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Algebraic & Geometric Topology 20 (2020) 3561–3588

Hyperbolicity of link complements in Seifert-fibered spaces

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Let $\overline{\gamma}$ be a link in a Seifert-fibered space M over a hyperbolic 2–orbifold $\mathcal{O}$ that projects injectively to a filling multicurve of closed geodesics γ in $\mathcal{O}$. We prove that the complement $M_{\overline{\gamma}}$ of $\overline{\gamma}$ in M admits a hyperbolic structure of finite volume, and we give combinatorial bounds of its volume.

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1 Introduction

Let Σ be a hyperbolic surface of finite type. In the projective unit tangent bundle $PT^1(\Sigma)$ there is a very special family of links $\widehat{\gamma} \doteq (\gamma, \dot{\gamma})$ coming from canonical lifts of a geodesic multicurve γ in Σ. These links correspond to the image under the map $T^1\Sigma \to PT^1(\Sigma)$ of a collection of periodic orbits of the geodesic flow. Foulon and Hasselblatt [9] gave a topological criterion, depending only on the immersion of γ in Σ, that guarantees the existence of a complete hyperbolic metric of finite volume in the canonical lift complement of γ in $PT^1(\Sigma)$.

Theorem 1.1 (Foulon–Hasselblatt [9]) *Let γ be a closed geodesic on a hyperbolic surface Σ. Then the complement of the canonical lift admits a finite-volume complete hyperbolic structure if and only if γ is filling.*

In [9] the previous theorem was stated in a more general setting. The authors considered any embedded lift $\overline{\gamma}$ in the unit tangent bundle of the hyperbolic surface as long as the projection was injective outside the double points of the closed geodesic γ. After reading their proof carefully, we noticed that an argument relative to the atoroidality of these knot complements was only stated for the particular case of knots coming from periodic orbits of the geodesic flow; on the other hand, the arguments for the other cases worked in greater generality.

This paper aims to prove the missing argument for the atoroidality of these link complements. This question was posed in a beautiful blog post of Calegari [6], where he gave

Published: 29 December 2020 DOI: 10.2140/agt.2020.20.3561

a geometric proof of Theorem 1.1. We also extend results of the second author from the unit tangent bundle to this setting. Moreover, we give sequences of closed filling geodesics $\{\gamma_n\}_{n\in\mathbb{N}}$ in Σ and topological lifts $\{\overline{\gamma}_n\}_{n\in\mathbb{N}}$ in $PT^1(\Sigma)$ whose associated knot complement volume is bounded linearly in terms of the self-intersection number of the closed geodesic.

One of the steps in the proof of the hyperbolicity of $M_{\widehat{\gamma}} \doteq PT^1(\Sigma) \setminus \widehat{\gamma}$ is to show that no essential torus $T \subset M_{\overline{\gamma}}$ is nullhomotopic in $PT^1(\Sigma)$. To do so, the authors of [9] argue that since the geodesic flow is product covered in the universal cover $\widetilde{PT^1(\Sigma)}$, the complement of all the lifts $\{\widetilde{\widehat{\gamma}}\}$ of $\widehat{\gamma}$ is homeomorphic to $(\mathbb{R}^2 \setminus X) \times \mathbb{R}$ for X a discrete set. Since $\pi_1((\mathbb{R}^2 \setminus X) \times \mathbb{R})$ is free and the essential torus T lifts to $\widetilde{PT^1(\Sigma)} \setminus \{\widetilde{\widehat{\gamma}}\}$, we reach a contradiction. This is because a free group does not contain any $\mathbb{Z}^2$ subgroup. To avoid using the geodesic flow we will directly show that $\pi_1(\widetilde{PT^1(\Sigma)} \setminus \{\widetilde{\overline{\gamma}}\})$ is free for any lift $\overline{\gamma}$ in $PT^1(\Sigma)$ of a geodesic multicurve on Σ:

Theorem 1.2 *Let M be a Seifert-fibered space over a hyperbolic surface Σ. Let $\overline{\gamma}$ be a link in M projecting injectively to a filling multicurve $\gamma \subset \Sigma$ of closed geodesics. Let $q\colon \widetilde{M} \to M$ be the universal covering map of M and $\{\widetilde{\overline{\gamma}}\}$ the total preimage of the link $\overline{\gamma}$ under q. Then the group $\pi_1(\widetilde{M} \setminus \{\widetilde{\overline{\gamma}}\})$ is free.*

By adding our argument to their proof we obtain a version of Theorem 1.1 in the more general setting of link complements in Seifert-fibered spaces whose projection to their hyperbolic 2–orbifold base is a filling geodesic multicurve. Our main result is:

Theorem 1.3 *Suppose $\mathcal{O}$ is a hyperbolic 2–orbifold and $\overline{\gamma}$ a link in an orientable Seifert-fibered space M over the orbifold $\mathcal{O}$ projecting injectively to a filling geodesic multicurve γ in $\mathcal{O}$. Then the complement of $\overline{\gamma}$ in M, denoted by $M_{\overline{\gamma}}$, is a hyperbolic manifold of finite volume.*

Once the hyperbolicity of $M_{\overline{\gamma}}$ is settled, by the Mostow rigidity theorem (see Benedetti and Petronio [4]) we can pursue the problem of estimating the volume of $M_{\overline{\gamma}}$. The volume invariant has been studied in the particular case of canonical lifts of geodesics in the projective unit tangent bundle $PT^1(\Sigma)$ of a hyperbolic surface Σ or the modular orbifold. Upper bounds have been found in terms of the geodesic length in Bergeron, Pinsky and Silberman [5] and a combinatorial lower bound by Rodríguez-Migueles [16].

In [16, Section 5], the second author noticed that the behavior of the volume of $M_{\overline{\gamma}}$ among different lifts of γ does not depend on the diagram given by the couple $(\gamma, \mathcal{O})$. More precisely, the second author proved the following result:

Proposition 1.4 *For any hyperbolic metric X on Σ, there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of filling closed geodesics and respective lifts $\{\overline{\gamma}_n\}_{n\in\mathbb{N}}$ in $PT^1(\Sigma)$ with $\ell_X(\gamma_n)\nearrow\infty$ such that*

$$k_X\ell_X(\gamma_n)\le \mathrm{Vol}(M_{\overline{\gamma}_n}),$$

where k_X is a positive constant that depends on the metric X. Moreover, there exists a constant $V_0>0$ such that $\mathrm{Vol}(M_{\widehat{\gamma}_n})<V_0$ for every $n\in\mathbb{N}$, where $\widehat{\gamma}_n$ is the canonical lift of γ_n on $PT^1(\Sigma)$.

By constructing a particular ideal triangulation on $M_{\overline{\gamma}}$ one can give a volume upper bound to $M_{\overline{\gamma}}$, independent of the lift $\overline{\gamma}$, which is linear in terms of the self-intersection number of γ.

Theorem 1.5 *Let M be a Seifert-fibered space over a hyperbolic 2–orbifold $\mathcal{O}$. Then, for any link $\overline{\gamma}\subset M$ projecting injectively to a filling geodesic multicurve γ on $\mathcal{O}$,*

$$\mathrm{Vol}(M_{\overline{\gamma}})<8v_3 i(\gamma,\gamma),$$

where v_3 is the volume of the regular ideal tetrahedron and $i(\gamma,\gamma)$ the self-intersection number of γ.

Furthermore, by Howie and Purcell [13, Theorem 1.1] one can construct a continuous lift inside the projective unit tangent bundle of a punctured hyperbolic surface over some closed geodesics such that the knot complement's hyperbolic volume is, up to a multiplicative factor, the self-intersection number of the geodesic multicurve. The sequences of geodesics, lifts and an estimate of the volume's lower bound for the corresponding knot complements are provided by the following result:

Corollary 1.6 *Let $\Sigma_{g,n}$ be an n–punctured hyperbolic surface with $n\ge 1$. Then there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of filling closed geodesics with $i(\gamma_n,\gamma_n)\nearrow\infty$, and respective lifts $\{\overline{\gamma}_n\}_{n\in\mathbb{N}}$ in $PT^1(\Sigma_{g,n})$, such that*

$$\tfrac{1}{2}v_8(i(\gamma_n,\gamma_n)-(2-2g))\le \mathrm{Vol}(M_{\overline{\gamma}_n})<8v_3 i(\gamma_n,\gamma_n),$$

where v_3 (resp. v_8) is the volume of the regular ideal tetrahedron (resp. octahedron) and $i(\gamma_n,\gamma_n)$ is the self-intersection number of γ_n.

The previous result shows that self-intersection is the optimal bound when considering general topological lifts. By generalizing arguments of [16, Theorem 1.5] to the Seifert-fibered setting we also give the following combinatorial lower bound:

Theorem 1.7 *Given a pants decomposition Π on a hyperbolic 2–orbifold $\mathcal{O}$, a Seifert-fibered space M over $\mathcal{O}$, and a filling geodesic multicurve γ on $\mathcal{O}$, for any closed continuous lift $\overline{\gamma}$ we have that*

$$\tfrac{1}{2}v_3 \sum_{P\in\Pi} \big(\sharp\{\textit{isotopy classes of } \overline{\gamma}\textit{–arcs in } p^{-1}(P)\}-3\big) \le \mathrm{Vol}(M_{\overline{\gamma}}),$$

where v_3 is the volume of a regular ideal tetrahedron.

Outline In Section 2 we recall some basic facts about Seifert-fibered spaces and orbifolds. In Section 3 we prove Theorems 1.3 and 1.2. In Section 4, by using results in [13] and [16], we prove some volume bounds.

Acknowledgements Cremaschi gratefully acknowledges support from the US National Science Foundation grants DMS 1107452, 1107263, 1107367, *RNMS: Geometric structures and representation varieties* (the GEAR Network) and also from the grant DMS-1564410 *Geometric structures on higher Teichmüller spaces*. Rodríguez-Migueles was generously supported by the Academy of Finland project #297258 *Topological geometric function theory*.

The authors would like to thank Ian Biringer, Martin Bridgeman, Juan Souto and Andrew Yarmola for helpful discussions. Moreover, we would like to thank the anonymous referee for many helpful comments and suggestions.

2 Seifert-fibered spaces and orbifolds

In this section, we recall some known facts about the topology of Seifert-fibered spaces and orbifolds. For more details see [14; 12].

Definition 2.1 A compact 3–manifold M is a *Seifert-fibered space* if M is the union of a collection $\{C_\alpha\}_{\alpha\in A}$ of pairwise disjoint simple closed curves called *fibers* such that every fiber C_α has a closed neighborhood V_α homeomorphic to a solid torus and a covering map $p_\alpha\colon \mathbb{D}^2\times\mathbb{S}^1\to V_\alpha$ satisfying:

(i) For all $x\in\mathbb{D}^2$ we have that $p_\alpha(\{x\}\times\mathbb{S}^1)=C_\beta$ for some $\beta\in A$, so that V_α is a union of fibers.

(ii) $p_\alpha^{-1}(C_\alpha)$ is connected.

(iii) The group of covering transformation is generated by $r_{n,m}$ for n and m relatively prime integers such that

$$r_{n,m}(re^{i\theta}, e^{i\phi}) \doteq (re^{i(\theta+2m\pi/n)}, e^{i(\phi+2\pi/n)}).$$

If $|n| = 1$ we have that p_α is a homeomorphism and we say that C_α is a *regular fiber*, otherwise we say it is a *singular fiber*.

Note that whenever $|n| > 1$, by (ii) $C_\alpha = p_\alpha(\{0\} \times \mathbb{S}^1)$ and for $x \neq 0$ we have that $p(\{x\} \times \mathbb{S}^1)$ is mapped to a fiber C_β which crosses the meridional disk $p_\alpha(\mathbb{D}^2 \times \{1\})$ n times and wraps m times around C_α. Also, since every fiber in a neighborhood of a singular fiber is regular, we get that if M is compact, it has finitely many singular fibers.

Definition 2.2 We say that a Hausdorff topological space $\mathcal{O}$ is an *orbifold* if we have a covering $\mathcal{U} \doteq \{U_i\}_{i \in \mathbb{N}}$ which is closed under finite intersections, and continuous maps $\phi_i\colon V_i \to U_i$ for V_i open subsets of $\mathbb{R}^2$, which are invariant under a faithful linear action of a finite group Γ_i such that $\phi_i\colon V_i / \Gamma_i \to U_i$ is a homeomorphism. Moreover, we say that the charts $\{U_i\}_{i \in \mathbb{N}}$ form an *orbifold atlas* if:

- For $U_i \subset U_j$ we have a monomorphism $f_{ij}\colon \Gamma_i \hookrightarrow \Gamma_j$.
- For $U_i \subset U_j$ we have a Γ_i–equivariant homeomorphism ψ_{ij}, called a gluing map, from V_i to an open subset of V_j.
- For all i and j we have $\phi_j \circ \psi_{ij} = \phi_i$.
- The gluing maps are unique up to compositions with group elements.

Remark Even though a general orbifold can have reflections, in the rest of this work we will only consider orbifolds with conical points. Therefore, the set of singular points in any orbifold $\mathcal{O}$ will always be a discrete set.

If M is a Seifert-fibered space we have a natural projection map $\pi\colon M \to \mathcal{O}$ obtained by mapping every fiber C_α to a point; the space $\mathcal{O}$ is called the *orbit-manifold*. Given a neighborhood of C_α, the map $\pi \circ p_\alpha\colon \mathbb{D}^2 \times \{1\} \to \mathcal{O}$ is an embedding if C_α is a regular fiber and is equivalent to the projection onto the orbit space of $\mathbb{D}^2 \times \{1\}$ under a periodic rotation otherwise. Therefore, the quotient space $\mathcal{O}$ is naturally an orbifold with discrete singular locus.

Remark From the classification theorem of Seifert-fibered spaces [19], it follows that any Seifert-fibered space M is homeomorphic to an $\mathbb{S}^1$–bundle over a compact surface S where we glue some singular neighborhoods along some tori boundary components. Equivalently, we can think of a Seifert-fibered space as an orientable $\mathbb{S}^1$–bundle over a compact orbifold $\mathcal{O}$.

3 Hyperbolicity of lift complements

For facts about the topology and geometry of 3-manifolds see [7; 15]. The aim of this section is to prove Theorem 1.3:

Theorem 1.3 *Suppose $\mathcal{O}$ is a hyperbolic 2–orbifold and $\overline{\gamma}$ a link in an orientable Seifert-fibered space M over the orbifold $\mathcal{O}$, which projects injectively to a filling geodesic multicurve γ in $\mathcal{O}$. Then $M_{\overline{\gamma}}$ is a hyperbolic manifold of finite volume.*

Definition 3.1 Given a Seifert-fibered space M with its bundle map $p\colon M \to \mathcal{O}$, we say that a link $\overline{\gamma} \subset M$ *projects injectively* to a multicurve $\gamma \subset \mathcal{O}$ if distinct components of $\overline{\gamma}$ map, under p, to distinct components of γ, and the projection p is injective except at self-intersection points of γ which have two preimages.

Let M be a Seifert-fibered space over a hyperbolic 2–orbifold $\mathcal{O}$, γ a geodesic multicurve on $\mathcal{O}$ and $\overline{\gamma}$ a link in M projecting injectively to γ under p. Then we have the commutative diagram

$$\begin{array}{ccc} & & M \\ & \overset{\overline{\gamma}}{\nearrow} & \big\downarrow p \\ \bigsqcup_{i=1}^{n} \mathbb{S}^1 & \xrightarrow[\gamma]{} & \mathcal{O} \end{array}$$

From now on we denote by $M_{\overline{\gamma}}$ the complement of a normal neighborhood of $\overline{\gamma}$ in M.

Definition 3.2 For a hyperbolic 2–orbifold $\mathcal{O}$ with a discrete set of singular points $\mathcal{S}$, we say that a multicurve γ of closed geodesics is *filling* if γ is disjoint from $\mathcal{S}$ and if $\mathcal{O} \setminus \gamma$ is a collection of disks, once-punctured disks or disks with one conical point. See Figure 1.

In order to prove Theorem 1.3, we will first reduce it to the case in which the orbifold $\mathcal{O}$ is a surface, ie to the case where the Seifert-fibered space has a hyperbolic surface Σ as base.

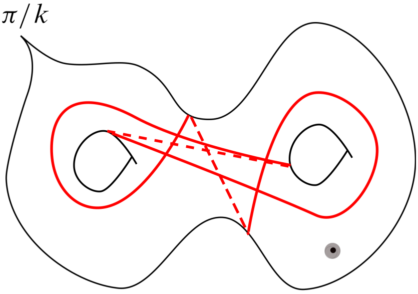

Figure 1: A filling geodesic on a 2–orbifold.

Lemma 3.3 *If $\Sigma \xrightarrow{q} \mathcal{O}$ is a finite cover from a surface Σ and γ is a filling geodesic multicurve in the orbifold $\mathcal{O}$, then the union of all lifts γ_0 is also a filling geodesic multicurve on Σ.*

Proof Let $\Sigma_0 \doteq \mathcal{O} \setminus \mathcal{S}$. Then γ is filling in Σ_0. Consider the induced cover $q\colon q^{-1}(\Sigma_0) \to \Sigma_0$ for $q^{-1}(\Sigma_0)$ a connected subsurface of Σ. Then $q^{-1}(\gamma)$ is filling in $q^{-1}(\Sigma_0)$. However, $\Sigma = q^{-1}(\Sigma_0) \cup \mathcal{D}$ for $\mathcal{D}$ a collection of disks each covering a disk with a cone point. Thus $q^{-1}(\gamma)$ is filling in Σ and since $q^{-1}(\gamma) = \gamma_0$, we are done. □

Lemma 3.4 *Given a finite cover $\pi\colon \widehat{M}_{\bar{\gamma}_0} \to M_{\bar{\gamma}}$, if $\widehat{M}_{\bar{\gamma}_0}$ is atoroidal, so is $M_{\bar{\gamma}}$.*

Proof Given an essential torus $T \subset M_{\bar{\gamma}}$, the restriction $\pi\colon \pi^{-1}(T) \to T$ is a finite cover, hence every component of $\pi^{-1}(T)$ is an essential torus $\widehat{T} \subset \widehat{M}_{\bar{\gamma}_0}$. Thus, since $\widehat{M}_{\bar{\gamma}_0}$ is atoroidal, the essential torus $\widehat{T}$ is homotopic into a torus component $\widehat{S}$ of $\partial \widehat{M}_{\bar{\gamma}_0}$. The torus component $\widehat{S}$ must cover a torus component S of $\partial M_{\bar{\gamma}}$. By pushing the homotopy via π we see that T is also homotopic into a torus component $\pi(\widehat{T})$ of $\partial M_{\bar{\gamma}}$. □

We now reduce the proof of the main theorem to the case in which the orbifold is an actual surface.

Proposition 3.5 *If Theorem 1.3 holds for hyperbolic surfaces, then it holds for orbifolds.*

Proof By the geometrization theorem (see [18]), the Seifert-fibered space M over a compact hyperbolic orbifold $\mathcal{O}$ has a geometry modeled on either $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}}_2$.

Assume that $G \cong \pi_1(M)$ is a discrete group of isometries of $\mathbb{H}^2 \times \mathbb{R}$ that acts freely and has quotient an orientable $\mathbb{S}^1$–bundle M. Notice that the isometry group of $\mathbb{H}^2 \times \mathbb{R}$ can be naturally identified with $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R})$ and we regard the factors as subgroups in the usual way. As G is discrete and M is a Seifert bundle, $G \cap \mathrm{Isom}(\mathbb{R}) = \mathbb{Z}$. Let Γ denote the image of the projection $G \xrightarrow{p} \mathrm{Isom}(\mathbb{H}^2)$. Then we have the exact sequence

$$1 \to \mathbb{Z} \to G \to \Gamma \to 1,$$

where Γ is a discrete group of isometries of $\mathbb{H}^2$.

On the other hand, if G is a discrete group of isometries of $\widetilde{\mathrm{SL}}_2$ acting freely and with quotient an orientable $\mathbb{S}^1$–bundle M, we have the exact sequence

$$0 \to \mathbb{R} \to \mathrm{Isom}(\widetilde{\mathrm{SL}}_2) \xrightarrow{p} \mathrm{Isom}(\mathbb{H}^2) \to 1.$$

If Γ denotes $p(G)$, we have the exact sequence

$$1 \to \mathbb{Z} \to G \to \Gamma \to 1$$

for Γ a discrete group of isometries of $\mathbb{H}^2$.

In either case Γ is a finitely generated subgroup of $\mathrm{Isom}(\mathbb{H}^2)$, thus by [3], Γ is residually finite. Hence, Γ has a torsion-free subgroup $\widehat{\Gamma}$ of finite index. Let $\widehat{G}$ be the subgroup of G projecting onto $\widehat{\Gamma}$ and let $\widehat{M} \doteq \widetilde{\mathrm{SL}}_2/\widehat{G}$ or $(\mathbb{H}^2 \times \mathbb{R})/\widehat{G}$. By the first isomorphism theorem (see [8]) we have that $G/\widehat{G} \cong \Gamma/\widehat{\Gamma}$, hence $\widehat{G}$ is also finite index in G. Therefore, we have the commutative diagram

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\pi} & M \\ \big\downarrow \hat{p} & & \big\downarrow p \\ \Sigma \doteq \mathbb{H}^2/\widehat{\Gamma} & \xrightarrow{\hat{\pi}} & \mathbb{H}^2/\Gamma \doteq \mathcal{O} \end{array}$$

where $\pi\colon \widehat{M} \to M$ is a finite-index cover. By lifting $\overline{\gamma} \subset M$ to $\overline{\gamma}_0 \doteq \pi^{-1}(\overline{\gamma}) \subset \widehat{M}$, we thus get a finite cover

$$\pi\colon \widehat{M}_{\overline{\gamma}_0} \to M_{\overline{\gamma}}.$$

Moreover, by the commutativity of the previous diagram, the link $\overline{\gamma}_0$ projects injectively onto the filling multicurve $\gamma_0 = \hat{\pi}^{-1}(\gamma)$. By Lemma 3.3 we get that γ_0 satisfies the conditions of Theorem 1.3 for Σ. Then by Proposition 3.5, if $\widehat{M}_{\overline{\gamma}_0}$ is atoroidal, we get that $M_{\overline{\gamma}}$ is also atoroidal. □

Therefore, to show Theorem 1.3 it suffices to prove:

Theorem 3.6 *Suppose Σ is a hyperbolic surface and $\overline{\gamma}$ is a link in an orientable Seifert-fibered space M over Σ, which projects injectively to a filling multicurve γ of closed geodesics in Σ. Then $M_{\overline{\gamma}}$ is a hyperbolic manifold of finite volume.*

3.1 Proof of Theorem 3.6

Before proving Theorem 3.6 we need to introduce some objects.

Definition 3.7 We say that a triangulation $\tau \doteq \{T_i\}_{1 \le i \le m}$ of a hyperbolic surface Σ is *simple* for a geodesic multicurve γ if

(1) the punctures of Σ are contained in the vertices of τ and every triangle $T \in \tau$ has at most one puncture;

(2) the edges and vertices of each element in τ are distinct;

(3) each edge of τ is a geodesic arc (or geodesic ray if one vertex is a puncture) transversal to γ;

(4) in every triangle $T \in \tau$ we have that if $\gamma \cap T \neq \varnothing$, then it contains either two intersecting subarcs of γ or a single subarc of γ; see Figure 2.

Lemma 3.8 *Let γ be a filling geodesic multicurve in a hyperbolic surface Σ. Then there exists a simple triangulation of Σ relative to γ.*

Proof We will build the triangulation in four steps.

(1) We start our triangulation around the self-intersection points of γ. Let $x \in \gamma$ be a self-intersection point and consider a small piecewise geodesic disk D around it which contains only two intersecting γ–arcs and no punctures. Choose 4 vertices in ∂D, one in each quadrant relative to the pair of γ–arcs, and the corresponding embedded geodesic quadrangle such that one of the diagonals does not pass through x. The quadrangle, with this diagonal, gives a triangulation τ around all self-intersection points x of γ. Since γ is filling, all connected components C of $\Sigma \setminus \gamma$ are punctured disks or disks and they all contain $k > 0$ vertices of τ, where k is equal to the number of geodesic arcs of ∂C. We denote by $\mathcal{M}$ all components that are monogons or bigons. In either case, every component $C \subset \mathcal{M}$ is homeomorphic to a punctured disk.

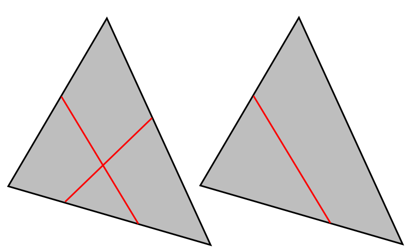

Figure 2: Two possible γ–arcs configuration inside T.

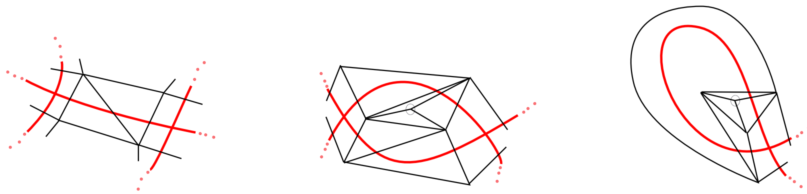

Figure 3: Triangulation τ of Σ along an embedded γ–arc, in red, of a $(2+k)$–gon (left), bigon (center), and monogon (right).

(2) Consider a connected component C of $\mathcal{M}$. If the component of $\Sigma\backslash\gamma$ is a monogon we add an extra vertex v' on the interior so that now we have 3 marked points in C: the ideal vertex, a vertex of τ and v'. Then, we extend τ as in Figure 3, right.

For bigons we have already two vertices of τ and one ideal vertex, so we extend τ as in Figure 3, center. We still denote this triangulation by τ and we note that all components of $\Sigma \setminus \gamma$ that are not triangulated are n–gons, with $n \geq 3$, containing n vertices of τ.

(3) Let C be a connected n–gon, $n \geq 3$, in $\Sigma \setminus \gamma$. Then we connect the n–vertices of τ in C by geodesic arcs to form a simple loop α, isotopic in C to ∂C. If C does not contain an ideal vertex w we add a vertex v and cone the vertices of α to either v or w. We still denote this triangulation by τ.

(4) The components of $\Sigma\setminus\tau$ that have not been triangulated are regular neighborhoods of γ–subarcs of the edges of the graph induced by γ and can be triangulated by geodesic arcs as in Figure 3, left.

Notice that by construction, our triangulation satisfies the properties of a simple triangulation in Definition 3.7. □

We will now build a partition, relative to a geodesic multicurve γ, of the universal cover $\widetilde{M}$ of M induced by a given simple triangulation τ on Σ. In our setting we have

$$\widetilde{M} \xrightarrow{q} M \xrightarrow{p} \Sigma.$$

Since p is a bundle map we have that for all $T_i \in \tau$ with $1 \leq i \leq m$, the preimage $p^{-1}(T_i)$ is homeomorphic to a solid torus $V_{T_i} \cong \mathbb{S}^1 \times \mathbb{D}^2$. Moreover, the solid torus V_{T_i} inherits from the triangulation $\{T_i\}_{1\leq i\leq m}$ a decomposition of ∂V_{T_i} into

(1) three *loops* w_i^1, w_i^2 and w_i^3 corresponding to the preimages of the vertices v_i^1, v_i^2 and v_i^3 of T_i;

(2) three *faces* F_i^1, F_i^2 and F_i^3 homeomorphic to annuli $I \times \mathbb{S}^1$ and corresponding to the preimages of the edges e_i^1, e_i^2 and e_i^3 of T_i.

By going to the universal cover $\widetilde{M}$ of M, each V_{T_i} lifts to a collection $\bigsqcup_{\alpha \in A_i} T_i^\alpha \times \mathbb{R}$ with $T_i^\alpha \cong \mathbb{D}^2$, and the previous decompositions of V_{T_i} induce a decomposition of each $T_i^\alpha \times \mathbb{R}$ into

(1) three *edges* $\widetilde{w}_i^1$, $\widetilde{w}_i^2$ and $\widetilde{w}_i^3$, each one homeomorphic to $\mathbb{R}$ and corresponding to the preimages of the vertices v_i^1, v_i^2 and v_i^3 of T_i;

(2) three *loops* $\widetilde{F}_i^1$, $\widetilde{F}_i^2$ and $\widetilde{F}_i^3$ homeomorphic to $I \times \mathbb{R}$ and corresponding to the preimages of the edges e_i^1, e_i^2 and e_i^3 of T_i.

Remark 1 By the above discussion, using the composition $\widetilde{M} \overset{q}{\twoheadrightarrow} M \overset{p}{\twoheadrightarrow} \Sigma$ we have the decomposition of $\widetilde{M}$ into *thick cylinders*

$$\widetilde{M} = \bigcup_{i=1}^{m} (p \circ q)^{-1}(T_i) = \bigcup_{i=1}^{m} \bigsqcup_{\alpha \in A_i} T_i^\alpha \times \mathbb{R}.$$

Lemma 3.9 *Let $\widetilde{M} \overset{q}{\twoheadrightarrow} M$ be the universal covering map of the Seifert-fibered manifold M, and let $M \overset{p}{\twoheadrightarrow} \Sigma$ be the Seifert map for Σ not a sphere. Given any simple triangulation τ on Σ we have that $\widetilde{M} = \bigcup_{n=1}^{\infty} K_n$, where each K_n is simply connected and $K_n = K_{n-1} \cup_{S_n} T_{j_n}^{\alpha_n} \times \mathbb{R}$ for S_n either one or two faces of $T_{j_n}^{\alpha_n} \times \mathbb{R}$.*

Proof By Remark 1, we have the decomposition $\widetilde{M} = \bigcup_{i=1}^{m} \bigsqcup_{\alpha \in A_i} T_i^\alpha \times \mathbb{R}$.

Claim 1 *For $i \neq j$, the thick cylinders $T_\alpha^i \times \mathbb{R}$ and $T_\beta^j \times \mathbb{R}$ are either disjoint, share at most two faces or share only one edge.*

Proof of claim Suppose that they are not disjoint, so that $T_i = p \circ q(T_\alpha^i \times \mathbb{R})$ and $T_j = p \circ q(T_\beta^j \times \mathbb{R})$ intersect in Σ. Then, since they are distinct elements of the triangulation τ, they must intersect in their boundary. Since Σ is not a sphere, it follows that T_i and T_j either intersect in a vertex, or they share at most two edges, and the result follows. □

We now claim:

Claim 2 *There are nested simply connected subsets $\{K_n\}_{n \in \mathbb{N}}$ of $\widetilde{M}$ such that $\widetilde{M} = \bigcup_{n=1}^{\infty} K_n$ and $K_n = K_{n-1} \cup_{S_n} T_{j_n}^{\alpha_n} \times \mathbb{R}$, where S_n is at most two faces of $T_{j_n}^{\alpha_n} \times \mathbb{R}$ sharing an edge.*

Proof of claim Pick $T_1 \in \tau$ and let K_1 be any component $T_1^\alpha \times \mathbb{R}$ of $(p \circ q)^{-1}(T_1)$ in $\widetilde{M}$. Then mark the edges $\widetilde{w}_1$, $\widetilde{w}_2$ and $\widetilde{w}_3$ in ∂K_1 and *develop* around them. That is, let $\{T_{i_k}^{\alpha_k} \times \mathbb{R}\}_{1 \leq k \leq n_1}$ be the finitely many components of the decomposition of $\widetilde{M}$

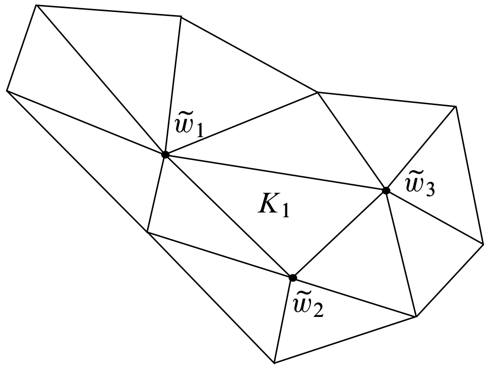

Figure 4: Schematic of the simply connected subset K_m engulfing K_1.

containing $\widetilde{w}_1$ as an edge. By Claim 1 at least one of the $\{T_{i_k}^{\alpha_k} \times \mathbb{R}\}_{1\le k\le n_1}$, say $T_{i_1}^{\alpha_k} \times \mathbb{R}$, shares one or two faces S with K_1. We then let $K_2 \doteq K_1 \cup_S T_{i_1}^{\alpha_k} \times \mathbb{R}$. By repeating this for all $\{T_{i_k}^{\alpha_k} \times \mathbb{R}\}_{1\le k\le n_1}$ we have added all solid tori $T_j^{\alpha} \times \mathbb{R}$ having $\widetilde{w}_1$ as an edge to K_1. The sets $\{K_n\}_{1\le n\le n_1+1}$ so constructed are all homeomorphic to $\mathbb{D}^2 \times \mathbb{R}$, hence simply connected. This is because at every stage we glue a thick cylinder to another thick cylinder along a simply connected subset of their boundary.

By repeating this with $\widetilde{w}_2$ and $\widetilde{w}_3$ we get new simply connected subsets $\{K_n\}_{n_1+2\le n\le m}$ for $m \in \mathbb{N}$; see Figure 4.

Moreover, all the K_n for $n \le m$ so constructed are simply connected and properly embedded in $\widetilde{M}$, and $\widetilde{w}_1$, $\widetilde{w}_2$ and $\widetilde{w}_3$ are contained in the interior of K_m. We then mark all edges $\widetilde{w}_1^m, \dots, \widetilde{w}_{n_m}^m$ of ∂K_m and repeat the previous construction by first adding all the thickened cylinders sharing an edge with $\widetilde{w}_1^m$ and so on.

This yields a collection $\{K_n\}_{n\in\mathbb{N}}$ of properly embedded nested simply connected subset of $\widetilde{M}$ such that for all $n \in \mathbb{N}$, $K_{n+1} = K_n \cup T_{i_n}^{\alpha_{k_n}} \times \mathbb{R}$ for some α_{k_n}, i_n. □

Moreover, since each $T_i^{\alpha} \times \mathbb{R}$ is in $\bigcup_{n=1}^{\infty} K_n$ and we have that the universal cover $\widetilde{M} = \bigcup_{i=1}^{m} \bigsqcup_{\alpha\in A_i} T_i^{\alpha} \times \mathbb{R}$, it follows that $\widetilde{M} = \bigcup_{n=1}^{\infty} K_n$. □

We now prove a key lemma.

Lemma 3.10 *Let M be a Seifert-fibered space over Σ with projection $p\colon M \to \Sigma$, and let $\bar{\gamma}$ be a link in M which projects injectively to a filling geodesic multicurve $\gamma \subset \Sigma$. Let τ be a simple triangulation for γ in Σ and let $\{\widetilde{\bar{\gamma}}\}$ be all the lifts in $\widetilde{M}$ of the link $\bar{\gamma}$. Then, for every $T \in \tau$, we have that $\pi_1\big(\widetilde{p^{-1}(T)} \setminus \{\widetilde{\bar{\gamma}}\}\big)$ is a free group. Moreover, the set of $\{\widetilde{\bar{\gamma}}\}$–arcs in $\widetilde{p^{-1}(T)}$ forms a free basis.*

Proof We claim:

Claim *For each $T \in \tau$, there exists a smooth lift $\overline{T}$ of T embedded in $p^{-1}(T) \subset M$ such that $\overline{\gamma} \cap \overline{T} = \varnothing$.*

Proof of claim Suppose γ_0 and γ_1 are two γ–arcs contained in T with unique intersection point $x \doteq \gamma_1(t) = \gamma_0(s)$. Consider a new lift $\overline{\gamma}_0'$ of the arc γ_0 in $p^{-1}(T)$ which is at positive constant $\mathbb{S}^1$–fiber distance from $\overline{\gamma}_0$ and does not intersect $\overline{\gamma}_1$. Let $\overline{\gamma}_1'$ be the unique lift passing through $y = \overline{\gamma}_0' \cap p^{-1}(x)$ and at $\mathbb{S}^1$–fiber distance $d_{\mathbb{S}^1}(y, \overline{\gamma}_1(s))$ from $\overline{\gamma}_1$. Consider any smooth lift $\overline{T}$ of T which contains $\overline{\gamma}_1' \cup \overline{\gamma}_0'$. As $\overline{\gamma}_1' \cup \overline{\gamma}_0'$ does not intersect $\overline{\gamma}$ and the projection of $\overline{T} \setminus (\overline{\gamma}_1' \cup \overline{\gamma}_0')$ under p is disjoint from $\gamma \cap T$, we get $\overline{\gamma} \cap \overline{T} = \varnothing$. The case of only one γ–arc inside T follows similarly from the previous case. □

By cutting $p^{-1}(T)$ along the lift $\overline{T}$ coming from the previous claim, we can associate to $p^{-1}(T) \setminus (\overline{T} \cup \overline{\gamma})$ a string diagram D_T on T such that D_T has at most one self-crossing. Following Wirtinger (see [17, Chapter 3, Section D]), we can give a presentation of the fundamental group $\pi_1(p^{-1}(T) \setminus (\overline{T} \cup \overline{\gamma}))$ using D_T and show that $\pi_1(p^{-1}(T) \setminus (\overline{T} \cup \overline{\gamma}))$ is a free group. Moreover, the generators are in bijection with the $\overline{\gamma}$–arcs inside $p^{-1}(T) \setminus \overline{T}$. Equivalently, the bijection is with γ–arcs inside T.

Lastly, since $\widetilde{p^{-1}(T)} \setminus \{\tilde{\overline{\gamma}}\}$ is obtained by translating one lift of $p^{-1}(T) \setminus (\overline{T} \cup \overline{\gamma})$ in $\widetilde{M}$ under the fiber action, meaning that we are gluing consecutive lifts along the common lift of $\overline{T}$ inside each one of them, and each lift of $\overline{T}$ is simply connected, by the Van Kampen theorem, we have that $\pi_1(\widetilde{p^{-1}(T)} \setminus \{\tilde{\gamma}'\})$ is a free product of free groups. Moreover, by the Van Kampen theorem the generators are in a one-to-one correspondence with the $\{\tilde{\overline{\gamma}}\}$–arcs inside $\widetilde{p^{-1}(T)}$. □

We can now prove:

Theorem 1.2 *Let M be a Seifert-fibered space over a hyperbolic surface Σ. Let $\overline{\gamma}$ be a link in M projecting injectively to a filling multicurve $\gamma \subset \Sigma$ of closed geodesics. Let $q\colon \widetilde{M} \to M$ be the universal covering map of M and $\{\tilde{\overline{\gamma}}\}$ the total preimage of the link $\overline{\gamma}$ under q. Then the group $\pi_1(\widetilde{M} \setminus \{\tilde{\overline{\gamma}}\})$ is free.*

Proof By Lemma 3.8, let τ be a simple triangulation of Σ and let $\bigcup_{i=1}^{\infty} K_n$ be the induced decomposition of $\widetilde{M}$ coming from Lemma 3.9. We define $C_{T_i^{\alpha}} \doteq T_i^{\alpha} \times \mathbb{R} \setminus \{\tilde{\overline{\gamma}}\}$ and let $C_n \doteq K_n \setminus \{\tilde{\overline{\gamma}}\}$.

Claim *For every $n \in \mathbb{N}$, we have that $\pi_1(C_n)$ is free. Moreover, the generators are in bijection with the $\{\widetilde{\overline{\gamma}}\}$–arcs inside C_n.*

Proof of claim The proof is by induction over n, where the base case is Lemma 3.10. Suppose that the claim is true for C_n. We will show it is also true for $C_{n+1} = C_n \cup_{Z_n} C_{T_j^\alpha}$. By Lemma 3.9 the intersection of C_n with $C_{T_j^\alpha}$ is either one or two punctured faces $Z_n = S_n \setminus \{\widetilde{\overline{\gamma}}\}$ such that each puncture comes from a subset of $\{\widetilde{\overline{\gamma}}\}$–arcs inside C_n and the same holds for $C_{T_j^\alpha}$. The natural inclusions

$$(i_1)_*\colon \pi_1(C_{T_j^\alpha} \cap C_n) \to \pi_1(C_{T_j^\alpha}) \quad \text{and} \quad (i_2)_*\colon \pi_1(C_{T_j^\alpha} \cap C_n) \to \pi_1(C_n)$$

map generators to generators. Thus, by Van Kampen's theorem, $\pi_1(C_{n+1})$ is also free because the new relations are given by

$$(i_1)_*(s)((i_2)_*(s))^{-1} = 1,$$

where s is a generating element of $\pi_1(C_{T_j^\alpha} \cap C_n)$. Thus, we are just pairing the generators of the two free groups. Therefore, the new relations either rename the generators of $\pi_1(C_{T_j^\alpha})$ with generators of $\pi_1(C_n)$, or reduce the number of generators of $\pi_1(C_n)$. □

By the previous claim, each $\pi_1(C_n)$ is free and the inclusions $j_n\colon C_n \to C_{n+1}$ induce maps $(j_n)_*\colon \pi_1(C_n) \to \pi_1(C_{n+1})$ mapping generators to generators. Therefore, the free basis of C_n is extended to a free basis of C_{n+1}, thus

$$\varinjlim_{(j_n)_*} \pi_1(C_n) \cong \pi_1(\widetilde{M} \setminus \{\widetilde{\overline{\gamma}}\})$$

is a free group as well. Moreover, the set $\{\widetilde{\overline{\gamma}}\}$ forms a generating set for $\pi_1(\widetilde{M} \setminus \{\widetilde{\overline{\gamma}}\})$. □

We say that a properly embedded arc α in $\mathbb{R}^3$ is *unknotted* if for any thickened cylinder V such that $\alpha \subset \operatorname{int}(V)$, we have that ∂V is isotopic to $\partial N_\epsilon(\alpha)$ in V. As a consequence of the previous proof we obtain:

Corollary 3.11 *Given a component $\alpha \in \{\widetilde{\overline{\gamma}}\}$, then α is unknotted in $\widetilde{M} \cong \mathbb{R}^2 \times \mathbb{R}$.*

We can now show:

Theorem 3.6 *Suppose Σ is a hyperbolic surface and $\overline{\gamma}$ is a link in an orientable Seifert-fibered space M over Σ projecting injectively to a filling multicurve γ of closed geodesics in Σ. Then the complement of $\overline{\gamma}$ in M, denoted by $M_{\overline{\gamma}}$, is a hyperbolic manifold of finite volume.*

Proof By Thurston's geometrization theorem [21], it suffices to show that $M_{\overline{\gamma}}$ is atoroidal, irreducible and has infinite π_1. The last two claims follow from standard arguments coming from 3–dimensional topology using the fact that $\gamma \neq 0$ in $\pi_1(\Sigma)$ and that Σ is not a 2–sphere, respectively. Thus, since $M_{\overline{\gamma}}$ is irreducible and $\pi_1(M_{\overline{\gamma}})$ is infinite, we only need to prove the atoroidality condition. The proof involves three cases, each of which will be proven by contradiction. Let $T \subset M_{\overline{\gamma}}$ be an incompressible torus not parallel to the boundary of $M_{\overline{\gamma}}$. Then, for $\iota\colon M_{\overline{\gamma}} \hookrightarrow M$, the subgroup $\pi_1(\iota(T))$ has either rank zero, one or two in $\pi_1(M)$.

Case 1 (rank of $\pi_1(\iota(T))$ is zero) This means that $\iota(T)$ is nullhomotopic in M. Hence, the map $\iota\colon T \to M$ lifts to an embedded torus $\widetilde{T}$ in $\widetilde{M}$. Moreover, for $\{\widetilde{\overline{\gamma}}\}$ the lifts of $\overline{\gamma} \subset M$, we get that $\widetilde{T} \subset \widetilde{M} \setminus \{\widetilde{\overline{\gamma}}\}$ is essential. However, by Theorem 1.2, $\pi_1(\widetilde{M} \setminus \{\widetilde{\overline{\gamma}}\})$ is a free group which does not contain any $\mathbb{Z}^2$ subgroup, giving us a contradiction.

Case 2 (rank of $\pi_1(\iota(T))$ is one) If $\operatorname{rank}(\pi_1(\iota(T)) = 1$ it means that $\iota(T)$ is compressible in M. Therefore we have a compression disk D such that compressing $\iota(T)$ along D gives us a 2–sphere $\mathbb{S}^2 \hookrightarrow M$. Since M is irreducible it means that $\mathbb{S}^2$ bounds a 3–ball $B \subset M$. Thus, we see that $\iota(T)$ bounds a solid torus V in M and by incompressibility of T in $M_{\overline{\gamma}}$ we must have that $\overline{\gamma} \cap V \neq \varnothing$. Since $T \cap \overline{\gamma} = \varnothing$ we have that every component $\overline{\gamma}_i \in \pi_0(\overline{\gamma})$ intersecting V is contained in V.

Claim *There is a unique component $\overline{\gamma}_i \in \pi_0(\overline{\gamma})$ contained in V and it is a generator of $\pi_1(V)$.*

Proof of claim Let α be a generator of $\pi_1(V)$ in $\pi_1(M)$. Then every component $\overline{\gamma}_i \subset V$ of $\overline{\gamma}$ is homotopic, in V, to α^{n_i} for some $n_i \in \mathbb{N}$. But every $\overline{\gamma}_i$ is the lift of a geodesic in Σ and so it is primitive. Hence, every $\overline{\gamma}_i \subset V$ generates $\pi_1(V)$. Thus, any two $\overline{\gamma}_i, \overline{\gamma}_j$ in V must be homotopic, contradicting the fact that $\overline{\gamma}$ projects injectively to a geodesic multicurve on Σ. Thus there is a unique component $\overline{\eta} \in \pi_0(\overline{\gamma})$ contained in V, and $[\overline{\eta}]$ generates $\pi_1(V)$. □

Claim *The torus T is boundary parallel in $M_{\overline{\gamma}}$.*

Proof of claim Consider a lift $\widetilde{V}$ of V in $\widetilde{M}$. Then $\widetilde{V}$ is homeomorphic to $\mathbb{D}^2 \times \mathbb{R}$ and it contains $\widetilde{\overline{\eta}}$. If $\widetilde{V}$ is not boundary parallel in $\widetilde{M} \setminus \{\widetilde{\overline{\gamma}}\}$, we have that the lift $\widetilde{\overline{\eta}}$ is knotted in $\widetilde{V}$, contradicting Corollary 3.11. Therefore, the infinite cylinder $\partial\widetilde{V}$ is isotopic into $\partial N_\epsilon(\widetilde{\overline{\eta}})$. Thus, $\pi_1(T)$ is conjugated into $\pi_1(\partial N_\epsilon(\widetilde{\overline{\eta}}))$, contradicting the fact that T was not parallel to the boundary of $M_{\overline{\gamma}}$. □

Case 3 (rank of $\pi_1(\iota(T))$ is two) If $\iota(T)$ is essential in M, by [11, Proposition 1.11], we must have that $\iota(T)$ is isotopic to either a horizontal surface or a vertical surface in M. If $\iota(T)$ is horizontal it means that the hyperbolic surface Σ is covered by a torus, which is impossible. Therefore, $\iota(T)$ is isotopic to a vertical torus T'. Then if we consider the projection $p\colon M \to \Sigma$, we see that $p(T')$ is an essential simple closed curve $\alpha \subset \Sigma$. Moreover, since $T' \cap \overline{\gamma} = \varnothing$, we have that $\alpha \cap \gamma = \varnothing$. However, this contradicts the fact that γ is a filling multicurve, giving us a contradiction.

Thus, $M_{\overline{\gamma}}$ is atoroidal, hence admits a complete hyperbolic metric of finite volume. □

4 Volume of $M_{\overline{\gamma}}$

Once the hyperbolicity of $M_{\overline{\gamma}}$ is settled, then by Mostow rigidity we can pursue the problem of estimating geometric invariants in terms of topological relations between the multicurve γ and the hyperbolic orbifold $\mathcal{O}$.

Specifically, we will show our volume upper bounds in terms of self-intersection, and extend the lower bound of the second author. Moreover, we will also construct continuous lifts inside the projective unit tangent bundle of a punctured hyperbolic surface over some closed geodesics so that the knot complement's hyperbolic volume is, up to a multiplicative factor, given by the self-intersection number of the geodesic multicurve.

4.1 Lifts $\overline{\gamma}$ whose volume complement is linear in $\iota(\gamma.\gamma)$

Recall the following definition:

Definition 4.1 Given a connected, orientable 3–manifold M with boundary, we let $S_k(M;\mathbb{R})$ be the *singular chain complex* of M. That is, $S_k(M;\mathbb{R})$ is the set of formal linear combinations of k–simplices, and we set as usual $S_k(M,\partial M;\mathbb{R}) = S_k(M;\mathbb{R})/S_k(\partial M;\mathbb{R})$. We denote by $\|c\|$ the l_1–norm of the k–chain c. If α is a homology class in $H_k^{\mathrm{sing}}(M,\partial M;\mathbb{R})$, the *Gromov norm of* α is defined as

$$\|\alpha\| = \inf_{[c]=\alpha}\left\{\|c\| = \sum_{\sigma}|r_\sigma| \;\middle|\; c = \sum_{\sigma} r_\sigma \sigma\right\}.$$

The *simplicial volume* of M is the Gromov norm of the fundamental class of $(M,\partial M)$ in $H_3^{\mathrm{sing}}(M,\partial M;\mathbb{R})$, and is denoted by $\|M\|$. In the special case in which ∂M has only tori boundary components, there is another similar definition of $\|M\|_0$, which coincides with $\|M\|$ whenever M is hyperbolic, and has the property that $\|N\|_0 = 0$ for any Seifert-fibered space N.

We recall the following results:

Proposition 4.2 [20, Proposition 6.5.2] *Let $(M, \partial M)$ be a compact, orientable 3–manifold with ∂M consisting of tori. If $(N, \partial N)$ is obtained from M by gluing pairs of tori in ∂M, then*

$$\|N\|_0 \le \|M\|_0.$$

Lemma 4.3 [20, Lemma 6.5.4] *Let M be a complete hyperbolic manifold of finite volume. Then $v_3\|M\| = v_3\|M\|_0 = \mathrm{Vol}(M)$.*

Then, we have:

Theorem 4.4 *Let $\Sigma_{g,n}$ be a hyperbolic surface and $M \cong \Sigma_{g,n} \times \mathbb{S}^1$. Then there exists a sequence of filling closed geodesics $\{\gamma_n\}_{n\in\mathbb{N}}$ with $i(\gamma_n, \gamma_n) \nearrow \infty$, and respective lifts $\{\overline{\gamma}_n\}_{n\in\mathbb{N}}$ in M such that*

$$\tfrac{1}{2} v_8(i(\gamma_n, \gamma_n) - (2-2g)) \le \mathrm{Vol}(M_{\overline{\gamma}_n}),$$

where v_8 is the volume of the regular ideal octahedron, and $i(\gamma_n, \gamma_n)$ is the self-intersection number of γ_n.

Proof For the sake of concreteness, we will first prove the result for the once-punctured torus $\Sigma_{1,1}$. Let γ_n be constructed as in Figure 5. That is, fix a simple closed geodesic s on $\Sigma_{1,1}$, and pick two distinct points p_1 and p_2 on s. Let α_1 and β_1 be two essential arcs linking in $\Sigma_{1,1} \setminus s$ linking p_1 with p_2 and such that $\iota(\alpha_1, \beta_1) = 1$. Note that $\alpha_1 \cup \beta_1$ gives us an essential loop in $\Sigma_{1,1}$. Let γ_1 be the closed geodesic representative of $\alpha_1 \cup \beta_1$. Then γ_n is obtained from γ_1 by Dehn-twisting β_1 $2n$–times along s.

Since $M \cong \Sigma_{1,1} \times \mathbb{S}^1$, consider a global section $S_{1,1}$ embedded in M. Let $\overline{\gamma}_n$ be constructed in $N_\varepsilon(S_{1,1})$, a normal ε–neighborhood of $S_{1,1}$, so that its corresponding diagram on $S_{1,1}$ is the alternating diagram in Figure 6.

By making a trivial Dehn filling around the torus coming from the puncture of $\Sigma_{1,1}$, the manifold $(M, S_{1,1})$ becomes $(\Sigma_1 \times \mathbb{S}^1, S_1)$ and the position of our knot $\overline{\gamma}_n$ does not change. By using [20, Lemma 6.5.4 and Proposition 6.5.2] and the fact that a solid torus V has $\|V\|_0 = 0$, we get

$$v_3\|(\Sigma_1 \times \mathbb{S}^1)_{\overline{\gamma}_n}\|_0 \le v_3\|(M_{\overline{\gamma}_n})\|_0 = \mathrm{Vol}(M_{\overline{\gamma}_n}),$$

where the last equality comes again from [20, Lemma 6.5.4].

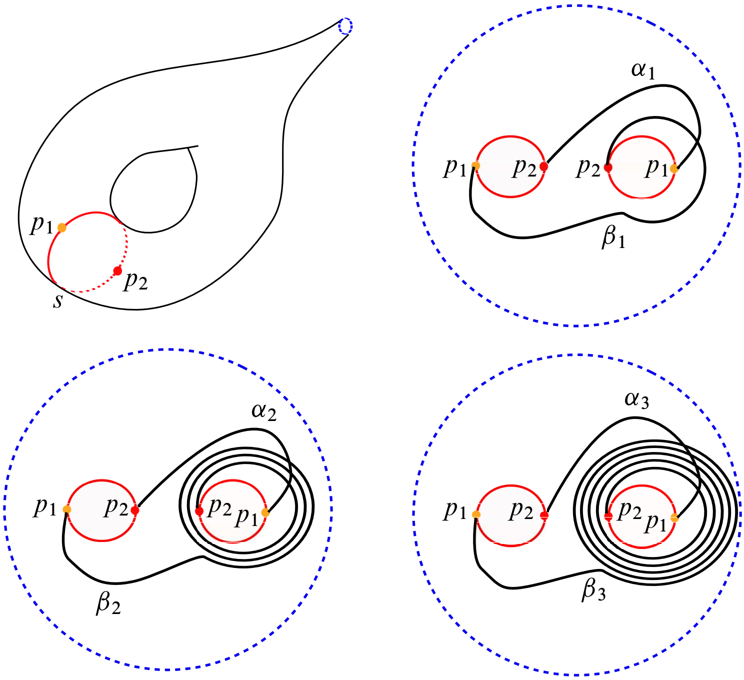

Figure 5: Top: s on $\Sigma_{1,1}$, left, and γ_1, right. Bottom: γ_2, left, and γ_3, right.

Notice that $(\Sigma_1 \times \mathbb{S}^1)_{\overline{\gamma}_n}$ is not hyperbolic, however it contains an hyperbolic piece given by $N_\varepsilon(S_1) \setminus \overline{\gamma}_n$. Since $N_\varepsilon(S_1) \setminus \overline{\gamma}_n$ is $(\Sigma_1 \times \mathbb{S}^1)_{\overline{\gamma}_n}$ split along an essential torus, by [20, Proposition 6.5.2] we have

$$\mathrm{Vol}(N_\varepsilon(S_1) \setminus \overline{\gamma}_n) = v_3 \| N_\varepsilon(S_1) \setminus \overline{\gamma}_n \|_0 \leq v_3 \| (\Sigma_1 \times \mathbb{S}^1)_{\overline{\gamma}_n} \|_0.$$

Furthermore, the projection of $\overline{\gamma}_n$ has a weakly twist-reduced, weakly generalized alternating diagram on a generalized projection surface (see [13, Section 2]) S_1 in

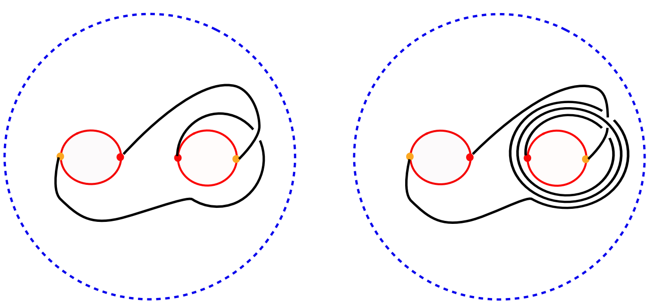

Figure 6: An alternating diagram for γ_1, left, and γ_2, right.

$N_\varepsilon(S_1)$. Since $N_\varepsilon(S_1)\setminus N_{\varepsilon/2}(S_1)$ is atoroidal and ∂–anannular,[1] $N_\varepsilon(S_1)$ is boundary incompressible in $N_\varepsilon(S_1)\setminus N_{\varepsilon/2}(S_1)$ and γ_n is filling in Σ_1, apply [13, Theorem 1.1] to get

$$\tfrac{1}{2}v_8(\mathrm{tw}(\gamma_n)-\chi(\Sigma_1)) \le \mathrm{Vol}(N_\varepsilon(S_1)\setminus \overline{\gamma}_n),$$

where tw is the number of twisting regions of the link diagram [13, Definition 6.4]. In the case of closed geodesics in minimal positions we do not have bigons in its diagram. Therefore, tw is equivalent to the self-intersection number of the corresponding geodesic.

To generalize this result to any hyperbolic surface $\Sigma_{g,n}$, notice that:

(1) The number of connected components of the sequence of $\{\Sigma_{1,1}\setminus\gamma_k\}_{k\in\mathbb{N}}$ tends to infinity. Then the previously constructed sequence has a subsequence $\{\gamma_k\}$ such that $\Sigma_{1,1}\setminus\gamma_k$ has more than n connected components. Then, by removing one puncture in n simply connected components of $\Sigma_{1,1}\setminus\gamma_k$, we can think of $\{\gamma_k\}$ as in $\Sigma_{1,n}$.

(2) It is a straightforward exercise to show that any projection of a link on the 2–sphere can be made alternating by changing crossings. Then any closed geodesic in $\Sigma_{0,n}$ admits an alternating diagram.

Let α_1 and α_2 be filling closed geodesics on $\Sigma_{g,1}$ and $\Sigma_{1,2}$, respectively. Let $\alpha_1 \,\overline{\star}\, \alpha_2$ be the closed geodesic homotopic to a closed curve obtained by surgering α_1 and α_2 along a simple arc transversely meeting one boundary component in each surface; see [16, Section 4.2]. To prove the $\Sigma_{g,1}$ case with $g\ge 2$, we can proceed by induction on the genus, using the following claim:

Claim *Let α_1 and α_2 be filling closed geodesics admitting an alternating diagram on $\Sigma_{g,1}$ and $\Sigma_{1,2}$, respectively. Then $\alpha_1 \,\overline{\star}\, \alpha_2$ is filling, and admits an alternating diagram on $\Sigma_{g+1,1}=\Sigma_{g,1}\cup_\partial \Sigma_{1,2}$.*

Proof of claim The filling property is proven in [16, Claim 4.13], and the existence of an alternating diagram follows from fixing an alternating diagram on each α_1 and α_2.

If, after connecting both geodesics, the corresponding diagram is not alternating (see Figure 7), then changing the crossing orientation of all crossings in one of the subarcs α_i makes the diagram of the geodesic corresponding to $\alpha_1 \,\overline{\star}\, \alpha_2$ alternating. □

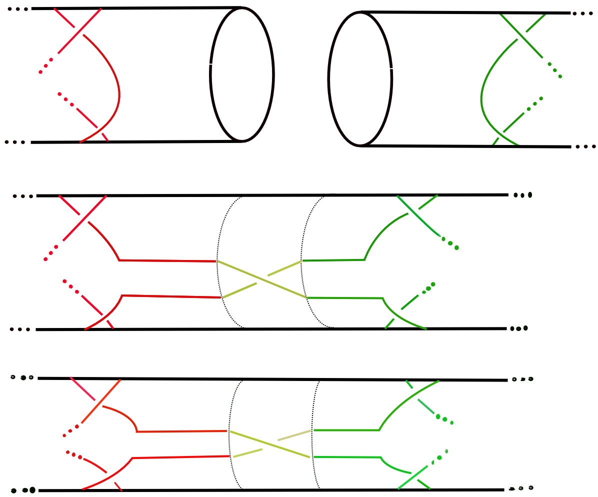

Figure 7: Top: α_1 and α_2 in a neighborhood of the glued boundary component. Middle: the induced projection diagram of $\alpha_1 \mathbin{\overline{\star}} \alpha_2$ around a neighborhood of the glued boundary component. Bottom: changing to the opposite crossing projection on one of the α_2 subarcs (green) to obtain an alternating diagram.

Finally, to find the sequences of geodesics for general hyperbolic surface $\Sigma_{g,n}$, we use the analogue of the argument used for the case of $\Sigma_{1,n}$ in (1) so that we could add or remove punctures. □

Remark 2 Not every closed geodesic on a surface of genus greater than or equal to 1 admits an alternating diagram; see Figure 8, center. Even though, for each hyperbolic surface, one can find an infinite number of distinct types of closed geodesics which admit an alternating diagram; see Figure 8, right.

We show now a general volume upper bound for any lift complement on Seifert-fibered spaces over a filling geodesic multicurve:

Theorem 1.5 *Let M be a Seifert-fibered space over a hyperbolic 2–orbifold $\mathcal{O}$. Then, for any link $\overline{\gamma} \subset M$ projecting injectively to a filling geodesic multicurve γ on $\mathcal{O}$,*

$$\mathrm{Vol}(M_{\overline{\gamma}}) < 8v_3 i(\gamma,\gamma),$$

where v_3 is the volume of the regular ideal tetrahedron and $i(\gamma,\gamma)$ the self-intersection number of γ.

[1] This means that it has no essential annulus whose boundary is not contained in the boundary components isotopic to the removed fiber.

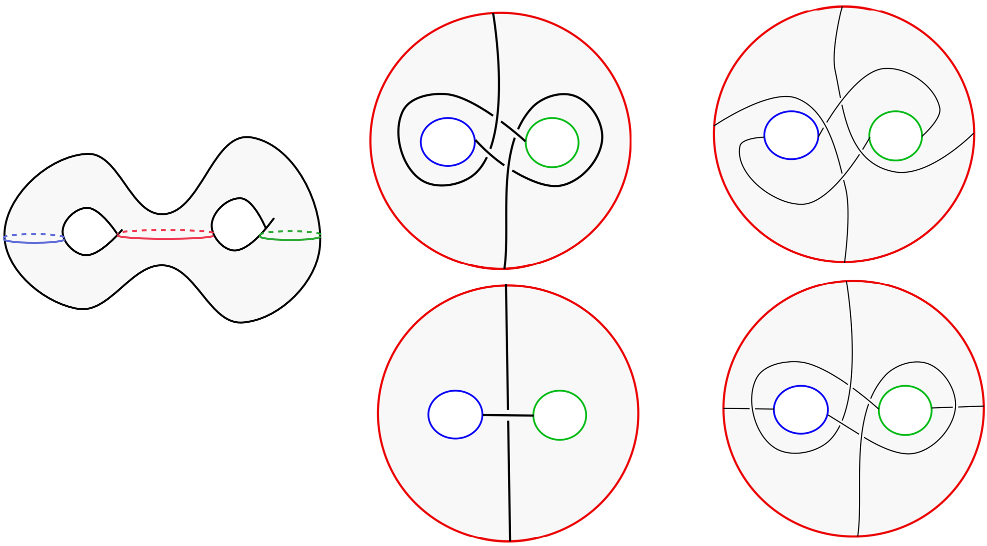

Figure 8: Left: pants decomposition on Σ_2. Center: closed geodesic not admitting an alternating diagram. Right: closed geodesic with an alternating diagram.

Proof The idea is to build a hyperbolic link L_γ inside M that reduces the *complexity* of $\overline{\gamma}$, in the sense that $M_{\overline{\gamma}}$ is obtained by performing Dehn filling along some components of L_γ. Since Dehn filling does not increase the volume [20, Theorem 6.5.6], and the number of tetrahedra in any ideal tetrahedral decomposition of a finite-volume hyperbolic manifold is an upper bound for its volume [20, Theorem 6.1.7], we have that

$$\mathrm{Vol}(M_{\overline{\gamma}}) \leq \mathrm{Vol}(M \setminus L_\gamma) \leq v_3 \,\sharp \mathcal{T}_{L_\gamma},$$

where $\mathcal{T}_{L_\gamma}$ is a decomposition of $M \setminus L_\gamma$ into ideal tetrahedra. That is, the vertices corresponds to the cusps of $M \setminus L_\gamma$. After constructing the link L_γ, we will argue that there exist $\mathcal{T}_{L_\gamma}$ with the number of tetrahedra comparable to the self-intersection number of γ.

Let $\mathcal{F}$ be the collection of fibers of M projecting under p to conical points of $\mathcal{O}$. For every simply connected region D of $\mathcal{O} \setminus \gamma$ not containing a conical point, we pick a regular fiber F_D whose projection lies in D, and call this collection of fibers $\mathcal{D}$. Let us denote by N the Seifert-fibered space obtained by removing $\mathcal{F} \cup \mathcal{D}$ from M. Since N has no singular fibers, let Σ be the Seifert surface of N. Note that Σ is homeomorphic to $\mathcal{O}$ minus the set of conical points and minus one point for each simply connected

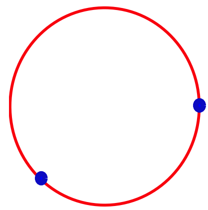

Figure 9: Triangulation of the fibers coming from self-intersection points of γ.

component of $\mathcal{O}\setminus\gamma$. Then we define $L_\gamma \doteq \overline{\gamma}\cup\mathcal{F}\cup\mathcal{D}$. By construction, $N_{\overline{\gamma}} = M\setminus L_\gamma$, and by Theorem 1.3 it admits a finite-volume hyperbolic structure.

To give a decomposition of $N_{\overline{\gamma}}$ into ideal tetrahedra, we start by taking a pair of ideal vertices in each fiber that project to a self-intersection point of γ and which connect the two points on $\overline{\gamma}$; see Figure 9. Moreover, let G_γ be the 4–valent graph induced by γ on Σ, and for α an edge of G_γ, let $A_\alpha \cong \mathbb{S}^1\times I$ be the preimage under $p|_N$.

By triangulating each annulus A_α, we extend this graph to an ideal triangulation of the CW–complex $p|_N^{-1}(\gamma) = \bigcup_{\alpha\in E(G_\gamma)} A_\alpha$. We do this by adding an ideal edge, which is an embedded arc connecting the other vertices in each boundary fiber that do not intersect the embedded $\overline{\gamma}$–arc in the annulus (up to isotopy this arc is unique) and then collapsing the $\overline{\gamma}$–arc in the annulus to a point. This induces an ideal triangular decomposition of each annulus by two ideal triangles; see Figure 10.

Let $H\subset N$ be a regular neighborhood of the triangulated CW–complex $p^{-1}|_N(\gamma)$. Then, H has a natural prism decomposition induced by the ideal triangulation of $p^{-1}|_N(\gamma)$, where some vertices correspond to $\overline{\gamma}$. Since γ fills Σ and we added a puncture in every complementary disk region, we have that N is homeomorphic to the interior of H. Moreover, the prism-decomposition of H induces a triangulation of ∂H, which are tori corresponding to fibers of punctures of Σ. Therefore, by collapsing

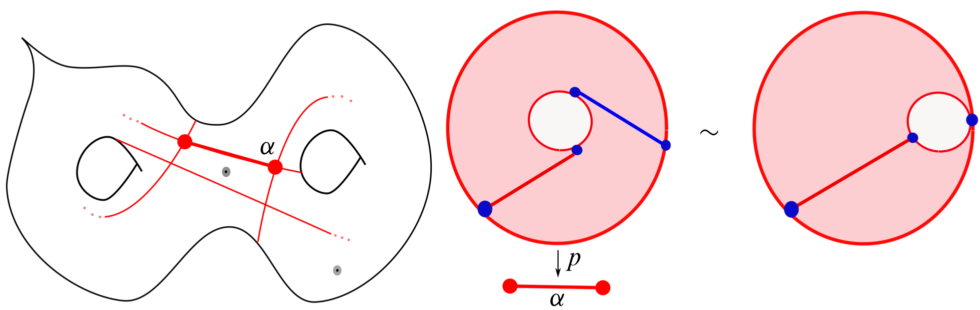

Figure 10: Triangular ideal decomposition of $p_{|N}^{-1}(\alpha)$, where the $\overline{\gamma}$–arc is in blue.

the boundary components of H to an ideal vertex, we obtain an ideal triangulation of $N_{\overline{\gamma}}$, because the ideal vertices of our ideal triangulation project precisely to the cusps of $N_{\overline{\gamma}}$.

Finally, the number of ideal tetrahedra used in this triangulation is four times the number of edges in the graph associated to γ. The number of edges is at most two times the self-intersection number of γ. Hence, we have at most eight ideal tetrahedra for each self-intersection point of γ. □

As a corollary of Theorems 4.4 and 1.5, and the fact that Seifert-fibered spaces over punctured surfaces are homeomorphic to trivial circle bundles, we obtain:

Corollary 1.6 *Let $\Sigma_{g,n}$ be an n–punctured hyperbolic surface with $n \geq 1$. Then there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of filling closed geodesics with $i(\gamma_n, \gamma_n) \nearrow \infty$, and respective lifts $\{\overline{\gamma}_n\}_{n\in\mathbb{N}}$ in $PT^1(\Sigma_{g,n})$, such that*

$$\tfrac{1}{2}v_8(i(\gamma_n,\gamma_n)-(2-2g)) \leq \mathrm{Vol}(M_{\overline{\gamma}_n}) < 8v_3 i(\gamma_n,\gamma_n),$$

where v_3 (resp. v_8) is the volume of the regular ideal tetrahedron (resp. octahedron), and $i(\gamma_n,\gamma_n)$ is the self-intersection number of γ_n.

Similarly to [16], given any geodesic multicurve γ and any continuous lift $\overline{\gamma}$, one has a combinatorial lower bound for the volume of $M_{\overline{\gamma}}$. Recall that a pants decomposition on an orbifold $\mathcal{O}$ is a maximal family of disjoint simple closed geodesics on the underlying topological surface $\Sigma_{\mathcal{O}}$ which do not intersect the singular points of $\mathcal{O}$. We will show:

Theorem 1.7 *Given a pants decomposition Π on a hyperbolic 2–orbifold $\mathcal{O}$, a Seifert-fibered space M over $\mathcal{O}$, and a filling geodesic multicurve γ on $\mathcal{O}$, for any closed continuous lift $\overline{\gamma}$ we have that*

$$\tfrac{1}{2}v_3\sum_{P\in\Pi}\big(\sharp\{\textit{isotopy classes of } \overline{\gamma}\textit{–arcs in } p^{-1}(P)\}-3\big) \leq \mathrm{Vol}(M_{\overline{\gamma}}),$$

where v_3 is the volume of the regular ideal tetrahedron.

Given a pair of pants P, we say that two arcs

$$\overline{\alpha},\overline{\beta}\colon [0,1]\to p^{-1}(P)$$

with $\overline{\alpha}(\{0,1\})\cup\overline{\beta}(\{0,1\})\subset\partial(p^{-1}(P))$ are in the same isotopy class in $p^{-1}(P)$ if there exists an isotopy $h\colon [0,1]_1\times[0,1]_2\to p^{-1}(P)$ such that

$$h_0(t_2)=\overline{\alpha}(t_2),\quad h_1(t_2)=\overline{\beta}(t_2)\quad\text{and}\quad h([0,1]_1\times\{0,1\})\subset\partial(p^{-1}(P)).$$

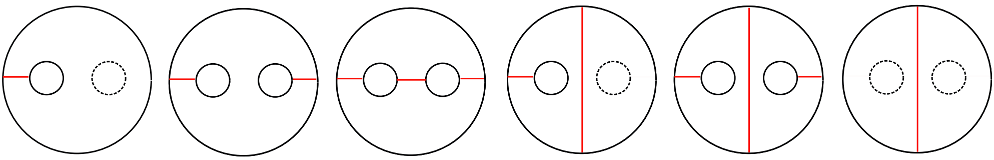

Figure 11: The projection on P of the only six $\overline{\gamma}$–arcs configuration, up to isotopy, whose $\overline{\gamma}$–arcs project to pairwise disjoint simple arcs in P.

Remark 3 Up to isotopy, for a family of simple arcs without intersection there are only six configurations of arcs in P. These are shown in Figure 11. The 3 in the lower bound of Theorem 1.7 comes from the fact that there are at most 3 isotopy classes of $\overline{\gamma}$–arcs on $p^{-1}(P)$ projecting to such a configuration.

Before stating the main result needed to prove Theorem 1.7, we recall some definitions.

If N is a hyperbolic 3–manifold and $S \subset N$ is an embedded incompressible surface, we will use $N|S$ to denote the manifold obtained from N by cutting along S. The manifold $N|S$ is homeomorphic to the complement in N of an open regular neighborhood of S. If one takes two copies of $N|S$, and glues them along their boundary by using the identity diffeomorphism, one obtains the double of $N|S$, which we denote by $D(N|S)$.

Definition 4.5 Let $p\colon N \to \mathcal{O}$ be a Seifert-fibered space. Let P be a pair of pants belonging to a pant decomposition of a orbifold $\mathcal{O}$, and let γ be a closed geodesic in $\mathcal{O}$ that is not isotopic into P. Moreover, assume that $P \cap \gamma$ is a finite set of geodesic arcs $\{\alpha_i\}_{i=1}^{n_P}$ connecting boundary components of P. We define $P_{\overline{\gamma}}$ to be the set

$$p^{-1}(P) \setminus \bigcup_{i=1}^{n_P} \overline{\alpha}_i.$$

We also define $D(P_{\overline{\gamma}})$ as the gluing, via the identity homeomorphism, of two copies of $P_{\overline{\gamma}}$ along the punctured tori coming from

$$\partial(p^{-1}(P)) \setminus \bigcup_{i=1}^{n_P} \overline{\alpha}_i.$$

Moreover, $D(P_{\overline{\gamma}})$ is a link complement in the Seifert-fibered space $D(p^{-1}(P))$, described as

$$D(p^{-1}(P)) \setminus \bigcup_{i=1}^{n_P} D(\overline{\alpha}_i),$$

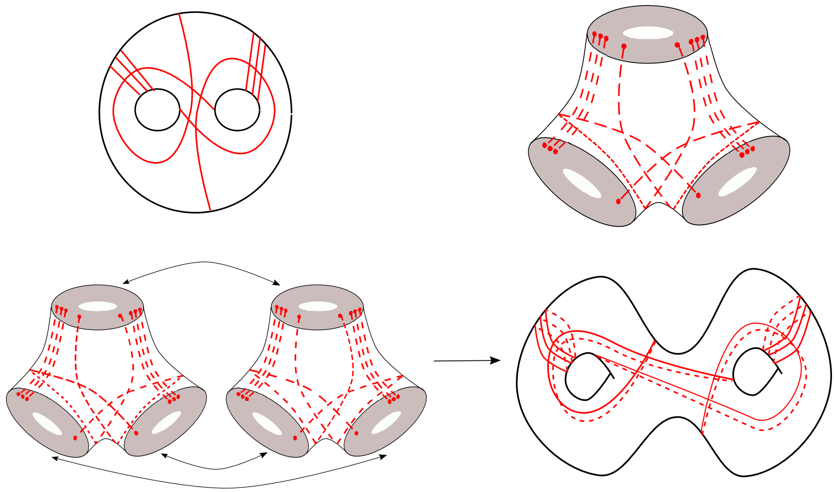

Figure 12: Top: A pair of pants and a set of geodesic arcs connecting the boundary, left, and the associated $P_{\overline{\gamma}}$, right. Bottom: $D(P_{\overline{\gamma}})$ with the induced projection to S^0.

where the projection orbifold of $D(p^{-1}(P))$, whose underlying surface will be denoted by S^0, is one of the following:

(1) a genus two surface if $\sharp(\partial(\Sigma_{\mathcal{O}}) \cap \partial P) = 0$;

(2) a surface of type $(1,2)$ if $\sharp(\partial(\Sigma_{\mathcal{O}}) \cap \partial P) = 1$;[2]

(3) a surface of type $(0,4)$ if $\sharp(\partial(\Sigma_{\mathcal{O}}) \cap \partial P) = 2$.

Each $D(\overline{\alpha}_i)$ is a knot in $D(p^{-1}(P))$ obtained by gluing $\overline{\alpha}_i$ along the two points $\partial(p^{-1}(P)) \cap \overline{\alpha}_i$ via the identity. See Figure 12.

The key ingredient to proving Theorem 1.7 is the following result due to Agol, Storm and Thurston; see [2, Theorem 9.1].

Theorem (Agol–Storm–Thurston) *Let N be a compact manifold with interior a hyperbolic 3–manifold of finite volume. Let S be a properly embedded incompressible surface in N. Then*

$$\tfrac{1}{2} v_3 \|D(N|S)\| \leq \mathrm{Vol}(N).$$

We now prove the lower bound for the volume of the canonical lift complement.

Proof of Theorem 1.7 Let $\{\eta_i\}_{i=1}^{3g+n-3}$ be the simple closed geodesics inducing the pants decomposition Π. Consider the incompressible surface $S \doteq \bigsqcup_{i=1}^{3g+n-3} (T_{\eta_i})_{\overline{\gamma}}$

[2] By a surface of type (n,m), we mean a genus n surface with m punctures.

in $M_{\overline{\gamma}}$, where $(T_{\eta_i})_{\overline{\gamma}}$ is the incompressible punctured torus defined by the full preimage $p^{-1}(\eta_i))\backslash(p^{-1}(\eta_i)\cap\overline{\gamma})$ of η_i in $M_{\overline{\gamma}}$; see [16, Lemma 2.5]. From [2, Theorem 9.1] we deduce that

$$\tfrac{1}{2}v_3 \sum_{P\in\Pi} \|D(P_{\overline{\gamma}})\|_0 = \tfrac{1}{2}v_3\|D(M_{\overline{\gamma}}|S)\|_0 \le \mathrm{Vol}(M_{\overline{\gamma}}).$$

For each pair of pants P, we have

$$v_3\,\sharp\{\text{cusps of } D(P_{\overline{\gamma}})^{\mathrm{hyp}}\} \le \mathrm{Vol}(D(P_{\overline{\gamma}})^{\mathrm{hyp}}) \le v_3\|D(P_{\overline{\gamma}})^{\mathrm{hyp}}\|_0 = v_3\|D(P_{\overline{\gamma}})\|_0,$$

where $D(P_{\overline{\gamma}})^{\mathrm{hyp}}$ is the atoroidal piece of $D(P_{\overline{\gamma}})$, ie the complement of the characteristic submanifold, with respect to its JSJ–decomposition. The first and second inequalities come from [1] and [10], respectively.

Let Ω be the subset of γ–arcs on P having one arc for each isotopy class of $\overline{\gamma}$–arcs on $p^{-1}(P)$. This means that $D(P_{\overline{\gamma}})^{\mathrm{hyp}} \cong D(P_{\overline{\Omega}})^{\mathrm{hyp}}$. Moreover, $D(P_{\overline{\Omega}})$ can be seen as a link complement in $D(p^{-1}(P))$ — see Definition 4.5 — whose projection to S^0 is a union of closed loops transversally homotopic to a union of closed loops in minimal position. The atoroidal piece of $D(P_{\overline{\Omega}})$ corresponds to the subsurface of S^0 which $D(\Omega)$ fills (Theorem 1.3).

(1) If the Ω–arc configuration on P is in the list of Remark 3, then by Theorem 1.3 we have that $D(P_{\overline{\gamma}})^{\mathrm{hyp}} = \varnothing$, and Remark 3 also gives us

$$v_3\big(\sharp\{\text{isotopy classes of } \overline{\gamma}\text{-arcs in } p^{-1}(P)\} - 3\big) \le v_3\,\sharp\{\text{cusps of } D(P_{\overline{\gamma}})^{\mathrm{hyp}}\}.$$

(2) If the Ω–arc configuration on P is not in the list of Remark 3, then there is at least one geometric intersection point on the projection of the link complement $D(P_{\overline{\Omega}})$ to S^0.

By Theorem 1.3 we conclude that $D(P_{\overline{\gamma}})^{\mathrm{hyp}} \ne \varnothing$. We will now define an injective function

$$\{\overline{\gamma}\text{–arcs in } p^{-1}(P)\} \xrightarrow{\varphi} \{\text{cusps of } D(P_{\overline{\gamma}})^{\mathrm{hyp}}\},$$

where the target can be decomposed as

$$\{\text{splitting tori of the JSJ–decomposition of } D(P_{\overline{\gamma}})\} \sqcup \{\text{cusp in } D(P_{\overline{\gamma}}) \cap D(P_{\overline{\gamma}})^{\mathrm{hyp}}\}.$$

The function φ is defined as follows: if the cusps in $D(P_{\overline{\gamma}})$ are induced by the $\overline{\gamma}$–arc in $p^{-1}(P)$ belonging to the characteristic submanifold of $D(P_{\overline{\gamma}})$, φ maps it to a splitting tori connecting the hyperbolic piece with the component of the characteristic submanifold where it is contained. Otherwise, the cusp belongs to $D(P_{\overline{\gamma}})^{\mathrm{hyp}}$ and

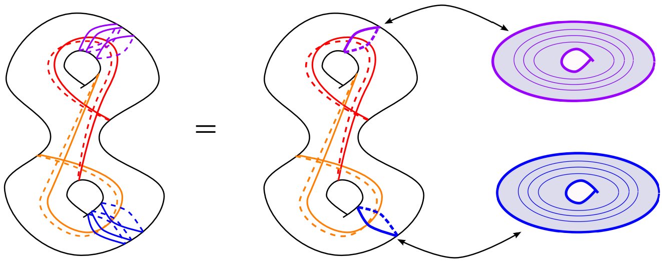

Figure 13: The JSJ–decomposition of $D(P_{\overline{\gamma}})$ of Figure 12.

φ sends it to itself; see Figure 13. Assume that there are more isotopy classes of $\overline{\gamma}$–arcs in $p^{-1}(P)$ than the number of cusps of $D(P_{\overline{\gamma}})^{\text{hyp}}$. Then there are two tori, associated with nonisotopic $\overline{\gamma}$–arcs in $p^{-1}(P)$, that belong to the same connected component of the characteristic submanifold. Since each component of the characteristic submanifold is a Seifert-fibered space over a punctured surface, we have that all such arcs correspond to regular fibers. Thus, they are isotopic in the corresponding component, hence isotopic in $p^{-1}(P)$, contradicting the fact that they were not isotopic. □

This result implies that there exists a filling geodesic multicurve γ on $\mathcal{O}$ with bounded components such that $\mathrm{Vol}(M_{\overline{\gamma}})$ can be as large as we want. Let us fix a pants decomposition on $\mathcal{O}$, then for any $N \in \mathbb{N}$ there exists a closed geodesic with at least N homotopy classes of geodesic arcs in one pair of pants. This is constructed by taking N nonhomotopic geodesic arcs in a pair of pants and linking them to form a filling geodesic multicurve on $\mathcal{O}$.

The lower bound of the volume of $M_{\overline{\gamma}}$ obtained in Theorem 1.7 does not have control on the length of the geodesic multicurve, even if each homotopy class of γ–arcs contributes to the length of γ.

Question 4.6 *Given a hyperbolic orbifold, estimate the volume of $M_{\overline{\gamma}}$ among the filling geodesic multicurves γ whose length is bounded by a fixed constant.*

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Received: 7 April 2019 Revised: 13 November 2019