq 1} \frac{1}{k!}\delta^1_k(\boldsymbol{s}a,\boldsymbol{s}a,\dots,\boldsymbol{s}a) \\ &= \ell_1(a) + \sum_{k\geq 2} (-1)^{k(k-1)/2}\frac{1}{k!}\ell_k(a,a,\dots,a) \in L_{-2}, \end{aligned}$$

where δ is the corresponding codifferential on $\overline{S}(\boldsymbol{s}L)$. For any $a \in L_{-1}$, the expression $\exp(\boldsymbol{s}a) - 1$ is a well-defined element of the completion of $\overline{S}(\boldsymbol{s}L)$ defined by the corresponding formal power series. By extending δ in the natural way, a straightforward calculation shows that

$$\text{(6-2)} \qquad \delta(\exp(\boldsymbol{s}a) - 1) = \exp(\boldsymbol{s}a)(\boldsymbol{s}\operatorname{curv}(a))$$

and hence

$$\text{(6-3)} \qquad \boldsymbol{s}\operatorname{curv}(a) = \operatorname{pr}_{\boldsymbol{s}L} \circ\, \delta(\exp(\boldsymbol{s}a) - 1),$$

where pr_{sL} denotes the canonical projection to the vector space sL. The *Maurer–Cartan elements* of L are the elements of the subset

$$\mathrm{MC}(L) := \{x \in L_{-1} \mid \mathrm{curv}(x) = 0\}.$$

Similarly, let $f\colon (L,\ell) \to (L',\ell')$ be a tame morphism. Such a morphism induces a function $f_*\colon L_{-1} \to L'_{-1}$ defined as

$$\begin{aligned} f_*(a) &:= s^{-1} \sum_{k \geq 1} \frac{1}{k!} F^1_k(sa, sa, \dots, sa) \\ &= f_1(a) + \sum_{k \geq 2} (-1)^{k(k-1)/2} \frac{1}{k!} f_k(a, a, \dots, a). \end{aligned} \tag{6-4}$$

As in (6-2), we extend F to the completions of $\bar{S}(sL)$ and $\bar{S}(sL')$, and a straightforward calculation shows that

$$F(\exp(sa) - 1) = \exp(s f_*(a)) - 1. \tag{6-5}$$

Remarks 6.2 (1) If $f\colon (L,\ell) \to (L',\ell')$ and $g\colon (L',\ell') \to (L'',\ell'')$ are tame morphisms, then it follows from the composition formula (2-5) for L_∞–morphisms that $gf\colon (L,\ell) \to (L'',\ell'')$ is also tame. Indeed, if f_k vanishes on $(L_{-1})^{\otimes k}$ for all $k \geq N_L$ and g_k vanishes on $(L'_{-1})^{\otimes k}$ for all $k \geq N_{L'}$, then $(gf)_k$ vanishes on $(L_{-1})^{\otimes k}$ for all $k \geq N_L N_{L'}$.

(2) The function f_* in (6-4) is also well defined for any *coalgebra* morphism $F\colon \bar{S}(sL) \to \bar{S}(sL')$ satisfying

$$F^1_k(sx_1, \dots, sx_k) = 0 \quad \text{for all } x_1, \dots, x_k \in L_{-1}$$

for $k \gg 1$. Compatibility of F with the codifferentials is obviously not necessary. We will use this in the proof of Proposition 6.5.

Remark 6.2(1) along with equations (6-2), (6-3) and (6-5) imply the following result:

Proposition 6.3 (see [5, Proposition 2.2]) *Let $f\colon (L,\ell) \to (L',\ell')$ be a tame L_∞–morphism between tame $\mathbb{Z}$–graded L_∞–algebras. Then the function (6-4) restricts to a well-defined function $f_*\colon \mathrm{MC}(L) \to \mathrm{MC}(L')$ between the corresponding Maurer–Cartan sets. Moreover, the assignment*

$$(L,\ell) \xrightarrow{f} (L',\ell') \longmapsto \mathrm{MC}(L) \xrightarrow{f_*} \mathrm{MC}(L')$$

is functorial.

6.1 "Deformation functors"

The following construction provides important examples of tame L_∞–algebras. We denote by $\mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$ the category whose objects are unital, nonnegatively and cohomologically graded commutative dg $\Bbbk$–algebras which are bounded from above. Morphisms in $\mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$ are unit-preserving cdga morphisms. Let $n \in \mathbb{N} \cup \{\infty\}$. Let $(L, \ell) \in \mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ be a finite-type Lie n–algebra and let $(B, d_B) \in \mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$. Denote by

$$(L \otimes B, \ell^B)$$

the $\mathbb{Z}$–graded L_∞–algebra whose underlying chain complex is $(L \otimes B, \ell^B_1)$, where

$$(L \otimes B)_m := \bigoplus_{i+j=m} L_i \otimes B^{-j}, \quad \ell^B_1(x \otimes b) := \ell_1 x \otimes b + (-1)^{|x|} x \otimes d_B b$$

and whose higher brackets are defined as

$$\ell^B_k(x_1 \otimes b_1, \dots, x_k \otimes b_k) := (-1)^{\varepsilon} \ell_k(x_1, \dots, x_k) \otimes b_1 b_2 \cdots b_k, \tag{6-6}$$

with

$$\varepsilon := \sum_{1 \leq i < j \leq k} |b_i||x_j|.$$

If $f\colon (L, \ell) \to (L', \ell')$ is a morphism of Lie n–algebras, then it is easy to verify that the maps $f^B_k\colon \Lambda^k(L \otimes B) \to L' \otimes B$ defined as

$$f^B_k(x_1 \otimes b_1, \dots, x_k \otimes b_k) := (-1)^{\varepsilon} f_k(x_1, \dots, x_k) \otimes b_1 b_2 \cdots b_k \tag{6-7}$$

assemble together to give an L_∞–morphism $f^B\colon (L \otimes B, \ell^B) \to (L' \otimes B, \ell'^B)$ in $\mathsf{L_\infty Alg}$.

Lemma 6.4 *Let $(L, \ell) \in \mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ be a finite-type Lie n–algebra and $(B, d_B) \in \mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$ a bounded cdga.*

(1) *If $f\colon (L, \ell) \to (L', \ell')$ is a morphism of Lie n–algebras, then the induced L_∞–morphism*

$$f^B\colon (L \otimes B, \ell^B) \to (L' \otimes B, \ell'^B)$$

is a tame morphism between tame L_∞–algebras.

(2) *The assignment*

$$(L \otimes B, \ell^B_k) \xrightarrow{f^B} (L' \otimes B, \ell'^B_k) \longmapsto \mathrm{MC}(L \otimes B) \xrightarrow{f^B_*} \mathrm{MC}(L' \otimes B)$$

defines a functor

$$\mathrm{MC}(-\otimes B)\colon \mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}} \to \mathsf{Set} \tag{6-8}$$

natural in $B \in \mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$.

Proof Statement (2) is straightforward. For statement (1), note that since the underlying cochain complex of B is bounded, there exists an $N \geq 0$ such that $B = \bigoplus_{i\geq 0}^{N} B^i$. Since the underlying chain complex of L is concentrated in nonnegative degrees we have $(L\otimes B)_{-1} = \bigoplus_{i\geq 0}^{N-1} L_i \otimes B^{i+1}$. It follows from (6-6) and (6-7) that for all $k \geq N$, and any $a_1,\dots,a_k \in (L\otimes B)_{-1}$, we have $\ell_k^B(a_1,\dots,a_k) = 0$ and $f_k^B(a_1,\dots,a_k) = 0$. A similar argument shows that $(L'\otimes B, \ell'^B)$ is also tame. □

We end this section by showing that the functor $\mathrm{MC}(-\otimes B)$ defined in (6-8) preserves certain pullback diagrams. We provide a careful detailed proof of this rather straightforward fact in order to have the analogous statement in the category of Banach manifolds follow as a simple corollary (Corollary 6.7).

Proposition 6.5 *Let* $B \in \mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$ *be a bounded cdga. Let* $f\colon (L,\ell) \to (L'',\ell'')$ *be a fibration and* $g\colon (L',\ell') \to (L'',\ell'')$ *be a morphism in* $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$. *Let* (L_P,ℓ_P) *be the pullback of the diagram* $(L',\ell') \xrightarrow{g} (L'',\ell'') \xleftarrow{f} (L,\ell)$. *Then the induced commutative diagram of sets*

$$\begin{tikzcd} \mathrm{MC}(L_P\otimes B) \arrow[r] \arrow[d] & \mathrm{MC}(L\otimes B) \arrow[d, "f_*^B"] \\ \mathrm{MC}(L'\otimes B) \arrow[r, "g_*^B"] & \mathrm{MC}(L''\otimes B) \end{tikzcd} \tag{6-9}$$

is a pullback square.

Proof We first consider the special case in which f is strict, and then use this to prove the general case.

Case 1 ($f = f_1\colon (L,\ell) \to (L'',\ell'')$ is a strict fibration) Let $g\colon (L',\ell') \to (L'',\ell'')$ be a morphism in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$. As in the proof of Proposition 4.1, we have the pullback diagram in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$

$$\begin{tikzcd} (\widetilde{L},\widetilde{\ell}) \arrow[r, "\mathrm{pr}\,h"] \arrow[d, "\mathrm{pr}'\,h"'] \arrow[dr, phantom, "\lrcorner"] & (L,\ell) \arrow[d, "f"] \\ (L',\ell') \arrow[r, "g"] & (L'',\ell'') \end{tikzcd} \tag{6-10}$$

where $\widetilde{L}$ is the pullback of diagram (4-6) in $\mathsf{Ch}_{\geq 0}$ and $h\colon (\widetilde{L},\widetilde{\ell}) \to (L'\oplus L, \ell'\oplus \ell)$ is the morphism in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ associated to the dg coalgebra morphism H defined in (4-7). Let $B \in \mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$. Then, via (6-4) and (6-7), the morphism h induces the function $h^B_*\colon \mathrm{MC}(\widetilde{L}\otimes B) \to \mathrm{MC}((L'\oplus L)\otimes B) \cong \mathrm{MC}(L'\otimes B)\times \mathrm{MC}(L\otimes B)$, where

$$h^B_*(a',a) = (a', a + (\boldsymbol{s}^{-1}\sigma\boldsymbol{s}\otimes \mathrm{id}_B)g^B_*(a')).$$

Applying the functor $\mathrm{MC}(-\otimes B)$ to (6-10) gives us the commutative diagram of sets

$$\begin{tikzcd}
\mathrm{MC}(\widetilde{L}\otimes B) \arrow[dr, "h^B_*"] \arrow[drr, bend left, "\mathrm{pr}^B_*\circ h^B_*"] \arrow[ddr, bend right, "\mathrm{pr}'^B_*\circ h^B_*"'] & & \\
& E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{MC}(L\otimes B) \arrow[d, "f^B_*"] \\
& \mathrm{MC}(L'\otimes B) \arrow[r, "g^B_*"] & \mathrm{MC}(L''\otimes B)
\end{tikzcd}$$

in which the pullback is denoted by

$$E := \mathrm{MC}(L'\otimes B)\times_{\mathrm{MC}(L''\otimes B)} \mathrm{MC}(L\otimes B). \tag{6-11}$$

We proceed by explicitly constructing the inverse of h^B_*.

Let $J\colon \overline{S}(\boldsymbol{s}L'\oplus \boldsymbol{s}L) \to \overline{S}(\boldsymbol{s}L'\oplus \boldsymbol{s}L)$ denote the inverse of the coalgebra morphism H, which was introduced in (4-8). Even though J may fail to preserve the codifferential, it follows from Remarks 6.2 and Lemma 6.4 that the function

$$j^B_*\colon (L'\otimes B\oplus L\otimes B)_{-1} \to (L'\otimes B\oplus L\otimes B)_{-1}$$

is well defined. We denote by $(L'\otimes B)_{-1}\times_{(L''\otimes B)_{-1}}(L\otimes B)_{-1}$ the pullback of the diagram of sets

$$(L'\otimes B)_{-1} \xrightarrow{g^B_*} (L''\otimes B)_{-1} \xleftarrow{f^B_*} (L\otimes B)_{-1}. \tag{6-12}$$

Now let

$$\varphi\colon (L'\otimes B)_{-1}\times_{(L''\otimes B)_{-1}}(L\otimes B)_{-1} \to (L'\otimes B\oplus L\otimes B)_{-1}$$

denote the function

$$\varphi(a',a) := j^B_*(a',a) = \big(a', a - (\boldsymbol{s}^{-1}\sigma\boldsymbol{s}\otimes\mathrm{id}_B)g^B_*(a')\big). \tag{6-13}$$

We claim that φ restricted to the pullback E is indeed the inverse to h^B_*. Let us first show that $\varphi|_E\colon E \to \mathrm{MC}(\widetilde{L} \otimes B)$ is a well-defined function. We break the calculation into two steps:

Step (i) $(\operatorname{im}\varphi \subseteq (\widetilde{L} \otimes B)_{-1})$ Suppose $(a', a) \in (L' \otimes B)_{-1} \times_{(L'' \otimes B)_{-1}} (L \otimes B)_{-1}$, so that $g^B_*(a') = f^B_*(a)$. It will be convenient to express the relevant terms involved as sums, ie $(a', a) = \sum_{i \geq 0} (a'_i, a_i)$ and

$$\varphi(a', a) = \sum_{i \geq 0} \varphi(a', a)_i, \quad g^B_*(a') = \sum_{i \geq 0} g^B_*(a')_i = \sum_{i \geq 0} f^B_*(a)_i = f^B_*(a),$$

where $(a'_i, a_i) \in (L'_i \oplus L_i) \otimes B^{i+1}$, $\varphi(a', a)_i \in (L'_i \oplus L_i) \otimes B^{i+1}$ and

$$g^B_*(a')_i = f^B_*(a)_i \in L''_i \otimes B^{i+1}.$$

We first consider the summand $\varphi(a', a)_0 = \big(a'_0, a_0 - (s^{-1}\sigma s \otimes \mathrm{id}_B) g^B_*(a')_0\big)$. Since all of the elements of B appearing in the term $a' \in \bigoplus_{i \geq 0} L'_i \otimes B^{i+1}$ are positively graded, equations (6-4) and (6-7) imply that

$$g^B_*(a')_0 = (g_1 \otimes \mathrm{id}_B)(a'_0) \in L''_0 \otimes B^1. \tag{6-14}$$

Recall from the definition of $\sigma\colon \bigoplus_{i \geq 1} sL''_i \to \bigoplus_{i \geq 1} sL_i$ in (4-4) that $\sigma s|_{L''_0} = \sigma|_{sL''_1} = 0$. Hence, we deduce from (6-14) that

$$\mathrm{pr}^B \varphi(a', a)_0 = a_0. \tag{6-15}$$

On the other hand, the morphism f is strict by hypothesis, and so from (6-15) we have

$$(f_1 \otimes \mathrm{id}_B)\, \mathrm{pr}^B \varphi(a', a)_0 = f^B_*(a)_0.$$

Therefore, by combining the above equality with (6-14), we obtain

$$(f_1 \otimes \mathrm{id}_B)\, \mathrm{pr}^B \varphi(a', a)_0 = g^B_*(a')_0 = (g_1 \otimes \mathrm{id}_B)\, \mathrm{pr}'^B \varphi(a', a)_0. \tag{6-16}$$

It follows from the definition of $s\widetilde{L}$ in (4-5) that $\widetilde{L}_0$ is the pullback of the linear maps f_1 and g_1. Therefore, since the functor $- \otimes_{\Bbbk} B^1$ is exact, we conclude from (6-16) that $\varphi(a', a)_0 \in \widetilde{L}_0 \otimes B^1$.

Now we consider the summands $\varphi(a', a)_i \in (L'_i \oplus L_i) \otimes B^{i+1}$ for $i \geq 1$. It follows from the definition $\widetilde{L}_{i \geq 1}$ that it is sufficient to verify that

$$\mathrm{pr}^B \varphi(a', a)_i = a_i - (s^{-1}\sigma s \otimes \mathrm{id}_B) g^B_*(a')_i \in \ker f_1 \otimes B.$$

Again, $f_*^B = f_1 \otimes \mathrm{id}_B$ by hypothesis. Therefore

$$\begin{aligned}(f_1 \otimes \mathrm{id}_B)(\mathrm{pr}^B\, \varphi(a',a)_i) &= f_*^B(a)_i - (f_1 s^{-1}\sigma s \otimes \mathrm{id}_B) g_*^B(a')_i \\ &= f_*^B(a)_i - g_*^B(a')_i \\ &= 0,\end{aligned}$$

where the last two equalities follow, respectively, from the definition of σ in (4-4), and the fact that (a',a) is an element of the pullback of (6-12).

Hence, we conclude that $\varphi(a',a) \in (\widetilde{L} \otimes B)_{-1}$.

Step (ii) $(\operatorname{im} \varphi|_E \subseteq \mathrm{MC}(\widetilde{L} \otimes B))$ To begin with, note that the pullback E in (6-11) is the equalizer of the functions

$$(L' \otimes B)_{-1} \times_{(L'' \otimes B)_{-1}} (L \otimes B)_{-1} \underset{0}{\overset{\mathrm{curv}'^B \times \mathrm{curv}^B}{\rightrightarrows}} (L' \otimes B)_{-2} \times (L \otimes B)_{-2},$$

where curv^B and curv'^B are the curvature functions (6-1) for the tame L_∞–algebras $L \otimes B$ and $L' \otimes B$, respectively.

Suppose $(a',a) \in E$. We wish to show that $\widetilde{\mathrm{curv}}^B(\varphi(a',a)) = 0$, where $\widetilde{\mathrm{curv}}^B$ is the curvature function for $\widetilde{L} \otimes B$. Let $\delta_\oplus$ denote the codifferential that encodes the product L_∞–structure on $L' \oplus L$. Then $(a',a) \in E$ implies that

$$\mathrm{curv}_\oplus^B(a',a) = (\mathrm{curv}'^B(a'), \mathrm{curv}^B(a)) = 0. \tag{6-17}$$

As in Claim 4.3, let $\widetilde{\delta} = J \circ \delta_\oplus \circ H$ denote the codifferential encoding the L_∞ structure on $\widetilde{L}$, and let $\widetilde{\delta}^B$ denote the induced codifferential for the L_∞ structure on $\widetilde{L} \otimes B$. Passing to the completions, equation (6-2) implies that

$$\widetilde{\delta}^B\big(\exp(s\varphi(a',a)) - 1\big) = \exp(s\varphi(a',a))\big(s\, \widetilde{\mathrm{curv}}^B(\varphi(a',a))\big), \tag{6-18}$$

while (6-5) gives us

$$\exp(s\varphi(a',a)) - 1 = \exp(s\, j_*^B(a',a)) - 1 = J^B\big(\exp(s(a',a)) - 1\big). \tag{6-19}$$

A straightforward calculation shows that $\widetilde{\delta}^B = J^B \circ \delta_\oplus^B \circ H^B$. Therefore, by combining Claim 4.2 with (6-19) we obtain

$$\begin{aligned}\widetilde{\delta}^B\big(\exp(s\varphi(a',a)) - 1\big) &= J^B \delta_\oplus^B\big(\exp(s(a',a)) - 1\big) \\ &= J^B\big(\exp(s(a',a))\big(s\, \mathrm{curv}_\oplus^B((a',a))\big)\big).\end{aligned}$$

It then follows from the above equality and (6-17) that $\widetilde{\delta}^B\big(\exp(s\varphi(a',a)) - 1\big) = 0$. Equation (6-18) then implies that $\widetilde{\mathrm{curv}}^B(\varphi(a',a)) = 0$, and so $\varphi(a',a) \in \mathrm{MC}(\widetilde{L} \otimes B)$.

Hence, we conclude that the function $\varphi|_E\colon E \to \mathrm{MC}(\widetilde{L} \otimes B)$ is well defined.

Finally, we verify that $\varphi|_E$ is the inverse to h_*^B. Keeping in mind Remarks 6.2, it is straightforward to show that the functoriality described in Proposition 6.3 and Lemma 6.4 holds for the assignment of the graded coalgebra morphisms

$$H, J\colon \bar{S}(sL' \oplus sL) \to \bar{S}(sL' \oplus sL)$$

to the functions

$$h_*^B, j_*^B\colon (L' \otimes B \oplus L \otimes B)_{-1} \to (L' \otimes B \oplus L \otimes B)_{-1}.$$

Hence, Claim 4.2 implies that $h_*^B \circ j_*^B = j_*^B \circ h_*^B = \mathrm{id}_{(L' \otimes B \oplus L \otimes B)_{-1}}$. It then follows from the definition of φ in (6-13) that $\varphi|_E$ is the inverse of h_*^B.

This concludes the proof of the proposition in the special case when f is a strict fibration.

Case 2 ($f\colon (L, \ell) \to (L'', \ell'')$ is an arbitrary fibration) First, we factor f into a strict fibration followed by an isomorphism. Indeed, by Lemma 3.11, there exist a Lie n–algebra $(\widehat{L}, \widehat{\ell})$ and an isomorphism $\psi\colon (\widehat{L}, \widehat{\ell}) \xrightarrow{\cong} (L, \ell)$ such that

$$f\psi\colon (\widehat{L}, \widehat{\ell}) \to (L'', \ell'')$$

is a strict fibration with $f\psi = (f\psi)_1 = f_1$. Let $g\colon (L', \ell') \to (L'', \ell'')$ be a morphism in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$. As in the proof of Corollary 4.4, we obtain a pair of pullback squares in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$,

$$\begin{tikzcd}
(\widetilde{L}, \widetilde{\ell}) \arrow[r, "\mathrm{pr}\, h"] \arrow[d, "\widetilde{\psi}", "\cong"'] \arrow[dd, bend right=60, "\mathrm{pr}'\, h"'] \arrow[dr, phantom, "\lrcorner"] & (\widehat{L}, \widehat{\ell}) \arrow[d, "\psi"', "\cong"] \arrow[dd, bend left=60, "f\psi = f_1"] \\
(L_P, \ell_P) \arrow[r, "q"] \arrow[d, "q'"'] \arrow[dr, phantom, "\lrcorner"] & (L, \ell) \arrow[d, "f"] \\
(L', \ell') \arrow[r, "g"] & (L'', \ell'')
\end{tikzcd}$$

where $(\widetilde{L}, \widetilde{\ell})$ is now the pullback of g and the strict fibration $f\psi\colon (\widehat{L}, \widehat{\ell}) \to (L'', \ell'')$, and (L_P, ℓ_P) is the pullback of g and f. Let

$$E := \mathrm{MC}(L' \otimes B) \times_{\mathrm{MC}(L'' \otimes B)} \mathrm{MC}(\widehat{L} \otimes B)$$

be the pullback of the diagram of sets $\mathrm{MC}(L' \otimes B) \xrightarrow{g_*^B} \mathrm{MC}(L'' \otimes B) \xleftarrow{(f\psi)_*^B} \mathrm{MC}(\widehat{L} \otimes B)$. Then our result from Case 1 implies that there exists a bijection $\varphi|_E\colon E \xrightarrow{\cong} \mathrm{MC}(\widetilde{L} \otimes B)$

such that the following diagram commutes:

$$
\begin{tikzcd}
E \arrow[dr, "\varphi|_E", "\cong"'] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & \\
& \mathrm{MC}(\widetilde{L} \otimes B) \arrow[rr, "(\mathrm{pr}\, h)^B_*"] \arrow[d, "\tilde{\psi}^B_*", "\cong"'] & & \mathrm{MC}(\widehat{L} \otimes B) \arrow[d, "\psi^B_*", "\cong"'] \\
& \mathrm{MC}(L_P \otimes B) \arrow[rr, "q^B_*"] \arrow[d, "q'^B_*"'] & & \mathrm{MC}(L \otimes B) \arrow[d, "f^B_*"] \\
& \mathrm{MC}(L' \otimes B) \arrow[rr, "g^B_*"] & & \mathrm{MC}(L'' \otimes B)
\end{tikzcd}
\tag{6-20}
$$

Now, let

$$E_P := \mathrm{MC}(L' \otimes B) \times_{\mathrm{MC}(L'' \otimes B)} \mathrm{MC}(L \otimes B)$$

be the pullback of the diagram of sets $\mathrm{MC}(L' \otimes B) \xrightarrow{g^B_*} \mathrm{MC}(L'' \otimes B) \xleftarrow{f^B_*} \mathrm{MC}(L \otimes B)$, and let

$$\varphi_{E_P} \colon E_P \to \mathrm{MC}(L_P \otimes B)$$

be the function

$$\varphi_{E_P}(a', a) := (\tilde{\psi}^B_* \circ \varphi|_E)(a', (\psi^B_*)^{-1}(a)) = (\tilde{\psi}^B_* \circ j^B_*)(a', (\psi^B_*)^{-1}(a)). \tag{6-21}$$

Note that φ_{E_P} is a bijection by construction. Using diagram (6-20), it is easy to see that $q^B_* \varphi_{E_P}(a', a) = a$ and $q'^B_* \varphi_{E_P}(a', a) = a'$. Hence, we conclude that (6-9) is a pullback square, and this completes the proof of the proposition. □

6.2 Maurer–Cartan sets with differentiable structure

We now consider the scenario in which all of the Maurer–Cartan sets in Proposition 6.5 have geometric structure. Specifically, we are interested in the case when the geometry arises from a dg Banach algebra structure on (B, d_B). Examples relevant to our applications in [16] include the cdgas $\Omega(\Delta^n)$ and $\Omega(\Lambda^k_j)$: the dg Banach algebras of r–times continuously differentiable forms on the geometric n–simplex Δ^n and the geometric horn $\Lambda^k_j \subseteq \Delta^k$, respectively. (See Section 5.1 of [10].)

So let $\Bbbk = \mathbb{R}$ and suppose $(B, d_B) \in \mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$ is a fixed cdga equipped with the structure of a dg Banach algebra. If $(L, \ell) \in \mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$, then since L is of finite type, the structure on B naturally makes $(L \otimes B)$ into a graded Banach space. From (6-1) and (6-6), we see that the curvature $\mathrm{curv}^B \colon (L \otimes B)_{-1} \to (L \otimes B)_{-2}$ is a polynomial and hence a smooth function between Banach manifolds, in the sense of [13, Chapter I.3]. Similarly, if $f\colon (L,\ell)\to(L',\ell')$ is a morphism in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$, then it follows from (6-4) and (6-7) that $f^B_*\colon (L\otimes B)_{-1}\to(L'\otimes B)_{-1}$ is a smooth function.

We now restrict our focus to those $(L,\ell)\in\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ which satisfy the following assumption:

Assumption 6.6 The Maurer–Cartan set $\mathrm{MC}(L\otimes B)\subseteq(L\otimes B)_{-1}$ is a Banach submanifold [13, Chapter II.2] and therefore

$$\mathrm{MC}(L\otimes B)\hookrightarrow(L\otimes B)_{-1}\overset{\mathrm{curv}^B}{\underset{0}{\rightrightarrows}}(L\otimes B)_{-2}$$

is an equalizer diagram in the category of Banach manifolds.

Corollary 6.7 *Suppose $B\in\mathsf{cdga}^{\mathrm{bnd}}_{\geq 0}$ has the structure of a dg Banach algebra. Let $f\colon(L,\ell)\to(L'',\ell'')$ be a fibration and $g\colon(L',\ell')\to(L'',\ell'')$ be a morphism in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$, and let (L_P,ℓ_P) be the pullback of the diagram $(L',\ell')\xrightarrow{g}(L'',\ell'')\xleftarrow{f}(L,\ell)$.*

Assume that all of the aforementioned Lie n–algebra satisfy Assumption 6.6. If the pullback of the diagram

$$\mathrm{MC}(L'\otimes B)\xrightarrow{g^B_*}\mathrm{MC}(L''\otimes B)\xleftarrow{f^B_*}\mathrm{MC}(L\otimes B)$$

exists as a Banach manifold, then the induced commutative diagram

$$\begin{array}{ccc}\mathrm{MC}(L_P\otimes B) & \longrightarrow & \mathrm{MC}(L\otimes B)\\ \big\downarrow & & \big\downarrow{\scriptstyle f^B_*}\\ \mathrm{MC}(L'\otimes B) & \xrightarrow{g^B_*} & \mathrm{MC}(L''\otimes B)\end{array}$$

is a pullback square in the category of Banach manifolds.

Proof First, we note that the underlying set of the pullback in the category of Banach manifolds is the usual fiber product [13, Chapter II.2]. Therefore, by hypothesis, the set $E_P:=\mathrm{MC}(L'\otimes B)\times_{\mathrm{MC}(L''\otimes B)}\mathrm{MC}(L\otimes B)$ admits the structure of a manifold.

In the proof of Proposition 6.5, we explicitly constructed in (6-21) a bijection

$$\varphi_{E_P}\colon E_P\xrightarrow{\cong}\mathrm{MC}(L_P\otimes B)$$

which identified $\mathrm{MC}(L_P \otimes B)$ as the pullback in the category of sets. We observe that all functions appearing in the construction of φ_{E_P}, its inverse and the relevant pullback diagrams are either

(1) polynomial functions between Banach spaces of the form $(V \otimes B)_{-1}$, where V is a finite-dimensional graded vector space, or

(2) polynomial functions between equalizers of polynomial functions of the above type.

By Assumption 6.6, the equalizers themselves are submanifolds of Banach spaces. Hence, the function φ_{E_P} respects the differentiable structures, as does its inverse. Therefore, φ_{E_P} is a diffeomorphism. □

We end this section with some remarks concerning the validity of our Assumption 6.6. It is not difficult to construct Lie n–algebras (L, ℓ) and dg Banach algebras (B, d_B) such that $\mathrm{MC}(L \otimes B)$ fails to satisfy this assumption. The following is a simple example:

Example 6.8 Let $L = L_1 \oplus L_2$ denote the graded vector space

$$L_1 := \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad |e_1| = |e_2| = 1, \qquad L_2 := \mathbb{R}\widetilde{e}, \quad |\widetilde{e}| = 2.$$

Equip L with the degree 0 graded skew-symmetric bracket $\ell_2\colon L \otimes L \to L$ whose nontrivial values on generators are

$$\ell_2(e_1, e_1) = \widetilde{e}, \quad \ell_2(e_2, e_2) = -\widetilde{e}.$$

It is easy to see that ℓ_2 satisfies the graded Jacobi identity. Therefore, ℓ_2 is an honest Lie bracket, and (L, ℓ_2) is a Lie 3–algebra.

Next, let (B, d_B) denote the cdga $B := \mathbb{R}[\theta]/(\theta^3)$, with $|\theta| = 2$, and trivial differential $d_B = 0$. As a graded vector space, $B = B_0 \oplus B_2 \oplus B_4$, where

$$B_0 = \mathbb{R}, \quad B_2 = \mathbb{R}\theta, \quad B_4 = \mathbb{R}\theta^2.$$

Since B is a finite-dimensional $\mathbb{R}$–algebra, it admits the structure of a Banach algebra (by embedding into $\mathrm{End}(\mathbb{R}^3)$, for example). Hence, the degree -1 piece of the tame L_∞–algebra $(L \otimes B, \ell_2^B)$ is just the Euclidean plane

$$(L \otimes B)_{-1} = \mathbb{R}(e_1 \otimes \theta) \oplus \mathbb{R}(e_2 \otimes \theta).$$

If $a = xe_1 \otimes \theta + ye_2 \otimes \theta \in (L \otimes B)_{-1}$, then $\mathrm{curv}^B(a) = 0$ if and only if

$$\tfrac{1}{2}(x^2 - y^2)(\widetilde{e} \otimes \theta^2) = 0.$$

Therefore $\mathrm{MC}(L \otimes B)$ is the zero locus of the polynomial $f(x, y) = x^2 - y^2$, and so it is not a submanifold of $(L \otimes B)_{-1} \cong \mathbb{R}^2$.

On the other hand, Assumption 6.6 is always satisfied in our main application of interest, when $B = \Omega(\Delta^n)$. Indeed, this follows from a result of Henriques [10, Theorem 5.10] and a result of Ševera and Širaň [20, Proposition 4.3]. Furthermore, the hypothesis in Corollary 6.7 concerning the existence of the pullback of Maurer–Cartan spaces is satisfied if the morphism $f\colon (L, \ell) \to (L'', \ell'')$ is a "quasisplit fibration" [16, Section 6]. This is a special kind of fibration, which we discuss in the next section.

7 Postnikov towers for Lie *n*–algebras

In this last section, we analyze the functorial aspects of Henriques' Postnikov construction for Lie n–algebras [10]. We show that in certain cases the Postnikov tower admits a convenient functorial decomposition. We use this in [16] to prove that the integration functor sends a certain distinguished class of fibrations in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ to fibrations between simplicial Banach manifolds. We call these distinguished fibrations "quasisplit".

Definition 7.1 A fibration of Lie n–algebras $f\colon (L, \ell) \to (L', \ell')$ is a *quasisplit fibration* if

(1) the induced map in homology $H(f_1)\colon H(L) \to H(L')$ is surjective in all degrees, and

(2) $H_0(L) \cong \ker H_0(f_1) \oplus H_0(L')$ in the category of Lie algebras.

Note that every acyclic fibration in $\mathsf{Lie}_n\mathsf{Alg}^{\mathrm{fin}}$ is a quasisplit fibration. More generally,

$$f\colon (L, \ell) \to (L', \ell')$$

is a quasisplit fibration if $\ker H(f_1)$ is central and $H(f_1)\colon H(L) \to H(L')$ is a split epimorphism in the category of $H_0(L)$–modules.

The string Lie 2–algebra $(\mathfrak{g}\oplus\mathbb{R}[-1],\{\ell_1,\ell_2,\ell_3\})$ associated to a simple Lie algebra $\mathfrak{g}$ of compact type was the original motivation for Henriques' work in [10]. It sits in a quasisplit fiber sequence of the form

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{R} \arrow[r] \arrow[d, "\ell_1 = 0"] & 0 \arrow[d] \\ 0 \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{g} \end{tikzcd}$$

This is a special case of a more general fiber sequence, called a "central n–extension". (See [16, Section 6].)

7.1 Morphisms between towers

Let (L,ℓ) be a Lie n–algebra. Following [10, Definition 5.6], we consider two different truncations of the underlying chain complex $(L, d=\ell_1)$. For any $m\geq 0$, denote by $\tau_{\leq m}L$ and $\tau_{<m}L$ the $(m+1)$–term complexes

$$(\tau_{\leq m}L)_i = \begin{cases} L_i & \text{if } i<m, \\ \operatorname{coker}(d_{m+1}) & \text{if } i=m, \\ 0 & \text{if } i>m, \end{cases} \qquad (\tau_{<m}L)_i = \begin{cases} L_i & \text{if } i<m, \\ \operatorname{im}(d_m) & \text{if } i=m, \\ 0 & \text{if } i>m. \end{cases}$$

In degree m, the differentials for $\tau_{\leq m}L$ and $\tau_{<m}L$ are $d_m\colon L_m/\operatorname{im}(d_{m+1})\to L_{m-1}$ and the inclusion $\operatorname{im}(d_m)\hookrightarrow L_{m-1}$, respectively. The homology complexes of $\tau_{\leq m}L$ and $\tau_{<m}L$ are

$$H_i(\tau_{\leq m}L) = \begin{cases} H_i(L) & \text{if } i\leq m, \\ 0 & \text{if } i>m, \end{cases} \qquad H_i(\tau_{<m}L) = \begin{cases} H_i(L) & \text{if } i<m, \\ 0 & \text{if } i\geq m. \end{cases}$$

We have the surjective chain maps

$$p_{\leq m}\colon L\to\tau_{\leq m}L, \quad p_{<m}\colon L\to\tau_{<m}L, \tag{7-1}$$

where in degree m, the map $p_{\leq m}$ is the surjection $L_m\to\operatorname{coker}(d_{m+1})$ and $p_{<m}$ is the differential $d_m\colon L_m\to\operatorname{im}(d_m)$. There are also the similarly defined surjective chain maps

$$q_{\leq m}\colon \tau_{\leq m}L\to\tau_{<m}L, \quad q_{<m+1}\colon \tau_{<m+1}L\xrightarrow{\sim}\tau_{\leq m}L. \tag{7-2}$$

The map $q_{\leq m}$ in degree m is the differential $d_m\colon \operatorname{coker} d_{m+1}\to\operatorname{im} d_m$ and the identity in all other degrees. The map $q_{<m+1}$ is the projection $L_m\to\operatorname{coker} d_{m+1}$ in degree m, the identity in all degrees $<m$, and the zero map in degree $m+1$. We note that $q_{<m+1}$ is a quasi-isomorphism of complexes.

Proposition 7.2 *Let (L,ℓ) be a Lie n–algebra.*

(1) *The Lie n–algebra structure on (L,ℓ) induces Lie $(m+1)$–structures on the complexes $\tau_{\le m}L$ and $\tau_{<m}L$ whose brackets are given by*

$$\tau_{\le m}\ell_k(\overline{x}_1,\dots,\overline{x}_k) := p_{\le m}\ell_k(x_1,\dots,x_k),$$
$$\tau_{<m}\ell_k(\overline{y}_1,\dots,\overline{y}_k) := p_{<m}\ell_k(y_1,\dots,y_k),$$

where $\overline{x}_i = p_{\le m}(x_i)$ and $\overline{y}_i = p_{<m}(y_i)$.

(2) *The assignments $(L,\ell)\mapsto(\tau_{\le m}L,\tau_{\le m}\ell)$ and $(L,\ell)\mapsto(\tau_{<m}L,\tau_{<m}\ell)$ are functorial.*

(3) *An L_∞–morphism $\phi\colon (L,\ell)\to(L',\ell')$ induces a morphism of towers of Lie n–algebras*

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\tau_{\le m-1}L \arrow[r] \arrow[d, "q_{\le m-1}"'] & \tau_{\le m-1}L' \arrow[d, "q'_{\le m-1}"] \\
\tau_{<m-1}L \arrow[r, "\tau_{<m-1}\phi"] \arrow[d, "q_{<m-1}"'] & \tau_{<m-1}L' \arrow[d, "q'_{<m-1}"] \\
\tau_{\le m-2}L \arrow[r, "\tau_{\le m-2}\phi"] \arrow[d, "q_{\le m-2}"'] & \tau_{\le m-2}L' \arrow[d, "q'_{\le m-2}"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\tau_{\le 1}L \arrow[r, "\tau_{\le 1}\phi"] \arrow[d, "q_{\le 1}"'] & \tau_{\le 1}L' \arrow[d, "q'_{\le 1}"] \\
\tau_{<1}L \arrow[r, "\tau_{<1}\phi"] \arrow[d, "q_{<1}"'] & \tau_{<1}L' \arrow[d, "q'_{<1}"] \\
\tau_{\le 0}L \arrow[r, "\tau_{\le 0}\phi"] & \tau_{\le 0}L'
\end{tikzcd} \tag{7-3}$$

in which the horizontal arrows are the strict L_∞–morphisms induced by the surjective chain maps (7-2).

Proof We prove (1) and (2) for $\tau_{\le m}L$. The same arguments apply for $\tau_{<m}L$. First, we verify that the brackets $\tau_{\le m}\ell_k$ are well defined. For degree reasons, the only nontrivial case to check is $\tau_{\le m}\ell_2(\overline{x}_1,\overline{x}_2)$ when $\overline{x}_1$ is in degree m and $\overline{x}_2$ is in degree 0. Suppose $x_1 = x_1' + d_{m+1}z$, where d_{m+1} is the differential ℓ_1 in degree $m+1$. The Jacobi-like identities (3-1) for the L_∞–structure imply that the degree 0 bracket ℓ_2

satisfies $\ell_1\ell_2(x_1,x_2) = \ell_2(\ell_1 x_1, x_2) + (-1)^m \ell_2(x_1, \ell_1 x_2)$. Hence, $\ell_2(x_1,x_2) = \ell_2(x_1', x_2) + d_{m+1}\ell_2(z, x_2)$, and so $\tau_{\leq m}\ell_2$ is well defined. The fact that the brackets ℓ_k satisfy the identities (3-1) immediately implies that the brackets $\tau_{\leq m}\ell_k$ satisfy them as well.

Next, let $\phi\colon (L,\ell) \to (L',\ell')$ be a morphism in $\mathsf{Lie}_n\mathsf{Alg}$. Define maps

$$\tau_{\leq m}\phi_k\colon \Lambda^k \tau_{\leq m} L \to \tau_{\leq m} L'$$

by

$$\tau_{\leq m}\phi_k(\bar{x}_1, \dots, \bar{x}_k) := p'_{\leq m}\phi_k(x_1, \dots, x_k),$$

where $p'_{\leq m}\colon L' \to \tau_{\leq m} L'$ is the projection (7-1). We verify that these are well defined. Again, for degree reasons, the only nontrivial case to check is $\tau_{\leq m}\phi_1(\bar{x})$ with $\bar{x}$ in degree m. Recall that ϕ_1 is a chain map (Remark 3.2). Hence, if $x_1 = x_1' + d_{m+1}z$, then $\tau_{\leq m}\phi_1(\bar{x}) = \tau_{\leq m}\phi_1(\bar{x}')$. The fact that the maps ϕ_k satisfy the defining equations (2-10) immediately implies that the maps $\tau_{\leq m}\phi_k$ form an L_∞–morphism $\tau_{\leq m}\phi\colon (\tau_{\leq m}L, \tau_{\leq m}\ell) \to (\tau_{\leq m}L', \tau_{\leq m}\ell')$.

For statement (3), since the L_∞ brackets for $\tau_{\leq m}L$ and $\tau_{< m}L$ are defined using the projection maps (7-1), it is easy to see that the horizontal projections in the diagram (7-3) are strict L_∞–morphisms. Since the vertical morphisms $\tau_{\leq m}\phi$ and $\tau_{< m}\phi$ are also defined using the projection maps (7-1), the diagram indeed commutes. □

Remark 7.3 The Lie n–algebra $\tau_{\leq 0}L$ is just the Lie algebra $H_0(L)$ concentrated in degree zero. Given a morphism of Lie n–algebras $f\colon (L,\ell) \to (L',\ell')$, the induced morphism $\tau_{\leq 0}f\colon H_0(L) \to H_0(L')$ of Lie algebras is the morphism $H_0(f_1)$ from Remark 3.2.

7.2 A functorial decomposition of towers

Let $f\colon (L,\ell) \to (L',\ell')$ be a quasisplit fibration (Definition 7.1). Our goal is to decompose the induced morphism between the Postnikov towers associated to L and L'. Let us make two simple initial observations. First, recall that every fibration in $\mathsf{Lie}_n\mathsf{Alg}$ can be factored into an isomorphism followed by a strict fibration (Proposition 3.8). Hence, we restrict our discussion here to strict quasisplit fibrations. Second, it follows directly from the definition that every quasisplit fibration is an L_∞–epimorphism (Definition 3.3). Therefore, we present our results below in a slightly more general context for the case when f is a strict L_∞–epimorphism.

We begin with the following useful lemma. A variation of this result arises in the construction of minimal models for L_∞–algebras.

Lemma 7.4 *Let* $f\colon (L,\ell)\to(L',\ell')$ *be an acyclic fibration in* $\mathsf{Lie}_n\mathsf{Alg}$. *Let* $(\ker f_1,\ell_1)$ *denote the kernel of the chain map* $f_1\colon (L,\ell_1)\to(L',\ell'_1)$ *considered as an abelian Lie* n*–algebra* (Example 3.4). *Then there exists an* L_∞*–morphism*

$$r\colon (L,\ell)\to(\ker f_1,\ell_1)$$

such that the morphism induced via the universal property of the product,

$$(f,r)\colon (L,\ell)\to(L'\oplus\ker f_1,\ell'\oplus\ell_{\ker f}),$$

is an isomorphism of Lie n*–algebras.*

Proof Since f is an acyclic fibration, the chain map $f_1\colon (L,\ell_1)\to(L',\ell'_1)$ is an acyclic fibration in $\mathsf{Ch}^{\mathrm{proj}}_{\geq 0}$. Therefore, there exists a chain map $\sigma\colon (L',\ell'_1)\to(L,\ell_1)$ such that $f_1\sigma=\mathrm{id}_{L'}$. Moreover, since the complexes (L,ℓ_1) and (L',ℓ'_1) are bifibrant in $\mathsf{Ch}^{\mathrm{proj}}_{\geq 0}$, there exists a chain homotopy $h\colon L\to L$ such that $\mathrm{id}_L-\sigma f_1=\ell_1 h+h\ell_1$. We consider the chain map

$$g_1\colon L\to\ker f_1,\quad g_1:=\mathrm{id}_L-\sigma f_1 \tag{7-4}$$

and, for $k\geq 2$, define the degree $k-1$ multilinear maps

$$g_k\colon \Lambda^k L\to\ker f_1,\quad g_k:=g_1\circ h\circ\ell_k. \tag{7-5}$$

Let $G\colon \overline{S}(sL)\to\overline{S}(s\ker f_1)$ denote the coalgebra map associated to the maps g_k. Let δ and $\delta_{\ker f_1}$ denote the codifferentials corresponding to the L_∞–structures on L and $\ker f_1$, respectively. We verify that $(\delta_{\ker f_1}\circ G)^1_m$ equals $(G\circ\delta)^1_m$ for all $m\geq 1$. Indeed, since g_1 is a chain map, we have $(\delta_{\ker f_1}\circ G)^1_1=(G\circ\delta)^1_1$. Now let $m\geq 2$. Equations (3-2) and (3-4) imply that

$$(\delta_{\ker f_1}\circ G)^1_m=(-1)^{m(m-1)/2}s(\ell_1 g_1 h\ell_m)(s^{-1})^{\otimes m} \tag{7-6}$$

and

$$(G\circ\delta)^1_m=G^1_1\delta^1_m+\sum_{k\geq 2}^{m}G^1_k\delta^k_m=G^1_1\delta^1_m+\sum_{k\geq 2}^{m}+sg_1\circ hs^{-1}\delta^1_k\delta^k_m.$$

Since $\delta\circ\delta=0$, we use (2-9) to rewrite the last term on the right-hand side:

$$(G\circ\delta)^1_m=G^1_1\delta^1_m-sg_1hs^{-1}\delta^1_1\delta^1_m=(-1)^{m(m-1)/2}s(g_1-g_1h\ell_1)\ell_m(s^{-1})^{\otimes m}.$$

By comparing the above equality with (7-6), and using the fact that $g_1 - g_1 h \ell_1 = \ell_1 g_1 h$, we conclude that $(\delta_{\ker f_1} G)^1_m = (G \circ \delta)^1_m$. Hence, $g\colon (L, \ell_k) \to (\ker f_1, \ell_1)$ is an L_∞–morphism. Finally, since the linear map $(f_1, g_1)\colon L \to L' \oplus \ker f_1$ is an isomorphism of complexes, it follows that the induced morphism

$$(F, G)\colon L \to L' \oplus \ker f_1$$

is an L_∞–isomorphism. □

Our first decomposition result involves the commuting squares in (7-3) whose left edges are the strict acyclic fibrations $q_{<m+1}\colon (\tau_{<m+1} L, \tau_{<m+1}\ell) \to (\tau_{\leq m} L, \tau_{\leq m}\ell)$ defined in (7-2). Let $\ker q_{<m+1}$ denote the kernel of the chain map $q_{<m+1}$. Then $\ker q_{<m+1}$ is an abelian Lie n–algebra concentrated in degrees m and $m+1$ with

$$(\ker q_{<m+1})_m = \operatorname{im} d_{m+1}, \quad (\ker q_{<m+1})_{m+1} = \operatorname{im} d_{m+1}[-1].$$

The induced differential $\ell^{\ker} = \ell_1$ on $\ker q_{<m+1}$ is simply the desuspension isomorphism.

Proposition 7.5 *Let $f\colon (L, \ell) \to (L', \ell')$ be a strict L_∞–epimorphism between Lie n–algebras. Then there exist morphisms in $\mathsf{Lie}_n\mathsf{Alg}$*

$$r\colon (\tau_{<m+1} L, \tau_{<m+1}\ell) \to (\ker q_{<m+1}, \ell^{\ker}),$$
$$r'\colon (\tau_{<m+1} L', \tau_{<m+1}\ell') \to (\ker q'_{<m+1}, \ell'^{\ker})$$

inducing L_∞–isomorphisms

$$(q_{<m+1}, r)\colon (\tau_{<m+1} L, \tau_{<m+1}\ell) \xrightarrow{\cong} (\tau_{\leq m} L \oplus \ker q_{<m+1}, \tau_{\leq m}\ell \oplus \ell^{\ker}),$$
$$(q'_{<m+1}, r')\colon (\tau_{<m+1} L', \tau_{<m+1}\ell') \xrightarrow{\cong} (\tau_{\leq m} L' \oplus \ker q'_{<m+1}, \tau_{\leq m}\ell' \oplus \ell'^{\ker})$$

such that the following diagram commutes in $\mathsf{Lie}_n\mathsf{Alg}$:

(7-7)
$$\begin{tikzcd}
\tau_{<m+1} L \arrow[r, "(q_{<m+1}, r)", "\cong"'] \arrow[d, "\tau_{<m+1} f"'] & \tau_{\leq m} L \oplus \ker q_{<m+1} \arrow[d, "\tau_{\leq m} f \oplus \tau_{<m+1} f|_{\ker}"] \\
\tau_{<m+1} L' \arrow[r, "(q'_{<m+1}, r')", "\cong"'] & \tau_{\leq m} L' \oplus \ker q'_{<m+1}
\end{tikzcd}$$

Proof Since f is strict, we have $f = f_1$ and so Proposition 7.2 implies that we have the commutative diagram in $\mathsf{Lie}_n\mathsf{Alg}$

$$\begin{tikzcd} \tau_{<m+1}L \arrow[r, "q_{<m+1}"] \arrow[d, "\tau_{<m+1}f_1"'] & \tau_{\leq m}L \arrow[d, "\tau_{\leq m}f_1"] \\ \tau_{<m+1}L' \arrow[r, "q'_{<m+1}"] & \tau_{\leq m}L' \end{tikzcd} \tag{7-8}$$

For the sake of brevity, let V and V' denote the abelian Lie n–algebras $\ker q_{<m+1}$ and $\ker q'_{<m+1}$, respectively.

Since $q_{<m+1}$ and $q'_{<m+1}$ are acyclic fibrations, Lemma 7.4 provides us with L_∞–isomorphisms $(q_{<m+1}, r)\colon \tau_{<m+1}L \xrightarrow{\cong} \tau_{\leq m}L \oplus V$ and $(q'_{<m+1}, r')\colon \tau_{<m+1}L' \xrightarrow{\cong} \tau_{\leq m}L' \oplus V'$. We will show that in the proof of Lemma 7.4, we can choose the morphisms r and r' such that diagram (7-7) commutes.

In degree m, (7-8) corresponds to the commutative diagram between short exact sequences of vector spaces

$$\begin{tikzcd} \operatorname{im} d_{m+1} \arrow[r, "i"] \arrow[d, "\tau_{<m+1}f_1|_{\operatorname{im} d}"'] & L_m \arrow[r, "\pi"] \arrow[d, "\tau_{<m+1}f_1"] & \operatorname{coker} d_{m+1} \arrow[d, "\tau_{\leq m}f_1"] \\ \operatorname{im} d'_{m+1} \arrow[r, "i'"] & L'_m \arrow[r, "\pi'"] & \operatorname{coker} d'_{m+1} \end{tikzcd}$$

Since f is an L_∞–epimorphism, the maps $\tau_{<m+1}f_1|_{\operatorname{im} d}$ and $\tau_{\leq m}f_1$ are surjections. This, along with the fact that the rows are exact, implies that there exist sections $s\colon \operatorname{coker} d_{m+1} \to L_m$ and $s'\colon \operatorname{coker} d'_{m+1} \to L'_m$ of π and π', respectively, such that

$$\tau_{<m+1}f_1 \circ s = s' \circ \tau_{\leq m}f_1.$$

The linear maps s and s' induce sections $\sigma\colon \tau_{\leq m}L \to \tau_{<m+1}L$ and $\sigma'\colon \tau_{\leq m}L' \to \tau_{<m+1}L'$ in $\mathsf{Ch}_{\geq 0}$ of the chain maps $q_{<m+1}$ and q'_{m+1}, respectively. Explicitly, we have

$$\sigma(x) := \begin{cases} x & \text{if } |x| < m, \\ 0 & \text{if } |x| > m, \\ s(x) & \text{if } |x| = m, \end{cases}$$

with an analogous formula for σ'. Moreover, it follows that

$$\tau_{<m+1}f_1 \circ \sigma = \sigma' \circ \tau_{\leq m}f_1. \tag{7-9}$$

We then construct chain homotopies $h\colon \tau_{<m+1}L \to \tau_{<m+1}L[1]$ and $h'\colon \tau_{<m+1}L' \to \tau_{<m+1}L'[1]$, as in the proof of Lemma 7.4. Explicitly, we have

$$h(x) := \begin{cases} 0 & \text{if } |x| < m \text{ or } |x| > m, \\ \boldsymbol{s}(x - s\pi(x)) \in \operatorname{im}(d_{m+1})[1] & \text{if } |x| = m, \end{cases}$$

with an analogous formula for h'. Hence, the homotopies satisfy

$$\tau_{<m+1} f_1 \circ h = h' \circ \tau_{<m+1} f_1. \tag{7-10}$$

We then use the chain maps σ and σ', the homotopies h and h', and the L_∞–structures $\tau_{<m+1}\ell$ and $\tau_{<m+1}\ell'$ to construct L_∞–morphisms

$$r\colon \tau_{<m+1}L \to V, \quad r'\colon \tau_{<m+1}L' \to V'$$

via the formulas (7-4) and (7-5) in the proof of Lemma 7.4.

Using (7-9), (7-10) and the fact that $\tau_{<m+1} f_1$ is an L_∞–morphism, a direct computation verifies that for all $k \geq 1$, we have

$$r'_k \circ (\tau_{<m+1} f_1)^{\otimes k} = \tau_{<m+1} f_1|_V \circ r_k.$$

Hence, the diagram (7-7) commutes. $\square$

Now let $m \geq 1$. We focus on those commuting squares in (7-3) whose left edges are the strict quasisplit fibrations $q_{\leq m}\colon (\tau_{\leq m}L, \tau_{\leq m}\ell) \to (\tau_{<m}L, \tau_{<m}\ell)$ defined in (7-2). As a chain map, $q_{\leq m}$ induces a short exact sequence of chain complexes

$$H_m \xrightarrow{i} \tau_{\leq m}L \xrightarrow{q_{\leq m}} \tau_{<m}L,$$

where H_n is the homology group $H_n(L)$ concentrated in degree n with trivial differential. Our second decomposition result is the following:

Proposition 7.6 *Let $m \geq 1$ and let $f\colon (L,\ell) \to (L',\ell')$ be a strict L_∞–epimorphism in $\mathsf{Lie}_n\mathsf{Alg}$ such that the induced map in homology*

$$H(f_1)\colon H_m \to H'_m$$

is surjective in degree m. Then there exist L_∞–structures $\widehat{\ell}$ and $\widehat{\ell}'$ on the graded vector spaces $\tau_{<m}L \oplus H_m$ and $\tau_{<m}L' \oplus H'_m$, respectively, and L_∞–isomorphisms

$$\widehat{q}\colon (\tau_{\leq m}L, \tau_{\leq m}\ell) \xrightarrow{\cong} (\tau_{<m}L \oplus H_m, \widehat{\ell}), \quad \widehat{q}'\colon (\tau_{\leq m}L', \tau_{\leq m}\ell') \xrightarrow{\cong} (\tau_{<m}L' \oplus H'_m, \widehat{\ell}')$$

such that the following diagram of strict L_∞–morphisms commutes:

$$
\begin{tikzcd}
(\tau_{\leq m} L, \tau_{\leq m}\ell) \arrow[r, "\widehat{q}", "\cong"'] \arrow[d, "\tau_{\leq m} f"'] & (\tau_{<m} L \oplus H_m, \widehat{\ell}) \arrow[d, "\tau_{<m} f \oplus H(f)"] \\
(\tau_{\leq m} L', \tau_{\leq m}\ell') \arrow[r, "\widehat{q}'", "\cong"'] & (\tau_{<m} L' \oplus H'_m, \widehat{\ell}')
\end{tikzcd}
\tag{7-11}
$$

Proof Since f is strict, we have $f = f_1$ and so Proposition 7.2 implies that we have the commutative diagram in $\mathsf{Lie}_n\mathsf{Alg}$

$$
\begin{tikzcd}
\tau_{\leq m} L \arrow[r, "q_{\leq m}"] \arrow[d, "\tau_{\leq m} f_1"'] & \tau_{<m} L \arrow[d, "\tau_{<m} f_1"] \\
\tau_{\leq m} L' \arrow[r, "q'_{\leq m}"] & \tau_{<m} L'
\end{tikzcd}
$$

In degree m, this corresponds to the commutative diagram between short exact sequences of vector spaces

$$
\begin{tikzcd}
H_m \arrow[r, "i"] \arrow[d, "H(f_1)"'] & \operatorname{coker} d_{m+1} \arrow[r, "d_m"] \arrow[d, "\tau_{\leq m} f_1"] & \operatorname{im} d_m[-1] \arrow[d, "\tau_{<m} f_1"] \\
H'_m \arrow[r, "i'"] & \operatorname{coker} d'_{m+1} \arrow[r, "d'_m"] & \operatorname{im} d'_m[-1]
\end{tikzcd}
$$

By hypothesis, the vertical maps are surjections. Let

$$\mu\colon H'_m \to H_m, \quad \nu\colon \operatorname{im} d'_m[-1] \to \operatorname{im} d_m[-1], \quad \psi\colon \operatorname{im} d_m[-1] \to \operatorname{coker} d_{m+1}$$

be sections of $H(f_1)$, $\tau_{<m} f_1$ and d_m, respectively. Consider the composition $s' := \tau_{\leq m} f_1 \circ \psi \circ \nu\colon \operatorname{im} d'_m[-1] \to \operatorname{coker} d'_{m+1}$. In general, the linear map $\tau_{\leq m} f_1 \psi - s' \tau_{<m} f_1$ will not equal zero. So let $s\colon \operatorname{im} d_m[-1] \to \operatorname{coker} d_{m+1}$ be the composition $s := \psi - i \circ \mu \circ (\tau_{\leq m} f_1 \psi - s' \tau_{<m} f_1)$. Then s and s' are sections of d_m and d'_m, respectively, and

$$\tau_{\leq m} f_1 \circ s = s' \circ \tau_{<m} f_1.$$

We use s and s' to define chain maps. Let $t\colon \tau_{<m} L \to \tau_{\leq m} L$ and $\widehat{r}\colon \tau_{\leq m} L \to H_m$ be the linear maps

$$t(x) := \begin{cases} s(x) & \text{if } |x| = m, \\ x & \text{if } |x| < m, \end{cases} \qquad \widehat{r} := \mathrm{id} - t q_{\leq m},$$

respectively. Then the isomorphism $\hat{q}\colon \tau_{\leq m} L \to \tau_{<m} L \oplus H_m$ is defined to be

$$\hat{q}(z) := (q_{\leq m}(z), \hat{r}(z)).$$

The map $\hat{q}'$ is defined in the analogous way, using the section s' instead of s. Hence (7-11) commutes as a diagram in the category $\mathsf{Ch}_{\geq 0}$.

Finally, we construct compatible L_∞–structures on $\tau_{<m} L \oplus H_m$ and $\tau_{<m} L' \oplus H'_m$ via transfer across the isomorphisms $\hat{q}$ and $\hat{q}'$. For each $k \geq 1$, we define

$$\hat{\ell}_k := \hat{q} \circ \tau_{\leq m} \ell_k \circ (\hat{q}^{-1})^{\otimes k}, \quad \hat{\ell}'_k := \hat{q}' \circ \tau_{\leq m} \ell'_k \circ (\hat{q}'^{-1})^{\otimes k}. \tag{7-12}$$

Hence, by construction, the chain maps $\hat{q}$ and $\hat{q}'$ lift to L_∞–isomorphisms. □

7.2.1 The structure of the "twisted product" $(\tau_{<m} L \oplus H_m, \hat{\ell})$ We emphasize that, in general, the Lie n–algebra $(\tau_{<m} L \oplus H_m, \hat{\ell})$ defined in the above proof of Proposition 7.6 is not the categorical product of the Lie n–algebra $(\tau_{<m} L, \tau_{<m}\ell)$ with the abelian Lie n–algebra H_m. This is in contrast with the decomposition $\tau_{<m+1} L \cong \tau_{\leq m} L \oplus \ker q_{<m+1}$ given in Proposition 7.5.

Indeed, it follows from the definition of $\hat{q}$ given in the above proof that the L_∞–structure maps $\hat{\ell}_k\colon \Lambda^k(\tau_{<m} L \oplus H_m) \to \tau_{<m} L \oplus H_m$ defined in (7-12) can be written as

$$\begin{aligned} &\hat{\ell}_k((x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)) \\ &\qquad = (\tau_{<m}\ell_k(x_1, x_2, \dots, x_k), \hat{r} \circ \tau_{\leq m}\ell_k(tx_1 + y_1, tx_2 + y_2, \dots, tx_k + y_k)). \end{aligned}$$

Note that there is only one nontrivial structure map $\hat{\ell}_k$ that involves nonzero inputs from H_m, since H_m is concentrated in top degree m. Namely,

$$\hat{\ell}_2((x, 0), (0, y)) = (0, \ell_2(x, y)),$$

where $x \in L_0 = \tau_{<m} L_0$ is an element of degree 0 and $y \in H_m$. This simple observation plays a key role in our study [16] of integrated quasisplit fibrations.

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