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Algebraic & Geometric Topology 20 (2020) 1431–1486

Relative recognition principle

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We prove the recognition principle for relative N–loop pairs of spaces for $3 \leq N \leq \infty$. If $3 \leq N < \infty$, this states that a pair of spaces homotopy equivalent to CW–complexes (X_c, X_o) is homotopy equivalent to $(Y^{\mathbb{S}^N}, \mathrm{HFib}(\iota)^{\mathbb{S}^{N-1}})$ for a functorially determined relative space $\iota\colon B \to Y$ if and only if (X_c, X_o) is a grouplike $\overline{\mathcal{SC}}_N$–space, where $\overline{\mathcal{SC}}_N$ is any cofibrant resolution of the Swiss-cheese relative operad $\mathcal{SC}_N$. The relative recognition principle for relative ∞–loop pairs of spaces states that a pair of spaces (X_c, X_o) homotopy equivalent to CW–complexes is homotopy equivalent to $(Y_0, \mathrm{HFib}(\iota_0))$ for a functorially determined relative spectrum $\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1}$ if and only if (X_c, X_o) is a grouplike $\mathcal{E}^{\to}$–algebra, where $\mathcal{E}^{\to}$ is a contractible cofibrant relative operad or equivalently a cofibrant resolution of the terminal relative operad $\mathrm{Com}^{\to}$ of continuous homomorphisms of commutative monoids. These principles are proved as equivalences of homotopy categories.

55P35, 55P48, 55R15; 55P42

1 Introduction

For $1 \leq N \leq \infty$ a recognition principle is a specification of conditions under which a space is of the homotopy type of an N–loop space. The concept of operad was central to establishing recognition principles. A topological operad $\mathcal{P}$ consists of a sequence of spaces $\mathcal{P}(\underline{n}) \in \prod_{\mathbb{N}} \mathtt{Top}$ and composition maps. The points $p \in \mathcal{P}(\underline{n})$ are abstract operations of n arguments, and a $\mathcal{P}$–algebra is a pointed space $X \in \mathtt{Top}_*$ equipped with concrete realizations of these operations as maps in $X^{X^{\underline{n}}}$. Formally a topological operad $\mathcal{P}$ determines a monad P in the category of pointed topological spaces $\mathtt{Top}_*$ via the coend construction.

If $N = 1$, Stasheff proved in [42; 43] that a pointed space X is of the weak homotopy type of a 1–loop space if and only if X is a grouplike $\mathcal{A}_\infty$–algebra, where $\mathcal{A}_\infty$ is the operad of associahedra. Though Stasheff didn't use the term "operad" for the sequence of associahedra, the structure he used was what is now called an operad structure.

In [31], May established the term "operad" and used the little N–cubes operad $\mathcal{C}_N$, introduced by Boardman and Vogt in [5] in the language of PROPs, in order to prove a recognition principle for connected N–loop spaces. All N–loop spaces are natural

Published: 27 May 2020 DOI: 10.2140/agt.2020.20.1431

$\mathcal{C}_N$–algebras. A sufficient condition for a pointed space of the homotopy type of a CW–complex X to be homotopic to an N–loop space is for it to be a grouplike $\mathcal{C}_N$–algebra. To get a necessary condition $\mathcal{C}_N$ must be substituted by a cofibrant resolution $\bar{\mathcal{C}}_N$. If $N = \infty$, an E_∞–operad $\mathcal{E}$ can be used, which is a cofibrant operad with each space contractible, or equivalently a cofibrant resolution of the terminal operad Com of commutative topological monoids. For an overview including the nonconnected cases, see Frankhuizen [12].

These recognition principles can be expressed as equivalences of homotopy categories. For $N \in \mathbb{N}$ the N–loop space functor Ω^N is part of an equivalence between the homotopy category of $(N-1)$–connected spaces and the homotopy category of grouplike $\bar{\mathcal{C}}_N$–algebras. The ∞–loop space functor Ω^∞ is part of an equivalence between the homotopy category of connective spectra and the homotopy category of grouplike $\mathcal{E}$–algebras for any E_∞–operad $\mathcal{E}$.

In this article a recognition principle for relative N–loop pairs of spaces for $3 \le N \le \infty$ is proved. The case $N = 1$ was proved by Hoefel, Livernet and Stasheff in [19]. For $\iota\colon B \to Y \in \mathtt{Top}_*^{\to}$ a map of pointed spaces the homotopy fiber $\mathrm{HFib}(\iota)$ is the space of pairs $(b,\gamma) \in B \times Y^I$ such that $\gamma(0) = \iota(b)$ and $\gamma(1) = y_0$. For $1 \le N < \infty$ the relative N–loop pair of spaces functor is $\Omega_2^N(\iota) := (Y^{\mathbb{S}^N}, \mathrm{HFib}(\iota)^{\mathbb{S}^{N-1}})$. The domain of the relative ∞–loop pair of spaces functor Ω_2^∞ is the category $\mathtt{Sp}^{\nearrow}$ of relative spectra $\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1}$, which are spectra maps with a shift in the degree, and it is $\Omega_2^\infty(\iota_\bullet) := \operatorname{colim}_{\bullet\to\infty}(Y_{\bullet+1}^{\mathbb{S}^{\bullet+1}}, \mathrm{HFib}(\iota_\bullet)^{\mathbb{S}^\bullet})$.

The theory of colored operads is a generalization of the theory of operads where operations on multiple spaces are allowed. In this article relative operads are used, which are a kind of operad in two colors. A topological relative operad $\mathcal{Q}$ consists of a sequence of spaces $\mathcal{Q}(\underline{l}) \in \prod_{\mathbb{N}} \mathtt{Top}$, a bisequence of spaces $\mathcal{Q}(\underline{m,n}) \in \prod_{\mathbb{N}^2} \mathtt{Top}$ and composition maps. A relative operad $\mathcal{Q}$ determines a monad Q on the category of pairs of pointed spaces $\mathtt{Top}_*^2$. The category of $\mathcal{Q}$–algebras is denoted by $\mathcal{Q}[\mathtt{Top}]$. Central to the relative recognition principle is the N–Swiss-cheese relative operad $\mathcal{SC}_N$. The underlying spaces of $\mathcal{SC}_N$ are $\mathcal{SC}_N(\underline{l}) = \mathcal{C}_N(\underline{l})$ and $\mathcal{SC}_N(\underline{m,n})$ is the subspace of $\mathcal{C}_N(\underline{m+n})$ with the first coordinate base of the N–dimensional cube embeddings indexed by the last n indices being mapped to the first coordinate base of the codomain N–dimensional cube (see Figure 1 for an example). All relative N–loop pairs of spaces are natural $\mathcal{SC}_N$–algebras.

The 2–Swiss-cheese relative operad $\mathcal{SC}_2$ was introduced by Voronov [45] as a model of the moduli space of genus-zero Riemann surfaces appearing in the open–closed string

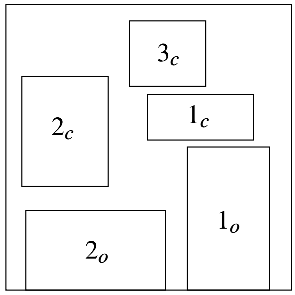


Figure 1: An element of $\mathcal{SC}_2(\underline{3,2})$.

theory studied by Zwiebach [47]. In Voronov's original version of the 2–Swiss-cheese relative operad the spaces $\mathcal{SC}_2(\underline{m,0})$ were taken to be empty, which is not assumed in the present article. Kontsevich used the Swiss-cheese relative operad in his work on deformation quantization to describe actions of $C_*(\mathcal{C}_N)$–algebras on $C_*(\mathcal{C}_{N-1})$–algebras [23]. Related to Kontsevich's approach to deformation quantization and Zwiebach's open–closed string field theory, Kajiura and Stasheff introduced open–closed homotopy algebras (OCHA) and strong homotopy Leibniz pairs (SHLP) in [22], which are the algebras over operads that can be obtained from the homology of the Swiss-cheese operad, as has been shown by Hoefel [16] and Hoefel and Livernet [18].

The Swiss-cheese relative operad itself has been the subject of intense study recently by several authors. Livernet has shown that unlike the little cubes operads the Swiss-cheese relative operads is not formal [27], and Willwacher has shown that extended Swiss-cheese relative operads are also not formal [46]. Idrissi has found a model of $\mathcal{SC}_2$ in the category of groupoids [21], and in general Quesney has found combinatorial models for $\mathcal{SC}_N$ in the category of sets and used them to exhibit models for relative loop spaces in dimension 2 [37]. Ducoulombier proved that totalizations of certain cosimplicial spaces are $\mathcal{SC}_2$–algebras and that there are $\mathcal{SC}_N$–algebra structures on pairs of spaces weakly equivalent to a pair composed of the space of embeddings of $\mathbb{R}^{N-1}$ in $\mathbb{R}^D$ and the manifold calculus limit of (l)–immersions of $\mathbb{R}^{N-1}$ in $\mathbb{R}^D$ [10; 11].

An E_∞^{rel}–operad is a cofibrant relative operad $\mathcal{E}^{\rightarrow}$ with each space contractible, or equivalently a cofibrant resolution of the terminal relative operad $\mathrm{Com}^{\rightarrow}$ of continuous homomorphisms of commutative topological monoids.

The relative recognition principle, Theorem 4.3.8, will be proved in the form of equivalences of homotopy categories. These equivalences are not induced by Quillen equivalences, as is common in model category theory, but by a weaker machinery introduced in Section 2.

Theorem *Let $3 \leq N < \infty$ and $\overline{\mathcal{SC}}_N$ be a cofibrant resolution of $\mathcal{SC}_N$. There is an equivalence between the homotopy category of grouplike $\overline{\mathcal{SC}}_N$–algebras and the homotopy category of relative spaces with domain $(N-2)$–connected and codomain $(N-1)$–connected.*

Let $\mathcal{E}^{\rightarrow}$ be an E_∞^{rel}–operad. There is an equivalence between the homotopy category of grouplike $\mathcal{E}^{\rightarrow}$–algebras and the homotopy category of relative spectra with connective domain and codomain.

1.1 Structure of the article

In Section 2 generalizations of Quillen adjunctions and idempotent Quillen (co)monads are introduced and will be the underlying machinery in the proof of the main theorem. The homotopy theory of relative loop spaces is presented in the language of monoidal model categories, in particular the category of relative spectra and its model structure are defined.

In Section 3 the theory of relative operads and their algebras is presented, including a description of the Swiss-cheese relative operads $\mathcal{SC}_N$, the $\mathcal{SC}_N$–algebra structure of relative N–loop pairs of spaces and some model-theoretical technical results.

The main results are proved in Section 4.

In the appendix a resolution of relative operads is introduced and used to construct explicit cofibrant resolutions of the Swiss-cheese relative operads out of the Fulton–MacPherson relative operads.

1.2 Notation and terminology

Throughout the language of category theory in Mac Lane [30] will be used. The theory of model categories as presented by Goerss and Jardine [14] and Hirschhorn [15], as well as the theory of monoidal model categories as presented by Hovey [20], is assumed.

By convention in diagrams in a model category $\mathcal{T}$ the morphisms in the class of weak equivalences W are denoted by arrows marked with a tilde $\xrightarrow{\sim}$, the ones in the class of cofibrations C by hooked arrows $\hookrightarrow$ and the ones in the class of fibrations F by double-headed arrows $\twoheadrightarrow$. The functorial weak factorization systems are denoted by $(\mathrm{Fat}_{C,F\cap W}, -_C, -_{F\cap W})$ and $(\mathrm{Fat}_{C\cap W,F}, -_{C\cap W}, -_F)$ such that a morphism $f \in \mathcal{T}(X,Y)$ is factored for instance as

$$X \xhookrightarrow{f_C} \mathrm{Fat}_{C,F\cap W}(f) \xrightarrow[\sim]{f_{F\cap W}} Y.$$

The notation $\mathfrak{C}\colon \mathcal{T} \to \mathcal{T}$ and cof: $\mathfrak{C} \Rightarrow \mathrm{Id}$ is used for the cofibrant resolution functor $\mathfrak{C}X = \mathrm{Fat}_{C,F\cap W}(\varnothing_X)$ and the associated natural trivial fibration $\mathrm{cof}_X = (\varnothing_X)_{F\cap W}$, and the notation $\mathfrak{F}\colon \mathcal{T} \to \mathcal{T}$ and fib: $\mathrm{Id} \Rightarrow \mathfrak{F}$ is used for the fibrant resolution functor $\mathfrak{F}X = \mathrm{Fat}_{C\cap W,F}(*_X)$ and the associated natural trivial cofibration $\mathrm{fib}_X = (*_X)_{C\cap W}$, where $\varnothing_X$ and $*_X$ are the unique morphisms associated with the initial and terminal objects, respectively.

The homotopy category of $\mathcal{T}$ with objects the bifibrant objects of $\mathcal{T}$ and with morphisms between bifibrant objects X and Y the set $\mathcal{T}(X,Y)/\simeq$ of homotopy classes of maps [20, Section 1.2] is denoted by $\mathcal{H}o\,\mathcal{T}$, and $\pi_{\mathcal{T}}\colon \mathcal{T} \to \mathcal{H}o\,\mathcal{T}$ denotes the *homotopy localization functor* with $\pi_{\mathcal{T}}X = \mathfrak{C}\mathfrak{F}X$.

The prototypical example of monoidal model category used is the closed cartesian category Top of compactly generated, weakly Hausdorff spaces as presented by Strickland [44] equipped with the mixed model structure where weak equivalences are weak homotopy equivalences, fibrations are Hurewicz fibrations and cofibrations are maps homotopy equivalent to retracts of relative CW–complexes in Cole [9, Theorem 2.1.16]. In Top all spaces are fibrant and cofibrant spaces are the spaces homotopy equivalent to CW–complexes. The cofibrantly generated Quillen model structure [38] will be assumed when defining the model structure for relative operads. When distinctions are necessary, (co)fibrations in the Strøm model structure are referred to as h–(co)fibrations and (co)fibrations in the Quillen model structure are referred to as q–(co)fibrations.

The theory of monads and their algebras as presented by May [31, Section 9] is assumed.

Acknowledgements

I would like to thank my PhD advisors, Dr Daciberg Lima Gonçalves and Dr Eduardo Hoefel, for their indispensable guidance during the research that resulted in this article. This study was financed in part by CNPq and by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance code 001.

2 Homotopy theory of relative loop spaces

In this section a generalization of the notion of Quillen adjunctions between model categories is introduced, which is referred to as weak Quillen quasiadjunctions, where instead of unit and counit morphisms there is a unit span and a counit cospan, and it is shown that they induce adjunctions on the homotopy categories. The relative N–loop pair of spaces functors Ω_2^N for $N \in \mathbb{N}$ are then defined and it is shown they admit a weak Quillen left adjoint functor $\Sigma_{\rightarrow}^N$, and that this adjunction induces a new model structure with cofibrant objects the $(N-1)$–connected relative CW–complexes. A generalization of left (right) Bousfield localizations via idempotent Quillen (co)monads is presented, which is used in the recognition principle to pick out the subcategory of grouplike algebras and connective relative spectra. The category of relative spectra, its stable model structure, the base pair of spaces functor Λ_2^∞ and the relative ∞–loop pair of spaces functor Ω_2^∞ are also defined and it is shown that Λ_2^∞ admits a weak Quillen left adjoint functor $\Sigma_{\rightarrow}^\infty$.

2.1 Weak Quillen quasiadjunctions

Functors between model categories that are compatible with the model structures induce functors between their respective homotopy categories. Let $\mathcal{T}$ and $\mathcal{A}$ be model categories and $S\colon \mathcal{T} \to \mathcal{A}$ and $\Lambda\colon \mathcal{A} \to \mathcal{T}$ be functors. If S preserves cofibrant objects and weak equivalences between cofibrant objects then S is *left derivable* and the right Kan extension $\mathbb{L}S\colon \mathcal{H}o\,\mathcal{T} \to \mathcal{H}o\,\mathcal{A}$ of $\pi_{\mathcal{A}} S \mathfrak{C}$ along $\pi_{\mathcal{T}}$ with $\mathbb{L}SX = \mathfrak{F}SX$ is the *left derived functor of* S. If Λ preserves fibrant objects and weak equivalences between fibrant objects then Λ is *right derivable* and the left Kan extension $\mathbb{R}\Lambda\colon \mathcal{H}o\,\mathcal{A} \to \mathcal{H}o\,\mathcal{T}$ of $\pi_{\mathcal{T}} \Lambda \mathfrak{F}$ along $\pi_{\mathcal{A}}$ with $\mathbb{R}\Lambda Y = \mathfrak{C}\Lambda Y$ is the *right derived functor of* Λ. An adjunction $(S \dashv \Lambda)\colon \mathcal{T} \rightleftharpoons \mathcal{A}$ is a *weak Quillen adjunction* if S is left derivable and Λ is right derivable.

The standard fact that weak Quillen adjunctions induce adjunctions between homotopy categories is a special case of Theorem 2.1.2. The more common notion of morphism between model categories in the literature, which by Ken Brown's lemma [20, Lemma 1.1.12] is strictly stronger than that of weak Quillen adjunction, is that of Quillen adjunction, which are adjunctions where the left adjoint preserves (trivial) cofibrations and the right adjoint preserves (trivial) fibrations. Not all adjunctions between homotopy categories are induced by Quillen adjunctions or even weak Quillen adjunctions. The following definition, which is central to the relative recognition principle, is a generalization where a pair of functors admit a unit span and a counit cospan:

Definition 2.1.1 Let $\mathcal{T}$ and $\mathcal{A}$ be model categories. A *weak Quillen quasiadjunction* between $\mathcal{T}$ and $\mathcal{A}$, denoted by $(S \dashv_{\mathscr{C},\mathscr{F}} \Lambda)\colon \mathcal{T} \rightleftharpoons \mathcal{A}$, is a quadruple of functors

$S\colon \mathcal{T} \to \mathcal{A}$, the *left Quillen quasiadjoint*, $\Lambda\colon \mathcal{A} \to \mathcal{T}$, the *right Quillen quasiadjoint*, $\mathscr{C}\colon \mathcal{T} \to \mathcal{T}$ and $\mathscr{F}\colon \mathcal{A} \to \mathcal{A}$ equipped with a natural span $\eta'_X\colon \mathscr{C}X \Rightarrow X$ and $\eta_X\colon \mathscr{C}X \Rightarrow \Lambda SX$ and a natural cospan $\epsilon'_Y\colon Y \Rightarrow \mathscr{F}Y$ and $\epsilon_Y\colon S\Lambda Y \Rightarrow \mathscr{F}Y$ such that

- (i) S is left derivable;
- (ii) Λ is right derivable;
- (iii) $\mathscr{C}$ and $\mathscr{F}$ preserve both cofibrant and fibrant objects;
- (iv) η' and ϵ' are natural weak equivalences;
- (v) if $X \in \mathcal{T}$ is cofibrant then $\epsilon_{SX} S\eta_X \simeq \epsilon'_{SX} S\eta'_X$;
- (vi) if $Y \in \mathcal{A}$ is fibrant then $\Lambda\epsilon_Y \eta_{\Lambda Y} \simeq \Lambda\epsilon'_Y \eta'_{\Lambda Y}$.

Note that the definition of weak Quillen adjunction is recovered if $\mathscr{C}$, $\mathscr{F}$, η' and ϵ' are all identities and if the homotopy equations (v) and (vi) are strict equations. Note also that since η' and ϵ' are natural weak equivalences the 2-out-of-3 property implies that $\mathscr{C}$ and $\mathscr{F}$ preserve weak equivalences.

Theorem 2.1.2 *A weak Quillen quasiadjunction* $(S \dashv_{\mathscr{C},\mathscr{F}} \Lambda)\colon \mathcal{T} \rightleftharpoons \mathcal{A}$ *induces a derived adjunction between the homotopy categories* $(\mathbb{L}S \dashv \mathbb{R}\Lambda)\colon \mathcal{H}o\,\mathcal{T} \rightleftharpoons \mathcal{H}o\,\mathcal{A}$.

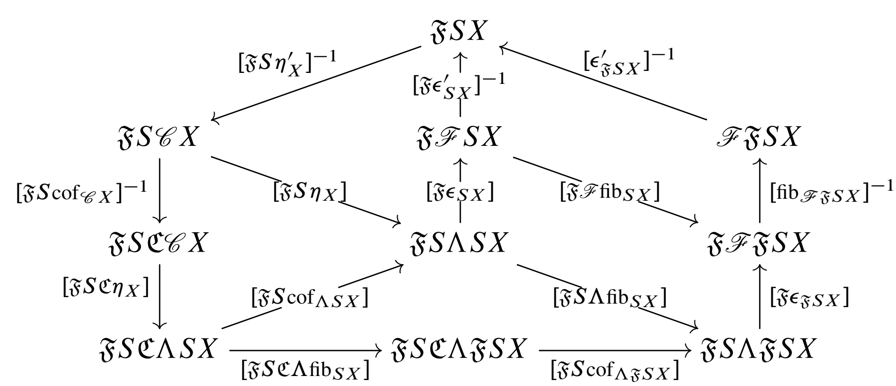
Proof The unit of the adjunction is the composition

$$[\widetilde{\eta}]_X := X \xRightarrow{[(\eta'\mathrm{cof}_{\mathscr{C}})_X]^{-1}} \mathbb{R}\mathscr{C}X \xRightarrow{[\mathfrak{C}(\Lambda\mathrm{fib}_S\,\eta)_X]} \mathbb{R}\Lambda\mathbb{L}SX$$

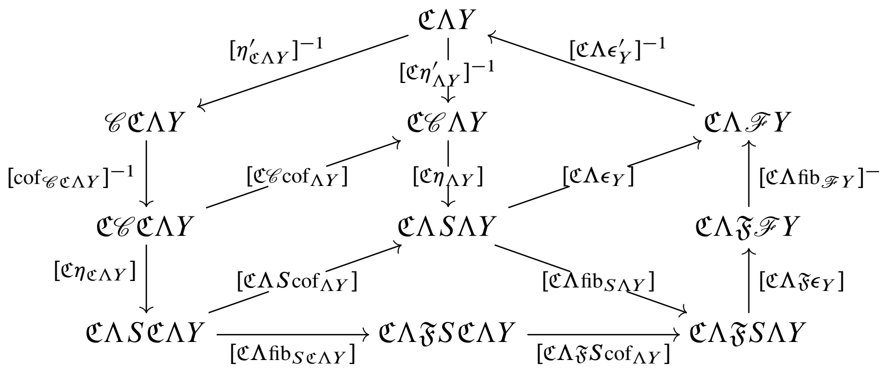
and the counit is the composition

$$[\widetilde{\epsilon}]_Y := \mathbb{L}S\mathbb{R}\Lambda Y \xRightarrow{[\mathfrak{F}(\epsilon S\mathrm{cof}_\Lambda)_Y]} \mathbb{L}\mathscr{F}Y \xRightarrow{[(\mathrm{fib}_{\mathscr{F}}\epsilon')_Y]^{-1}} Y.$$

That the unit–counit equations hold follows from the commutativity of the diagrams



in $\mathcal{Ho}\,\mathcal{A}$ for all $X \in \mathcal{Ho}\,\mathcal{T}$ and



in $\mathcal{Ho}\,\mathcal{T}$ for all $Y \in \mathcal{Ho}\,\mathcal{A}$. □

The following corollary follows immediately:

Corollary 2.1.3 *Let* $(S \dashv_{\mathscr{C},\mathscr{F}} \Lambda)\colon \mathcal{T} \rightleftharpoons \mathcal{A}$ *be a weak Quillen quasiadjunction. If* $\Lambda\mathrm{fib}_{SX}\,\eta_X \in W_{\mathcal{T}}$ *for every cofibrant* $X \in \mathcal{T}$ *and* $\epsilon_Y\,S\mathrm{cof}_{\Lambda Y} \in W_{\mathcal{A}}$ *for every fibrant* $Y \in \mathcal{A}$*, then*

$$(\mathbb{L}S \dashv \mathbb{R}\Lambda)\colon \mathcal{Ho}\,\mathcal{T} \rightleftharpoons \mathcal{Ho}\,\mathcal{A}$$

is an equivalence of categories.

2.2 Finite relative loop pairs of spaces

For $\mathcal{T}$ a monoidal model category the pointed category $\mathcal{T}_*$ of objects under the terminal object inherits a monoidal model structure with tensor product the smash product $Y \wedge X = Y \otimes X \sqcup_{Y \otimes * \sqcup * \otimes X} *$ and exponential the pointed exponential $Y^X = Y^X \times_{Y^*} *$ with the pushout and the pullback taken in $\mathcal{T}$. The unit of the smash product is $* \sqcup \mathbb{1}$. A morphism f is in $W_{\mathcal{T}_*}$, $C_{\mathcal{T}_*}$ or $F_{\mathcal{T}_*}$ if it is in $W_{\mathcal{T}}$, $C_{\mathcal{T}}$ or $F_{\mathcal{T}}$, respectively. In the induced model structure on $\mathtt{Top}_*$ all pointed spaces are fibrant and the cofibrant pointed spaces are the based spaces that are homotopy equivalent to CW–complexes and with the basepoint map a Hurewicz cofibration.

As a special case of the fact that for $X \in \mathcal{T}$ a cofibrant object in a monoidal model category the adjunction $(-\otimes X \dashv -^X)$ is a Quillen adjunction the N–suspension/N–loop space adjunction

$$(\Sigma^N \dashv \Omega^N) := (- \wedge \mathbb{S}^N \dashv -^{\mathbb{S}^N})\colon \mathtt{Top}_* \rightleftharpoons \mathtt{Top}_*$$

is a Quillen adjunction, where $\mathbb{S}^N := I^N/\partial I^N$ with I the real interval $[0,1]$ and the basepoint the equivalence class of the border. Note that trivially Σ^N preserves fibrant objects and that Ω^N preserves cofibrant objects by Fritsch and Piccinini [13, Corollary 5.3.7]. This is a nice advantage of working in the mixed model structure when studying loop spaces.

If $\mathcal{T}$ is a model category then the category of morphisms and commutative squares $\mathcal{T}^{\rightarrow}$ inherits the projective model structure where a commutative square

$$(e,f) \in \mathcal{T}^{\rightarrow}(\kappa\colon A \to X,\, \iota\colon B \to Y)$$

is in $W_{\mathcal{T}^{\rightarrow}}$ or $F_{\mathcal{T}^{\rightarrow}}$ if e and f are both in $W_{\mathcal{T}}$ or $F_{\mathcal{T}}$, respectively, and (e,f) is in $C_{\mathcal{T}^{\rightarrow}}$ if e and $(f,\iota) \in \mathcal{T}(X \sqcup_A B, Y)$ are both in $C_{\mathcal{T}}$. An object in $\mathcal{T}^{\rightarrow}$ is fibrant if it is a morphism between fibrant objects and it is cofibrant if it is a cofibration between cofibrant objects. For instance, in $\mathtt{Top}_*^{\rightarrow}$ all maps are fibrant and the cofibrant maps are homotopy equivalent to inclusions of pointed CW–pairs.

For $N \in \mathbb{N}$, relative N–loop spaces are $(N-1)$–loop spaces of homotopy fibers of maps. They come equipped with a natural inclusion of the N–loop space of the codomain. Let $I \in \mathtt{Top}_*$ be the interval with 1 as the basepoint. For $\iota\colon B \to Y \in \mathtt{Top}_*^{\rightarrow}$ the *homotopy fiber* of ι is the pullback $\mathrm{HFib}(\iota) := B \times_Y Y^I$ in $\mathtt{Top}_*$ induced by ι and the evaluation at 0 map, ie $(b,\gamma) \in \mathrm{HFib}(\iota)$ if $b \in B$ and $\gamma \in Y^I$ (in particular, $\gamma(1) = y_0$) and $\gamma(0) = \iota(b)$.

Definition 2.2.1 The *relative N–loop pair of spaces functor* is

$$\Omega_2^N = (\Omega_c^N, \Omega_o^N)\colon \mathtt{Top}_*^{\rightarrow} \to \mathtt{Top}_*^2, \quad (\iota\colon B \to Y) \mapsto (Y^{\mathbb{S}^N}, \mathrm{HFib}(\iota)^{\mathbb{S}^{N-1}}).$$

Note that there is always a natural map

$$\theta \in \mathtt{Top}_*(Y^{\mathbb{S}^N}, \mathrm{HFib}(\iota)^{\mathbb{S}^{N-1}}), \quad \theta(\gamma)(s) := (b_0, s' \mapsto \gamma(s', s)).$$

This map will be studied as part of the structure described by the Swiss-cheese relative operads. The functor Ω_2^N admits a weak Quillen left adjoint.

Definition 2.2.2 The *relative N–suspension functor* is

$$\Sigma_{\rightarrow}^N\colon \mathtt{Top}_*^2 \to \mathtt{Top}_*^{\rightarrow},$$
$$(X_c, X_o) \mapsto \begin{pmatrix} j_{X_o}^0 \wedge 1_{\mathbb{S}^{N-1}}\colon X_o \wedge \mathbb{S}^{N-1} \to ((X_o \wedge I) \vee (X_c \wedge \mathbb{S}^1)) \wedge \mathbb{S}^{N-1} \\ [x_o, s] \mapsto [x_o, 0, s] \end{pmatrix}.$$

Proposition 2.2.3 *There is a weak Quillen adjunction* $(\Sigma^N_{\to} \dashv \Omega^N_2)$.

Proof The unit is

$$\eta^N_{(X_c,X_o)} \in \mathtt{Top}^2_*((X_c,X_o),\Omega^N_2\Sigma^N_{\to}(X_c,X_o)),$$

$$X_c \xrightarrow{\eta^N_{X_c}} \big(((X_o\wedge I)\vee(X_c\wedge\mathbb{S}^1))\wedge\mathbb{S}^{N-1}\big)^{\mathbb{S}^N}, \quad \eta^N_{X_c}(x_c)(t) := [x_c,t],$$

$$X_o \xrightarrow{\eta^N_{X_o}} \mathrm{HFib}(j^0_{X_o}\wedge 1_{\mathbb{S}^{N-1}})^{\mathbb{S}^{N-1}}, \qquad \eta^N_{X_o}(x_o)(s) := ([x_o,s], s'\mapsto[x_o,s',s]),$$

and the counit is

$$\epsilon^N_\iota \in \mathtt{Top}^{\to}(\Sigma^N_{\to}\Omega^N_2\iota,\iota),$$

$$\begin{array}{ccc} \mathrm{HFib}(\iota)^{\mathbb{S}^{N-1}}\wedge\mathbb{S}^{N-1} & \xrightarrow{\epsilon^N_B} & B \\ \Big\downarrow{\scriptstyle j^0_{\mathrm{HFib}(\iota)^{\mathbb{S}^{N-1}}}\wedge 1_{\mathbb{S}^{N-1}}} & & \Big\downarrow{\scriptstyle\iota} \\ ((\mathrm{HFib}(\iota)^{\mathbb{S}^{N-1}}\wedge I)\vee(Y^{\mathbb{S}^N}\wedge\mathbb{S}^1))\wedge\mathbb{S}^{N-1} & \xrightarrow[\epsilon^N_Y]{} & Y \end{array}$$

$$\epsilon^N_B([(\beta,\gamma),s]) := \beta(s), \qquad \begin{cases} \epsilon^N_Y([(\beta,\gamma),s',s]) := \gamma(s)(s'), \\ \epsilon^N_Y([\alpha,s'',s]) := \alpha(s'',s). \end{cases}$$

It is trivial to check that the unit–counit equations are satisfied.

Wedging by cofibrant objects preserves cofibrations, so the functors $-\wedge\mathbb{S}^{N-1}$, $-\wedge\mathbb{S}^N$ and $-\wedge I$ in particular preserve cofibrant objects, and therefore since cofibrations are closed under coproducts the images of $\Sigma^N_{\to}$ on cofibrant objects are maps between cofibrant objects. The inclusion of a cofibrant object into the base of its cone is a cofibration, therefore $\Sigma^N_{\to}$ preserves cofibrant objects. Ken Brown's lemma and the fact that wedging by cofibrant objects preserves trivial cofibrations imply that $-\wedge\mathbb{S}^N$ preserves weak equivalences between cofibrant objects, and closure of trivial cofibrations under coproducts and Ken Brown's lemma imply that weak equivalences between cofibrant objects are closed under coproducts, and therefore $\Sigma^N_{\to}$ preserves weak equivalences between cofibrant objects.

Clearly Ω^N_2 preserves fibrant objects. There is a natural exact sequence of pointed spaces

$$B^{\mathbb{S}^N} \to Y^{\mathbb{S}^N} \to \mathrm{HFib}(\iota)^{\mathbb{S}^{N-1}} \to B^{\mathbb{S}^{N-1}} \to Y^{\mathbb{S}^{N-1}}$$

for all relative space $\iota\colon B\to Y$ that induces a natural exact sequence of homotopy groups [36, Section 8.6], and therefore Ω^N_2 preserves weak equivalences by the five lemma and the fact that the functors $-^{\mathbb{S}^q}$ preserve weak equivalences. □

Note that $\Sigma^N_{\to}$ preserves cofibrations and trivial cofibrations between cofibrant objects. The adjunction $(\Sigma^N_{\to} \dashv \Omega^N_2)$ and the cofibrantly generated Quillen model structure induces a new model structure on $\mathtt{Top}^{\to}_*$ as in Berger and Moerdijk [1, page 6].

Definition 2.2.4 Let $1 \leq m < \infty$. A relative spaces map

$$(e, f) \in \mathtt{Top}^{\to}_*(\iota\colon A \to X,\ \iota'\colon B \to Y)$$

is a *relative* $(m-1)$*–weak homotopy equivalence* if the induced homomorphisms $f_* \in \mathtt{Grp}(\pi_q X, \pi_q Y)$ are isomorphisms for all $q \geq m$ and if the induced homomorphisms $(e, f^I)_* \in \mathtt{Grp}(\pi_q(A \times_X X^I), \pi_q(B \times_Y Y^I))$ are isomorphisms for all $q \geq m-1$.

This definition is equivalent to $\Omega^N_2(e, f)$ being a pair of weak homotopy equivalences. A pointed space Y is m–connected if $\pi_q(Y)$ is trivial for $0 \leq q \leq m$.

Definition 2.2.5 A relative space $\iota\colon B \to Y \in \mathtt{Top}^{\to}_*$ is m–connected if B is $(m-1)$–connected and Y is m–connected.

The images of $\Sigma^N_{\to}$ are always $(N-1)$–connected. The adjunction $(\Sigma^N_{\to} \dashv \Omega^N_2)$ transfers a model structure on $\mathtt{Top}^{\to}_*$ from the Quillen model structure where the weak equivalences are the relative $(N-1)$–weak homotopy equivalences, all objects are fibrant and cofibrant objects are $(N-1)$–connected relative CW–pairs. When equipped with the mixture of this model structure with the Strøm model structure the category of relative spaces is denoted by $\mathtt{Top}^{\to}_{N-1}$.

Proposition 2.2.6 *There is a weak Quillen adjunction*

$$(\mathrm{Id} \dashv \mathrm{Id})\colon \mathtt{Top}^{\to}_{N-1} \rightleftharpoons \mathtt{Top}^{\to}_*$$

that induces the inclusion of the coreflective homotopy subcategory of $(N-1)$*–connected relative spaces.*

Proof The identity functor trivially preserves fibrant objects and weak homotopy equivalences are $(N-1)$–weak homotopy equivalences, so Id: $\mathtt{Top}^{\to}_* \to \mathtt{Top}^{\to}_{N-1}$ is right derivable. The generating cofibrations of $\mathtt{Top}^{\to}_{N-1}$ are cofibrations in $\mathtt{Top}^{\to}_*$, and $(N-1)$–weak homotopy equivalences between $(N-1)$–connected relative spaces are weak homotopy equivalences, so Id: $\mathtt{Top}^{\to}_{N-1} \to \mathtt{Top}^{\to}_*$ is left derivable. Also every cylinder object in $\mathtt{Top}^{\to}_{N-1}$ is a cylinder object in $\mathtt{Top}^{\to}_*$; therefore, the homotopy relations coincide and $\mathbb{L}\mathrm{Id}$ is full. □

2.3 Bousfield localizations

A left (right) Bousfield localization is a model categorical version of a (co)reflective subcategory, ie, a full subcategory whose inclusion admits a left (right) adjoint, the (co)reflector. In this section results related to left Bousfield localizations are presented explicitly, but they can all be dualized into results about right Bousfield localizations. Bousfield localizations through a generalization of Bousfield and Friedlander's method of Quillen idempotent monads [7, Theorem A.7] will now be constructed. This generalization is required for the same reason weak Quillen quasiadjunctions are needed, which is because in the context of the relative recognition theorem there is a unit, and in the infinite case a counit, only up to resolutions. All proofs in this section are adaptations of the original arguments for Quillen idempotent monads.

Definition 2.3.1 Let $\mathcal{T}$ be a right proper model category. A *Quillen idempotent quasimonad on* $\mathcal{T}$ is a pair of endofunctors $Q\colon \mathcal{T} \to \mathcal{T}$ and $\overline{\mathscr{C}}\colon \mathcal{T} \to \mathcal{T}$ equipped with a natural span $\eta'\colon \overline{\mathscr{C}} \Rightarrow \mathrm{Id}$ and $\eta\colon \overline{\mathscr{C}} \Rightarrow Q$ such that

(i) η' is a natural weak equivalence;

(ii) Q preserves weak equivalences;

(iii) $Q\eta$ and η_Q are natural weak equivalences;

(iv) if $X \in \mathcal{T}$, $p \in F(E,B)$ and $f \in \mathcal{T}(X,B)$ are such that $\eta_E, \eta_B, Qf \in W$, then $Q(\pi_E) \in W(Q(X \times_B E), QE)$;

(v) if $X, K \in \mathcal{T}$ and $\iota \in C(\overline{\mathscr{C}}X, K)$, then $i_K \in W(K, X \sqcup_{\overline{\mathscr{C}}X} K)$.

If $\overline{\mathscr{C}}$ and η' are both identities, then the first and last conditions are trivial and Bousfield and Friedlander's original definition of Quillen idempotent monad is recovered.

That $\iota \in \mathcal{T}^{\to}$ has the left lifting property with regard to $p \in \mathcal{T}^{\to}$, or equivalently that p has the right lifting property with regards to ι, is denoted by $\iota \boxslash p$. If M is a class of morphisms of a category $\mathcal{T}$, the subclass of morphisms with the left lifting property with regards to morphisms in M is denoted by ${}^{\boxslash}M$, and the subclass of morphisms with the right lifting property with regards to morphisms in M is denoted by $M^{\boxslash}$.

Definition 2.3.2 Let $\mathcal{T}$ be a right proper model category and $(Q, \overline{\mathscr{C}}, \eta', \eta)$ be a Quillen idempotent quasimonad on $\mathcal{T}$. The Q*–weak equivalences* are defined as $W_Q := Q^{-1}(W)$, the Q*–cofibrations* as $C_Q := C$ and the Q*–fibrations* as $F_Q := (C_Q \cap W_Q)^{\boxslash}$.

Lemma 2.3.3 *Let $\mathcal{T}$ be a right proper model category and $(Q, \overline{\mathscr{C}}, \eta', \eta)$ be a Quillen idempotent quasimonad on $\mathcal{T}$; then $W_Q \cap F_Q = W \cap F$.*

Proof Let $p \in \mathcal{T}(E,B)$. If $p \in F \cap W$ then $p \in W_Q$ by Definition 2.3.1(ii) and $p \in F_Q$, since $C_Q = C$ implies $p \in C_Q^{\boxslash} \subset (C_Q \cap W_Q)^{\boxslash}$.

Suppose now that $p \in F_Q \cap W_Q$. In the factorization $p = p_{F\cap W} p_C$ it is the case that $p_C \in W_Q$ by Definition 2.3.1(ii) and the fact that W_Q satisfies the 2-out-of-3 property, so $p_C \in C_Q \cap W_Q$. Therefore $p_C \boxslash p$ and so p is a retract of $p_{F\cap W}$ by the retract argument [20, Lemma 1.1.9]. Therefore $p \in F \cap W$ since trivial fibrations are closed under retracts. □

Lemma 2.3.4 *Let $\mathcal{T}$ be a right proper model category, $(Q, \overline{\mathscr{C}}, \eta', \eta)$ be a Quillen idempotent quasimonad on $\mathcal{T}$ and $p \in F(E,B)$. If $\eta_E, \eta_B \in W$ then $p \in F_Q$.*

Proof Let $\kappa \in C_Q \cap W_Q(A,X)$ and $(e,f) \in \mathcal{T}^{\to}(\kappa, p)$. Consider the factorization of $Q(e,f)$

$$\begin{array}{ccccc} QA & \xrightarrow[\sim]{Qe_{C\cap W}} & \mathrm{Fat}_{C\cap W,F}(Qe) & \xrightarrow{Qe_F} & QE \\ {\scriptstyle Q\kappa}\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\downarrow{\scriptstyle \mathrm{Fat}_{C\cap W,F}(Qe,Qf)} & & \downarrow{\scriptstyle Qp} \\ QX & \xrightarrow[Qf_{C\cap W}]{\sim} & \mathrm{Fat}_{C\cap W,F}(Qf) & \xrightarrow[Qf_F]{} & QB \end{array}$$

By Definition 2.3.1(ii) and the 2-out-of-3 property, $\mathrm{Fat}_{C\cap W,F}(Qe,Qf) \in W$. Taking the pullback of the right square along (η_E, η_B), there is the commutative diagram

$$\begin{array}{ccccc} \overline{\mathscr{C}}A & \xrightarrow{(Qe_{C\cap W}\eta_A, \overline{\mathscr{C}}e)} & \mathrm{Fat}_{C\cap W,F}(Qe) \times_{QE} \overline{\mathscr{C}}E & \xrightarrow{\pi_{\overline{\mathscr{C}}E}} & \overline{\mathscr{C}}E \\ {\scriptstyle \overline{\mathscr{C}}\kappa}\downarrow & & \downarrow{\scriptstyle (\mathrm{Fat}_{C\cap W,F}(Qe,Qf), \overline{\mathscr{C}}p)} & & \downarrow{\scriptstyle \overline{\mathscr{C}}p} \\ \overline{\mathscr{C}}X & \xrightarrow[(Qf_{C\cap W}\eta_L, \overline{\mathscr{C}}f)]{} & \mathrm{Fat}_{C\cap W,F}(Qf) \times_{QB} \overline{\mathscr{C}}B & \xrightarrow[\pi_{\overline{\mathscr{C}}B}]{} & \overline{\mathscr{C}}B \end{array}$$

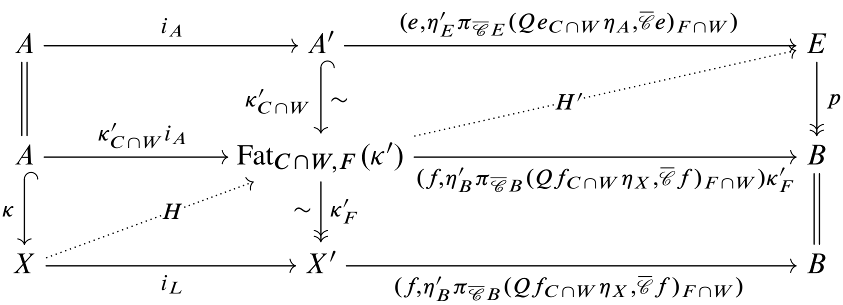
with $(\mathrm{Fat}_{C\cap W,F}(Qe,Qf), \overline{\mathscr{C}}p) \in W$ by the 2-out-of-3 property, $\eta_E, \eta_B \in W$, $\mathcal{T}$ being right proper and the existence of the commutative square

$$\begin{array}{ccc} \mathrm{Fat}_{C\cap W,F}(Qe) \times_{QE} \overline{\mathscr{C}}E & \xrightarrow[\sim]{\pi_{\mathrm{Fat}_{C\cap W,F}(Qe)}} & \mathrm{Fat}_{C\cap W,F}(Qe) \\ {\scriptstyle (\mathrm{Fat}_{C\cap W,F}(Qe,Qf), \overline{\mathscr{C}}p)}\downarrow & & {\scriptstyle\sim}\downarrow{\scriptstyle \mathrm{Fat}_{C\cap W,F}(Qe,Qf)} \\ \mathrm{Fat}_{C\cap W,F}(Qf) \times_{QB} \overline{\mathscr{C}}B & \xrightarrow[\pi_{\mathrm{Fat}_{C\cap W,F}(Qf)}]{\sim} & \mathrm{Fat}_{C\cap W,F}(Qf) \end{array}$$

Now, taking the cofibration/trivial fibration factorization of the left square and then taking the pushout along (η'_A, η'_X), define

$$\begin{aligned} A' &:= A \sqcup_{\overline{\mathscr{C}}A} \mathrm{Fat}_{C,F\cap W}(Qe_{C\cap W}\eta_A, \overline{\mathscr{C}}e), \\ X' &:= X \sqcup_{\overline{\mathscr{C}}X} \mathrm{Fat}_{C,F\cap W}(Qf_{C\cap W}\eta_X, \overline{\mathscr{C}}f), \\ \kappa' &:= \big(\kappa, \mathrm{Fat}_{C,F\cap W}\big(\overline{\mathscr{C}}\kappa, (\mathrm{Fat}_{C\cap W,F}(Qe, Qf), \overline{\mathscr{C}}p)\big)\big)\colon A' \to X' \end{aligned}$$

such that $\kappa' \in W$ by the 2-out-of-3 property and Definition 2.3.1(v). Therefore there is the commutative diagram



with $\kappa'_F \in W$ by the 2-out-of-3 property. Then $H'H \in \mathcal{T}(X, E)$ is a lift of $(e, f) \in \mathcal{T}^{\to}(\kappa, p)$. Therefore $p \in F_Q$. $\square$

Theorem 2.3.5 *Let $\mathcal{T}$ be a right proper model category and $(Q, \overline{\mathscr{C}}, \eta', \eta)$ a Quillen idempotent quasimonad on $\mathcal{T}$; then (W_Q, C_Q, F_Q) is a left Bousfield localization of $\mathcal{T}$.*

Proof Since Q is a functor it preserves composition and isomorphisms, therefore W_Q contains isomorphisms and satisfies the 2-out-of-3 property. It is the case that $(C_Q, F_Q \cap W_Q) = (C, F \cap W)$ by Lemma 2.3.3, so the weak factorization system $(\mathrm{Fat}_{C_Q,F_Q\cap W_Q}, -_{C_Q}, -_{F_Q}) := (\mathrm{Fat}_{C,F\cap W}, -_C, -_F)$ is well defined.

There is also a trivial Q–cofibration/Q–fibration weak factorization system. Since by definition $F_Q = (C_Q \cap W_Q)^{\boxslash}$, and so ${}^{\boxslash}F_Q = {}^{\boxslash}((C_Q \cap W_Q)^{\boxslash}) \supset W_Q \cap C_Q$, the existence of this weak factorization system implies ${}^{\boxslash}F_Q \subset W_Q \cap C_Q$ by the retract argument, the closure of trivial cofibrations under retracts and the fact Q preserves retracts.

Defining $Q'X := \mathrm{Fat}_{C,F\cap W}(\eta_X)$, then $i_{Q'X} \in W(Q'X, X \sqcup_{\overline{\mathscr{C}}X} Q'X)$ by Definition 2.3.1(v). Therefore $\eta_{X\sqcup_{\overline{\mathscr{C}}X} Q'X} \in W$ for all $X \in \mathcal{T}$ since $\overline{\mathscr{C}}$ and Q preserve weak equivalences, W satisfies the 2-out-of-3 property and the following diagram commutes:

$$\begin{array}{ccccc}
\overline{\mathscr{C}}QX & \xleftarrow[\sim]{\overline{\mathscr{C}}\eta_{XF\cap W}} & \overline{\mathscr{C}}Q'X & \xrightarrow[\sim]{\overline{\mathscr{C}}i_{Q'X}} & \overline{\mathscr{C}}(X\sqcup_{\overline{\mathscr{C}}X} Q'X)\\
{\scriptstyle \eta_{QX}}\big\downarrow{\scriptstyle\sim} & & {\scriptstyle \eta_{Q'X}}\big\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle \eta_{X\sqcup_{\overline{\mathscr{C}}X}Q'X}}\\
QQX & \xleftarrow[\sim]{Q\eta_{XF\cap W}} & QQ'X & \xrightarrow[\sim]{Qi_{Q'X}} & Q(X\sqcup_{\overline{\mathscr{C}}X} Q'X)
\end{array}$$

Let $f\in\mathcal{T}(X,Y)$ and apply the trivial cofibration/fibration factorization on $(f,Q'f)$ and the natural span of the Quillen idempotent quasimonad to obtain the diagram

$$\begin{array}{ccccc}
X\sqcup_{\overline{\mathscr{C}}X} Q'X & \xhookrightarrow[\sim]{(f,Q'f)_{C\cap W}} & \mathrm{Fat}_{C\cap W,F}(f,Q'f) & \xrightarrow{(f,Q'f)_F} & Y\sqcup_{\overline{\mathscr{C}}Y} Q'Y\\
{\scriptstyle\sim}\big\uparrow{\scriptstyle \eta'_{X\sqcup_{\overline{\mathscr{C}}X}Q'X}} & & {\scriptstyle\sim}\big\uparrow{\scriptstyle \eta'_{\mathrm{Fat}_{C\cap W,F}(f,Q'f)}} & & {\scriptstyle\sim}\big\uparrow{\scriptstyle \eta'_{Y\sqcup_{\overline{\mathscr{C}}Y}Q'Y}}\\
\overline{\mathscr{C}}(X\sqcup_{\overline{\mathscr{C}}X} Q'X) & \xrightarrow[\sim]{\overline{\mathscr{C}}(f,Q'f)_{C\cap W}} & \overline{\mathscr{C}}\mathrm{Fat}_{C\cap W,F}(f,Q'f) & \xrightarrow{\overline{\mathscr{C}}(f,Q'f)_F} & \overline{\mathscr{C}}(Y\sqcup_{\overline{\mathscr{C}}Y} Q'Y)\\
{\scriptstyle\sim}\big\downarrow{\scriptstyle \eta_{X\sqcup_{\overline{\mathscr{C}}X}Q'X}} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle \eta_{\mathrm{Fat}_{C\cap W,F}(f,Q'f)}} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle \eta_{Y\sqcup_{\overline{\mathscr{C}}Y}Q'Y}}\\
Q(X\sqcup_{\overline{\mathscr{C}}X} Q'X) & \xrightarrow[\sim]{Q(f,Q'f)_{C\cap W}} & Q\mathrm{Fat}_{C\cap W,F}(f,Q'f) & \xrightarrow{Q(f,Q'f)_F} & Q(Y\sqcup_{\overline{\mathscr{C}}Y} Q'Y)
\end{array}$$

By the previous observations, $\eta_{X\sqcup_{\overline{\mathscr{C}}X}Q'X}, \eta_{Y\sqcup_{\overline{\mathscr{C}}Y}Q'Y}\in W$, and by the 2-out-of-3 property $\eta_{\mathrm{Fat}_{C\cap W,F}(f,Q'f)}\in W$, therefore $(f,Q'f)_F\in F_Q$ by Lemma 2.3.4. Also $(f,Q'f)_{C\cap W}\in C_Q\cap W_Q$ since $C=C_Q$ and $W\subset W_Q$.

By Definition 2.3.1(iii) and the 2-out-of-3 property, $Q\eta_{XC}\in W$. Therefore, by the 2-out-of-3 property, $Qi_X\in W$, and so $i_X\in W_Q$ for all $X\in\mathcal{T}$:

$$\begin{array}{ccc}
Q\overline{\mathscr{C}}X & \xrightarrow[\sim]{Q\eta'_X} & QX\\
{\scriptstyle Q\eta_{XC}}\big\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle Qi_X}\\
QQ'X & \xrightarrow[Qi_{Q'X}]{\sim} & Q(X\sqcup_{\overline{\mathscr{C}}X} Q'X)
\end{array}$$

By taking the pullback of $(f,Q'f)_F$ along i_Y, Definition 2.3.1(iv) implies that $\pi_{\mathrm{Fat}_{C\cap W,F}(f,Q'f)}\in W_Q$, which implies $((f,Q'f)_{C\cap W}i_X,f)\in W_Q$ by the 2-out-of-3 property and so $((f,Q'f)_{C\cap W}i_X,f)_C\in C_Q\cap W_Q$ in the diagram

$$\begin{array}{ccccc}
& \mathrm{Fat}_{C,F\cap W}((f,Q'f)_{C\cap W}i_X,f) & & & \\
{\scriptstyle ((f,Q'f)_{C\cap W}i_X,f)_C}\nearrow & & \searrow{\scriptstyle ((f,Q'f)_{C\cap W}i_X,f)_{F\cap W},\ \sim} & & \\
X & \xrightarrow{((f,Q'f)_{C\cap W}i_X,f)} & \mathrm{Fat}_{C\cap W,F}(f,Q'f)\times_{Y\sqcup_{\overline{\mathscr{C}}Y}Q'Y}Y & \xrightarrow{\pi_Y} & Y\\
{\scriptstyle i_X}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathrm{Fat}_{C\cap W,F}(f,Q'f)}} & & \big\downarrow{\scriptstyle i_Y}\\
X\sqcup_{\overline{\mathscr{C}}X}Q'X & \xhookrightarrow[(f,Q'f)_{C\cap W}]{\sim} & \mathrm{Fat}_{C\cap W,F}(f,Q'f) & \xrightarrow[(f,Q'f)_F]{} & Y\sqcup_{\overline{\mathscr{C}}Y}Q'Y
\end{array}$$

Also $\pi_Y \in F_Q$ since fibrations are closed under pullbacks, and so $F \cap W = F_Q \cap W_Q$ implies $\pi_Y((f, Q'f)_{C\cap W} i_X, f)_{F\cap W} \in F_Q$ by Lemma 2.3.3.

Therefore the upper morphisms in the above diagram present a trivial Q–cofibration/Q–fibration weak factorization system. $\square$

Setting $Q'X := \mathrm{Fat}_{C,F\cap W}(\eta_X)$ and $Q''X := X \sqcup_{\overline{\mathscr{C}}X} Q'X$, the following characterization of Q–fibrations can be derived. Homotopy pullback results from [14, Section II.9] will be used, which are presented there in the context of simplicial sets but are valid in any right proper model category.

Proposition 2.3.6 *Let $\mathcal{T}$ be a right proper model category, $(Q, \overline{\mathscr{C}}, \eta', \eta)$ be a Quillen idempotent quasimonad on $\mathcal{T}$ and $p \in \mathcal{T}(E, B)$; then $p \in F_Q$ if and only if the following conditions are satisfied:*

(i) *$p \in F$.*

(ii) *The commutative square*

$$\begin{array}{ccc} E & \xrightarrow{\;i_E\;} & Q''E \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle (p,Q'p)} \\ B & \xrightarrow[\;i_B\;]{} & Q''B \end{array}$$

is a homotopy pullback. Equivalently, because $\mathcal{T}$ is a right proper model category, $(p, (p, Q'p)_W i_E) \in W(E, B \times_{Q''B} K)$ for all $((p, Q'p)_W, (p, Q'p)_F) \in W(Q''E, K) \times F(K, Q''B)$ such that $(p, Q'p) = (p, Q'p)_F (p, Q'p)_W$.

Proof Suppose p satisfies conditions (i) and (ii). There is the factorization $(p, Q'p) = (p, Q'p)_F (p, Q'p)_{C\cap W}$, and by the previous proof $(p, Q'p)_F \in F_Q$. Also $\pi_B \in F_Q(B \times_{Q''B} \mathrm{Fat}_{C\cap W,F}((p, Q'p)), B)$ since fibrations are closed under pullbacks:

$$\begin{array}{ccccc} E & & \xrightarrow{\qquad\qquad i_E \qquad\qquad} & & Q''E \\ & \searrow{\scriptstyle (p,(p,Q'p)_{C\cap W} i_E)} & & & \big\downarrow{\scriptstyle (p,Q'p)_{C\cap W}}\ \sim \\ {\scriptstyle p}\big\downarrow & & B \times_{Q''B} \mathrm{Fat}_{C\cap W,F}((p, Q'p)) \xrightarrow{\pi_{\mathrm{Fat}_{C\cap W,F}((p,Q'p))}} \mathrm{Fat}_{C\cap W,F}((p, Q'p)) & & \\ & \swarrow{\scriptstyle \pi_B} & & & \big\downarrow{\scriptstyle (p,Q'p)_F} \\ B & & \xrightarrow[\qquad\qquad i_B \qquad\qquad]{} & & Q''B \end{array}$$

From the hypothesis $(p, (p, Q'p)_{C\cap W} i_E) \in W$ and the 2-out-of-3 property, it follows that $(p, (p, Q'p)_{C\cap W} i_E)_C \in C \cap W$, and so from the hypothesis $p \in F$ it follows that

$(p,(p,Q'p)_{C\cap W}i_E)_C \boxslash p$. Therefore p is a retract of $\pi_B(p,(p,Q'p)_{C\cap W}i_E)_{F\cap W}$ by the retract argument, and $p \in F_Q$ since $F \cap W = F_Q \cap W_Q$.

Suppose now that $p \in F_Q$. Since $F_Q \subset F$, (i) holds. Let $((p,Q'p)_W,(p,Q'p)_F))$ be as in condition (ii). There is a natural span of weak equivalences $Q'' \overset{\sim}{\Longleftarrow} Q' \overset{\sim}{\Longrightarrow} Q$ by the definition of Q' and by Definition 2.3.1(v), and so $Q''W_Q \subset W$ by the 2-out-of-3 property. Consider the commutative diagram

$$\begin{array}{ccccc}
E & \xrightarrow{\;i_E\;} & Q''E & \xrightarrow[\sim]{Q''i_E} & Q''Q''E \\
{\scriptstyle (p,(p,Q'p)_W i_E)}\big\downarrow & & \sim\big\downarrow{\scriptstyle (p,Q'p)_W} & & \sim\big\downarrow{\scriptstyle Q''(p,Q'p)_W} \\
B \times_{Q''B} K & \xrightarrow{\;\pi_K\;} & K & \xrightarrow[\sim]{\;i_K\;} & Q''K \\
{\scriptstyle \pi_B}\big\downarrow & & \big\downarrow{\scriptstyle (p,Q'p)_F} & & \big\downarrow{\scriptstyle Q''(p,Q'p)_F} \\
B & \xrightarrow[\;i_B\;]{} & Q''B & \xrightarrow[Q''i_B]{\sim} & Q''Q''B
\end{array}$$

By the previous proof, $i_E, i_B \in W_Q$ and $(p,Q'p)_F \in F_Q$. From the 2-out-of-3 property follows $i_K \in W$, and from the naturality of η and the 2-out-of-3 property follows $\eta_K \in W$. Therefore $\pi_K \in W_Q$ by Definition 2.3.1(iv). Also $\pi_B \in F_Q$ by closure under pullbacks. So $(p,(p,Q'p)_W i_E) \in W_Q$ by the 2-out-of-3 property of W_Q. Since every commutative square that contains parallel weak equivalences is a homotopy pullback and pullbacks along fibrations are also homotopy pullbacks, the two lower squares are homotopy pullbacks, and so the lower rectangle is also a homotopy pullback. This lower rectangle is equal to the rectangle

$$\begin{array}{ccccc}
B \times_{Q''B} K & \xrightarrow{\;i_{B\times_{Q''B}K}\;} & Q''(B \times_{Q''B} K) & \xrightarrow[\sim]{Q''\pi_K} & Q''K \\
{\scriptstyle \pi_B}\big\downarrow & & \big\downarrow{\scriptstyle Q''\pi_B} & & \big\downarrow{\scriptstyle Q''(p,Q'p)_F} \\
B & \xrightarrow[\;i_B\;]{} & Q''B & \xrightarrow[Q''i_B]{\sim} & Q''Q''B
\end{array}$$

and since the right commutative square contains parallel weak equivalences it is a homotopy pullback and so the left square is also a homotopy pullback.

Consider the factorization

$$(p,(p,Q'p)_W i_E) = (p,(p,Q'p)_W i_E)_{F\cap W}(p,(p,Q'p)_W i_E)_C.$$

From the 2-out-of-3 property, $(p,(p,Q'p)_W i_E) \in W_Q$ and $F \cap W = F_Q \cap W_Q$, it follows that $(p,(p,Q'p)_W i_E)_C \in C_Q \cap W_Q$. Therefore $(p,(p,Q'p)_W i_E)_C \boxslash p$ and $p \in \mathrm{Ret}\big(\pi_B(p,(p,Q'p)_W i_E)_{F\cap W}\big)$ by the retract argument.

The commutative square of condition (ii) is a retract of

$$
\begin{array}{ccc}
\mathrm{Fat}_{C,F\cap W}\big((p,(p,Q'p)_W i_E)\big) & \xrightarrow{\mathrm{Fat}_{C,F\cap W}(i_E,\, i_{B\times_{Q''B}K})} & \mathrm{Fat}_{C,F\cap W}\big((p,(p,Q'p)_W i_E),\ Q'(p,(p,Q'p)_W i_E)\big) \\
{\scriptstyle\sim}\big\downarrow{\scriptstyle (p,(p,Q'p)_W i_E)_{F\cap W}} & & {\scriptstyle ((p,(p,Q'p)_W i_E),Q'(p,(p,Q'p)_W i_E))_{F\cap W}}\big\downarrow{\scriptstyle\sim} \\
B\times_{Q''B}K & \xrightarrow{\quad i_{B\times_{Q''B}K}\quad} & Q''(B\times_{Q''B}K) \\
{\scriptstyle \pi_B}\big\downarrow & & \big\downarrow{\scriptstyle Q''\pi_B} \\
B & \xrightarrow[\quad i_B\quad]{} & Q''B
\end{array}
$$

with the upper square a homotopy pullback since it contains parallel weak equivalences, and so the whole commutative rectangle is a homotopy pullback. Since homotopy pullbacks are closed under retracts, condition (ii) is satisfied. □

For every weak Quillen quasiadjunction $(S \dashv_{\mathscr{C},\mathscr{F}} \Lambda)\colon \mathcal{T} \rightleftharpoons \mathcal{A}$ there are the associated endofunctor $\Lambda\mathfrak{F}S\mathfrak{C}$ that preserves weak equivalences equipped with the natural span

$$X \xLeftarrow[\sim]{(\mathrm{cof}\eta'_{\mathfrak{C}})_X} \mathscr{C}\mathfrak{C}X \xRightarrow{(\Lambda\mathrm{fib}_S\eta)_{\mathfrak{C}X}} \Lambda\mathfrak{F}S\mathfrak{C}X,$$

and the endofunctor $S\mathfrak{C}\Lambda\mathfrak{F}$ that preserves weak equivalences equipped with the natural cospan

$$S\mathfrak{C}\Lambda\mathfrak{F}Y \xRightarrow{(\epsilon S\mathrm{cof}_\Lambda)_{\mathfrak{F}Y}} \mathscr{F}\mathfrak{F}Y \xLeftarrow[\sim]{(\epsilon'_{\mathfrak{F}}\mathrm{fib})_Y} Y.$$

Definition 2.3.7 Let $(S \dashv_{\mathscr{C},\mathscr{F}} \Lambda)\colon \mathcal{T} \rightleftharpoons \mathcal{A}$ be a weak Quillen quasiadjunction; then it is *idempotent* if $\mathcal{T}$ is right proper, $\mathcal{A}$ left proper and the associated endofunctors and natural transformations above form an idempotent Quillen quasimonad and an idempotent Quillen quasicomonad.

The following theorem is a model-theoretic version of the fact that an adjunction $(S \dashv \Lambda)$ with ΛS a reflector and $S\Lambda$ a coreflector induces an equivalence between the respective subcategories:

Theorem 2.3.8 *Let* $(S \dashv_{\mathscr{C},\mathscr{F}} \Lambda)\colon \mathcal{T} \rightleftharpoons \mathcal{A}$ *be an idempotent weak Quillen quasiadjunction*; *then the adjunction* $(\mathbb{L}S \dashv \mathbb{R}\Lambda)$ *can be factored into an equivalence between the reflective subcategory* $\mathcal{H}o\,\mathcal{T}_{\Lambda\mathfrak{F}S\mathfrak{C}}$ *and the coreflective subcategory* $\mathcal{H}o\,\mathcal{A}_{S\mathfrak{C}\Lambda\mathfrak{F}}$,

$$\mathcal{H}o\,\mathcal{T} \underset{\mathbb{R}\mathrm{Id}}{\overset{\mathbb{L}\mathrm{Id}}{\rightleftarrows}} \mathcal{H}o\,\mathcal{T}_{\Lambda\mathfrak{F}S\mathfrak{C}} \underset{\mathbb{R}\Lambda}{\overset{\mathbb{L}S}{\rightleftarrows}} \mathcal{H}o\,\mathcal{A}_{S\mathfrak{C}\Lambda\mathfrak{F}} \underset{\mathbb{R}\mathrm{Id}}{\overset{\mathbb{L}\mathrm{Id}}{\rightleftarrows}} \mathcal{H}o\,\mathcal{A}.$$

Proof By the definitions, $(\Lambda \mathrm{fib}_S \eta)_X$ is a $\Lambda\mathfrak{F}S\mathfrak{C}$–weak equivalence and $(\epsilon S \mathrm{cof}_\Lambda)_Y$ is an $S\mathfrak{C}\Lambda\mathfrak{F}$–weak equivalence. Therefore the equivalence follows from Corollary 2.1.3. □

The infinite relative recognition theorem, Theorem 4.3.8, is a particular case of this theorem.

2.4 Relative spectra and infinite relative loop pairs of spaces

Infinite loop spaces and their homotopy structure can be described via the category of spectra introduced by Lima [26] equipped with the stable model structure introduced by Bousfield and Friedlander [7]. See also Schwede [40]. The category of spectra is denoted by $\mathtt{Sp}$.

By the closed monoidal structure of $\mathtt{Top}_*$, for every spectrum $Y_\bullet \in \mathtt{Sp}$ with structural maps $\sigma_{\bullet,m} \in \mathtt{Top}_*(Y_\bullet \wedge \mathbb{S}^m, Y_{\bullet+m})$ there is are adjoint structural maps $\sigma^\dagger_{\bullet,m} \in \mathtt{Top}_*(Y_\bullet, Y_{\bullet+m}^{\mathbb{S}^m})$. The q^{th} stable homotopy group of $Y_\bullet$ for $q \in \mathbb{Z}$ is

$$\pi_q^S(Y_\bullet) := \operatorname*{colim}_{\bullet\to\infty} \pi_{q+\bullet}(Y_\bullet),$$

where the colimit is induced by the adjoint structural maps $\sigma^\dagger_{\bullet,m}$. A spectrum $Y_\bullet$ is *connective* if $\pi_q^S(Y_\bullet)$ is trivial for all $q < 0$. A *stable weak homotopy equivalence* is a spectra map that induces an isomorphism on all stable homotopy groups.

The spectra of interest for stable homotopy theory are the Ω–spectra, which are the ones such that the adjoint structural maps $\sigma^\dagger_{\bullet,m}$ are all weak equivalences. These are the fibrant objects of the stable model structure on spectra. The cofibrant spectra are the ones composed of cofibrant spaces with cofibrant structural maps.

In order to define infinite relative loop spaces, the category of maps of spectra that shift the index by 1 is required.

Definition 2.4.1 A *relative spectrum* is a pair of spectra

$$(B_\bullet, Y_\bullet) \in \mathtt{Sp} \times \mathtt{Sp}$$

equipped with a sequence of maps

$$\iota_\bullet \in \prod_{\mathbb{N}} \mathtt{Top}_*(B_\bullet, Y_{\bullet+1})$$

such that the sequence of squares

$$\begin{array}{ccc} B_\bullet \wedge \mathbb{S}^1 & \xrightarrow{\sigma_{\bullet,1}} & B_{\bullet+1} \\ {\scriptstyle \iota_\bullet \wedge 1_{\mathbb{S}^1}}\downarrow & & \downarrow{\scriptstyle \iota_{\bullet+1}} \\ Y_{\bullet+1} \wedge \mathbb{S}^1 & \xrightarrow[\sigma_{\bullet+1,1}]{} & Y_{\bullet+2} \end{array}$$

commute. A relative spectrum is denoted by either $\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1}$ or simply $\iota_\bullet$. A *relative spectra map* between relative spectra $\kappa_\bullet\colon A_\bullet \nearrow X_{\bullet+1}$ and $\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1}$ is a pair of spectra maps $(e_\bullet, f_\bullet) \in \mathtt{Sp}(A_\bullet, B_\bullet) \times \mathtt{Sp}(X_\bullet, Y_\bullet)$ such that the sequence of squares

$$\begin{array}{ccc} A_\bullet & \xrightarrow{e_\bullet} & B_\bullet \\ {\scriptstyle \kappa_\bullet}\downarrow & & \downarrow{\scriptstyle \iota_\bullet} \\ X_{\bullet+1} & \xrightarrow[f_{\bullet+1}]{} & Y_{\bullet+1} \end{array}$$

commute. The category of relative spectra is denoted by $\mathtt{Sp}^{\nearrow}$.

Note that for every relative spectrum $\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1}$ there is an associated exact sequence of groups

$$\cdots \to \pi_q^S Y_\bullet \xrightarrow{(b_{0\bullet},\sigma^\dagger_{\bullet,1})_*} \pi_q^S \mathrm{HFib}(\iota_\bullet) \xrightarrow{(\pi_{B_\bullet})_*} \pi_q^S B_\bullet \xrightarrow{(\iota_\bullet)_*} \pi_{q-1}^S Y_\bullet \to \cdots .$$

As is the case with $\mathtt{Sp}$, the category $\mathtt{Sp}^{\nearrow}$ is bicomplete and the limits and colimits are computed in each index. A model structure on $\mathtt{Sp}^{\nearrow}$ can be built in an analogous way to the stable model structure on $\mathtt{Sp}$. This model structure is a left Bousfield localization through a Quillen idempotent monad on a strict model structure induced by the pointed mixed model structure on topological spaces. In the following the convention that spectra are the terminal space $*$ in negative degrees is used.

Definition 2.4.2 If $\kappa_\bullet\colon A_\bullet \nearrow X_{\bullet+1}$, $\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1} \in \mathtt{Sp}^{\nearrow}$, then a relative spectra map $(e_\bullet, f_\bullet) \in \mathtt{Sp}^{\nearrow}(\kappa_\bullet, \iota_\bullet)$ is a

- *strict weak equivalence* if $e_\bullet, f_\bullet \in W$ for all $\bullet \in \mathbb{N}$;
- *strict cofibration* if
  (i) $(e_\bullet, \sigma_{\bullet-1,1}) \in C(A_\bullet \vee_{A_{\bullet-1}\wedge\mathbb{S}^1} (B_{\bullet-1} \wedge \mathbb{S}^1), B_\bullet)$ for all $\bullet \in \mathbb{N}$, and
  (ii) $((f_\bullet, \iota_{\bullet-1}), \sigma_{\bullet-1,1}) \in C((X_\bullet \vee_{A_{\bullet-1}} B_{\bullet-1}) \vee_{(X_{\bullet-1}\vee_{A_{\bullet-2}} B_{\bullet-2})\wedge\mathbb{S}^1} Y_{\bullet-1} \wedge \mathbb{S}^1, Y_\bullet)$ for all $\bullet \in \mathbb{N}$;
- *strict fibration* if $e_\bullet, f_\bullet \in F$ for all $\bullet \in \mathbb{N}$.

These classes are part of the *strict model structure* on relative spectra and are denoted by $(\overline{W}, \overline{C}, \overline{F})$. The weak factorization systems can be defined inductively from the mixed model structure on $\mathtt{Top}_*$.

The cofibrant objects of the strict model structure are relative spectra such that $\sigma_{\bullet-1,1} \in C(B_{\bullet-1} \wedge \mathbb{S}^1, B_\bullet)$ and $(\iota_{\bullet-1}, \sigma_{\bullet-1,1}) \in C(B_{\bullet-1} \vee_{B_{\bullet-2} \wedge \mathbb{S}^1} (Y_{\bullet-1} \wedge \mathbb{S}^1), Y_\bullet)$ for all $\bullet \in \mathbb{N}$. In particular the underlying spaces of a cofibrant relative spectrum are cofibrant and the structural maps are cofibrations. All relative spectra are fibrant in the strict model structure.

Definition 2.4.3 The *relative spectrification functor* is

$$\widetilde{\Omega}\colon \mathtt{Sp}^{\nearrow} \to \mathtt{Sp}^{\nearrow}, \quad (\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1}) \mapsto \operatorname*{colim}_{q\to\infty}(\iota^{\mathbb{S}^q}_{\bullet+q}\colon B^{\mathbb{S}^q}_{\bullet+q} \nearrow Y^{\mathbb{S}^q}_{\bullet+1+q}),$$

and it is equipped with the natural inclusions $l\colon \mathrm{Id} \Rightarrow \widetilde{\Omega}$ into the colimits.

Proposition 2.4.4 *The functor $\widetilde{\Omega}$ equipped with the natural map l is an idempotent Quillen monad on $\mathtt{Sp}^{\nearrow}$ with the strict model structure.*

A relative spectra map $(e_\bullet, f_\bullet) \in \mathtt{Sp}^{\nearrow}(\kappa_\bullet\colon A_\bullet \nearrow X_{\bullet+1}, \iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1})$ is an $\widetilde{\Omega}$–weak equivalence if $\pi^S_q e_\bullet$ and $\pi^S_q f_\bullet$ are isomorphisms for all $q \in \mathbb{Z}$, and it is an $\widetilde{\Omega}$–fibration if $(e_\bullet, f_\bullet) \in \overline{F}$ and

$$(\sigma^\dagger_{\bullet,1}, e_\bullet) \in W(A_\bullet, A^{\mathbb{S}^1}_{\bullet+1} \times_{B^{\mathbb{S}^1}_{\bullet+1}} B_\bullet) \quad \textit{for all } \bullet \in \mathbb{N},$$

$$(\sigma^\dagger_{\bullet,1}, f_\bullet) \in W(X_\bullet, X^{\mathbb{S}^1}_{\bullet+1} \times_{Y^{\mathbb{S}^1}_{\bullet+1}} Y_\bullet) \quad \textit{for all } \bullet \in \mathbb{N}.$$

A relative spectrum $\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1}$ is $\widetilde{\Omega}$–fibrant if $B_\bullet$ and $Y_\bullet$ are both Ω–spectra.

Proof The conditions in Definition 2.3.1 are satisfied. The strict model structure is right proper since all objects are fibrant. Conditions 2.3.1(i) and 2.3.1(v) are trivially satisfied since $\overline{\mathscr{C}} = \mathrm{Id}$ and $\eta' = 1_{\mathrm{Id}}$. Since each $-^{\mathbb{S}^q}$ preserves weak equivalences, $\widetilde{\Omega}$ preserves weak equivalences and so condition 2.3.1(ii) is satisfied. Condition 2.3.1(iii) is satisfied because $\widetilde{\Omega} = \widetilde{\Omega}\widetilde{\Omega}$. Condition 2.3.1(iv) is satisfied since the mixed model structure on $\mathtt{Top}_*$ is right proper and $\widetilde{\Omega}$ preserves limits and fibrations. That $\widetilde{\Omega}$–weak equivalences are as described follows from the fact that for every spectrum $X_\bullet$ the equations

$$\pi_{\bullet+p} \operatorname*{colim}_{q\to\infty} X^{\mathbb{S}^q}_{\bullet+q} = \operatorname*{colim}_{q\to\infty} \pi_{\bullet+p+q} X_{\bullet+q} = \pi^S_p X_\bullet$$

hold. The description of $\widetilde{\Omega}$–fibrations and $\widetilde{\Omega}$–fibrant relative spectra follows from the characterization of fibrations in Proposition 2.3.6. □

Definition 2.4.5 The *relative base pair of spaces functor* is

$$\Lambda_2^\infty = (\Lambda_c^\infty, \Lambda_o^\infty)\colon \mathtt{Sp}^{\nearrow} \to \mathtt{Top}_*^2, \quad (\iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1}) \mapsto (Y_0, \mathrm{HFib}(\iota_0)).$$

The *relative ∞–loop pair of spaces functor* is $\Omega_2^\infty := \Lambda_2^\infty \widetilde{\Omega}$.

Definition 2.4.6 The *relative suspension spectra functor* is

$$\Sigma_\to^\infty\colon \mathtt{Top}_*^2 \to \mathtt{Sp}^{\nearrow}, \quad (X_c, X_o) \mapsto \begin{pmatrix} j_{X_o}^0 \wedge \mathbb{S}^\bullet\colon X_o \wedge \mathbb{S}^\bullet \nearrow ((X_o \wedge I) \vee (X_c \wedge \mathbb{S}^1)) \wedge \mathbb{S}^\bullet \\ [x_o, s] \mapsto [x_o, 0, s] \end{pmatrix},$$

with $((X_o \wedge I) \vee (X_c \wedge \mathbb{S}^1)) \wedge \mathbb{S}^{-1} := X_c$.

Proposition 2.4.7 *There is a weak Quillen adjunction* $(\Sigma_\to^\infty \dashv \Lambda_2^\infty)$.

Proof The unit is

$$\eta_{(X_c, X_o)}^\infty \in \mathtt{Top}_*^2((X_c, X_o), \Lambda_2^\infty \Sigma_\to^\infty (X_c, X_o)),$$
$$X_c \xrightarrow{\eta_{X_c}^\infty = 1_{X_c}} X_c, \qquad \eta_{X_c}^\infty(x_c) := x_c,$$
$$X_o \xrightarrow{\eta_{X_o}^\infty} \mathrm{HFib}(j_{X_o}^0), \quad \eta_{X_o}^\infty(x_o) := (x_o, s \mapsto (x_o, s)),$$

and the counit is

$$\epsilon_{\iota_\bullet}^\infty \in \mathtt{Sp}^{\nearrow}(\Sigma_\to^\infty \Lambda_2^\infty \iota_\bullet, \iota_\bullet),$$

$$\begin{CD} \mathrm{HFib}(\iota_0) \wedge \mathbb{S}^\bullet @>{\epsilon_{B_\bullet}^\infty}>> B_\bullet \\ @V{j_{\mathrm{HFib}(\iota_0)}^0 \wedge \mathbb{S}^\bullet}VV @VV{\iota_\bullet}V \\ \big((\mathrm{HFib}(\iota_0) \wedge I) \vee (Y_0 \wedge \mathbb{S}^1)\big) \wedge \mathbb{S}^\bullet @>>{\epsilon_{Y_{\bullet+1}}^\infty}> Y_{\bullet+1} \end{CD}$$

$$\epsilon_{B_\bullet}^\infty\big([(b, \gamma), s]\big) := \sigma_{0,\bullet}(b, s),$$
$$\begin{cases} \epsilon_{Y_{\bullet+1}}^\infty([(b, \gamma), s', s]) := \sigma_{1,\bullet+1}(\gamma(s'), s), \\ \qquad \epsilon_{Y_{\bullet+1}}^\infty([y, s', s]) := \sigma_{0,\bullet+1}(y, (s', s)). \end{cases}$$

That the functors are derivable follows from the same arguments as in the proof of Definition 2.2.2. □

3 Relative operads

In this section the theory of relative operads and their algebras is presented. The relative operad $\mathrm{Com}^{\rightarrow}$ of homomorphisms of commutative monoids and the Swiss-cheese relative operads $\mathcal{SC}_N$ are presented, and also the $\mathcal{SC}_N$–algebra structure of relative N–loop pairs of spaces. The Quillen model structure on topological spaces can be transferred to the category of relative operads and their algebras [3], and for algebras over topological relative operads there is also a Strøm model structure and therefore a mixed model structure on the categories of algebras. With this model structure it is shown that some functors constructed with the bar construction and the monads associated with relative operads are left derivable. This follows from a useful result on the compatibility of Reedy model structures, monoidal model structures and coends.

The bar construction will be central to the main result. Let $\mathcal{T}$ and $\mathcal{A}$ be categories. The category $B(\mathcal{T},\mathcal{A})$ is defined as follows: the objects of $B(\mathcal{T},\mathcal{A})$ are triples (F,C,X), with C a monad in $\mathcal{T}$, F a C–functor in $\mathcal{A}$ and X a C–algebra, and morphisms are triples $(\alpha,\phi,f)\in B(\mathcal{T},\mathcal{A})((F,C,X),(F',C',X'))$ with ϕ a monad morphism, f a C–morphism and α a C–functors morphism. The *two-sided bar construction* is the functor

$$B_*\colon B(\mathcal{T},\mathcal{A})\to\mathcal{A}^{\Delta^{\mathrm{op}}},\quad (F,C,X)\mapsto\left(FC^*X,\ \begin{matrix}\partial_i=\begin{cases}\lambda_{C^{*-1}}, & i=0,\\ FC^{i-1}\mu_{C^{*-i}}, & 0<i<*,\\ FC^{*-1}\xi, & i=*,\end{cases}\\ s_i=FC^i\eta_{C^{*-i+1}},\quad 0\le i\le *\end{matrix}\right).$$

The geometric realization $|B_*(F,C,X)|$ is denoted by $B(F,C,X)$. From [31, Sections 9.2 and 11.8] any map $f\in\mathcal{A}(Y,FX)$ determines a map $\tau(f)\in\mathcal{A}(Y,B(F,C,X))$ and any map $g\in\mathcal{A}(FX,Y)$ such that $g\partial_0=g\partial_1\in\mathcal{A}(FCX,Y)$ determines a map $\varepsilon(g)\in\mathcal{A}(B(F,C,X),Y)$. Maps of these types are central to the original recognition theorem and the relative version in Section 4.

3.1 Relative operads and their algebras

Colored operads are a generalization of operads where operations on multiple objects are considered. Relative operads are a kind of colored operads on two colors $\{c,o\}$ that were first defined by Voronov [45].

Definition 3.1.1 A *relative set* is a set A equipped with a function

$$\mathrm{cor}_A\in\mathtt{Set}(A\sqcup\{A\},\{c,o\}),$$

the *coloring of* A, such that if $\operatorname{cor}(A)=c$ then $\operatorname{cor}(a)=c$ for all $a\in A$. A relative set is simply denoted by A and the coloring cor_A simply as cor when its relative set A is obvious from context. A relative set A is *open* if $\operatorname{cor}(A)=o$ and is *closed* if $\operatorname{cor}(A)=c$. The subset of closed and open elements of a relative set A are denoted respectively as A_c and A_o.

A *relative function* f between relative sets A and A' of the same color is a function $f\in \mathtt{Set}(A,A')$ such that $\operatorname{cor}_A=\operatorname{cor}_{A'}f$. There are no relative functions between relative sets of different colors.

Definition 3.1.2 Let $\mathbb{S}_{\mathrm{rel}}$ be the category with objects the closed finite relative sets $\underline{l}:=\{1_c,\dots,l_c\}$ for $l\in\mathbb{N}$ and the open relative sets $\underline{m,n}:=\{1_c,\dots,m_c,1_o,\dots,n_o\}$ for $(m,n)\in\mathbb{N}^2$, and with morphisms the bijective relative functions. The relative sets $\underline{0}$ and $\underline{0,0}$ are respectively the closed and open empty sets. Denote by $\mathbb{S}_c$ and $\mathbb{S}_o$ the full subcategories of respectively closed and open relative sets in $\mathbb{S}_{\mathrm{rel}}$.

Let $\mathbb{S}_{\mathrm{rel}}^{\mathrm{inj}}$ be the category with the same objects as $\mathbb{S}_{\mathrm{rel}}$ but with morphisms the injective relative functions. Denote by S_c^{inj} and $\mathbb{S}_o^{\mathrm{inj}}$ the full subcategories of closed and open objects, respectively.

If $(A,(B^a))\in\mathbb{S}_{\mathrm{rel}}\times\prod_A\mathbb{S}_{\operatorname{cor}(a)}$ then $\Sigma_A B^a$ is the relative set composed of pairs (a,b) with $a\in A$ and $b\in B^a$ equipped with the coloring $\operatorname{cor}(a,b)=\operatorname{cor}(b)$ and $\operatorname{cor}(\Sigma_A B^a)=\operatorname{cor}(A)$. The relative set $\Sigma_A B^a$ is considered an object of $\mathbb{S}_{\mathrm{rel}}$ by equipping it with the linear order $(a,b)<(a',b')$ if either $\operatorname{cor}(b)=c$ and $\operatorname{cor}(b')=o$, or $\operatorname{cor}(b)=\operatorname{cor}(b')$ and either $a<a'$, or $a=a'$ and $b<b'$.

The relative bijections $\mathbb{S}_{\underline{l}}:=\mathbb{S}_{\mathrm{rel}}(\underline{l},\underline{l})$ form a group isomorphic to the symmetric group with l elements and the relative bijections $\mathbb{S}_{\underline{m,n}}:=\mathbb{S}_{\mathrm{rel}}(\underline{m,n},\underline{m,n})$ is a group isomorphic to $\mathbb{S}_{\underline{m}}\times\mathbb{S}_{\underline{n}}$.

Definition 3.1.3 Let $\mathcal{T}$ be a bicomplete symmetric monoidal category. An $\mathbb{S}_{\mathrm{rel}}$*–object in* $\mathcal{T}$ is a functor $\mathcal{Q}\colon\mathbb{S}_{\mathrm{rel}}^{\mathrm{op}}\to\mathcal{T}$ such that $\mathcal{Q}(\underline{0})=\mathbb{1}=\mathcal{Q}(\underline{0,0})$. The category of $\mathbb{S}_{\mathrm{rel}}$–objects in $\mathcal{T}$ is denoted by $\mathbb{S}_{\mathrm{rel}}$–$\mathcal{T}$.

For each $\mathcal{Q}\in\mathbb{S}_{\mathrm{rel}}-\mathcal{T}$ and $A\in\mathbb{S}_{\mathrm{rel}}$, there is a right $\mathbb{S}_A$–action on $\mathcal{Q}(A)$.

Definition 3.1.4 $\mathcal{Q}\in\mathbb{S}_{\mathrm{rel}}-\mathtt{Top}$ is $\mathbb{S}_{\mathrm{rel}}$–free if for each $A\in\mathbb{S}_{\mathrm{rel}}$ the $\mathbb{S}_A$–action on $\mathcal{Q}(A)$ is free.

Definition 3.1.5 A (unital symmetric) relative operad $\mathcal{Q}$ in $\mathcal{T}$ is an $\mathbb{S}_{\mathrm{rel}}$–object in $\mathcal{T}$ equipped with units

$$\eta_c^{\mathcal{Q}} \in \mathcal{T}(\mathbb{1}, \mathcal{Q}(\underline{1})), \quad \eta_o^{\mathcal{Q}} \in \mathcal{T}(\mathbb{1}, \mathcal{Q}(\underline{0,1})),$$

and for each $(A, (B^a)) \in \mathbb{S}_{\mathrm{rel}} \times \prod_A \mathbb{S}_{\mathrm{cor}(a)}$ a composition morphism

$$\circ^{\mathcal{Q}}_{A,(B^a)} \in \mathcal{T}\Big(\mathcal{Q}(A) \otimes \bigotimes_A \mathcal{Q}(B^a), \mathcal{Q}(\Sigma_A B^a)\Big)$$

satisfying associativity, unit and equivariance conditions [35].

For every relative operad $\mathcal{Q}$ in $\mathcal{T}$ there is an associated monad defined by a coend on the category $\mathcal{T}^2_{\mathbb{1}/}$ of pairs of objects under the monoidal unit. For a survey of the many applications of coends see Loregian [28].

Definition 3.1.6 Let $\mathcal{T}$ be a category, $\mathcal{C}$ a small category and $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{T}$ a bifunctor. The *coend* of F is the coequalizer

$$\int^{\mathcal{C}} F(C,C) := \mathrm{Coeq}\Big(\coprod_{\mathcal{C}(C,C')} F(C',C) \rightrightarrows \coprod_{\mathcal{C}} F(C,C)\Big).$$

The underlying functor of $\mathcal{Q}$ can be extended to a functor on $\mathbb{S}^{\mathrm{inj,op}}_{\mathrm{rel}}$. For $\tau \in \mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}(A,B)$ define $(B^b_\tau) \in \bigotimes_B \mathbb{S}_{\mathrm{cor}(b)}$ as $B^b_\tau = \underline{0}$ if $\mathrm{cor}(b) = c$ and $b \notin \mathrm{Im}\,\tau$, $B^b_\tau = \underline{1}$ if $\mathrm{cor}(b) = c$ and $b \in \mathrm{Im}\,\tau$, $B^b_\tau = \underline{0,0}$ if $\mathrm{cor}(b) = o$ and $b \notin \mathrm{Im}\,\tau$, and $B^b_\tau = \underline{0,1}$ if $\mathrm{cor}(b) = o$ and $b \in \mathrm{Im}\,\tau$. Then the right action $\cdot\tau \in \mathcal{T}(\mathcal{Q}(B), \mathcal{Q}(A))$ is the composition of the morphisms $\mathcal{Q}(B) \to \mathcal{Q}(B) \otimes \big(\bigotimes_B \mathcal{Q}(B^b_\tau)\big) \to \mathcal{Q}(A)$ induced by the units and compositions. These morphisms are the *degenerations* of the relative operad. For every $(X_c, X_o) \in \mathcal{T}^2_{\mathbb{1}/}$ define the functor

$$(X_c, X_o)^{\otimes -}\colon \mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}} \to \mathcal{T}, \quad A \mapsto X_c^{\otimes A_c} \otimes X_o^{\otimes A_o},$$

with left action $\tau\cdot \in \mathcal{T}((X_c,X_o)^A, (X_c,X_o)^B)$ for each $\tau \in \mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}(A,B)$ the composition $(X_c,X_o)^A \to (X_c,X_o)^{\mathrm{Im}\,\tau} \otimes \big(\bigotimes_{B\setminus \mathrm{Im}\,\tau} \mathbb{1}\big) \to (X_c,X_o)^B$ induced by the rearrangements of the coordinates and the basepoint maps. A relative operad $\mathcal{Q}$ then defines the monad

$$Q = (Q^c, Q^o)\colon \mathcal{T}^2_{\mathbb{1}/} \to \mathcal{T}^2_{\mathbb{1}/},$$

$$(X_c, X_o) \mapsto \Big(\int^{\mathbb{S}^{\mathrm{inj}}_c} \mathcal{Q}(\underline{l}) \otimes X_c^{\otimes \underline{l}}, \int^{\mathbb{S}^{\mathrm{inj}}_o} \mathcal{Q}(\underline{m,n}) \otimes (X_c, X_o)^{\otimes \underline{m,n}}\Big),$$

with unit and multiplication induced by the operadic unit and composition.

Definition 3.1.7 Let $\mathcal{Q} \in \mathrm{Op}_{\mathrm{rel}}(\mathcal{T})$; then a $\mathcal{Q}$–algebra is an algebra over the associated monad Q. The category of $\mathcal{Q}$–algebras is denoted by $\mathcal{Q}[\mathcal{T}]$.

The terminal topological relative operad is $\mathrm{Com}^{\rightarrow}$ with underlying $\mathbb{S}_{\mathrm{rel}}$–space

$$\mathrm{Com}^{\rightarrow}\colon \mathbb{S}_{\mathrm{rel}}^{\mathrm{op}} \to \mathtt{Top}, \quad A \mapsto *.$$

The structural maps are obvious since $*$ is the terminal space. Algebras $(X_c, X_o) \in \mathrm{Com}^{\rightarrow}[\mathtt{Top}]$ are pairs of topological commutative monoids equipped with a continuous homomorphism from X_c to X_o induced by $* \in \mathrm{Com}^{\rightarrow}(\underline{1,0})$.

The N–Swiss-cheese relative operad $\mathcal{SC}_N$ has as underlying $\mathbb{S}_{\mathrm{rel}}$–space

$$\mathcal{SC}_N\colon \mathbb{S}_{\mathrm{rel}}^{\mathrm{op}} \to \mathtt{Top},$$
$$A \mapsto \big\{d_A = (d_a) \in (I^N)^{\bigsqcup_A I^N} \mid \forall a \in A\, \exists (m^i_{d_a}) \in (0,1]^N, (c^i_{d_a}) \in I^N \big(d_a(t^i_a) = (m^i_{d_a} t_a + c^i_{d_a})\big)$$
$$\text{and } d_a(\mathring{I}^N_a) \cap d_{a'}(\mathring{I}^N_{a'}) = \varnothing \text{ if } a \neq a';\, c^1_{d_a} = 0 \text{ if } \mathrm{cor}(a) = o\big\},$$

ie $\mathcal{SC}_N(A)$ is the subspace of maps in $(I^N)^{\bigsqcup_A I^N}$ defined in each N–cube by coordinatewise dilations and translation with disjoint interiors such that the bases in the first coordinate of the N–cubes with open indices are mapped inside the first coordinate base. The $\mathbb{S}_A$–actions shuffle the map indices. The relative operad structural maps are defined by composition, ie $c_A \circ (d^a_{B^a}) = (c_a d^a_b)$, with the degenerations deleting the respective N–cubes. For example see Figure 2. Define $\mathcal{SC}_\infty := \mathrm{colim}_{\bullet \to \infty} \mathcal{SC}_\bullet$ with the colimit taken over the natural relative operad inclusions $- \times 1_I \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})(\mathcal{SC}_\bullet, \mathcal{SC}_{\bullet+1})$.

Proposition 3.1.8 *Let $1 \leq N \leq \infty$; then the images of Ω^N_2 are naturally $\mathcal{SC}_N$–algebras.*

Proof First let $N < \infty$. Define

$$\theta^N_{2,\iota} \in \mathtt{Top}^2_*(SC_N \Omega^N_2(\iota\colon B \to Y), \Omega^N_2(\iota\colon B \to Y)),$$
$$\theta^N_{c,\iota}([d_{\underline{l}}, \gamma^{\underline{l}}])(t) := \begin{cases} \gamma^i(d_i^{-1}(t)), & t \in d_i(I^N), \\ y_0, & t \notin d_{\underline{l}}\big(\bigsqcup_{\underline{l}} I^N\big), \end{cases}$$
$$\theta^N_{o,\iota}([d_{\underline{m,n}}, ((\alpha,\beta)^{\underline{m}}, \gamma^{\underline{n}})])(s) := \begin{cases} (\alpha^j(d_j^{-1}(0,s)), \Psi), & (0,s) \in d_k(I^N), \\ (b_0, \Xi), & (0,s) \notin d_{\underline{n}}\big(\bigsqcup_{\underline{n}} I^N\big), \end{cases}$$

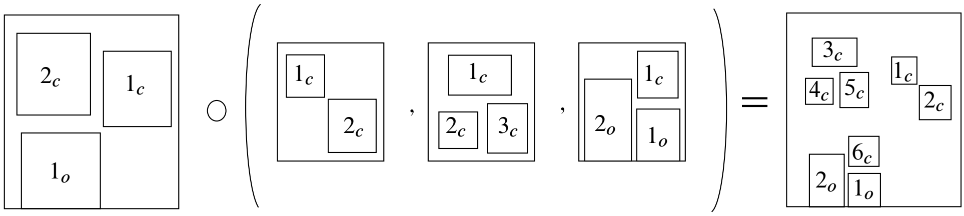


Figure 2: Composition in $\mathcal{SC}_2$.

where

$$\Psi\colon s' \mapsto \begin{cases} \beta^j(d_j^{-1}(0,s))(s'), & (s',s)\in d_j(I^N),\\ \gamma^k(d_k^{-1}(s',s)), & (s',s)\in d_k(I^N),\\ y_0, & (s',s)\notin d_{\underline{m,n}}\big(\bigsqcup_{\underline{m,n}} I^N\big),\end{cases}$$

$$\Xi\colon s' \mapsto \begin{cases} \gamma^k(d_k^{-1}(s',s)), & (s',s)\in d_k(I^N),\\ y_0, & (s',s)\notin d_{\underline{m,n}}\big(\bigsqcup_{\underline{m,n}} I^N\big).\end{cases}$$

It's easy to verify that θ_2^N is compatible with the relative operad structural maps of $\mathcal{SC}_N$.

The $\mathcal{SC}_\infty$–algebra structure on the images of Ω_2^∞ are induced by the finite cases, since by definition an element $d_A \in \mathcal{SC}_\infty(A)$ belongs to some $\mathcal{SC}_N(A)$ with $N<\infty$. □

3.2 Homotopical properties of relative operads and their algebras

Conditions for the existence of a model structure on colored operads in a symmetric model category $\mathcal{T}$ and their algebras are given in [3, Theorem 2.1]. The category $\mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$ admits a cofibrantly generated model structure induced by the Quillen model structure, and for any $\mathcal{Q}\in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$ the category $\mathcal{Q}[\mathtt{Top}]$ admits a model structure with weak equivalences and fibrations being the pairs of maps that are respectively pairs of weak equivalences and pairs of fibrations of the mixed model structure when the $\mathcal{Q}$–algebra structure is forgotten.

A q–cofibrant relative operad $\mathcal{Q}$ satisfies the *homotopy invariance property* [1, Theorem 3.5; 6, Theorem 4.58], which in particular states that if $(X_c,X_o),(Y_c,Y_o)\in\mathcal{T}^2_{\mathbb{1}/}$ are bifibrant, (Y_c,Y_o) admits a $\mathcal{Q}$–algebra structure and $(f_c,f_o)\in\mathcal{T}^2_{\mathbb{1}/}((X_c,X_o),(Y_c,Y_o))$ is a weak equivalence, then (X_c,X_o) admits a $\mathcal{Q}$–algebra structure such that (f_c,f_o) preserves the $\mathcal{Q}$–algebra structure up to homotopy.

The following class of relative operads will be central to the infinite relative recognition principle:

Definition 3.2.1 A topological relative operad $\mathcal{E}^{\to}$ is an $E^{\infty}_{\mathrm{rel}}$*–operad* if it is a q–cofibrant resolution of the terminal relative operad $\mathrm{Com}^{\to}$. Equivalently $\mathcal{E}^{\to}$ is an $E^{\infty}_{\mathrm{rel}}$*–operad* if it is a q–cofibrant relative operad with all underlying spaces contractible.

For $\mathcal{E}^{\to}$ an $E^{\mathrm{rel}}_{\infty}$–operad the homotopy invariance property implies the pair $\Lambda^{\infty}_2 \iota_\bullet = (Y_0, \mathrm{HFib}(\iota_0))$ is an $\mathcal{E}^{\to}$–algebra for bifibrant relative spectra $\iota_\bullet$, and so there is a well-defined functor $\mathbb{R}\Lambda^{\infty}_2\colon \mathcal{H}o\,\mathtt{Sp}^{\nearrow} \to \mathcal{H}o\,\mathcal{E}^{\to}[\mathtt{Top}]$ isomorphic to $\mathbb{R}\Omega^{\infty}_2$.

If $\mathcal{T}$ is a G–projective model category for all small groups G and $\mathcal{R}$ is a dualizable generalized Reedy category, then the functor categories $\mathcal{T}^{\mathcal{R}}$ and $\mathcal{T}^{\mathcal{R}^{\mathrm{op}}}$ admit the Reedy model structure [4, Theorem 1.6]. The categories $\mathbb{S}_{\mathrm{inj}}$ of the last subsection and Δ of isomorphism classes of finite ordinals are dualizable generalized Reedy categories, and $\mathtt{Top}$ is G–projective for all groups G. The following result can be proved by an argument similar to the one for Lurie [29, Proposition A.2.9.26]. The operation $\square$ is the pushout-product of morphisms defined as $\kappa \,\square\, \iota := (\kappa \otimes 1_Y, 1_X \otimes \iota)$.

Proposition 3.2.2 *Let $\mathcal{T}$ be a monoidal model category, $\mathcal{R}$ a dualizable generalized Reedy category such that if $R \in \mathcal{R}$ then $\mathcal{T}$ is $\mathcal{R}_R$–projective and $\kappa \in \mathcal{T}^{\mathcal{R}^{\mathrm{op}}}(A, X)$ and $\iota \in \mathcal{T}^{\mathcal{R}}(B, Y)$ cofibrations; then the morphism*

$$\int^{\mathcal{R}} \kappa^R \,\square\, \iota^R \in \mathcal{T}\left(\int^{\mathcal{R}} A^R \otimes Y^R \sqcup_{A^R \otimes B^R} X^R \otimes B^R, \int^{\mathcal{R}} X^R \otimes Y^R\right)$$

is a cofibration. Also if κ or ι is a trivial cofibration then $\int^{\mathcal{R}} \kappa^R \,\square\, \iota^R \in \mathcal{T}$ is a trivial cofibration.

In the following proposition for $X \in \mathcal{T}^{\mathcal{R}}$ and $R \in \mathcal{R}$ the latching object of X at R is denoted by LX^R [4, page 4]. For the conditions on $\mathcal{T}$ see [3, Theorem 2.1].

Proposition 3.2.3 *Let $\mathcal{T}$ be a symmetric, cofibrantly generated model category equipped with a symmetric monoidal fibrant resolution functor $\mathfrak{F}$ and a cocommutative coalgebra interval object I, $\mathcal{Q}$ a relative operad on $\mathcal{T}$ with cofibrant underlying $\mathbb{S}_{\mathrm{rel}}$–object and cofibrations as units, $\mathcal{A}$ a model category and S a left derivable Q–functor in $\mathcal{A}$ that preserves cofibrations between cofibrant objects; then*

$$B_*(S, Q, -)\colon \mathcal{Q}[\mathcal{T}] \to \mathcal{A}^{\Delta^{\mathrm{op}}}$$

is left derivable.

Proof Let $\overline{\mathbb{1}}$ be the initial relative operad with $\overline{\mathbb{1}}(\underline{0}) = \overline{\mathbb{1}}(\underline{1}) = \overline{\mathbb{1}}(\underline{0,0}) = \overline{\mathbb{1}}(\underline{1,0}) = \mathbb{1}$ and $\overline{\mathbb{1}}(A) := \varnothing$ in all other cases. A $\overline{\mathbb{1}}$–algebra is a pair of objects under $\mathbb{1}$ with no other structure.

If $(\kappa_c, \kappa_o) \in C_{\mathcal{T}^2_{\mathbb{1}/}}((A_c, A_o), (X_c, X_o))$ and $(A_c, A_o) \in \mathcal{T}^2_{\mathbb{1}/}$ is cofibrant, then

$$(\kappa_c, \kappa_o)^{\otimes -} \in C_{\mathcal{T}^{\mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}}}((A_c, A_o)^{\otimes -}, (X_c, X_o)^{\otimes -}),$$

and, by the hypothesis on $\mathcal{Q}$,

$$\eta^{\mathcal{Q}} \in C_{\mathcal{T}^{\mathbb{S}^{\mathrm{inj,op}}_{\mathrm{rel}}}}(\overline{\mathbb{1}}, \mathcal{Q}).$$

Therefore Proposition 3.2.2 applied on each coordinate implies

$$(Q(\kappa_c, \kappa_o), \eta^{\mathcal{Q}}_{(X_c, X_o)}) \in C_{\mathcal{T}^2_{\mathbb{1}/}}(Q(A_c, A_o) \sqcup_{(A_c, A_o)} (X_c, X_o), Q(X_c, X_o)).$$

Also Q preserves cofibrations by closure under compositions and pushouts. Analogously, $\mathcal{Q}$ preserves trivial cofibrations. By Ken Brown's lemma, $\mathcal{Q}$ preserves weak equivalences between cofibrant objects.

Let $(X_c, X_o) \in \mathcal{Q}[\mathcal{T}]$ be cofibrant. Since Q preserves cofibrations, every cofibrant $\mathcal{Q}$–algebra is cofibrant in $\mathcal{T}^2_{\mathbb{1}/}$, and therefore

$$l_0 \in C_{\mathcal{T}^2_{\mathbb{1}/}}\big(LB_0(\mathrm{Id}, Q, (X_c, X_o)) = (\mathbb{1}, \mathbb{1}), (X_c, X_o)\big).$$

Note that for all $q \in \mathbb{N}$ the square in the diagram below is a pushout:

$$\begin{array}{ccc}
LB_{q-1}(\mathrm{Id}, Q, (X_c, X_o)) & \xrightarrow{\;l_{q-1}\;} & Q^{q-1}(X_c, X_o) \\
\downarrow {\scriptstyle \eta^{\mathcal{Q}}_{LB_{q-1}(\mathrm{Id}, Q, -)}} & & \downarrow \\
QLB_{q-1}(\mathrm{Id}, Q, (X_c, X_o)) & \longrightarrow & LB_q(\mathrm{Id}, Q, (X_c, X_o)) \\
 & & \downarrow {\scriptstyle l_q = (Ql_{q-1}, \eta^{\mathcal{Q}}_{Q^{q-1}})} \\
 & & Q^q(X_c, X_o)
\end{array}$$

$$\eta^{\mathcal{Q}}_{Q^{q-1}} : Q^{q-1}(X_c, X_o) \to Q^q(X_c, X_o), \qquad Ql_{q-1} : QLB_{q-1}(\mathrm{Id}, Q, (X_c, X_o)) \to Q^q(X_c, X_o)$$

and therefore, by the argument in the beginning of this proof and by induction, $B_*(\mathrm{Id}, Q, (X_c, X_o)) \in (\mathcal{T}^2_{\mathbb{1}/})^{\Delta^{\mathrm{op}}}$ is cofibrant. Since Q preserves weak equivalences between cofibrant objects, $B_*(\mathrm{Id}, Q, -)$ also preserves weak equivalences between cofibrant objects. Therefore $B_*(\mathrm{Id}, Q, -)$ is left derivable, and the hypothesis on S imply that $B_*(S, Q, -)$ is also left derivable. $\square$

Geometric realization is left derivable, so under the conditions of the proposition if $\mathcal{A}$ is any of the topological categories of Section 2 $B(S, Q, -)\colon \mathcal{Q}[\mathcal{T}] \to \mathcal{A}$ is left derivable.

The following proposition follows by an argument analogous to the one for [31, Proposition 3.4].

Proposition 3.2.4 *Let* $\psi \in \mathrm{Op}(\mathtt{Top})(\mathcal{Q}, \mathcal{Q}')$ *be a weak equivalence between* $\mathbb{S}_{\mathrm{rel}}$*–free relative operads and* $(X_c, X_o) \in \mathtt{Top}^2_*$ *be cofibrant; then*

$$\int^{\mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}} \psi(A) \otimes 1^{\otimes A}_{(X_c, X_o)} \in \mathtt{Top}^2_*(Q(X_c, X_o), Q'(X_c, X_o))$$

is a weak equivalence.

4 Relative recognition principle

The main results are proved in this section. A relative version of the approximation theorem follows from the existence of a quasifibration $p^N_o \in \mathtt{Top}_*(\mathcal{SC}^o_N(X_c, X_o), \mathcal{C}_{N-1}X_o)$ and a commutative square from p^N_o to a fibration

$$\partial \in \mathtt{Top}_*(\Omega^N_o \Sigma^N_{\to}(X_c, X_o), \Omega^{N-1}\Sigma^{N-1}X_o)$$

which is α^{N-1} on the base spaces and homotopy equivalent to α^N on the fibers, from which follows that the total space map is a group completion. Since α^1 is not a group completion, the proof in this article does not apply for the nonconnected cases when $N = 2$ and $N = 1$. After some technical results on the compatibility of geometric realization with Ω^N_2, $\Sigma^N_{\to}$ and the monads associated to relative operads are presented, the relative recognition principle will follow from the relative approximation theorem. The general case for $N = 1$ was proved by Hoefel, Livernet and Stasheff [19].

4.1 Relative approximation theorem

Let's recall the approximation theorem in May [31; 32] and Cohen [8].

Definition 4.1.1 An $\mathcal{H}$–space is a pointed space $X \in \mathtt{Top}_*$ equipped with a map

$$\mu \in \mathtt{Top}_*(X \times X, X)$$

such that

$$\mu(x_0, -) \simeq 1_X \simeq \mu(-, x_0) \in \mathtt{Top}_*(X, X).$$

An $\mathcal{H}$–map between $\mathcal{H}$–spaces X and Y is a pointed map $f \in \mathtt{Top}_*$ such that $f\mu = \mu(f \times f)$. The category of $\mathcal{H}$–spaces is denoted by $\mathcal{H}\text{–}\mathtt{Top}$.

For an $\mathcal{H}$–space (X, μ) the homology groups $H_*(X; k)$ for any commutative coefficient ring k equipped with the Pontryagin product μ_* and the unit $[x_0]$ is a graded k–algebra.

Definition 4.1.2 An $\mathcal{H}$–space X is *homotopy associative* if

$$\mu(-, \mu(-, -)) \simeq \mu(\mu(-, -), -) \in \mathtt{Top}_*(X^{\times 3}, X).$$

The k–algebra structure on $H_*(X; k)$ for a homotopy associative $\mathcal{H}$–space X is associative.

For every $d_{\underline{2}} \in \mathcal{C}_N(\underline{2})$ and $\mathcal{C}_N$–algebras (X, ξ), the map $\xi([d_{\underline{2}}, (-, -)])$ endows X with a homotopy associative $\mathcal{H}$–space structure.

Definition 4.1.3 An $\mathcal{H}$–space X is *admissible* if it is homotopy associative and

$$\mu(x, -) \simeq \mu(-, x) \in \mathtt{Top}_*(X, X)$$

for all $x \in X$.

The k–algebra structure on $H_*(X; k)$ for an admissible $\mathcal{H}$–space X is associative and graded commutative. For $2 \leq N \leq \infty$ the $\mathcal{H}$–space structures on $\mathcal{C}_N$–algebras are admissible.

Definition 4.1.4 A homotopy associative $\mathcal{H}$–space X is *grouplike* if the monoid $\pi_0 X$ is a group. A pair of homotopy associative $\mathcal{H}$–spaces (X_c, X_o) is grouplike if the $\mathcal{H}$–space structures on both X_c and X_o are grouplike.

The N–loop spaces and the relative N–loop pairs of spaces for $2 \leq N \leq \infty$ are grouplike.

Definition 4.1.5 A homological group completion of an admissible $\mathcal{H}$–space X is a grouplike admissible $\mathcal{H}$–space G equipped with an $\mathcal{H}$–map $g \in \mathcal{H}\text{–}\mathtt{Top}(X, G)$ such that for every commutative ring k the induced homomorphism

$$\overline{g}_* \in \mathtt{GrAlg}_k(H_*(X, k)[\pi_0 X^{-1}], H_*(G, k))$$

is an isomorphism, where $[\pi_0 X^{-1}]$ denotes the localization at the subring of connected components.

A homological group completion of a pair of admissible $\mathcal{H}$–spaces (X_c, X_o) is a pair of homological group completions.

For every grouplike homotopy associative $\mathcal{H}$–space X there is a homotopy equivalence between X and $X_0 \times \pi_0 X$, where X_0 is the connected component containing the basepoint [8, Lemma I.4.6]. Since every $\mathcal{H}$–space X is simple, that is, $\pi_1 X$ is abelian and acts trivially on $\pi_q X$ for all q, the dual Whitehead theorem for connected $\mathcal{H}$–spaces implies that a group completion of a grouplike admissible $\mathcal{H}$–space is a weak equivalence.

Let α^N be the composition of the natural transformations $C_N \eta^N\colon C_N \Rightarrow C_N \Omega^N \Sigma^N$ and $\theta^N_{\Sigma^N}\colon C_N \Omega^N \Sigma^N \Rightarrow \Omega^N \Sigma^N$.

Theorem 4.1.6 (approximation theorem) *If $X \in \mathtt{Top}_*$ is connected, α^1_X is a weak equivalence. If $2 \le N \le \infty$ then α^N_X is a homological group completion for all $X \in \mathtt{Top}_*$.*

Corollary 4.1.7 *If $2 \le N \le \infty$ and $X \in \mathcal{C}_N[\mathtt{Top}]$ is grouplike, then α^N_X is a weak equivalence.*

A relative version of these results also holds. Define α^N_2 as the composition of the natural transformations $SC_N \eta^N\colon SC_N \Rightarrow SC_N \Omega^N_2 \Sigma^N_\to$ and $\theta^N_{\Sigma^N_\to}\colon SC_N \Omega^N_2 \Sigma^N_\to \Rightarrow \Omega^N_2 \Sigma^N_\to$, which is explicitly given by

$$\alpha^N_{2(X_c,X_o)} \in (\mathcal{H}\text{–}\mathtt{Top})^2(SC_N(X_c,X_o), \Omega^N_2 \Sigma^N_\to (X_c,X_o)),$$

$$\alpha^N_{c(X_c,X_o)}([d_{\underline{l}}, x^{\underline{l}}])(t) = \begin{cases} [x^i, d_i^{-1}(t)], & t \in d_i(I^N), \\ x^c_0, & t \notin d_{\underline{l}}\big(\bigsqcup_{\underline{l}} I^N\big), \end{cases}$$

$$\alpha^N_{o(X_c,X_o)}([d_{\underline{m,n}}, x^{\underline{m,n}}])(s) = \begin{cases} ([x^k, d_k^{-1}(0,s)], \Psi), & (0,s) \in d_k(I^N),\ k \in \underline{n}, \\ (x^c_0, \Xi), & (0,s) \notin d_{\underline{n}}\big(\bigsqcup_{\underline{n}} I^N\big), \end{cases}$$

where now

$$\Psi\colon s' \mapsto \begin{cases} [x^a, d_a^{-1}(s',s)], & (s',s) \in d_a(I^N),\ a \in \underline{m,n}, \\ x^o_0, & (s',s) \notin d_{\underline{m,n}}\big(\bigsqcup_{\underline{m,n}} I^N\big), \end{cases}$$

$$\Xi\colon s' \mapsto \begin{cases} [x^j, d_j^{-1}(s',s)], & (s',s) \in d_j(I^N),\ j \in \underline{m}, \\ x^o_0, & (s',s) \notin d_{\underline{m,n}}\big(\bigsqcup_{\underline{m,n}} I^N\big). \end{cases}$$

Note that α_c^N is the composition of α^N with the inclusion of the deformation retract $(X_c \wedge \mathbb{S}^N)^{\mathbb{S}^N}$ in $\big(((X_o \wedge I) \sqcup (X_c \wedge \mathbb{S}^1)) \wedge \mathbb{S}^{N-1}\big)^{\mathbb{S}^N}$.

Let $\pi_c, \pi_o\colon \mathtt{Top}_*^2 \to \mathtt{Top}_*$ be the projection functors. The functors SC_N^c and SC_N^o are SC_N–functors, with $\lambda^{SC_N^o} := \pi_o \circ^{SC_N}$ and $\lambda^{SC_N^c} := \pi_c \circ^{SC_N}$. The functor $C_{N-1}\pi_o$ is also an SC_N–functor, with structural map defined as

$$\lambda^{C_{N-1}\pi_o}_{(X_c,X_o)}\big([d_{\underline{l}}, ([d_{\underline{m^i,n^i}}, x^{\underline{m^i,n^i}}])]\big) := [d_{\underline{l}} \circ (d_{\underline{n}^i}\restriction_{\{0\}\times I^{N-1}}), x^{\Sigma_{\underline{l}}\underline{n}^i}].$$

There is also the SC_N–functor map $p_o^N\colon SC_N^o \Rightarrow C_{N-1}\pi_o$ defined as

$$p^N_{o(X_c,X_o)}([d_{\underline{m,n}}, x^{\underline{m,n}}]) := [d_{\underline{n}}\restriction_{\{0\}\times I^{N-1}}, x^{\underline{n}}].$$

Note that $SC_N^o(X_c, X_o)$ is a $\mathcal{C}_{N-1}$–algebra and that p_o^N is also a C_{N-1}–map, and therefore when $N \geq 2$ it is an $\mathcal{H}$–map and when $N > 2$ it is an $\mathcal{H}$–map between admissible $\mathcal{H}$–spaces. Under some mild conditions on $(X_c, X_o) \in \mathtt{Top}_*^2$ the map $p^N_{o(X_c,X_o)}$ is a quasifibration.

Definition 4.1.8 A map $p \in \mathtt{Top}(E, B)$ is a *quasifibration* if the natural inclusions $i_{p^{-1}(b)} \in \mathtt{Top}\big(p^{-1}(b), \{(e,\gamma) \in E \times_B B^I \mid \gamma(1) = b\}\big)$ defined as $i_{p^{-1}(b)}(e) = (e, t \mapsto b)$ are weak equivalences for all $b \in B$. A subspace $U \subset B$ is *distinguished* if $p\restriction_{p^{-1}(U)}$ is a quasifibration.

From [34, Section 2.7] there is the following criterion for a map to be a quasifibration, which considers both h–cofibrations and q–fibrations. Note that these aren't part of the distinguished classes of the mixed model structure.

Proposition 4.1.9 *Let $\phi \in \mathtt{Top}(E, B)$ be a map of filtered spaces such that $F^qE = \phi^{-1}F^qB$ for each $q \in \mathbb{N}$. If for each $q \geq 1$ the map $F^q\phi$ is obtained by pushouts from a commutative diagram of the form*

$$\begin{array}{ccccc} F^{q-1}E & \xleftarrow{g_q} & D^q & \xrightarrow{j_q} & E^q \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi_q} & & \downarrow{\scriptstyle\phi_q} \\ F^{q-1}B & \xleftarrow[f_q]{} & A^q & \xrightarrow[i_q]{} & B^q \end{array}$$

such that

(i) *F^0B is distinguished,*

(ii) *ϕ_q is a q–fibration,*

(iii) i_q *and* j_q *are* h*–cofibrations*,

(iv) *the right square is a pullback*, *and*

(v) $g_q\restriction_{\psi_q^{-1}(a)} \in \mathtt{Top}\big(\psi_q^{-1}(a), \phi^{-1}(f_q(a))\big)$ *are weak equivalences for all* $a \in A^k$,

then each $F^q B$ *is distinguished and* ϕ *is a quasifibration.*

For every topological relative operad $\mathcal{Q}$ and $(X_c, X_o) \in \mathcal{T}_*^2$ the pair of spaces $Q(X_c, X_o)$ admits a natural double filtration. Let $F^{p,q}\mathbb{S}_{\mathrm{rel}}^{\mathrm{inj}}$ be the full subcategory of $\mathbb{S}_{\mathrm{rel}}^{\mathrm{inj}}$ containing the $\underline{l}$ with $l \le p$ and the $\underline{m,n}$ with $m \le p$ and $n \le q$. Define $F^{p,q}Q(X_c, X_o)$ as the images of the natural inclusions of

$$\left(\int^{F^{p,q}\mathbb{S}_c^{\mathrm{inj}}} \mathcal{Q}(\underline{l}) \otimes X_c^{\otimes \underline{l}}, \int^{F^{p,q}\mathbb{S}_o^{\mathrm{inj}}} \mathcal{Q}(\underline{m,n}) \otimes (X_c, X_o)^{\otimes \underline{m,n}}\right)$$

in $Q(X_c, X_o)$. This defines a filtration $F^q Q(X_c, X_o) := \bigcup_{p \in \mathbb{N}} F^{p,q}Q(X_c, X_o)$.

Theorem 4.1.10 *Let* $(X_c, X_o) \in \mathtt{Top}_*^2$ *with* X_o h*–cofibrant*; *then* $p^N_{o(X_c,X_o)}$ *is a quasifibration with fiber* $C_N(X_c)$.

Proof The map p_o^N and the natural filtrations on $C_{N-1}X_o$ and $SC_N(X_c, X_o)$ satisfy the conditions of Proposition 4.1.9.

Let $q \in \mathbb{N}$, $\mathbb{S}_{c,q}$ be the full subcategory of $\mathbb{S}_c$ containing only $\underline{q}$ and $\mathbb{S}_{o,q}$ be the full subcategory of $\mathbb{S}_o^{\mathrm{inj}}$ containing only the $\underline{m,q}$ with $m \in \mathbb{N}$. Define

$$SC_N^q(X_c, X_o) := \int^{\mathbb{S}_{o,q}} \mathcal{SC}_N(\underline{m,q}) \times (X_c, X_o)^{\underline{m,q}}$$

and

$$C_{N-1}^q X_o := \int^{\mathbb{S}_{c,q}} \mathcal{C}_{N-1}(\underline{q}) \times X_o^{\underline{q}}.$$

Define also

$$D_N^q := \{[d_{\underline{m,q}}, x^{\underline{m,q}}] \in SC_N^q(X_c, X_o) \mid x^k = x_0^o \text{ for some } k \in \underline{q}\}$$

and

$$A_N^q := \{[d_{\underline{q}}, x^{\underline{q}}] \in C_{N-1}^q X_o \mid x^k = x_0^o \text{ for some } k \in \underline{q}\}.$$

The maps $F^q p_o^N$ are then the pushouts of

$$\begin{array}{ccccc} F^{q-1}SC_N^o(X_c, X_o) & \xleftarrow{g_q} & D_N^q & \xrightarrow{j_q} & SC_N^q(X_c, X_o) \\ \Big\downarrow{\scriptstyle p_o^N} & & \Big\downarrow{\scriptstyle \psi_q} & & \Big\downarrow{\scriptstyle \phi_q} \\ F^{q-1}C_{N-1}X_o & \xleftarrow[f_q]{} & A_N^q & \xrightarrow[i_q]{} & C_{N-1}^q X_o \end{array}$$

where i_q and j_q are the inclusions, g_q and f_q are induced by the degeneracy of the little cubes with the same index as the x^k that are equal to the basepoint, and ψ_q and ϕ_k are defined in the same way as p_o^N. These diagrams satisfy the conditions in Proposition 4.1.9:

(i) $F^0 C_{N-1} X_o = *$. Since every space is fibrant and every fibration is a quasifibration $F^0 C_{N-1} X_o$ is distinguished.

(ii) By definition, ϕ_q is a q–fibration if for every commutative square

$$\begin{array}{ccc} I^r & \xrightarrow{[d_{\underline{m,q}},\, x^{\underline{m,q}}]} & SC_N^q(X_c, X_o) \\ {\scriptstyle \iota_0^r}\big\downarrow & \overset{\widetilde{H}}{\nearrow} & \big\downarrow{\scriptstyle \phi_q} \\ I^r \times I & \xrightarrow[{[\delta_{\underline{q}},\, \xi^{\underline{q}}]}]{} & C_{N-1}^q X_o \end{array}$$

there is a lift $\widetilde{H}$ that makes the diagram commute.

An element d_A in $\mathcal{SC}_N(A)$ or $\mathcal{C}_{N-1}(A)$ is of the form $d_A(s_a^i) = (m_{d_a}^i s_a^i + c_{d_a}^i)$ for some $(m_{d_a}^i) \in (0,1]^N$ and some $(c_{d_a}^i) \in I^N$ for each $a \in A$. For $(d_{\underline{m,n}}, \delta_{\underline{n}}) \in \mathcal{SC}_N(\underline{m,n}) \times_{\mathcal{C}_{N-1}(\underline{n})} \mathcal{C}_{N-1}(\underline{n})^I$ and $v \in (0,1]$, define

$$\gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}}} \in \mathtt{Top}(I, (I^N)^{\bigsqcup_{a \in \underline{m,n}} I_a^N}),$$

$$\gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}}}(t)(s_a^i) := \begin{cases} \begin{cases} \left(\frac{2v-t}{2v} m^1_{d_a} s^1_a + \frac{(2v-t)c^1_{d_a}+t}{2v},\ \frac{2v-t}{2v} m^i_{d_a} s^i_a + c^i_{d_a}\right), & 0 \le t \le v, \\ \left(\frac{1}{2} m^1_{d_a} s^1_a + \frac{1+c^1_{d_a}}{2},\ \frac{1}{2} m^i_{d_a} s^i_a + c^i_{d_a}\right), & v \le t \le 1, \end{cases} & \operatorname{cor}(a) = c; \\ \begin{cases} \left(\left(\frac{2v-t}{2v}\right) m^1_{d_a} s^1_a + \left(\frac{2v-t}{2v}\right) c^1_{d_a},\ m^i_{\delta_a(t)} s^i_a + c^i_{\delta_a(t)}\right), & 0 \le t \le v, \\ \left(\frac{1}{2} m^1_{d_a} s^1_a + \frac{1}{2} c^1_{d_a},\ m^i_{\delta_a(t)} s^i_a + c^i_{\delta_a(t)}\right), & v \le t \le 1, \end{cases} & \operatorname{cor}(a) = o, \end{cases}$$

ie $\gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}}}$ is the path in $(I^N)^{\bigsqcup_{a \in \underline{m,n}} I_a^N}$ such that

$$\gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}}}(0) = d_{\underline{m,n}}, \quad p_o \gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}}} = \delta_{\underline{n}},$$

the heights in the first coordinate of the open little cubes are linearly halved in the interval $[0,\nu]$ and remain constant in the interval $[\nu,1]$, and such that the side lengths in all coordinates of the closed little cubes and the distance in the first coordinate from the center of the little cube to 1 are also linearly halved in the interval $[0,\nu]$ and remain constant in the interval $[\nu,1]$.

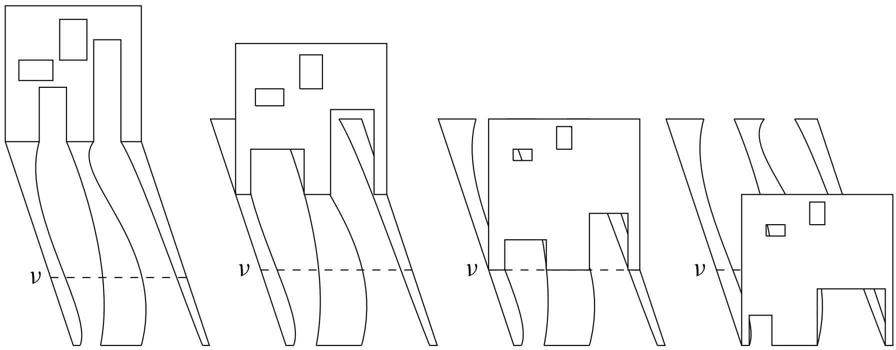


Figure 3: Cross-sections of $\gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}}}$.

For some $(d_{\underline{m,n}}, \delta_{\underline{n}})$ and ν it might be the case that $\gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}}} \notin \mathcal{SC}_N(\underline{m,n})^I$, because there might be $a, a' \in \underline{m,n}$ and $t \in I$ such that $a \neq a'$ and

$$\gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}},a}(t)(\mathring{I}^N) \cap \gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}},a'}(t)(\mathring{I}^N) \neq \varnothing,$$

but there is always some $\nu' \in (0,1]$ such that for $\nu \in (0,\nu']$ these intersections are empty. Define for each $\underline{m,n} \in \mathbb{S}_o$ the map

$$\nu_{m,n} \in \mathtt{Top}(\mathcal{SC}_N(\underline{m,n}) \times_{\mathcal{C}_{N-1}(\underline{n})} \mathcal{C}_{N-1}(\underline{n})^I, (0,1]),$$
$$\nu_{m,n}(d_{\underline{m,n}}, \delta_{\underline{n}}) := \max\{\nu \in (0,1] \mid \gamma^{\nu}_{d_{\underline{m,n}},\delta_{\underline{n}}} \in \mathcal{SC}_N(\underline{m,n})^I\}.$$

For any commutative square as above there is always $\overline{\nu} \in (0,1]$ such that $\gamma^{\overline{\nu}}_{d_{\underline{m,q}}(s),\delta_{\underline{q}}(s)} \in \mathcal{SC}_N(\underline{m,q})^I$ for all $s \in I^q$. Let $F^p\mathbb{S}_{o,q}$ be the full subcategory of $\mathbb{S}_{o,q}$ containing only the $\underline{m,q}$ with $m \le p$. The natural inclusion of $\int^{F^p\mathbb{S}_{o,q}} \mathcal{SC}_N(\underline{m,q}) \times (X_c, X_o)^{\underline{m,q}}$ in $SC^q_N(X_c, X_o)$ induces a filtration $F^pSC^q_N(X_c, X_o)$. Any compact subspace K of $SC^q_N(X_c, X_o)$ is contained in $F^pSC^q_N(X_c, X_o)$ for some $p \in \mathbb{N}$. To see this, assume to the contrary that there is an infinite sequence of points $(z_i) \in K^{\mathbb{N}}$ all lying in distinct $F^pSC^q_N(X_c, X_o)$. Consider the subset $S := \bigsqcup_{\mathbb{N}}\{z_i\} \subset K$. In order to show that S is closed, assume that $S \cap F^{p-1}SC^q_N(X_c, X_o)$ is closed. Then $S \cap F^pSC^q_N(X_c, X_o)$ contains at most one more point. The space $SC^q_N(X_c, X_o)$ is weakly Hausdorff, so points are closed and therefore $S \cap F^pSC^q_N(X_c, X_o)$ is closed. It follows that S is closed. The same argument shows that any subset of S is closed, so S has the discrete topology. Being a closed subset of a compact set, S must be compact. Therefore, S has to be finite, a contradiction.[1] Now the filtration

[1] I thank Eduardo Hoefel for providing the argument in this paragraph through private communication.

$F^pSC_N^q(X_c,X_o)$ and the map $[d_{\underline{m,q}}, x^{\underline{m,q}}]$ from the commutative diagram induce a filtration $F^pI^r := [d_{\underline{m,q}}, x^{\underline{m,q}}]^{-1}(F^pSC_N^q(X_c,X_o))$. Define

$$\nu^p \in \mathtt{Top}(\overline{F^pI^r - F^{p-1}I^r},(0,1]), \quad \nu^p(s) := \nu_{p,q}(d_{\underline{m,q}}(s), \delta_{\underline{q}}(s)).$$

Since I^r is compact, $\overline{F^pI^r - F^{p-1}I^r}$ is compact, and therefore the image of ν^p has a positive minimum. By the two previous observations the minimum

$$\overline{\nu} := \min\{\nu^l(s) \mid p \in \mathbb{N},\ s \in \overline{F^pI^r - F^{p-1}I^r}\}$$

exists. Thus there is

$$\widetilde{H} \in \mathtt{Top}(I^r \times I, SC_N^q(X_c,X_o)), \quad \widetilde{H}(s,t) := [\gamma^{\overline{\nu}}_{d_{\underline{m,q}}(s),\delta_{\underline{q}}(s)}(t), (x^{\underline{m}}(s), \xi^{\underline{q}}(s,t))],$$

which is a lift of $([d_{\underline{m,q}}, x^{\underline{m,q}}], [\delta_{\underline{q}}, \xi^{\underline{q}}])$.

(iii) Since X_o is h–cofibrant, there are maps $u \in \mathtt{Top}(X_o, I)$ and $H \in \mathtt{Top}(X_o \times I, X_o)$ making (x_0^o, X_o) an NDR pair, and so there are the maps $u' \in \mathtt{Top}(C_{N-1}^q X_o, I)$ with $u'([d_{\underline{q}}, x^{\underline{q}}]) := \min\{u(x^i) \mid i \in \underline{q}\}$ and $H' \in \mathtt{Top}(C_{N-1}^q X_o \times I, C_{N-1}^q X_o)$ with $H'([d_{\underline{q}}, x^{\underline{q}}], t) := [d_{\underline{q}}, (H(x^i,t))_{i \in \underline{q}}]$ which makes $(A_N^q, C_{N-1}^q X_o)$ an NDR pair. The pair $(D_N^q, SC_N^q(X_c,X_o))$ is an NDR pair by an analogous argument.

(iv) It is trivial to check that the right square is a pullback.

(v) Fix $[d_{\underline{q}}, x^{\underline{q}}] \in A_N^q$ and define the subspaces P_N^q as

$$\left\{ [d_{\underline{m,q}}, x^{\underline{m,q}}] \in \psi_q^{-1}([d_{\underline{q}}, x^{\underline{q}}]) \;\middle|\; \begin{array}{l} d_j(I) \subset \left[\tfrac{1}{2},1\right] \times I^{N-1} \text{ for all } j \in \underline{m}, \\ d_k(I) \subset \left[0,\tfrac{1}{2}\right] \times I^{N-1} \text{ for all } k \in \underline{q} \end{array} \right\}$$

and Q_N^q as

$$\left\{ [d_{\underline{m,q-1}}, x^{\underline{m,q-1}}] \in (p_o^N)^{-1}(f_q([d_{\underline{q}}, x^{\underline{q}}])) \;\middle|\; \begin{array}{l} d_j(I) \subset \left[\tfrac{1}{2},1\right] \times I^{N-1} \text{ for all } j \in \underline{m}, \\ d_k(I) \subset \left[0,\tfrac{1}{2}\right] \times I^{N-1} \text{ for all } k \in \underline{q-1} \end{array} \right\}.$$

Then P_N^q and Q_N^q are deformation retracts of

$$\psi_q^{-1}([d_{\underline{q}}, x^{\underline{q}}]) \quad \text{and} \quad (p_o^N)^{-1}(f_q([d_{\underline{q}}, x^{\underline{q}}])),$$

respectively, and the restriction $g_q\restriction_{P_q^N}$ is a fibration with contractible fiber, and therefore a weak equivalence. This implies $g_q\restriction_{\psi_q^{-1}([d_{\underline{q}}, x^{\underline{q}}])}$ is also a weak equivalence.

Therefore, by Proposition 4.1.9, p_o^N is a quasifibration. That $C_N X_c$ is the fiber follows easily from the definitions. □

Corollary 4.1.11 *Let* $(X_c, X_o) \in \mathtt{Top}^2_*$ *with* X_o *a cofibrant space. If* $N = 1$ *and* X_c *is connected,* $N = 2$ *and* X_o *is connected, or if* $3 \leq N \leq \infty$*, then* $\alpha_2^N{}_{(X_c, X_o)}$ *is a homological group completion.*

Proof Consider the commutative diagram

$$\begin{array}{ccccccc}
 & C_N X_c & \longrightarrow & SC^o_N(X_c, X_o) & \xrightarrow{p^N_o} & C_{N-1}X_o \\
\Omega^N\Sigma^N X_c \; (\xleftarrow{\alpha^N} C_N X_c) & \downarrow{\scriptstyle \alpha^N_c} & & \downarrow{\scriptstyle \alpha^N_o} & & \downarrow{\scriptstyle \alpha^{N-1}} \\
\Omega^N\Sigma^N X_c \xrightarrow{\sim} & \Omega^N_c \Sigma^N_{\rightarrow}(X_c, X_o) & \longrightarrow & \Omega^N_o \Sigma^N_{\rightarrow}(X_c, X_o) & \xrightarrow[\partial]{} & \Omega^{N-1}\Sigma^{N-1}X_o
\end{array}$$

The maps α^N and α^{N-1} are group completions by Theorem 4.1.6, and therefore α^N_c also is. Note that $H(SC^o_N(X_c, X_o)) \cong H(C_N X_c) \otimes H(C_{N-1}X_o)$, that $\pi_1(C_{N-1}X_o)$ acts trivially on $H(C_N X_c)$, that $\pi_1(\Omega^{N-1}\Sigma^{N-1}X_o)$ acts trivially on $H(\Omega^N\Sigma^N X_c)$ and that both $C_{N-1}X_o$ and $\Omega^{N-1}\Sigma^{N-1}X_o$ are cofibrant spaces. Therefore the induced map on the Serre spectral sequences, which exists by Theorem 4.1.10, implies α^N_o is a group completion. □

4.2 Compatibility of the geometric realization functor

In this section some compatibility results of the geometric realization functor analogous to the ones in [31, Section 12] are stated. Proofs are only sketched since they are simple adaptations of the arguments there.

Proposition 4.2.1 *Let* $(X_c, X_o)_* \in (\mathtt{Top}^2_*)^{\Delta^{\mathrm{op}}}$*. There are natural homeomorphisms*

$$\tau^N \in \mathtt{Top}^{\rightarrow}_*\big(|\Sigma^N_{\rightarrow}(X_c, X_o)_*|, \Sigma^N_{\rightarrow}|(X_c, X_o)_*|\big)$$

for $N < \infty$ *and also*

$$\tau^\infty \in \mathtt{Sp}^{\nearrow}\big(|\Sigma^\infty_{\rightarrow}(X_c, X_o)_*|, \Sigma^\infty_{\rightarrow}|(X_c, X_o)_*|\big).$$

Proof Define

$$\tau^N([[x, s], u]) := [[x, u], s];$$

it can be directly checked that this defines a homeomorphism. □

Proposition 4.2.2 *Let* $(X_c, X_o)_* \in (\mathtt{Top}^2_*)^{\Delta^{op}}$ *and* $\mathcal{Q} \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$*. There is a natural homeomorphism*

$$\nu^{\mathcal{Q}} \in \mathtt{Top}^2_*(|Q(X_c, X_o)_*|, Q|(X_c, X_o)_*|)$$

such that the following diagrams commute:

$$\begin{array}{ccc} |(X_c, X_o)_*| & \xrightarrow{|\eta_*|} & |Q(X_c, X_o)_*| \\ & \searrow^{\eta} & \downarrow \nu^{\mathcal{Q}} \\ & & Q|(X_c, X_o)_*| \end{array} \qquad \begin{array}{ccc} |QQ(X_c, X_o)_*| & \xrightarrow{\nu^{\mathcal{Q}}\nu^{\mathcal{Q}}} & QQ|(X_c, X_o)_*| \\ \downarrow |\mu_*| & & \downarrow \mu \\ |Q(X_c, X_o)_*| & \xrightarrow[\nu^{\mathcal{Q}}]{} & Q|(X_c, X_o)_*| \end{array}$$

If $(X_c, X_o)_* \in \mathcal{Q}[\mathtt{Top}]^{\Delta^{op}}$*, then* $|(X_c, X_o)_*| \in \mathcal{Q}[\mathtt{Top}]$*; therefore, geometric realization defines a functor* $\mathcal{Q}[\mathtt{Top}]^{\Delta^{op}} \to \mathcal{Q}[\mathtt{Top}]$*.*

Proof Define

$$\nu^{\mathcal{Q}}([[q, (x^a)], u]) = [q, ([x^a, u])];$$

it can be directly checked that this defines a homeomorphism and that the diagrams commute. The second statement follows directly from the first. □

Proposition 4.2.3 *Let* $1 \le N < \infty$ *and* $\iota_*\colon B_* \to Y_* \in (\mathtt{Top}^{\to}_*)^{\Delta^{op}}$ *be* h*–cofibrant with each* ι_* $(N-1)$*–connected; then there is a weak equivalence*

$$\gamma^N_2 \in \mathtt{Top}^2_*(|\Omega^N_2 \iota_*|, \Omega^N_2 |\iota_*|).$$

Let $\iota_{\bullet,*}\colon B_{\bullet,*} \nearrow Y_{\bullet+1,*} \in (\mathtt{Sp}^{\nearrow})^{\Delta^{op}}$ *be* h*–cofibrant with each* $\iota_{\bullet,*}$ $\bullet$*–connected; then there is a weak equivalence*

$$\gamma^\infty_2 \in \mathtt{Top}^2_*(|\Omega^\infty_2 \iota_{\bullet,*}|, \Omega^\infty_2 |\iota_{\bullet,*}|).$$

Proof Define

$$\gamma^N_c([\alpha, u])(t) := [\alpha(t), u], \quad \gamma^N_o\big([(\beta, \gamma), u]\big)(s) := [(\beta(s), s' \mapsto \gamma(s)(s')), u].$$

These are pairs of maps of fibers of simplicial quasifibrations over the same simplicial spaces with contractible total spaces, with the contraction compatible with the simplicial structure. □

Proposition 4.2.4 *Let* $(X_c, X_o)_* \in (\mathtt{Top}^2_*)^{\Delta^{op}}$ *and* $1 \le N \le \infty$*. The maps* γ^N_2 *are* $\mathcal{SC}_N$*–maps and the following diagram commutes:*

$$\begin{array}{ccc} |SC_N(X_c,X_o)_*| & \xrightarrow{\nu^{SC_N}} & SC_N|(X_c,X_o)_*| \\ {\scriptstyle |\alpha^N_{2*}|}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^N_2} \\ |\Omega^N_2\Sigma^N_\to(X_c,X_o)_*| & \xrightarrow[\tau^N\gamma^N_2]{} & \Omega^N_2\Sigma^N_\to|(X_c,X_o)_*| \end{array}$$

Proof This can be checked directly from the definitions. □

4.3 Recognition principle for relative loop pairs of spaces

The relative recognition principle will be proved by showing that the relative N–loop pairs of spaces functors are part of weak Quillen quasiadjunctions that induce equivalences of certain homotopy subcategories.

Definition 4.3.1 Let $N \in \mathbb{N}$ and $\mathcal{Q}_N \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$ be equipped with a weak equivalence $\mathfrak{v}^N \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})(\mathcal{Q}_N, \mathcal{SC}_N)$. The *relative N–delooping functor of $\mathcal{Q}_N$–algebras* is

$$B^N_\to\colon \mathcal{Q}_N[\mathtt{Top}] \to \mathtt{Top}^\to_*, \quad (X_c,X_o) \mapsto B(\Sigma^N_\to, \mathcal{Q}_N, (X_c,X_o)).$$

with the $\mathcal{Q}_N$–functor structure of $\Sigma^N_\to$ induced by $\mathfrak{v}^N$, α^N_2 and the $(\Sigma^N_\to \dashv \Omega^N_2)$ adjunctions.

Let $\mathcal{Q}_\infty \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$ be equipped with a sequence $\mathcal{Q}_\bullet \in \prod_{\mathbb{N}} \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$ of relative operads, a sequence of weak equivalences $\mathfrak{v}^\bullet \in \prod_{\mathbb{N}\sqcup\{\infty\}} \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})(\mathcal{Q}_\bullet, \mathcal{SC}_\bullet)$, a sequence of maps $i^\bullet \in \prod_{\mathbb{N}} \mathrm{Op}_{\mathrm{rel}}(\mathcal{Q}_\bullet, \mathcal{Q}_{\bullet+1})$ that commute up to homotopy with the inclusions $- \times 1_I \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})(\mathcal{SC}_\bullet, \mathcal{SC}_{\bullet+1})$, and a homotopy equivalence $\mathfrak{u} \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})(\mathrm{colim}_{\bullet\to\infty} \mathcal{Q}_\bullet, \mathcal{Q}_\infty)$. The *relative infinite delooping functor of $\mathcal{Q}_\infty$–algebras* is

$$B^\infty_\to\colon \mathcal{Q}_\infty[\mathtt{Top}] \to \mathtt{Sp}^\nearrow, \quad (X_c,X_o) \mapsto B(\Sigma^{\bullet+1}_\to, \mathcal{Q}_{\bullet+1}, (X_c,X_o)),$$

with the structural spectra maps induced by the $i^\bullet$.

The q–cofibrant resolutions $\mathfrak{W}_I\mathcal{FM}_N$ of $\mathcal{SC}_N$ in the appendix give explicit examples of relative operads that can be equipped as in this definition.

Definition 4.3.2 Let $\mathcal{Q} \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$. The *bar resolution of $\mathcal{Q}$–algebras* is

$$\overline{B}\colon \mathcal{Q}[\mathtt{Top}] \to \mathcal{Q}[\mathtt{Top}], \quad (X_c,X_o) \mapsto B(\mathcal{Q},\mathcal{Q},(X_c,X_o)).$$

Theorem 4.3.3 *Let $3 \leq N \leq \infty$, $\mathcal{Q}_N \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$ be $\mathbb{S}_{\mathrm{rel}}$–free and equipped as in Definition 4.3.1, and $((X_c, X_o), \xi) \in \mathcal{Q}_N[\mathtt{Top}]$ with X_c h–cofibrant and X_o m–cofibrant in $\mathtt{Top}_*$. Consider the following diagram of $\mathcal{Q}_N$–maps, with the $\mathcal{Q}_N$–algebra structures on the images of Ω_2^N induced by $\mathfrak{v}^N$:*

$$\begin{array}{ccc} \bar{B}(X_c, X_o) & \xrightarrow{B(\alpha_2^N \mathfrak{v}^N, 1, 1)} & B(\Omega_2^N \Sigma_\to^N, \mathcal{Q}_N, (X_c, X_o)) \\ {\scriptstyle \varepsilon(\xi)}\downarrow & & \downarrow{\scriptstyle \gamma_2^N} \\ (X_c, X_o) & & \Omega_2^N B_\to^N (X_c, X_o) \end{array}$$

Then:

(i) *$\varepsilon(\xi)$ is a strong deformation retract with right inverse $\tau(\eta^{\mathcal{Q}_N})$, and therefore a weak equivalence.*

(ii) *$B(\alpha_2^N \mathfrak{v}^N, 1, 1)$ is a group completion, and therefore a weak equivalence if (X_c, X_o) are grouplike.*

(iii) *γ_2^N is a weak equivalence.*

(iv) *In $\mathtt{Top}_*^2$ the composition $\gamma_2^N B(\alpha_2^N \mathfrak{v}^N, 1, 1)\tau(\eta^{\mathcal{Q}_N})$ coincides with the composition $\Omega_2^N(\tau(1_{\Sigma_\to^N}))\eta_2^N$ and it is a weak equivalence if (X_c, X_o) is grouplike.*

(v) *$B_\to^N(X_c, X_o)$ is $(N-1)$–connected if $N < \infty$ and is connective if $N = \infty$.*

(vi) *If $N < \infty$ and $\iota \in \mathtt{Top}_*^\to$ is an $(N-1)$–connected relative space then*

$$\varepsilon(\epsilon_\iota^N) \in W_{\mathtt{Top}_*^\to}(B_\to^N \Omega_2^N \iota, \iota).$$

In general the following diagram is commutative:

$$\begin{array}{ccc} \bar{B}\Omega_2^N \iota & \xrightarrow{B(\alpha_2^N \mathfrak{v}^N, 1, 1)} & B(\Omega_2^N \Sigma_\to^N, \mathcal{Q}_N, \Omega_2^N \iota) \\ {\scriptstyle \varepsilon(\theta_2^N)}\downarrow & \swarrow{\scriptstyle \varepsilon(\Omega_2^N \epsilon_\iota^N)} & \downarrow{\scriptstyle \gamma_2^N} \\ \Omega_2^N \iota & \xleftarrow[\Omega_2^N \varepsilon(\epsilon_\iota^N)]{} & \Omega_2^N B_\to^N \Omega_2^N \iota \end{array}$$

and $\Omega_2^N \varepsilon(\epsilon_\iota^N)$ is a retraction with right inverse $\Omega_2^N(\tau(1_{\Sigma_\to^N \Omega_2^N}))\eta_{\Omega_2^N}^N$.
If $N = \infty$ and $\iota_\bullet \in \mathtt{Sp}^\nearrow$ is connective then

$$\varepsilon(\epsilon_{\widetilde{\Omega}\iota_\bullet}^\infty) \in W_{\mathtt{Sp}^\nearrow}(B_\to^\infty \Omega_2^\infty \iota_\bullet, \widetilde{\Omega}\iota_\bullet).$$

In general the following diagram is commutative:

$$\begin{array}{ccc}
\bar{B}\Omega_2^\infty \iota_\bullet & \xrightarrow{B(\alpha_2^\infty \mathfrak{v}^\infty,1,1)} & B(\Omega_2^\infty \Sigma_\to^\infty, Q_\infty, \Omega_2^\infty \iota_\bullet) \\
{\scriptstyle \varepsilon(\theta_2^\infty)}\downarrow & \swarrow{\scriptstyle \varepsilon(\Omega_2^\infty \epsilon^\infty_{\tilde{\Omega}\iota_\bullet})} & \downarrow{\scriptstyle \gamma_2^\infty} \\
\Omega_2^\infty \iota_\bullet & \xleftarrow[\Omega_2^\infty \varepsilon(\epsilon^\infty_{\tilde{\Omega}\iota_\bullet})]{} & \Omega_2^\infty B_\to^\infty \Omega_2^\infty \iota_\bullet
\end{array}$$

and $\Omega_2^\infty \varepsilon(\epsilon^\infty_{\tilde{\Omega}\iota_\bullet})$ *is a retraction with right inverse* $\Omega_2^\infty(\tau(1_{\Sigma_\to^\infty \Omega_2^\infty}))\eta^\infty_{\Omega_2^\infty}$.

(vii) *For all* $(Y_c, Y_o) \in \mathtt{Top}_*^2$ *the map*

$$\varepsilon(\epsilon^N_{\Sigma_\to^N(Y_c,Y_o)} \Sigma_\to^N B(\alpha_2^N \mathfrak{v}^N, 1, 1)) \in \begin{cases} \mathtt{Top}_*^\to(B_\to^N Q_N(Y_c, Y_o), \Sigma_\to^N(Y_c, Y_o)), & N < \infty, \\ \mathtt{Sp}^\nearrow(B_\to^\infty Q_\infty(Y_c, Y_o), \Sigma_\to^\infty(Y_c, Y_o)), & N = \infty, \end{cases}$$

is a strong deformation retract with right inverse $\tau(\Sigma_\to^N \eta^{Q_N})$.

Proof The maps $\varepsilon(\xi)$ and $B(\alpha_2^N \mathfrak{v}, 1, 1)$ are realizations of simplicial Q_N–maps, therefore by Proposition 4.2.2 they are Q_N–maps, and γ_2^N is a Q_N–map by Proposition 4.2.4.

(i) and (vii) hold on the level of simplicial spaces by [31, Theorems 9.10 and 9.11] and therefore hold after realization by [31, Corollary 11.10].

(ii) holds on the level of simplicial spaces by Proposition 3.2.4 and Corollary 4.1.11 and therefore holds after realization by the argument in [32, Theorem 2.3.ii)].

(iii) follows from Proposition 4.2.3.

(iv) follows from (i), (ii) and (iii).

(v) follows from [31, Theorem 11.12; 32, Remark A.5].

The upper triangle in (vi) commutes by the naturality of ε, and the lower triangle by [31, Theorem 9.11]. The fact that $\varepsilon(\epsilon^N)$ is a weak equivalence under the stated connectivity conditions follows from the commutativity of the diagram and the previous items. □

The relative delooping of relative N–loop pairs of spaces is unique up to weak equivalence among $(N-1)$–connected relative spaces if $N < \infty$ and among connective relative spectra if $N = \infty$.

Corollary 4.3.4 *Under the hypothesis of the theorem consider a span of $\mathcal{Q}_N$–weak equivalences*

$$(X_c, X_o) \xleftarrow[\sim]{f} (Y_c, Y_o) \xrightarrow[\sim]{g} \Omega_2^N \iota.$$

If $N < \infty$ and $\iota \in \mathtt{Top}_*^{\rightarrow}$ is $(N-1)$–connected, then the diagram of relative spaces

$$B_{\rightarrow}^N(X_c, X_o) \xleftarrow[\sim]{B(1,1,f)} B_{\rightarrow}^N(Y_c, Y_o) \xrightarrow[\sim]{\varepsilon(\epsilon_\iota^N)B(1,1,g)} \iota$$

displays a weak equivalence between ι and $B_{\rightarrow}^N(X_c, X_o)$.

If $N = \infty$ and $\iota_\bullet \in \mathtt{Sp}^{\nearrow}$ is connective, then the diagram of relative spectra

$$B_{\rightarrow}^\infty(X_c, X_o) \xleftarrow[\sim]{B(1,1,f)} B_{\rightarrow}^\infty(Y_c, Y_o) \xrightarrow[\sim]{\varepsilon(\epsilon_{\tilde{\Omega}\iota_\bullet}^\infty)B(1,1,g)} \tilde{\Omega}\iota_\bullet \xleftarrow[\sim]{l_{\iota_\bullet}} \iota_\bullet$$

displays a stable weak equivalence between $\iota_\bullet$ and $B_{\rightarrow}^N(X_c, X_o)$.

Proof The map $\varepsilon(\epsilon_{\tilde{\Omega}\iota_\bullet}^N)$ is a weak equivalence by Theorem 4.3.3(vi), and $B(1,1,f)$ and $B(1,1,g)$ are weak equivalences on the level of simplicial spaces by Proposition 3.2.3, and so their realizations are weak equivalences by [31, Theorem 11.13]. □

The relative operads $\mathcal{Q}_N$ must be assumed to be q–cofibrant in order for the functors involved to be compatible with the model structures and $\mathcal{Q}_N[\mathtt{Top}]$ to be homotopy invariant. In particular $\mathcal{Q}_\infty$ must be an E_∞^{rel}–operad.

Theorem 4.3.5 *For $1 \le N < \infty$ and $\overline{\mathcal{SC}}_N$ a cofibrant resolution of $\mathcal{SC}_N$, there is a weak Quillen quasiadjunction*

$$(B_{\rightarrow}^N \dashv_{\bar{B},\mathrm{Id}} \Omega_2^N)\colon \overline{\mathcal{SC}}_N[\mathtt{Top}] \rightleftharpoons \mathtt{Top}_*^{\rightarrow}$$

that induces an adjunction of homotopy categories

$$(\mathbb{L}B_{\rightarrow}^N \dashv \mathbb{R}\Omega_2^N)\colon \mathcal{H}o\,\overline{\mathcal{SC}}_N[\mathtt{Top}] \rightleftharpoons \mathcal{H}o\,\mathtt{Top}_*^{\rightarrow}.$$

For $\mathcal{E}^{\rightarrow}$ an E_∞^{rel}–operad there is a weak Quillen quasiadjunction

$$(B_{\rightarrow}^\infty \dashv_{\bar{B},\tilde{\Omega}} \Omega_2^\infty)\colon \mathcal{E}^{\rightarrow}[\mathtt{Top}] \rightleftharpoons \mathtt{Sp}^{\nearrow}$$

that induces an adjunction

$$(\mathbb{L}B_{\rightarrow}^\infty \dashv \mathbb{R}\Lambda_2^\infty)\colon \mathcal{H}o\,\mathcal{E}^{\rightarrow}[\mathtt{Top}] \rightleftharpoons \mathcal{H}o\,\mathtt{Sp}^{\nearrow}$$

of homotopy categories.

Proof The indicated functors equipped with the natural span

$$\mathrm{Id} \xLeftarrow{\varepsilon(\xi)} \overline{B} \xRightarrow{\gamma_2^N B(\alpha_2^N \mathfrak{v}^N,1,1)} \Omega_2^N B_{\to}^N,$$

the natural map $\varepsilon(\epsilon^N)\colon B_{\to}^N \Omega_2^N \Rightarrow \mathrm{Id}$ if $N < \infty$ and the cospan

$$B_{\to}^{\infty} \Omega_2^{\infty} \xRightarrow{\varepsilon(\epsilon_{\widetilde{\Omega}}^{\infty})} \widetilde{\Omega} \xLeftarrow{l} \mathrm{Id}$$

if $N = \infty$ satisfy the conditions of Definition 2.1.1. Condition (i) holds by Proposition 3.2.3. Condition (ii) holds by Propositions 2.2.3 and 2.4.7 and the construction of the model structures. Proposition 3.2.3 also implies one part of condition (iii) and the other follows from the fact $\widetilde{\Omega}$ preserves cofibrant objects in the mixed model structure and the construction of the stable model structure. One part of condition (iv) is Theorem 4.3.3(i) and the other follows from the construction of the stable model structure. Conditions (v) and (vi) follow from Theorems 4.3.3(iv) and 4.3.3(vi), respectively.

The last statement follows from the existence of the natural isomorphism between $\mathbb{R}\Lambda_2^N$ and $\mathbb{R}\Omega_2^N$. □

There is a Bousfield localization of $\overline{\mathcal{SC}}_N$–algebras where the fibrant objects are precisely the grouplike algebras.

Theorem 4.3.6 *For $3 \leq N \leq \infty$ and $\overline{\mathcal{SC}}_N$ a cofibrant resolution of $\mathcal{SC}_N$, the endofunctor $\Omega_2^N B_{\to}^N \mathfrak{C}$ is part of a Quillen idempotent quasimonad such that the $\Omega_2^N B_{\to}^N \mathfrak{C}$–weak equivalences are the maps*

$$(f_c, f_o) \in \overline{\mathcal{SC}}_N[\mathtt{Top}]((X_c, X_o), (Y_c, Y_o))$$

such that the homomorphisms

$$\begin{aligned}\overline{f}_{c*} &\in \mathtt{GrAlg}_k\big(H_*(X_c,k)[\pi_0 X_c^{-1}], H_*(Y_c,k)[\pi_0 Y_c^{-1}]\big),\\ \overline{f}_{o*} &\in \mathtt{GrAlg}_k\big(H_*(X_o,k)[\pi_0 X_o^{-1}], H_*(Y_o,k)[\pi_0 Y_o^{-1}]\big)\end{aligned}$$

are isomorphisms for all commutative rings k and the $\Omega_2^N B_{\to}^N \mathfrak{C}$–fibrant objects are the grouplike $\overline{\mathcal{SC}}_N$–algebras.

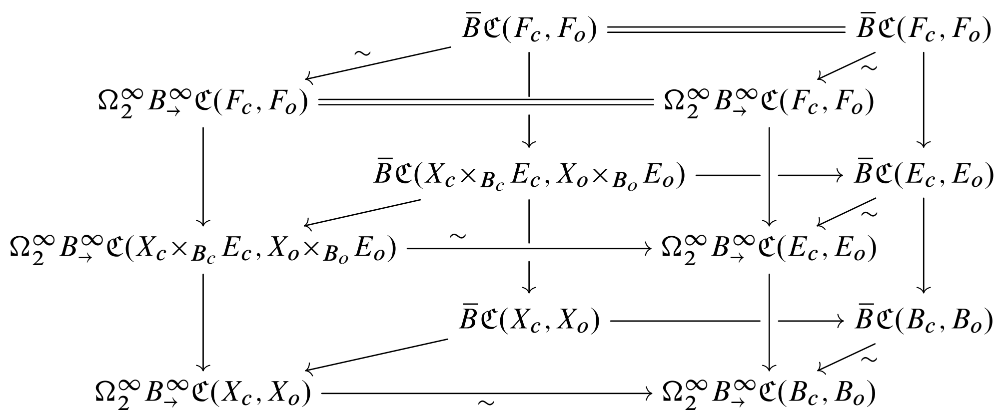
Proof First note that the following diagram of $\overline{\mathcal{SC}}_N$–maps commutes:

$$\begin{array}{ccc}
(X_c, X_o) & \xrightarrow{(f_c, f_o)} & (Y_c, Y_o) \\
{\scriptstyle \mathrm{cof}\varepsilon(\xi_{\mathfrak{C}})} \uparrow \wr & & \wr \uparrow {\scriptstyle \mathrm{cof}\varepsilon(\xi'_{\mathfrak{C}})} \\
\bar{B}\mathfrak{C}(X_c, X_o) & \xrightarrow{B(1,1,\mathfrak{C}(f_c,f_o))} & \bar{B}\mathfrak{C}(Y_c, Y_o) \\
{\scriptstyle \gamma_2^N B(\alpha_2^N \mathfrak{v}^N, 1, \mathfrak{C})} \downarrow & & \downarrow {\scriptstyle \gamma_2^N B(\alpha_2^N \mathfrak{v}^N, 1, \mathfrak{C})} \\
\Omega_2^N B_{\rightarrow}^N \mathfrak{C}(X_c, X_o) & \xrightarrow[\Omega_2^N B_{\rightarrow}^N \mathfrak{C}(f_c, f_o)]{} & \Omega_2^N B_{\rightarrow}^N \mathfrak{C}(Y_c, Y_o)
\end{array}$$

By Theorem 4.3.3 and the dual Whitehead theorem [33], the map $\Omega_2^N B_{\rightarrow}^N \mathfrak{C}(f_c, f_o)$ is a weak equivalence if and only if the homomorphisms induced by (f_c, f_o) in the statement of the theorem are isomorphisms.

The functor $\Omega_2^N B_{\rightarrow}^N \mathfrak{C}$ equipped with the endofunctor $\bar{B}\mathfrak{C}$ and the natural maps $\mathrm{cof}\varepsilon(\xi_{\mathfrak{C}})\colon \bar{B}\mathfrak{C} \Rightarrow \mathrm{Id}$ and $\gamma_2^N B(\alpha_2^N \mathfrak{v}^N, 1, \mathfrak{C})\colon \bar{B}\mathfrak{C} \Rightarrow \Omega_2^N B_{\rightarrow}^N \mathfrak{C}$ satisfies the conditions of Definition 2.3.1.

That condition (i) holds follows from Theorem 4.3.3(i), the fact that cof is a trivial fibration and the 2-out-of-3 property. That (ii) holds follows from Propositions 3.2.3 and 2.4.7 and the construction of the stable model structure. From the first part of this proof and Theorem 4.3.3(ii), condition (iii) holds. Since fibrations are preserved by pullbacks, fibrations induce long exact sequences of homotopy groups in $\mathtt{Top}_*$ and the diagram



commutes, condition (iv) holds. Condition (v) holds since the images of $\bar{B}\mathfrak{C}$ are cofibrant $\overline{\mathcal{SC}}_N$–algebras and since pushouts of weak equivalences along cofibrations with cofibrant domain in the category of algebras over a cofibrant operad in a left proper model category are weak equivalences, as Spitzweck proves in [41, Theorem 4.4].

That the $\Omega_2^N B_{\to}^N \mathfrak{C}$–fibrant objects are the grouplike $\overline{\mathcal{SC}}_N$–algebras follows from Proposition 2.3.6. □

The category of $\overline{\mathcal{SC}}_N$–algebras equipped with the left Bousfield localization induced by the Quillen idempotent quasimonad above is denoted by $\overline{\mathcal{SC}}_N[\mathtt{Top}]_{\mathrm{Grp}}$.

For $N < \infty$ the category $\mathtt{Top}_*^{\to}$ can be equipped with the $(N-1)$–connected model structure and the proof of Theorem 4.3.5 would still follow. The existence of long exact sequences of stable homotopy groups associated to spectra cofibrations implies there is a right Bousfield localization that presents the homotopy subcategory of connective relative spectra.

Theorem 4.3.7 *For $\mathcal{E}^{\to}$ an E_∞^{rel}–operad, the endofunctor $B_{\to}^\infty \Omega_2^\infty$ is part of a Quillen idempotent quasicomonad such that the $B_{\to}^\infty \Omega_2^\infty$–weak equivalences are the relative spectra maps*

$$(e_\bullet, f_\bullet) \in \mathtt{Sp}^{\nearrow}(\kappa_\bullet\colon A_\bullet \nearrow X_{\bullet+1},\ \iota_\bullet\colon B_\bullet \nearrow Y_{\bullet+1})$$

such that the homomorphisms

$$(e_\bullet)_* \in \mathtt{AbGrp}(\pi_q^S A_\bullet, \pi_q^S B_\bullet), \quad (f_\bullet)_* \in \mathtt{AbGrp}(\pi_q^S X_\bullet, \pi_q^S Y_\bullet)$$

are isomorphisms for all $q \geq 0$ and the $B_{\to}^\infty \Omega_2^\infty$–cofibrant objects are the connective relative spectra.

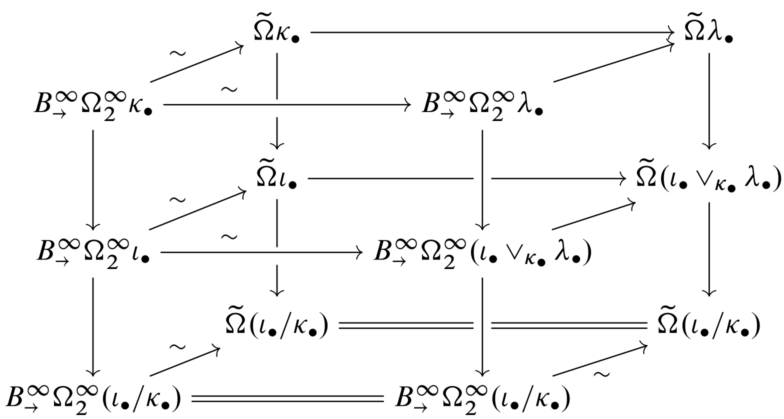
Proof First note that the following diagram of relative spectra commutes:

$$\begin{array}{ccc}
\kappa_\bullet & \xrightarrow{\quad (e_\bullet, f_\bullet) \quad} & \iota_\bullet \\
{\scriptstyle l_{\kappa_\bullet}}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle l_{\iota_\bullet}} \\
\widetilde{\Omega}\kappa_\bullet & \xrightarrow{\quad \widetilde{\Omega}(e_\bullet, f_\bullet) \quad} & \widetilde{\Omega}\iota_\bullet \\
{\scriptstyle \varepsilon(\epsilon^\infty_{\widetilde{\Omega}\kappa_\bullet})}\big\uparrow & & \big\uparrow{\scriptstyle \varepsilon(\epsilon^\infty_{\widetilde{\Omega}\iota_\bullet})} \\
B_{\to}^\infty \Omega_2^\infty \kappa_\bullet & \xrightarrow[\quad B_{\to}^\infty \Omega_2^\infty (e_\bullet, f_\bullet) \quad]{} & B_{\to}^\infty \Omega_2^\infty \iota_\bullet
\end{array}$$

By Theorem 4.3.3(vi), the long exact sequence of stable homotopy groups associated with a relative spectrum, and the five lemma, the map $B_{\to}^\infty \Omega_2^\infty (e_\bullet, f_\bullet)$ is a weak equivalence if and only if the homomorphisms induced by $(e_\bullet, f_\bullet)$ in the statement of the theorem are isomorphisms.

The functor $B^{\infty}_{\rightarrow}\Omega^{\infty}_{2}$ equipped with the endofunctor $\widetilde{\Omega}$ and the natural maps $l\colon \mathrm{Id} \Rightarrow \widetilde{\Omega}$ and $\varepsilon(\epsilon^{\infty}_{\widetilde{\Omega}})\colon B^{\infty}_{\rightarrow}\Omega^{\infty}_{2} \Rightarrow \widetilde{\Omega}$ satisfy the conditions dual to the ones in Definition 2.3.1.

First note that conditions (i), (ii) and (iii) are self-dual. Condition (i) holds by the definition of the stable model structure. That (ii) holds follows from Propositions 3.2.3 and 2.4.7. From the first part of this proof and Theorem 4.3.3(vi), condition (iii) holds. Since cofibrations are preserved by pushouts, cofibrations induce long exact sequences of stable homotopy groups in Sp and the diagram



commutes, the dual of condition (iv) holds. The dual of (v) holds since $\mathrm{Sp}^{\nearrow}$ is right proper.

That the $B^{\infty}_{\rightarrow}\Omega^{\infty}_{2}$–cofibrant objects are the connective relative spectra follows from the dual of Proposition 2.3.6. □

The category of relative spectra equipped with the model structure given by the right Bousfield localization associated with the Quillen idempotent quasicomonad above is denoted by $\mathrm{Sp}^{\nearrow}_{\mathrm{Con}}$.

Theorem 4.3.8 *For $2 < N < \infty$ and $\overline{\mathcal{SC}}_N$ a cofibrant resolution of $\mathcal{SC}_N$, the adjunction*

$$(\mathbb{L}B^{N}_{\rightarrow} \dashv \mathbb{R}\Omega^{N}_{2})\colon \mathcal{H}o\,\overline{\mathcal{SC}}_N[\mathrm{Top}]_{\mathrm{Grp}} \rightleftharpoons \mathcal{H}o\,\mathrm{Top}^{\rightarrow}_{N-1}$$

is an equivalence.

For $\mathcal{E}^{\rightarrow}$ an $E^{\mathrm{rel}}_{\infty}$–operad, the adjunction

$$(\mathbb{L}B^{\infty}_{\rightarrow} \dashv \mathbb{R}\Lambda^{\infty}_{2})\colon \mathcal{H}o\,\mathcal{E}^{\rightarrow}[\mathrm{Top}]_{\mathrm{Grp}} \rightleftharpoons \mathcal{H}o\,\mathrm{Sp}^{\nearrow}_{\mathrm{Con}}$$

is an equivalence.

Proof With these model structures the conditions of Corollary 2.1.3 are satisfied. For $N=\infty$ the conditions of Theorem 2.3.8 are satisfied. □

The relative recognition principle for $N=1$ was proved in [19]. The reason the argument in this section doesn't work in general if $N=1$ or $N=2$ comes from the fact that $\mathcal{C}_1$–algebras aren't necessarily admissible and so the natural map $\alpha^1\colon \mathcal{C}_1 \Rightarrow \Omega^1\Sigma^1$ isn't necessarily a natural group completion. It is the case that $\Omega^2_2 B^2_{\rightarrow}(X_c, X_o)$ as constructed in this section is a group completion if X_o is connected. The general argument should involve a modification of the relative delooping functor $B^2_{\rightarrow}$ similar to the delooping functor in [19].

Appendix Cofibrant resolutions of the Swiss-cheese relative operads

In this appendix a resolution of relative operads $\mathfrak{W}_I\colon \mathrm{Op}_{\mathrm{rel}}(\mathcal{T}) \to \mathrm{Op}_{\mathrm{rel}}(\mathcal{T})$ is introduced which takes relative operads with cofibrant underlying nonunital relative operads to cofibrant unital relative operads. The resolution of the Fulton–MacPherson relative operads $\mathcal{FM}_N$ are then shown to be cofibrant resolutions of $\mathcal{SC}_N$ by adapting an argument in Hoefel [17].

A.1 A resolution for relative operads

Nonunital relative operads are defined as in Definitions 3.1.3 and 3.1.5 except it is assumed that $\mathcal{Q}(\underline{0}) = \varnothing = \mathcal{Q}(\underline{0,0})$. They have a similar structure to unital relative operads except they don't admit degeneracies and they induce monads in $\mathcal{T}^2$ instead of $\mathcal{T}^2_{\mathbb{1}/}$. The category of nonunital relative operads is denoted by $\mathrm{Op}^{\circ}_{\mathrm{rel}}(\mathcal{T})$ and the forgetful functor $U^{\circ}\colon \mathrm{Op}_{\mathrm{rel}}(\mathcal{T}) \to \mathrm{Op}^{\circ}_{\mathrm{rel}}(\mathcal{T})$ admits a left adjoint *free unital operad functor* $F^*\colon \mathrm{Op}^{\circ}_{\mathrm{rel}}(\mathcal{T}) \to \mathrm{Op}_{\mathrm{rel}}(\mathcal{T})$.

The category of objects under A in $\mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}$ has as objects relative injections $\tau \in \mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}(A,B)$ and morphisms between $\tau \in \mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}(A,B)$ and $\tau' \in \mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}(A,B')$ are morphisms $\sigma \in \mathbb{S}^{\mathrm{inj}}_{\mathrm{rel}}(B,B')$ such that $\tau' = \sigma\tau$. The category of objects under A is denoted by $\mathbb{S}_{A/}$. For $\mathcal{Q} \in \mathrm{Op}^{\circ}_{\mathrm{rel}}(\mathcal{T})$ the underlying $\mathbb{S}_{\mathrm{rel}}$–object of $F^*\mathcal{Q}$ is

$$F^*\mathcal{Q}\colon \mathbb{S}^{\mathrm{op}}_{\mathrm{rel}} \to \mathcal{T}, \quad A \mapsto \int^{\mathbb{S}_{A/}} \mathcal{Q}(B),$$

and the units and the nondegeneracy compositions are induced by those of $\mathcal{Q}$. The degeneracies are given by the natural inclusions into the colimits. The unit of the adjunction is induced by inclusion into the colimits, and the counit by degeneracies.

If $\mathcal{T}$ is a symmetric model category that satisfies the conditions in [3, Theorem 2.1], the category $\mathrm{Op}^{\circ}_{\mathrm{rel}}(\mathcal{T})$ admits a model structure such that $(F^* \dashv U^{\circ})$ is a Quillen adjunction, and so in particular F^* preserves cofibrant objects. One of the conditions on $\mathcal{T}$ is that it contains an *interval object*, which is an object I equipped with maps $0, 1 \in \mathcal{T}(\mathbb{1}, I)$ and $\varepsilon \in \mathcal{T}(I, \mathbb{1})$ such that $(0,1) \in \mathcal{T}(\mathbb{1} \sqcup \mathbb{1}, I)$ is a cofibration and ε is a weak equivalence. In $\mathtt{Top}$ the interval $I = [0,1]$ is an interval object. The following resolution functor is similar to the Boardman–Vogt resolution [6; 2]. Let

$$I\colon \mathbb{S}_{A/} \to \mathcal{T}, \quad (\tau\colon A \to B) \mapsto I^{\otimes B \setminus \operatorname{Im}\tau},$$

with the map $\sigma\cdot \in \mathcal{T}(I^{\otimes B\setminus \operatorname{Im}\tau}, I^{\otimes B'\setminus \operatorname{Im}\tau'})$ associated to $\sigma \in \mathbb{S}_{A/}(\tau\colon A \to B, \tau'\colon A \to B')$ being induced by the map 0 in the coordinates in $B' \setminus \operatorname{Im}\sigma$.

Definition A.1.1 Let $\mathcal{T}$ be a symmetric model category and $I \in \mathcal{T}$ an interval object. The *degeneration resolution* is the endofunctor $\mathfrak{W}_I\colon \mathrm{Op}_{\mathrm{rel}}(\mathcal{T}) \to \mathrm{Op}_{\mathrm{rel}}(\mathcal{T})$ with underlying $\mathbb{S}_{\mathrm{rel}}$–object

$$\mathfrak{W}_I\,\mathcal{Q}\colon \mathbb{S}_{\mathrm{rel}} \to \mathcal{T}, \quad A \mapsto \int^{\mathbb{S}_{A/}} \mathcal{Q}(B) \otimes I(\tau),$$

for each $\mathcal{Q} \in \mathrm{Op}_{\mathrm{rel}}(\mathcal{T})$. The units are the inclusions into the colimit of the units in $\mathcal{Q}$. The relative operad compositions that don't involve degenerations are defined by the composition in $\mathcal{Q}$ and a reordering of the copies of I, and the degenerations are induced by the morphism 1.

In $\mathtt{Top}$ an element $[q, (t^b)]_\tau \in \mathfrak{W}_I\,\mathcal{Q}(A)$ is represented by a relative injection $\tau\colon A \to B$, a $q \in \mathcal{Q}(B)$ and a point in the cube $I^{B\setminus \operatorname{Im}\tau}$, and these elements satisfy the relation that $[q\cdot\sigma, (t^b)]_\tau = [q, \sigma\cdot(t^b)]_{\tau'}$ for all $\sigma \in \mathbb{S}_{A/}(\tau,\tau')$. This means there is always a representative such that (t^b) doesn't contain any zeros.

The full subcategories $F^q\mathbb{S}_{A/}$ of $\mathbb{S}_{A/}$ for $q \in \mathbb{N}$ containing the injections $\tau\colon A \to B$ with $B \setminus \operatorname{Im}\tau \le q$ induce a filtration $\mathfrak{W}^q_I\,\mathcal{Q}$ in $\mathbb{S}_{\mathrm{rel}} - \mathcal{T}$. The inclusions of this filtration are denoted by $\mathfrak{w}^q \in \mathbb{S}_{\mathrm{rel}} - \mathcal{T}(\mathfrak{W}^q_I\,\mathcal{Q}, \mathfrak{W}^{q+1}_I\,\mathcal{Q})$. Note that $\mathfrak{W}^0_I\,\mathcal{Q}$ is isomorphic to $\mathcal{Q}$.

The following result is similar to the result that the Boardman–Vogt resolution is a cofibrant resolution for relative operads with cofibrant underlying $\mathbb{S}_{\mathrm{rel}}$–objects [2, Theorem 5.1]. For the conditions on $\mathcal{T}$ see [3, Theorem 2.1].

Theorem A.1.2 *Let $\mathcal{T}$ be a symmetric cofibrantly generated model category equipped with a symmetric monoidal fibrant resolution functor $\mathfrak{F}$ and a cocommutative coalgebra interval object I. For all unital relative operads $\mathcal{Q}$ such that $U^\circ\mathcal{Q}$ is a cofibrant nonunital relative operad, the counit of the adjunction $(F^* \dashv U^\circ)$ admits a factorization*

$$F^*U^\circ\mathcal{Q} \xhookrightarrow{\delta^{\mathcal{Q}}} \mathfrak{W}_I\,\mathcal{Q} \xrightarrow[\sim]{\gamma^{\mathcal{Q}}} \mathcal{Q}$$

into a cofibration $\delta^{\mathcal{Q}}$ followed by a weak equivalence $\gamma^{\mathcal{Q}}$. In particular, $\mathfrak{W}_I\,\mathcal{Q}$ is a cofibrant resolution of $\mathcal{Q}$.

Proof The last assertion follows from the fact that F^* preserves cofibrant objects. The map ε and the degenerations of $\mathcal{Q}$ define morphisms $\gamma_v^{\mathcal{Q}} \in \mathcal{T}(\mathcal{Q}(B)\otimes I^{\otimes B\setminus \operatorname{Im}\tau}, \mathcal{Q}(A))$ which fit together into a relative operad morphism $\gamma^{\mathcal{Q}} \in \mathrm{Op}_{\mathrm{rel}}(\mathcal{T})(\mathfrak{W}_I\,\mathcal{Q}, \mathcal{Q})$ such that the composition $\mathfrak{W}_I^0\mathcal{Q} \xhookrightarrow{\sim} \mathfrak{W}_I\,\mathcal{Q} \xrightarrow{\gamma^{\mathcal{Q}}} \mathcal{Q}$ of $\mathbb{S}_{\mathrm{rel}}$–objects is the identity. Since the inclusion of $\mathfrak{W}_I^0\mathcal{Q}$ into $\mathfrak{W}_I\,\mathcal{Q}$ is a trivial cofibration, $\gamma^{\mathcal{Q}}$ is a weak equivalence.

The morphism 1 defines morphisms $\delta_v^{\mathcal{Q}} \in \mathcal{T}(\mathcal{Q}(B), \mathcal{Q}(B)\otimes I(\tau))$ which fit together into a relative operad morphism $\delta^{\mathcal{Q}} \in \mathrm{Op}_{\mathrm{rel}}(\mathcal{T})(F^*U^\circ\mathcal{Q}, \mathfrak{W}_I\,\mathcal{Q})$. Clearly $\gamma^{\mathcal{Q}}\delta^{\mathcal{Q}}$ equals the counit of the adjunction $(F^* \dashv U^\circ)$. It remains to show that $\delta^{\mathcal{Q}}$ is a cofibration, ie that for any commutative square of unital relative operads

$$\begin{array}{ccc} F^*U^\circ\mathcal{Q} & \xrightarrow{\phi} & \mathcal{E} \\ {\scriptstyle \delta^{\mathcal{Q}}}\downarrow & \overset{\overline{\psi}}{\nearrow} & \downarrow{\scriptstyle p}\ \sim \\ \mathfrak{W}_I\,\mathcal{Q} & \xrightarrow[\psi]{} & \mathcal{B} \end{array}$$

where p is a trivial fibration, the lift $\overline{\psi}$ exists. By the $(F^* \dashv U^\circ)$ adjunction a morphism $\overline{\psi}$ is a lift of the square above if and only if $U^\circ\overline{\psi}$ is a lift of the square of nonunital operads

$$\begin{array}{ccc} U^\circ\mathcal{Q} & \xrightarrow{\phi_0} & U^\circ\mathcal{E} \\ {\scriptstyle U^\circ \operatorname{colim}_{q\to\infty} \mathfrak{w}^q}\downarrow & \overset{U^\circ\overline{\psi}}{\nearrow} & \downarrow{\scriptstyle U^\circ p}\ \sim \\ U^\circ\mathfrak{W}_I\,\mathcal{Q} & \xrightarrow[U^\circ\psi]{} & U^\circ\mathcal{B} \end{array}$$

We construct $\overline{\psi}$ inductively on the filtration degree of $\mathfrak{W}_I\,\mathcal{Q}$. Define $\overline{\psi}_0 := \phi_0 \in \mathbb{S}_{\mathrm{rel}} - \mathcal{T}(\mathfrak{W}_I^0\mathcal{Q}, \mathcal{E})$. Now assume constructed $\overline{\psi}_\bullet \in \mathbb{S}_{\mathrm{rel}} - \mathcal{T}(\mathfrak{W}_I^\bullet\mathcal{Q}, \mathcal{E})$ for $\bullet \leq q-1$ such that, for $A \in \mathbb{S}_{\mathrm{rel}}$, $(B^a) \in \Pi_A\mathbb{S}_{\mathrm{cor}(a)}$ and $(i, (j^a)) \in \mathbb{N}\times\Pi_A\mathbb{N}$ with $j^a = 1$ if $B^a = \varnothing$ and $i + \Sigma_A j^a \leq q-1$, the square below commutes:

$$\begin{array}{ccc} \mathfrak{W}^i_I \mathcal{Q}(A)\otimes\big(\bigotimes_A \mathfrak{W}^{j^a}_I \mathcal{Q}(B^a)\big) & \xrightarrow{\ \circ\ } & \mathfrak{W}^{q-1}_I(\Sigma_A B^a) \\ {\scriptstyle (\bar\psi_i,(\bar\psi_{j^a}))}\downarrow & & \downarrow{\scriptstyle \bar\psi_{q-1}} \\ \mathcal{E}(A)\otimes\big(\bigotimes_A \mathcal{E}(B^a)\big) & \xrightarrow[\ \circ\ ]{} & \mathcal{E}(\Sigma_A B^a)\end{array}$$

Let

$$\bar I\colon \mathbb{S}_{A/}\to\mathcal{T},\qquad (\tau\colon A\to B)\mapsto \operatorname*{colim}_{D\subseteq B\setminus \operatorname{Im}\tau,\, D\neq\varnothing} I_D(\tau):=\bigotimes_{B\setminus\operatorname{Im}\tau}\begin{cases}\mathbb{1} & \text{if } b\in D,\\ I & \text{if } b\notin D,\end{cases}$$

and

$$\tilde I\colon \mathbb{S}_{A/}\to\mathcal{T},\qquad (\tau\colon A\to B)\mapsto \operatorname*{colim}_{D\subseteq B\setminus \operatorname{Im}\tau,\, D\neq\varnothing} I^+_D(\tau):=\bigotimes_{B\setminus\operatorname{Im}\tau}\begin{cases}\mathbb{1}\sqcup\mathbb{1} & \text{if } b\in D,\\ I & \text{if } b\notin D.\end{cases}$$

By the pushout-product axiom, $\bar I(\nu)\hookrightarrow\tilde I(\nu)\hookrightarrow I(\nu)$ are cofibrations. There is a factorization of the inclusion of the filtration $\mathfrak{w}^q$ by the sequence of pushouts

$$\begin{array}{ccc} \int^{\mathbb{S}^q_{-/}} \mathcal{Q}(B)\otimes\bar I(\tau) & \longrightarrow & \mathfrak{W}^{q-1}_I\mathcal{Q} \\ \downarrow & & \downarrow \\ \int^{\mathbb{S}^q_{-/}} \mathcal{Q}(B)\otimes\tilde I(\tau) & \longrightarrow & \widetilde{\mathfrak{W}}^{q-1}_I\mathcal{Q} \\ \downarrow & & \downarrow \\ \int^{\mathbb{S}^q_{-/}} \mathcal{Q}(B)\otimes I(\tau) & \longrightarrow & \mathfrak{W}^{q}_I\mathcal{Q} \end{array}$$

with the vertical natural morphisms cofibrations of $\mathbb{S}_{\mathrm{rel}}$–objects since $\mathcal{Q}$ is in particular a cofibrant $\mathbb{S}_{\mathrm{rel}}$–object.

The morphism $\bar\psi_{q-1}\in\mathbb{S}_{\mathrm{rel}}-\mathcal{T}(\mathfrak{W}^{q-1}_I\mathcal{Q},\mathcal{E})$ extends uniquely to a morphism $\tilde\psi_{q-1}\in\mathbb{S}_{\mathrm{rel}}-\mathcal{T}(\widetilde{\mathfrak{W}}^{q-1}_I\mathcal{Q},\mathcal{E})$ that satisfies the inductive hypothesis. This gives us a commutative square

$$\begin{array}{ccc} \widetilde{\mathfrak{W}}^{q-1}_I\mathcal{Q} & \xrightarrow{\tilde\psi_{q-1}} & \mathcal{E} \\ \downarrow & {\scriptstyle \bar\psi_q}\nearrow & \downarrow{\scriptstyle p}\,\sim \\ \mathfrak{W}^{q}_I\mathcal{Q} & \xrightarrow[\psi_q]{} & \mathcal{B} \end{array}$$

The lift $\bar\psi_q\in\mathbb{S}_{\mathrm{rel}}-\mathcal{T}(\mathfrak{W}^q_I\mathcal{Q},\mathcal{E})$ exists by the model structure on $\mathbb{S}_{\mathrm{rel}}-\mathcal{T}$ and it satisfies the induction hypothesis. The operad map $\bar\psi\in\mathrm{Op}_{\mathrm{rel}}(\mathcal{T})(\mathfrak{W}_I\mathcal{Q},\mathcal{E})$ is then defined as the colimit of the $\bar\psi_q$. □

A.2 Cofibrant resolution of the Swiss-cheese relative operads

Let $N \in \mathbb{N}$ and $\mathcal{FM}_N \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})$ be the Fulton–MacPherson relative operads of compactifications of normalized configuration spaces [24; 17]; see also [25] for the uncolored version. Let $\mathrm{Conf}_N(\underline{l})$ be the configuration space of l points in the euclidean space $\mathbb{R}^N$ and $\mathrm{Conf}_N(\underline{m,n})$ be the configuration space of $m+n$ points in the euclidean half-space $\mathbb{H}^N$ such that the first m points are in the interior of the half-space and the last n points are in the border of the half-space. The interior of $\mathcal{FM}_N(\underline{l})$ is homeomorphic to the orbit space $\mathring{\mathcal{FM}}_N(\underline{l}) = \mathrm{Conf}_N(\underline{l})/\mathbb{R}^N \rtimes \mathbb{R}^+$ of the action given by translation and dilation, and the interior of $\mathcal{FM}_N(\underline{m,n})$ is homeomorphic to the orbit space $\mathring{\mathcal{FM}}_N(\underline{m,n}) = \mathrm{Conf}_N(\underline{m,n})/\mathbb{R}^{N-1} \rtimes \mathbb{R}^+$ of the action given by translation parallel to the border of $\mathbb{H}^N$ and dilation. For $A \in \mathbb{S}_{\mathrm{rel}}$ define $\widetilde{A} \in \mathbb{S}_{\mathrm{rel}}$ as $\widetilde{A} = A$ if $\mathrm{cor}(A) = c$ and $\widetilde{A} = A \sqcup A'_c$ if $\mathrm{cor}(A) = o$, with A'_c a copy of A_c. Let ρ be the reflection of $\mathbb{R}^N$ across the hyperplane $\partial\mathbb{H}^N$. For $x \in \mathrm{Conf}_N(A)$ and $a \in \widetilde{A}$ define $\tilde{x}(a) = x(a)$ if $a \in A$ and $\tilde{x}(a) = \rho x(a)$ if $a \in A'_c$. For $k \in \mathbb{N}$ let

$$\binom{\widetilde{A}}{k} = \{(a_i) \in \widetilde{A}^k \mid a_i \neq a_{i'} \text{ if } i \neq i'\};$$

then each $\mathring{\mathcal{FM}}_N(A)$ can be embedded into $(\mathbb{S}^{N-1})^{\binom{\widetilde{A}}{2}} \times [0,\infty]^{\binom{\widetilde{A}}{3}}$ by the maps

$$\theta_{(a_1,a_2)} \in (\mathbb{S}^{N-1})^{\mathring{\mathcal{FM}}_N}, \qquad \theta_{(a_1,a_2)}([x]) := \frac{\tilde{x}(a_2) - \tilde{x}(a_1)}{\|\tilde{x}(a_2) - \tilde{x}(a_1)\|},$$
$$\delta_{(a_1,a_2,a_3)} \in [0,\infty]^{\mathring{\mathcal{FM}}_N}, \qquad \delta_{(a_1,a_2,a_3)}([x]) := \frac{\tilde{x}(a_2) - \tilde{x}(a_1)}{\|\tilde{x}(a_3) - \tilde{x}(a_1)\|},$$

and $\mathcal{FM}_N(A)$ is the closure of the image of these embeddings. Elements of $\mathcal{FM}_N(A)$ can intuitively be visualized as normalized virtual configurations of points which can be infinitesimally close but with the directions between two points and the ratio of distances between three points well defined by the natural extensions of the maps $\theta_{(a^1,a^2)}$ and $\delta_{(a^1,a^2,a^3)}$ to $\mathcal{FM}_N$. The relative operad structural maps are defined by the inclusion of infinitesimal configurations, and the degenerations delete the respective points of the configurations. Define $\mathcal{FM}_\infty := \mathrm{colim}_{\bullet\to\infty} \mathcal{FM}_\bullet$ with the colimit induced by the natural inclusions $i_c^\bullet \in \mathtt{Top}(\mathbb{R}^\bullet, \mathbb{R}^{\bullet+1})$ and $i_o^\bullet \in \mathtt{Top}(\mathbb{H}^\bullet, \mathbb{H}^{\bullet+1})$ into the first N coordinates and with the last coordinate 0.

Salvatore used the Boardman–Vogt resolution to prove that the nonunital Fulton–MacPherson operads are cofibrant [39]. The unital versions fail to be cofibrant because the degenerations aren't cellular, but the resolution in the previous subsection fixes this.

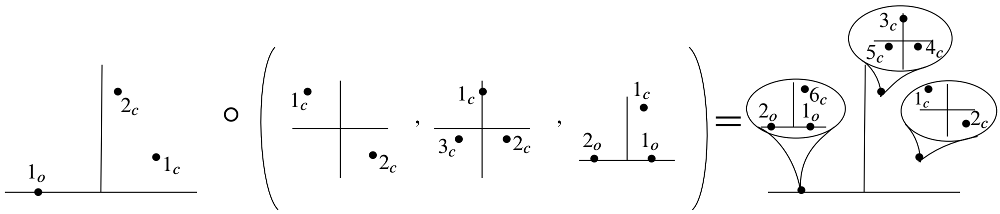


Figure 4: Composition in $\mathcal{FM}_2$.

Consider the subrelative operad $\mathcal{SC}_N^{\square} \subset \mathcal{SC}_N$ with $d_A \in \mathcal{SC}_N^{\square}(A)$ if the $(m^i_{d_a})$ associated to each $a \in A$ has constant entries, or equivalently such that their images have all sides of the same length. Note that the inclusion of this subrelative operad is a weak equivalence of relative operads. Let $c^c_{1/2} \in I^N$ be the center of the N–cube and $c^o_{1/2}$ be the projection of $c^c_{1/2}$ onto $\{0\} \times I^{N-1}$. Define $\pi \in \mathbb{S}_{\mathrm{rel}} - \mathtt{Top}(\mathcal{SC}_N^{\square}, \mathcal{F}\mathring{\mathcal{M}}_N)$ as $\pi_A(d_A) = [(d_a(c^{\mathrm{cor}(a)}_{1/2}))]$. The following lemma is a cubical equivalent of [17, Lemma 3.1.1].

Lemma A.2.1 *Let* $1 \leq N \leq \infty$, $A \in \mathbb{S}_{\mathrm{rel}}$ *and* $x \in \mathcal{F}\mathring{\mathcal{M}}_N(A)$; *then the inverse image* $\pi^{-1}(x)$ *is convex in* $\mathcal{SC}_N^{\square}(A)$.

From this follow cubical equivalents of [17, Corollary 3.1.2, Theorem 3.1.3 and Corollary 3.1.5]. In particular, $U^\circ\mathcal{FM}_N$ are cofibrant resolutions of $U^\circ\mathcal{SC}_N$, ie explicit weak equivalences of nonunital relative operads can be constructed.

Theorem A.2.2 *Let* $1 \leq N \leq \infty$; *then there is a cofibrant resolution*

$$\mathfrak{v}^N \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})(\mathfrak{W}_I\mathcal{FM}^N, \mathcal{SC}^N).$$

Proof From [17, Theorem 3.1.3 and Corollary 3.1.5] there is a cofibrant resolution of nonunital relative operads $\nu^N \in \mathrm{Op}^\circ_{\mathrm{rel}}(\mathtt{Top})(U^\circ\mathcal{FM}_N, U^\circ\mathcal{SC}_N^{\square})$.

The map $\mathfrak{v}^N \in \mathrm{Op}_{\mathrm{rel}}(\mathtt{Top})(\mathfrak{W}_I\mathcal{FM}_N, \mathcal{SC}_N)$ is defined as a colimit of inductively defined maps $\mathfrak{v}^N_q \in \mathbb{S}_{\mathrm{rel}} - \mathtt{Top}(\mathfrak{W}^q_I\mathcal{FM}_N, \mathcal{SC}_N)$. Set $\mathfrak{v}^N_0 := \nu^N$. Suppose that $\mathfrak{v}^N_{q-1}$ is well defined. Let $A \in \mathbb{S}_{\mathrm{rel}}$ and $\tau\colon A \to B \in \mathbb{S}^q_{A/}$. Define

$$\mathfrak{v}^N_\tau \in \mathtt{Top}(\mathcal{FM}_N(B) \times I(\tau), \mathcal{SC}_N(A))$$

as $\mathfrak{v}^N_{q-1}$ on the subspace $\mathcal{FM}_N(B) \times \overline{I}(\tau)$ and as $(\nu^N \mathrm{proj}_{\mathcal{FM}_N(B)}(-)) \cdot \tau$ on the subspace $\mathcal{FM}_N(B) \times \{(1, \dots, 1)\}$. Since the degenerations in $\mathcal{FM}_N$ preserve relative positions of the points in the configurations that are not deleted by Lemma A.2.1, $\mathfrak{v}^N_\sigma$ can be extended to the whole $\mathcal{FM}_N(B) \times I(\tau)$. This construction is compatible with the identifications in $\mathfrak{W}^q_I\mathcal{F}\mathcal{M}_N$ and therefore $\mathfrak{v}^N_q$ is well defined. Set $\mathfrak{v}^N := \operatorname{colim}_{q\to\infty} \mathfrak{v}^N_q$, which is a relative operad map. Since each $\mathfrak{v}^N_A$ is homotopic to the composition of the weak equivalences $\gamma_A^{\mathcal{F}\mathcal{M}_N}$ and ν^N_A, then, $\mathfrak{v}^N$ is a weak equivalence. The theorem therefore follows from Theorem A.1.2. □

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Received: 26 September 2018 Revised: 21 February 2019