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Shadows of acyclic 4–manifolds with sphere boundary

YUYA KODA
HIRONOBU NAOE

In terms of Turaev's shadows, we provide a sufficient condition for a compact, smooth, acyclic 4–manifold with boundary the 3–sphere to be diffeomorphic to the standard 4–ball. As a consequence, we prove that if a compact, smooth, acyclic 4–manifold with boundary the 3–sphere has shadow-complexity at most 2, then it is diffeomorphic to the standard 4–ball.

57N13; 57M20, 57R55, 57R65

Introduction

In [27; 28], Turaev introduced the notion of *shadow* as a combinatorial tool to present smooth 3– and 4–manifolds. A shadow of a 4–manifold M with boundary is a simple polyhedron X properly embedded in M so that M collapses onto X, and X is locally flat in M. The polyhedron X is also called a shadow of the 3–manifold ∂M. By counting the minimum number of vertices of a shadow of a given 4– or 3–manifold, we get a (nonnegative) integer-valued invariant, called the *shadow-complexity*.

In the 3–manifold topology, shadows are used to study quantum invariants; see for example Turaev [27; 28], Burri [1], Shumakovitch [25], Thurston [26] and Carrega and Martelli [2]. Moreover, it was revealed that the shadow-complexity of a 3–manifold M is strongly related to the Gromov norm and the minimum number of codimension-2 singular fibers of a stable map $M \to \mathbb{R}^2$; see Costantino and Thurston [9], Costantino, Frigerio, Martelli and Petronio [8] and Ishikawa and Koda [13].

In the dimension 4, shadows allow us to classify 4–manifolds experimentally according to increasing complexity. Costantino [5] studied closed 4–manifolds of shadow-complexity 0 or 1 in a special case. Here, a shadow of a closed 4–manifold is a shadow of the union of 0–, 1– and 2–handles of its handle decomposition. In [17], Martelli gave a complete classification of the closed 4–manifolds of shadow-complexity 0. A very interesting consequence of that paper is that a simply connected, closed 4–manifold has complexity zero if and only if it is a connected sum of copies of the standard S^4, $S^2 \times S^2$, $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$. This implies in particular that the shadow-complexity

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detects the exotic structures on those manifolds. Recently, the closed 4–manifolds of shadow-complexity 1 were also classified by Koda, Martelli and Naoe [15]. For the other studies of 4–manifolds using shadows see eg Costantino [3; 6; 7; 4], Martelli [16], Naoe [20; 21] and Ishikawa and Naoe [14].

In the present paper, we consider the following naive question:

Question Let M be an acyclic 4–dimensional 2–handlebody with boundary the 3–sphere. Then is M diffeomorphic to the standard 4–ball?

Here, recall that a compact, oriented 4–manifold is called a 2–handlebody if it is made of finitely many handles of index at most 2. Note that the manifold M in the above question is at least *homeomorphic* to the 4–ball. Indeed, it is easy to see that M is simply connected; thus, M is homeomorphic to the 4–ball by Freedman's classification theorem [10]. A negative answer to the above question implies the existence of an exotic 4–sphere. The following theorem gives an affirmative answer to the question when the (special) shadow-complexity of M is very small. Here, for the definition of the *special shadow-complexity*, see Section 1.

Theorem 0.1 (1) (Costantino [5]) *Every acyclic 4–manifold of special shadow-complexity 0 or 1 with boundary the 3–sphere is diffeomorphic to the standard 4–ball.*

(2) (Naoe [19]) *Every acyclic 4–manifold of shadow-complexity 0 is diffeomorphic to the standard 4–ball.*

In this paper, using shadows we provide a sufficient condition for a compact, smooth, acyclic 4–manifold with boundary the 3–sphere to be diffeomorphic to the standard 4–ball (Theorem 2.11). As a direct consequence, we show (in Theorem 2.1) that every acyclic 4–manifold M of shadow-complexity at most 2 with $\partial M \cong S^3$ is diffeomorphic to the standard 4–ball. Precisely speaking, we show the same thing for a wider class of 4–manifolds, that is, 4–manifolds of *connected shadow-complexity* at most 2. See Section 1 for the definition.

Throughout the paper, we will work in the smooth category unless otherwise mentioned.

1 Shadows

A compact and connected polyhedron X is called a *simple polyhedron* if every point of X has a star neighborhood homeomorphic to one of the five models shown in

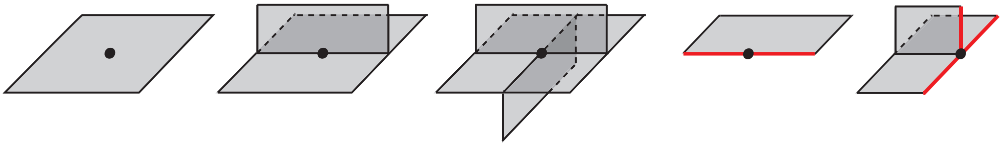


Figure 1: The local models of a simple polyhedron.

Figure 1. A point whose star neighborhood is shaped on the model (iii) is called a *vertex* of X, and we denote the set of vertices of X by $V(X)$. The set of points whose star neighborhoods are shaped on the models (ii), (iii) or (v) is called the *singular set* of X, and we denote it by $S(X)$. The set of points whose star neighborhoods are shaped on the models (iv) or (v) is called the *boundary* of X and we denote it by ∂X. Each component of $X \setminus S(X)$ is called a *region*, and we denote the set of regions of X by $R(X)$. The number of vertices of X is called the *complexity* of X. In [15], the *connected complexity* of X was defined to be the maximum number of vertices that are contained in some connected component of $S(X)$. A simple polyhedron X is said to be *closed* if $\partial X = \varnothing$. A simple polyhedron X is said to be *special* if each region of X is an open disk. We note that if X is special, X is closed and $S(X)$ is connected.

Definition A simple polyhedron X embedded in a compact oriented smooth 4–manifold M is called a *shadow* of M if

- M collapses onto X after equipping the natural PL structure on M;
- X is *locally flat*, that is, each point x of X has a neighborhood $\mathrm{Nbd}(x; X)$ that lies in a 3–dimensional submanifold of M; and
- $\partial M \cap X = \partial X$.

Note that ∂X is a *knotted trivalent graph*, ie a smooth graph in ∂M with only vertices of valence 3, where we admits knot components as well. For $k \in \{0, 1, 2, 3\}$, a k–*handlebody* is defined to be an oriented 4–manifold made of finitely many handles of index at most k. In [27; 28], Turaev proved that any 2–handlebody has a (special) shadow. In [3; 9], the *shadow-complexity* (resp. *special shadow-complexity*) of a 2–handlebody M, denoted by $\mathrm{sc}(M)$ (resp. $\mathrm{sc}^{\mathrm{sp}}(M)$), was defined to be the minimum complexity of any shadow (resp. special shadow) of M. In [15], the *connected shadow-complexity* of M, denoted by $\mathrm{sc}^*(M)$, was defined to be the minimum connected complexity of any shadow of M. Note that the shadow-complexity of M is 0 if and only if the connected shadow-complexity of M is 0. In general, $\mathrm{sc}^*(M) \le \mathrm{sc}(M) \le \mathrm{sc}^{\mathrm{sp}}(M)$.

A *framed knotted trivalent graph* is a knotted trivalent graph equipped with a framing, ie an orientable surface thickening of the graph considered up to isotopy. Let M be a compact oriented smooth 4–manifold, and let $X \subset M$ be a shadow. Fix a framing of the knotted trivalent graph $\Gamma := \partial X$. To each region R of X, we may assign a half-integer $\mathrm{gl}(R)$, called a *gleam*, as follows. Let $\iota\colon R \hookrightarrow M$ be the inclusion. Let $\overline{R}$ be the metric completion of R with the path metric inherited from a Riemannian metric on R. Suppose for simplicity that the natural extension $\bar{\iota}\colon \overline{R} \to M$ is injective. The boundary $\partial\overline{R}$ of $\overline{R}$ consists of simple closed curves. The framing of Γ and the germs $\mathrm{Nbd}(\overline{R}; X) \setminus R$ of the remaining regions near $\partial\overline{R}$ provide the structure of an interval bundle over $\partial\overline{R}$, which is a subbundle of the normal bundle of $\partial\overline{R}$ in M. Let $\overline{R}'$ be a generic small perturbation of $\overline{R}$ such that $\partial\overline{R}'$ lies in the interval bundle. The gleam $\mathrm{gl}(R)$ is then (well) defined by counting the finitely many isolated intersections of $\overline{R}$ and $\overline{R}'$ with signs as follows:

$$\mathrm{gl}(R) = \tfrac{1}{2}\,\#(\partial\overline{R} \cap \partial\overline{R}') + \#(\mathrm{Int}\,\overline{R} \cap \mathrm{Int}\,\overline{R}') \in \tfrac{1}{2}\mathbb{Z}.$$

We call a polyhedron X equipped with a gleam on each region a *shadowed polyhedron*. In [27; 28], Turaev showed that the 4–manifold M and the framed knotted trivalent graph $\Gamma \subset \partial M$ are recovered from a shadowed polyhedron (X, gl) in a canonical way.

Let K_i be a framed oriented knot in the boundary of a compact oriented 4–manifold M_i for $i = 1, 2$. Let B_i be a 3–ball in ∂M_i such that $B_i \cap K_i$ is a properly embedded trivial arc in B_i for $i = 1, 2$. Let $g\colon (B_2, B_2 \cap K_2) \to (B_1, B_1 \cap K_1)$ be a diffeomorphism such that

- $g\colon B_2 \to B_1$ is orientation-reversing;
- $g|_{B_2 \cap K_2}\colon B_2 \cap K_2 \to B_1 \cap K_1$ is orientation-reversing; and
- g respects (the corresponding parts of) the framings.

We denote the framed knot $K := (K_1 \setminus \mathrm{Int}\, B_1) \cup_{g|_{K_2 \cap \partial B_2}} (K_2 \setminus \mathrm{Int}\, B_2)$ in $\partial(M_1 \natural M_2) = \partial M_1 \# \partial M_2$ by $K_1 \# K_2$, and call it the *connected sum* of K_1 and K_2. The following two lemmas are straightforward from the definition:

Lemma 1.1 *Let K_i be a framed oriented knot in the boundary of a compact oriented 4–manifold M_i for $i = 1, 2$. Let $f\colon \mathrm{Nbd}(K_2; \partial M_2) \to \mathrm{Nbd}(K_1; \partial M_1)$ be an orientation-reversing diffeomorphism such that $f|_{K_2}\colon K_2 \to K_1$ is orientation-reversing and f respects the framings. Then the 4–manifold $M_1 \cup_f M_2$ is obtained from $M_1 \natural M_2$ by attaching a 2–handle along the framed knot $K_1 \# K_2$.*

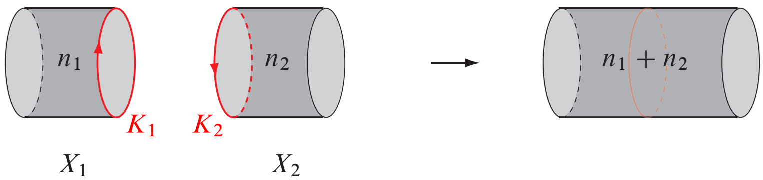


Figure 2: This move gives a gluing formula for shadows.

Lemma 1.2 *Let X_i be a shadowed polyhedron of a compact 4–manifold M_i for $i = 1, 2$. Let K_i be a (framed) knot component of ∂X_i. Fix an orientation of each of K_1 and K_2. Let $f\colon \mathrm{Nbd}(K_2; \partial M_2) \to \mathrm{Nbd}(K_1; \partial M_1)$ be a diffeomorphism such that $f|_{K_2}\colon K_2 \to K_1$ is orientation-reversing and f respects the framings. Then the shadowed polyhedron obtained by the move shown in Figure 2 is a shadow of $M_1 \cup_f M_2$.*

Let X be a simple polyhedron. In general, a polyhedron obtained by collapsing X might be no longer a simple polyhedron but an *almost-simple polyhedron*, ie a compact polyhedron where the link of each point can be embedded into the complete graph Γ_4 with 4 vertices; see Matveev [18] for the details. A point of an almost-simple polyhedron is called a *true vertex* if its link is Γ_4, or, equivalently, the star neighborhood of the point is shaped on the model of Figure 1(iii). An almost-simple polyhedron is said to be *minimal with respect to collapsing* if it cannot be collapsed onto any proper subpolyhedron. Up to a small perturbation, each point of such a polyhedron has a star neighborhood of one of Figures 1(i)–(iii) and 3(i)–(iv). Note that a simple polyhedron is minimal if and only if it is closed.

Lemma 1.3 *Let M be a 4–manifold of shadow-complexity (resp. connected shadow-complexity) $n \geq 1$. Then M admits a closed shadow of complexity (resp. connected complexity) exactly n.*

Proof Let M be a 4–manifold of shadow-complexity $n \geq 1$. Let X be a shadow of M with exactly n vertices. Then X collapses onto an almost-simple polyhedron Y

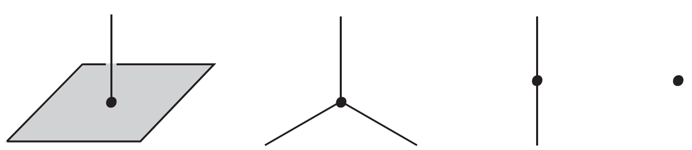


Figure 3: Local models of an almost-simple polyhedron.

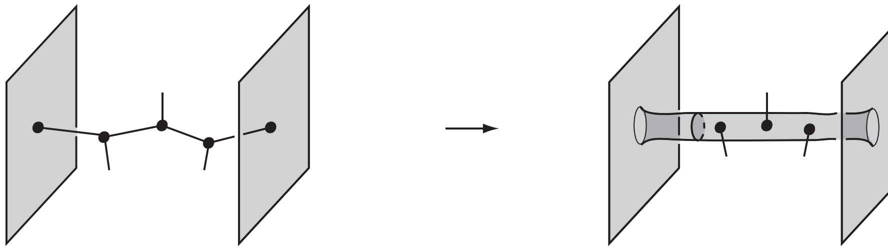


Figure 4: This move reduces the number of edges of the graph part, and does not produce true vertices.

that is minimal with respect to collapsing and has at most n true vertices. If Y remains a simple polyhedron, there is nothing to prove. If Y is a graph, ie a 0– or 1–dimensional complex, then M is a 1–handlebody, which contradicts the assumption that the shadow-complexity n of M is at least 1. Suppose that Y is not a graph. Then Y is the union of a simple polyhedron Y' and a graph Γ.

If there exists a path, ie a subgraph homeomorphic to $[0,1]$, in Γ that connects two different points of Y', we apply the move shown in Figure 4. Otherwise, there exists a *racket*, ie a subgraph homeomorphic to

$$\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}\cup\{(x,0)\in\mathbb{R}^2 \mid 1\le x\le 2\},$$

in Y' with the (unique) univalent vertex on Y'. In this case, we apply the move shown in Figure 5. Note that the polyhedra before and after each of the above two moves have the same regular neighborhood in M. Further, the resulting polyhedron remains minimal with respect to collapsing, and the number of true vertices does not increase by each of the moves. Since the number of edges of Γ is finite, by applying these moves finitely many times we finally end with a closed shadow Z of M with at most

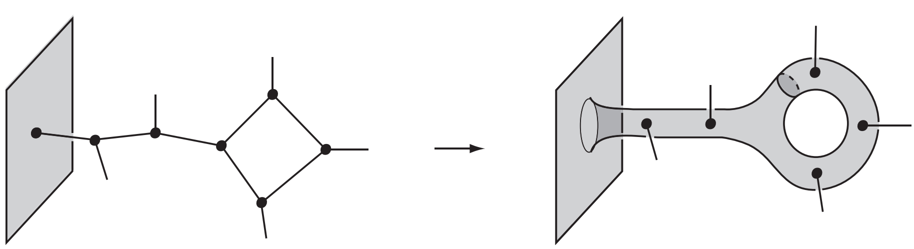


Figure 5: This move reduces the number of edges of the graph part, and does not produce true vertices.

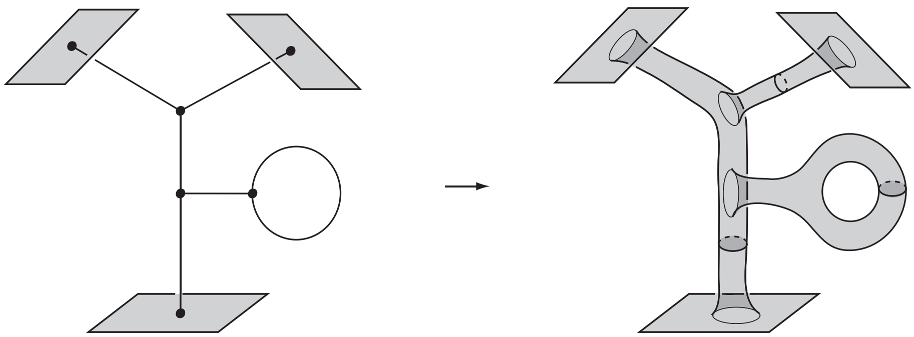


Figure 6: From an almost-simple polyhedron to a simple polyhedron.

n vertices (see Figure 6). Since the shadow-complexity of M is n, Z (and so Y') has exactly n (true) vertices.

The argument for the connected shadow-complexity runs exactly the same way. □

Remark It is easy to see that the gleams of the small disks region produced by the moves in Figures 4 and 5 are zero. (We do not use this fact in this paper.)

1.1 Diagrams of special polyhedra

Let X be a special polyhedron having at least one vertex. Recall that the singular part $S(X)$ of X is a connected (possibly nonsimple) 4–regular graph. Let $\{v_1, \dots, v_k\}$ be the set of vertices and $\{e_1, \dots, e_{2k}\}$ the set of edges of $S(X)$. Set $X' := \mathrm{Nbd}(S(X); X)$. Note that the closure of $X \setminus X'$ consists of disks, hence the topological type of X is uniquely recovered from that of X'. At each vertex v_i, choose a neighborhood $V_i := \mathrm{Nbd}(v_i; X')$ homeomorphic to Figure 1(iii) so that each component of the closure of $X \setminus \bigcup_{i=1}^k V_i$ is homeomorphic to $Y \times [0,1]$, where Y is the cone over 3 points. Let E_j be the component of $X \setminus \bigcup_{i=1}^k V_i$ corresponding to the edge e_j. We call each of V_i and E_j a *block* of X'.

The diagram of X is obtained as follows. Draw a diagram (with only normal crossings) of the graph $S(X)$ on $\mathbb{R}^2$. In the diagram, replace each vertex v_i of $S(X)$ with the local diagram (which describes the block V_i) shown in Figure 7, left. Replace each edge e_j of $S(X)$ in the diagram with one of the four local diagrams (which describes the block E_j) shown in Figure 7, right, so that the gluing of the end of the strands matches the combinatorial structure of X'. Apparently, one simple polyhedron admits many diagrams, but each diagram defines a unique special polyhedron.

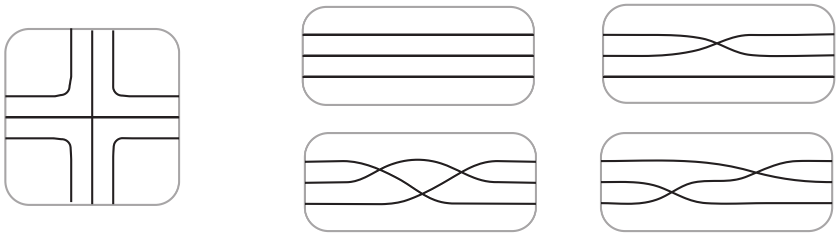


Figure 7: Diagrams corresponding to blocks.

The above decomposition of a neighborhood of the singular part of a special polyhedron into blocks allows us to enumerate all special polyhedron with a given number of vertices systematically. The tables in Appendix B list all the special polyhedra with 1 or 2 vertices. Note that the special polyhedron n^i_j in Appendix B has n vertices and i regions. Table B.1 was already given in [15] with different names for the polyhedra. The special polyhedra 1^1_1, 1^1_2, 1^2_1, 1^2_2, 1^2_3, 1^2_4, 1^2_5, 1^3_1, 1^3_2, 1^3_3, 1^4_1 in this paper are $X_1, X_2, \dots, X_{11}$ in [15], respectively.

1.2 From special shadows to Kirby diagrams

Let X be a special shadow of a 4–manifold M. We can construct a Kirby diagram of M as follows. Draw a diagram of X as explained in Section 1.1. The diagram can be regarded as immersed circles on $\mathbb{R}^2$ with normal crossings. At each crossing, choose over/under crossings in an arbitrary way. Choose a maximal tree T of the graph $S(X)$. Encircle with a dotted circle the 3 strands of the diagram corresponding to each edge of $S(X)$ not contained in T.

Let X be a special polyhedron with m regions. For each region R of X, let $\iota_R\colon R \hookrightarrow M$ be the inclusion. Let $\overline{R}$ be the metric completion of R with the path metric inherited from a Riemannian metric on R. Let $\bar{\iota}_R\colon \overline{R} \to M$ be the natural extension of ι. Let T be a maximal tree in $S(X)$.

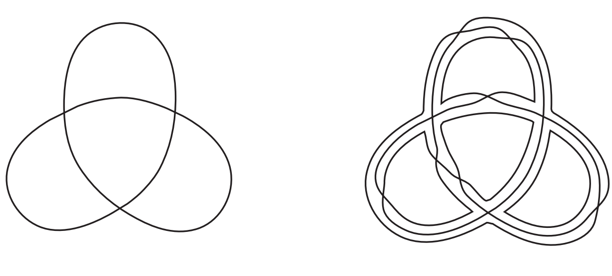


Figure 8: An example of a diagram of a special shadow.

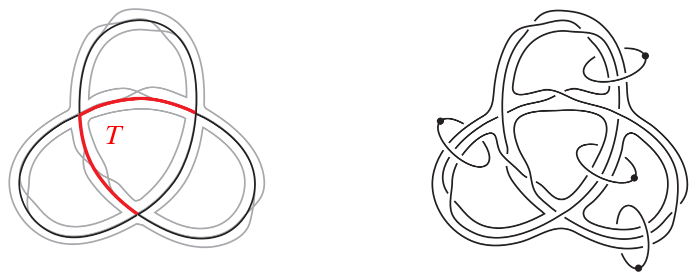


Figure 9: From a diagram of a shadow to a Kirby diagram.

Definition We say that X admits k $(\leq m)$ *canceling pairs* (with respect to T) if there exist

- an ordered subset $\{e_1, \dots, e_k\}$ of the edges of $S(X)$ not contained in T, and
- an ordered subset $\{R_1, \dots, R_k\}$ of the regions of X

such that for each $i \in \{1, \dots, k\}$, $\partial \overline{R}_i$ passes through e_i exactly once, and $\partial \overline{R}_i$ does not pass through e_j (for $i < j \leq k$).

Note that if a special polyhedron X admits k canceling pairs with respect to T then a Kirby diagram obtained from X by using T as above admits k canceling pairs of 1– and 2–handles (a dotted circle and a component of the framed link).

1.3 Graphs encoding simple polyhedra without vertices

Let Y be the cone over 3 points. We denote by Y_{111}, Y_{12} and Y_3 the three Y–bundles over S^1 such that $\#\partial Y_{111} = 3$, $\#\partial Y_{12} = 2$ and $\#\partial Y_3 = 1$.

Every simple polyhedron X whose singular set is a disjoint union of circles is decomposed into pieces each homeomorphic to a disk D^2, a pair of pants P, a Möbius band Y_2, Y_{111}, Y_{12} or Y_3. Such a decomposition induces a graph having one vertex for each piece or a component of ∂X as shown in Figure 10, and one edge for each circle along which X is decomposed. This graph is introduced by Martelli in [17] for the classification of closed 4–manifolds with shadow-complexity 0. Let G be a graph encoding X. We note that there is a natural (but not unique) embedding $G \hookrightarrow X$ such that G is a retract of X. In particular, we have the following:

Lemma 1.4 *Let G be a graph encoding a simple polyhedron X whose singular set is a disjoint union of circles. Then a natural embedding $G \hookrightarrow X$ induces an injection of the fundamental groups.*

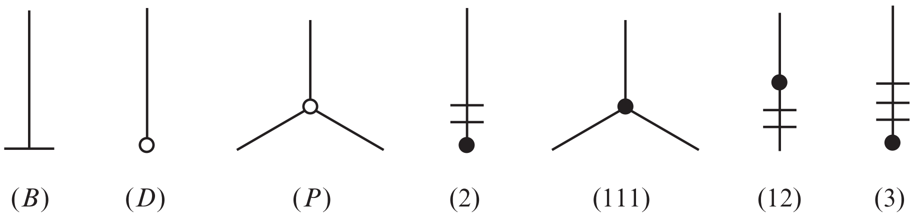


Figure 10: Vertices of a graph encoding the pieces B, D^2, P, Y_2, Y_{111}, Y_{12} and Y_3 of a simple polyhedron, where B is a component of the boundary.

As was mentioned in [17, Remark 2.3], the simple polyhedron X is recovered from a pair of a graph G and a cocycle in $H^1(G;\mathbb{Z}/2\mathbb{Z})$.

2 Main theorem

We prove the following:

Theorem 2.1 *Every acyclic 4–manifold of connected shadow-complexity at most 2 with boundary the 3–sphere is diffeomorphic to the standard 4–ball. In other words, there exists no acyclic 4–manifold of connected shadow-complexity 1 or 2 with boundary the 3–sphere.*

The key ingredients in the proof are the Property R theorem by Gabai [11] and detailed analyses of the combinatorial structures of acyclic polyhedra. In fact, we prove the same thing as above in a more general setting in Theorem 2.11. We note that the nature of acyclic polyhedra with at least one vertex is completely different from that of acyclic polyhedra without vertices. Indeed, it was shown in [19] that every acyclic simple polyhedron without vertices collapses onto D^2. In contrast, as we will see in the following arguments, there exist infinitely many closed acyclic simple polyhedra with a given (positive) number of vertices.

2.1 Acyclic special polyhedra

In this subsection, we focus on special polyhedra, and prove a special case (Lemma 2.6) of Theorem 2.1.

Lemma 2.2 *Let X be an acyclic special polyhedron with n vertices. Then the number of regions of X is exactly $n+1$.*

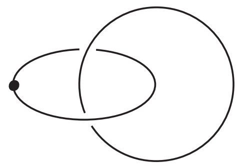


Figure 11: A Kirby diagram of D^4.

Proof Since X is acyclic, the Euler characteristic $\chi(X)$ is 1. The number of edges of X is $2n$ as $S(X)$ is a 4–regular graph. Thus, by the assumption that X is special, we have

$$1 = \chi(X) = \#V(X) - \#E(X) + \#R(X) = n - 2n + \#R(X),$$

which implies $\#R(X) = n + 1$. □

Lemma 2.3 *An acyclic special polyhedron with at most* 2 *vertices is one of* 1^2_i *with* $i = 1, 2$ *and* 2^3_i *with* $i = 1, 2, \ldots, 16$.

Proof This is easily checked by Lemma 2.2 and Tables B.1 and B.4. □

Recall the following famous Property R theorem by Gabai [11]:

Theorem 2.4 (Gabai [11]) *Any* 3*–manifold obtained by* 0*–surgery on a nontrivial knot in* S^3 *is irreducible. In particular, nontrivial surgery on a nontrivial knot in* S^3 *does not yield* $S^2 \times S^1$.

Theorem 2.4 implies that a Kirby diagram with a (framed) knot and a dotted circle of a 4–manifold M with $\partial M \cong S^3$ is nothing but the one shown in Figure 11. In particular, M is diffeomorphic to D^4. The following is the direct consequence of Theorem 2.4:

Lemma 2.5 *Let* M *be an acyclic* 4*–manifold with* $\partial M \cong S^3$. *Let* X *be a special shadow of* M *with* n *vertices. If* X *admits* n *canceling pairs,* M *is diffeomorphic to* D^4.

Proof By Lemma 2.2 a Kirby diagram of M corresponding to X is an $(n+1)$–component (framed) link L with $n + 1$ dotted circles. If that diagram admits n canceling pairs of 1– and 2–handles, after canceling them, we get a (framed) knot K with a dotted circle. Hence, the assertion follows from Theorem 2.4. □

Lemma 2.6 *Every acyclic* 4*–manifold* M *of special shadow-complexity at most* 2 *with* $\partial M \cong S^3$ *is diffeomorphic to* D^4*.*

Proof The case of special shadow-complexity 0 is due to Theorem 0.1(1). Let M be an acyclic 4–manifold with special shadow-complexity n, where n is 1 or 2. Let X be a special shadow of M with n vertices. By Lemma 2.3, X is one of 1_i^2 with $i = 1, 2$ and 2_i^3 with $i = 1, 2, \dots, 16$. For each case, we can easily check that all of them admit n canceling pairs. Thus, by Lemma 2.5, M is diffeomorphic to D^4. □

Remark Every shadow X of a 4–manifold M with $\partial M \cong S^3$ is simply connected. In fact, the restriction of a projection $M \searrow X$ to ∂M induces a surjective homomorphism $\pi_1(\partial M) \to \pi_1(X)$. This fact does not give any restriction for the shadows in the above argument. That is, every acyclic special polyhedron with vertices up to 2 is simply connected. Note that by the Poincaré conjecture proved by Perelman [22; 24; 23] any special spine of any integral homology 3–sphere except S^3 is an acyclic special polyhedron that is not simply connected. By Matveev [18], such a special polyhedron has at least 5 vertices. Indeed, the Poincaré homology sphere admits a special spine with 5 vertices, and that spine has the minimum number of vertices among the special spines of integral homology 3–spheres that are not homeomorphic to S^3. We do not know at present whether there exists an acyclic special polyhedron with 3 or 4 vertices that is not simply connected.

2.2 Acyclic simple polyhedra

We are going to extend Lemma 2.6 to the 4–manifolds of connected shadow-complexity at most 2. We begin with two lemmas that will be used repeatedly in the remaining part of the paper.

Lemma 2.7 (Ikeda [12]) *Let* X *be an acyclic simple polyhedron. Then the following holds*:

(1) *Every region of* X *is a* 2*–sphere with holes.*
(2) *If* X *has no vertices, then* $\partial X \neq \varnothing$*.*

Lemma 2.8 (Naoe [19]) *Let* X *be an acyclic simple polyhedron. Let* $\gamma \subset X \setminus S(X)$ *be a simple closed curve. Then* γ *splits* X *into* A *and* B *such that*

(1) A *is acyclic*; *and*
(2) B *is a homology-*S^1 *and* $H_1(B; \mathbb{Z})$ *is generated by the cycle represented by* γ*.*

We note that if the piece B in Lemma 2.8 has no vertices, then it has at least one boundary component other than γ. Otherwise, the polyhedron $\widehat{B}$ obtained from B by capping off γ by a disk is an acyclic polyhedron with $\#V(\widehat{B}) = 0$ and $\partial\widehat{B} = \varnothing$, which contradicts Lemma 2.7. In particular, an acyclic simple polyhedron does not contain a piece homeomorphic to Y_2 or Y_3 (recall Section 1.3).

The following lemma is a generalization of Lemma 2.2:

Lemma 2.9 *Let X be an acyclic simple polyhedron with at least one vertex. Suppose that there exists a component S' of $S(X)$ containing $n \geq 1$ vertices. Set $X' := \mathrm{Nbd}(S'; X)$ and $m := \#\partial X'$. Then $m \geq n+1$ and the simple closed curves $\partial X'$ split X into X', $n+1$ acyclic pieces $A_1, \dots, A_{n+1}$ and $m-n-1$ homology-S^1 pieces $B_1, \dots, B_{m-n-1}$. Further, if X is closed, each B_i contains at least one vertex.*

Proof By Lemma 2.8 and the Euler characteristic computation, it is straightforward to see that $\partial X'$ splits X into X', $n+1$ acyclic pieces $A_1, \dots, A_{n+1}$ and $m-n-1$ homology-S^1 pieces $B_1, \dots, B_{m-n-1}$. Suppose that some B_i contains no vertices. Then, as we noted right after Lemma 2.8, ∂B_i consists of at least two components. This implies that ∂X is not empty. □

Remark If an acyclic piece A_i in Lemma 2.9 contains no vertices, we can describe its explicit shape, which is of independent interest, as in Appendix A.

Lemma 2.10 *Let M be an acyclic 4–manifold with $\partial M \cong S^3$. Let $X \subset M$ be a shadow with n vertices. Suppose that there exists a component S' of $S(X)$ containing all vertices of X. Set $X' := \mathrm{Nbd}(S'; X)$. Suppose that $\#\partial X' = n+1$. Let $\widehat{X}'$ be the special polyhedron obtained from X' by capping off the boundary components by disks. If $\widehat{X}'$ admits n canceling pairs, then M is diffeomorphic to D^4.*

Proof By Lemma 2.9, the simple closed curves $\partial X'$ splits X into X' and $n+1$ acyclic pieces $A_1, \dots, A_{n+1}$. Set $\gamma_i = X' \cap A_i$ for $i \in \{1, \dots, n+1\}$. The decomposition of X into X' and $A_1, \dots, A_{n+1}$ naturally induces a decomposition of M. Let $M_{X'}$ and M_{A_i} be the pieces of the decomposition of M corresponding to X' and A_i. Note that $M_{X'}$ is diffeomorphic to $\natural_{n+1}(S^1 \times D^3)$. Since A_i does not contain vertices, M_{A_i} is diffeomorphic to D^4 by Theorem 0.1(2). Let K_i be the framed knot in ∂M_{A_i} ($\cong S^3$) corresponding to γ_i.

Since $\widehat{X}'$ admits n canceling pairs, some Kirby diagram obtained from a diagram of $\widehat{X}'$ as explained in Section 1.2 admits n canceling pairs. That Kirby diagram consists of an $(n+1)$–component framed link $L = L_1 \sqcup \cdots \sqcup L_{n+1}$, where L_i corresponds to γ_i, together with $n+1$ dotted circles $U = U_1 \sqcup \cdots \sqcup U_{n+1}$. By Lemmas 1.1 and 1.2, the Kirby diagram obtained from $L \cup U$ by replacing L_i with $L_i \# K_i$ represents the 4–manifold M. Hence, that Kirby diagram of M can be simplified to a (framed) knot K with a single dotted circle by handle-canceling. Since $\partial M \cong S^3$, the Kirby diagram thus obtained is the one shown in Figure 11 by Theorem 2.4. This implies that M is diffeomorphic to D^4. $\square$

Let X be a closed acyclic simple polyhedron. We say that X satisfies the *cancellation condition* if the following holds: for each component S' of $S(X)$ such that $\#V(X') \geq 1$ and $\#\partial X' = \#V(X')+1$, where $X' := \mathrm{Nbd}(S'; X)$, the special polyhedron $\widehat{X}'$ obtained from X' by capping off the boundary components by disks admits $\#V(X')$ canceling pairs.

Theorem 2.1 is a direct consequence of the following theorem:

Theorem 2.11 *Let M be an acyclic 4–manifold with $\partial M \cong S^3$. If M admits a closed shadow satisfying the cancellation condition, then M is diffeomorphic to D^4.*

Proof of Theorem 2.1 from Theorem 2.11 Let M be an acyclic 4–manifold of connected shadow-complexity n with $\partial M \cong S^3$, where $n = 0, 1$ or 2. The case where $n = 0$ is due to Theorem 0.1 (2). In the following we assume that $n = 1$ or 2. By Lemma 1.3, M admits a closed shadow X of connected complexity n. Let S' be a component of $S(X)$ such that $\#V(X') \geq 1$ and $\#\partial X_i = \#V(X') + 1$, where $X' := \mathrm{Nbd}(S'; X)$. Then the special polyhedron $\widehat{X}'$ obtained from X' by capping off the boundary components by disks remains acyclic. As we have seen in the proof of Lemma 2.6, $\widehat{X}'$ admits $\#V(X') + 1$ canceling pairs. Therefore, X satisfies the cancellation condition. Consequently, M is diffeomorphic to D^4 by Theorem 2.11. $\square$

Proof of Theorem 2.11 Let M be an acyclic 4–manifold with $\partial M \cong S^3$. Let X be a closed shadow of X satisfying the cancellation condition. If X contains no vertices, then the assertion follows from Theorem 0.1(2). In the following we assume that X contains vertices. Let $S_1, S_2, \ldots, S_k$ be the connected components of $S(X)$ having at least one vertex. We use induction on k.

Let $k = 1$. Set $X' := \mathrm{Nbd}(S_1; X)$. By Lemma 2.9, we have $\#\partial X' = \#V(X')+1$. Since X satisfies the cancellation condition, M is diffeomorphic to D^4 by Lemma 2.10.

Let $k \geq 2$ and assume that the conclusion holds for all $k' < k$. Set $X_i := \mathrm{Nbd}(S_i; X)$ and $n_i := \#V(X_i)$ for $i = 1, 2, \ldots, k$. By Lemma 2.9, for each i the simple closed curves ∂X_i split X into X_i, $n_i + 1$ acyclic pieces and several (possibly no) homology-S^1 pieces. Further, here if there exists a homology-S^1 piece, then it contains a vertex. The argument is divided into two cases.

Case 1 Suppose first that there exists $i \in \{1,2,\dots,k\}$ such that $\#\partial X_i = n_i + 1$. Without loss of generality, we can assume that $\#\partial X_1 = n_1 + 1$. By Lemma 2.9, ∂X_1 splits X into X_1 and $n_1 + 1$ acyclic pieces $A_1,\dots,A_{n_1+1}$. The decomposition of X into $X_1, A_1,\dots,A_{n_1+1}$ naturally induces a decomposition of M into $M_{X_1}, M_{A_1},\dots,M_{A_{n_1+1}}$. Since, for each $j \in \{1,\dots,n_1+1\}$, the number of connected components of A_j having at least 1 vertex is fewer than k, we have $M_{A_j} \cong D^4$ for each $j \in \{1,\dots,n_1+1\}$ by the assumption of the induction. Now the rest of the proof for this case runs as in Lemma 2.10.

Case 2 Suppose that $\#\partial X_i > n_i + 1$ for all $i \in \{1,2,\dots,k\}$. We are going to show that in this case X is not closed, which is a contradiction. Let $Y_1, Y_2,\dots,Y_l$ be the connected components of $X - \bigcup_{i=1}^k X_i$. Let $\widehat{G}$ be the bipartite graph whose vertices are $\{X_1, X_2,\dots,X_k\} \sqcup \{Y_1,Y_2,\dots,Y_l\}$ such that two vertices X_i and Y_j span an edge if and only if $X_i \cap Y_j \neq \varnothing$. Note that since X is acyclic, $\widehat{G}$ is a tree. Note also that the set of edges of $\widehat{G}$ one-to-one corresponds to the set of simple closed curves of $\bigcup_{i=1}^k \partial X_i$. Each edge of $\widehat{G}$ is labeled by 0 or 1 as follows. Let e be the edge of $\widehat{G}$ corresponding to a simple closed curve γ of ∂X_i. The curve γ separates X into two components X_γ and Y_γ, where $X_i \subset X_\gamma$. See Figure 12. By Lemma 2.8, Y_γ is acyclic or a homology-S^1. We assign 0 to e if Y_γ is acyclic, and 1 to e if Y_γ is a homology-S^1. Note that for each vertex X_i, there exists at least one 1–labeled edge connected to X_i by the assumption of Case 2.

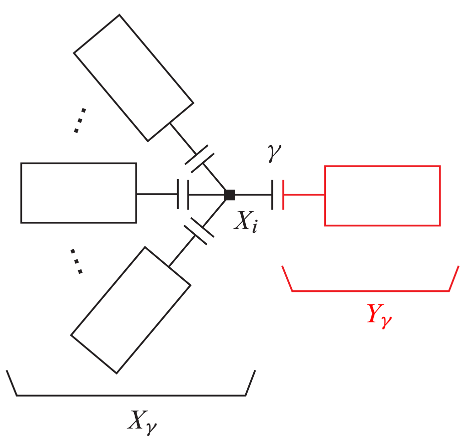


Figure 12: The decomposition of X into X_γ and Y_γ.

Claim *The graph $\widehat{G}$ contains a vertex Y_i such that every edge connected to Y_i is labeled by* 1.

Proof Suppose for a contradiction that for each vertex Y_i, there exists a 0–labeled edge connected to Y_i. Choose an arbitrary vertex X_{k_1} of $\widehat{G}$. Then X_{k_1} is connected to a vertex Y_{l_1} of $\widehat{G}$ by a 1–labeled edge. By the assumption, Y_{l_1} is connected to a vertex X_{k_2} of $\widehat{G}$ by a 0–labeled edge. Then X_{k_2} is connected to a vertex Y_{l_2} by a 1–labeled edge. In this way, we can make a path of arbitrarily large length. During the process, we do not pass the same vertex more than once because $\widehat{G}$ is a tree. This contradicts the finiteness of $\widehat{G}$. □

By the above claim, without loss of generality, we can assume that all the edges, say $\gamma_1, \dots, \gamma_m$, connected to Y_1 is labeled by 1. This implies that, for each i,

- X_{γ_i} is acyclic; and
- Y_{γ_i} is a homology-S^1 and $H_1(Y_{\gamma_i}; \mathbb{Z})$ is generated by the cycle represented by γ_i.

Let Z_1 be the polyhedron obtained from Y_{γ_1} by capping off γ_1 by a disk. Then Z_1 is acyclic. Construct inductively a sequence of polyhedra $Z_1, Z_2, \dots, Z_m$, where Z_{i+1} is the polyhedron obtained from $\overline{Z_i \setminus X_{\gamma_{i+1}}}$ by capping off γ_{i+1} by a disk. Since each X_{γ_i} is acyclic, Z_i is again acyclic. Since Z_m contains no vertex, by Lemma 2.7(2), Z_m has at least one boundary component. This implies that X is not closed, which is a contradiction. □

Appendix A Acyclic simple polyhedron without vertices and with a single boundary circle

Let X be an acyclic simple polyhedron without vertices and with a single boundary circle γ. In this appendix, we are going to describe a specific shape of X. The results here are not necessary for the proof of the main theorem. Since X is acyclic, X cannot contain a piece homeomorphic to Y_2 or Y_3. Further, X does not contain a piece homeomorphic to Y_{111} as well by the following lemma:

Lemma A.1 (Naoe [19]) *Let X be an acyclic simple polyhedron without vertices. Fix a component γ of ∂X. Then X collapses onto a subpolyhedron X' fixing γ such that X' does not contain a piece homeomorphic to Y_{111} and $\partial X' = \gamma$.*

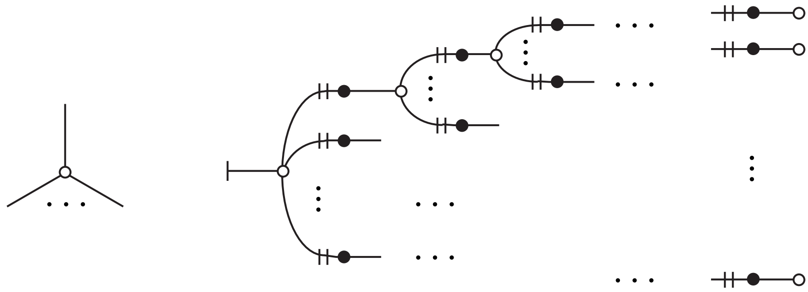


Figure 13: Left: a vertex encoding a sphere with a finite number of holes. Right: the graph G.

Let G be a graph G encoding X, which is a tree by Lemma 1.4. Let v_0 be the unique vertex of type (B) (recall Figure 10), which corresponds to the unique boundary component $\partial X = \gamma$, in G. Let v_1 be a vertex of type (12) in G. Since G is a tree, there exists a unique path from v_0 to v_1. Let e be an edge in the path incident to v_1.

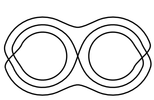
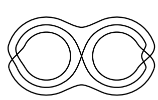
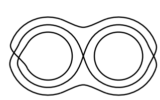
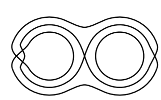
Lemma A.2 *In the above setting, the edge e is marked with two lines.*

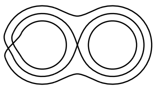
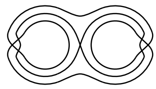
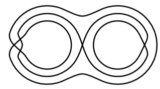
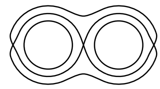
Proof Suppose not for a contradiction. By the simple closed curve α corresponding to e, the polyhedron X decomposes into 2 polyhedra X_1 and X_2, where X_1 contains γ. We note that $\partial X_2 = \alpha$. By Lemma 2.8, one of them is acyclic, and the other is a homology-S^1. Since X_2 has no boundary component other than α, X_2 cannot be a homology-S^1. Thus X_2 is acyclic. By collapsing X_2 from α, we obtain an acyclic simple polyhedron containing a Möbius band in a region. This contradicts Lemma 2.7. □

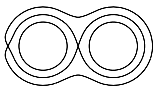
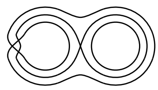
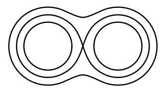
Now we are ready to describe the shape of X. For convenience, as a generalization of a vertex of type (P), we introduce a white vertex of degree $d \geq 3$ as shown in Figure 13, left, to encode a piece of a simple polyhedron homeomorphic to the sphere with d holes. By the previous observation and Lemma A.2, the shape of G can thus be described as in Figure 13, right.

Appendix B Table of special polyhedra of complexity up to 2

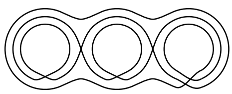
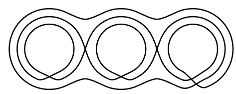
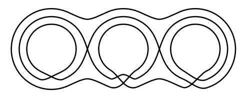
B.1 Special polyhedra with 1 vertex

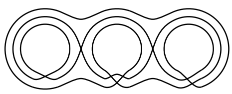
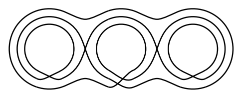
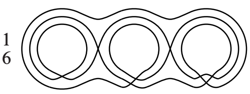
1^1_1  1^1_2  1^2_1  1^2_2 

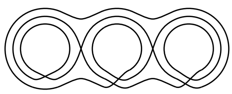
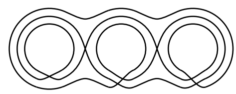
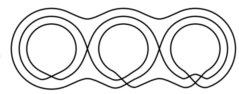
1^2_3  1^2_4  1^2_5  1^3_1 

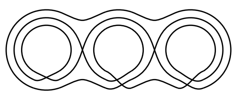
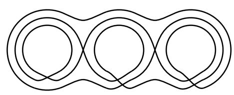
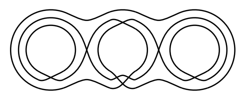
1^3_2  1^3_3  1^4_1 

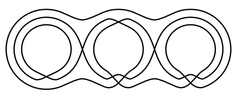
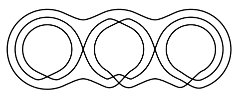
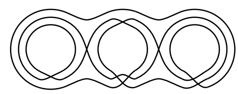
B.2 Special polyhedra with 2 vertices and 1 region

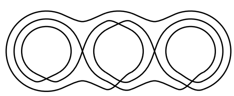
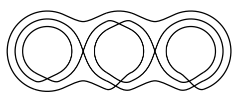
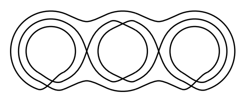
2^1_1  2^1_2  2^1_3 

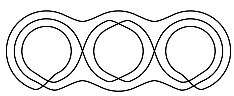
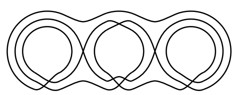
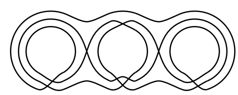
2^1_4  2^1_5  2^1_6 

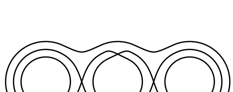
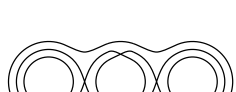
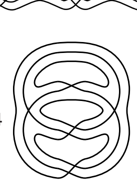
2^1_7  2^1_8  2^1_9 

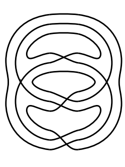
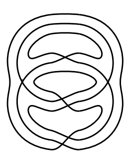
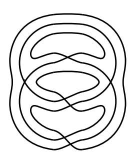
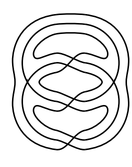
2^1_{10}  2^1_{11}  2^1_{12} 

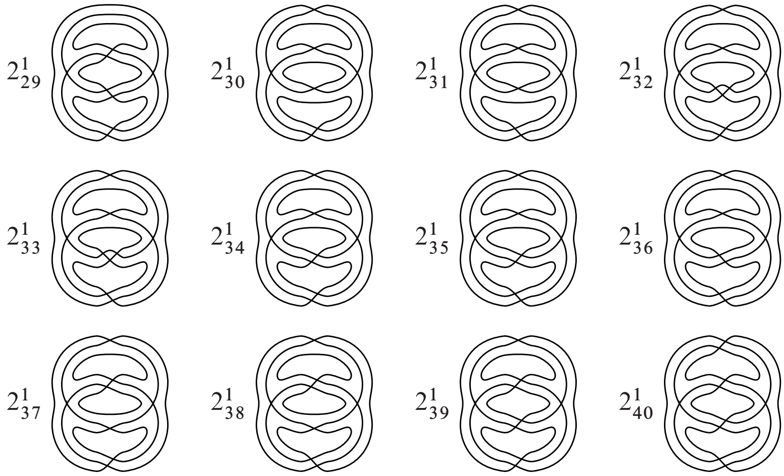
2^1_{13}  2^1_{14}  2^1_{15} 

2^1_{16}  2^1_{17}  2^1_{18} 

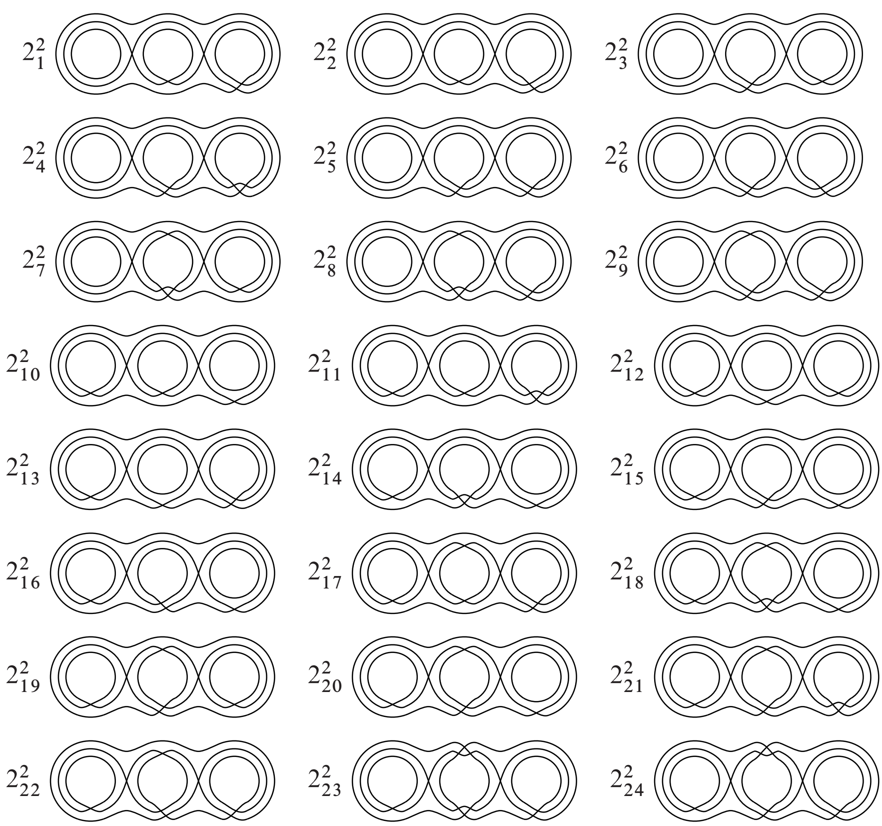
2^1_{19}  2^1_{20}  2^1_{21} 

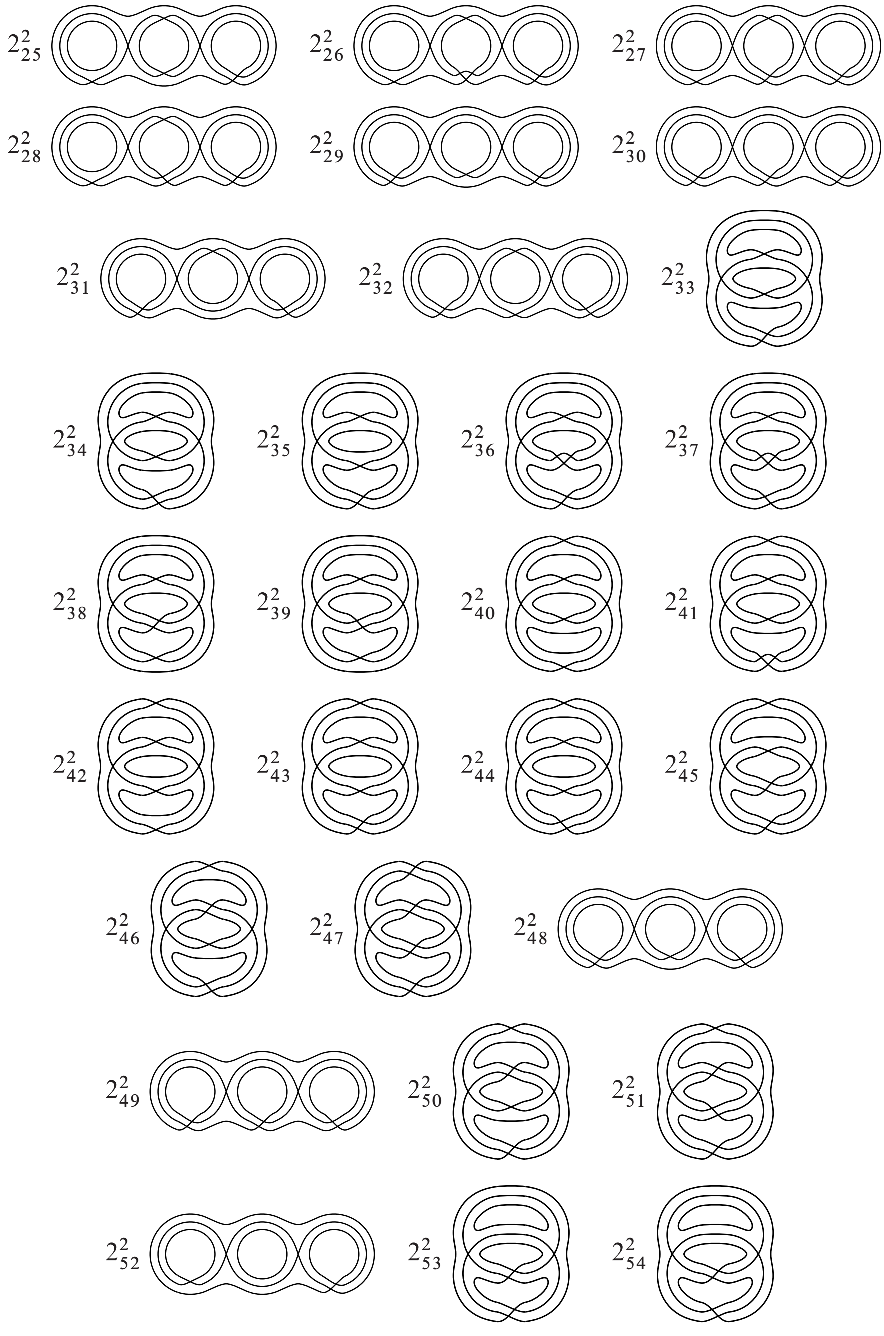
2^1_{22}  2^1_{23}  2^1_{24} 

2^1_{25}  2^1_{26}  2^1_{27}  2^1_{28} 

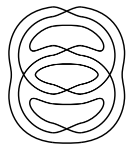
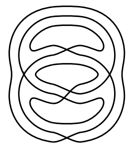


B.3 Special polyhedra with 2 vertices and 2 regions

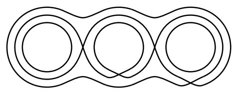
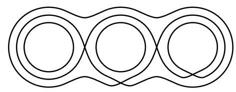
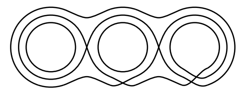


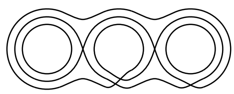
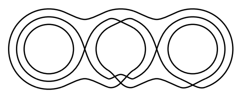
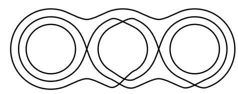


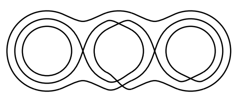
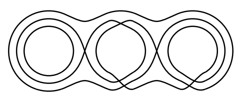
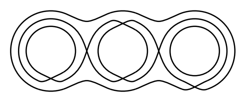
2^2_{25}
2^2_{26}
2^2_{27}
2^2_{28}
2^2_{29}
2^2_{30}
2^2_{31}
2^2_{32}
2^2_{33}
2^2_{34}
2^2_{35}
2^2_{36}
2^2_{37}
2^2_{38}
2^2_{39}
2^2_{40}
2^2_{41}
2^2_{42}
2^2_{43}
2^2_{44}
2^2_{45}
2^2_{46}
2^2_{47}
2^2_{48}
2^2_{49}
2^2_{50}
2^2_{51}
2^2_{52}
2^2_{53}
2^2_{54}
2^2_{55}
2^2_{56}

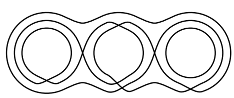
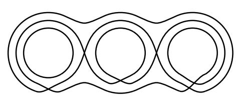
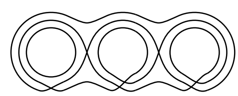
2^2_{57}  2^2_{58} 

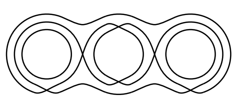
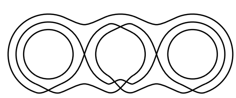
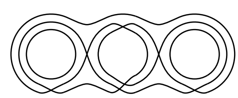
B.4 Special polyhedra with 2 vertices and 3 regions

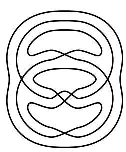
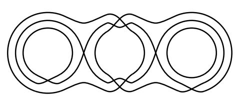
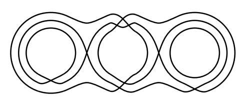
2^3_{1}  2^3_{2}  2^3_{3} 

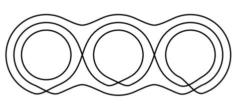
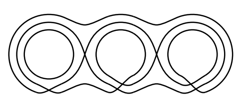
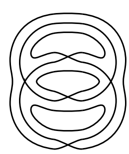
2^3_{4}  2^3_{5}  2^3_{6} 

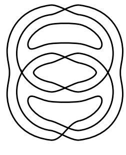
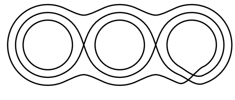
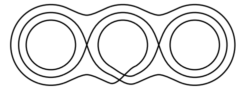
2^3_{7}  2^3_{8}  2^3_{9} 

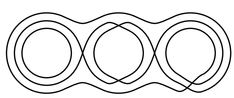
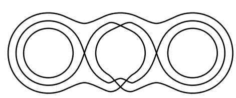
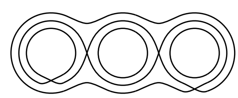
2^3_{10}  2^3_{11}  2^3_{12} 

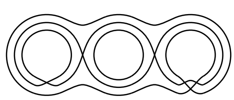
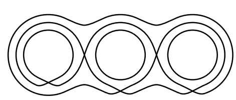
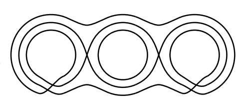
2^3_{13}  2^3_{14}  2^3_{15} 

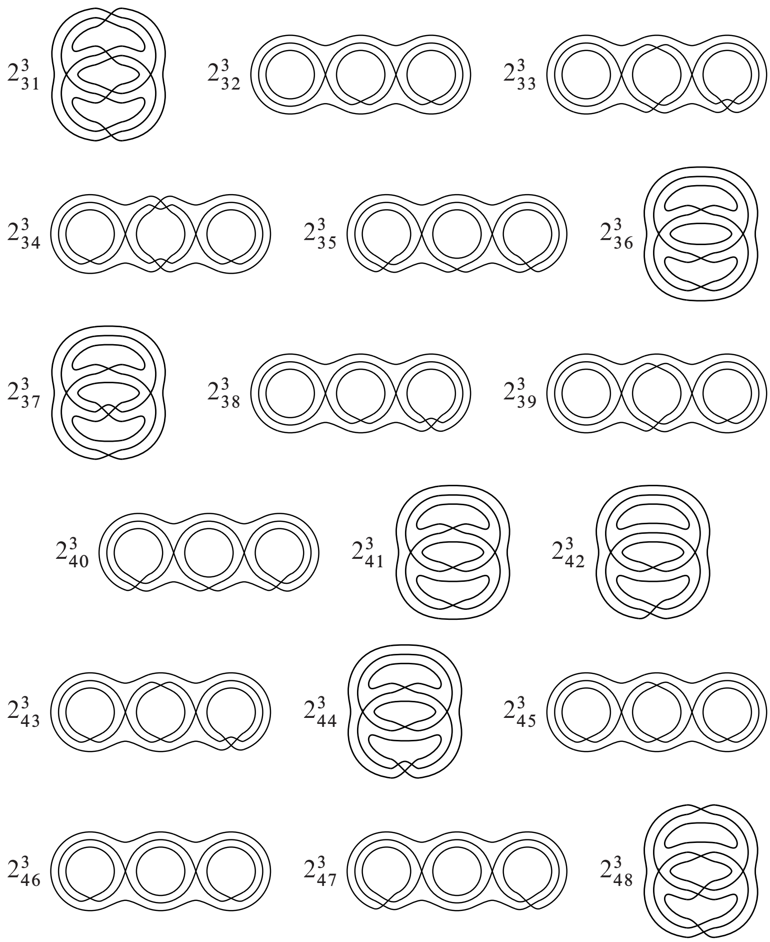
2^3_{16}  2^3_{17}  2^3_{18} 

2^3_{19}  2^3_{20}  2^3_{21} 

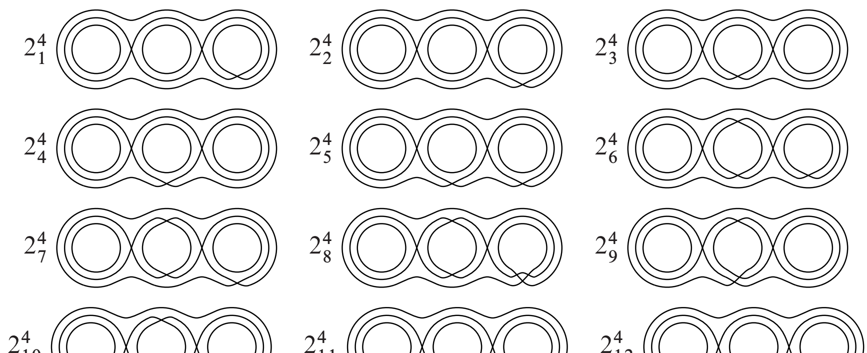
2^3_{22}  2^3_{23}  2^3_{24} 

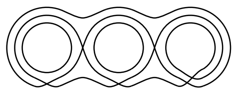
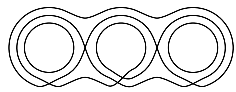
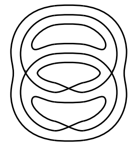
2^3_{25}  2^3_{26}  2^3_{27} 

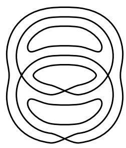
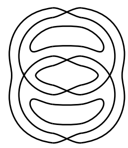
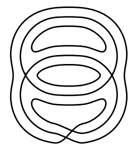
2^3_{28}  2^3_{29}  2^3_{30} 



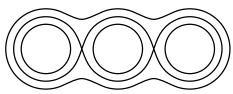
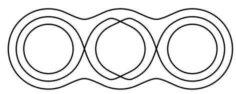
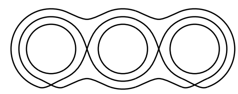
B.5 Special polyhedra with 2 vertices and 4 regions

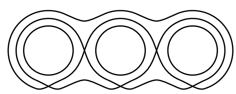
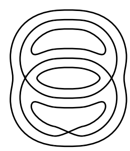


2^4_{13}  2^4_{14}  2^4_{15} 

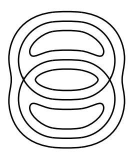
2^4_{16}  2^4_{17}  2^4_{18} 

B.6 Special polyhedra with 2 vertices and 5 regions

2^5_1  2^5_2  2^5_3 

2^5_4  2^5_5 

B.7 Special polyhedra with 2 vertices and 6 regions

2^6_1 

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Department of Mathematics, Hiroshima University
Hiroshima, Japan

Department of Mathematics, Chuo University
Tokyo, Japan

ykoda@hiroshima-u.ac.jp, naoe@math.chuo-u.ac.jp

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