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Trees, dendrites and the Cannon–Thurston map

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When $1 \to H \to G \to Q \to 1$ is a short exact sequence of three word-hyperbolic groups, Mahan Mj (formerly Mitra) has shown that the inclusion map from H to G extends continuously to a map between the Gromov boundaries of H and G. This boundary map is known as the Cannon–Thurston map. In this context, Mj associates to every point z in the Gromov boundary of Q an "ending lamination" on H which consists of pairs of distinct points in the boundary of H. We prove that for each such z, the quotient of the Gromov boundary of H by the equivalence relation generated by this ending lamination is a dendrite, that is, a tree-like topological space. This result generalizes the work of Kapovich and Lustig and Dowdall, Kapovich and Taylor, who prove that in the case where H is a free group and Q is a convex cocompact purely atoroidal subgroup of $\mathrm{Out}(F_N)$, one can identify the resultant quotient space with a certain $\mathbb{R}$–tree in the boundary of Culler and Vogtmann's Outer space.

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1 Introduction

In [7], Cannon and Thurston showed that when $M = (S \times [0,1])/\big((x,0) \sim (\phi(x),1)\big)$ is the mapping torus of a closed hyperbolic surface S by a pseudo-Anosov homeomorphism ϕ of S, the inclusion $i\colon \pi_1 S \to \pi_1 M$ extends to a continuous, surjective, $\pi_1(S)$–equivariant map

$$\mathbb{S}^1 = \partial\mathbb{H}^2 = \partial\pi_1 S \xrightarrow{\partial i} \partial\pi_1 M = \partial\mathbb{H}^3 = \mathbb{S}^2.$$

Although published in 2007, this work has sparked much consideration since its circulation as a preprint in 1984. In modern terminology, if H and G are word-hyperbolic groups with $H \le G$ and the inclusion map $i\colon H \to G$ extends to a (necessarily unique and H–equivariant) continuous map $\partial i\colon \partial H \to \partial G$ between the Gromov boundaries of H and G, then ∂i is called the *Cannon–Thurston map*. This definition naturally extends to the more general setting of hyperbolic metric spaces. Such a map automatically exists when H is an undistorted (ie quasiconvex) subgroup of G, since in that case the inclusion map is a quasi-isometric embedding. The 1984 result of Cannon and Thurston gave the first nontrivial example of the existence of such a map.

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Since then, Mj (formerly Mitra) has studied the existence of Cannon–Thurston maps in settings which involve distorted subgroups of hyperbolic groups [27; 28; 29]. In particular, Mj showed in [27] that when

$$(*) \qquad 1 \to H \to G \to Q \to 1$$

is a short exact sequence of infinite word-hyperbolic groups, the Cannon–Thurston map $\partial i \colon \partial H \to \partial G$ exists. Since an infinite normal subgroup of infinite index in a word-hyperbolic group G is not quasiconvex [16], this result gives another nontrivial example of the existence of Cannon–Thurston maps. It has been shown by Kapovich and Short [24] that when H is an infinite normal subgroup of a hyperbolic group G, the limit set of H in ∂G is all of ∂G. As this limit set is precisely the image of ∂H under ∂i, it follows that the Cannon–Thurston map is surjective in this setting.

In [26], Mj developed a theory of "algebraic ending laminations" for hyperbolic group extensions to describe when points in ∂H are identified under the Cannon–Thurston map ∂i in the setting described above. This work provides an analog of the theory of ending laminations in the context of pseudo-Anosov homeomorphisms of surfaces developed by Thurston; see Fathi, Laudenbach and Poénaru [15]. To each point $z \in \partial Q$, Mj associates an "algebraic ending lamination" on H, $\Lambda_z \subseteq \partial^2 H$, where $\partial^2 H = (\partial H \times \partial H) - \mathrm{diag}$. The main result of [27] states that two distinct points $p, q \in \partial H$ are identified under the Cannon–Thurston map if and only if there exists some $z \in \partial Q$ for which (p, q) is a leaf of the ending lamination Λ_z.

If H is a torsion-free, infite-index, word-hyperbolic, normal subgroup of a word-hyperbolic group G, it follows from combined work of Mosher [32], Paulin [33], Rips and Sela [34] and Bestvina and Feighn [3] that H must be a free product of free groups and surface groups. For a brief explanation of this, see [26]. Suppose H is the fundamental group of a closed hyperbolic surface S and Γ is a convex cocompact subgroup of $\mathrm{Mod}(S)$ (and hence Γ is word-hyperbolic; see Farb and Mosher [14]). Then, Γ naturally gives rise to a short exact sequence $1 \to H \to E_\Gamma \to \Gamma \to 1$ coming from Birman's short exact sequence for S. Hamenstädt [19] has shown that in this setting, the extension group E_Γ is hyperbolic and the orbit map of Γ into the curve complex of S is a quasi-isometric embedding. Since the boundary of the curve complex consists of ending laminations on S — see Klarreich [25] — it follows that to each point $z \in \partial\Gamma$ there is an associated ending lamination L_z on the surface S. Mj and Rafi [30] showed that the algebraic ending lamination Λ_z is the same as the diagonal closure of the surface lamination L_z. To each such ending lamination L_z, there is an associated dual $\mathbb{R}$–tree T_z which can be constructed by lifting L_z to $\widetilde{S}$ and collapsing each leaf and complementary component to a point. For more details, see for example Bestvina [2] and Coulbois, Hilion and Lustig [10].

In the free group setting, Mj's algebraic ending laminations for hyperbolic extensions of free groups are closely related to the theory of algebraic laminations on free groups developed by Coulbois, Hilion and Lustig in [10]. For any subgroup $\Gamma \leq \mathrm{Out}(F_N)$, the full preimage of Γ under the quotient map $\mathrm{Aut}(F_N) \to \mathrm{Out}(F_N)$, also denoted by E_Γ, fits into the short exact sequence $1 \to F_N \to E_\Gamma \to \Gamma \to 1$. The main result of Dowdall and Taylor [12] states that whenever $\Gamma \leq \mathrm{Out}(F_N)$ is a convex cocompact and purely atoroidal subgroup, the extension group, E_Γ, is word-hyperbolic. In [11], Dowdall, Kapovich and Taylor study the fibers of the Cannon–Thurston map $\partial i \colon \partial F_N \to \partial E_\Gamma$ in the case where $\Gamma \leq \mathrm{Out}(F_N)$ is convex cocompact and purely atoroidal. Since Γ is convex cocompact, the orbit map to the free factor complex, $\mathcal{F}$, is a quasi-isometric embedding — see Hamenstädt and Hensel [21] — and hence extends to a continuous embedding $\partial\Gamma \to \partial\mathcal{F}$. By work of Bestvina and Reynolds [4] and Hamenstädt [20], $\partial\mathcal{F}$ consists of equivalence classes of arational F_N–trees. Therefore, there is a class of arational F_N–trees, T_z, associated to each point $z \in \partial\Gamma$. Moreover, each such tree T_z comes equipped with the "dual lamination" $L(T_z)$, defined by Coulbois, Hilion and Lustig in [10]. A key result of [11] states that $\Lambda_z = L(T_z)$ for each $z \in \partial\Gamma$. This theorem extends the result of Kapovich and Lustig [23], who prove this equality for the specific case where $\Gamma = \langle\varphi\rangle$ is the cyclic group generated by a fully irreducible, atoroidal automorphism of F_N.

Given an $\mathbb{R}$–tree T, Coulbois, Hilion and Lustig define a suitable topology on $\widehat{T} = \overline{T} \cup \partial T$, where $\overline{T}$ denotes the metric completion of T and ∂T is the Gromov boundary. This topology, known as the "observers' topology", is coarser than the Gromov topology and ensures that $\widehat{T}$ is compact. Recall that a *dendrite* is a compact, connected, locally connected metrizable space which contains no simple closed curves. Coulbois, Hilion and Lustig [9] show that, for any $\mathbb{R}$–tree T, $\widehat{T}$ equipped with the "observers' topology" is a dendrite, as well as a proper, Hausdorff metric space. Dendrites naturally arise from this compactification of simplicial trees, but in general can be much more complicated spaces such as certain Julia sets. Combining the result of [11] with a general result from [9] implies that for each $z \in \partial\Gamma$, for Γ a convex-cocompact and purely atoroidal subgroup of $\mathrm{Out}(F_N)$, $\partial F_N/\Lambda_z$ equipped with the quotient topology is homeomorphic to $\widehat{T}_z$ equipped with the "observers' topology". In particular, $\partial F_N/\Lambda_z$ is homeomorphic to a dendrite. Here, $\partial F_N/\Lambda_z$ means the quotient space of ∂F_N by the equivalence relation on ∂F_N generated by $\Lambda_z \subseteq \partial F_n \times \partial F_n$. The main result of the present paper extends this result as follows:

Theorem A *Let $1 \to H \to G \to Q \to 1$ be a short exact sequence of infinite, finitely generated, word-hyperbolic groups. For each $z \in \partial Q$, let Λ_z denote the algebraic ending lamination on H associated to z. Then, for each $z \in \partial Q$, the space $\partial H/\Lambda_z$ is homeomorphic to a dendrite.*

We now sketch the proof of Theorem A. Let $P\colon \Gamma_G \to \Gamma_Q$ denote the map which is induced by the quotient map $P\colon G \to Q$. Let $z \in \partial Q$ be arbitrary and take any $z' \in \partial Q$ with $z' \neq z$. Consider a bi-infinite geodesic $\gamma = (z', z) \subseteq \Gamma_Q$ and define the space $X(\gamma)$ to be the subgraph of Γ_G given by $X(\gamma) = P^{-1}(\gamma)$. We show that $X(\gamma)$ satisfies the properties of being a metric graph bundle, as defined by Mj and Sardar [31], and that $X(\gamma)$ is hyperbolic (Proposition 3.10). We go on to show that $X(\gamma)$ also satisfies the properties of being a bi-infinite hyperbolic stack, as defined by Bowditch [5], with fibers being copies of the Cayley graph of H (Proposition 4.6). We then look at the semi-infinite stack $X(\gamma)^+$ which lies over the geodesic ray $\gamma^+ = [z_0, z)$, where $z_0 \in (z', z)$. We denote the natural "0^{th} slice" map from $\Gamma_H \to X(\gamma)^+$ by i_γ^+, and also refer to the continuous extension of this map to $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ as the Cannon–Thurston map. We then show the following:

Theorem B *Let $1 \to H \to G \to Q \to 1$ be a short exact sequence of infinite, finitely generated, word-hyperbolic groups. Let $z, z' \in \partial Q$ be distinct and let $\gamma \subseteq \Gamma_Q$ be a bi-infinite geodesic in Γ_Q between z and z'. Let $i_\gamma^+\colon \Gamma_H \to X(\gamma)^+$ be the inclusion of Γ_H into the semi-infinite stack $X(\gamma)^+$ over $\gamma^+ = [z_0, z)$ for some $z_0 \in \gamma$, and let $i_\gamma\colon \Gamma_H \to X(\gamma)$ be the inclusion of Γ_H into the bi-infinite stack $X(\gamma)$ over γ. Then*

(1) *the Cannon–Thurston map $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ is surjective; and*

(2) *the Cannon–Thurston map $\partial i_\gamma\colon \partial H \to \partial X(\gamma)$ is surjective.*

Using the work of Mj from [26], we show that the following holds:

Theorem C *Let $1 \to H \to G \to Q \to 1$ be a short exact sequence of infinite, finitely generated, word-hyperbolic groups. Let $z, z' \in \partial Q$ be distinct and let $\gamma \subseteq \Gamma_Q$ be a bi-infinite geodesic between z and z'. Let $i_\gamma^+\colon \Gamma_H \to X(\gamma)^+$ be the inclusion of Γ_H into the semi-infinite stack $X(\gamma)^+$ over $\gamma^+ = [z_0, z)$ for some $z_0 \in \gamma$, and let $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ be the Cannon–Thurston map.*

Then, for any distinct $u, v \in \partial H$, we have $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$ if and only if (u, v) is a leaf of the ending lamination Λ_z.

To finish the proof of Theorem A, note that by a general result of Bowditch [5], $\partial X(\gamma)^+$ is a dendrite (Proposition 6.1). Theorem C implies that the Cannon–Thurston map $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ quotients through to an injective map $\tau_z\colon \partial H/\Lambda_z \to \partial X(\gamma)^+$. Since ∂i_γ^+ is continuous, the map τ_z is also continuous. By Theorem B, τ_z is also surjective. Thus, $\tau_z\colon \partial H/\Lambda_z \to \partial X(\gamma)^+$ is a continuous bijection between two compact topological spaces, where $\partial X(\gamma)^+$ is Hausdorff. Therefore, τ_z is a homeomorphism.

In Section 2, we provide background on hyperbolic metric spaces and hyperbolic groups. The space $X(\gamma)$ is introduced in Section 3 and is shown to be hyperbolic. In Section 4, we show that $X(\gamma)$ is a bi-infinite, hyperbolic stack and use this to prove Theorem B. The ending lamination Λ_z is defined in Section 5 and several technical results are given which lead to the proof of Theorem C. Finally, Theorem A is proved in Section 6.

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2 Background

In this section, we will discuss some basic definitions and facts about hyperbolic metric spaces and hyperbolic groups. For general references on hyperbolic spaces, groups and their boundaries, see [1; 6; 8; 17; 16; 18; 22].

2.1 Hyperbolic metric spaces

Let (X,d) be a geodesic metric space. For any $x, y \in X$, we will denote a geodesic between x and y by $[x,y]_X$, or by $[x,y]$ if the space is clear. Given any three points $x, y, z \in X$, the *Gromov product* of x and y relative to z is defined to be

$$(x,y;X)_z := \tfrac{1}{2}(d(x,z)+d(y,z)-d(x,y)).$$

If the space X is clear, we will simply write $(x,y)_z$ for $(x,y;X)_z$.

Let $\delta \geq 0$. A geodesic metric space (X, d) is called *δ–hyperbolic* if for any $x, y, z \in X$ and any geodesics $[z, x]$ and $[z, y]$ in X the following holds. Let $x' \in [z, x]$ and $y' \in [z, y]$ be any points such that $d(z, x') = d(z, y') \leq (x, y)_z$. Then $d(x', y') \leq \delta$. Note that this property implies that for any geodesic triangle $\Delta = [x, y] \cup [y, z] \cup [z, x]$ in X, each side of Δ is contained in the δ–neighborhood of the union of the other two sides. See [1; 6] for more details and other equivalent definitions of hyperbolicity. The metric space (X, d) is said to be *hyperbolic* if it is δ–hyperbolic for some $\delta \geq 0$. Note that in a hyperbolic metric space, the Gromov product $(x, y)_z$ measures how closely the geodesics $[z, x]$ and $[z, y]$ travel.

A sequence of points $(x_n)_{n\in\mathbb{N}} \in X$ is said to *converge to infinity* if, for some basepoint $x \in X$,

$$\liminf_{i,j\to\infty} (x_i, x_j)_x = \infty.$$

It is known that this definition is independent of basepoint. Two sequences (x_n) and (y_n) in X which converge to infinity are said to be *equivalent* if

$$\liminf_{i,j\to\infty} (x_i, y_j)_x = \infty.$$

We denote the equivalence class of a sequence (x_n) converging to infinity by $[(x_n)]$ and again note that this equivalence is independent of chosen basepoint. The *Gromov boundary* of X is defined to be

$$\partial X := \{[(x_n)] \mid (x_n) \text{ is a sequence converging to infinity in } X\}.$$

We can also represent ∂X by equivalence classes of geodesic rays, where two rays represent the same point at infinity if they have bounded Hausdorff distance.

If X is a proper hyperbolic metric space, then ∂X is known to be compact, and so the space $\widehat{X} = X \cup \partial X$ can be considered a compactification of X. There is a natural topology that is carried by ∂X which can be extended to a topology on $\widehat{X}$. Fix a basepoint $x \in X$ and, for any $p \in \partial X$ and $r \geq 0$, define the set

$$U(p, r) := \Big\{q \in \partial X \mid \text{there exist sequences } (x_n) \text{ and } (y_n) \text{ with } [(x_n)] = p \\ \text{and } [(y_n)] = q \text{ such that } \liminf_{i,j\to\infty} (x_i, y_j)_x \geq r\Big\}.$$

The topology on ∂X is then generated by $\{U(p, r) \mid r \geq 0\}$. To get a topology on $\widehat{X}$, we define for each $p \in \partial X$ and $r \geq 0$ the additional sets

$$U'(p, r) := \Big\{y \in X \mid \liminf_{i\to\infty} (x_i, y)_x \geq r \text{ for some sequence } (x_n) \text{ with } [(x_n)] = p\Big\}.$$

For each $p \in \partial X$ we put the basis of neighborhoods for $p \in \widehat{X}$ to be

$$\{U(p,r) \cup U'(p,r) \mid r \geq 0\}.$$

For each $y \in X$, we use the same neighborhood basis as in X. For a proper hyperbolic space, these topologies can be equivalently defined in terms of geodesic rays. Informally, two points $a, b \in \widehat{X}$ are close if geodesic rays which begin at some basepoint x and end at a and b stay uniformly Hausdorff close for a long time. Both formulations of ∂X are known to be independent of basepoint. For more details, see [22].

Let (X, d_X) and (Y, d_Y) be metric spaces, and let $\kappa \geq 1$ and $\epsilon \geq 0$. A map $f\colon X \to Y$ is said to be a (κ,ϵ)*–quasi-isometric embedding* if for all $x_1, x_2 \in X$,

$$\frac{1}{\kappa} d_X(x_1,x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \kappa d_X(x_1,x_2) + \epsilon.$$

A (κ,ϵ)*–quasigeodesic* in a metric space (X,d) is the image of a (κ,ϵ)–quasi-isometric embedding $f\colon I \to X$, where $I \subseteq \mathbb{R}$ is a subinterval. The map f itself is also referred to as a (κ,ϵ)–quasigeodesic. It is known that quasigeodesics "diverge exponentially" in a hyperbolic metric space:

Proposition 2.1 (Mj [26, Proposition 2.4]) *Let (X,d) be a δ–hyperbolic, geodesic metric space. Given $K \geq 1$, $\epsilon \geq 0$ and $\alpha \geq 0$, there exist $b > 1$, $A > 0$ and $C > 0$ such that the following holds:*

If r_1 and r_2 are two (K,ϵ)–quasigeodesics in X with $d(r_1(0), r_2(0)) \leq \alpha$ and there exists $T \geq 0$ with $d(r_1(T), r_2(T)) \geq C$, then any path joining $r_1(T+t)$ to $r_2(T+t)$ and lying outside the union of the $(T+t-1)/(K+\epsilon)$–balls around $r_1(0)$ and $r_2(0)$ has length greater than Ab^t for all $t \geq 0$.

The following are basic facts that we will need later about hyperbolic metric spaces:

Proposition 2.2 *Let (X,d) be a δ–hyperbolic metric space and let $A \geq 0$. If $x, y, z \in X$ are such that $(x,z)_y \leq A$, then $[x,y] \cup [y,z]$ is a $(1,2A)$–quasigeodesic.*

Proof Suppose that x, y and z are such that $(x,z)_y \leq A$. We need to show that $d(p,y) + d(y,q) \leq d(p,q) + 2A$ for all $p \in [x,y]$ and $q \in [y,z]$. By the triangle inequality,

$$\begin{aligned}(p,q)_y &= \tfrac{1}{2}(d(p,y) + d(q,y) - d(p,q)) \\ &\leq \tfrac{1}{2}\big(d(p,y) + d(q,y) - (d(p,z) - d(q,z))\big) \\ &= \tfrac{1}{2}(d(p,y) + d(y,z) - d(p,z)) = (p,z)_y.\end{aligned}$$

Similarly, $(p,z)_y \le (x,z)_y$. Therefore, $(p,q)_y \le A$ by hypothesis, and so

$$d(p,q) + d(y,q) = d(p,q) + 2(p,q)_y \le d(p,q) + 2A.$$

Hence, $[x,y] \cup [y,z]$ is a $(1,2A)$–quasigeodesic. □

The next proposition says that geodesic quadrilaterals in hyperbolic metric spaces must either be "tall and thin" or "short and long".

Proposition 2.3 *Let (X,d) be a δ–hyperbolic metric space and let $x,y,z,w \in X$. Then, either there are points $a \in [x,y]$ and $a' \in [z,w]$ with $d(a,a') \le 2\delta$, or there are points $b \in [x,w]$ and $b' \in [y,z]$ with $d(b,b') \le 2\delta$.*

Proof Consider the geodesic quadrilateral with sides $[x,y]$, $[y,z]$, $[z,w]$ and $[x,w]$. Draw in the diagonal $[y,w]$ and consider the two triangles $xyw = [x,y] \cup [y,w] \cup [w,x]$ and $ywz = [y,w] \cup [w,z] \cup [z,y]$. Mark internal points $p \in [x,y]$, $q \in [x,w]$ and $r \in [y,w]$ such that $d(x,p) = d(x,q)$, $d(w,q) = d(w,r)$ and $d(y,p) = d(y,r)$. Similarly, mark internal points $q' \in [y,z]$, $p' \in [z,w]$ and $r' \in [y,w]$ such that $d(z,q') = d(z,p')$, $d(w,p') = d(w,r')$ and $d(y,q') = d(y,r')$. Note that, since X is δ–hyperbolic, we have that $\max\{d(p,q), d(q,r), d(p,r)\} \le \delta$ and $\max\{d(p',q'), d(q',r'), d(p',r')\} \le \delta$. There are two cases to consider.

First, suppose that $d(y,r) \le d(y,r')$. In this case, there exists some point $s \in [y,w]$ between r and r' such that $d(s,[x,w]) \le \delta$ and $d(s,[y,z]) \le \delta$. Hence, there exist $b \in [x,w]$ and $b' \in [y,z]$ such that $d(b,b') \le d(b,s) + d(s,b') \le 2\delta$.

Now, suppose that $d(y,r) > d(y,r')$. In this case, there is some point $s' \in [y,w]$ between r' and r such that $d(s',[x,y]) \le \delta$ and $d(s',[z,w]) \le \delta$. So, there is some $a \in [x,y]$ and $a' \in [z,w]$ with $d(a,a') \le 2\delta$. □

Proposition 2.4 *Let (X,d) be a δ–hyperbolic metric space and let $A \ge 0$. If $x,y,z,w \in X$ are such that $(x,z)_y \le A$, $(y,w)_z \le A$ and $d(y,z) > 10\delta + 2A$, then $[x,y] \cup [y,z] \cup [z,w]$ is a $(1, 4\delta + 4A)$–quasigeodesic.*

Proof Fix $x,y,z,w \in X$ such that $(x,z)_y \le A$, $(y,w)_z \le A$ and $d(y,z) > 10\delta + 2A$. We need to show that for all $p,q \in [x,y] \cup [y,z] \cup [z,w]$, the distance between p and q along $[x,y] \cup [y,z] \cup [z,w]$ is at most $d(p,q) + 4\delta + 4A$. This statement is certainly true if p and q are on the same geodesic segment, and the proof of Proposition 2.2

shows that it also holds if p and q are on adjacent segments. So, it remains to show that if $p \in [x,y]$ and $q \in [z,w]$, then $d(p,y)+d(y,z)+d(z,q) \le d(p,q)+4\delta+4A$.

So, fix $p \in [x,y]$ and $q \in [z,w]$ and let $[p,q]$ denote the geodesic segment between p and q. Since $d(y,z) > 10\delta + 2A$, there exists a point $r \in [y,z]$ such that $d(r,y) > 5\delta + A$ and $d(r,z) > 5\delta + A$. As geodesic quadrilaterals are 2δ–thin, there exists some $r' \in [y,p] \cup [p,q] \cup [q,z]$ at distance at most 2δ from r. We claim that $r' \in [p,q]$. Suppose instead that $r' \in [p,y]$. Then, since $(p,r)_y \le (x,z)_y \le A$, we have that $d(y,[p,r]) \le A+\delta$. So, $d(z,y) \le d(p,x)+A+\delta$. But then

$$\begin{aligned} d(p,x)+A+\delta-d(x,y) &\le d(p,r')+d(r',x)+A+\delta-d(x,r')-d(r',y) \\ &= d(p,r')+A+\delta-d(r',y) \\ &\le d(p,r')+A+\delta-[d(y,p)-d(p,r')] \\ &= 2d(p,r')+A+\delta-d(y,p) \\ &< 4\delta+A+\delta-(5\delta+A)=0, \end{aligned}$$

which is a contradiction. Similarly, we cannot have that $r' \in [z,q]$ and hence our claim that $r' \in [p,q]$ must be true.

As $(p,r)_y \le A$ and $(q,r)_z \le A$, we have that $d(p,y)+d(y,r) \le d(p,r)+2A$ and $d(r,z)+d(z,q) \le d(r,q)+2A$. By the triangle inequality, $d(p,r) \le d(p,r')+2\delta$ and $d(q,r) \le d(q,r')+2\delta$. Therefore, $d(p,y)+d(y,z)+d(z,q) \le d(p,q)+4\delta+4A$. □

Lemma 2.5 *Let (X,d) be a δ–hyperbolic metric space and let $x,y,z,w \in X$. If there exist points $a \in [x,w]$ and $b \in [y,z]$ such that $d(a,b) \le 2\delta$, then $[x,y] \cup [y,z] \cup [z,w]$ is a $(1, 4\delta+4d(y,z))$–quasigeodesic.*

Proof Let $x,y,z,w \in X$ be as above and consider the geodesic quadrilateral with edges $[x,y]$, $[y,z]$, $[z,w]$ and $[x,w]$. Note that both $(x,z)_y$ and $(y,w)_z$ are bounded by $d(y,z)$. So, the proof of Proposition 2.4 shows that if $p \in [x,y]$ and $q \in [y,z]$, then $d(p,y)+d(y,q) \le d(p,q)+2d(y,z)$. If $p \in [x,y]$ and $q \in [z,w]$, then there exist points $u \in [p,q]$ and $v \in [y,z]$ with $d(u,v) \le 2\delta$. Thus, $d(p,v) \le d(p,u)+2\delta$ and $d(q,v) \le d(q,u)+2\delta$. Additionally, $d(p,y)+d(y,v) \le d(p,v)+2d(y,z)$ and $d(q,z)+d(z,v) \le d(q,v)+2d(y,z)$. Therefore,

$$\begin{aligned} d(p,y)+d(y,z)+d(z,q) &\le d(p,v)+d(q,v)+4d(y,z) \\ &\le d(p,q)+4\delta+4d(y,z). \end{aligned}$$ □

Let $\mathcal{N}_r(U)$ denote the r–neighborhood around a subset U of X. It is known that in a hyperbolic metric space, any quasigeodesic stays near the geodesic between its endpoints:

Proposition 2.6 [18, Proposition 7.2.A; 8, Théorème 3.1.3; 17, Théorèmes 5.6, 5.11] *Let (X,d) be a δ–hyperbolic metric space and let $x,y \in X$. For any $\kappa \geq 1$ and $\epsilon \geq 0$, there exists $L = L(\delta,\kappa,\epsilon) \geq 0$ such that if α is a (κ,ϵ)–quasigeodesic between x and y, then, for any geodesic $\beta = [x,y]$, we have that $\alpha \subset \mathcal{N}_L(\beta)$ and $\beta \subset \mathcal{N}_L(\alpha)$.*

2.2 Hyperbolic groups

A finitely generated group H is said to be *word-hyperbolic* if for some — equivalently any — finite generating set of H, there exists $\delta \geq 0$ such that the Cayley graph of H with respect to the word metric is δ–hyperbolic. Let H be a word-hyperbolic group and fix a finite generating set S_H for H. We will denote the Cayley graph of H with respect to S_H by Γ_H and let d_H, or simply d, denote the word metric. Let ∂H denote the Gromov boundary of Γ_H and let $\widehat{\Gamma}_H = \partial H \cup \Gamma_H$ be the Gromov compactification of Γ_H. Then $\widehat{\Gamma}_H$ is a compact, Hausdorff topological space. It is known that for a word-hyperbolic group, ∂H is independent of choice of finite generating set.

We will now introduce some terminology and facts that we will use throughout this paper. Given a group H with finite generating set S_H, let $\Sigma_H := S_H \cup S_H^{-1}$ denote the alphabet of H. A *word* w over the alphabet Σ_H is an expression $s_1 \cdots s_n$, where $s_i \in \Sigma_H$ and $n \geq 0$ (the case $n = 0$ represents the empty word). We will denote the set of all finite words over Σ_H by Σ_H^*, and will think of a word as the label of some (not necessarily geodesic) path in Γ_H.

If $w \in \Sigma_H^*$ is the label of some path in Γ_H from a vertex a to b, then we will denote the group element $a^{-1}b \in H$ representing the word w by $\overline{w}$. Given any element $h \in H$, we will denote the conjugacy class of h in H by $[h]_H$ (or simply by $[h]$ if the ambient group is clear). For a word $w \in \Sigma_H^*$, $|w|_H$ denotes the length of any path labeled by w in Γ_H. The length of an element $h \in H$, also denoted by $|h|_H$, is defined to be the length of any geodesic from the identity 1_H to the vertex h in Γ_H. We will drop the subscript if the group we are working in is clear.

For the remainder of this section, we assume that H is a word-hyperbolic group with a fixed finite generating set S_H. We will also usually abbreviate $|h|_H$ by $|h|$ for $h \in H$. The following definitions generalize the notion of words in a free group being cyclically and almost cyclically reduced to the context of a general word-hyperbolic group.

Definition 2.7 Let $\kappa \geq 0$. An element $h \in H$ is said to be *κ–almost conjugacy minimal* in H if $|h|_H \leq |h'|_H + \kappa$ for all $h' \in [h]_H$. If $\kappa = 0$, then h is said to be *conjugacy minimal*. A geodesic $[a, aw] \subseteq \Gamma_H$ is said to be a *κ–almost conjugacy minimal representative* if $\overline{w} \in H$ is κ–almost conjugacy minimal. If $[a, aw] \subseteq \Gamma_H$ is a κ–almost conjugacy minimal representative, then we will also refer to the word w labeling this geodesic as a *κ–almost conjugacy minimal representative*. If $\kappa = 0$, then $[a, aw]$ and w are said to be *conjugacy minimal representatives*.

Lemma 2.8 *Fix an element $h \in H$ and any constant $\kappa \geq 0$. If h is κ–almost conjugacy minimal, then $(1, hh)_h \leq \frac{1}{2}(\kappa + \delta)$.*

Proof Let $h \in H$ be κ–almost conjugacy minimal and suppose the geodesic $[1, h] \subseteq \Gamma_H$ is labeled by $\alpha h' \beta$, where $|\alpha| = |\beta| = (1, hh)_h$ and $\beta\alpha = s$, with $|s| \leq \delta$. Then $h =_H \alpha h' s \alpha^{-1}$, and so $h's \in [h]_H$. As h is κ–almost conjugacy minimal, we have that $|h| \leq |h's| + \kappa \leq |h'| + \delta + \kappa$. Finally, $|h| = |\alpha| + |h'| + |\beta| = 2(1, hh)_h + |h'|$, and so $(1, hh)_h \leq \frac{1}{2}(\kappa + \delta)$. □

Lemma 2.9 *There exists a constant $C \geq 0$ such that, for any element $h \in H$, the following holds: Suppose that $u, c \in H$ are such that $h = c^{-1}uc$, where $u \in [h]$ is conjugacy minimal and $|c|$ is the smallest element conjugating h to any conjugacy minimal representative. Then the path $[c, 1] \cup [1, u] \cup [u, uc] \subseteq \Gamma_H$ is a $(1, C)$–quasigeodesic.*

Proof Let $h, u, c \in H$ be as in the hypothesis above and consider the quadrilateral in Γ_H with vertices 1, c, u and uc, and edges $[1, u]$ labeled by u, $[c, uc]$ labeled by h, and $[1, c]$ and $[u, uc]$ both labeled by c. We want to show the path $\gamma = [c, 1] \cup [1, u] \cup [u, uc]$ is a $(1, C)$–quasigeodesic for some constant $C \geq 0$.

Let $p \in [1, c]$ and $q \in [1, u]$ be such that $d(1, p) = d(1, q) = (c, u)_1$. As $d(p, q) \leq \delta$, we must have that $d(1, p)$ is also at most δ. Otherwise, $q^{-1}c$ would be a shorter word conjugating h to a cyclic conjugate of u, contradicting the minimality of $|c|$. Similarly, $(1, uc)_u \leq \delta$. If $|u| > 12\delta$, then, by Proposition 2.4, γ is a $(1, 8\delta)$–quasigeodesic.

If $|u| \leq 12\delta$, then, since H is finitely generated, there are only finitely many possibilities for such u. Hence, there are only finitely many cases to consider and the result holds by taking C to be, for instance, the length of the longest path γ that we get in this setting. □

Corollary 2.10 *For any $\kappa \geq 0$, there exists a constant $M > 0$ such that if $h \in H$ is κ–almost conjugacy minimal, then there is an element $c \in H$ with $|c| \leq M$ and a conjugacy minimal element $u \in [h]$ such that $h = c^{-1}uc$.*

Proof Let $c \in H$ be a shortest-length element conjugating h to any conjugacy minimal element in $[h]$. By Lemma 2.9, there exists some constant $C > 0$ such that $[c,1] \cup [1,u] \cup [u,uc]$ is a $(1,C)$–quasigeodesic. So $2|c| + |u| \leq |h| + C$. Since h is κ–almost conjugacy minimal, $|h| \leq |u| + \kappa$. Thus, $|c| \leq \frac{1}{2}(C+\kappa)$. □

Lemma 2.11 *For any $\kappa \geq 0$ there exists a constant $A \geq 0$ such that if $h \in H$ satisfies $(1,hh)_h \leq A$, then h is κ–almost conjugacy minimal.*

Proof Let $h \in H$ be such that $(1,hh)_h \leq A$. Then, by Proposition 2.2, the path $[1,h] \cup [h,hh]$ is a $(1,2A)$–quasigeodesic. Additionally, all subpaths of $[1,h] \cup [h,hh]$ are $(1,2A)$–quasigeodesics. In particular, any (nonreduced) edge-path representing a cyclic conjugate of h is a $(1,2A)$–quasigeodesic. Choose a cyclic conjugate, h' of h such that $ch'c^{-1} = u$, where $u \in [h]_H$ is conjugacy minimal and $|c|$ is smallest.

Consider the points 1, c, u and uc; geodesics $[1,c]$, $[1,u]$ and $[u,uc]$, and the $(1,2A)$–quasigeodesic path between c and uc — call it γ' — labeled by the (nonreduced) word h'. By Lemma 2.9, there is some constant C for which $\gamma = [c,1] \cup [1,u] \cup [u,uc]$ is a $(1,C)$–quasigeodesic. As γ' and γ are quasigeodesics sharing the same endpoints, Proposition 2.6 implies that γ' and γ live in a D–neighborhood of each other for some constant $D \geq 0$ depending only on the quasi-isometry constants and δ.

We will now show that $|c|$ is bounded. If $|u| \leq 12\delta$, then there are only finitely many cases to check and we can take maximum length we get in these cases. So, suppose that $|u| > 12\delta$. Note that the distance between any point on $[1,c]$ must be at least $|u|$ from a point on $[u,uc]$ as otherwise we would get a contradiction with u being conjugacy minimal. So by Proposition 2.3, there must exist points $x \in [1,u]$ and $x' \in [c,uc]$ such that $d(x,x') \leq 2\delta$. Let x_0 denote the point along γ' where the two paths labeled by h meet. As the triangle with vertices c, x_0 and uc is δ–thin, there must exist a point $x'' \in [c,x_0] \cup [x_0,uc] = \omega'$ such that $d(x',x'') \leq \delta$. Therefore, $d(x,x'') \leq 3\delta$. Now, consider the word c' which labels the path from x to x'' and note that c' conjugates a cyclic conjugate of u to a cyclic conjugate of h'. Therefore, by the minimality of c, we must have in this case that $|c| \leq |c'| \leq 3\delta$.

We now want to show that the distance between x_0 and $[1,u]$ is bounded. Consider the point $y_0 \in \gamma$ which is closest to x_0. Without loss of generality, we may assume that either $y_0 \in [1,u]$ or $y_0 \in [c,1]$. If $y_0 \in [1,u]$, then $d(x_0,[1,u]) \leq M$. If $y_0 \in [c,1]$, then $d(x_0,[1,u]) \leq M + |c|$. As $|c|$ is bounded by some constant, the distance between x_0 and $[1,u]$ is also bounded by some constant. Therefore, h is κ–almost conjugacy minimal for some $\kappa \geq 0$ independent of h. □

For the purpose of this paper, if X is a graph, then we will assume that any quasi-isometry or quasi-isometric embedding takes vertices to vertices and edges to edge-paths. The following lemma follows from Proposition 2.6:

Lemma 2.12 *Let $K \geq 1$ and $C \geq 0$. Then, for any $\kappa \geq 0$, there exists $\kappa' \geq 0$ such that if $w \in \Sigma_H^*$ is a κ–almost conjugacy minimal representative and $\psi\colon \Gamma_H \to \Gamma_H$ is any (K, C)–quasi-isometry, then $\psi(w)$ is a κ'–almost conjugacy minimal representative.*

3 Metric graph bundles

In [3], Bestvina and Feighn explored the question of when a space which results from the combination of Gromov-hyperbolic spaces will itself be hyperbolic. They introduced the notion of a graph of spaces and provided a "flaring" condition which gives a sufficient condition for the hyperbolicity of a graph of hyperbolic spaces. Mj and Sardar generalized this work in [31], where they introduced the notion of a metric graph bundle and defined the following flaring condition.

Let X and B be connected graphs, each equipped with the path metric where each edge has length 1, and let $p\colon X \to B$ be a simplicial surjection. For the purpose of this paper, we will consider $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

Definition 3.1 X is said to be a *metric graph bundle* over B if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that:

(B1) For each vertex $b \in V(B)$, the fiber $F_b := p^{-1}(b)$ is a connected subgraph of X, and, for all vertices $u, v \in V(F_b)$, the induced path metric d_b on F_b satisfies $d_b(u, v) \leq f(d_X(u, v))$.

(B2) If $b_1, b_2 \in V(B)$ are any two adjacent vertices and if $x_1 \in V(F_{b_1})$ is any vertex, then there is some vertex $x_2 \in V(F_{b_2})$ adjacent to x_1 in X.

Remark 3.2 If $p\colon X \to B$ is a metric graph bundle and $W \subseteq B$ is any connected subgraph, then $p\colon p^{-1}(W) \to W$ is again a metric graph bundle.

Given any metric graph bundle $p\colon X \to B$ and a connected, closed interval $I \subseteq \mathbb{R}$, a (k, k)*–quasi-isometric lift* of a geodesic $\gamma\colon I \to B$ is any (k, k)–quasigeodesic $\tilde{\gamma}\colon I \to X$ for which $p(\tilde{\gamma}(n)) = \gamma(n)$ for all $n \in I \cap \mathbb{Z}$.

Definition 3.3 The metric graph bundle $p\colon X \to B$ is said to satisfy the *flaring condition* if for all $k \geq 1$, there exists $\lambda_k > 1$ and $n_k, M_k \in \mathbb{N}$ such that the following holds: if $\gamma\colon [-n_k, n_k] \to B$ is any geodesic and $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are any two (k,k)–quasi-isometric lifts of γ in X which satisfy $d_{\gamma(0)}(\tilde{\gamma}_1(0), \tilde{\gamma}_2(0)) \geq M_k$, then

$$\lambda_k \cdot d_{\gamma(0)}(\tilde{\gamma}_1(0), \tilde{\gamma}_2(0)) \leq \max\{d_{\gamma(-n_k)}(\tilde{\gamma}_1(-n_k), \tilde{\gamma}_2(-n_k)), d_{\gamma(n_k)}(\tilde{\gamma}_1(n_k), \tilde{\gamma}_2(n_k))\}.$$

The following are two theorems of Mj and Sardar which we will use later. The first is their combination theorem for metric graph bundles, which generalizes the combination theorem of Bestvina and Feighn [3]. The second shows that flaring is a necessary condition for the hyperbolicity of a metric graph bundle.

Theorem 3.4 (Mj and Sardar [31]) *Suppose that $p\colon X \to B$ is a metric graph bundle which satisfies*

(1) *B is a δ–hyperbolic metric space*;

(2) *for each $b \in V(B)$, the fiber F_b is δ–hyperbolic with respect to d_b, the path metric induced by X*;

(3) *for each $b \in V(B)$, the set of barycenters of ideal triangles in F_b is D–dense*; *and*

(4) *the flaring condition is satisfied.*

Then X is a hyperbolic metric space.

Theorem 3.5 (Mj and Sardar [31]) *Suppose that $p\colon X \to B$ is a metric graph bundle which satisfies*

(1) *X is δ–hyperbolic*; *and*

(2) *for each $b \in V(B)$, the fiber F_b is δ–hyperbolic with respect to d_b, the path metric induced by X.*

Then the metric bundle satisfies the flaring condition.

Throughout the remainder of this paper, we will use the following conventions:

Convention 3.6 Unless otherwise specified, let $1 \to H \xrightarrow{i} G \xrightarrow{P} Q \to 1$ be a short exact sequence of three infinite, word-hyperbolic groups. Fix finite, symmetric generating sets S_H, S_G and S_Q for H, G and Q, respectively, so that $i(S_H) \subseteq S_G$ and $S_Q := P(S_G)$. Let Γ_H, Γ_G and Γ_Q denote the Cayley graphs with respect to these generating sets. Let $P\colon \Gamma_G \to \Gamma_Q$ also denote the map on the Cayley graphs induced by $P\colon G \to Q$ which is given as follows. If $v \in \Gamma_G$ is a vertex labeled by the element $g \in G$, then v will get sent to the vertex in Γ_Q labeled by the element $P(g) \in Q$. Suppose $e = [g_1, g_2] \in \Gamma_G$ is an edge between adjacent vertices $g_1, g_2 \in \Gamma_G$. If g_1 and g_2 are in the same coset of H in G, then e will get collapsed to the vertex $P(g_1) = P(g_2)$ in Γ_Q. Otherwise, e will get mapped to the edge between $P(g_1)$ and $P(g_2)$ in Γ_Q labeled by $P(g_1^{-1}g_2) \in S_Q$.

Convention 3.7 Suppose $\gamma = (z', z)$ is a bi-infinite geodesic in Γ_Q between $z', z \in \partial Q$ with $z' \neq z$, and let $z_0 \in V(\gamma)$ be a vertex of γ which minimizes $d_Q(1, \gamma)$. Label the sequence of vertices in order along the portion of γ from z_0 to z by $z_0, z_1, z_2, \dots,$ and, similarly, label the sequence of vertices in order along the portion of γ from z_0 to z' by $z_0, z_{-1}, z_{-2}, \dots$. Let $\gamma^+ = [z_0, z)$ and $\gamma^- = (z', z_0]$.

Definition 3.8 The *subgraph of Γ_G corresponding to γ* is

$$X(\gamma) := P^{-1}(\gamma).$$

Note that we can think of $X(\gamma)$ as the subgraph of Γ_G with vertical fibers that are copies of Γ_H corresponding to the cosets $g_i H$, where $g_i \in P^{-1}(z_i)$ for each $z_i \in V(\gamma)$. Since $S_Q = P(S_G)$, there are edges between adjacent cosets $g_i H$ and $g_{i+1} H$ between any vertex $g_i h$ and the vertex $g_i h P^{-1}([z_i, z_{i+1}])$, where $[z_i, z_{i+1}]$ is the edge in γ between z_i and z_{i+1}. Let $P_\gamma\colon X(\gamma) \to \gamma$ denote the restriction of P to $X(\gamma)$.

Mj and Sardar showed in [31] that $P\colon \Gamma_G \to \Gamma_Q$ is a metric graph bundle. The same reasoning shows that the restricted map P_γ is a metric graph bundle as well. We include the argument below for completeness.

Proposition 3.9 *Given $P\colon \Gamma_G \to \Gamma_Q$ as in Convention 3.6, the map $P\colon \Gamma_G \to \Gamma_Q$ and the restricted map $P_\gamma\colon X(\gamma) \to \gamma$ are metric graph bundles.*

Proof For each vertex $q \in V(\Gamma_Q)$, $P^{-1}(q) = F_q$ is a copy of Γ_H, and so the induced path metric d_q is equal to d_H for all q. Hence, condition (B1) is satisfied by the function $f(n) := \max\{d_H(1, g) \mid d_G(1, g) \leq n\}$. Now, suppose $q_1, q_2 \in \Gamma_Q$ are adjacent vertices where $P(g_1 H) = q_1$ and $P(g_2 H) = q_2$. Since P maps edges between distinct cosets of Γ_H in Γ_G isometrically onto edges in Γ_Q, there exist some $h_1, h_2 \in H$ such that $g_1 h_1$ and $g_2 h_2$ are adjacent in Γ_G. Therefore, $s = (g_1 h_1)^{-1} g_2 h_2 \in S_G$. Hence, for all $x_1 = g_1 h \in V(F_{q_1})$, x_1 is adjacent to $x_1 s = g_1 h s = (g_1 h h_1^{-1} g_1^{-1}) g_2 h_2$ in Γ_G. This element is contained in the coset $g_2 H = F_{q_2}$ since H is normal in G, and so condition (B2) is satisfied. By Remark 3.2, $P_\gamma\colon X(\gamma) \to \gamma$ is also a metric graph bundle. □

Condition (B2) says that if we choose any lift g_0 of z_0, there exists $g_1 \in P^{-1}(z_1)$ such that $d_{X(\gamma)}(g_0, g_1) = 1$. Continuing in this fashion, we get a lift $\sigma\colon \gamma \to X(\gamma)$, where $\sigma(z_i) = g_i$, such that $d_{X(\gamma)}(g_i, g_{i+1}) = 1$ for all i. By the triangle inequality and the fact that γ is a geodesic in Γ_Q, we have that $d_{X(\gamma)}(g_i, g_j) \le d_Q(Pg_i, Pg_j) = d_Q(z_i, z_j)$. But, as every path in Γ_G projects to a path in Γ_Q of no greater length and as $X(\gamma) \subseteq \Gamma_G$, we also have that $d_Q(Pa, Pb) \le d_G(a,b) \le d_{X(\gamma)}(a,b)$. Hence, $d_{X(\gamma)}(g_i, g_j) = d_G(g_i, g_j) = d_Q(z_i, z_j)$ for all $g_i, g_j \in \sigma(\gamma)$.

Proposition 3.10 *The space $X(\gamma)$ is hyperbolic.*

Proof By Theorem 3.5, Γ_G satisfies the flaring condition since Γ_G and Γ_H are both hyperbolic and for each $q \in \Gamma_Q$, $F_q := p^{-1}(q)$ is a copy of Γ_H. Suppose that σ is a (K,C)–quasi-isometric lift of γ to $X(\gamma)$. Note that, for all $a, b \in \gamma$, $d_Q(a,b) = d_Q(P\cdot\sigma(a), P\cdot\sigma(b)) \le d_G(\sigma(a), \sigma(b))$. Also, since $X(\gamma) \subseteq \Gamma_G$,

$$d_G(\sigma(a), \sigma(b)) \le d_{X(\gamma)}(\sigma(a), \sigma(b)).$$

So, any quasi-isometric lift of a portion of γ to $X(\gamma)$ is also a quasi-isometric lift when considered as a path in Γ_G. Thus, $X(\gamma)$ satisfies the flaring condition. Additionally, the barycenters of ideal triangles in Γ_H are dense since the H–orbit of the barycenter of any ideal triangle in Γ_H is dense in Γ_H. Therefore, by Theorem 3.4, $X(\gamma)$ is hyperbolic. □

4 Stacks of spaces

In [5], Bowditch defines the notion of a stack of spaces. We will show that the bundle $X(\gamma)$ described above can be thought of as a hyperbolic stack of spaces.

Definition 4.1 Let (X, d_X) and (Y, d_Y) be path-metric spaces. A map $f\colon X \to Y$ is said to be *straight* if there exist functions $F_1, F_2\colon [0,\infty) \to [0,\infty)$ such that, for all $x, x' \in X$,

$$F_1(d_X(x,x')) \le d_Y(f(x), f(x')) \le F_2(d_X(x,x')),$$

where $F_1(t) \to \infty$ as $t \to \infty$. If $X \subseteq Y$, we say that X is a *straight* subspace if the inclusion map $i\colon X \to Y$ is a straight map with respect to the induced path metric on X.

Definition 4.2 Let $(\mathcal{X}, \rho)$ be a geodesic space, and let $((X_i, \rho_i))_{i\in\mathbb{Z}}$ be a sequence of geodesic subspaces, $X_i \subseteq \mathcal{X}$, called the *sheets* of $\mathcal{X}$ with uniform quasi-isometries

$f_i\colon X_i \to X_{i+1}$. The space $(\mathcal{X}, \rho)$ is said to be a *bi-infinite hyperbolic stack* if it satisfies the conditions (S1)–(S6) stated below:

(S1) Each of the spaces (X_i, ρ_i) is uniformly straight in $\mathcal{X}$, and $\rho(X_i, X_j)$ is bounded away from 0 for $i \neq j$.

(S2) For all $i, j \in \mathbb{Z}$, $\rho(X_i, X_j)$ is bounded below by an increasing linear function of $|i - j|$.

(S3) For all $i \in \mathbb{Z}$, $\mathrm{haus}(X_i, X_{i+1})$ is bounded above.

(S4) The spaces (X_i, ρ_i) are uniformly hyperbolic geodesic spaces.

(S5) The space $(\mathcal{X}, \rho)$ is hyperbolic.

(S6) The union $\bigcup_{i\in\mathbb{Z}} X_i$ is quasidense in $\mathcal{X}$.

Given a bi-infinite stack $\mathcal{X}$, denote by $\mathcal{X}^+$ and $\mathcal{X}^-$ the subsets of $\mathcal{X}$ which consist of the sheets $(X_i)_{i\in\mathbb{N}}$ and $(X_i)_{i\in-\mathbb{N}}$, respectively. Here, $\mathbb{N} = \mathbb{Z}_{\geq 0}$ and $-\mathbb{N} = \mathbb{Z}_{\leq 0}$. We will refer to $\mathcal{X}^+$ and $\mathcal{X}^-$ as *semi-infinite* stacks.

4.1 General background on stacks

We first give some general background on stacks of spaces which we will later apply to the space $X(\gamma)$. Bowditch proves the following about stacks of hyperbolic spaces indexed by any subset $I \subseteq \mathbb{Z}$ of consecutive integers:

Proposition 4.3 (Bowditch [5, Proposition 2.1.7]) *Suppose $\mathcal{X}$ is a bi-infinite stack with uniformly hyperbolic sheets $(X_i)_{i\in\mathbb{Z}}$. If $\mathcal{X}$ is hyperbolic, then so is $\mathcal{X}(I)$, where $I \subseteq \mathbb{Z}$ is any set of consecutive integers. In particular, the semi-infinite stacks $\mathcal{X}^+$ and $\mathcal{X}^-$ are hyperbolic whenever $\mathcal{X}$ is hyperbolic.*

Given a (bi-infinite) stack $\mathcal{X}$, Bowditch defines an *r–chain*, $(x_i)_{i\in\mathcal{I}}$, to be a sequence of points, $x_i \in X_i$, such that $\rho(x_i, x_{i+1}) \leq r$ for all $i \in \mathcal{I}$. *Bi-infinite*, *positive* and *negative* r–chains are defined to be r–chains indexed by $\mathbb{Z}$, $\mathbb{N}$ and $-\mathbb{N}$, respectively. Bowditch notes that each r–chain interpolates a quasigeodesic in $\mathcal{X}$. If $\mathcal{X}$ is a hyperbolic stack, it comes equipped with its Gromov boundary, $\partial\mathcal{X}$. Thus, when $\mathcal{X}$ is a proper, hyperbolic stack, each positive and negative chain determines a point of $\partial\mathcal{X}$. In this setting, there is a fixed r_0 depending on the hyperbolicity constant of $\mathcal{X}$ for which each point in $\mathcal{X}$ is contained in some r_0–chain. Bowditch defines $\partial^+\mathcal{X}$ (respectively $\partial^-\mathcal{X}$) to be those subsets of $\partial\mathcal{X}$ which are determined by positive (respectively negative) r_0–chains.

Note that the positive chains in $\mathcal{X}^+$ are exactly the positive chains in $\mathcal{X}$, and the negative chains in $\mathcal{X}^-$ are exactly the negative chains in $\mathcal{X}$. Furthermore, two chains determine the same point in $\partial\mathcal{X}^+$ or $\partial\mathcal{X}^-$ if and only if those two chains determine the same point in $\partial\mathcal{X}$. Hence, on the level of sets, we can identify $\partial^+\mathcal{X}^+$ with $\partial^+\mathcal{X}$ and $\partial^-\mathcal{X}^-$ with $\partial^-\mathcal{X}$.

All of the sheets X_i are quasi-isometric to one another, and so we get a homeomorphism from ∂X_i to ∂X_j for all $i, j \in \mathbb{Z}$. We will let ∂X_0 denote this space which is homeomorphic to ∂X_i for all $i \in \mathbb{Z}$. The notion of the Cannon–Thurston map, as defined earlier between the boundaries of hyperbolic groups, can be extended in the natural way to be defined between the boundaries of hyperbolic spaces. Bowditch proves the following statements about the Cannon–Thurston maps in this setting of stacks of spaces.

Proposition 4.4 (Bowditch [5, Propositions 2.3.2 and 2.3.3]) *Let $\mathcal{X}$ be a bi-infinite hyperbolic stack, let $\mathcal{X}^+$ and $\mathcal{X}^-$ be semi-infinite proper hyperbolic stacks, and let ω, ω^+ and ω^- denote the inclusions of X_0 into $\mathcal{X}$, $\mathcal{X}^+$ and $\mathcal{X}^-$, respectively. Then*

1. *the continuous Cannon–Thurston maps $\partial\omega\colon \partial X_0 \to \partial\mathcal{X}$, $\partial\omega^+\colon \partial X_0 \to \partial\mathcal{X}^+$ and $\partial\omega^-\colon \partial X_0 \to \partial\mathcal{X}^-$ exist;*
2. *$\partial\mathcal{X} = \partial^+\mathcal{X} \cup \partial^-\mathcal{X} \cup \partial\omega(\partial X_0)$; and*
3. *$\partial\mathcal{X}^+ = \partial^+\mathcal{X} \cup \partial\omega^+(\partial X_0)$ and $\partial\mathcal{X}^- = \partial^-\mathcal{X} \cup \partial\omega^-(\partial X_0)$.*

Given the Cannon–Thurston maps $\partial\omega$ and $\partial\omega^\pm$, denote by $\widehat{\omega}$ and $\widehat{\omega}^\pm$ the continuous extensions of the inclusion maps. Bowditch defines the maps $\partial\tau^\pm\colon \partial\mathcal{X}^\pm \to \partial\mathcal{X}$, which extend to continuous maps $\widehat{\tau}^\pm\colon \widehat{\mathcal{X}^\pm} \to \widehat{\mathcal{X}}$ such that $\widehat{\omega} = \widehat{\tau}^\pm \circ \widehat{\omega}^\pm$. For $y \in \partial^+\mathcal{X}^+ = \partial^+\mathcal{X}$, the map $\partial\tau^+$ is given by $\partial\tau^+(y) = y$; and for $a \in \partial X_0$, we have that $\partial\tau^+ \circ \partial\omega^+(a) = \partial\omega(a)$. The map $\partial\tau^-$ is defined similarly. Bowditch proves that $\partial\tau^\pm$ are continuous maps. Using this structure, Bowditch shows the following:

Lemma 4.5 (Bowditch [5, Lemmas 2.3.5, 2.3.6, 2.3.7 and 2.3.9]) *Let $\mathcal{X}$ be a bi-infinite, proper, hyperbolic stack.*

1. *Suppose $a \in \partial X_0$ and $y \in \partial^+\mathcal{X}$. Then $\partial\omega(a) = y$ if and only if there is a sequence $(\underline{x}^n)_{n\in\mathbb{N}}$ of positive chains, $\underline{x}^n = (x_i^n)_{i\in\mathbb{N}}$, each converging to y, and with x_0^n converging to $a \in \partial X_0$.*
2. *Given $a \in \partial X_0$ and $y \in \partial^\pm\mathcal{X}$, we have $\partial\omega^\pm(a) = y$ if and only if $\partial\omega(a) = y$.*

(3) *Suppose $a, b \in \partial X_0$ are distinct. If $\partial\omega^+(a) = \partial\omega^+(b) = y$, then $y \in \partial^+\mathcal{X}$; and if $\partial\omega^-(a) = \partial\omega^-(b) = y$, then $y \in \partial^-\mathcal{X}$.*

(4) *If $a, b \in \partial X_0$ and $\partial\omega(a) = \partial\omega(b)$, then either $\partial\omega^+(a) = \partial\omega^+(b)$ or $\partial\omega^-(a) = \partial\omega^-(b)$.*

4.2 Application of stacks

We now apply this work of Bowditch to our setting of hyperbolic group extensions. Let γ be as in Convention 3.7, and recall that $P\colon \Gamma_G \to \Gamma_Q$ is the projection map and $X(\gamma) := P^{-1}(\gamma)$.

Proposition 4.6 *The space $X(\gamma)$ with the induced path metric $d_{X(\gamma)}$ from Γ_G is a hyperbolic stack.*

Proof We need to show that $X(\gamma)$ satisfies conditions (S1)–(S6). For each vertex $z_i \in \gamma$, choose some $g_i \in G$ such that $P(g_i) = z_i$. For each $i \in \mathbb{Z}$, the sheet X_i of $X(\gamma)$ is the copy of Γ_H which corresponds to the coset $g_i\Gamma_H$ of H in G. Since X_i and X_j represent different cosets of Γ_H in Γ_G for $i \neq j$, we have that $d_G(X_i, X_j) \le d_{X(\gamma)}(X_i, X_j)$ is bounded away from 0 for $i \neq j$. Now, for all $i \in \mathbb{Z}$, let $\beta_i(n) := \max\{d_{X_i}(a,b) \mid d_{X(\gamma)}(a,b) \le n\}$. Then $\beta_i^{-1}(d_{X_i}(a,b)) \le d_{X(\gamma)}(a,b) \le d_{X_i}(a,b)$, and so condition (S1) is satisfied.

We see that condition (S2) is satisfied since $d_{X(\gamma)}(X_i, X_j) \ge d_Q(z_i, z_j) = |i-j|$. Similarly, the Hausdorff distance between X_i and X_{i+1} in $X(\gamma)$ is at most 2, and so condition (S3) is satisfied. As each X_i is a copy of Γ_H, which is δ–hyperbolic, (S4) holds. Additionally, $\bigcup_{i\in\mathbb{Z}} X_i$ is in the 1–neighborhood of $X(\gamma)$, and so (S6) is satisfied. Finally, we have by Proposition 3.10 that condition (S5) is satisfied. Therefore, $X(\gamma)$ is a bi-infinite hyperbolic stack. □

Recall, as in Convention 3.7, that z_0 denotes a point on γ closest to the identity in Γ_Q, the vertices along γ between z_0 and z are labeled by $z_1, z_2, \dots$, and the vertices along γ between z_0 and z' are labeled by $z_{-1}, z_{-2}, \dots$. Then, for all $x_i \in X_i$, $Px_i = z_i$. Since $X(\gamma)$ satisfies property (B2) of being a metric graph bundle, every vertex in $X(\gamma)$ is contained in some 1–chain. Let $y \in \partial X(\gamma)$, and let $y_n \in X(\gamma)$ be a sequence of vertices in $X(\gamma)$ which converge to y. As every vertex in $X(\gamma)$ is contained in some 1–chain, for each $n \in \mathbb{N}$ we can construct a 1–chain $\underline{x}^n = (x_i^n)_{i=0}^{m_n}$ in $X(\gamma)$ with terminal point $x_{m_n}^n := y_n$ as follows. Without loss of generality, assume that $y_n \in X(\gamma)^+$. Then, there exists some $m_n \in \mathbb{N}$ and some $h \in H$ such that $y_n = g_{m_n} h \in X_{m_n} = g_{m_n}\Gamma_H$, where $g_i = \sigma(z_i)$. Set $x^n_{m_n} := y_n$ and define $x^n_{m_n-1} := g_{m_n} h g_{m_n}^{-1} g_{m_n-1}$. Given the point $x^n_{m_n-j}$, where $j \in \{1, 2, \dots, m_n - 1\}$, set $x^n_{m_n-j-1} := x^n_{m_n-j} g^{-1}_{m_n-j} g_{m_n-j-1}$. Note that $x^n_i \in X_i = g_i\Gamma_H$ for each i, and so $\underline{x}^n$ defined in this fashion is a 1–chain in $X(\gamma)$ with terminal point y_n.

We now have a sequence of 1–chains $\underline{x}^n$ with terminal points y_n converging to $y \in \partial X(\gamma)$. Passing to a subsequence, we may assume that x^n_0 converges to $x_0 \in X_0 \cup \partial X_0$. Suppose first that $x_0 \in X_0$. Then, since the points x^n_1 remain in a compact subset of X_1, they subconverge on a point $x_1 \in X_1$ with $d_{X(\gamma)}(x_0, x_1) = 1$. Continuing on in this fashion, we can pass to a subsequence of our partial chains to get an infinite 1–chain $\underline{x} = \{x_0, x_1, \dots, x_n, \dots\}$ in $X(\gamma)$, where x^n_i converges to $x_i \in X_i$ for all i. Note that for large enough n, x^n_i remains uniformly close to x_i for arbitrarily many i. Hence, we must have that the terminal points of the chains $\underline{x}^n$ converge to the terminal point of $\underline{x}$ in $X(\gamma) \cup \partial X(\gamma)$. Since the chains $\underline{x}^n$ each have terminal point y_n, we therefore have that $y \in \partial X(\gamma)$ is the terminal point of a 1–chain in $X(\gamma)$, and so $y \in \partial^+ X(\gamma)$. Suppose now that $x_0 \in \partial X_0$. Then, by Lemma 4.5(1), $\partial\omega_\gamma(x_0) = y$, where $\partial\omega_\gamma\colon \partial X_0 \to \partial X(\gamma)$. Therefore, for all $y \in \partial X(\gamma)$, either y is the endpoint of a 1–chain in $X(\gamma)$, or $y \in \omega_\gamma(\partial X_0)$.

So, suppose that $(x_i)_{i\in\mathbb{Z}}$ is a 1–chain. For all $i, j \in \mathbb{Z}$ with $i < j$,

$$\begin{aligned} d_Q(Px_i, Px_j) &\le d_{X(\gamma)}(x_i, x_j) \\ &\le d_{X(\gamma)}(x_i, x_{i+1}) + d_{X(\gamma)}(x_{i+1}, x_{i+1}) + \cdots + d_{X(\gamma)}(x_{j-1}, x_j) \\ &= d_Q(Px_i, Px_{i+1}) + \cdots + d_Q(Px_{j-1}, Px_j) \\ &= d_Q(z_i, z_j). \end{aligned}$$

Hence, every 1–chain in $X(\gamma)$ interpolates a geodesic in $X(\gamma)$ which is an isometric lift of γ. Furthermore, as $d_Q(Px_i, Px_j) \le d_G(x_i, x_j) \le d_{X(\gamma)}(x_i, x_j)$ for all $i, j \in \mathbb{Z}$, every 1–chain interpolates a geodesic in Γ_G as well. Therefore, if $y \in \partial^+ X(\gamma)$ is the terminal point in $X(\gamma)$ of the positive 1–chain $(y_i)_{i\in\mathbb{N}}$, then the terminal point of this chain in Γ_G will determine a point of ∂G as well. As the only r–chains we will be considering in $X(\gamma)$ are 1–chains, all 1–chains in this space will now simply be referred to as chains.

Convention 4.7 Given $P\colon \Gamma_G \to \Gamma_Q$ as in Convention 3.6 and γ as in Convention 3.7, let $\sigma\colon \gamma \to \Gamma_G$ denote an isometric lift of γ such that $P(\sigma(z_i)) = z_i$ for all $z_i \in \gamma$, and set $g_i := \sigma(z_i)$. Let $X(\gamma) := P^{-1}(\gamma)$ and $X(\gamma)^+ := P^{-1}(\gamma^+)$ be the stacks which consist of the sheets $X_i = g_i\Gamma_H$ for all $i \in \mathbb{Z}$ and $i \in \mathbb{N}$, respectively. Denote by $\omega_\gamma\colon X_0 \to X(\gamma)$ and $\omega_\gamma^+\colon X_0 \to X(\gamma)^+$ the inclusions of the sheet $X_0 = g_0\Gamma_H$ into $X(\gamma)$ and $X(\gamma)^+$, respectively. Define $i_{X_0}\colon \Gamma_H \to X_0$ as follows. Set $i_{X_0}(h) := g_0 \cdot g_0^{-1} h g_0 = h g_0$ for all vertices $h \in \Gamma_H$. Extend i_{X_0} to a map on all of Γ_H by sending an edge $[a,b]$ to a shortest path between ag_0 and bg_0. Now let $i_\gamma\colon \Gamma_H \to X(\gamma)$ be given by $i_\gamma := \omega_\gamma \circ i_{X_0}$ and $i_\gamma^+\colon \Gamma_H \to X(\gamma)^+$ be given by $i_\gamma^+ := \omega_\gamma^+ \circ i_{X_0}$. Note that if $1 \in \gamma$, then i_{X_0} and i_γ^+ are simply the identity inclusion map $i\colon \Gamma_H \to \Gamma_G$.

Lemma 4.8 *The maps $i_\gamma\colon \Gamma_H \to X(\gamma)$ and $i_\gamma^+\colon \Gamma_H \to X(\gamma)^+$ as given in Convention 4.7 extend continuously to the maps $\hat{\imath}_\gamma\colon \widehat{\Gamma}_H \to \widehat{X(\gamma)}$ and $\hat{\imath}_\gamma^+\colon \widehat{\Gamma}_H \to \widehat{X(\gamma)^+}$, respectively.*

Proof Given $\omega_\gamma\colon X_0 \to X(\gamma)$ and $\omega_\gamma^+\colon X_0 \to X(\gamma)^+$ as in Convention 4.7, note that Proposition 4.4 gives that the Cannon–Thurston maps $\partial\omega_\gamma\colon \partial X_0 \to \partial X(\gamma)$ and $\partial\omega_\gamma^+\colon \partial X_0 \to \partial X(\gamma)$ both exist. Let $\widehat{\omega}_\gamma\colon \widehat{X}_0 \to \widehat{X(\gamma)}$ and $\widehat{\omega_\gamma^+}\colon \widehat{X}_0 \to \widehat{X(\gamma)^+}$ denote the continuous extensions of ω_γ and ω_γ^+. For all $g \in G$, conjugation by g gives an automorphism of H which takes $h \in H$ to $g^{-1}hg$. This automorphism is a quasi-isometry from Γ_H to itself. So, $i_{X_0}\colon \Gamma_H \to X_0$ is a quasi-isometry from Γ_H to $X_0 = g_0\Gamma_H$, and so extends to a homeomorphism $\partial i_{X_0}\colon \partial H \to \partial X_0$. Hence, $\partial i_\gamma := \partial\omega_\gamma \circ \partial i_{X_0}$ and $\partial i_\gamma^+\colon \partial\omega_\gamma^+ \circ \partial i_{X_0}$ exist, are continuous, and extend i_γ and i_γ^+ continuously to the maps $\hat{\imath}_\gamma$ and $\widehat{i_\gamma^+}$, respectively. □

Lemma 4.8 allows us to now refer to the maps ∂i_γ and ∂i_γ^+ as Cannon–Thurston maps. The goal of the remainder of this section is to show that the maps ∂i_γ and ∂i_γ^+ are surjective. We will first show surjectivity for the case where the geodesic γ lives over the identity in Γ_Q.

Convention 4.9 Let $\gamma = (z', z)$ be as in Convention 3.7 and let $\gamma' := z_0^{-1} \cdot \gamma = (z_0^{-1}z', z_0^{-1}z)$. Note that $1 \in V(\gamma')$. For each $z_i \in \gamma$, let $z_i' := z_0^{-1} \cdot z_i$. Given $\sigma\colon \gamma \to \Gamma_G$ as in Convention 4.7, let $\sigma'\colon \gamma' \to \Gamma_G$ be such that $\sigma' := g_0^{-1} \cdot \sigma$. Set $g_i' := \sigma'(z_i')$, and denote the sheet $g_i'\Gamma_H$ by X_i'. Note that the sheet X_0' is the identity coset $1 \cdot \Gamma_H$, and so the map $i_{X_0'}\colon \Gamma_H \to X_0'$ is the identity map.

In a similar manner as Bowditch [5], we define a map $\hat{\tau}_{\gamma'}\colon \widehat{X(\gamma')} \to \widehat{\Gamma}_G$ with $\hat{\imath} = \hat{\tau}_{\gamma'} \circ \hat{\imath}_{\gamma'}$ and will later show that $\partial\tau_{\gamma'}\colon \partial X(\gamma') \to \partial G$ is continuous. Let $\tau_{\gamma'} := \hat{\tau}_{\gamma'}|_{X(\gamma')}$ be the identity inclusion of $X(\gamma')$ into Γ_G given by $\tau_{\gamma'}(g) = g$. Note that for all $h \in H$, $\tau_{\gamma'} \circ i_{\gamma'}(h) = h = i(h)$.

As the map $i_{X_0'}$ is the identity map, $\partial\omega_{\gamma'} = \partial i_{\gamma'}$. So, by Proposition 4.4, we have that $\partial X(\gamma') = \partial i_{\gamma'}(\partial H) \cup \partial^{\pm} X(\gamma')$. If $(y_i)_{i\in\mathbb{N}}$ is a positive 1–chain in $X(\gamma')$ with endpoint $y \in \partial^+ X(\gamma')$, then $(\tau_{\gamma'}(y_i))_{i\in\mathbb{N}}$ interpolates a geodesic ray in Γ_G with the same label as the geodesic ray interpolated by (y_i) in $X(\gamma')$. Denote the endpoint of this geodesic ray in Γ_G by $\overline{y} \in \partial G$, and for all $y \in \partial^{\pm} X(\gamma')$ define $\partial\tau_{\gamma'}(y) := \overline{y}$. Finally, for all $a \in \partial H$, define $\partial\tau_{\gamma'}(\partial i_{\gamma'}(a)) := \partial i(a)$. Note that if $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are distinct but equivalent 1–chains in $X(\gamma')$, then the geodesic rays interpolated by these chains are Hausdorff close in both $X(\gamma')$ and Γ_G. Hence, $\tau_{\gamma'}$ is well-defined on equivalence classes of chains. To finish showing that $\hat{\tau}_{\gamma'}$ is well-defined, we need the following lemma:

Lemma 4.10 *Let γ' be as in Convention 4.9. Suppose $(\underline{x}^n)_{n\in\mathbb{N}}$ is a sequence of positive chains in $X(\gamma')$, where $\underline{x}^n = (x_i^n)_{i\in\mathbb{N}}$ is a positive chain with terminal point $y_n \in \partial^+ X(\gamma')$. Suppose also that in $\widehat{X(\gamma')}$, $y_n \to y \in \partial X(\gamma')$ and in $\widehat{X_0'}$, $x_0^n \to \partial i_{X_0'}(a) \in \partial X_0'$. Then, in $\widehat{\Gamma}_G$, $\hat{\tau}_{\gamma'}(y_n) \to \hat{\imath}(a)$.*

Proof Let $f(n) = \max\{d_{X_0'}(a,b) \mid d_G(\tau_{\gamma'}(a), \tau_{\gamma'}(b)) \le n\}$. Note that since Γ_G is finitely generated, such a maximum exists, and that $f(n) \to \infty$ as $n \to \infty$. For each $n \in \mathbb{N}$, there exists $a_n \in \Gamma_H$ such that $x_0^n = i_{X_0'}(a_n) = a_n$. As $x_0^n \to \partial i_{X_0'}(a)$, this implies that $a_n \to a \in \partial H$ in $\widehat{\Gamma}_H$. Let $\lambda_n = [\hat{\tau}_{\gamma'}(x_0^n), \hat{\tau}_{\gamma'}(y_n))_G = [i(a_n), \hat{\tau}_{\gamma'}(y_n))_G$ be the geodesic ray in Γ_G interpolated by $(\tau_{\gamma'}(x_i^n))_{i\in\mathbb{N}}$ for each $n \in \mathbb{N}$.

Suppose that in $\widehat{\Gamma}_G$, $\lim_{n\to\infty} \hat{\tau}_{\gamma'}(y_n) \ne \lim_{n\to\infty} i(a_n)$. Then there exist constants $R, N > 0$ such that $d_G(1, \lambda_n) \le R$ for all $n \ge N$. So, for each $n \ge N$, there exists some point $x_{i_n}^n$ in the chain $\underline{x}^n$ such that $d_G(1, \tau_{\gamma'}(x_{i_n}^n)) \le R$. Then

$$d_G(1, \tau_{\gamma'}(x_{i_n}^n)) \ge d_Q(P\cdot 1, P\cdot\tau_{\gamma'}(x_{i_n}^n)) = d_Q(1, P\cdot x_{i_n}^n) = |i_n|.$$

As $d_G(1, \tau_{\gamma'}(x_{i_n}^n)) \le R$, this means that $|i_n| \le R$. Note that since $\underline{x}^n$ is a 1–chain, $d_G(\tau_{\gamma'}(x_0^n), \tau_{\gamma'}(x_{i_n}^n)) = d_{X(\gamma')}(x_0^n, x_{i_n}^n) = |i_n| \le R$. So,

$$\begin{aligned} d_{X_0'}(1, x_0^n) &\le f\big(d_G(\tau_{\gamma'}(1), \tau_{\gamma'}(x_0^n))\big) \\ &= f\big(d_G(1, \tau_{\gamma'}(x_0^n))\big) \\ &\le f\big(d_G(1, \tau_{\gamma'}(x_{i_n}^n)) + d_G(\tau_{\gamma'}(x_{i_n}^n), \tau_{\gamma'}(x_0^n))\big) \\ &\le f(2R). \end{aligned}$$

But, $d_{X_0'}(1, x_0^n) \to \infty$ as $n \to \infty$ since $x_0^n \to \partial i_{X_0'}(a) \in \partial X_0'$, and so we have a contradiction. Therefore, $d_G(1, \lambda_n) \to \infty$ as $n \to \infty$. Hence, in $\widehat{\Gamma}_G$, $\lim_{n\to\infty} \tau_{\gamma'}(x_0^n) =$

$\lim_{n\to\infty} \hat{\tau}_{\gamma'}(y_n)$. As $\tau_{\gamma'}(x_0^n) = i(a_n)$ and $i(a_n) \to \hat{\imath}(a)$ as $n \to \infty$, we have that $\hat{\tau}_{\gamma'}(y_n) \to \hat{\imath}(a)$, as desired. □

Lemma 4.11 *The map* $\hat{\tau}_{\gamma'}\colon \widehat{X(\gamma')} \to \hat{\Gamma}_G$ *is well-defined and satisfies*

$$\hat{\tau}_{\gamma'} \circ \hat{\imath}_{\gamma'} = \hat{\imath}\colon \hat{\Gamma}_G \to \hat{\Gamma}_H.$$

Proof If $x \in \Gamma_H$, then $\tau_{\gamma'} \circ i_{\gamma'}(x) = x = i(x)$. Similarly, $\partial\tau_{\gamma'} \circ \partial i_{\gamma'}(a) = \partial i(a)$ if $a \in \partial H$. So, $\hat{\imath} = \hat{\tau}_{\gamma'} \circ \hat{\imath}_{\gamma'}$. Now it suffices to show that $\partial\tau_{\gamma'}\colon \partial X(\gamma') \to \partial G$ is well-defined. First, we need to show that if $y \in \partial^+ X(\gamma')$ and $a \in \partial H$ are such that $\partial i_{\gamma'}(a) = y$, then $\partial\tau_{\gamma'}(y) = \partial\tau_{\gamma'}(\partial i_{\gamma'}(a))$. So, suppose that $y \in \partial^+ X(\gamma')$ and $a \in \partial H$ are such that $\partial i_{\gamma'}(a) = y$. Since $\partial i_{\gamma'}(a) = y$ and $\partial i_{X_0'}$ is the identity, this implies that $\partial i_{\gamma'}(a) = \partial\omega_{\gamma'} \circ \partial i_{X_0'}(a) = \partial\omega_{\gamma'}(a) = y$. By Lemma 4.5(1), there exists a sequence $(\underline{x}^n)_n$ of positive chains, each converging to y, with x_0^n converging to $a = \partial i_{X_0'}(a) \in \partial X_0'$. By Lemma 4.10, the existence of such a sequence of chains implies that $\partial\tau_{\gamma'}(y) = \partial i(a)$. Hence, $\partial\tau_{\gamma'}(y) = \partial\tau_{\gamma'}(\partial i_{\gamma'}(a))$.

Now suppose that $a, b \in \partial H$ with $a \neq b$ are such that $\partial i_{\gamma'}(a) = \partial i_{\gamma'}(b)$. Since $\partial i_{X_0'}$ is the identity, this implies that $\partial\omega_{\gamma'}(a) = \partial\omega_{\gamma'}(b)$. By Lemma 4.5(4), we may assume without loss of generality that $\partial\omega_{\gamma'}^+(a) = \partial\omega_{\gamma'}^+(b)$. Since a and b are distinct, we have by Lemma 4.5(3) that $\partial\omega_{\gamma'}^+(a) = \partial\omega_{\gamma'}^+(b) = y \in \partial^+ X(\gamma')$. By Lemma 4.5(2), we now have that $\partial\omega_{\gamma'}(a) = \partial\omega_{\gamma'}(b) = y$. So, by the same reasoning as above, Lemmas 4.5(1) and 4.10 give that $\partial\tau_{\gamma'}(\partial i_{\gamma'}(a)) = \partial\tau_{\gamma'}(\partial i_{\gamma'}(b)) = \partial\tau_{\gamma'}(y) = \bar{y}$. □

Corollary 4.12 *If* $a, b \in \partial H$ *are such that* $\partial i_{\gamma'}^+(a) = \partial i_{\gamma'}^+(b)$, *then* $\partial i(a) = \partial i(b)$.

Proof Suppose $a, b \in \partial H$ are such that $\partial i_{\gamma'}^+(a) = \partial i_{\gamma'}^+(b)$. If $a = b$, then $\partial i(a) = \partial i(b)$. So suppose $a \neq b$. As $\partial i_{X_0'}$ is the identity, $\partial\omega_{\gamma'}^+(a) = \partial\omega_{\gamma'}^+(b)$. By Lemma 4.5(3), there exists $y \in \partial^+ X(\gamma')$ such that $\partial\omega_{\gamma'}^+(a) = \partial\omega_{\gamma'}^+(b) = y$. By Lemma 4.5(2), this implies that $\partial\omega_{\gamma'}(a) = \partial\omega_{\gamma'}(b)$. So $\partial i_{\gamma'}(a) = \partial\omega_{\gamma'} \circ \partial i_{X_0'}(a) = \partial\omega_{\gamma'} \circ \partial i_{X_0'}(b) = \partial i_{\gamma'}(b)$. As $\partial\tau_{\gamma'}$ is well-defined by Lemma 4.11, $\partial\tau_{\gamma'}(\partial i_{\gamma'}(a)) = \partial\tau_{\gamma'}(\partial i_{\gamma'}(b))$, and so $\partial i(a) = \partial i(b)$. □

The goal of the remainder of this section is to use this work of Bowditch to prove that the Cannon–Thurston map $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ is surjective.

Lemma 4.13 *Fix* $\gamma = (z', z) \subseteq \Gamma_Q$ *as in Convention 3.7 and let* $X(\gamma)^+$ *be as described above. Let* $(y_n)_{n\in\mathbb{N}}$ *be a* 1*–chain in* $X(\gamma)^+$ *and denote the word which labels the geodesic from* y_0 *to* y_n *in* $X(\gamma)^+$ *by* α_n. *Fix some* $h \in H$ *of infinite order and let* ρ_n *denote any path in* $X(\gamma)^+$ *which is the concatenation of a path labeled by* α_n *followed* *by a path labeled by h and finally a path labeled by α_n^{-1}. Then there exists some constant $C \geq 0$ independent of n (but dependent on h) such that for all n, ρ_n is a $(1, C)$–quasigeodesic in $X(\gamma)^+$.*

Proof Let $(y_n)_{n\in\mathbb{N}}$ be a 1–chain in $X(\gamma)^+$, and for each $n \geq 0$ let α_n denote the word which labels the geodesic from y_0 to y_n. Given $h \in H$, let β denote any quasigeodesic in $X(\gamma)^+$ labeled by h. Since $h \in H$ is fixed, there exists some constant $C' \geq 0$ such that β is a $(1, C')$–quasigeodesic. For each $n \geq 0$, let $[x_n = y_0, y_n]$ be the geodesic in $X(\gamma)^+$ from $x_n = y_0$ to y_n labeled by α_n, let $z_n \in X(\gamma)^+$ be a point such that β is a quasigeodesic in $X(\gamma)^+$ from y_n to z_n, and let $[z_n, w_n]$ be the geodesic in $X(\gamma)^+$ labeled by α_n^{-1}. Denote by δ_n the label of the geodesic in $X(\gamma)^+$ between x_n and w_n.

For each $n \geq 0$, consider the quadrilateral in $X(\gamma)^+$ with vertices $y_0 = x_n$, y_n, z_n and w_n, and with sides $[x_n, y_n]$ labeled by α_n, β labeled by h, $[z_n, w_n]$ labeled by α_n^{-1} and $[x_n, w_n]$ labeled by δ_n. Unless otherwise specified, we will denote $d_{X(\gamma)^+}$ simply by d, and all geodesic and quasigeodesic segments considered are geodesics or quasigeodesics in $X(\gamma)^+$.

As before, we need to show that if p and q are points on $\rho_n = [x_n, y_n] \cup \beta \cup [z_n, w_n]$, then the distance between p and q along ρ_n is at most $d(p, q) + C$. There are two cases to consider. By Proposition 2.3, either there is a point on $[x_n, w_n]$ at most distance 2δ in $X(\gamma)^+$ from a point on $[y_n, z_n]$, or there is a point on the side $[x_n, y_n]$ at most distance 2δ in $X(\gamma)^+$ from a point on the side $[z_n, w_n]$. If there is some point on the side $[x_n, w_n]$ within 2δ of a point on the side $[y_n, z_n]$, then Lemma 2.5 gives that $[x_n, y_n] \cup [y_n, z_n] \cup [z_n, w_n]$ is a $(1, 4\delta + 4d(y_n, z_n))$–quasigeodesic. Since β is a $(1, C')$–quasigeodesic between y_n and z_n, this gives that ρ_n is a $(1, C)$–quasigeodesic for some $C \geq 0$.

So, suppose now that the two sides labeled by α_n and α_n^{-1} come within 2δ of each other in $X(\gamma)^+$. We make the following claim:

Claim *If $a \in [x_n, y_n]$ and $a' \in [z_n, w_n]$ are the furthest points in $X(\gamma)^+$ from y_n and z_n, respectively, such that $d(a, a') \leq 2\delta$, then there is some constant $K > 0$ dependent on h but independent of n such that $\max\{d(a, y_n), d(a', z_n)\} \leq K$.*

Assuming this claim, we will now show that ρ_n is a $(1, C)$–quasigeodesic in $X(\gamma)^+$. First fix $p \in [x_n, y_n]$ and $q \in \beta$. Since $(p, q; X(\gamma)^+)_{y_n}$ is bounded by $|\beta|_{X(\gamma)^+} \leq |h|_H$ in $X(\gamma)^+$,

$$d(p, y_n) + d(y_n, q) = d(p, q) + 2(p, q; X(\gamma)^+)_{y_n} \leq d(p, q) + 2|h|_H.$$

So, suppose $p \in [x_n, y_n]$ and $q \in [z_n, w_n]$. If $p \in [a, y_n]$ and $q \in [a', z_n]$, then $d(p, y_n) + |\beta|_{X(\gamma)^+} + d(z_n, q) \le d(p,q) + |h|_H + 2K$. Now suppose $p \in [x_n, a]$ and $q \in [a', z_n]$. Since $d(a, a') \le 4\delta$, we have by the triangle inequality that

$$d(q,a) \le d(q,a') + d(a',a) \le K + 2\delta,$$
$$d(p,a) \le d(p,q) + d(q,a).$$

Therefore,

$$\begin{aligned} d(p,a) + d(a, y_n) + |\beta|_{X(\gamma)^+} + d(z_n, q) &\le d(p,q) + d(q,a) + K + |h|_H + K \\ &\le d(p,q) + 3K + 2\delta + |h|_H. \end{aligned}$$

The final case to consider is when $p \in [x_n, a]$ and $q \in [w_n, a']$. In this case, there must be points $u \in [p,q]$ and $v \in [a, a']$ such that $d(u,v) \le 2\delta$. This is because, by choice of a and a', there are no points at which $[q, a']$ is within a distance of 2δ of $[p, a]$ in $X(\gamma)^+$. So $d(q,v) \le d(q,u) + d(u,v)$ and $d(p,v) \le d(p,u) + d(u,v)$. Additionally, $d(p,a) \le d(p,v) + d(v,a)$ and $d(q,a') \le d(q,v) + d(v,a')$. Hence,

$$\begin{aligned} d(p,a) + d(a, y_n) + |\beta|_{X(\gamma)^+} &+ d(z_n, a') + d(a', q) \\ &\le d(p,a) + K + |h|_H + K + d(a',q) \\ &\le d(p,v) + d(v,a) + 2K + |h|_H + d(q,v) + d(v,a') \\ &\le d(p,v) + d(q,v) + 2K + |h|_H + 2\delta \\ &\le d(p,u) + d(q,u) + 2d(u,v) + 2K + |h|_H + 2\delta \\ &\le d(p,q) + 2K + |h|_H + 6\delta. \end{aligned}$$

Proof of claim Suppose to the contrary that there is no such bound on how long the sides labeled by α_n and α_n^{-1} stay uniformly close in $X(\gamma)^+$. Let S_Q be the generating set for Q and let $L = \{w \in \Sigma_Q^* \mid w \text{ a geodesic in } Q\}$. Since Q is a hyperbolic group, the language L of geodesic words is a regular language for Q (see [13]) which is accepted by some finite state automaton, $\mathcal{A}$, with start state s_0. Then $\gamma^+ = [z_0, z) \subseteq \Gamma_Q$ gives an infinite path from s_0 in $\mathcal{A}$ such that all states are accept states. Let γ_n denote the initial portion of the path γ^+ of length n, ie $\gamma_n := P([y_0 = x_n, y_n])$.

For each n, assume without loss of generality that the side of ρ_n labeled by α_n begins at the vertex y_0 and ends at the vertex y_n. Let y_{i_n} denote the vertex along the side α_n where the side labeled by α_n and the side labeled by α_n^{-1} begin to be 2δ–close. Note that after the point y_{i_n}, the sides labeled by α_n and α_n^{-1} will continue to travel within a distance of $|h|_{X(\gamma)^+}$ of each other in $X(\gamma)^+$. Project the $X(\gamma)^+$–geodesic $[y_{i_n}, y_n]$ to Q and feed this geodesic, $P([y_{i_n}, y_n])$, into $\mathcal{A}$. By assumption, the length of these geodesics go to infinity as $n \to \infty$. So, there will be some $n > 0$ for which some state in $\mathcal{A}$ repeats more times than the number of words in G of length at most $|h|_{X(\gamma)^+}$. Note that the label of any loop in $\mathcal{A}$ is a periodic Q–geodesic word. Since there is a state that repeats more times than the number of words in G of length at most $|h|_{X(\gamma)^+}$, it follows that there is some subpath of $[y_{i_n}, y_n]$ labeled by a word $v \in \Sigma_Q^*$ which has infinite order in Q and some word $m \in \Sigma_G^*$ of length at most $|h|_{X(\gamma)^+}$ such that, in G, $P^{-1}(v)m(P^{-1}(v))^{-1} = m$ and such that h is conjugate to m in G. As h has infinite order in G and h is conjugate to m, it follows that m has infinite order in G as well. As $P^{-1}(v)$ and m commute in G, this implies that $(P^{-1}(v))^p = m^q$ for some $p, q \neq 0$. But then $v^p = 1$ in Q, because h projects to the identity in Q which means that m projects to the identity in Q as well. The fact that $v^p = 1$ contradicts v being a periodic geodesic in Q. This completes the proof of the claim and the lemma. □

Theorem B *Let $1 \to H \to G \to Q \to 1$ be a short exact sequence of infinite, finitely generated, word-hyperbolic groups. Let $z, z' \in \partial Q$ be distinct and let $\gamma \subseteq \Gamma_Q$ be a bi-infinite geodesic in Γ_Q between z and z'. Let $i_\gamma^+\colon \Gamma_H \to X(\gamma)^+$ be the inclusion of Γ_H into the semi-infinite stack $X(\gamma)^+$ over $\gamma^+ = [z_0, z)$, and let $i_\gamma\colon \Gamma_H \to X(\gamma)$ be the inclusion of Γ_H into the bi-infinite stack $X(\gamma)$, as in Convention 4.7. Then*

(1) *the Cannon–Thurston map $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ is surjective; and*

(2) *the Cannon–Thurston map $\partial i_\gamma\colon \partial H \to \partial X(\gamma)$ is surjective.*

Proof Let $\gamma = (z', z) \subseteq \Gamma_Q$ be as in Convention 3.7 and let $\gamma' := z_0^{-1} \cdot \gamma$ be as in Convention 4.9. We will first show that the Cannon–Thurston maps $\partial i_{\gamma'}^+\colon \partial H \to \partial X(\gamma')^+$ and $\partial i_{\gamma'}\colon \partial H \to \partial X(\gamma')$ are surjective.

Consider first the map $\partial i_{\gamma'}^+\colon \partial H \to \partial X(\gamma)^+$. Since $\partial i_{\gamma'}^+ = \partial \omega_{\gamma'}^+ \circ \partial i_{X_0'}$ and $\partial i_{X_0'}$ is the identity, it suffices to show that $\partial \omega_{\gamma'}^+$ is surjective. By Proposition 4.4(3), we need only show that if $y \in \partial^+ X(\gamma')^+$, then there exists $a \in \partial X_0'$ such that $\partial i_{\gamma'}^+(a) = y$. So, suppose that $y \in \partial^+ X(\gamma')^+$ is the endpoint of the chain (y_n) and fix some $h \in H$ of infinite order. Let α_n be the word which labels the path from y_0 to y_n in $X(\gamma')^+$, and consider the path ρ_n in $X(\gamma')^+$ which is labeled by the word $\alpha_n h \alpha_n^{-1}$.

By Lemma 4.13, ρ_n is a $(1, C)$–quasigeodesic in $X(\gamma')^+$ for some C independent of n. Let h_n be the word which labels the geodesic in X_0' between the endpoints of ρ_n. Since $|h_n|_H \to \infty$, there exists a subsequence h_{n_i} such that $y_0 h_{n_i} \to a \in \partial X_0'$. Since $\partial \omega_{\gamma'}^+$ is a continuous extension of $\omega_{\gamma'}^+$,

$$\lim_{n_i \to \infty} \omega_{\gamma'}^+(y_0 h_{n_i}) = \partial \omega_{\gamma'}^+ \lim_{n_i \to \infty} y_0 h_{n_i} = \partial \omega_{\gamma'}^+(a).$$

Since $y_{n_i} \to y$ and since ρ_{n_i} is a quasigeodesic and $y_{n_i} \in \rho_{n_i}$, it follows that $\lim_{n_i\to\infty} y_{n_i} = \lim_{n_i\to\infty} \omega_\gamma^+(y_0 h_{n_i}) = y$ in $\widehat{X(\gamma')^+}$.

To see that $\partial i_{\gamma'}\colon \partial H \to \partial X(\gamma')$ is surjective, note that, by Proposition 4.4(2), $\partial X(\gamma') = \partial^+ X(\gamma') \cup \partial^- X(\gamma') \cup \partial i_{\gamma'}(\partial H)$. Note that the map $\hat{\imath}_{\gamma'}\colon \widehat{H} \to \widehat{X(\gamma')}$ is defined in the same way as $\hat{\imath}^+_{\gamma'}$. So, to show the surjectivity of $\partial i_{\gamma'}$, it suffices to note that in the above argument, we can replace $y \in \partial^+ X(\gamma')^+$ with $y' \in \partial^- X(\gamma')^-$. As the same reasoning holds, $\partial i_{\gamma'}\colon \partial H \to \partial X(\gamma')$ is surjective as well.

Now, let $t^H_{g_0}\colon \Gamma_H \to g_0\Gamma_H$, $t_{g_0}\colon X(\gamma') \to X(\gamma)$ and $t^+_{g_0}\colon X(\gamma')^+ \to X(\gamma)^+$ denote the maps induced by left-translation of the vertices of Γ_H, $X(\gamma')$ and $X(\gamma')^+$, respectively, by the element $g_0 = \sigma(z_0)$. Note that $\omega_\gamma \circ t^H_{g_0}(h) = t_{g_0} \circ i_{\gamma'}(h)$ and $\omega^+_\gamma \circ t^H_{g_0}(h) = t^+_{g_0} \circ i^+_{\gamma'}(h)$ for all $h \in H$. Since $t^H_{g_0}$, t_{g_0} and $t^+_{g_0}$ are isometries, these maps extend continuously to the boundary maps $\partial t^H_{g_0}\colon \partial H \to \partial g_0 H$, $\partial t_{g_0}\colon \partial X(\gamma') \to \partial X(\gamma)$ and $\partial t^+_{g_0}\colon \partial X(\gamma')^+ \to \partial X(\gamma)$, respectively, which are homeomorphisms. Hence, we have that $\partial\omega_\gamma \circ \partial t^H_{g_0}(a) = \partial t_{g_0} \circ \partial i_{\gamma'}(a)$ and $\partial\omega^+_\gamma \circ \partial t^H_{g_0}(a) = \partial t^+_{g_0} \circ \partial i^+_{\gamma'}(a)$ for all $a \in \partial H$. As $\partial i_{\gamma'}$ and $\partial i^+_{\gamma'}$ are surjective by the above argument and as $\partial t^H_{g_0}$, ∂t_{g_0} and $\partial t^+_{g_0}$ are homeomorphisms, this implies that $\partial\omega_\gamma$ and $\partial\omega^+_\gamma$ are surjective.

As noted previously, each $g \in G$ gives rise to an automorphism ϕ_g of H with $\phi_g(h) = g^{-1}hg$. This automorphism of H induces a quasi-isometry of Γ_H taking an edge $[u, v]$ to a shortest edge-path between $\phi_g(u)$ and $\phi_g(v)$. As $\phi_g\colon \Gamma_H \to \Gamma_H$ is a quasi-isometry, it extends to a homeomorphism $\partial\phi_g\colon \partial H \to \partial H$. Recall that $i_\gamma = \omega_\gamma \circ t^H_{g_0} \circ \phi_{g_0}$ and $i^+_\gamma = \omega^+_\gamma \circ t^H_{g_0} \circ \phi_{g_0}$. So, $\partial i_\gamma = \partial\omega_\gamma \circ \partial t^H_{g_0} \circ \partial\phi_{g_0}$ and $\partial i^+_\gamma = \partial\omega^+_\gamma \circ \partial t^H_{g_0} \circ \partial\phi_{g_0}$. As $\partial\omega_\gamma$ and $\partial\omega^+_\gamma$ are surjective, and as $\partial t^H_{g_0}$ and $\partial\phi_{g_0}$ are homeomorphisms, ∂i_γ and ∂i^+_γ are surjective. □

Recall that given the maps $\partial i^+_\gamma\colon \partial H \to \partial X(\gamma)^+$ and $\partial i_\gamma\colon \partial H \to \partial X(\gamma)$, Bowditch defines a map $\partial\tau^+\colon \partial X(\gamma)^+ \to \partial X(\gamma)$ with $\partial i_\gamma = \partial\tau^+ \circ \partial i^+_\gamma$. This map is given by $\partial\tau^+(y) = y$ for all $y \in \partial^+ X(\gamma)^+$, and $\partial\tau^+ \circ \partial i^+_\gamma(a) = \partial i_\gamma(a)$ for all $a \in \partial H$. We can now show the following about the map $\partial\tau^+$:

Corollary 4.14 *The map $\partial\tau^+\colon \partial X(\gamma)^+ \to \partial X(\gamma)$ as defined above is surjective.*

Proof By Proposition 4.4(2) and Theorem B(2), we have that $\partial X(\gamma) = \partial i_\gamma(\partial H)$. Suppose $y \in \partial X(\gamma)$. By Theorem B(2), there exists $a \in \partial H$ such that $\partial i_\gamma(a) = y$. Then, by definition of τ^+, we have that $\tau^+(\partial i^+_\gamma(a)) = \partial i_\gamma(a) = y$. □

5 Ending laminations

Recall that by Convention 3.6 we have fixed a short exact sequence $1 \to H \to G \to Q \to 1$ of three infinite word-hyperbolic groups with Cayley graphs Γ_H, Γ_G and Γ_Q, respectively. For each $g \in G$, conjugation by g gives an automorphism ϕ_g of H defined by $\phi_g(h) = g^{-1}hg$. Note that ϕ_g provides a bijection of the vertices of Γ_H which is a quasi-isometry of Γ_H with parameters depending on $|g|$. As such, ϕ_g extends to a homeomorphism of ∂H that coincides with the action of left-multiplication by g^{-1}. We will also denote this homeomorphism by ϕ_g. When $\lambda = [a,b]$ is a geodesic segment in Γ_H, we will denote a geodesic in Γ_H between $\phi_g(a)$ and $\phi_g(b)$ by λ_g. Similarly, if $\lambda = (u,v)$ is a bi-infinite geodesic in Γ_H with endpoints in ∂H, then $\lambda_g = (\phi_g(u), \phi_g(v)) = (g^{-1}u, g^{-1}v)$ also denotes the bi-infinite geodesic in Γ_H between the images of the endpoints of λ under the homeomorphism ϕ_g.

Given $\kappa \geq 1$ and $\epsilon \geq 0$, define a (κ,ϵ)*–quasi-isometric section* to be a (κ,ϵ)–quasi-isometric embedding $\sigma\colon \Gamma_Q \to \Gamma_G$ such that $P \cdot \sigma$ is the identity map on Γ_Q. The existence of such a quasi-isometric section in the setting of Convention 3.6 is guaranteed by Mosher [32]. If $\gamma \subseteq \Gamma_Q$ is a bi-infinite geodesic or a geodesic ray, we will also refer to a (κ,ϵ)–quasi-isometric embedding $\sigma\colon \gamma \to \Gamma_G$ as a quasi-isometric section. All sections we consider in this paper are assumed to take vertices to vertices and edges to edge-paths.

Definition 5.1 An *algebraic lamination* on H is defined to be a nonempty subset L of the double boundary $\partial^2 H$ which is closed, symmetric (flip-invariant) and H–invariant. If $L \subseteq \partial^2 H$ is an algebraic lamination, an element $(p,q) \in L$ will be referred to as a *leaf* of the lamination. As each point $(p,q) \in \partial^2 H$ can be represented by a bi-infinite geodesic λ in Γ_H from p to q, we will sometimes refer to the geodesic λ as a leaf of the lamination as well.

In [26], Mj describes a set of algebraic ending laminations on Γ_H associated to the hyperbolic group extension $(*)$ which are parametrized by points in the Gromov boundary of Γ_Q. These algebraic ending laminations are defined below.

Convention 5.2 Fix $\kappa \geq 1$ and $\epsilon \geq 0$, and let $\sigma\colon \Gamma_Q \to \Gamma_G$ be a quasi-isometric section of Γ_Q into Γ_G. For a fixed $z \in \partial Q$, let $[1,z) \subseteq \Gamma_Q$ be a geodesic ray from the identity to z. Denote the n^{th} vertex along $[1,z)$ by z_n, and set $g_n := \sigma(z_n)$.

Definition 5.3 (Mj [26]) Let $z \in \partial Q$.

(1) Let $h \in H$ be an element of infinite order. Choose a geodesic $[1,z)$ as in Convention 5.2. Define $R_{z,h}$ to be the set of all pairs $(a, aw) \in H \times H$ such that

there is some $n \geq 0$ for which $w \in [g_n h g_n^{-1}]_H$ and w is a conjugacy minimal representative of $g_n h g_n^{-1}$ in H. Let $\overline{R}_{z,h}$ denote the closure of $R_{z,h}$ in $\widehat{H} \times \widehat{H}$, and set

$$\Lambda_{z,h} := \overline{R}_{z,h} \cap \partial^2 H.$$

So, $\Lambda_{z,h}$ consists of all points $(p,q) \in \partial^2 H$ for which there exists a sequence $(a_{n_i}, a_{n_i} w_{n_i}) \in H \times H$ such that $(a_{n_i}, a_{n_i} w_{n_i})$ converges to (p,q) in $\widehat{H} \times \widehat{H}$ as $n_i \to \infty$, where w_{n_i} is some conjugacy minimal representative of $g_{n_i} h g_{n_i}^{-1}$ in H.

(2) The *algebraic ending lamination* corresponding to z is

$$\Lambda_z := \bigcup_{\substack{h \in H \\ h \text{ infinite order}}} \Lambda_{z,h}.$$

(3) The *algebraic ending lamination* for the short exact sequence $(*)$ is

$$\Lambda := \bigcup_{z \in \partial Q} \Lambda_z.$$

Note that Λ_z is H–invariant and nonempty. While $\Lambda_{z,h}$ is not necessarily symmetric as defined, $\Lambda_{z,h} \cup \Lambda_{z,h^{-1}}$ is symmetric. Moreover, by Theorem C the subset $\Lambda_z \subseteq \partial^2 H$ is closed and therefore Λ_z is an algebraic lamination on H. Mj explained in [26] that in Definition 5.3(2), it suffices to choose a finite collection of elements $h \in H$. Since $\Lambda_{z,h}$ is a closed subset of $\partial^2 H$ for each $h \in H$, this also shows that Λ_z is closed.

Remark 5.4 We note the following about Definition 5.3 and the laminations Λ_z and Λ:

(1) In Definition 5.3, the quasi-isometric section σ only needs to be defined on the ray $[1, z)$ rather than on all of Γ_Q.

(2) The lamination Λ_z is independent of choice of quasi-isometric section, since if σ and σ' are two quasi-isometric sections on $[1, z)$, then $[\sigma(z_n) h \sigma(z_n)^{-1}]_H = [\sigma'(z_n) h \sigma'(z_n)^{-1}]_H$.

(3) The lamination Λ_z is independent of geodesic ray $[1, z)$ by Mj's Lemma 3.3 of [26].

(4) The definitions of Λ_z and Λ are independent of the choice of generating set for Q. This follows from the proof of [26, Lemma 3.3], which can be adapted to show that Λ_z is actually independent of quasigeodesic ray from 1 to z.

(5) Fix $z_0 \in \Gamma_Q$ and $z \in \partial Q$, and let $\gamma = [z_0, z)$ be a geodesic ray in Γ_Q with vertices $z'_n \in \gamma$ such that $d_Q(z_0, z_n) = n$. Let $\sigma' \colon [z_0, z) \to \Gamma_G$ be a quasi-isometric section with $\sigma'(z'_n) = g'_n$ and let Λ'_z be the algebraic ending lamination obtained by considering conjugacy minimal representatives of $g'_n h (g'_n)^{-1}$. The proof of [26, Lemma 3.3] also shows that $\Lambda_z = \Lambda'_z$. So, when defining Λ_z, we can consider a geodesic ray from any basepoint $z_0 \in \Gamma_Q$ converging to $z \in \partial Q$.

The next proposition shows how leaves of the lamination Λ_z behave under the action of conjugation by elements of G.

Proposition 5.5 *Let $1 \to H \to G \to Q \to 1$ be as in Convention 3.6 and let $P \colon \Gamma_G \to \Gamma_Q$ be the induced map. Then, for all $g \in G$, $z \in \partial Q$ and $(u, v) \in \partial^2 H$, we have that (u, v) is a leaf of Λ_z if and only if $(g^{-1}u, g^{-1}v)$ is a leaf of $\Lambda_{P(g)^{-1}z}$.*

Proof Fix $z \in \partial Q$ and $g \in G$, and set $q_0 := P(g)$. Let $\lambda = (u, v)$ be a leaf of Λ_z. If $[1, z)$ is a geodesic ray in Γ_Q with vertices $1, z_1, z_2, \dots$, then $q_0^{-1} \cdot [1, z) = [q_0^{-1}, q_0^{-1} z)$ is a geodesic ray in Γ_Q with vertices $q_0^{-1}, q_0^{-1} z_1, q_0^{-1} z_2, \dots$. Since Λ_z is independent of quasi-isometric section, we may assume that σ is a quasi-isometric section with $\sigma(q_0) = g$. As in Convention 5.2, we will denote $\sigma(z_i)$ by g_i.

Since $(u, v) \in \Lambda_z$, there is some sequence $(a_i, a_i w_i) \in H \times H$ such that $w_i \in [g_{n_i} h g_{n_i}^{-1}]_H$ is a conjugacy minimal representative of $g_{n_i} h g_{n_i}^{-1}$ in H for some $n_i \geq 0$ and such that $a_i \to u$ and $a_i w_i \to v$ in $\widehat{\Gamma}_H$ as $i \to \infty$. Note that the sequence

$$(\phi_g(a_i), \phi_g(a_i w_i)) = (\phi_g(a_i), \phi_g(a_i)\phi_g(w_i))$$

converges to $(\phi_g(u), \phi_g(v)) = (g^{-1}u, g^{-1}v)$ in $\widehat{H} \times \widehat{H}$.

Since $w_i \in [g_{n_i} h g_{n_i}^{-1}]_H$, we have that $\phi_g(w_i) \in [g^{-1} g_{n_i} h g_{n_i}^{-1} g]_H$. As mentioned earlier, there exist constants $K \geq 1$ and $C \geq 0$ such that ϕ_g is a (K, C)–quasi-isometry. Since for each $i \geq 0$ we have that w_i is a conjugacy minimal representative, Lemma 2.12 implies that there exists some $\kappa \geq 0$ such that, for all $i \geq 0$, $\phi_g(w_i)$ is a κ–almost conjugacy minimal representative of $[g^{-1} g_{n_i} h g_{n_i}^{-1} g]_H$ in H. So, for each $i \geq 0$, there exists some $c_i \in H$ with $|c_i|_H \leq \kappa$ such that $c_i^{-1} \phi_g(w_i) c_i$ is a conjugacy minimal representative of $[g^{-1} g_{n_i} h g_{n_i}^{-1} g]_H$. As $(\phi_g(a_i), \phi_g(a_i)\phi_g(w_i)) \to (g^{-1}u, g^{-1}v)$ and $|c_i| \leq \kappa$ for all $i \geq 0$, we must also have that

$$(\phi_g(a_i)c_i, \phi_g(a_i)\phi_g(w_i)c_i) = (\phi_g(a_i)c_i, \phi_g(a_i)c_i c_i^{-1}\phi_g(w_i)c_i) \to (g^{-1}u, g^{-1}v).$$

For each $n_i \geq 0$, the element $g^{-1}g_{n_i}$ is in the same coset of H in G as $\sigma(q_0^{-1}z_{n_i})$. So, $c_i^{-1}\phi_g(w_i)c_i$ is a conjugacy minimal representative of $[\sigma(q_0^{-1}z_{n_i})h\sigma(q_0^{-1}z_{n_i})^{-1}]_H$. Therefore, by definition of $\Lambda_{q_0^{-1}z}$ and Remark 5.4(5), $\lambda_g = (g^{-1}u, g^{-1}v)$ is a leaf of $\Lambda_{q_0^{-1}z}$.

Now suppose that $\lambda_g = (g^{-1}u, g^{-1}n)$ is a leaf of $\Lambda_{P(g)^{-1}z}$. Let $g^{-1}u = u'$, $g^{-1}v = v'$ and let $\lambda' = (u', v')$. Then the forward direction of this proposition shows that $\lambda'_{g^{-1}} \in \Lambda_{P(g^{-1})^{-1}P(g)^{-1}z} = \Lambda_z$. As $\lambda'_{g^{-1}} = (u, v) = \lambda$, the reverse direction of this proposition follows. □

The main result of Mj in [26] is the following:

Theorem 5.6 (Mj [26, Theorem 4.11]) *Suppose that $1 \to H \to G \to Q \to 1$ is as in Convention 3.6 and $\partial i\colon \partial H \to \partial G$ is the Cannon–Thurston map. Then, for distinct points $u, v \in \partial H$, we have $\partial i(u) = \partial i(v)$ if and only if $(u, v) \in \Lambda$.*

The goal of the remainder of this section is to prove Theorem C. We first show that if $\lambda = (u, v)$ is a leaf of Λ_z, then ∂i_γ^+ identifies the endpoints u and v.

Proposition 5.7 *Let $1 \to H \to G \to Q \to 1$ be as in Convention 3.6, γ be as in Convention 3.7, i_γ^+ be as in Convention 4.7 and let $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ denote the Cannon–Thurston map. If $\lambda = (u, v)$ is a leaf of Λ_z, then $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$.*

Proof Let $\lambda = (u, v) \in \Lambda_z$ and suppose that $h \in H$ is such that λ is a leaf of $\Lambda_{z,h}$. By Remark 5.4, we can consider $\Lambda_{z,h}$ defined by the geodesic ray $[z_0, z)$. If $\sigma'\colon \Gamma_Q \to \Gamma_G$ is any quasi-isometric section, then $[\sigma'(z_i)h\sigma'(z_i)^{-1}]_H = [g_i h g_i^{-1}]_H$ for all $z_i \in [z_0, z)$. Hence, there exist elements $a_i \in H$ and conjugacy minimal representatives $w_i \in [g_{n_i} h g_{n_i}^{-1}]_H$ for some $n_i \geq 0$ such that $a_i \to u$ and $a_i w_i \to v$ as $i \to \infty$. Since w_i is conjugacy minimal, $[a_i w_i^{-1}, a_i] \cup [a_i, a_i w_i] \cup [a_i w_i, a_i w_i^2]$ is a $(1, C_1)$–quasigeodesic for $C_1 = C_1(\delta)$ by Lemma 2.8 and Proposition 2.4. So $a_i w_i^{-1} \to u$ and $a_i w_i^2 \to v$ as well.

Suppose for each $i \geq 0$ that $g_{n_i} h g_{n_i}^{-1} =_H c_i^{-1} w_i' c_i$, where $c_i \in H$ is a minimal length element conjugating $g_{n_i} h g_{n_i}^{-1}$ to a cyclic conjugate w_i' of w_i. Mark vertices p_i on $[a_i w_i^{-1}, a_i]$ and q_i on $[a_i w_i, a_i w_i^2]$ where the path labeled by $(w_i')^2$ begins and ends. Let $x_i = p_i c_i$ and $y_i = q_i c_i$ denote the vertices at the end of the paths labeled by c_i which start at p_i and q_i, respectively. Now, as in the proof of Lemma 2.9 the minimality of $|c_i|$ requires that $(x_i, q_i; \Gamma_H)_{p_i} \leq \delta$ and $(p_i, y_i; \Gamma_H)_{q_i} \leq \delta$. Hence, by Proposition 2.4, $[x_i, p_i] \cup [p_i, q_i] \cup [q_i, y_i]$ is a $(1, 8\delta)$–quasigeodesic in Γ_H. So, we must have that $x_i \to u$ and $y_i \to v$ in $\widehat{\Gamma}_H$. Note that the geodesic in Γ_H between x_i and y_i is labeled by the word $c_i^{-1}(w_i')^2 c_i =_H g_{n_i} h^2 g_{n_i}^{-1}$.

Recall that $i_\gamma^+(x_i) = x_i g_0$ and $i_\gamma^+(y_i) = y_i g_0$. So, the geodesic between $i_\gamma^+(x_i)$ and $i_\gamma^+(y_i)$ in $X(\gamma)$ is labeled by a word representing the element $g_0^{-1} g_{n_i} h^2 g_{n_i}^{-1} g_0$. To show that $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$, we must show that, in $X(\gamma)^+$, the distance between some fixed point and the geodesic between $i_\gamma^+(x_i)$ and $i_\gamma^+(y_i)$ goes to infinity as $i \to \infty$. For each $i > 0$, consider the path $\rho_i \in X(\gamma)^+$ from $i_\gamma^+(x_i)$ to $i_\gamma^+(y_i)$ which consists of the geodesic $[x_i g_0, x_i g_{n_i}]$ labeled by $g_0^{-1} g_{n_i}$, followed by the quasigeodesic from $x_i g_{n_i}$ to $x_i g_{n_i} h^2$ labeled by h^2, followed by the geodesic $[x_i g_{n_i} h^2, y_i g_0]$ labeled by $g_{n_i}^{-1} g_0$. This path is a $(1, C)$–quasigeodesic in $X(\gamma)^+$ by Lemma 4.13 for some constant $C \geq 0$ independent of i.

So, take an arbitrary point $p \in \rho_n$. We will show that p is far from g_0 in $X(\gamma)^+$, and so the distance in $X(\gamma)^+$ from a quasigeodesic between $i_\gamma^+(x_n)$ and $i_\gamma^+(y_n)$ to g_0 goes to infinity as n goes to infinity. Note that since $d_H(1, x_n) \to \infty$, we must have that $d_{X(\gamma)^+}(g_0, i_\gamma^+(x_n)) \to \infty$.

Suppose first that the point p belongs to the initial part of ρ_n which is labeled by $g_0^{-1} g_n$. In this case, $p = i_\gamma^+(x_n) g_0^{-1} g_j$ for some $0 \leq j \leq n$. There are two cases for us to consider:

(1) If $j \leq \frac{1}{2} d_{X(\gamma)^+}(g_0, i_\gamma^+(x_n))$, then

$$\begin{aligned} d_{X(\gamma)^+}(p, g_0) &= d_{X(\gamma)^+}(i_\gamma^+(x_n) g_0^{-1} g_j, g_0) \\ &\geq d_{X(\gamma)^+}(i_\gamma^+(x_n), g_0) - j \\ &\geq \tfrac{1}{2} d_{X(\gamma)^+}(g_0, i_\gamma^+(x_n)). \end{aligned}$$

(2) If $j > \frac{1}{2} d_{X(\gamma)^+}(g_0, i_\gamma^+(x_n))$, then

$$d_{X(\gamma)^+}(p, g_0) = d_{X(\gamma)^+}(i_\gamma^+(x_n) g_j, g_0) \geq j > \tfrac{1}{2} d_{X(\gamma)^+}(g_0, i_\gamma^+(x_n)).$$

In both cases, $d_{X(\gamma)^+}(p, g_0) \to \infty$ as $n \to \infty$. The case where p belongs to the terminal part of ρ_n which is labeled by $g_n^{-1} g_0$ is handled similarly.

Finally, if p is a vertex in the portion of ρ_n which is labeled by h^2, then since $h^2 \in H$ is fixed, in $X(\gamma)^+$, p must lie a bounded distance away from the element $i_\gamma^+(x_n) g_0^{-1} g_n$. In this case, $d_{X(\gamma)^+}(p, g_0) \geq d_{X(\gamma)^+}(i_\gamma^+(x_n) g_0^{-1} g_n, g_0) - |h^2|_H$, and $d_{X(\gamma)^+}(i_\gamma^+(x_n) g_0^{-1} g_n, g_0) - |h^2|_H \to \infty$ as $n \to \infty$. Therefore, the distance between $[i_\gamma(x_n)^+, i_\gamma^+(y_n)]_{X(\gamma)^+}$ and g_0 in $X(\gamma)^+$ goes to infinity as $n \to \infty$. Hence, $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$. □

The following several lemmas from Mj [26] will allow us to show that certain geodesics are conjugacy minimal representatives. We have stated and proved these results in the setting where $\gamma = (z', z) \subseteq \Gamma_Q$ does not necessarily go through the identity. However, we will apply these results in a simpler setting where γ does go through the identity. We have included the more general statements here to illuminate what happens in the general setting. The following three results are used to prove Corollary 5.12, which is key to the proof of Theorem C.

Lemma 5.8 (see Mj [26, Lemma 4.2]) *There exists $\kappa \geq 0$ such that, for any $(u, v) \in \partial^2 H$ with $\partial i(u) = \partial i(v)$, any geodesic subsegment $[p, q]$ of $\lambda = (u, v)$ has an extension $[r, q]$ in λ with $d_H(p, r)$ equal to 0 or 1 such that $[r, q]$ is a κ–almost conjugacy minimal representative.*

The next lemma is proved in a similar manner to Mj's Lemma 4.3 in [26].

Lemma 5.9 *Given $\kappa \geq 0$, there exists $C \geq 1$ such that for any distinct $z, z' \in \partial Q$ and for any geodesic $\gamma = (z', z) \subset \Gamma_Q$ with $z_0 \in \gamma$, the following holds:*

If $\lambda = [1, h] \subseteq \Gamma_H$ and λ_{g_0} is a κ–almost conjugacy minimal representative for some $g_0 \in P^{-1}(z_0)$, then there exists a $(C, 0)$–quasi-isometric section σ_0 of (z', z) into $X(\gamma)$ containing g_0 such that, for all $g \neq g_0$ in $\sigma_0((z', z))$, λ_g is a conjugacy minimal representative.

Proof Let $\gamma = (z', z)$ be as in Convention 3.7. Let $\sigma\colon (z', z) \to X(\gamma)$ be an isometric lift of (z', z) into $X(\gamma)$ with $\sigma(z_0) = g_0$ and such that λ_{g_0} is a κ–almost conjugacy minimal representative for some $\kappa \geq 0$. We will construct the quasi-isometric section σ_0 satisfying the conclusions of the lemma inductively.

Set $\sigma_0(z_0) = g_0$. For each $n \geq 0$ set $s_n := \sigma(z_n)^{-1}\sigma(z_{n+1})$, and for each $n \leq 0$ set $s_{n-1} = \sigma(z_n)^{-1}\sigma(z_{n-1})$. Note that since σ is an isometric embedding, $|s_n| = 1$ for all n. So, there exists some $K_1 \geq 1$ and $\epsilon_1 \geq 0$ such that $\phi_{s_n}\colon \Gamma_H \to \Gamma_H$ is a (K_1, ϵ_1)–quasi-isometry for all $n \geq 0$. As λ_{g_0} is a κ–almost conjugacy minimal representative, there exists $\kappa' \geq 0$ such that $\phi_{s_0}(\lambda_{g_0}) = \lambda_{g_0 s_0}$ is a κ'–conjugacy minimal representative by Lemma 2.12. By Corollary 2.10, there exists $c_0 \in H$ and $M' \geq 0$ with $|c_0|_H \leq M'$ such that $\lambda_{g_0 s_0 c_0}$ is a conjugacy minimal representative. Set $\sigma_0(z_1) := g_0 s_0 c_0$. We can similarly define $\sigma_0(z_{-1})$.

Suppose that $\sigma_0(z_j)$ has been constructed satisfying the conclusions of the lemma for all $-m \leq j \leq n$. By assumption, $\lambda_{\sigma_0(z_n)}$ is a conjugacy minimal representative, and so by Lemma 2.12 there exists $\kappa'' \geq 0$ such that $\lambda_{\sigma_0(z_n)s_n}$ is a κ''–almost conjugacy minimal representative. Then, by Corollary 2.10, there exists $c_n \in H$ and $M'' \geq 0$ with $|c_n|_H \leq M''$ such that $\lambda_{\sigma_0(z_n)s_n c_n}$ is a conjugacy minimal representative. Set $\sigma_0(z_{n+1}) := \sigma_0(z_n)s_n c_n$. We can similarly define $\sigma_0(z_{-m-1})$. Note that $d_{X(\gamma)}(\sigma_0(z_i), \sigma_0(z_{i+1})) \leq \max\{M', M''\}$, and so σ_0 is a $(C,0)$–quasi-isometric section, where $C := \max\{M', M''\}$ and λ_g is a conjugacy minimal representative for all $g \neq g_0$ in $\sigma_0((z', z))$. □

The following corollary is obtained from the previous lemma by translating the quasi-isometric section by an element of G. Here, we choose the quasi-isometric section σ_0 to go through the point $g_0 \in \Gamma_G$ rather than the identity.

Corollary 5.10 (see Mj [26, Corollary 4.4]) *Given $\kappa \geq 0$, there exists $C \geq 1$ such that for any geodesic ray $[z_0, z)$ in Γ_Q and any $g \in P^{-1}([z_0, z))$ the following holds:*

If $\lambda = [1, h] \subseteq \Gamma_H$ and λ_{g_0} is a κ–almost conjugacy minimal representative for some $g_0 \in P^{-1}(z_0)$, then there exists a $(C,0)$–quasi-isometric section σ_0 of $[z_0, z)$ into Γ_G containing $g \in \Gamma_G$ such that, for all $g' \neq g$ in $\sigma_0([z_0, z))$, $\lambda_{g_0 g^{-1} g'}$ is a conjugacy minimal representative.

Proof By Lemma 5.9, there exists a $(C,0)$–quasi-isometric section $\sigma'\colon (z', z) \to X(\gamma)$ with $\sigma'(z_0) = g_0$ such that, for all $g' \neq g_0$ in $\sigma'((z', z))$, $\lambda_{g'}$ is a conjugacy minimal representative. Suppose that $g \in P^{-1}(z_n)$ and set $\sigma_0(z_n) := g$. For each integer i with $i \geq -n$, set $\sigma_0(z_{n+i}) := t_{gg_0^{-1}} \cdot \sigma'(z_i)$. Now, $\sigma_0\colon [z_0, z) \to X(\gamma)^+$ is a $(C,0)$–quasi-isometric section since it is a left-translate of σ' by $gg_0^{-1} \in G$. Also, note that for all $g' \neq g$ in $\sigma([z_0, z))$, we have that $g' = t_{gg_0^{-1}} \cdot \sigma'(z_i)$ for some $i \geq -n$ with $i \neq 0$. Then $\lambda_{g_0 g^{-1} g'} = \lambda_{g_0 g^{-1} g g_0^{-1} \sigma'(z_i)} = \lambda_{\sigma'(z_i)}$ is a conjugacy minimal representative by Lemma 5.9. □

The following lemma will allow us to reduce to the simpler setting where $\gamma = (z', z) \subseteq \Gamma_Q$ passes through the identity in Γ_Q.

Lemma 5.11 *Suppose $\gamma = (z', z)$ is as in Convention 3.7 and let $\gamma' := z_0^{-1} \cdot \gamma = (z_0^{-1} z', z_0^{-1} z)$ be as in Convention 4.9. Let $X(\gamma)$ and $X(\gamma')$ be the stacks as in Convention 4.7, where the section $\sigma\colon \gamma \to X(\gamma)$ is such that $\sigma(z_0) = g_0$ and $\sigma'\colon \gamma' \to X(\gamma')$ is chosen so that $\sigma' = g_0^{-1} \cdot \sigma$. Let i_γ^+ and $i_{\gamma'}^+$ be as in Convention 4.7, and let $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ and $\partial i_{\gamma'}^+\colon \partial H \to \partial X(\gamma')^+$ be the Cannon–Thurston maps.*

Then, for any two distinct points $u, v \in \partial H$, we have $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$ if and only if $\partial i_{\gamma'}^+(\phi_{g_0}(u)) = \partial i_{\gamma'}^+(\phi_{g_0}(v))$, where $g_0 \in P^{-1}(z_0)$.

Proof Let $\gamma = (z', z)$ be as in Convention 3.7, let $\gamma' := z_0^{-1} \cdot \gamma = (z_0^{-1} z', z_0^{-1} z)$ and fix some $g_0 \in P^{-1}(z_0)$. Recall that i_γ^+ is given by $i_\gamma^+(h) = t_{g_0} \cdot \phi_{g_0}(h) = hg_0$ and $i_{\gamma'}^+$ is given by $i_{\gamma'}(h) = h$. Suppose first that $u, v \in \partial H$ are distinct points such that $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$. Then, for any sequences $(u_n), (v_n) \in \Gamma_H$ with $u_n \to u$ and $v_n \to v$ in $\widehat{\Gamma}_H$, we have that, in $\widehat{X(\gamma)^+}$, $\lim_{n\to\infty} i_\gamma^+(u_n) = \lim_{i\to\infty} i_\gamma^+(v_n)$. So, in $\widehat{X(\gamma)^+}$ we have that $\lim_{i\to\infty} u_n g_0 = \lim_{i\to\infty} v_n g_0$. Note that $X(\gamma)^+ = g_0 X(\gamma')^+$ and so left-translation by g_0^{-1} gives an isometry from $X(\gamma)^+$ to $X(\gamma')^+$. Therefore, in $\widehat{X(\gamma')^+}$ we have that $\lim_{i\to\infty} g_0^{-1} u_n g_0 = \lim_{i\to\infty} g_0^{-1} v_n g_0$. So, by definition of $i_{\gamma'}^+$, we have that $\lim_{i\to\infty} i_{\gamma'}^+(\phi_{g_0}(u_n)) = \lim_{i\to\infty} i_{\gamma'}^+(\phi_{g_0}(v_n))$ in $\widehat{X(\gamma')^+}$. Since, in $\widehat{\Gamma}_H$, $\phi_{g_0}(u_n) \to \phi_{g_0}(u)$ and $\phi_{g_0}(v_n) \to \phi_{g_0}(v)$ as $n \to \infty$, we have that $\partial i_{\gamma'}^+(\phi_{g_0}(u)) = \partial i_{\gamma'}^+(\phi_{g_0}(v))$ by the continuity of $\hat{i}_{\gamma'}^+$ (Lemma 4.8). The reverse implication follows in the same manner by noting that left-translation by g_0 gives an isometry from $X(\gamma')^+$ to $X(\gamma)^+$. □

The following result follows directly from Lemma 5.8 and Corollary 5.10. This corollary will be used in the proof of Theorem C to construct a sequence of conjugacy minimal representatives which converge to some bi-infinite geodesic $\lambda \subseteq \partial^2 H$ whose endpoints are identified by ∂i_γ^+.

Corollary 5.12 (see Mj [26, Lemma 4.5]) *There exists C' such that, for any $\lambda = (u, v)$, $u, v \in \partial H$ with $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$, any geodesic ray $[z_0, z)$ in Γ_Q and any geodesic subsegment $[p, q]$ of λ_g, for some $g \in P^{-1}([z_0, z))$ the following holds:*

There exists an extension $[r, q] = \mu$ of $[p, q]$ in λ_g with $d_H(p, r)$ equal to 0 or 1 and a $(C', 0)$–quasi-isometric section $\sigma\colon [z_0, z) \to X(\gamma)$ such that $gr \in \sigma([z_0, z))$ and $\mu_{g_0 r^{-1} g^{-1} g'}$ is a conjugacy minimal representative for all $g' \neq gr$ in $\sigma([z_0, z))$.

Proof Let $\lambda = (u, v)$ be such that $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$, let $[z_0, z) \in \Gamma_Q$ be a geodesic ray, let $g \in P^{-1}([z_0, z))$ and let $[p, q]$ be any geodesic subsegment of $\lambda_g = (\phi_g(u), \phi_g(v))$. By Lemma 5.11, $\partial i_{\gamma'}^+(\phi_{g_0}(u)) = \partial i_{\gamma'}^+(\phi_{g_0}(v))$. So, by Corollary 4.12, $\partial i(\phi_g(u)) = \partial i(\phi_g(v))$. So, by Lemma 5.8, there exists an extension $[r, q] = \mu$ of $[p, q]$ in λ_g with $d_H(p, r)$ equal to 0 or 1 and such that $[r, q]$ is a κ–almost conjugacy minimal representative for some $\kappa \geq 0$. Let $\mu' = [1, r^{-1} q]$ and note that μ' is also a κ–almost conjugacy minimal representative since it has the same label as μ. By Lemma 2.12, μ_{g_0} and μ'_{g_0} are κ'–almost conjugacy minimal representatives for some $\kappa' \geq 0$ depending on g_0. So, by Corollary 5.10, there exists $C' \geq 1$ and a $(C', 0)$–quasi-isometric section $\sigma\colon [z_0, z) \to \Gamma_G$ containing $gr \in \Gamma_G$ such that, for all $g' \neq gr \in \sigma([z_0, z))$,

$\mu'_{g_0 r^{-1} g^{-1} g'}$ is a conjugacy minimal representative. Therefore, $\mu_{g_0 r^{-1} g^{-1} g'}$ is also a conjugacy minimal representative. □

For the next portion of this section, we will assume that the bi-infinite geodesic $\gamma = (z', z) \subseteq \Gamma_Q$ goes through the identity in Q, and so $\gamma^+ = [1, z)$. Note that several of the previous lemmas simplify in this case. We now make the following convention:

Convention 5.13 Let $\gamma = (z', z)$ be a bi-infinite geodesic in Γ_Q between $z', z \in \partial Q$ with $z' \neq z$ and assume that $1 \in \gamma$. Label the sequence of vertices in order along the portion of γ from 1 to z by $1 = z_0, z_1, z_2, \dots$. Similarly, label the sequence of vertices in order along the portion of γ from 1 to z' by $1 = z_0, z_{-1}, z_{-2}, \dots$. Let $\sigma_0 \colon \gamma \to \Gamma_G$ denote an isometric lift of γ through the identity in Γ_G, ie such that $\sigma_0(1) = 1$, and set $g_i := \sigma_0(z_i)$. Let $X(\gamma)$ and $X(\gamma)^+$ denote the stacks over $\gamma = (z', z)$ and $\gamma^+ = [1, z)$, respectively. Finally, let $i_\gamma \colon \Gamma_H \to X(\gamma)$ and $i_\gamma^+ \colon \Gamma_H \to X(\gamma)^+$ be the respective inclusion maps given by $i_\gamma(h) = h$ and $i_\gamma^+(h) = h$ for all $h \in H$.

Before proving Theorem C, we will first introduce some necessary terminology as well as some lemmas which were first stated by Mj in [26].

Given a (finite or infinite) geodesic $\lambda \subset \widehat{\Gamma}_H$ with endpoints $a, b \in \widehat{\Gamma}_H$ and an element $g \in G$, recall that $\lambda_g \subset \widehat{\Gamma}_H$ denotes the geodesic joining $\phi_g(a) = g^{-1} a g$ and $\phi_g(b) = g^{-1} b g$. For any quasi-isometric section $\sigma \colon \Gamma_Q \to \Gamma_G$ and geodesic λ, Mj defines the set

$$B(\lambda, \sigma) := \bigcup_{g \in \sigma(Q)} t_g \cdot i(\lambda_g),$$

where t_g denotes left-translation by the element $g \in G$. For our purposes, we will consider the subset of $B(\lambda, \sigma)$ which lives in $X(\gamma)^+$,

$$B_{\gamma^+}(\lambda, \sigma) = \bigcup_{g \in \sigma([1, z))} t_g \cdot i_\gamma^+(\lambda_g).$$

Note that $B_{\gamma^+}(\lambda, \sigma) = B(\lambda, \sigma) \cap P^{-1}([1, z))$ and that if λ is a bi-infinite geodesic, then $B_{\gamma^+}(\lambda, \sigma)$ is independent of quasi-isometric section σ for the same reason Mj uses to show $B(\lambda, \sigma)$ is independent of quasi-isometric section [26].

On the vertices of Γ_H, define the map $\pi_{g,\lambda} \colon \Gamma_H \to \lambda_g$ by sending $h \in H$ to a closest vertex on λ_g. We will now define a projection map to the set $B_{\gamma^+}(\lambda, \sigma)$. As σ is a quasi-isometric section, for each $g' \in X(\gamma)^+$ there are unique $g \in \sigma([1,z))$ and $h \in H$ such that $g' = t_g \cdot i_\gamma^+(h)$. So, define

$$\Pi_\lambda^\sigma(g') = \Pi_\lambda^\sigma \cdot t_g \cdot i_\gamma^+(h) := t_g \cdot i_\gamma^+ \cdot \pi_{g,\lambda}(h).$$

The following statements are versions of the analogous statements from Mj [26] which apply to the setting in which we are working. In most cases, the proofs that Mj provided go through with no changes to the reasoning. We provide details of the necessary modifications where they are needed.

The same proof of Mj's Theorem 3.7 of [27] verifies the following statement. In particular, this lemma will be used to show that if $\sigma\colon [1,z) \to X(\gamma)^+$ is a (K,ϵ)–quasi-isometric section, then the projection of σ to $B_{\gamma^+}(\lambda,\sigma)$ is also a quasi-isometric section.

Lemma 5.14 (see Mj [26, Theorem 4.6]) *For all $K \geq 1$ and $\epsilon \geq 0$, there exists a constant $C \geq 1$ such that, if $\sigma\colon [1,z) \to X(\gamma)^+$ is any (K,ϵ)–quasi-isometric section and $\lambda \subseteq \Gamma_H$ is any bi-infinite geodesic, then $d_{X(\gamma)^+}(\Pi_\lambda^\sigma(x), \Pi_\lambda^\sigma(y)) \leq C d_{X(\gamma)^+}(x,y)$ for all $x, y \in X(\gamma)^+$.*

Lemma 5.15 (see Mj [26, Lemma 4.7]) *For all $K \geq 1$ and $\epsilon \geq 0$, there exists $A \geq 1$ such that, if $\sigma\colon [1,z) \to X(\gamma)^+$ is a (K,ϵ)–quasi-isometric section, then for all $p,q \in \sigma([1,z))$ and $x \in t_p \cdot i_\gamma^+(\lambda_p)$ there exists $y \in t_q \cdot i_\gamma^+(\lambda_q)$ such that $d_{X(\gamma)^+}(x,y) \leq A d_Q(Px,Py) = A d_Q(Pp,Pq)$.*

Proof Let $\sigma\colon [1,z) \to X(\gamma)^+$ be a (K,ϵ)–quasi-isometric section, $p,q \in \sigma([1,z))$, $x \in t_p \cdot i_\gamma^+(\lambda_p)$ and set $y = \Pi_\lambda^\sigma(xp^{-1}q)$. Note that $y \in t_q \cdot i_\gamma^+(\lambda_q)$. Then, by Lemma 5.14, there exists a constant $C \geq 1$ such that $d_{X(\gamma)^+}(\Pi_\lambda^\sigma(x), \Pi_\lambda^\sigma(xp^{-1}q)) = d_{X(\gamma)^+}(x,y) \leq C d_{X(\gamma)^+}(x, xp^{-1}q)$. Since $p,q \in \sigma([1,z))$ and σ is a (K,ϵ)–quasi-isometric section, $|p^{-1}q| \leq K d_Q(Pp,Pq) + \epsilon$. Therefore, $d_{X(\gamma)^+}(x, xp^{-1}q) = |p^{-1}q| \leq K d_Q(Pp,Pq) + \epsilon$. So, let $A = C(K+\epsilon)$. As $Px = Pp$ and $Py = Pq$, we have finally that $d_{X(\gamma)^+}(x,y) \leq A d_Q(Px,Py) = A d_Q(Pp,Pq)$, as required. □

The following is the version of [26, Lemma 4.8] that we need for our purposes. It is proved by an argument similar to the one given by Mj using the previous lemma.

Lemma 5.16 (see Mj [26, Lemma 4.8]) *For all $K \geq 1$ and $\epsilon \geq 0$, there exists $M \geq 0$ such that the following holds. Suppose λ is a bi-infinite geodesic in Γ_H and a is a* *vertex on λ splitting λ into semi-infinite geodesics λ^- and λ^+. Suppose further that $\sigma\colon [1,z)\to X(\gamma)^+$ is a (K,ϵ)–quasi-isometric section such that $\sigma([1,z))\subseteq B_{\gamma^+}(\lambda,\sigma)$ and $i_\gamma^+(a)\in\sigma([1,z))$. Then any geodesic in $X(\gamma)^+$ joining a point in $B_{\gamma^+}(\lambda^-,\sigma)$ to a point in $B_{\gamma^+}(\lambda^+,\sigma)$ passes through an M–neighborhood of $\sigma([1,z))$.*

Lemma 5.17 (see Mj [26, Corollary 4.10]) *Given $K\geq 1$ and $\epsilon\geq 0$, there exists α such that, if $\lambda=(u,v)$ is such that $\partial i_\gamma^+(u)=\partial i_\gamma^+(v)$, then the following is satisfied:*

If σ and σ' are (K,ϵ)–quasi-isometric sections such that $B_{\gamma^+}(\lambda,\sigma)=B_{\gamma^+}(\lambda,\sigma')$ and σ and σ' are contained in $B_{\gamma^+}(\lambda,\sigma)$, then there exists $N\geq 0$ such that, for all $n\geq N$,

$$d_{X(\gamma)^+}(\sigma(z_n),\sigma'(z_n))\leq\alpha.$$

Proof Let $\lambda=(u,v)$ be such that $\partial i_\gamma^+(u)=\partial i_\gamma^+(v)$ and let σ and σ' be (K,ϵ)–quasi-isometric sections satisfying the hypotheses of the lemma. Let (p_n) and (q_n) be a sequence of vertices on λ such that $p_n\to u$ and $q_n\to v$ as $n\to\infty$. For each $n\geq 0$, Lemma 5.16 guarantees there exist points $z_{n'},z_{n''}\in[1,z)$ such that any geodesic in $X(\gamma)^+$ joining $i_\gamma^+(p_n)$ to $i_\gamma^+(q_n)$ passes through an M–neighborhood of both $\sigma(z_{n'})$ and $\sigma'(z_{n''})$. Since $\partial i_\gamma^+(u)=\partial i_\gamma^+(v)$ and $\widehat{i_\gamma^+}$ is continuous, we must have that the sequences $\{i_\gamma^+(p_n)\}$, $\{i_\gamma^+(q_n)\}$, $\{\sigma(z_{n'})\}$ and $\{\sigma'(z_{n''})\}$ all converge to the same point in $\partial X(\gamma)^+$. Since σ and σ' are quasi-isometric sections of $[1,z)$ into $X(\gamma)^+$ and as $d_{X(\gamma)^+}(1,[i_\gamma^+(p_n),i_\gamma^+(q_n)])\to\infty$, we must have that $z_{n'}\to z$ and $z_{n''}\to z$. Therefore, $\sigma([1,z))$ and $\sigma'([1,z))$ are asymptotic quasigeodesic rays in $X(\gamma)^+$ and, for all $n\geq N$,

$$\max\bigl\{d_{X(\gamma)^+}\bigl(\sigma(z_n),\sigma'([1,z))\bigr),d_{X(\gamma)^+}\bigl(\sigma([1,z)),\sigma'(z_n)\bigr)\bigr\}\leq\alpha'.$$

But, since σ and σ' are (K,ϵ)–quasi-isometric sections, if $z_{n'}$ is such that

$$d_{X(\gamma)^+}(\sigma(z_n),\sigma'(z_{n'}))\leq\alpha',$$

then

$$\begin{aligned}d_{X(\gamma)^+}(\sigma(z_n),\sigma'(z_n))&\leq d_{X(\gamma)^+}(\sigma(z_n),\sigma'(z_{n'}))+d_{X(\gamma)^+}(\sigma'(z_{n'}),\sigma'(z_n))\\&\leq\alpha'+K|n-n'|+\epsilon\\&\leq\alpha'+K\alpha'+\epsilon=\alpha.\end{aligned}$$

Thus, $d_{X(\gamma)^+}(\sigma(z_n),\sigma'(z_n))\leq\alpha$ for all $n\geq N$. □

We are now ready to prove the main theorem of this section, which is reminiscent of Mj's Theorem 4.11 of [26].

Theorem C *Let $1 \to H \to G \to Q \to 1$ be a short exact sequence of infinite, finitely generated, word-hyperbolic groups. Let $z, z' \in \partial Q$ be distinct and let $\gamma \subseteq \Gamma_Q$ be a bi-infinite geodesic in Γ_Q between z and z'. Let $i_\gamma^+ \colon \Gamma_H \to X(\gamma)^+$ be the inclusion of Γ_H into the semi-infinite stack $X(\gamma)^+$ over $\gamma^+ = [z_0, z)$ as in Convention 4.7, and let $\partial i_\gamma^+ \colon \partial H \to \partial X(\gamma)^+$ be the Cannon–Thurston map.*

Then, for any distinct $u, v \in \partial H$, we have $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$ if and only if (u, v) is a leaf of the ending lamination Λ_z.

Proof Suppose first that $\gamma = (z', z)$ is as in Convention 5.13 with $1 \in \gamma$. By Proposition 5.7, it suffices to show that if $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$, then $\lambda = (u, v) \in \Lambda_z$. So, let $u, v \in \partial H$ be distinct points such that $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$. As the set of leaves of $\partial^2 H$ whose endpoints are identified under ∂i_γ^+ is H–invariant, we may assume that $\lambda = (u, v)$ passes through $1 \in \Gamma_H$.

Let $\sigma_0 \colon [1, z) \to X(\gamma)^+$ be the isometric lift of γ^+ into Γ_G through the identity as in Convention 5.13. Let $\sigma_e := \Pi_\lambda^{\sigma_0} \cdot \sigma_0$ be the projection of σ_0 onto $B_{\gamma^+}(\lambda, \sigma_0)$ and set $g'_n := \sigma_e(z_n)$. By Lemma 5.14, σ_e is a $(C, 0)$–quasi-isometric section of $[1, z)$ into $B_{\gamma^+}(\lambda, \sigma_0)$ for some $C \geq 1$.

By Corollary 5.12, there exists $C' \geq 1$ such that, for any $g \in \sigma_0([1, z))$ and any $[p, q] \subseteq \lambda_g$, there exists an extension $[r, q] =: \mu$ of $[p, q]$ in λ_g with $d_H(p, r) = 0$ or 1 and a $(C', 0)$–quasi-isometric section σ such that $gr \in \sigma([1, z))$ and $\mu_{r^{-1}g^{-1}g'}$ is a conjugacy minimal representative for all $g' \neq gr$ in $\sigma([1, z))$. Projecting σ to $B_{\gamma^+}(\lambda, \sigma_0)$ yields, by Lemma 5.14, a $(C_2, 0)$–quasi-isometric section for some $C_2 \geq 1$.

If σ' is any $(C_2, 0)$–quasi-isometric section, Lemma 5.17 gives that there is some $\alpha > 0$ such that, if $\sigma' \subseteq B_{\gamma^+}(\lambda, \sigma_0)$, then there exists some $N \geq 0$ such that $d_{X(\gamma)^+}(g'_n, \sigma'(z_n)) \leq \alpha$ for all $n \geq N$. Given this α, Proposition 2.1 guarantees there are some $b > 1$, $A > 0$ and $\eta > 0$ depending on α and C_2 such that, if $\sigma'([1, z))$ is a $(C_2, 0)$–quasi-isometric section of $[1, z)$ into $X(\gamma)^+$ with $d_{X(\gamma)^+}(\sigma'(z_n), g'_n) \geq \eta$, then any path in $i_\gamma^+(\Gamma_H)$ joining $\sigma'(1)$ and $\sigma_e(1)$ has length greater than or equal to Ab^n.

Now let λ^+ and λ^- denote the two closures of the components of $\lambda \setminus \{1\}$. Note that $g'_n \in t_{g_n} \cdot i_\gamma^+(\lambda_{g_n})$ for each $n > 0$. Hence, for all $n > 0$ there exists $p_n \in \lambda_{g_n}^-$ and $q_n \in \lambda_{g_n}^+$ such that $d_{X(\gamma)^+}(t_{g_n} \cdot i_\gamma^+(p_n), g'_n) = d_{X(\gamma)^+}(g_n p_n, g'_n) = \eta + 1$ and $d_{X(\gamma)^+}(t_{g_n} \cdot i_\gamma^+(q_n), g'_n) = d_{X(\gamma)^+}(g_n q_n, g'_n) = \eta$. By Corollary 5.12, for each $n > 0$

there exists $r_n \in \lambda^-_{g_n}$ with $d_H(r_n, p_n) = 0$ or 1 and a $(C', 0)$–quasi-isometric section σ_n of $[1, z)$ into $X(\gamma)^+$ satisfying the following two conditions:

(1) $g_n r_n = \sigma_n(z_n)$.

(2) If $\mu^{(n)}$ is the subsegment of λ_{g_n} in Γ_H joining r_n and q_n, then $\mu^{(n)}_{r_n^{-1} g_n^{-1} \sigma_n(z_m)}$ is a conjugacy minimal representative for all $z_m \neq z_n$.

For each $n > 0$, define a new quasi-isometric section $\tau_n(z_i) := t_{g_n q_n r_n^{-1} g_n^{-1}} \cdot \sigma_n(z_i)$ which is obtained by left-translating σ_n to go through the point $g_n q_n \in t_{g_n} \cdot i^+_\gamma(\lambda_{g_n})$. We will now project σ_n and τ_n to the set $B_{\gamma^+}(\lambda, \sigma_0)$ to get new quasi-isometric sections which satisfy the hypotheses of Lemma 5.17. Denote these new $(C_2, 0)$–quasi-isometric sections by $\sigma'_n := \Pi^{\sigma_0}_\lambda \cdot \sigma_n$ and $\tau'_n := \Pi^{\sigma_0}_\lambda \cdot \tau_n$.

By Lemma 5.17, there is some α such that for every index $n > 0$, $d_{X(\gamma)^+}(g'_k, \sigma'_n(z_k)) \leq \alpha$ as long as $k \geq N$ for some constant $N = N(n)$. So, the $(C_2, 0)$–quasigeodesic rays interpolated by σ'_n and σ_e satisfy the hypotheses of Proposition 2.1 since there is some point along these rays where $d_{X(\gamma)^+}(g'_k, \sigma'_n(z_k)) \leq \alpha$ and since the rays were defined so that $d_{X(\gamma)^+}(g'_n, \sigma'_n(z_n)) = d_{X(\gamma)^+}(g'_n, g_n r_n) \geq \eta$. As any path in $i^+_\gamma(\Gamma_H)$ has distance at least n/C_2 from any path in $t_{g_n} \cdot i^+_\gamma(\Gamma_H)$, there exists $b > 1$ and $A > 0$ such that the portion of $i^+_\gamma(\lambda)$ between $\sigma'_n(1)$ and $\sigma_e(1) = g'_0$ has length at least Ab^n. As the same holds true for the quasigeodesic rays interpolated by τ'_n and σ_e, the portion of $i^+_\gamma(\lambda)$ between $\tau'_n(1)$ and $\sigma_e(1) = g'_0$ also has length greater than or equal to Ab^n.

Note that, for all $n \geq 0$, $\sigma_n(1)$, $\sigma'_n(1)$, $\tau_n(1)$ and $\tau'_n(1)$ all lie in $i^+_\gamma(\Gamma_H)$. Let $[\sigma'_n(1)^*, \tau'_n(1)^*]$ denote the subsegment of λ joining $(i^+_\gamma)^{-1} \cdot \sigma'_n(1)$ and $(i^+_\gamma)^{-1} \cdot \tau'_n(1)$. Then the sequence $\{[\sigma'_n(1)^*, \tau'_n(1)^*]\}$ converges to λ in $\widehat{\Gamma}_H$.

Since $d_{X(\gamma)^+}(g_n r_n, g_n q_n) \leq 2\eta + 2$, there exists $\rho > 0$ such that $r_n^{-1} q_n$ is an element of H with $|r_n^{-1} q_n|_H \leq \rho$. Since there are only finitely many of these, we may pass to a subsequence n_j such that $r_{n_j}^{-1} q_{n_j} = h$, where h is some fixed element of H. Note that the subsequence $\{[\sigma'_{n_j}(z_0)^*, \tau'_{n_j}(z_0)^*]\}$ also converges to λ in $\widehat{\Gamma}_H$.

Let $[\sigma_n(1)^*, \sigma'_n(1)^*]$ denote a geodesic segment in Γ_H joining $(i^+_\gamma)^{-1} \cdot \sigma_n(1)$ and $(i^+_\gamma)^{-1} \cdot \sigma'_n(1)$ and define $[\tau'_n(1)^*, \tau_n(1)^*]$ similarly. Since

$$\sigma'_n(1)^* = (i^+_\gamma)^{-1} \cdot \Pi^{\sigma_0}_\lambda \cdot i^+_\gamma(\sigma_n(1)),$$

we must have that $(\sigma_n(1)^*, \tau'_n(1)^*)_{\sigma'_n(1)^*} \leq 2\delta$ in Γ_H. Otherwise, there would be a point on $i^+_\gamma(\lambda)$ closer to $\sigma_n(1)$ than $\sigma'_n(1)$, contradicting the definition of $\sigma'_n(1)$ as the projection of $\sigma_n(1)$ to $i^+_\gamma(\lambda)$. For a similar reason, $(\sigma'_n(1)^*, \tau_n(1)^*)_{\tau'_n(1)^*} \leq 2\delta$.

So, by Proposition 2.4, for all n sufficiently large (so that $d_H(\sigma_n'(1)^*, \tau_n'(1)^*) > 14\delta$), $[\sigma_n(1)^*, \sigma_n'(1)^*] \cup [\sigma_n'(1)^*, \tau_n'(1)^*] \cup [\tau_n'(1)^*, \tau_n(1)^*]$ is a $(1, 12\delta)$–quasigeodesic. Thus, for all n sufficiently large, there is some constant $B > 0$ depending only on δ such that $[\sigma_n(1)^*, \sigma_n'(1)^*] \cup [\sigma_n'(1)^*, \tau_n'(1)^*] \cup [\tau_n'(1)^*, \tau_n(1)^*]$ lies in a B–neighborhood of the geodesic $[\sigma_n(1)^*, \tau_n(1)^*]$ in Γ_H.

As the sequence $\{[\sigma_{n_j}'(1)^*, \tau_{n_j}'(1)^*]\}$ converges to λ, we must also have that the sequence

$$\{[\sigma_{n_j}(1)^*, \sigma_{n_j}'(1)^*] \cup [\sigma_{n_j}'(1)^*, \tau_{n_j}'(1)^*] \cup [\tau_{n_j}'(1)^*, \tau_{n_j}(1)^*]\}$$

converges to λ. In particular, $\{\sigma_{n_j}(1)^*\}$ and $\{\tau_{n_j}(1)^*\}$ must converge to the endpoints of λ in Γ_H. Recall that σ_n was chosen so that, in particular, $\mu^{(n)}_{r_n^{-1} g_n^{-1} \sigma_n(1)}$ is a conjugacy minimal representative. Since $\mu^{(n)}_{r_n^{-1} g_n^{-1} \sigma_n(1)}$ is the label of the geodesic in Γ_H between $\sigma_n(1)^*$ and $(g_n q_n r_n^{-1} g_n^{-1} \sigma_n(1))^* = \tau_n(1)^*$, we have that $\{[\sigma_{n_j}(1)^*, \tau_{n_j}(1)^*]\}$ is a sequence of conjugacy minimal representatives of $\phi_{r_{n_j}^{-1} g_{n_j}^{-1} \sigma_{n_j}(1)}(h)$.

Let $\sigma''\colon [1, z) \to \Gamma_G$ be any quasi-isometric section. Note that, for all $n \geq 0$, $\sigma''(z_n)$ and $\sigma_n(1)^{-1} g_n r_n$ are in the same coset of H in G. Therefore, $\phi_{r_{n_j}^{-1} g_{n_j}^{-1} \sigma_{n_j}(1)}(h)$ and $\phi_{(\sigma''(z_n))^{-1}}(h)$ have the same conjugacy minimal representatives. Hence, $\lambda = (u, v) \in \Lambda_{z,h} \subseteq \Lambda_z$.

Finally, suppose that $\gamma = (z', z)$ goes through $z_0 \in \Gamma_Q$ rather than the identity. Then, $\gamma' := z_0^{-1}\gamma = (z_0^{-1}z', z_0^{-1}z)$ does go through the identity. If $\partial i_\gamma^+(u) = \partial i_\gamma^+(v)$, Lemma 5.11 implies that $\partial i_{\gamma'}^+(\phi_{g_0}(u)) = \partial i_{\gamma'}^+(\phi_{g_0}(v))$. By the above, this implies that $\lambda_{g_0} = (\phi_{g_0}(u), \phi_{g_0}(v)) \in \Lambda_{z_0^{-1}z}$. Finally, by Proposition 5.5, this implies that $\lambda \in \Lambda_z$, as desired. □

6 Proof of the main result

We can now prove the main result of the paper, Theorem A from the introduction. Recall that a *dendrite* is a compact, connected, locally connected metrizable space which contains no simple closed curves.

Proposition 6.1 (Bowditch [5, Proposition 2.5.2]) *Let $\mathcal{X}$ be a bi-infinite hyperbolic stack and let $\mathcal{X}^+$ be the corresponding semi-infinite stack. Then the Gromov boundary $\partial\mathcal{X}^+$ is a dendrite.*

If $L \subseteq \partial^2 H$ is an algebraic lamination on H, then $\partial H / L$ denotes the quotient space of ∂H by the equivalence relation generated by $L \subseteq \partial^2 H$.

Theorem A *Let $1 \to H \to G \to Q \to 1$ be a short exact sequence of infinite, finitely generated, word-hyperbolic groups and choose $z \in \partial Q$. Then the space $\partial H/\Lambda_z$ is homeomorphic to a dendrite.*

Proof Let $\gamma = (z', z)$ be as in Convention 3.7 and let $X(\gamma)^+$ and $i_\gamma^+\colon \Gamma_H \to X(\gamma)^+$ be as in Convention 4.7. Denote by $\pi_z\colon \partial H \to \partial H/\Lambda_z$ the quotient map. If $a, b \in \partial H$ are such that $\pi_z(a) = \pi_z(b)$, then $\partial i_\gamma^+(a) = \partial i_\gamma^+(b)$ by Proposition 5.7. So, the Cannon–Thurston map $\partial i_\gamma^+\colon \partial H \to \partial X(\gamma)^+$ quotients through to a map $\tau_z\colon \partial H/\Lambda_z \to \partial X(\gamma)^+$ with $\partial i_\gamma^+ = \tau_z \circ \pi_z$. We will show that τ_z is a continuous bijection from a compact topological space to a Hausdorff topological space, and thus is a homeomorphism.

Note that the Gromov boundary of a proper hyperbolic space is compact and metrizable (see for instance [22]), and so $\partial X(\gamma)^+$ is compact Hausdorff and $\partial H/\Lambda_z$ is compact. As ∂i_γ^+ is continuous by virtue of being a Cannon–Thurston map (Lemma 4.8) and the quotient map π_z is also continuous, the map τ_z must be continuous. By Theorem B, ∂i_γ^+ is surjective and so τ_z must also be surjective. If $a', b' \in \partial H/\Lambda_z$ are such that $\tau_z(a') = \tau_z(b') = u \in \partial X(\gamma)^+$, then, since ∂i_γ^+ is surjective, there must exist $a, b \in \partial H$ such that $\partial i_\gamma^+(a) = \partial i_\gamma^+(b) = u$. But, by Theorem C, this implies that $(a, b) \in \Lambda_z$, and so τ_z is injective. It now follows that $\tau_z\colon \partial H/\Lambda_z \to \partial X(\gamma)^+$ is a homeomorphism. Therefore, by Proposition 6.1, $\partial H/\Lambda_z$ is a dendrite. □

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