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Finite rigid sets in arc complexes

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For any compact, connected, orientable, finite-type surface with marked points other than the sphere with three marked points, we construct a finite rigid set of its arc complex: a finite simplicial subcomplex of its arc complex such that any locally injective map of this set into the arc complex of another surface with arc complex of the same or lower dimension is induced by a homeomorphism of the surfaces, unique up to isotopy in most cases. It follows that if the arc complexes of two surfaces are isomorphic, the surfaces are homeomorphic. We also give an exhaustion of the arc complex by finite rigid sets. This extends the results of Irmak and McCarthy (Turkish J. Math. 34 (2010) 339–354).

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1 Introduction

Let S and S' be compact, connected, orientable, finite-type surfaces (with empty boundary) with marked points. The *arc complex* $\mathcal{A}(S)$ of S, first defined by Harer in [9], is the simplicial complex whose vertices correspond to the isotopy classes of arcs on S and whose k–simplices ($k>0$) correspond to collections of $k+1$ distinct isotopy classes of arcs which are pairwise disjoint. Homeomorphisms of S induce simplicial automorphisms of $\mathcal{A}(S)$. Conversely, in [21], Irmak and McCarthy prove that every simplicial automorphism (in fact, every injective simplicial self-map) of $\mathcal{A}(S)$ arises from a homeomorphism of S, unique up to isotopy in most cases; see also Disarlo [8]. In these cases $\mathrm{Aut}(\mathcal{A}(S))$ is isomorphic to $\mathrm{Mod}^{\pm}(S)$, the extended mapping class group of S.

Theorems of this type were first proved by Ivanov, who showed in [23] that any automorphism of the curve complex of a surface with genus $g \geq 2$ is induced by a homeomorphism of the surface. Korkmaz [25] and Luo [28] cover the cases where $g<2$. Further, Shackleton [36] and Hernández Hernández [11] give conditions such that maps between a priori different curve complexes are induced by homeomorphisms. Aramayona and Leininger construct finite rigid sets in the curve complex in [3] and

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provide an exhaustion of the curve complex by finite rigid sets in [4]. In this paper, we adapt the arguments of Irmak and McCarthy to prove analogues of Aramayona and Leininger's results for the arc complex.

Recall that a simplicial map is *locally injective* if the restriction to the star of every vertex is injective.

Theorem 1.1 *Let $S = S_{g,n}$ be a compact, connected, orientable surface of genus g with $n \geq 1$ marked points. If $S \neq S_{0,3}$, then there exists a finite simplicial subcomplex $\mathcal{X}$ of $\mathcal{A}(S)$ such that for any compact, connected, orientable, finite-type surface S' with marked points and with $\dim(\mathcal{A}(S)) \geq \dim(\mathcal{A}(S'))$ and for any locally injective simplicial map*

$$\lambda\colon \mathcal{X} \to \mathcal{A}(S'),$$

there is a homeomorphism $H\colon S \to S'$ which induces λ. Moreover, if $S \neq S_{0,1}$, $S_{0,2}$ or $S_{1,1}$, then H is unique up to isotopy, and in these exceptional cases, H is unique up to $Z(\mathrm{Mod}^{\pm}(S))$.

We refer to any simplicial subcomplex of $\mathcal{A}(S)$ with this property as *rigid*. Theorem 1.1 implies the following generalization of Irmak and McCarthy's result (cf Theorem 1.1 of [21]):

Corollary 1.2 *Let S and S' be compact, connected, orientable, finite-type surfaces with marked points such that $\dim(\mathcal{A}(S)) \geq \dim(\mathcal{A}(S'))$ and $S \neq S_{0,3}$. Then, for any locally injective simplicial map $\phi\colon \mathcal{A}(S) \to \mathcal{A}(S')$, there is a homeomorphism $H\colon S \to S'$ which induces ϕ. Moreover, if $S \neq S_{0,1}$, $S_{0,2}$ or $S_{1,1}$, then H is unique up to isotopy, and in these exceptional cases, H is unique up to $Z(\mathrm{Mod}^{\pm}(S))$.*

We provide a counterexample to demonstrate why Theorem 1.1 and Corollary 1.2 do not hold for $S = S_{0,3}$. However, we use a cardinality argument to show that the following theorem holds even if $S = S_{0,3}$:

Corollary 1.3 *Let S and S' be compact, connected, orientable, finite-type surfaces with marked points. If $\mathcal{A}(S)$ and $\mathcal{A}(S')$ are isomorphic, then S and S' are homeomorphic.*

Corollary 1.3 was also shown by Bell, Disarlo and Tang in [6] using a result about the flip graph. Our proof relies only upon results about the arc complex.

Finally, we give an exhaustion of $\mathcal{A}(S)$ by finite rigid sets.

Theorem 1.4 *For any compact, connected, orientable, finite-type surface S with marked points such that $S \neq S_{0,3}$, there exists a sequence $(\mathcal{X}_i)_{i\in\mathbb{N}}$ of finite rigid sets of $\mathcal{A}(S)$ such that $\mathcal{X}_0 \subseteq \mathcal{X}_1 \subseteq \cdots \subseteq \mathcal{A}(S)$ and $\bigcup_{i\in\mathbb{N}} \mathcal{X}_i = \mathcal{A}(S)$.*

Remark Irmak and McCarthy prove their results for surfaces with n boundary components rather than n marked points; however, they consider arcs up to isotopy not necessarily fixing the endpoints of the arcs. This implies that Dehn twists around boundary components act trivially on $\mathcal{A}(S)$. Hence we can think of these boundary components as marked points instead. In [8], Disarlo considers surfaces with nonempty boundary, with at least one marked point on each boundary component and with a finite number of punctures in the interior. In contrast to Irmak and McCarthy, she considers arcs up to isotopy fixing the endpoints; hence, in this setting Dehn twists around boundary components act nontrivially on $\mathcal{A}(S)$. Disarlo proves that in this context, isomorphisms between arc complexes are induced by homeomorphisms of the associated surfaces.

Results of this type are often referred to as *Ivanov-style rigidity* or simply *rigidity*. Since Ivanov's seminal work, rigidity results have been proved for many other complexes, including the pants complex (Margalit [29] and Aramayona [1]), the arc and curve complex (Korkmaz and Papadopoulos [26]), the ideal triangulation graph or flip graph (Korkmaz and Papadopoulos [27], Aramayona, Koberda and Parlier [2]), the Schmutz graph of nonseparating curves (Schaller [35]), the complex of nonseparating curves (Irmak [14]), the Hatcher–Thurston complex (Irmak and Korkmaz [20]) and the polygonalization complex (Bell, Disarlo and Tang [6]), among others. See [32] by McCarthy and Papadopoulos for a survey. Rigidity results exist for complexes of nonorientable surfaces as well, such as the curve complex (Atalan and Korkmaz [5] and Irmak [17; 16]), the arc complex (Irmak [15]) and the two-sided curve complex (Irmak and Paris [22]). In [24], Ivanov conjectured that every object naturally associated to a surface with sufficiently rich structure has the extended mapping class group as its group of automorphisms. Brendle and Margalit [7] prove general rigidity results about subcomplexes of the complex of domains (introduced by McCarthy and Papadopoulos in [32]) for closed surfaces and McLeay [33] proves general rigidity results for subcomplexes of the complex of domains of punctured surfaces. These imply rigidity for a large class of complexes towards Ivanov's conjecture.

Finite rigidity results are less plentiful but some exist. Maungchang [31] proves finite rigidity for the pants graph of a punctured sphere and Hernández Hernández, Leininger and Maungchang extend the result for any surface $S_{g,n}$ in [12]. Maungchang also gives an exhaustion of the pants graph of punctured spheres by finite rigid sets in [30]. For nonorientable surfaces, Ilbira and Korkmaz construct finite rigid sets of the curve complex in [13] and Irmak gives an exhaustion of the curve complex by finite rigid sets in [18], strengthening her exhaustion by finite superrigid sets in [19].

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2 The arc complex

Let $S = S_{g,n}$ denote a compact, connected, orientable surface (with empty boundary) of genus g with $n \geq 1$ marked points. Let $\mathscr{P}_S$ be the set of marked points of S. Homeomorphisms between surfaces will be assumed to map marked points to marked points and isotopies between these homeomorphisms will be relative to the marked points. The *extended mapping class group* $\mathrm{Mod}^{\pm}(S)$ of S is the group of isotopy classes of homeomorphisms of S.

An *arc* on S is a map $\gamma\colon [0,1] \to S$ such that $\gamma(0), \gamma(1) \in \mathscr{P}_S$, $\gamma((0,1)) \cap \mathscr{P}_S = \varnothing$ and $\gamma|_{(0,1)}$ is injective. We will identify an arc γ with its image $\gamma([0,1])$ on S and we call $\gamma((0,1))$ the *interior* of the arc. We call two arcs isotopic only if there exists an isotopy between them such that each of the transition maps is also an arc. In particular, isotopies of arcs will be relative to the endpoints and are not permitted to pass through marked points. We will assume that all arcs are *essential*, ie cannot be isotoped into a regular neighborhood of a marked point.

If a and b are isotopy classes of arcs on S, then the *geometric intersection number* $i(a,b)$, or *intersection number*, is the minimum number of intersection points of the interiors of representatives of a and b. The *arc complex* $\mathcal{A}(S)$ is the simplicial complex whose vertices correspond to the isotopy classes of arcs on S and whose k–simplices ($k > 0$) correspond to collections of $k+1$ distinct isotopy classes of arcs which have pairwise disjoint representatives. Unless necessary, we will not distinguish between an isotopy class of arcs, a representative of the class and the corresponding vertex of the arc complex. We say two arcs are *disjoint* if $i(a,b) = 0$. By *distinct arcs*, we mean distinct isotopy classes of arcs.

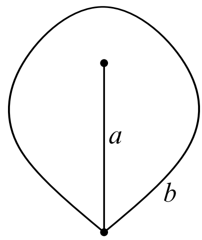

Figure 1: A nonembedded triangle.

We state explicitly the results of Irmak and McCarthy as we will reference them later. Recall that $Z(G)$ refers to the center of the group G.

Theorem 2.1 [21, Theorem 1.1] *Let S be a compact, connected, orientable, finite-type surface with marked points. If $\varphi\colon \mathcal{A}(S) \to \mathcal{A}(S)$ is an injective simplicial map then φ is induced by a homeomorphism $H\colon S \to S$.*

Theorem 2.2 [21, Theorem 1.2] *Let S be a compact, connected, orientable, finite-type surface with marked points. If S is not $S_{0,1}$, $S_{0,2}$, $S_{0,3}$ or $S_{1,1}$, then $\mathrm{Aut}(\mathcal{A}(S))$ is naturally isomorphic to $\mathrm{Mod}^{\pm}(S)$. For each of the special cases, $\mathrm{Aut}(\mathcal{A}(S))$ is naturally isomorphic to $\mathrm{Mod}^{\pm}(S)/Z(\mathrm{Mod}^{\pm}(S))$.*

A key object in the proofs of these results and our results is triangulations of surfaces. We dedicate the remainder of this section to describing necessary background information regarding triangulations. If S is neither $S_{0,1}$ nor $S_{0,2}$, it has at least three distinct, disjoint arcs, and we call a maximal collection of distinct pairwise disjoint arcs on S a *triangulation* of S. There is a natural Δ–complex associated to a triangulation with the arcs as its 1–skeleton, hence the name "triangulation" (see eg Hatcher [10, Section 2.1] for more on Δ–complexes). We will refer to the 2–cells as *triangles* of T. We say that an arc a is a *side* of a triangle Δ in T if a is contained in $\partial\Delta$. We say a triangle is *embedded* if its sides are distinct arcs on S and that it is *nonembedded* otherwise. Note that an embedded triangle is not required to have distinct vertices. All nonembedded triangles are of the form pictured in Figure 1. Call the side of a nonembedded triangle which joins two distinct punctures the *inner arc* (eg arc a in Figure 1). Call the other arc the *outer arc* (eg arc b in Figure 1).

We can use the Euler characteristic to see that the number of arcs in a triangulation of S is $6g+3n-6$, hence $\dim(\mathcal{A}(S)) = 6g+3n-7$. If $\dim(\mathcal{A}(S)) = \dim(\mathcal{A}(S'))$, a locally injective simplicial map from a subcomplex $\mathcal{Y}$ of $\mathcal{A}(S)$ into $\mathcal{A}(S')$ sends any triangulation T contained in $\mathcal{Y}$ to a triangulation T' of S'. The number of triangles in a triangulation is $4g+2n-4$. Hence, if $\dim(\mathcal{A}(S))=\dim(\mathcal{A}(S'))$, then a triangulation of S has the same number of triangles as a triangulation of S'.

We say that two triangulations are obtained from each other by a *flip* if they differ by exactly one arc. In this case, the distinct arcs have intersection number one. Conversely, if two distinct arcs a and b have intersection number one, there exist triangulations T_a and T_b containing a and b, respectively, such that $T_a \setminus \{a\} = T_b \setminus \{b\}$.

We will need the following result of Mosher regarding triangulations later:

Proposition 2.3 [34, Connectivity theorem for elementary moves, page 36] *Let S be a compact, connected, orientable, finite-type surface with marked points. Any two triangulations of S differ by a finite number of flips.*

3 Exceptional cases

In this section, we dispense with the cases where $\mathcal{A}(S)$ is empty or has dimension ≤ 2, ie if S is $S_{0,1}$, $S_{0,2}$, $S_{0,3}$ or $S_{1,1}$. On $S_{0,1}$, there are no essential arcs, hence $\mathcal{A}(S_{0,1})=\varnothing$. On $S_{0,2}$, there is only one essential arc, hence $\mathcal{A}(S_{0,2})$ is a single point and has dimension 0. All other surfaces have arc complex of dimension >0, hence for $S_{0,1}$ and $S_{0,2}$, Theorem 1.1 follows from setting $\mathcal{X}=\mathcal{A}(S)$ and applying the results of Irmak and McCarthy (Theorems 2.1 and 2.2).

Now suppose $\dim(\mathcal{A}(S))=2$. Then S is $S_{0,3}$ or $S_{1,1}$. It is well known that $\mathcal{A}(S_{0,3})$ is isomorphic to a regular tessellation of a triangle by four triangles (see eg [21; 32]) and that $\mathcal{A}(S_{1,1})$ is isomorphic to the flag complex of the *Farey graph*, a decomposition by ideal triangles of $\mathbb{H}^2 \cup \mathbb{QP}^1$ (see eg [21]). Figure 2 shows a simplicial embedding of $\mathcal{A}(S_{0,3})$ into $\mathcal{A}(S_{1,1})$ but $S_{0,3}$ and $S_{1,1}$ are not homeomorphic. Hence, Theorem 1.1

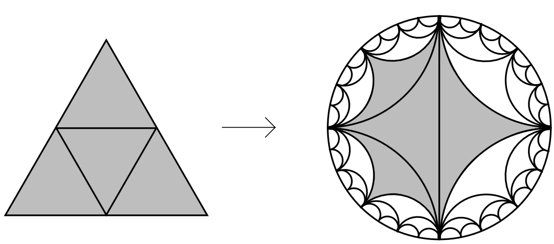

Figure 2: A simplicial embedding of $\mathcal{A}(S_{0,3})$ into $\mathcal{A}(S_{1,1})$.

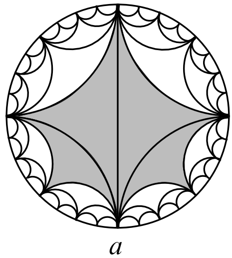

Figure 3: A rigid set in $\mathcal{A}(S_{1,1})$.

and Corollary 1.2 cannot be extended for $S_{0,3}$. However, we can find a finite rigid set in $\mathcal{A}(S_{1,1})$ and we prove later that Corollary 1.2 holds for $S_{1,1}$ as well.

Proof of Theorem 1.1 for $S = S_{1,1}$ Let $\mathcal{X}$ be the simplicial subcomplex of $\mathcal{A}(S)$ indicated in gray in Figure 3. Let S' be such that $\dim(\mathcal{A}(S)) \geq \dim(\mathcal{A}(S'))$ and let $\lambda\colon \mathcal{X} \to \mathcal{A}(S')$ be a locally injective simplicial map. Observe that $\mathcal{X}$ is contained in the closure of the star of the vertex marked a, hence λ is injective. Since λ injectively maps a simplex of dimension 2 into $\mathcal{A}(S')$, $\mathcal{A}(S')$ must not have dimension less than 2. Further, since there are five vertices in $\mathcal{X}$ connected to a by an edge in $\mathcal{X}$ and since every vertex in $\mathcal{A}(S_{0,3})$ has degree at most four, $S' \neq S_{0,3}$, hence $S' = S_{1,1}$. Now, $\mathcal{A}(S)$ is connected and each edge borders exactly two triangles, so an induction argument shows that λ can be extended uniquely to an automorphism of $\mathcal{A}(S)$. Then we again apply the results of Irmak and McCarthy. □

Now we can give an exhaustion of $\mathcal{A}(S)$ by finite rigid sets in the cases that S is $S_{0,1}$, $S_{0,2}$ and $S_{1,1}$.

Proof of Theorem 1.4 for $S_{0,1}$, $S_{0,2}$ and $S_{1,1}$ Suppose S is $S_{0,1}$ or $S_{0,2}$. Then, as discussed above, $\mathcal{A}(S)$ is finite, so we can take $\mathcal{X}_i = \mathcal{A}(S)$ for all $i \in \mathbb{N}$. Now suppose S is $S_{1,1}$. Let $\mathcal{X}_0$ be the finite rigid set of $\mathcal{A}(S)$ from the proof above. For $i \in \mathbb{N}$, let $\mathcal{X}_{i+1}$ be the simplicial subcomplex of $\mathcal{A}(S)$ containing $\mathcal{X}_i$ and any triangles which share a side with a triangle in $\mathcal{X}_i$. Then $(\mathcal{X}_i)_{i\in\mathbb{N}}$ is an exhaustion of $\mathcal{A}(S)$ by finite rigid sets by the same argument as above. □

4 The general case

In this section, S and S' will be surfaces with

$$\dim(\mathcal{A}(S)) \geq \dim(\mathcal{A}(S')) \quad \text{and} \quad \dim(\mathcal{A}(S)) > 2.$$

Throughout the section, we frequently assume the existence of a local injection from some subset of $\mathcal{A}(S)$ into $\mathcal{A}(S')$. Note that if this subset contains a maximal simplex of $\mathcal{A}(S)$, ie a triangulation of S, then the dimension of $\mathcal{A}(S')$ cannot be less than that of $\mathcal{A}(S)$, hence $\dim(\mathcal{A}(S)) = \dim(\mathcal{A}(S'))$. We will utilize this fact without making any further note of it.

Suppose $V = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is a collection of simplices of a simplicial complex C. The *span of* V, $\mathrm{Span}_C(V)$, refers to the set of all simplices τ of C such that each vertex of τ is the vertex of a simplex in V. Observe that V finite implies that $\mathrm{Span}_C(V)$ is finite as well.

Roughly speaking, the proof of Theorem 1.1 will proceed as follows: We will include in $\mathcal{X}$ (the span of) a triangulation T of S which maps to a triangulation T' of S' under any locally injective map. We first show that, by adding finitely many arcs to $\mathcal{X}$, we can guarantee that triangles of T map to triangles of the same type (embedded or nonembedded) in T'. Then we show that, by adding finitely many more arcs to $\mathcal{X}$, we can guarantee that the orientations of adjacent triangles in T' match, so that the map $T \to T'$ can be extended to a homeomorphism $H\colon S \to S'$. We use Proposition 2.3 to show that by including finitely many more arcs in $\mathcal{X}$ we can guarantee that any locally injective simplicial map $\lambda\colon \mathcal{X} \to \mathcal{A}(S')$ agrees with the induced map H_* on all of $\mathcal{X}$. Finally, we show uniqueness by proving that any other such homeomorphism H' has induced map equal to H_* and then applying the results of Irmak and McCarthy.

First, we need the following (cf [21, Proposition 3.1]):

Lemma 4.1 *If (a, b) is a pair of arcs with $i(a, b) = 1$, then there exists a finite simplicial subcomplex $\mathcal{B}$ of $\mathcal{A}(S)$ containing a and b with the following property: If $\mathcal{Y}$ is a simplicial subcomplex of $\mathcal{A}(S)$ which contains $\mathcal{B}$ and $\lambda\colon \mathcal{Y} \to \mathcal{A}(S')$ a locally injective simplicial map, then $i(\lambda(a), \lambda(b)) = 1$.*

Proof There exist triangulations T_a and T_b of S which share all arcs except for $a \in T_a$ and $b \in T_b$. Let $T_0 = T_a \cap T_b$. We define

$$\mathcal{B} = \mathrm{Span}_{\mathcal{A}(S)}(T_0 \cup \{a, b\}).$$

Now suppose $\mathcal{Y}$ is a simplicial subcomplex of $\mathcal{A}(S)$ which contains $\mathcal{B}$ and $\lambda\colon \mathcal{Y} \to \mathcal{A}(S')$ a locally injective simplicial map. Note that $\lambda|_{\mathcal{B}}$ is injective since λ is locally injective. Since the simplex spanned by T_0 has codimension one in $\mathcal{A}(S)$, $T_0' = \lambda(T_0)$ has codimension one in $\mathcal{A}(S')$. Let $a' = \lambda(a)$ and $b' = \lambda(b)$. Then $\lambda(T_a) = \{a'\} \cup T_0'$ and $\lambda(T_b) = \{b'\} \cup T_0'$ are both triangulations of S'. It follows that $i(a', b') = 1$. □

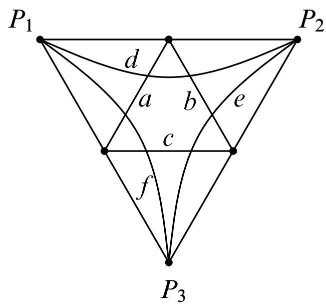

Figure 4: Arc configurations. Vertices and arcs around the perimeter may be identified.

We will also use the following result of Irmak and McCarthy:

Proposition 4.2 [21, Proposition 3.2] *Let Δ be an embedded triangle on S with sides a, b and c. Then there exists a triangulation T on S containing a, b and c such that the unique triangles Δ_a, Δ_b and Δ_c of T on S which are different from Δ and have, respectively, a as a side, b as a side and c as a side, are distinct triangles of T on S.*

Applying Proposition 4.2, we deduce the following (cf [21, Proposition 3.3]):

Lemma 4.3 *If a, b and c are the edges of an embedded triangle on S, then there exists a finite simplicial subcomplex $\mathcal{C}$ of $\mathcal{A}(S)$ containing a, b and c with the following property: if $\mathcal{Y}$ is a simplicial subcomplex of $\mathcal{A}(S)$ which contains $\mathcal{C}$ and $\lambda\colon \mathcal{Y} \to \mathcal{A}(S')$ a locally injective simplicial map, then $\lambda(a)$, $\lambda(b)$ and $\lambda(c)$ are the edges of an embedded triangle on S'.*

Proof By Proposition 4.2, there exists a triangulation T of S containing $\{a, b, c\}$ such that the unique triangles Δ_a, Δ_b and Δ_c of T on S which are different from Δ and have, respectively, a as a side, b as a side and c as a side, are distinct triangles of T on S.

Let P_1 be the vertex of Δ_a opposite a, P_2 the vertex of Δ_b opposite b, and P_3 the vertex of Δ_c opposite c (see Figure 4). There exists an arc d connecting P_1 to P_2 which intersects a and b once and is disjoint from all other arcs in T. Similarly, there exists an arc e connecting P_2 to P_3 which intersects b and c once and is disjoint from all other arcs in T. And finally, there is an arc f connecting P_1 to P_3 which intersects a and c once and is disjoint from all other arcs in T. Note that d, e and f are pairwise disjoint.

Now, consider the six pairs of arcs $\mathscr{R}_1=(a,d)$, $\mathscr{R}_2=(a,f)$, $\mathscr{R}_3=(b,d)$, $\mathscr{R}_4=(b,e)$, $\mathscr{R}_5=(c,e)$ and $\mathscr{R}_6=(c,f)$. Recall that each of these pairs has intersection number one. Let $\mathcal{B}_i$ be the finite simplicial complex of $\mathcal{A}(S)$ from Lemma 4.1 corresponding to $\mathscr{R}_i$ for each $1\le i\le 6$.

We define

$$\mathcal{C}=\operatorname{Span}_{\mathcal{A}(S)}\Bigl(T\cup\{d,e,f\}\cup\bigcup_{1\le i\le 6}\mathcal{B}_i\Bigr).$$

Suppose $\mathcal{Y}$ is any simplicial subcomplex of $\mathcal{A}(S)$ which contains $\mathcal{C}$ and $\lambda\colon\mathcal{Y}\to\mathcal{A}(S')$ is a locally injective simplicial map. Let $T'=\lambda(T)$; note that it is a triangulation of S'. Additionally, let $a'=\lambda(a)$, $b'=\lambda(b)$, $c'=\lambda(c)$, $d'=\lambda(d)$, $e'=\lambda(e)$ and $f'=\lambda(f)$. Lemma 4.1 and the local injectivity of λ guarantee each pair in $\{a',b',c',d',e',f'\}$ has the same intersection number as its preimage under λ.

Since d' intersects a' and b' once each and is disjoint from all other arcs in the triangulation T', it must be the case that a' and b' border an embedded triangle in T' — call it Δ_1. Analogously, since e' intersects b' and c' once each and is disjoint from all other arcs in the triangulation T', there is an embedded triangle Δ_2 in T' with sides b' and c' and, since f' intersects a' and c' once each and is disjoint from all other arcs in the triangulation T', there is an embedded triangle Δ_3 in T' with sides a' and c'. Suppose that the third side of Δ_1 is r', the third side of Δ_2 is s' and the third side of Δ_3 is t'. If $r'=c'$, $s'=a'$ or $t'=b'$, then we are done.

Suppose $r'\neq c'$, $s'\neq a'$ and $t'\neq b'$, hence $\Delta_1\neq\Delta_2$, $\Delta_2\neq\Delta_3$ and $\Delta_3\neq\Delta_1$. Up to homeomorphism (and ignoring admissible identifications among arcs and among vertices of the triangles), there are four configurations, and these depend on the relative orientations of Δ_1 and Δ_2, and the relative orientation of Δ_3 with respect to Δ_1 and Δ_2. See Figure 5. In each of these cases, d' and e' must intersect, which is a contradiction. □

Now we can use Lemma 4.3 to prove the corresponding result for nonembedded triangles (cf [21, Proposition 3.4]).

Lemma 4.4 *If a and b border a nonembedded triangle on S with a the inner arc, then there exists a finite simplicial subcomplex $\mathcal{D}$ of $\mathcal{A}(S)$ containing a and b with the following property: if $\mathcal{Y}$ is a simplicial subcomplex of $\mathcal{A}(S)$ which contains $\mathcal{D}$ and $\lambda\colon\mathcal{Y}\to\mathcal{A}(S')$ a locally injective simplicial map, then $\lambda(a)$ and $\lambda(b)$ border a nonembedded triangle on S' with $\lambda(a)$ as the inner arc.*

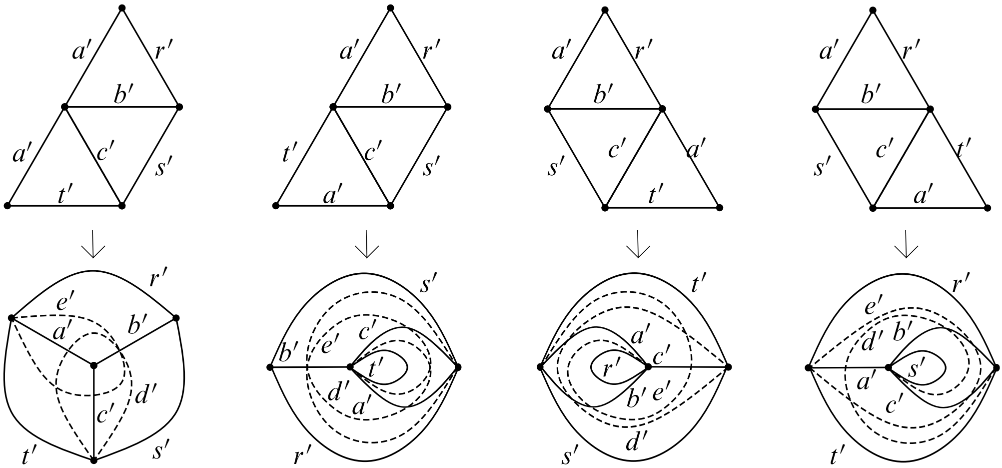

Figure 5: Four possible arrangements of triangles in the proof of Lemma 4.3. Some additional identifications of vertices and arcs may be made.

Proof Since S is not $S_{0,3}$, there exists an embedded triangle having b as a side. Call the other sides of this triangle c and d. Let e be the arc pictured in Figure 6.

We can see that arcs a, c and e border an embedded triangle on S, as do arcs a, d and e. As noted previously, the arcs b, c and d also border an embedded triangle. Let $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively, be the finite simplicial subcomplexes of $\mathcal{A}(S)$ from Lemma 4.3 corresponding to these triples of arcs. Also, observe that $i(b,e)=1$. Then let $\mathcal{B}$ be the corresponding finite simplicial subcomplex of $\mathcal{A}(S)$ from Lemma 4.1. We define

$$\mathcal{D} = \mathrm{Span}_{\mathcal{A}(S)}(\{a,b,c,d,e\} \cup \mathcal{B} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3).$$

Let $\mathcal{Y}$ be any simplicial subcomplex of $\mathcal{A}(S)$ which contains $\mathcal{D}$ and $\lambda\colon \mathcal{Y} \to \mathcal{A}(S')$ a locally injective simplicial map. Let $a' = \lambda(a)$, $b' = \lambda(b)$, $c' = \lambda(c)$, $d' = \lambda(d)$

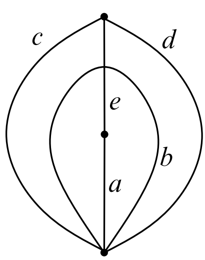

Figure 6: Arc configurations. Outer vertices may be identified.

and $e' = \lambda(e)$. Lemma 4.1 and the local injectivity of λ guarantee each pair in $\{a', b', c', d', e'\}$ has the same intersection number as the preimage under λ. By Lemma 4.3, we know that a', c' and e' border an embedded triangle on S', as do a', d' and e', and also b', c' and d'. Since b' is disjoint from a', c' and d' and intersects e' once, the only possible arrangement of arcs then guarantees that a' and b' border a nonembedded triangle on S' with a' as the inner arc. □

The two lemmas above give conditions under which a locally injective simplicial map takes a triangulation T of S to a triangulation T' of S' in such a way that triangles in T are sent to triangles of the same type (embedded or nonembedded) in T'. The following lemma provides conditions under which two triangles in T which share an edge map to consistently oriented triangles in T' (cf [21, Propositions 3.5–3.7]).

If Δ is a triangle, let $\mathring{\Delta}$ denote $\Delta \setminus \mathscr{P}_S$, the triangle minus its vertex set. The edges of $\mathring{\Delta}$ are the interiors of arcs; however, for simplicity, we will not use a separate notation for them.

Lemma 4.5 *Suppose Δ_1 is an embedded triangle on S with sides a, b and c and Δ_2 an embedded triangle with sides c, d and e, as shown in Figure 7. Then there exists a finite simplicial subcomplex $\mathcal{E}$ of $\mathcal{A}(S)$ containing a, b, c, d and e with the following property: Let $\mathcal{Y}$ be any simplicial subcomplex of $\mathcal{A}(S)$ containing $\mathcal{E}$ and $\lambda\colon \mathcal{Y} \to \mathcal{A}(S')$ be a locally injective simplicial map. Let $a' = \lambda(a)$, $b' = \lambda(b)$, $c' = \lambda(c)$, $d' = \lambda(d)$ and $e' = \lambda(e)$. Then there exist an embedded triangle Δ'_1 on S' with sides a', b' and c' and an embedded triangle Δ'_2 on S' with sides c', d' and e', and the natural homeomorphisms $F_1\colon (\mathring{\Delta}_1, a, b, c) \to (\mathring{\Delta}'_1, a', b', c')$ and $F_2\colon (\mathring{\Delta}_2, c, d, e) \to (\mathring{\Delta}'_2, c', d', e')$ can be made to agree along c.*

Proof Let f be the arc shown in Figure 7. Then f is an arc on S which intersects c exactly once and is disjoint from a, b, d and e. Then a, d and f are the sides of a

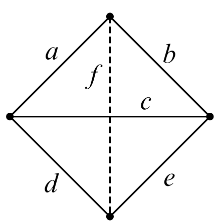

Figure 7: Two embedded triangles sharing a side. We allow for the possibility that $a = d$, $a = e$, $b = d$ or $b = e$. We also allow identifications among vertices.

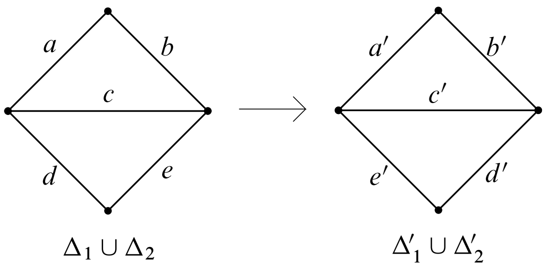

Figure 8: Triangle configurations reversing relative orientation.

triangle on S. This triangle is nonembedded if $a = d$, and embedded otherwise. Let $\mathcal{K}$ be the finite simplicial subcomplex of $\mathcal{A}(S)$ from either Lemma 4.3 or Lemma 4.4 corresponding to this triangle. Further, let $\mathcal{B}$ be the finite simplicial complex of $\mathcal{A}(S)$ from Lemma 4.1 corresponding to the pair (c, f) and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the finite simplicial complexes of $\mathcal{A}(S)$ from Lemma 4.3 corresponding to Δ_1 and Δ_2, respectively. We define

$$\mathcal{E} = \mathrm{Span}_{\mathcal{A}(S)}(\{a, b, c, d, e, f\} \cup \mathcal{B} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{K}).$$

Let $\mathcal{Y}$ be any simplicial subcomplex of $\mathcal{A}(S)$ containing $\mathcal{E}$ and $\lambda\colon \mathcal{Y} \to \mathcal{A}(S')$ a locally injective simplicial map. By Lemma 4.3 there is an embedded triangle Δ_1' on S' with sides a', b' and c' and an embedded triangle Δ_2' on S' with sides c', d' and e'. Further, by Lemma 4.1, we know that $i(c', f') = 1$. Since λ is simplicial, a', b', d' and e' are disjoint from f'. Depending on the identifications of sides, either Lemma 4.3 or Lemma 4.4 implies that a', d' and f' border a triangle. By inspection, we see that these conditions hold simultaneously only if the orientation of Δ_1' relative to Δ_2' is the same as the orientation of Δ_1 relative to Δ_2. Figure 8, for example, shows the case where no sides are identified and the relative orientations are reversed. As another example, Figure 9 shows the case where $b = e$ and the relative orientations

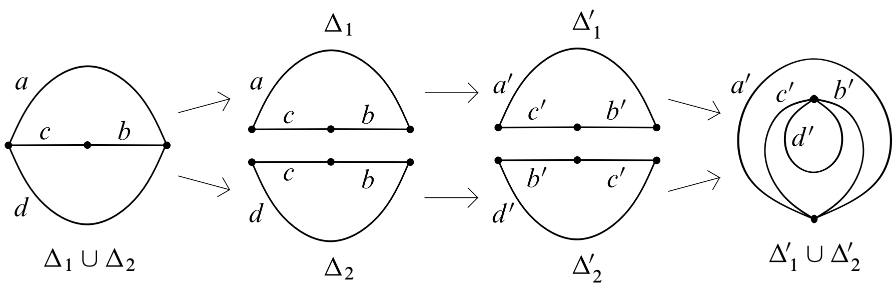

Figure 9: Triangle configurations reversing relative orientation.

are reversed. In both cases, there is no possible placement of f' which satisfies all the conditions above. All possible identifications of the sides of Δ_1 and Δ_2 yield this result, hence the natural homeomorphisms $F_1\colon (\mathring{\Delta}_1, a, b, c) \to (\mathring{\Delta}'_1, a', b', c')$ and $F_2\colon (\mathring{\Delta}_2, c, d, e) \to (\mathring{\Delta}'_2, c', d', e')$ can be made to agree along c. □

Now we apply the above results to find a candidate homeomorphism (cf Proposition 3.8 of [21]).

Proposition 4.6 *Let T be a triangulation of S. Then there exists a finite simplicial subcomplex $\mathcal{F}$ of $\mathcal{A}(S)$ containing T with the following property: if $\mathcal{Y}$ is any simplicial subcomplex of $\mathcal{A}(S)$ containing $\mathcal{F}$ and $\lambda\colon \mathcal{Y} \to \mathcal{A}(S')$ a locally injective simplicial map, there exists a homeomorphism $H\colon S \to S'$ whose induced map on $\mathcal{A}(S)$ agrees with λ on T.*

Proof There are $N = 4g + 2n - 4$ triangles in T. Denote them by $\{\Delta_i : 1 \le i \le N\}$. Since S is neither $S_{0,3}$ nor $S_{1,1}$, no two components Δ_i and Δ_j share the same three sides. Suppose that M is the number of embedded triangles in T. Then, after reordering, we may assume that $\{\Delta_i : 1 \le i \le M\}$ are embedded triangles and $\{\Delta_j : M+1 \le j \le N\}$ are nonembedded triangles. Then, for each $1 \le i \le M$, let $\mathcal{C}_i$ be the finite simplicial subcomplex of $\mathcal{A}(S)$ from Lemma 4.3 corresponding to Δ_i, and for each $M+1 \le i \le N$, let $\mathcal{D}_i$ be the finite simplicial subcomplex of $\mathcal{A}(S)$ from Lemma 4.4 corresponding to Δ_i.

Suppose there are K arcs in T which border two distinct triangles of T (the inner arc of a nonembedded triangle does not have this property, so it may be that $K < N$.) Let $\mathcal{E}_i$ for $1 \le i \le K$ be the finite simplicial complexes of $\mathcal{A}(S)$ from Lemma 4.5 corresponding to each pair. We define

$$\mathcal{F} = \operatorname{Span}_{\mathcal{A}(S)}\Bigl(T \cup \bigcup_{1\le i\le M} \mathcal{C}_i \cup \bigcup_{M+1\le i\le N} \mathcal{D}_i \cup \bigcup_{1\le i\le K} \mathcal{E}_i\Bigr).$$

Let $\mathcal{Y}$ be any simplicial subcomplex of $\mathcal{A}(S)$ containing $\mathcal{F}$ and let $\lambda\colon \mathcal{Y} \to \mathcal{A}(S')$ be a locally injective simplicial map. Let $T' = \lambda(T)$; note that it is a triangulation of S'. Since $\dim(\mathcal{A}(S)) = \dim(\mathcal{A}(S'))$, there are also N triangles in T'. Denote them by $\{\Delta'_i : 1 \le i \le N\}$. Suppose Δ_i is a triangle in S with sides a, b and c. Then Lemmas 4.3 and 4.4 ensure that $\lambda(a)$, $\lambda(b)$ and $\lambda(c)$ border a triangle in T' of the same type (embedded or nonembedded). Then after reordering, we may assume that Δ'_i corresponds to Δ_i in this way.

Suppose Δ_i is embedded with sides a_i, b_i and c_i. Let $a_i' = \lambda(a_i)$, $b_i' = \lambda(b_i)$ and $c_i' = \lambda(c_i)$. Then there exists a homeomorphism $F_i\colon (\mathring{\Delta}_i, a_i, b_i, c_i) \to (\mathring{\Delta}_i', a_i', b_i', c_i')$. This homeomorphism is well defined up to relative isotopies and its orientation type is fixed. Suppose Δ_i is nonembedded with inner arc a_i and outer arc b_i. Then $a_i' = \lambda(a_i)$ is the inner arc of a triangle Δ_i' with outer arc $b_i' = \lambda(b_i)$. Further, there exist two homeomorphisms $F_i, F_i^*\colon (\mathring{\Delta}_i, a_i, b_i) \to (\mathring{\Delta}_i', a_i', b_i')$ with opposite orientation types.

Now suppose Δ_i and Δ_j are two distinct triangles in a triangulation T of S which share the side s. Since S is not $S_{0,3}$, it cannot be the case that Δ_i and Δ_j are both nonembedded triangles. Suppose Δ_i is nonembedded and Δ_j is embedded. The shared side s must be the outer arc of Δ_i. Then one of F_i or F_i^* can be made to agree with F_j along s and we may assume it is the one called F_i. If instead Δ_i and Δ_j are both embedded, then Lemma 4.5 guarantees that F_i and F_j can be made to agree along s.

Then there is a homeomorphism of the punctured surfaces $F\colon S \setminus \mathscr{P}_S \to S' \setminus \mathscr{P}_{S'}$ whose restriction to $\mathring{\Delta}_i$ is equal to F_i for $1 \le i \le N$. This can be extended uniquely to a homeomorphism $H\colon S \to S'$, and by construction the induced map by H on $\mathcal{A}(S)$ agrees with λ on T. □

We can now prove the general case of Theorem 1.1 (cf [21, Proposition 3.11]).

Proof of Theorem 1.1 We dispensed with the cases where $\dim(\mathcal{A}(S)) \le 2$ in Section 3. Suppose $\dim(\mathcal{A}(S)) > 2$.

Let T be a triangulation of S and let $\mathcal{F}$ be as in Proposition 4.6. Suppose $y_1, \dots, y_k$ are the vertices of $\mathcal{F}$. Then, for each i, we fix a triangulation T^i of S containing y_i. By Proposition 2.3, there exists a finite sequence of triangulations $T = T_0^i, T_1^i, \dots, T_{m_i-1}^i, T_{m_i}^i = T^i$ such that for each $0 \le j \le m_i - 1$, T_j^i and T_{j+1}^i differ by a flip. We define

$$\mathcal{X} = \mathrm{Span}_{\mathcal{A}(S)}\Bigl(\mathcal{F} \cup \bigcup_{1 \le i \le k} \bigcup_{0 \le j \le m_i} \{T_j^i\}\Bigr).$$

Then $\mathcal{X}$ is a finite simplicial subcomplex of $\mathcal{A}(S)$ and has the property that any vertex $y \in \mathcal{X}$ is contained in a triangulation T^* of vertices in $\mathcal{X}$ such that there exists a finite sequence of simplices $T = T_0, T_1, \dots, T_{m-1}, T_m = T^*$, all contained in $\mathcal{X}$, where for each $0 \le j \le m-1$, T_j and T_{j+1} differ by a flip.

Let $\lambda\colon \mathcal{X} \to \mathcal{A}(S')$ be any locally injective simplicial map and $H\colon S \to S'$ be as in Proposition 4.6. Let $H_*\colon \mathcal{A}(S) \to \mathcal{A}(S')$ be the induced map of H. Proposition 4.6

says that $H_*(x) = \lambda(x)$ for $x \in T$. We now show that this is true for any vertex $y \in \mathcal{X}$, so that $H_*|_{\mathcal{X}} = \lambda$.

Suppose $\phi = (H_*)^{-1} \circ \lambda\colon \mathcal{X} \to \mathcal{A}(S)$. We already know that ϕ is the identity on T. Suppose y is a vertex in $\mathcal{X}$. Let $T = T_0, T_1, \dots, T_{m-1}, T_m = T^*$ be the finite sequence of triangulations, all contained in $\mathcal{X}$, such that $y \in T^*$ and for each $0 \le j \le m-1$, T_j and T_{j+1} differ by a flip. We will show that if ϕ is the identity on T_i, this implies that it is the identity on T_{i+1}. Now $T_i \cap T_{i+1}$ has codimension one in $\mathcal{A}(S)$. Say a is the single arc in $T_i \setminus T_{i+1}$ and b the single arc in $T_{i+1} \setminus T_i$. In general, a collection of arcs corresponding to a codimension one simplex in $\mathcal{A}(S)$ has either one or two arcs disjoint from all arcs in the collection, ie one or two ways to complete the collection into a triangulation of S. For $T_i \cap T_{i+1}$, there are two arcs — a and b. We know that $\phi(a) = a$ and $\phi(T_i \cap T_{i+1}) = T_i \cap T_{i+1}$ by our hypothesis. Then, since ϕ is a locally injective simplicial map, it must be the case that $\phi(b) = b$, hence ϕ is the identity on T_{i+1}. Then, by induction, it follows that $\phi(y) = y$ and hence that ϕ is the identity on all of $\mathcal{X}$. This implies that $H_*|_{\mathcal{X}} = \lambda$. If H' is another homeomorphism whose induced map agrees with λ on $\mathcal{X}$, then H and H' agree on the triangulation T, thus they are homotopic. □

We can now prove the corollaries.

Proof of Corollary 1.2 Fix S, S' and ϕ. Let $\mathcal{X}$ be the finite rigid set from Theorem 1.1. Then $\phi|_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{A}(S')$ meets the conditions described in Theorem 1.1, so there is a homeomorphism $H\colon S \to S'$ which induces $\phi|_{\mathcal{X}}$. Then it remains to show that ϕ and H agree on all of $\mathcal{A}(S)$, which can be done by employing a near identical argument to the one used in the proof of Theorem 1.1. □

Proof of Corollary 1.3 Suppose $\varphi\colon \mathcal{A}(S) \to \mathcal{A}(S')$ is an isomorphism. If $S \neq S_{0,3}$, we apply Theorem 1.1 to the injective simplicial map $\varphi|_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{A}(S')$, where $\mathcal{X}$ is the finite rigid set given in the theorem. Suppose $S = S_{0,3}$. Then $\dim(\mathcal{A}(S)) = 2$, so S' is either $S_{0,3}$ or $S_{1,1}$. However, $\mathcal{A}(S_{0,3})$ is finite and $\mathcal{A}(S_{1,1})$ is infinite, so it must be that S' is $S_{0,3}$, hence S and S' are homeomorphic. □

Finally, we extend the proof of Theorem 1.1 given above to construct an exhaustion of $\mathcal{A}(S)$ by finite rigid sets.

Proof of Theorem 1.4 We dispensed with the cases where $\dim(\mathcal{A}(S)) \le 2$ in Section 3, so suppose $\dim(\mathcal{A}(S)) > 2$. Let $\mathcal{X}_0$ be the finite rigid set of $\mathcal{A}(S)$ from Theorem 1.1, which by construction contains a triangulation T of S. It is well known that $\mathcal{A}(S)$ has countable vertex set, so we can enumerate the vertices $\mathcal{A}^{(0)}(S) = \{x_1, x_2, \dots\}$. As above, for each $i \in \mathbb{N}$, there is a triangulation T^i of S containing x_i and a finite sequence of simplices $T = T^i_0, T^i_1, \dots, T^i_{m_i-1}, T^i_{m_i} = T^i$ where for each $0 \le j \le m_i - 1$, T^i_j and T^i_{j+1} differ by a flip. Then, for $i \ge 1$, we define

$$\mathcal{X}_i = \mathrm{Span}_{\mathcal{A}(S)}\Bigg(\mathcal{X}_{i-1} \cup \bigcup_{0 \le j \le m_i} T^i_j\Bigg).$$

Observe that $\mathcal{X}_i$ is finite. An identical argument to the one employed in the proof of Theorem 1.1 above shows that $\mathcal{X}_i$ is rigid. It is clear that $\mathcal{X}_0 \subseteq \mathcal{X}_1 \subseteq \cdots \subseteq \mathcal{A}(S)$ and $\bigcup_{i \in \mathbb{N}} \mathcal{X}_i = \mathcal{A}(S)$. □

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