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On the KO–groups of toric manifolds

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We consider the real topological K–groups of a toric manifold M, which turns out to be closely related to the topology of the small cover $M_{\mathbb{R}}$, the fixed points under the canonical conjugation on M. Following the work of Bahri and Bendersky (2000), we give an explicit formula for the KO–groups of toric manifolds, and then we characterize the two extreme classes of toric manifolds according to their mod 2 cohomology groups as $\mathcal{A}(1)$–modules.

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1 Introduction

Following Davis and Januszkiewicz [11], by a toric manifold we mean a smooth manifold, with locally standard torus action, such that the orbit space is homeomorphic to a simple convex polytope (see (8) for details). Toric manifolds include all projective nonsingular toric varieties, and they generate the complex cobordism ring (see Buchshtaber and Raĭ [6]). We refer the readers to Buchstaber and Panov [5, Chapter 5] for more details.

Let $c\colon KO^*(M) \to K^*(M)$ and $r\colon K^*(M) \to KO^*(M)$ be the complexification and realification maps, respectively, and let the cokernel

$$W^i(M) = KO^{2i}(M)/r$$

of r be the i^{th} *Witt group*, which is clearly 4–periodic. Since $r \circ c = 2$, Witt groups consist of only 2–torsion elements.

Following Astey, Bahri, Bendersky, Cohen, Davis, Gitler, Mahowald, Ray and Wood [1] and Zibrowius [19], consider the Bott sequence

$$\cdots \to KO^{i+1}(M) \xrightarrow{\cdot\eta} KO^i(M) \xrightarrow{c} K^i(M) \xrightarrow{r\circ t^{-1}} KO^{i+2}(M) \to \cdots,$$

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where $t\colon K^0(M) \to K^{-2}(M)$ is the isomorphism given by multiplying the Bott element t and $\eta\colon KO^0(M) \to KO^{-1}(M)$ is the map given by multiplying the generator $\eta \in KO^{-1}(\mathrm{pt})$ from the Hopf bundle.

Since the integral cohomology $H^*(M;\mathbb{Z})$ concentrates in even dimensions, by the Atiyah–Hirzebruch spectral sequence, the complex topological K–group $K^0(M)$ is free and $K^1(M) = 0$. The Bott sequence above splits into the form

$$0 \to KO^{2i+1}(M) \xrightarrow{\eta} KO^{2i}(M) \xrightarrow{c} K^{2i}(M) \xrightarrow{r \circ t^{-1}} KO^{2i+2}(M) \xrightarrow{\eta} KO^{2i+1}(M) \to 0,$$

so we see that η induces an isomorphism

$$\eta\colon W^{i+1}(M) \xrightarrow{\cong} KO^{2i+1}(M). \tag{1}$$

Since the image of $KO^{2i}(M)$ under c is a subgroup of the free abelian group $K^{2i}(M)$, it is also free and the sequence

$$0 \to KO^{2i+1}(M) \xrightarrow{\eta} KO^{2i}(M) \xrightarrow{c} \operatorname{im} c \to 0$$

splits. By (1) we have that $\eta^2\colon W^{i+1}(M) \to KO^{2i}(M)$ induces an isomorphism onto its image and

$$KO^{2i}(M) = W^{i+1}(M) \oplus \text{free part}. \tag{2}$$

Next we determine the Witt group

$$W^*(M) = \bigoplus_{i=0}^{3} W^i(M).$$

Let $\mathcal{A}(1)$ be the subalgebra of the mod 2 Steenrod algebra $\mathcal{A}$ generated by Sq^1 and Sq^2. Bahri and Bendersky [2] considered the reduced mod 2 cohomology $\widetilde{H}^*(M;\mathbb{Z}_2)$ as an $\mathcal{A}(1)$–module, which turns out to be a direct sum

$$\widetilde{H}^*(M;\mathbb{Z}_2) = \underline{\mathcal{S}} \oplus \underline{\mathcal{M}}, \tag{3}$$

where

$$\underline{\mathcal{S}} = \bigoplus_i \mathbb{Z}_2\{x_i\}$$

is a trivial $\mathcal{A}(1)$–module generated by elements x_i such that $\mathrm{Sq}^1 x_i = \mathrm{Sq}^2 x_i = 0$, and

$$\underline{\mathcal{M}} = \bigoplus_j \mathbb{Z}_2\{y_j, z_j\}$$

is generated by pairs of elements y_j and z_j such that $\mathrm{Sq}^1 y_j = \mathrm{Sq}^1 z_j = 0$ and $\mathrm{Sq}^2 y_j = z_j$. By considering the Adams spectral sequence converging to $ko_*(M)$, with E_2–term

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\widetilde{H}^*(M;\mathbb{Z}_2),\mathbb{Z}_2) = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\underline{\mathcal{S}},\mathbb{Z}_2) \oplus \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\underline{\mathcal{M}},\mathbb{Z}_2),$$

they showed that the Adams spectral sequence collapses at the E_2–page. Here is a summary of their results, in which the free part follows easily from the Atiyah–Hirzebruch spectral sequence with rational coefficients $KO^*(\mathrm{pt};\mathbb{Q})$.

Theorem 1.1 [2] *All torsion elements in $KO^*(M)$ have order 2, and they come from the module $\underline{\mathcal{S}}$. More precisely, each element $x \in \underline{\mathcal{S}}$ in dimension n_x contributes two 2–torsion elements in dimensions $n_x - 1$ and $n_x - 2$, respectively. The free part of the group $KO^{2i}(M)$ has rank n_{2i}, where*

$$n_0 = n_4 = \dim_{\mathbb{Q}} \bigoplus_{k=0}^{\infty} H^{4k}(M;\mathbb{Q}), \quad n_2 = n_6 = \dim_{\mathbb{Q}} \bigoplus_{k=0}^{\infty} H^{4k+2}(M;\mathbb{Q}).$$

1.1 Main results

In order to determine the Witt group $W^*(M)$ completely, it remains to determine the submodule $\underline{\mathcal{S}}$. We will see that it is closely related to the cohomology of the small cover $M_{\mathbb{R}}$, which is a submanifold of M comprising the fixed points of a special involution, the "conjugation" on M (see (11) for details).

Here is our trivial observation, which is the starting point:

Lemma 1.2 *There is a canonical isomorphism*

$$\phi\colon H^*(M;\mathbb{Z}_2) \xrightarrow{\cong} H^*(M_{\mathbb{R}};\mathbb{Z}_2)$$

of rings such that $\phi \circ \mathrm{Sq}^2 = \mathrm{Sq}^1 \circ \phi$, where the image of an element in dimension $2i$ is in dimension i for $i = 0, \dots, n$.

Therefore, we only need to consider those Sq^1–cocycles in $H^*(M_{\mathbb{R}};\mathbb{Z}_2)$. In this direction we have a complete answer with the help of the previous work of Trevisan [18], Choi and Park [8] and Cai and Choi [7] on the cohomology groups of $M_{\mathbb{R}}$. Let λ_2 be the mod 2 reduction of the characteristic matrix λ of M (see (7)), and let $\mathrm{Row}(\lambda_2)$ be the subspace of $\mathbb{Z}_2^m$ spanned by the row vectors of λ_2. Let K be the simplicial complex dual to the boundary of P, and let K_ω be the full subcomplex with vertex set $\{i \mid w_i = 1\}$ for $\omega = (w_i)_{i=1}^m \in \mathrm{Row}(\lambda_2)$ (see (9) and (22) for details).

Theorem 1.3 *Let M be the toric manifold over a simple convex polytope P with characteristic matrix λ. Let K be the simplicial complex dual to the boundary of P. Then, for $i = 0, 1, 2, 3$, we have additive isomorphisms*

$$W^{i+1}(M) = \bigoplus_{k=0}^{\infty} \bigoplus_{\omega \in \mathrm{Row}(\lambda_2)} \widetilde{H}^{i+4k}(K_\omega; \mathbb{Z}_2). \tag{4}$$

Combining these results, for $i = 0, 1, 2, 3$, we have the additive isomorphisms

$$KO^{2i}(M) = \bigoplus_{k=0}^{\infty} \Big[H^{4k+1-(-1)^i}(M; \mathbb{Z}) \oplus \bigoplus_{\omega \in \mathrm{Row}(\lambda_2)} \widetilde{H}^{i+4k}(K_\omega; \mathbb{Z}_2) \Big], \tag{5}$$

$$KO^{2i+1}(M) = \bigoplus_{k=0}^{\infty} \bigoplus_{\omega \in \mathrm{Row}(\lambda_2)} \widetilde{H}^{i+4k}(K_\omega; \mathbb{Z}_2). \tag{6}$$

A geometric construction of bundles realizing part of these elements is given by Civan and Ray [9].

Next we consider two extreme cases in (3). We say that a toric manifold M is of $\mathcal{S}$–type if $\widetilde{H}^*(M; \mathbb{Z}_2) = \underline{\mathcal{S}}$ and it is of $\mathcal{M}$–type if $\widetilde{H}^*(M; \mathbb{Z}_2) = \underline{\mathcal{M}}$. For the first case we have the following result, in which (S4) $\Longrightarrow$ (S3) is proved in [11].

Theorem 1.4 *Let M be a toric manifold with characteristic matrix λ. The following are equivalent*:

- (S1) *M is of $\mathcal{S}$–type*;
- (S2) *the corresponding small cover $M_{\mathbb{R}}$ has no 2–torsion elements in its integral cohomology $H^*(M_{\mathbb{R}}; \mathbb{Z})$*;
- (S3) *$H^*(M_{\mathbb{R}}; \mathbb{Z})$ is torsion-free*;
- (S4) (*up to a basis change*) *the mod 2 reduction of λ has only one nonzero entry in each column* (*ie $M_{\mathbb{R}}$ is a pullback of the linear model in the sense of* [11]).

There are many interesting examples in this case, including the isospectral manifolds of tridiagonal hermitian matrices (see Bloch, Flaschka and Ratiu [4] and Davis [10]); the class of torsion-free small covers includes Tomei manifolds [17], the isospectral manifolds of tridiagonal real symmetric matrices.

For the second case, we make a conclusion here, where the equivalence of (M1) and (M2) is proved in [2].

Theorem 1.5 *The following are equivalent*:

(M1) *M is of $\mathcal{M}$–type.*

(M2) *$KO^*(M)$ is torsion-free* (*ie* $W^*(M)=0$).

(M3) *$\widetilde{H}^*(M_{\mathbb{R}};\mathbb{Z})$ has only 2–torsion elements.*

(M4) *$\widetilde{H}^*(K_\omega;\mathbb{Z}_2)=0$ for all nonzero $\omega \in \mathrm{Row}(\lambda_2)$.*

For a toric manifold M of $\mathcal{M}$–type, the ring structure $KO^*(M)$ is given explicitly in [1, Theorem 6.5]. It is observed by Park [15, Proposition 5.1] that if the property (M3) holds, then the dimension of $M_{\mathbb{R}}$ must be even.

Finally, we consider the problem of how to construct toric manifolds of $\mathcal{M}$–type. Nishimura [14] gives some interesting examples, and a general idea to construct such kind of manifolds is given in [1], using the wedge construction given by Bahri, Bendersky, Cohen and Gitler [3]. Following their idea, we have the result below (see Section 4 for details).

Theorem 1.6 *Let M be a toric manifold over a simple convex polytope P of dimension n, and let K be the simplicial complex dual to the boundary of P. If vertices $v_1,\dots,v_n$ of K span a* (*maximal*) *simplex in K, then the toric manifold $M(v_1,\dots,v_n)$ obtained from the simplicial wedge construction is of $\mathcal{M}$–type.*

The paper is organized as follows. The calculation of the Witt group $W^*(M)$ is done in Section 2, where Lemma 1.2 and Theorem 1.3 are proved. The characterizations of $\mathcal{S}$–type and $\mathcal{M}$–type toric manifolds (ie Theorems 1.4 and 1.5) are proved in Sections 3 and 4, respectively. Theorem 1.6 is also proved in Section 4.

2 The cohomology of small covers

Let P be a simple convex polytope of dimension n and let $\mathcal{F}=\{F_1,\dots,F_m\}$ be the set of facets (ie codimension 1 faces) of P. Since P is simple, each face $F \subset P$ of codimension l is an intersection of exactly l facets. Suppose

$$\lambda\colon \mathcal{F} \to \mathbb{Z}^n$$

is a *characteristic function* in the sense that whenever $F_{i_1},\dots,F_{i_n}$ intersect at a vertex, their images $\lambda(F_{i_1}),\dots,\lambda(F_{i_n})$ span the integer lattice $\mathbb{Z}^n$ of rank n. Occasionally we treat λ as an integral matrix of size $n \times m$ in the form

$$\lambda = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{21} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \tag{7}$$

and call it a *characteristic matrix*.

Let $T^n = S^1 \times \ldots \times S^1$ be the n–dimensional compact torus whose Lie algebra is generated by $e_1, \ldots, e_n$, an additive basis of the lattice $\mathbb{Z}^n$. For a subset $I \subset [m] = \{1, 2, \ldots, m\}$, let $T_I \subset T^n$ be the subtorus with its Lie algebra generated by $\{\lambda(F_i)\}_{i \in I}$. Then the corresponding toric manifold M is given by

$$M = T^n \times P/{\sim}, \tag{8}$$

where $(t, p) \sim (t', p')$ if and only if $p = p'$ and $t' t^{-1} \in T_I$, where $I = \{i \in [m] \mid p \in F_i\}$ (if p is in the interior of P, then $I = \varnothing$ with $T_\varnothing$ the identity of T^n).

Following [11], let $\tau \colon M \to M$ be the involution given by $\tau([t, p]) = [t^{-1}, p]$. The fixed points of τ is a submanifold of dimension n, called the small cover associated to M, and is denoted by $M_{\mathbb{R}}$ (if M is a projective nonsingular toric variety, then τ can be realized as complex conjugation).

Recall that the simplicial complex dual to the boundary of P is given by

$$K_P = \{\{i_1, i_2, \ldots, i_l\} \subset [m] \mid F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l} \neq \varnothing\}. \tag{9}$$

Theorem 2.1 [11] *Let M be a toric manifold and $M_{\mathbb{R}}$ be the corresponding small cover. Then we have the isomorphism*

$$H^*(M; \mathbb{Z}) = \mathbb{Z}[v_1, \ldots, v_m]/(I + J)$$

of graded rings, where the degree of each generator v_k is 2, I is the Stanley–Reisner ideal generated by square-free monomials $v_{i_1} v_{i_2} \cdots v_{i_s}$ such that $i_1, \ldots, i_s$ does not span a simplex in K_P, and J is generated by n linear elements

$$\sum_{k=1}^{m} a_{ik} v_k,$$

where $(a_{ik})_{k=1}^m$ is the i^{th} row of λ for $i = 1, \ldots, n$.

Likewise, we have the isomorphism

$$H^*(M_{\mathbb{R}};\mathbb{Z}_2) = \mathbb{Z}_2[x_1,\dots,x_m]/(I_{\mathbb{R}} + J_{\mathbb{R}})$$

of graded rings, where the degree of each generator x_k *is* 1*, and* $I_{\mathbb{R}}$ *and* $J_{\mathbb{R}}$ *are defined as above by replacing* v_i *with* x_i*, respectively.*

Proof of Lemma 1.2 Let ϕ: $H^*(M;\mathbb{Z}_2) \to H^*(M_{\mathbb{R}};\mathbb{Z}_2)$ be the map sending each mod 2 reduction $[v_k]$ of v_k to x_k for $k = 1,\dots,m$. Since

$$\phi(\mathrm{Sq}^2([v_k])) = \phi([v_k]^2) = x_k^2 = \mathrm{Sq}^1 x_k,$$

we have $\phi \circ \mathrm{Sq}^2 = \mathrm{Sq}^1 \circ \phi$, as desired. □

First note that since $H^*(M;\mathbb{Z}_2)$ is a trivial Sq^1–module, a pair of generators in $\underline{\mathcal{M}}$ connected by Sq^2 corresponds bijectively to a pair of elements in $H^*(M_{\mathbb{R}};\mathbb{Z}_2)$ connected by Sq^1, namely a $\mathbb{Z}_2$ summand in $H^*(M_{\mathbb{R}};\mathbb{Z})$.

In what follows, a nontrivial element in an abelian group is *primitive* if it is not divisible by an integer greater than 1. To complete the proof of Theorem 1.3, we need the following result, proved in [7, Corollary 5.3]:

Proposition 2.2 *Let* G_1^i *be the group* $\bigoplus_{\omega \in \mathrm{Row}(\lambda_2)} \widetilde{H}^i(K_\omega;\mathbb{Z})$ *and let* G_2^i *be the group* $H^{i+1}(M_{\mathbb{R}};\mathbb{Z})$ *for each dimension* $i \geq 0$*. We have a correspondence between them such that*

(1) *the free part of* G_1^i *is isomorphic to that of* G_2^i*, and*

(2) *primitive* 2^j*–torsion elements in* G_1^i *correspond bijectively to primitive* 2^{j+1}*–torsion elements in* G_2^i *for all* $j \geq 1$*.*

2.1 Proof of Theorem 1.3

By Theorem 1.1, the Witt group $W^*(M)$ of a toric manifold comes from the module $\underline{\mathcal{S}}$. Let

$$\underline{\mathcal{S}} = \bigoplus_{i=0}^{\infty} \underline{\mathcal{S}}^{2i}$$

be graded by cohomology dimensions. We claim that for each $i \geq 1$ there is an isomorphism

$$\underline{\mathcal{S}}^{2i} = \bigoplus_{w \in \mathrm{Row}(\lambda_2)} \widetilde{H}^{i-1}(K_\omega;\mathbb{Z}_2). \tag{10}$$

Let $j \geq 1$ be an integer. By Lemma 1.2, an element as a Sq^2–cocycle in $\underline{\mathcal{S}}^{2i}$ corresponds bijectively to a Sq^1–cocycle in $H^i(M_{\mathbb{R}}; \mathbb{Z}_2)$, and under the mod 2 reduction, such an element comes from $H^i(M_{\mathbb{R}}; \mathbb{Z})$, being either

(1) a torsion-free generator,

(2) a primitive 2^{j+1}–torsion element in $H^i(M_{\mathbb{R}}; \mathbb{Z})$ (after the mod 2 reduction, a primitive 2^{j+1}–torsion element in dimension i becomes a pair of Sq^1–cocycles in dimensions i and $i-1$, respectively), or

(3) a primitive 2^{j+1}–torsion element in $H^{i+1}(M_{\mathbb{R}}; \mathbb{Z})$.

To prove the claim (10), let G_1' be the group on the right-hand side. Notice that an element in G_1' comes from either

(1′) a torsion-free generator in G_1^{i-1},

(2′) a primitive 2^j–torsion element in G_1^{i-1}, or

(3′) a primitive 2^j–torsion element in G_1^i.

Comparing the two situations above, it follows from Proposition 2.2 that we get a bijection between $\underline{\mathcal{S}}^{2i}$ and G_1'.

By formulas (1)–(2), the Witt group $W^{i+1}(M)$ is isomorphic to the torsion part of $KO^{2i+1}(M)$ or $KO^{2i}(M)$, where the latter is isomorphic to $\bigoplus_k \underline{\mathcal{S}}^{2i+2+8k}$ by Theorem 1.1 and the 8–periodicity of KO–groups. Finally, formula (4) follows from (10), and the proof is completed.

Example 2.3 Let P be the n–simplex with characteristic matrix

$$\lambda = \begin{array}{c} \begin{array}{cc} 1,\dots,n & n+1 \end{array} \\ \begin{pmatrix} \quad I_n \quad & \quad * \quad \end{pmatrix} \end{array}$$

of size $n \times (n+1)$, in which I_n is the identity submatrix of size $n \times n$ and all entries in the last column are -1. We have $M = \mathbb{C}P^n$. As the dual of the boundary of P, K is the boundary of an n–simplex, thus K_ω is contractible except when n is odd and $\omega = \sum_{i=1}^n r_i$ with r_i the i^{th} row of λ_2. We see that in this case $K_\omega = K$, and

$$W^*(\mathbb{C}P^n) = W^{n_4}(\mathbb{C}P^n) = \begin{cases} \widetilde{H}^{n-1}(K; \mathbb{Z}_2) = \mathbb{Z}_2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

where $n \equiv n_4 \bmod 4$, giving the two 2–torsion elements in $KO^{2n_4-1}(\mathbb{C}P^n)$ and $KO^{2n_4-2}(\mathbb{C}P^n)$, respectively, when n is odd (see [12]).

3 Toric manifolds of $\mathcal{S}$–type

To prove Theorem 1.4, we need a detailed information of the cell structure of a small cover $M_{\mathbb{R}}$. Recall that a nontrivial element is primitive if it is not divisible by an integer greater than 1.

Lemma 3.1 *Let X be a CW complex with $Y \subset X$ a subcomplex. Suppose that each cell in either Y or X is a generator in their mod 2 (co)homology groups, respectively. Then every primitive 2–torsion element in $H^*(Y;\mathbb{Z})$ is the image of a primitive 2–torsion element in $H^*(X;\mathbb{Z})$ under the map $i^*\colon H^*(X;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ induced by the inclusion $i\colon Y \to X$.*

Proof Since a cell of Y is also a cell of X, by assumption, the inclusion induces a surjection $i_2^*\colon H^*(X;\mathbb{Z}_2) \to H^*(Y;\mathbb{Z}_2)$ in mod 2 cohomology. We see that a primitive 2–torsion element $\alpha \in H^*(Y;\mathbb{Z})$ corresponds to a pair in $H^*(Y;\mathbb{Z}_2)$ connected by the Bockstein Sq^1, which is functorial, thus there is a pair in $H^*(X;\mathbb{Z}_2)$ connected by Sq^1, corresponding to a primitive 2–torsion element $\beta \in H^*(X;\mathbb{Z})$ such that $i^*(\beta) = \alpha$. □

3.1 Cellular decompositions of small covers

In this part we recall the necessary parts we need to prove Theorem 1.4. All results in this part are essentially known and we refer the readers to [11] for more details.

We consider the mod 2 reduction of the characteristic matrix given in (7) as a function $\lambda_2\colon \mathcal{F} \to \mathbb{Z}_2^n$. Let $\mathbb{F}^n$ be the vector space $\mathbb{Z}_2^n$. For a subset $I \subset [m] = \{1, \dots, m\}$, let $\mathbb{F}_I^n$ be the subspace generated by the image of $\{\lambda_2(F_i)\}_{i \in I}$. Then the small cover is given by

$$M_{\mathbb{R}} = \mathbb{F}^n \times P/\sim, \tag{11}$$

where $(g, p) \sim (g', p')$ if and only if $p = p'$ and $g - g' \in \mathbb{F}_I^n$ with $I = \{i \in [m] \mid p \in F_i\}$. Clearly $M_{\mathbb{R}}$ admits a piecewise linear structure induced from P. As an additive group, $\mathbb{F}^n$ acts on $M_{\mathbb{R}}$ via

$$\mathbb{F}^n \times M_{\mathbb{R}} \to M_{\mathbb{R}}$$

sending $(g', [g, p])$ to $[g' + g, p]$.

Now we embed the simple convex n–polytope P linearly into the Euclidean space $\mathbb{R}^n$, and let

$$f\colon \mathbb{R}^n \to \mathbb{R} \tag{12}$$

be a linear function, being generic in the sense that f is injective on the set of vertices of P (for example, the inner product with a vector which is not orthogonal to any line in $\mathbb{R}^n$ connecting two vertices of P). In this way we have an ordering $v_1 < v_2 < \cdots < v_l$ of all vertices of P, such that $f(v_i) < f(v_j)$ if and only if $v_i < v_j$.

Since P is simple, for each vertex v_i above there are n edges connected to it. Let E_i be the set of these edges, and let E_i^- be the subset of E_i such that $\sup_{x\in e} f(x) \le f(v_i)$ for all $e \in E_i^-$. If we treat an edge $e \in E_i$ as a vector $\boldsymbol{e}$ pointing away from v_i and suppose $\boldsymbol{v}_i$ is the position vector of v_i, then

$$D_i = \left\{ \boldsymbol{v}_i + \sum_{e\in E_i} t_e \boldsymbol{e} \;\middle|\; t_e \in \left[0, \tfrac{1}{2}\right] \right\} \tag{13}$$

gives a closed neighborhood of v_i in P.

Let

$$\pi\colon M_{\mathbb{R}} \to P$$

be the orbit map sending $[g, p]$ to p. Now we have a filtration

$$P_1 \subset P_2 \subset \cdots \subset P_l = P \tag{14}$$

of polyhedra $P_i = \bigcup_{j\le i} D_j$, giving a filtration

$$M_1 \subset M_2 \subset \cdots \subset M_l = M_{\mathbb{R}} \tag{15}$$

such that $M_i = \pi^{-1}(P_i)$.

Let $I = [0, 1]$ be the unit interval. Recall that for $0 \le k \le n$, an n–dimensional k–handle (k is called the *index* of the handle) W on a piecewise linear manifold X is a copy (up to piecewise linear homeomorphism) of $I^k \times I^{n-k}$, attached to the boundary ∂X by a piecewise linear embedding

$$\varphi\colon (\partial I^k) \times I^{n-k} \to \partial X.$$

It turns out that the union $X \cup_\varphi W$ is again a piecewise linear manifold (see [16, Chapter 6]).

Proposition 3.2 *The filtration* (15) *gives a handlebody decomposition of* $M_{\mathbb{R}}$. *More precisely, the handlebody* $M_i = M_{i-1} \cup \pi^{-1}(D_i)$ *is a piecewise linear manifold* (*possibly with boundary*), *where the index of the handle*

$$\pi^{-1}(D_i) = \mathbb{F}^n \times D_i/\sim$$

is Card E_i^-.

Proof We prove by induction. Consider $M_1 = \pi^{-1}(D_1)$; under a change of basis, we may assume that $v_1 = F_1 \cap F_2 \cap \cdots \cap F_n$ and $\lambda(F_i) = e_i$, the i^{th} canonical basis element of $\mathbb{F}^n$, for $i = 1, \ldots, n$. Using coordinates $(t_e)_{e \in E_1}$ in (13) of D_1, we see that by definition (11),

$$\pi^{-1} D_1 = \mathbb{F}^n \times D_1/\sim$$

is piecewise linearly homeomorphic to the cube $(t'_e)_{e \in E_1}$, where $t'_e \in \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Now suppose M_{i-1} is a piecewise linear manifold for $2 \le i \le n$. Let $(t_e)_{e \in E_i}$ be the coordinates (13) of D_i. Since D_i is connected to P_{i-1} by edges from E_i^-, we see that the attaching part A_i of D_i along P_{i-1} consists of those points with coordinates $(t_e)_{e \in E_i}$ such that at least one component t_e is $\frac{1}{2}$ for all $e \in E_i^-$.

Since in the neighborhood D_i of v_i the coordinates are also locally standard (the images of the n facets near v_i under λ_2 span $\mathbb{F}^n$), up to a change of basis we can still use the coordinates of $\pi^{-1} D_1$ above, and it can be checked directly that

$$\pi^{-1} A_i = \mathbb{F}^n \times A_i/\sim,$$

being the attaching part of $\pi^{-1} D_i$ along M_{i-1}, is in the form $\partial(I^k) \times I^{n-k}$, where $k = \text{Card}\, E_i^-$. Therefore, by induction, M_i is a piecewise linear manifold and the proof is completed. □

Proposition 3.3 *Up to homotopy,* $M_{\mathbb{R}}$ *admits a CW decomposition*

$$M_1 \subset M_2 \subset \cdots \subset M_l = M_{\mathbb{R}}$$

in which M_i *is obtained from* M_{i-1} *by attaching a cell* e_i *of dimension* $k_i = \text{Card}\, E_i^-$ *for* $i = 1, \ldots, l$. *If* $H^*(M_s; \mathbb{Z})$ *has a primitive* 2*–torsion element, then so does* $H^*(M_{\mathbb{R}}; \mathbb{Z})$.

Proof It is well known that, up to homotopy, attaching a k–handle is equivalent to attaching a k–cell. Therefore the first statement follows from Proposition 3.2 above. In [11, Theorem 3.1] it is proved that each e_i is a generator in $H^*(M_{\mathbb{R}}; \mathbb{Z}_2)$, therefore the second statement follows from Lemma 3.1. □

Lemma 3.4 *Let F be any given facet of P with s vertices. Then we can choose a generic linear function f* (see (12)) *such that, in the ordering $v_1 < v_2 < \cdots < v_l$ of vertices of P induced from f, the first s vertices $v_1, \dots, v_s$ are those of F.*

Proof Suppose v_0 is a vertex of F. Let P be embedded linearly into $\mathbb{R}^n$ such that

(1) v_0 coincides with the origin;

(2) the n edges connecting to v_0 coincide with the n coordinate axes;

(3) P lies in the first orthant

$$\{(x_i)_{i=1}^n \mid x_i \geq 0 \text{ for } i = 1, \dots, n\}$$

and F lies in the hyperplane $\{(x_i)_{i=1}^n \mid x_n = 0\}$ such that any other vertex of P has the last coordinate $x_n > 0$.

Now we choose the n^{th} basis vector $\boldsymbol{e} = (0, 0, \dots, 0, 1)$. The function $f \colon \mathbb{R}^n \to \mathbb{R}$ given by the standard inner product with $\boldsymbol{e}$ has the property that $f(v) = 0$ for all vertices v of F, and $f(v') \geq \delta > 0$ for all vertices v' not in F. We may perturb $\boldsymbol{e}$ into the form $\boldsymbol{e}_\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}, 1)$ so that the new function f_ε given by inner product with $\boldsymbol{e}_\varepsilon$ is generic, and $f_\varepsilon(v) < \frac{\delta}{2}$ $\left(\text{resp. } f_\varepsilon(v') > \frac{\delta}{2}\right)$ for all vertices v of F (resp. for all vertices v' not in F). Clearly the function f_ε has the desired property. □

Proposition 3.5 *Let $v_1 < \dots < v_l$ be an ordering of vertices of P induced by a generic linear function f, in which $v_1, \dots, v_s$ are all vertices of a facet F. Let M_s be the handlebody in the filtration* (15). *Then we have a fiber bundle*

$$I \to M_s \to \pi^{-1} F \tag{16}$$

with $I = [-1, 1]$ the closed interval. In particular, M_s is homotopy equivalent to $\pi^{-1} F$. Moreover, $\pi^{-1} F$ is a small cover of dimension $n - 1$.

Proof Consider the polyhedron P_s in the filtration (14). By definition, F is a facet of P_s, and $M_s = \pi^{-1} P_s$. Let $F, F_1, \dots, F_r$ be all facets of P whose intersection with P_s is nonempty. It is easy to see that $f_i = F_i \cap F$ for $i = 1, \dots, r$ give all facets of F, and P_s is a tubular neighborhood of F in P. More precisely, there is a piecewise linear projection $\rho \colon P_s \to F$ such that, for all $x \in F$, $\rho^{-1}(x)$ is canonically identified with the closed interval $[0, 1]$, and

$$\rho^{-1} f_i = F_i \quad \text{for } i = 1, \dots, r. \tag{17}$$

By construction (11), we have

$$M_s = \pi^{-1} P_s = \mathbb{F}^n \times P_s/\sim, \quad \pi^{-1} F = \mathbb{F}^n \times F/\sim.$$

Let $\widetilde{\rho}\colon M_s \to \pi^{-1} F$ be the map sending $[g, p]$ to $[g, \rho(p)]$. By (17), $\widetilde{\rho}$ is a well-defined surjection, and it can be checked that $\pi^{-1}([g, x])$ is the disjoint union of two pieces of intervals $[g, \rho^{-1}(x)]$ and $[g + \lambda_2(F), \rho^{-1}(x)]$, glued along their common boundary $[g, x]$, therefore $\pi^{-1}([g, x])$ is canonically identified with the interval I. As a conclusion, we have the fiber bundle (16), as desired.

It remains to show that $\pi^{-1} F$ is a small cover. Let $\mathbb{F} \cong \mathbb{Z}_2$ be the subspace of $\mathbb{F}^n$ generated by the element $\lambda_2(F)$, and let $\mathbb{F}^{n-1}$ be the quotient $\mathbb{F}^n/\mathbb{F}$. Since every point $[g, p] \in \pi^{-1} F$ is identified with $[g + \lambda_2(F), p]$, we see that

$$\pi^{-1} F = \mathbb{F}^n \times F/\sim = (\mathbb{F} \oplus \mathbb{F}^{n-1}) \times F/\sim = \mathbb{F}^{n-1} \times F/\sim',$$

where $(g, p) \sim (g', p')$ if and only if $p = p'$ and $g - g' \in \mathbb{F}_I^{n-1}$, where $\mathbb{F}_I^{n-1}$ is generated by $\{\lambda_2(F_i) \mid p \in F_i\}$ in the quotient group $\mathbb{F}^{n-1}$. □

The corollary below follows directly from the proof above.

Corollary 3.6 *Let $F, F_1, \dots F_r$ be facets of P such that $f_i = F_i \cap F$ give all facets of F for $i = 1, \dots, r$. Suppose that when restricted to $\{F_i\}_{i=1}^r$, the characteristic matrix λ_2 has the form*

$$\lambda_2|_{P_s} = \begin{array}{c} \begin{matrix} F & F_1 & F_2 & \cdots & F_r \end{matrix} \\ \begin{pmatrix} 1 & a_{11} & a_{12} & \cdots & a_{1r} \\ 0 & a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n1} & a_{n2} & \cdots & a_{nr} \end{pmatrix} \end{array} \tag{18}$$

in which $\lambda_2(F) = e_1$ is the canonical basis element. Then the small cover $\pi^{-1} F$ has the characteristic matrix

$$\lambda_2|_F = \begin{array}{c} \begin{matrix} f_1 & f_2 & \cdots & f_r \end{matrix} \\ \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} \end{pmatrix} \end{array}. \tag{19}$$

by removing the first row and the first column of (18). *More generally, if $\lambda(F)$ has only one nonzero entry in the i^{th} row, then the characteristic matrix of $\pi^{-1} F$ is obtained by removing the column $\lambda(F)$ together with the i^{th} row.*

3.2 Proof of Theorem 1.4

By Lemma 1.2, (S1) is equivalent to the condition that there are no pairs in $H^*(M_{\mathbb{R}};\mathbb{Z}_2)$ connected by Sq^1, namely there are no primitive 2–torsion elements in $H^*(M_{\mathbb{R}};\mathbb{Z})$. We denote the latter condition by (S1′). Since (S4) $\Longrightarrow$ (S3) is proved in [11] and (S3) $\Longrightarrow$ (S2) $\Longrightarrow$ (S1′) is obvious, we only need to prove (S1′) $\Longrightarrow$ (S4).

The proof is an induction on the dimension n. A 1–dimensional small cover is a circle, and (S4) holds trivially. When $n=2$, (S1′) implies that $M_{\mathbb{R}}$ is orientable, and (S4) follows from the classification of small covers in dimension 2 (see [11, page 427, Example 1.20]).

Consider $n=3$. Let v and v' be two vertices connected by an edge e. Up to a basis change of $\mathbb{F}^n$ and relabel of facets, suppose $v = F\cap F_1\cap F_2$, $v' = F\cap F_1\cap F_3$, $e = F\cap F_1$, and $\lambda(F)$, $\lambda(F_1)$ and $\lambda(F_2)$ form the standard basis of $\mathbb{F}^3$. Observe that, by the nondegeneracy of λ_2, near e it has the form

$$\begin{array}{cccc} F & F_1 & F_2 & F_3 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & a_{13} \\ 0 & 1 & 0 & a_{23} \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{20}$$

We will show that $a_{13}=a_{23}=0$.

By Lemma 3.4, starting from the facet F, we get a filtration

$$M_1 \subset M_2 \subset \cdots \subset M_s \subset \cdots \subset M_l = M_{\mathbb{R}}$$

as a handle decomposition of $M_{\mathbb{R}}$, in which M_s is homotopy equivalent to the 2–dimensional small cover $\pi^{-1}F$. We see that, when restricted to $\{F_i\}_{i=1}^r$, $\lambda_2|_{P_s}$ has the form (18) and $\pi^{-1}F$ has its characteristic matrix in the form (19). Notice that

$$H^*(\pi^{-1}F;\mathbb{Z}) = H^*(M_s;\mathbb{Z})$$

does not have primitive 2–torsion elements, otherwise so would $H^*(M_{\mathbb{R}};\mathbb{Z})$ by Proposition 3.3, a contradiction to (S1′). As a conclusion, by the induction hypothesis, we have $a_{23}=0$ and

$$\lambda_2|_F = \begin{array}{c} \begin{array}{cccccc} f_1 & f_2 & f_3 & \cdots & f_{r-1} & f_r \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & * & * \\ 0 & 1 & 1 & \cdots & * & * \end{pmatrix} \end{array},$$

where there is only one nonzero entry in each column. Therefore, by Corollary 3.6,

$$\lambda_2|_{P_s} = \begin{array}{c} \begin{array}{cccccc} F & F_1 & F_2 & F_3 & \cdots & F_r \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & a_{13} & \cdots & a_{1r} \\ 0 & 1 & 0 & 0 & \cdots & * \\ 0 & 0 & 1 & 1 & \cdots & * \end{pmatrix} \end{array}.$$

It remains to show that $a_{13} = 0$. Now, replacing F by F_1, we use Lemma 3.4 again to get a filtration (ie a different cell decomposition of $M_{\mathbb{R}}$)

$$M_1' \subset M_2' \subset \cdots M_{s_1}' \subset \cdots \subset M_l' = M_{\mathbb{R}}$$

with $M_{s_1}' = \pi^{-1} P_{s_1}'$ homotopy equivalent to the 2–dimensional small cover $\pi^{-1} F_1$, which has no primitive 2–torsion elements by Proposition 3.3. By Corollary 3.6, the characteristic matrix of $\pi^{-1} F_1$ is in the form

$$\lambda_{F_1} = \begin{array}{c} \begin{array}{cccc} f' & f_2' & f_3' & \cdots \end{array} \\ \begin{pmatrix} 1 & 0 & a_{13} & \cdots \\ 0 & 1 & 1 & \cdots \end{pmatrix} \end{array},$$

where $f' = F \cap F_1$, $f_2' = F_2 \cap F_1$ and $f_3' = F_3 \cap F_1$; thus we get $a_{13} = 0$ by the induction hypothesis.

As a conclusion, for any pair v and v' of vertices connected by an edge e, in the four columns of λ_2 near e, if the three of them corresponding to v have only one nonzero entry, so does the fourth one. In this way we start from a chosen vertex v (where λ_2 is standard) and run through all other vertices connected to v by edges; we see that (S4) holds when $n = 3$.

In a higher dimension n the proof is similar. Given any two vertices v and v' of P connected by an edge $e = F \cap F_1 \cap \cdots \cap F_{n-2}$, where $v = F \cap F_1 \cap \cdots \cap F_{n-1}$ and $v' = F \cap F_1 \cap \cdots \cap F_{n-2} \cap F_n$, we assume that, near e, λ_2 is in the form

$$\begin{array}{c} \begin{array}{cccccc} F & F_1 & \cdots & F_{n-2} & F_{n-1} & F_n \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & a_{n1} \\ 0 & 1 & \cdots & 0 & 0 & * \\ 0 & 0 & \cdots & 0 & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & * \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \end{array}. \tag{21}$$

By the induction hypothesis and repeating the argument above, we can show that the last column has only one nonzero entry. Therefore, in the $n+1$ columns of λ_2 associated to any pair of vertices connected by an edge, if n of them corresponding to one of the vertices have only one nonzero entry, so does the last column. Since all vertices are connected by edges, it follows that, near all edges, λ_2 has only one nonzero entry in each column, namely (S4) holds. The proof is completed.

4 Toric manifolds of $\mathcal{M}$–type

Proof of Theorem 1.5 The equivalence of (M1) and (M2) is given in Theorem 1.1. By Lemma 1.2, (M1) is equivalent to the condition that the Sq^1–cohomology of $\widetilde{H}^*(M_{\mathbb{R}};\mathbb{Z}_2)$ is trivial, which is (M3). The equivalence of (M2) and (M4) is given by Theorem 1.3. □

The remaining part of this section is devoted to a proof of Theorem 1.6.

4.1 Toric manifolds from simplicial wedge constructions [3]

Let $K \subset 2^V$ be an abstract simplicial complex with vertex set V. For a simplex $\sigma \in K$, the *link* of σ is the subcomplex

$$\mathrm{Link}_K \sigma = \{\tau \in K \mid \sigma \cup \tau \in K,\ \sigma \cap \tau = \varnothing\}$$

and the *join* of two simplicial complexes K_1 and K_2 with disjoint vertex sets V_1 and V_2, respectively, is given by

$$K_1 * K_1 = \{\sigma \subset V_1 \cup V_2 \mid \sigma = \sigma_1 \cup \sigma_2,\ \sigma_1 \in K_1,\ \sigma_2 \in K_2\}.$$

For a subset $S \subset V$, let K_S be the *full subcomplex*

$$K_S = \{\sigma \subset S \mid \sigma \in K\}. \tag{22}$$

Suppose the vertex set of K is $V = \{v_i\}_{i=1}^m$. The *simplicial wedge construction* on the j^{th} vertex v_j is a new simplicial complex $K(v_j)$ with vertex set $\{v_i\}_{i=1}^m \cup \{v_j'\}$, which is given by

$$K(v_j) = \{v_j, v_j'\} * \mathrm{Link}_K(v_j) \cup \{v_j\} * K_{V\setminus\{v_j\}} \cup \{v_j'\} * K_{V\setminus\{v_j\}}. \tag{23}$$

For a simple convex polytope P with facet set $\{F_i\}_{i=1}^m$, let $K_P \subset 2^V$ be the dual to the boundary of P, namely $V = \{v_i\}_{i=1}^m$ and $\{v_{i_t}\}_t \in K$ if and only if $\bigcap_t F_{i_t}$ is nonempty.

Let M be a toric $2n$–manifold over P with characteristic matrix (7). It is proved in [3] that there is a new toric manifold $M(v_j)$ of dimension $2n+2$ over a simple convex polytope $P(v_j)$ with its boundary dual to $K_P(v_j)$ (see [13] for an explicit construction from P to $P(v_j)$), and the characteristic matrix of $M(v_j)$ is

$$\lambda(v_j) = \begin{array}{c} \begin{array}{ccccccccc} v_1 & v_2 & \cdots & v_{j-1} & v_j & v_{j+1} & \cdots & v_m & v_j' \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1m} & 0 \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2j} & a_{2,j+1} & \cdots & a_{2m} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{nj} & a_{n,j+1} & \cdots & a_{nm} & 0 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \end{pmatrix} \end{array}, \tag{24}$$

which is obtained from λ by adding the last row and the last column with nonzero entries ± 1.

4.2 Proof of Theorem 1.6

Suppose $v_1, \dots, v_n$ span a simplex in K_P. Let $M(v_1, \dots, v_n)$ (resp. $M_{\mathbb{R}}(v_1, \dots, v_n)$) be the toric manifold (resp. small cover) obtained by consecutive simplicial wedge constructions from M (resp. $M_{\mathbb{R}}$). After a basis change and the mod 2 reduction, the characteristic matrix $\lambda_2(v_1, \dots, v_n)$ of $M_{\mathbb{R}}(v_1, \dots, v_n)$ is in the form

$$\lambda_2(v_1, \dots, v_n) = \begin{array}{c} \begin{array}{ccc} v_1, \dots, v_n & v_{n+1}, \dots, v_m & v_1' \dots, v_n' \end{array} \\ \begin{pmatrix} I_n & * & \mathbf{0} \\ I_n & \mathbf{0} & I_n \end{pmatrix} \end{array} \tag{25}$$

with size $2n \times (m+n)$, in which I_n the identity matrix of size $n \times n$.

Let $\omega' = (w_i')_{i=1}^{m+n}$ be a nonzero vector from the row space $\mathrm{Row}(\lambda_2(v_1, \dots, v_n))$ spanned by the rows $r_1, \dots, r_{2n}$ of $\lambda_2(v_1, \dots, v_n)$, and let $K'_{\omega'}$ be the full subcomplex of $K' = K(v_1, \dots, v_n)$. By definition, the vertex set of $K'_{\omega'}$ is

$$\{v_i \mid w_i' = 1,\ 1 \le i \le m\} \cup \{v_j' \mid w_j' = 1,\ m+1 \le j \le m+n\}.$$

By Theorem 1.5, we only need to show that $K'_{\omega'}$ is contractible.

By construction (23), for the n pairs of vertices $\{v_i, v_i'\}_{i=1}^n$, if the full subcomplex $K'_{\omega'}$ contains only one vertex $v \in \{v_i, v_i'\}$ but does not contain the whole edge $\{v_i, v_i'\}$, then it is a cone with apex v, which is contractible. If we write $\omega' = \omega_\alpha' + \omega_\beta'$, where ω_α' (resp. ω_β') is a linear sum of $r_1, \dots, r_n$ (resp. $r_{n+1}, \dots, r_{2n}$), we see that this is the case, except when ω_α' is zero. However, in the last case, $K'_{\omega'}$ is the simplex spanned by $\{v_i, v_i' \mid w_i' = 1\}$, which is clearly contractible.

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