Algebra & Number Theory

Volume 14

2020

No. 7

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Hung M. Bui, Alexandra Florea, Jonathan P. Keating and Edva Roditty-Gershon

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ALGEBRA AND NUMBER THEORY 14:7 (2020)
https://doi.org/10.2140/ant.2020.14.1853

Moments of quadratic twists of elliptic curve L-functions over function fields

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We calculate the first and second moments of L-functions in the family of quadratic twists of a fixed elliptic curve E over $\mathbb{F}_q[x]$, asymptotically in the limit as the degree of the twists tends to infinity. We also compute moments involving derivatives of L-functions over quadratic twists, enabling us to deduce lower bounds on the correlations between the analytic ranks of the twists of two distinct curves.

1. Introduction and statement of results

The values of L-functions at the central point of the critical strip have been the subject of considerable interest in recent years. One way to study these central values is by considering moments in families of L-functions. There are now precise conjectured asymptotic formulas for such moments motivated by analogies with random matrix theory [Keating and Snaith 2000a; 2000b]. More precise asymptotic formulas containing lower order terms were conjectured in [Conrey et al. 2003; 2005; Diaconu et al. 2003]. In the case of the Riemann zeta-function, the analogue of these conjectures is now relatively well understood in terms of correlations of the divisor function [Conrey and Keating 2015a; 2015b; 2015c; 2016; 2019]. The moments of other degree-one L-functions have also been investigated intensively. It remains a challenge to extend these calculations to L-functions of degree two and higher.

The first moment of the family of derivatives of L-functions of quadratic twists of a fixed modular form was studied in [Bump et al. 1990b; Iwaniec 1990; Murty and Murty 1991]. Questions related to the nonvanishing of L-functions in this family were considered in [Bump et al. 1990a; 1990b; Murty and Murty 1991]. For example, it is shown independently in [Murty and Murty 1991] and [Bump et al. 1990a], using different techniques, that for a fixed elliptic curve with root number equal to 1 there are infinitely many fundamental discriminants $d < 0$ such that its twist by d has analytic rank equal to 1.

The second moment of the family was considered by Soundararajan and Young [2010]. Unconditionally, they obtained a lower bound for the second moment which matches the asymptotic formula conjectured by Keating and Snaith [2000b] and assuming the generalized Riemann hypothesis (GRH) they established the conjectured formula. Using similar ideas, again under GRH, Petrow [2014] obtained several asymptotic formulas for moments of derivatives of these GL(2) L–functions when the sign of the functional equation is -1.

MSC2010: primary 11M06; secondary 11M38.
Keywords: elliptic curve, L-function, rank, finite field, correlation.

While no asymptotic formulas for moments larger than the second are known for this family, there are lower and upper bounds of the right order of magnitude. Rudnick and Soundararajan [2005; 2006] established unconditional lower bounds for all moments larger than the first, and, assuming GRH, the work of Soundararajan [2009] and its refinement by Harper [2013] produced upper bounds of the conjectured order of magnitude. Radziwiłł and Soundararajan [2015] proved upper bounds for moments below the first in this family of L-functions. Their techniques also allow them to obtain a one-sided central limit for the distribution of the logarithm of these central L-values. Their result supports a conjecture by Keating and Snaith [2000a] which can be viewed as the analogue of Selberg's central limit theorem for the distribution of $\log\left|\zeta\left(\frac{1}{2}+it\right)\right|$.

In this paper we study several moment problems of comparable difficulty to the moment computation of Soundararajan and Young in the function field family of quadratic twists of an elliptic curve. Since we are working over function fields, the results we obtain are unconditional due to the availability of GRH in this setting.

Recently there has been a good deal of work on computing moments of L-functions in the function field setting. Andrade and Keating [2012] obtained an asymptotic formula for the first moment in the symplectic family of quadratic L-functions when the degree of the L-functions (which is a polynomial in this case) goes to infinity and the size of the finite field is fixed (see also [Hoffstein and Rosen 1992] for a similar result). A lower order term of size approximately the cube root of the main term was computed in [Florea 2017c]. The second, third and fourth moments were computed in [Florea 2017b; 2017a] (see also [Diaconu 2019]). We note that the asymptotic formula for the fourth moment does not have a power savings error term, but recovers several of the expected leading order terms in the conjectured formula [Andrade and Keating 2014]. Obtaining an asymptotic formula with the leading order term for the fourth moment in the family of quadratic L-functions is comparable in difficulty to establishing an asymptotic formula for the second moment of L-functions of quadratic twists of an elliptic curve, and is one of the problems we consider in this paper.

We note that for all of our results, we fix the size q of the finite field we work in and let the degree of the L-functions go to infinity. If instead one fixes the degree and lets $q\to\infty$, then Katz and Sarnak [1999] showed that the L-functions become equidistributed in the orthogonal group, and hence computing the various moments reduces to computing several random matrix integrals (see for example [Keating and Snaith 2000a]). In the case of elliptic curve L-functions, the relevant equidistribution results were established in [Hall et al. 2017].

To state our results we first need some notation. Fix a prime power q with

$$(q,6)=1 \quad \text{and} \quad q\equiv 1 \pmod 4.$$

Let $K=\mathbb{F}_q(t)$ be the rational function field and $\mathcal{O}_K=\mathbb{F}_q[t]$. Let E/K be an elliptic curve defined by $y^2=x^3+ax+b$, with $a,b\in\mathcal{O}_K$ and discriminant $\Delta=4a^3+27b^2$ such that $\deg_t(\Delta)$ is minimal among models of E/K of this form.

The normalized L-function associated to the elliptic curve E/K has the following Euler product and Dirichlet series, which converge for $\Re(s) > 1$,

$$\begin{aligned} L(E,s) &:= \mathcal{L}(E,u) \\ &= \sum_{f \in \mathcal{M}} \lambda(f) u^{\deg(f)} \\ &= \prod_{P \mid \Delta} (1 - \lambda(P) u^{\deg(P)})^{-1} \prod_{P \nmid \Delta} (1 - \lambda(P) u^{\deg(P)} + u^{2\deg(P)})^{-1}, \end{aligned} \tag{1-1}$$

where we set $u := q^{-s}$, and $\mathcal{M}$ denotes the set of monic polynomials over $\mathbb{F}_q[t]$. For a more detailed discussion of the coefficients $\lambda(f)$ and how to define the L-function, see Section 2.2 in [Baig and Hall 2012]. The L-function is a polynomial in u with integer coefficients of degree

$$\mathfrak{n} := \deg(\mathcal{L}(E,u)) = \deg(M) + 2\deg(A) - 4, \tag{1-2}$$

where for simplicity we denote by M the product of the finite primes where E has multiplicative reduction and by A the product of the finite primes where E has additive reduction. Moreover, the L-function satisfies a functional equation; namely, there exists $\epsilon(E) \in \{\pm 1\}$ such that

$$\mathcal{L}(E,u) = \epsilon(E)(\sqrt{q}u)^{\mathfrak{n}} \mathcal{L}\left(E, \frac{1}{qu}\right).$$

For a more precise formula for the sign of the functional equation, see Lemma 2.3 in [Baig and Hall 2012]. Now for $D \in \mathcal{O}_K$ with D square-free, monic of odd degree and $(D, \Delta) = 1$, we consider the twisted elliptic curve $E \otimes \chi_D/K$ with the affine model $y^2 = x^3 + D^2ax + D^3b$. Then the L-function corresponding to the twisted elliptic curve has the following Dirichlet series and Euler product

$$\begin{aligned} \mathcal{L}(E \otimes \chi_D, u) &= \sum_{f \in \mathcal{M}} \lambda(f) \chi_D(f) u^{\deg(f)} \\ &= \prod_{P \mid \Delta} (1 - \lambda(P) \chi_D(P) u^{\deg(P)})^{-1} \prod_{P \nmid \Delta D} (1 - \lambda(P) \chi_D(P) u^{\deg(P)} + u^{2\deg(P)})^{-1}. \end{aligned}$$

The new L-function is a polynomial of degree $(\mathfrak{n} + 2\deg(D))$ and satisfies the functional equation

$$\mathcal{L}(E \otimes \chi_D, u) = \epsilon (\sqrt{q}u)^{\mathfrak{n} + 2\deg(D)} \mathcal{L}\left(E \otimes \chi_D, \frac{1}{qu}\right), \tag{1-3}$$

where

$$\epsilon = \epsilon(E \otimes \chi_D) = \epsilon_{\deg(D)} \epsilon(E) \chi_D(M).$$

Here $\epsilon_{\deg(D)} \in \{\pm 1\}$ is an integer which only depends on the degree of D (see Proposition 4.3 in [Baig and Hall 2012]).

Let $\mathcal{H}^*_{2g+1}$ denote the set of monic, square free polynomials of degree $(2g+1)$ coprime to Δ. Our first two theorems concern the first moments of $L\big(E \otimes \chi_D, \frac{1}{2}\big)$ and $L'\big(E \otimes \chi_D, \frac{1}{2}\big)$.

Theorem 1.1. *Unless $\epsilon_{2g+1}\epsilon(E) = -1$ and $M = 1$, we have*

$$\frac{1}{|\mathcal{H}^*_{2g+1}|} \sum_{D \in \mathcal{H}^*_{2g+1}} L\big(E \otimes \chi_D, \tfrac{1}{2}\big) = c_1(M) + O_\varepsilon(q^{-g+\epsilon g}),$$

where the value $c_1(M)$ is defined in (5-6) *and* (5-2). *In particular, the constant $c_1(M) \neq 0$ in this case and we obtain an asymptotic formula.*

Theorem 1.2. *Unless $\epsilon_{2g+1}\epsilon(E) = 1$ and $M = 1$, we have*

$$\frac{1}{|\mathcal{H}^*_{2g+1}|} \sum_{D \in \mathcal{H}^*_{2g+1}} \epsilon^- L'\big(E \otimes \chi_D, \tfrac{1}{2}\big) = c_2(M) L(\operatorname{Sym}^2 E, 1) g + c_3(M) + O_\varepsilon(q^{-g+\epsilon g}),$$

where the values $c_2(M)$ and $c_3(M)$ are defined in (6-2), (6-3) *and* (5-2). *In particular, the constant $c_2(M) \neq 0$ in this case and we obtain an asymptotic formula.*

Theorem 1.2 above should be compared to the number field analogous result in [Iwaniec 1990]. In the following theorems we obtain asymptotic formulas for the second moments of $L\big(E \otimes \chi_D, \tfrac{1}{2}\big)$ and $L'\big(E \otimes \chi_D, \tfrac{1}{2}\big)$.

Theorem 1.3. *Unless $\epsilon_{2g+1}\epsilon(E) = -1$ and $M = 1$, we have*

$$\frac{1}{|\mathcal{H}^*_{2g+1}|} \sum_{D \in \mathcal{H}^*_{2g+1}} L\big(E \otimes \chi_D, \tfrac{1}{2}\big)^2 = c_4(M) L(\operatorname{Sym}^2 E, 1)^3 g + O_\varepsilon(g^{1/2+\varepsilon}),$$

where the value $c_4(M)$ is defined in (7-2), (3-8) *and* (3-9). *In particular, the constant $c_4(M) \neq 0$ in this case and we obtain an asymptotic formula.*

Theorem 1.4. *Unless $\epsilon_{2g+1}\epsilon(E) = 1$ and $M = 1$, we have*

$$\frac{1}{|\mathcal{H}^*_{2g+1}|} \sum_{D \in \mathcal{H}^*_{2g+1}} \epsilon^- L'\big(E \otimes \chi_D, \tfrac{1}{2}\big)^2 = c_5(M) L(\operatorname{Sym}^2 E, 1)^3 g^3 + O_\varepsilon(g^{2+\epsilon}),$$

where the value $c_5(M)$ is defined in (8-2), (3-8) *and* (3-9). *In particular, the constant $c_5(M) \neq 0$ in this case and we obtain an asymptotic formula.*

Note that Theorem 1.3 is the function field analogue of Theorem 1.2 in [Soundararajan and Young 2010]. Considering the smoothed second moment, Soundararajan and Young obtain an error term of size $(\log X)^{3/4+\epsilon}$, which would translate to $g^{3/4+\epsilon}$ in the function field setting. Using slightly different techniques, Petrow [2014] states that the error term could be improved to $(\log X)^{1/2+\epsilon}$ which is of the same quality we obtain in the result above.

Our Theorem 1.3 should also be compared to the asymptotic formula for the fourth moment of quadratic L-functions over function fields in [Florea 2017a]. We remark that for the symplectic family of quadratic L-functions, one can obtain lower order terms in the asymptotic formula by using an inductive argument and then obtaining upper bounds for moments of L-functions evaluated at points far from the critical point. The fact that one can compute a few lower order terms can be explained by the gap between powers of g coming from evaluating moments at the critical point versus evaluating moments far from the critical point. When computing the fourth moment of quadratic L-functions close to the central point, one expects to obtain a power of g^{10}. As we move away from the central point, the family starts to behave like a family with unitary symmetry and one expects an upper bound of the magnitude g^4. The difference in powers of g gives one room to use a repetitive argument to rigorously compute lower order terms down to g^4. In the case of the orthogonal family we consider in this paper, note that the main term in Theorem 1.3 is of size g, and the error term has size $g^{1/2}$ coming from obtaining an upper bound for the second moment evaluated at a point far from the central point. The small difference between these powers of g does not give us enough room to compute a lower order term in this case.

We can also study the moment of the product of the quadratic twists of two elliptic curve L-functions. Let E_1 and E_2 be two elliptic curves over K. Let $\Delta = \Delta_1\Delta_2$, where, for $i = 1, 2$, Δ_i denotes the discriminant of E_i. Let M_i denote the product of the finite primes where E_i has multiplicative reduction and

$$\epsilon_i = \epsilon_{\deg(D)}\epsilon(E_i)\chi_D(M_i).$$

Define

$$\epsilon_i^+ := \frac{1+\epsilon_i}{2} \quad \text{and} \quad \epsilon_i^- := \frac{1-\epsilon_i}{2}.$$

Theorem 1.5. *Unless $\epsilon_{2g+1}\epsilon(E_1) = -1$ and $M_1 = 1$, or $\epsilon_{2g+1}\epsilon(E_2) = 1$ and $M_2 = 1$, or $\epsilon(E_1) = \epsilon(E_2)$ and $M_1 = M_2$, we have*

$$\frac{1}{|\mathcal{H}^*_{2g+1}|}\sum_{D\in\mathcal{H}^*_{2g+1}} \epsilon_2^- L\big(E_1\otimes\chi_D, \tfrac12\big)L'\big(E_2\otimes\chi_D, \tfrac12\big)$$
$$= c_6(M_1, M_2)L(\mathrm{Sym}^2 E_1, 1)L(\mathrm{Sym}^2 E_2, 1)L(E_1\otimes E_2, 1)g + O_\varepsilon(g^{1/2+\varepsilon}),$$

where the value $c_6(M_1, M_2)$ is defined in (9-2), (3-8) *and* (3-10). *In particular, the constant $c_6(M_1, M_2)\neq 0$ in this case and we obtain an asymptotic formula.*

We are not aware of the analogous number field result in the literature. We also prove the following.

Theorem 1.6. *Unless $\epsilon_{2g+1}\epsilon(E_1) = 1$ and $M_1 = 1$, or $\epsilon_{2g+1}\epsilon(E_2) = 1$ and $M_2 = 1$, or $\epsilon(E_1) = -\epsilon(E_2)$ and $M_1 = M_2$, we have*

$$\frac{1}{|\mathcal{H}^*_{2g+1}|}\sum_{D\in\mathcal{H}^*_{2g+1}} \epsilon_1^-\epsilon_2^- L'\big(E_1\otimes\chi_D, \tfrac12\big)L'\big(E_2\otimes\chi_D, \tfrac12\big)$$
$$= c_7(M_1, M_2)L(\mathrm{Sym}^2 E_1, 1)L(\mathrm{Sym}^2 E_2, 1)L(E_1\otimes E_2, 1)g^2 + O_\varepsilon(g^{1+\varepsilon}),$$

where the value $c_7(M_1, M_2)$ is defined in (10-2), (3-8) *and* (3-10). *In particular, the constant $c_7(M_1, M_2)\neq 0$ in this case and we obtain an asymptotic formula.*

Note that this result is the analogue of Theorem 2.2 in [Petrow 2014]. An interesting problem would be to compute the average of $L\big(E_1\otimes\chi_D, \tfrac12\big)L\big(E_2\otimes\chi_D, \tfrac12\big)$ for distinct elliptic curves E_1 and E_2. This would have applications to the question of simultaneous nonvanishing of $L\big(E_1\otimes\chi_D, \tfrac12\big)$ and $L\big(E_2\otimes\chi_D, \tfrac12\big)$.

However our techniques do not allow us to obtain such an asymptotic formula, as the error term (coming from Proposition 4.7) would dominate the main term which would have constant size.

Define the analytic rank of a quadratic twist of an elliptic curve L-function $L(E\otimes\chi_D, s)$ by

$$r_{E\otimes\chi_D} := \operatorname{ord}_{s=1/2} L(E\otimes\chi_D, s).$$

Combining the upper bounds for moments of elliptic curve L-functions (see Section 4) with Theorems 1.5 and 1.6 leads to the following corollary.

Corollary 1.7. *Unless $\epsilon_{2g+1}\epsilon(E_1) = -1$ and $M_1 = 1$, or $\epsilon_{2g+1}\epsilon(E_2) = 1$ and $M_2 = 1$, or $\epsilon(E_1) = \epsilon(E_2)$ and $M_1 = M_2$, we have*

$$\#\{D \in \mathcal{H}^*_{2g+1} : r_{E_1\otimes\chi_D} = 0, r_{E_2\otimes\chi_D} = 1\} \gg_\varepsilon \frac{q^{2g}}{g^{6+\varepsilon}}$$

as $g\to\infty$. Also, unless $\epsilon_{2g+1}\epsilon(E_1) = 1$ and $M_1 = 1$, or $\epsilon_{2g+1}\epsilon(E_2) = 1$ and $M_2 = 1$, or $\epsilon(E_1) = -\epsilon(E_2)$ and $M_1 = M_2$, we have

$$\#\{D \in \mathcal{H}^*_{2g+1} : r_{E_1\otimes\chi_D} = r_{E_2\otimes\chi_D} = 1\} \gg_\varepsilon \frac{q^{2g}}{g^{6+\varepsilon}}$$

as $g\to\infty$.

As far as we are aware, Corollary 1.7 is the first result in literature where explicit lower bounds concerning the correlations between the ranks of two twisted elliptic curves are obtained. Following Harper's argument [2013], for the upper bounds for moments of L-functions one may remove the exponents ε in Corollary 1.7. We fail to obtain positive proportions in the above results because we are not able to use a mollifier. Note that the results of Heath-Brown [2004] adapted to the function field setting do not lead to positive proportions either.

2. Some useful lemmas

In this section we will gather a few useful lemmas we will need throughout the paper.

Recall that q is a prime power with $q\equiv 1 \pmod 4$ and $(q, 6) = 1$. Let $\mathcal{M}$ denote the set of monic polynomials over $\mathbb{F}_q[t]$ and $\mathcal{H}$ be the set of monic, square-free polynomials. Let $\mathcal{M}_n$ denote the set of monic polynomials of degree n over $\mathbb{F}_q[t]$ and $\mathcal{M}_{\le n}$ be the set of monic polynomials of degree less than or equal to n. Let $\mathcal{H}_n$ denote the monic, square-free polynomials of degree n and recall that $\mathcal{H}^*_n$ denotes the set of monic, square-free polynomials of degree n coprime to Δ. The norm of a polynomial f is defined by $|f| = q^{\deg(f)}$. Let ϕ denote the Euler totient function and τ the divisor function.

We define the zeta-function as

$$\zeta_q(s) = \sum_{f\in\mathcal{M}} \frac{1}{|f|^s}$$

for $\Re(s) > 1$. By counting monic polynomials of a given degree, one can easily show that

$$\zeta_q(s) = \frac{1}{1-q^{1-s}},$$

and this provides a meromorphic continuation of ζ_q with a simple pole at $s = 1$. As before, we will make the change of variables $u = q^{-s}$ and so the zeta-function becomes

$$\mathcal{Z}(u) = \zeta_q(s) = \sum_{f \in \mathcal{M}} u^{\deg(f)} = \frac{1}{1-qu},$$

with a simple pole at $u = 1/q$. Note that $\mathcal{Z}(u)$ can also be written as an Euler product

$$\mathcal{Z}(u) = \prod_P (1 - u^{\deg(P)})^{-1},$$

where the product is over monic, irreducible polynomials in $\mathbb{F}_q[t]$.

The quadratic character over $\mathbb{F}_q[t]$ is defined as follows. For P a monic, irreducible polynomial let

$$\left(\frac{f}{P}\right) = \begin{cases} 1 & \text{if } P \nmid f,\ f \text{ is a square modulo } P, \\ -1 & \text{if } P \nmid f,\ f \text{ is not a square modulo } P, \\ 0 & \text{if } P \mid f. \end{cases}$$

We extend the definition of the quadratic residue symbol above to any monic $D \in \mathbb{F}_q[t]$ by multiplicativity, and define the quadratic character χ_D by

$$\chi_D(f) = \left(\frac{D}{f}\right).$$

Since we assumed that $q \equiv 1 \pmod 4$, note that the quadratic reciprocity law takes the following form: if A and B are two monic coprime polynomials, then

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right).$$

Throughout the paper, we will often make use of the Perron formula over function fields. If the series $\sum_{f \in \mathcal{M}} a(f) u^{\deg(f)}$ is absolutely convergent for $|u| \le r < 1$ then

$$\sum_{f \in \mathcal{M}_n} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \left(\sum_{f \in \mathcal{M}} a(f) u^{\deg(f)} \right) \frac{du}{u^{n+1}},$$

and

$$\sum_{f \in \mathcal{M}_{\le n}} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \left(\sum_{f \in \mathcal{M}} a(f) u^{\deg(f)} \right) \frac{du}{u^{n+1}(1-u)}.$$

Recall that the twisted elliptic curve L-function $\mathcal{L}(E\otimes\chi_D, u)$ is a polynomial of degree $(\mathfrak{n}+2\deg(D))$, with $\mathfrak{n}$ being defined in (1-2). Thus we can write

$$\mathcal{L}(E\otimes\chi_D, u)=\sum_{n=0}^{\mathfrak{n}+2\deg(D)} c_n u^n,$$

where $c_n=\sum_{f\in M_n}\lambda(f)\chi_D(f)$.

Lemma 2.1. *The coefficients c_n of $\mathcal{L}(E\otimes\chi_D, u)$ satisfy the following relation*

$$c_n=\epsilon q^{n-\mathfrak{n}/2-\deg(D)}c_{\mathfrak{n}+2\deg(D)-n},$$

with ϵ as in (1-3). *In particular, if $\epsilon=-1$ and $\mathfrak{n}$ is even, then $c_{\mathfrak{n}/2+\deg(D)}=0$.*

Proof. From the functional equation (1-3) we have

$$\sum_{n=0}^{\mathfrak{n}+2\deg(D)} c_n u^n=\epsilon\sum_{n=0}^{\mathfrak{n}+2\deg(D)} c_n q^{\mathfrak{n}/2+\deg(D)-n}u^{\mathfrak{n}+2\deg(D)-n}.$$

By setting $k:=\mathfrak{n}+2\deg(D)-n$ we get

$$\sum_{n=0}^{\mathfrak{n}+2\deg(D)} c_n u^n=\epsilon\sum_{k=0}^{\mathfrak{n}+2\deg(D)} c_{\mathfrak{n}+2\deg(D)-k}q^{k-\mathfrak{n}/2-\deg(D)}u^k.$$

Comparing the coefficients we obtain the lemma. □

For $D\in\mathcal{H}^*_{2g+1}$, we can obtain the following exact formulas for $L\big(E\otimes\chi_D,\frac{1}{2}\big)$ and $L'\big(E\otimes\chi_D,\frac{1}{2}\big)$. These are the analogues of the approximate functional equations in the number field setting.

Lemma 2.2. *Let $D\in\mathcal{H}^*_{2g+1}$. Then*

$$L\big(E\otimes\chi_D,\tfrac{1}{2}\big)=\sum_{f\in\mathcal{M}_{\le[\mathfrak{n}/2]+\deg(D)}}\frac{\lambda(f)\chi_D(f)}{\sqrt{|f|}}+\epsilon\sum_{f\in\mathcal{M}_{\le[(\mathfrak{n}-1)/2]+\deg(D)}}\frac{\lambda(f)\chi_D(f)}{\sqrt{|f|}},$$

with ϵ as in (1-3).

Proof. We use Lemma 2.1 to get

$$\begin{aligned}\mathcal{L}(E\otimes\chi_D, u)&=\sum_{n=0}^{\mathfrak{n}+2\deg(D)} c_n u^n\\&=\sum_{n=0}^{[\mathfrak{n}/2]+\deg(D)} c_n u^n+\sum_{n=[\mathfrak{n}/2]+\deg(D)+1}^{\mathfrak{n}+2\deg(D)} c_n u^n\\&=\sum_{n=0}^{[\mathfrak{n}/2]+\deg(D)} c_n u^n+\epsilon\sum_{n=[\mathfrak{n}/2]+\deg(D)+1}^{\mathfrak{n}+2\deg(D)} c_{\mathfrak{n}+2\deg(D)-n}q^{n-\mathfrak{n}/2-\deg(D)}u^n.\end{aligned}$$

Changing the summation variable in the second sum leads to

$$\mathcal{L}(E\otimes\chi_D,u)=\sum_{n=0}^{[\mathfrak{n}/2]+\deg(D)} c_n u^n+\epsilon\sum_{n=0}^{[(\mathfrak{n}-1)/2]+\deg(D)} c_n (qu^2)^{\mathfrak{n}/2+\deg(D)-n}u^n.$$

Taking $u=q^{-1/2}$ and recalling that $c_n=\sum_{f\in M_n}\lambda(f)\chi_D(f)$ concludes the proof. □

Lemma 2.3. *Let $D\in\mathcal{H}^*_{2g+1}$. If $\epsilon=-1$, then*

$$L'\big(E\otimes\chi_D,\tfrac{1}{2}\big)=2(\log q)\sum_{f\in\mathcal{M}_{\le[\mathfrak{n}/2]+\deg(D)}}\frac{\big([\mathfrak{n}/2]+\deg(D)-\deg(f)\big)\lambda(f)\chi_D(f)}{\sqrt{|f|}}.$$

Proof. The above formula follows simply by differentiating the last equation in the proof of Lemma 2.2. Just note that as remarked in Lemma 2.1, if $\epsilon=-1$ and $\mathfrak{n}$ is even, then $c_{\mathfrak{n}/2+\deg(D)}=0$. □

We also have the following lemma which expresses a character sum over square-free polynomials in terms of sums over monics.

Lemma 2.4. *We have*

$$\sum_{D\in\mathcal{H}^*_{2g+1}}\chi_D(f)=\sum_{C_1\mid\Delta}\mu(C_1)\chi_{C_1}(f)\sum_{C_2\mid(\Delta f)^\infty}\sum_{R\in\mathcal{M}_{2g+1-\deg(C_1)-2\deg(C_2)}}\chi_R(f)$$
$$-q\sum_{C_1\mid\Delta}\mu(C_1)\chi_{C_1}(f)\sum_{C_2\mid(\Delta f)^\infty}\sum_{R\in\mathcal{M}_{2g-1-\deg(C_1)-2\deg(C_2)}}\chi_R(f),$$

where by $C_2\mid(\Delta f)^\infty$ we mean that the prime factors of C_2 divide Δf.

Proof. Let

$$\mathcal{A}(u)=\sum_{\substack{D\in\mathcal{H}\\(D,\Delta)=1}}\chi_D(f)u^{\deg(D)}.$$

Then

$$\begin{aligned}\mathcal{A}(u)&=\prod_{P\nmid\Delta f}(1+\chi_P(f)u^{\deg(P)})\\&=\prod_{P\nmid\Delta f}(1-u^{2\deg(P)})(1-\chi_P(f)u^{\deg(P)})^{-1}\\&=(1-qu^2)\mathcal{L}(u,\chi_f)\prod_{P\mid\Delta f}(1-u^{2\deg(P)})^{-1}\prod_{\substack{P\mid\Delta\\P\nmid f}}(1-\chi_P(f)u^{\deg(P)}).\end{aligned}$$

Writing

$$\prod_{P\mid\Delta f}(1-u^{2\deg(P)})^{-1}=\sum_{C_2\mid(\Delta f)^\infty}u^{2\deg(C_2)},\quad\prod_{\substack{P\mid\Delta\\P\nmid f}}(1-\chi_P(f)u^{\deg(P)})=\sum_{C_1\mid\Delta}\mu(C_1)\chi_{C_1}(f)u^{\deg(C_1)},$$

and comparing the coefficients of u^{2g+1}, the conclusion follows. □

As in [Hayes 1966] we define the exponential over function fields as follows. For $a \in \mathbb{F}_q((1/t))$ let

$$e_q(a) = \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_1)}{p}\right),$$

where a_1 is the coefficient of $1/t$ in the Laurent expansion of a and q is a power of the prime p. We define the generalized quadratic Gauss sum as

$$G(V, f) = \sum_{u \pmod f} \chi_f(u) e_q\left(\frac{uV}{f}\right),$$

where χ_f is the quadratic character defined before. We gather here a few useful facts about $G(V, f)$ whose proofs can be found in [Florea 2017c].

Lemma 2.5. (1) *If $(f, h) = 1$, then $G(V, fh) = G(V, f)G(V, h)$.*

(2) *Write $V = V_1 P^\alpha$ where $P \nmid V_1$. Then*

$$G(V, P^j) = \begin{cases} 0 & \text{if } j \le \alpha \text{ and } j \text{ odd}, \\ \phi(P^j) & \text{if } j \le \alpha \text{ and } j \text{ even}, \\ -|P|^{j-1} & \text{if } j = \alpha + 1 \text{ and } j \text{ even}, \\ \chi_P(V_1)|P|^{j-1/2} & \text{if } j = \alpha + 1 \text{ and } j \text{ odd}, \\ 0 & \text{if } j \ge 2 + \alpha. \end{cases}$$

The following Poisson summation formula in function fields holds.

Lemma 2.6. *Let $f \in \mathcal{M}_n$. If n is even, then*

$$\sum_{R \in \mathcal{M}_m} \chi_R(f) = \frac{q^m}{|f|}\left(G(0, f) + (q-1) \sum_{V \in \mathcal{M}_{\le n-m-2}} G(V, f) - \sum_{V \in \mathcal{M}_{n-m-1}} G(V, f)\right),$$

otherwise

$$\sum_{R \in \mathcal{M}_m} \chi_R(f) = \frac{q^m \overline{\tau(q)}}{|f|} \sum_{V \in \mathcal{M}_{n-m-1}} G(V, f),$$

where

$$\tau(q) = \sum_{c=1}^{q-1} \chi_f(c) \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(c)}{p}\right)$$

is the usual Gauss sum over $\mathbb{F}_q$.

2A. ***Outline of the proof.*** We will use the approximate functional equations for the L-functions involved in the moment computations and then truncate the Dirichlet series close to the endpoint. For the longer Dirichlet series, we will use Poisson summation and standard techniques to compute the main terms. For the tails, we will go back and write the Dirichlet series in terms of expressions involving moments and then use upper bounds for moments. The key in bounding the tails is the fact that the moments behave differently depending on the points where we evaluate them (the power of g gets smaller in different ranges).

3. Main proposition

For $N \mid \Delta^\infty$, let

$$S_{E_1,E_2}(N, X, Y; \alpha, \beta) := \sum_{D\in\mathcal{H}^*_{2g+1}} \sum_{\substack{f\in\mathcal{M}_{\le X}\\ h\in\mathcal{M}_{\le Y}}} \frac{\lambda_1(f)\lambda_2(h)\chi_D(Nfh)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}}, \tag{3-1}$$

where λ_i are the Dirichlet coefficients of $L(E_i, s)$ as in (1-1).

Proposition 3.1. *Assume $X \ge Y$. We have*

$$\begin{aligned} &S_E(N, X, Y)\\ &\quad := S_{E,E}(N, X, Y; 0, 0)\\ &\quad = |\mathcal{H}_{2g+1}|\mathcal{C}_E(N; 1, 1, 1)L(\mathrm{Sym}^2 E, 1)^3 Y + O(q^{2g}) + O(q^{2g-Y/5}g^2) + O(q^{g/2+3(X+Y)/8}g^{30}), \end{aligned}$$

and if $E_1 \ne E_2$, then

$$\begin{aligned} &S_{E_1,E_2}(N, X, Y; \alpha, \beta)\\ &\quad = |\mathcal{H}_{2g+1}|\mathcal{C}_{E_1,E_2}(N; 1, 1, 1, \alpha, \beta)L(\mathrm{Sym}^2 E_1, 1+2\alpha)L(\mathrm{Sym}^2 E_2, 1+2\beta)L(E_1\otimes E_2, 1+\alpha+\beta)\\ &\qquad + O(q^{2g-Y/5}g^2) + O(q^{g/2+3(X+Y)/8}g^{30}) \end{aligned}$$

uniformly for $|\alpha|, |\beta| \le 1/g$, where the values $\mathcal{C}_E(N; 1, 1, 1)$ and $\mathcal{C}_{E_1,E_2}(N; 1, 1, 1, \alpha, \beta)$ are defined in (3-8), (3-9) *and* (3-10).

We begin the proof of the proposition by applying Lemma 2.4 and rewriting $S_{E_1,E_2}(N, X, Y; \alpha, \beta)$ as

$$\begin{aligned} &\sum_{\substack{f\in\mathcal{M}_{\le X}\\ h\in\mathcal{M}_{\le Y}}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \sum_{C_1\mid\Delta} \mu(C_1)\chi_{C_1}(Nfh) \sum_{C_2\mid(\Delta fh)^\infty} \sum_{R\in\mathcal{M}_{2g+1-\deg(C_1)-2\deg(C_2)}} \chi_R(Nfh)\\ &\quad - q \sum_{\substack{f\in\mathcal{M}_{\le X}\\ h\in\mathcal{M}_{\le Y}}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \sum_{C_1\mid\Delta} \mu(C_1)\chi_{C_1}(Nfh) \sum_{C_2\mid(\Delta fh)^\infty} \sum_{R\in\mathcal{M}_{2g-1-\deg(C_1)-2\deg(C_2)}} \chi_R(Nfh)\\ &\qquad = S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta) + T_{E_1,E_2}(N, X, Y, Z; \alpha, \beta), \end{aligned} \tag{3-2}$$

where $S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta)$ denotes the contribution of the terms with $\deg(C_2) \le Z$ and where $T_{E_1,E_2}(N, X, Y, Z; \alpha, \beta)$ denotes that with $\deg(C_2) > Z$ for some $Z \le g$. We first estimate $T_{E_1,E_2}(N, X, Y, Z; \alpha, \beta)$, which is easier.

3A. *The term $T_{E_1,E_2}(N, X, Y, Z; \alpha, \beta)$.*

Lemma 3.2. *We have*

$$T_{E_1,E_2}(N, X, Y, Z; \alpha, \beta) \ll_\varepsilon q^{2g-3Z/2}g^{6+\varepsilon}$$

uniformly for $|\alpha|, |\beta| \le 1/g$.

Proof. It suffices to prove the bound for $T^{'}_{E_1,E_2}$, which is

$$\sum_{\substack{f\in\mathcal{M}_{\le X}\\ h\in\mathcal{M}_{\le Y}}}\frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}}\sum_{C_1\mid\Delta}\mu(C_1)\chi_{C_1}(Nfh)\sum_{\substack{C_2\mid(\Delta fh)^\infty\\ \deg(C_2)>Z}}\sum_{R\in\mathcal{M}_{2g+1-\deg(C_1)-2\deg(C_2)}}\chi_R(Nfh).$$

We use the Perron formula for the sum over f and h. We write $g(C_2)/(\mathrm{rad}(C_2),\Delta)=C=AB$, and replace f,h by Af and Bh, respectively, where $g(f)$ is defined to be $g(f)=\prod_{P\mid f}P$. Then

$$\begin{aligned}T^{'}_{E_1,E_2}=&\sum_{\substack{c_2>Z\\ c_1+2c_2\le 2g+1}}\sum_{\substack{C_1\in\mathcal{M}_{c_1}\\ C_2\in\mathcal{M}_{c_2}\\ C_1\mid\Delta,C=AB}}\frac{\mu(C_1)\chi_{C_1}(NAB)}{|A|^{1/2+\alpha}|B|^{1/2+\beta}}\sum_{R\in\mathcal{M}_{2g+1-c_1-2c_2}}\chi_R(NAB)\\ &\times\frac{1}{(2\pi i)^2}\oint_{|u|=r}\oint_{|v|=r}\sum_{f,h\in\mathcal{M}}\frac{\chi_{C_1R}(fh)\lambda_1(Af)\lambda_2(Bh)u^{\deg(f)}v^{\deg(h)}}{|f|^{1/2+\alpha}|h|^{1/2+\beta}}\\ &\times\frac{du\,dv}{u^{X-\deg(A)+1}v^{Y-\deg(B)+1}(1-u)(1-v)}\end{aligned}$$

for any $r<1$. The sum over f and h may be written as

$$\mathcal{D}_1(A,B,C_1R;u,v,\alpha,\beta)\mathcal{L}\Big(E_1\otimes\chi_{C_1R},\frac{u}{q^{1/2+\alpha}}\Big)\mathcal{L}\Big(E_2\otimes\chi_{C_1R},\frac{v}{q^{1/2+\beta}}\Big),$$

where $\mathcal{D}_1(A,B,C_1R;u,v,\alpha,\beta)$ is some Euler product which is uniformly convergent provided that $|u|,|v|\le q^{-1/g}$, and satisfies

$$\mathcal{D}_1(A,B,C_1R;u,v,\alpha,\beta)\ll\tau(AB)$$

uniformly in this region. Moving the u and v contours to $|u|=|v|=q^{-1/g}$ and using the bound

$$\begin{aligned}&\Big|\mathcal{L}\Big(E_1\otimes\chi_{C_1R},\frac{u}{q^{1/2+\alpha}}\Big)\mathcal{L}\Big(E_2\otimes\chi_{C_1R},\frac{v}{q^{1/2+\beta}}\Big)\Big|\\ &\qquad\ll\Big|\mathcal{L}\Big(E_1\otimes\chi_{C_1R},\frac{u}{q^{1/2+\alpha}}\Big)\Big|^2+\Big|\mathcal{L}\Big(E_2\otimes\chi_{C_1R},\frac{v}{q^{1/2+\beta}}\Big)\Big|^2\end{aligned}$$

we get

$$\begin{aligned}T^{'}_{E_1,E_2}\ll g^2&\sum_{\substack{c_2>Z\\ c_1+2c_2\le 2g+1}}\sum_{\substack{C_1\in\mathcal{M}_{c_1}\\ C_2\in\mathcal{M}_{c_2}\\ C_1\mid\Delta}}\frac{\tau(C)^2}{\sqrt{|C|}}\oint_{|u|=q^{-1/g}}\oint_{|v|=q^{-1/g}}\\ &\sum_{R\in\mathcal{M}_{2g+1-c_1-2c_2}}\bigg(\Big|\mathcal{L}\Big(E_1\otimes\chi_{C_1R},\frac{u}{q^{1/2+\alpha}}\Big)\Big|^2+\Big|\mathcal{L}\Big(E_2\otimes\chi_{C_1R},\frac{v}{q^{1/2+\beta}}\Big)\Big|^2\bigg)du\,dv.\end{aligned}$$

Now let $D=(C_1,R)$. Write $R=DR_1=DEH^2$, where E is square-free, and let $C_1=DC_3$. Then

$$\Big|\mathcal{L}\Big(E_1\otimes\chi_{C_1R},\frac{u}{q^{1/2+\alpha}}\Big)\Big|\ll_\varepsilon|DH|^\varepsilon\Big|\mathcal{L}\Big(E_1\otimes\chi_{C_3E},\frac{u}{q^{1/2+\alpha}}\Big)\Big|.$$

Using upper bounds for moments (see Remark 4.2 after Theorem 4.1), we get that

$$T'_{E_1,E_2} \ll_\varepsilon q^{2g} g^{3+\varepsilon} \sum_{\substack{C_2\in\mathcal{M} \\ Z<\deg(C_2)\le g}} \frac{\tau(C_2)^2}{\sqrt{|g(C_2)|C_2^2}} \ll_\varepsilon q^{2g-3Z/2} g^{6+\varepsilon},$$

and this finishes the proof of Lemma 3.2. □

3B. ***The term $S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta)$.*** Define $S'_{E_1,E_2}(N, X, Y, Z; \alpha, \beta)$ to be

$$\sum_{\substack{f\in\mathcal{M}_{\le X} \\ h\in\mathcal{M}_{\le Y}}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \sum_{C_1\mid\Delta} \mu(C_1)\chi_{C_1}(Nfh) \sum_{\substack{C_2\mid(\Delta fh)^\infty \\ \deg(C_2)\le Z}} \sum_{R\in\mathcal{M}_{2g+1-\deg(C_1)-2\deg(C_2)}} \chi_R(Nfh),$$

and $S''_{E_1,E_2}(N, X, Y, Z; \alpha, \beta)$ to be the same sum with g being replaced by $(g-1)$. Then

$$S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta) = S'_{E_1,E_2}(N, X, Y, Z; \alpha, \beta) - qS''_{E_1,E_2}(N, X, Y, Z; \alpha, \beta).$$

Using Lemma 2.6 on the sum over R, it follows that $S'_{E_1,E_2}(N, X, Y, Z; \alpha, \beta)$ equals

$$\begin{aligned} q^{2g+1} &\sum_{\substack{f\in\mathcal{M}_{\le X} \\ h\in\mathcal{M}_{\le Y} \\ \deg(Nfh)\text{ even}}} \frac{\lambda_1(f)\lambda_2(h)}{|N||f|^{3/2+\alpha}|h|^{3/2+\beta}} \sum_{\substack{C_1\mid\Delta \\ C_2\mid(\Delta fh)^\infty \\ \deg(C_2)\le Z \\ \deg(C_1)+2\deg(C_2)\le 2g+1}} \frac{\mu(C_1)\chi_{C_1}(Nfh)}{|C_1||C_2|^2} \\ &\times\Bigg(G(0, Nfh) + (q-1)\sum_{V\in\mathcal{M}_{\le\deg(Nfh)+\deg(C_1)+2\deg(C_2)-2g-3}} G(V, Nfh) - \sum_{V\in\mathcal{M}_{\deg(Nfh)+\deg(C_1)+2\deg(C_2)-2g-2}} G(V, Nfh)\Bigg) \\ &+ q^{2g+1}\overline{\tau(q)} \sum_{\substack{f\in\mathcal{M}_{\le X} \\ h\in\mathcal{M}_{\le Y} \\ \deg(Nfh)\text{ odd}}} \frac{\lambda_1(f)\lambda_2(h)}{|N||f|^{3/2+\alpha}|h|^{3/2+\beta}} \sum_{\substack{C_1\mid\Delta \\ C_2\mid(\Delta fh)^\infty \\ \deg(C_2)\le Z \\ \deg(C_1)+2\deg(C_2)\le 2g+1}} \frac{\mu(C_1)\chi_{C_1}(Nfh)}{|C_1||C_2|^2} \\ &\times \sum_{V\in\mathcal{M}_{\deg(Nfh)+\deg(C_1)+2\deg(C_2)-2g-2}} G(V, Nfh). \end{aligned}$$

Let $S'_{E_1,E_2}(V=0)$ denote the terms with $V=0$ above and $S'_{E_1,E_2}(V\neq 0)$ be the terms with nonzero V. The terms $S''_{E_1,E_2}(V=0)$ and $S''_{E_1,E_2}(V\neq 0)$ are similarly defined. Let

$$S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V=0) = S'_{E_1,E_2}(V=0) - qS''_{E_1,E_2}(V=0)$$

and

$$S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V\neq 0) = S'_{E_1,E_2}(V\neq 0) - qS''_{E_1,E_2}(V\neq 0)$$

so that we have

$$S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta) = S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V=0) + S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V\neq 0). \tag{3-3}$$

We shall evaluate $S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V=0)$ in Section 3C and bound $S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V\neq 0)$ in Section 3D.

3C. *The $V = 0$ terms.*

Lemma 3.3. *We have*

$$\begin{aligned}S_E(N, X, Y, Z; V=0) &:= S_{E,E}(N, X, Y, Z; 0, 0; V=0)\\ &= |\mathcal{H}_{2g+1}|\mathcal{C}_E(N; 1, 1, 1)L(\mathrm{Sym}^2 E, 1)^3 Y + O(q^{2g}) + O(q^{2g-Y/5}g^2),\end{aligned}$$

and if $E_1 \neq E_2$, then

$$\begin{aligned}&S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V=0)\\ &\quad= |\mathcal{H}_{2g+1}|\mathcal{C}_{E_1,E_2}(N; 1, 1, 1, \alpha, \beta)L(\mathrm{Sym}^2 E_1, 1+2\alpha)L(\mathrm{Sym}^2 E_2, 1+2\beta)L(E_1\otimes E_2, 1+\alpha+\beta)\\ &\qquad + O(q^{2g-Y/5}g^2) + O(q^{2g-3Z/2})\end{aligned}$$

uniformly for $|\alpha|, |\beta| \leq 1/g$, where the values $\mathcal{C}_E(N; 1, 1, 1)$ and $\mathcal{C}_{E_1,E_2}(N; 1, 1, 1, \alpha, \beta)$ are defined in (3-8), (3-9) *and* (3-10).

Proof. Note that $G(0, Nfh) \neq 0$ if and only if Nfh is a square polynomial, and in this case $G(0, Nfh) = \phi(Nfh)$. Hence

$$S'_{E_1,E_2}(V=0) = q^{2g+1} \sum_{\substack{f\in\mathcal{M}_{\leq X}\\ h\in\mathcal{M}_{\leq Y}\\ Nfh=\square}} \frac{\lambda_1(f)\lambda_2(h)\phi(Nfh)}{|N||f|^{3/2+\alpha}|h|^{3/2+\beta}} \sum_{\substack{C_2\,|\,(\Delta fh)^\infty\\ \deg(C_2)\leq Z}} \frac{1}{|C_2|^2} \sum_{\substack{C_1\,|\,\Delta\\ (C_1,Nfh)=1\\ \deg(C_1)\leq 2g+1-2\deg(C_2)}} \frac{\mu(C_1)}{|C_1|}. \tag{3-4}$$

We have

$$\begin{aligned}\sum_{\substack{C_1\,|\,\Delta\\ (C_1,Nfh)=1\\ \deg(C_1)\leq 2g+1-2\deg(C_2)}} \frac{\mu(C_1)}{|C_1|} &= \sum_{\substack{C_1\,|\,\Delta\\ (C_1,Nfh)=1}} \frac{\mu(C_1)}{|C_1|} - \sum_{\substack{C_1\,|\,\Delta\\ (C_1,Nfh)=1\\ \deg(C_1)> 2g+1-2\deg(C_2)}} \frac{\mu(C_1)}{|C_1|}\\ &= \prod_{\substack{P\,|\,\Delta\\ P\nmid Nfh}} \left(1-\frac{1}{|P|}\right) + O(q^{-2g}|C_2|^2)\\ &= \frac{|Nfh|}{\phi(Nfh)} \prod_{P\,|\,\Delta fh} \left(1-\frac{1}{|P|}\right) + O(q^{-2g}|C_2|^2).\end{aligned} \tag{3-5}$$

Note that

$$\sum_{\substack{C_2\in\mathcal{M}_n\\ C_2\,|\,(\Delta fh)^\infty}} 1 \ll_\varepsilon q^{\varepsilon n}. \tag{3-6}$$

Let $N = N_1^2 N_2$ with N_2 being square-free. The condition $Nfh = \square$ is equivalent to $fh = N_2\ell^2$ for some polynomial ℓ. Then we can write $f = N_2' A$ and $h = N_2'' B$, with $N_2' N_2'' = N_2$ and $AB = \ell^2$. It follows that the contribution of the error term in (3-5) to (3-4) will be

$$\ll_\varepsilon q^{\varepsilon g} \sum_{\ell \in \mathcal{M}_{\leq (X+Y)/2}} \frac{1}{|\ell|} \sum_{AB=\ell^2} |\lambda_1(N_2'A)\lambda_2(N_2''B)| \ll_\varepsilon q^{\varepsilon g},$$

by using the bound $|\lambda_i(f)| \leq \tau(f) \ll_\varepsilon |f|^\varepsilon$. Thus we can rewrite (3-4) as

$$S'_{E_1,E_2}(V=0) = q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ Nfh = \square}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \prod_{P \mid \Delta fh} \left(1 - \frac{1}{|P|}\right) \sum_{\substack{C_2 \mid (\Delta fh)^\infty \\ \deg(C_2) \leq Z}} \frac{1}{|C_2|^2} + O_\varepsilon(q^{\varepsilon g}).$$

We obtain a similar estimate for $S''_{E_1,E_2}(V=0)$ with g being replaced by $(g-1)$, and hence

$$\begin{aligned} &S_{E_1,E_2}(N,X,Y,Z;\alpha,\beta;V=0) \\ &\qquad = |\mathcal{H}_{2g+1}| \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ Nfh = \square}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \prod_{P \mid \Delta fh} \left(1 - \frac{1}{|P|}\right) \sum_{\substack{C_2 \mid (\Delta fh)^\infty \\ \deg(C_2) \leq Z}} \frac{1}{|C_2|^2} + O_\varepsilon(q^{\varepsilon g}). \end{aligned} \tag{3-7}$$

From the Perron formula for the sum over C_2,

$$\sum_{\substack{C_2 \mid (\Delta fh)^\infty \\ \deg(C_2) \leq Z}} \frac{1}{|C_2|^2} = \frac{1}{2\pi i} \oint_{|w|=r} \prod_{P \mid \Delta fh} \left(1 - \frac{w^{\deg(P)}}{|P|^2}\right)^{-1} \frac{dw}{w^{Z+1}(1-w)}$$

for any $r < 1$, it follows that

$$\begin{aligned} &S_{E_1,E_2}(N,X,Y,Z;\alpha,\beta;V=0) \\ &= \frac{|\mathcal{H}_{2g+1}|}{(2\pi i)^3} \oint_{|u|=r} \oint_{|v|=r} \oint_{|w|=r} \mathcal{A}_{E_1,E_2}(N;u,v,w,\alpha,\beta) \frac{du\,dv\,dw}{u^{X+1}v^{Y+1}w^{Z+1}(1-u)(1-v)(1-w)} + O_\varepsilon(q^{\varepsilon g}), \end{aligned}$$

where

$$\mathcal{A}_{E_1,E_2}(N;u,v,w,\alpha,\beta) = \sum_{\substack{f,h \in \mathcal{M} \\ Nfh = \square}} \frac{\lambda_1(f)\lambda_2(h)u^{\deg(f)}v^{\deg(h)}}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \prod_{P \mid \Delta fh} \left(1 - \frac{1}{|P|}\right)\left(1 - \frac{w^{\deg(P)}}{|P|^2}\right)^{-1}.$$

We can write down an Euler product for $\mathcal{A}_{E_1,E_2}(N;u,v,w,\alpha,\beta)$ as follows.

$$\begin{aligned} &\mathcal{A}_{E_1,E_2}(N;u,v,w,\alpha,\beta) \\ &\quad = \prod_{P \nmid \Delta} \left(1 + \left(1 - \frac{1}{|P|}\right)\left(1 - \frac{w^{\deg(P)}}{|P|^2}\right)^{-1} \sum_{i+j \text{ even} \geq 2} \frac{\lambda_1(P^i)\lambda_2(P^j)u^{i\deg(P)}v^{j\deg(P)}}{|P|^{(1/2+\alpha)i+(1/2+\beta)j}}\right) \\ &\qquad \times \prod_{P \mid \Delta} \left(\left(1 - \frac{1}{|P|}\right)\left(1 - \frac{w^{\deg(P)}}{|P|^2}\right)^{-1} \sum_{\substack{i,j \\ i+j+\mathrm{ord}_P(N) \text{ even}}} \frac{\lambda_1(P^i)\lambda_2(P^j)u^{i\deg(P)}v^{j\deg(P)}}{|P|^{(1/2+\alpha)i+(1/2+\beta)j}}\right). \end{aligned} \tag{3-8}$$

Then

$$\begin{aligned}\mathcal{A}_E(N; u, v, w, 0, 0)&\\ &:= \mathcal{A}_{E,E}(N; u, v, w, 0, 0)\\ &= \mathcal{C}_E(N; u, v, w)\mathcal{L}\left(\mathrm{Sym}^2 E, \frac{u^2}{q}\right)\mathcal{L}\left(\mathrm{Sym}^2 E, \frac{v^2}{q}\right) \times \mathcal{L}\left(\mathrm{Sym}^2 E, \frac{uv}{q}\right)\mathcal{Z}\left(\frac{uv}{q}\right),\end{aligned} \tag{3-9}$$

and

$$\begin{aligned}&\mathcal{A}_{E_1,E_2}(N; u, v, w, \alpha, \beta)\\ &\quad= \mathcal{C}_{E_1,E_2}(N; u, v, w, \alpha, \beta)\mathcal{L}\left(\mathrm{Sym}^2 E_1, \frac{u^2}{q^{1+2\alpha}}\right)\mathcal{L}\left(\mathrm{Sym}^2 E_2, \frac{v^2}{q^{1+2\beta}}\right)\mathcal{L}\left(E_1 \otimes E_2, \frac{uv}{q^{1+\alpha+\beta}}\right),\end{aligned} \tag{3-10}$$

if $E_1 \neq E_2$, where $\mathcal{C}_E(N; u, v, w)$ and $\mathcal{C}_{E_1,E_2}(N; u, v, w, \alpha, \beta)$ are some Euler products which are uniformly bounded for example when $|u|, |v| \leq q^{1/5}$, $|w| \leq q^{3/2}$.

Consider the case $E_1 = E_2 = E$. We have

$$\begin{aligned}&S_E(N, X, Y, Z; V = 0)\\ &\quad= \frac{|\mathcal{H}_{2g+1}|}{(2\pi i)^3}\oint_{|u|=r}\oint_{|v|=r}\oint_{|w|=r} \mathcal{C}_E(N; u, v, w)\mathcal{L}\left(\mathrm{Sym}^2 E, \frac{u^2}{q}\right)\\ &\qquad\times\mathcal{L}\left(\mathrm{Sym}^2 E, \frac{v^2}{q}\right)\mathcal{L}\left(\mathrm{Sym}^2 E, \frac{uv}{q}\right)\frac{du\,dv\,dw}{u^{X+1}v^{Y+1}w^{Z+1}(1-u)(1-v)(1-w)(1-uv)} + O_\varepsilon(q^{\varepsilon g})\end{aligned} \tag{3-11}$$

for any $r < 1$. We choose $r = q^{-1/g}$ and move the u contour to $|u| = q^{1/5}$, encountering two simple poles at $u = 1$ and $u = 1/v$. The new integral is trivially bounded by $O(q^{2g-X/5}g^2)$.

Furthermore, the contribution from the residue at $u = 1/v$ is

$$\begin{aligned}-|\mathcal{H}_{2g+1}|L(\mathrm{Sym}^2 E, 1)\frac{1}{(2\pi i)^2}\oint_{|v|=q^{-1/g}}\oint_{|w|=q^{-1/g}} &\mathcal{C}_E(N; 1/v, v, w)\\ &\times \mathcal{L}\left(\mathrm{Sym}^2 E, \frac{1}{v^2 q}\right)\mathcal{L}\left(\mathrm{Sym}^2 E, \frac{v^2}{q}\right)\frac{dv\,dw}{v^{Y-X}w^{Z+1}(1-v)^2(1-w)},\end{aligned}$$

which is $O(q^{2g})$. This can be seen by first moving the v contour to $|v| = q^{-1/5}$, creating no poles, and then moving the w contour to $|w| = q^{3/2}$, crossing a simple pole at $w = 1$. Both the new integral and the residue at $w = 1$ are $O(q^{2g})$ as $X \geq Y$. So

$$\begin{aligned}S_E(N, X, Y, Z; V = 0) = |\mathcal{H}_{2g+1}|L(\mathrm{Sym}^2 E, 1)\frac{1}{(2\pi i)^2}\oint_{|v|=q^{-1/g}}\oint_{|w|=q^{-1/g}} &\mathcal{C}_E(N; 1, v, w)\\ \times \mathcal{L}\left(\mathrm{Sym}^2 E, \frac{v^2}{q}\right)\mathcal{L}\left(\mathrm{Sym}^2 E, \frac{v}{q}\right)&\frac{dv\,dw}{v^{Y+1}w^{Z+1}(1-v)^2(1-w)}\\ &+ O(q^{2g}) + O(q^{2g-X/5}g^2).\end{aligned}$$

We now move the v contour to $|v| = q^{1/5}$, encountering a double pole at $v = 1$. The new integral is bounded by $O(q^{2g-Y/5}g)$, and an argument similar to the above implies that the residue at $v = 1$ is

$$|\mathcal{H}_{2g+1}|L(\mathrm{Sym}^2 E, 1)^3 Y\frac{1}{2\pi i}\oint_{|w|=q^{-1/g}} \mathcal{C}_E(N; 1, 1, w)\frac{dw}{w^{Z+1}(1-w)} + O(q^{2g}).$$

Hence

$$S_E(N, X, Y, Z; V=0) = |\mathcal{H}_{2g+1}| \mathcal{C}_E(N; 1, 1, 1) L(\mathrm{Sym}^2 E, 1)^3 Y + O(q^{2g}) + O(q^{2g-Y/5} g^2).$$

For $E_1 \neq E_2$, we have that

$$\begin{aligned} S_{E_1,E_2}&(N, X, Y, Z; \alpha, \beta; V=0) \\ &= \frac{|\mathcal{H}_{2g+1}|}{(2\pi i)^3} \oint_{|u|=r} \oint_{|v|=r} \oint_{|w|=r} \mathcal{C}_{E_1,E_2}(N; u, v, w, \alpha, \beta) \mathcal{L}\Big(\mathrm{Sym}^2 E_1, \frac{u^2}{q^{1+2\alpha}}\Big) \mathcal{L}\Big(\mathrm{Sym}^2 E_2, \frac{v^2}{q^{1+2\beta}}\Big) \\ &\qquad \times \mathcal{L}\Big(E_1 \otimes E_2, \frac{uv}{q^{1+\alpha+\beta}}\Big) \frac{du\,dv\,dw}{u^{X+1} v^{Y+1} w^{Z+1} (1-u)(1-v)(1-w)} + O_\varepsilon(q^{\varepsilon g}) \end{aligned}$$

for any $r < 1$. We choose $r = q^{-1/g}$ and first shift the u contour to $|u| = q^{1/5}$, encountering a pole at $u = 1$. The new integral over $|u| = q^{1/5}$, $|v| = |w| = q^{-1/g}$ is bounded by $q^{2g-X/5} g^2$. To calculate the residue at $u = 1$, we move the v contour to $|v| = q^{1/5}$, crossing a pole at $v = 1$. The new integral is $O(q^{2g-Y/5} g)$. For the residue at $u = v = 1$, we move the w contour to $|w| = q^{3/2}$. In doing so we obtain

$$\begin{aligned} S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V=0) = |\mathcal{H}_{2g+1}| \mathcal{C}_{E_1,E_2}(N; 1, 1, 1, \alpha, \beta) L(\mathrm{Sym}^2 E_1, 1+2\alpha) \\ \times L(\mathrm{Sym}^2 E_2, 1+2\beta) L(E_1 \otimes E_2, 1+\alpha+\beta) \\ + O(q^{2g-Y/5} g^2) + O(q^{2g-3Z/2}), \end{aligned}$$

and this concludes the proof of the lemma. □

3D. ***The $V \neq 0$ terms.***

Lemma 3.4. *We have*

$$S_{E_1,E_2}(N, X, Y, Z; \alpha, \beta; V \neq 0) \ll q^{(X+Y+Z)/2} g^{30}$$

uniformly for $|\alpha|, |\beta| \leq 1/g$.

Proof. We will prove the bound for the term

$$\begin{aligned} S(V \neq 0) = q^{2g+1} \overline{\tau(q)} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ \deg(Nfh) \text{ odd}}} \frac{\lambda_1(f) \lambda_2(h)}{|N||f|^{3/2+\alpha} |h|^{3/2+\beta}} \sum_{\substack{C_1 \mid \Delta \\ C_2 \mid (\Delta f h)^\infty \\ \deg(C_2) \leq Z \\ \deg(C_1) + 2\deg(C_2) \leq 2g+1}} \frac{\mu(C_1) \chi_{C_1}(Nfh)}{|C_1||C_2|^2} \\ \times \sum_{V \in \mathcal{M}_{\deg(Nfh)+\deg(C_1)+2\deg(C_2)-2g-2}} G(V, Nfh), \qquad (3\text{-}12) \end{aligned}$$

the treatment of the other terms being similar. We also assume for simplicity that $\deg(N)$, X and Y are all odd. The other cases can be done similarly.

We use the Perron formula in the forms

$$\sum_{\substack{f\in\mathcal{M}_{\leq X}\\ \deg(f)\ \mathrm{odd}}} a(f)=\frac{1}{2\pi i}\oint_{|u|=r}\bigg(\sum_{f\in\mathcal{M}} a(f)u^{\deg(f)}\bigg)\frac{du}{u^{X+1}(1-u^2)}$$

and

$$\sum_{\substack{f\in\mathcal{M}_{\leq X}\\ \deg(f)\ \mathrm{even}}} a(f)=\frac{1}{2\pi i}\oint_{|u|=r}\bigg(\sum_{f\in\mathcal{M}} a(f)u^{\deg(f)}\bigg)\frac{du}{u^{X}(1-u^2)}$$

for the sums over f and h. We write $V=V_1V_2^2$ with V_1 being a square-free polynomial and $V_2\in\mathcal{M}$, $g(C_2)/(\mathrm{rad}(C_2),\Delta)=C=AB$, and replace f,h by Af and Bh, respectively. We then see that

$$\begin{aligned}S(V\neq 0)=q^{2g+1}\overline{\tau(q)}&\sum_{\substack{c_2\leq Z\\ c_1+2c_2\leq 2g+1}}\ \sum_{\substack{C_1\in\mathcal{M}_{c_1}\\ C_2\in\mathcal{M}_{c_2}\\ C_1\,|\,\Delta, C=AB}}\frac{\mu(C_1)\chi_{C_1}(NAB)}{|N||A|^{3/2+\alpha}|B|^{3/2+\beta}|C_1||C_2|^2}\\ &\times\frac{1}{(2\pi i)^3}\oint_{|u|=r}\oint_{|v|=r}\oint_{|w|=r}\sum_{V_1\in\mathcal{H}}\mathcal{B}(N,A,B,C_1,V_1;u/w,v/w,w,\alpha,\beta)\\ &\times\frac{(1+uv)\,du\,dv\,dw}{u^{X-\deg(A)+1}v^{Y-\deg(B)+1}w^{\deg(NAB)-\deg(V_1)+c_1+2c_2-2g-1}(1-u^2)(1-v^2)}\end{aligned}\tag{3-13}$$

for any $r<1$, where $\mathcal{B}(N,A,B,C_1,V_1;u,v,w,\alpha,\beta)$ equals

$$\sum_{f,h,V_2\in\mathcal{M}}\frac{\chi_{C_1}(fh)\lambda_1(Af)\lambda_2(Bh)u^{\deg(f)}v^{\deg(h)}w^{2\deg(V_2)}G(V_1V_2^2,NABfh)}{|f|^{3/2+\alpha}|h|^{3/2+\beta}}.$$

To proceed we need to study the function $\mathcal{B}(N,A,B,C_1,V_1;u,v,w,\alpha,\beta)$.

Lemma 3.5. *The function $\mathcal{B}(N,A,B,C_1,V_1;u,v,w,\alpha,\beta)$ defined above may be written as*

$$\mathcal{D}_2(N,A,B,C_1,V_1;u,v,w,\alpha,\beta)\mathcal{L}\bigg(E_1\otimes\chi_{C_1V_1},\frac{u}{q^{1+\alpha}}\bigg)\mathcal{L}\bigg(E_2\otimes\chi_{C_1V_1},\frac{v}{q^{1+\beta}}\bigg),$$

where $\mathcal{D}_2(N,A,B,C_1,V_1;u,v,w,\alpha,\beta)$ is some Euler product which is uniformly convergent provided that $|u|,|v|\leq q^{1/2-2/g}$, $|w|\leq q^{-1/2-\varepsilon}$, and satisfies

$$\mathcal{D}_2(N,A,B,C_1,V_1;u,v,w,\alpha,\beta)\ll g^{10}\tau_{10}(AB)\sqrt{AB}$$

uniformly in this region.

Proof. It is easy to see that $\mathcal{B}(N,A,B,C_1,V_1;u,v,w,\alpha,\beta)$ converges absolutely if $|u|,|v|\leq q^{-\varepsilon}$ and $|w|\leq q^{-1/2-\varepsilon}$. We claim that the sum over f,h and V_2 is triply multiplicative. Indeed, one can easily see that the double sum over f, h is multiplicative, so

$$\sum_{f,h\in\mathcal{M}} \frac{\chi_{C_1}(fh)\lambda_1(Af)\lambda_2(Bh)u^{\deg(f)}v^{\deg(h)}G(V_1V_2^2, NABfh)}{|f|^{3/2+\alpha}|h|^{3/2+\beta}}$$
$$= \prod_P \biggl(\sum_{i,j} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^{i+a_P})\lambda_2(P^{j+b_P})u^{i\deg(P)}v^{j\deg(P)}G(V_1V_2^2, P^{i+j+a_P+b_P+n_P})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}}\biggr),$$

where a_P, b_P and n_P denote the orders of A, B and N with respect to P respectively. Let $A_P(V_2)$ denote the Euler product above. Note that when $P\nmid V_2$, we have $A_P(V_2) = A_P(1)$. Then we rewrite the double sum over f, h as

$$\prod_P A_P(1) \prod_{P\mid V_2} \frac{A_P(V_2)}{A_P(1)}.$$

We introduce the sum over V_2 and use the observation that for $(V_2, V_3) = 1$ and $P\nmid V_3$ we have $A_P(V_2V_3) = A_P(V_2)$. Then

$$\prod_P A_P(1) \sum_{V_2\in\mathcal{M}} w^{2\deg(V_2)} \prod_{P\mid V_2} \frac{A_P(V_2)}{A_P(1)}$$
$$= \prod_P A_P(1) \prod_P \biggl(1 + \frac{1}{A_P(1)} \sum_k A_P(P^k) w^{2k\deg(P)}\biggr)$$
$$= \prod_P \biggl(\sum_{i,j,k} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^{i+a_P})\lambda_2(P^{j+b_P})u^{i\deg(P)}v^{j\deg(P)}w^{2k\deg(P)}G(V_1P^{2k}, P^{i+j+a_P+b_P+n_P})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}}\biggr),$$

and hence the generating series for f, h, V_2 is indeed triply multiplicative.

Now we rewrite $\mathcal{B}(N, A, B, C_1, V_1; u, v, w, \alpha, \beta)$ as

$$\prod_{P\nmid C_1} \biggl(\sum_{i,j,k} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^{i+a_P})\lambda_2(P^{j+b_P})u^{i\deg(P)}v^{j\deg(P)}w^{2k\deg(P)}G(V_1P^{2k}, P^{i+j+a_P+b_P+n_P})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}}\biggr)$$
$$\times \prod_{P\mid C_1} \biggl(\sum_k \lambda_1(P^{a_P})\lambda_2(P^{b_P})w^{2k\deg(P)}G(V_1P^{2k}, P^{a_P+b_P+n_P})\biggr).$$

We next compute the Euler factors at an irreducible P in the region $|u|, |v| \le q^{1/2-2/g}$, $|w| \le q^{-1/2-\varepsilon}$. Note that in this region, $w^{2k\deg(P)} \ll_\varepsilon |P|^{-1-\varepsilon}$ if $k \ge 1$.

Consider first the case when $P\nmid NABC_1V_1$. The contribution of such an Euler factor is

$$\sum_{i,j,k} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^i)\lambda_2(P^j)u^{i\deg(P)}v^{j\deg(P)}w^{2k\deg(P)}G(V_1P^{2k}, P^{i+j})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}}.$$

In view of Lemma 2.5, this is equal to

$$1 + \frac{\chi_{C_1V_1}(P)\lambda_1(P)u^{\deg(P)}}{|P|^{1+\alpha}} + \frac{\chi_{C_1V_1}(P)\lambda_2(P)v^{\deg(P)}}{|P|^{1+\beta}} + O\biggl(\frac{1}{|P|^{1+\varepsilon}}\biggr),$$

which justifies the two L-functions.

In the case $P \mid V_1$ but $P \nmid NABC_1$, the Euler factor equals

$$\sum_{i,j} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^i)\lambda_2(P^j)u^{i\deg(P)}v^{j\deg(P)}G(V_1, P^{i+j})}{|P|^{(3/2+\alpha)i+(3/2+beta)j}} + O\biggl(\frac{1}{|P|^{1+\varepsilon}}\biggr)$$
$$= 1 - \frac{\lambda_1(P^2)u^{2\deg(P)}}{|P|^{2+2\alpha}} - \frac{\lambda_1(P)\lambda_2(P)(uv)^{\deg(P)}}{|P|^{2+\alpha+\beta}} - \frac{\lambda_2(P^2)v^{2\deg(P)}}{|P|^{2+2\beta}} + O\biggl(\frac{1}{|P|^{1+\varepsilon}}\biggr)$$
$$= 1 + O\biggl(\frac{1}{|P|^{1+2/g}}\biggr).$$

Similarly, the corresponding Euler factor is

$$= \begin{cases} O(|P|^{1/2}) & \text{if } P \mid AB \text{ and } P \nmid NC_1V_1, \\ O(1) & \text{if } P \mid AB,\ P \mid V_1 \text{ and } P \nmid NC_1, \\ 1 + O(1/|P|^{1+\varepsilon}) & \text{if } P \mid C_1 \text{ and } P \nmid NAB, \\ O(|P|^{1/2}) & \text{if } \ P \mid C_1,\ P \mid AB \text{ and } P \nmid NV_1, \\ 0 & \text{if } \ P \mid C_1,\ P \mid AB,\ P \mid V_1 \text{ and } P \nmid N, \end{cases}$$

and is

$$\ll_\varepsilon \sum_{i+j \le 2k} |P|^{1-(1+\varepsilon)k-(i+j)/2} \ll_\varepsilon |P|,$$

if $P \mid N$.

The lemma easily follows by combining these estimates. □

We now return to (3-13). In view of Lemma 3.5, we take $r \le q^{-1/2-\varepsilon}$ and move the u and v contours to $|u| = |v| = rq^{1/2-2/g}$. This creates no poles. Then, by the above result,

$$\mathcal{B}(N, A, B, C_1, V_1; u/w, v/w, w, \alpha, \beta)$$
$$\ll g^{10}\tau_{10}(AB)\sqrt{AB}\biggl|\mathcal{L}\biggl(E_1 \otimes \chi_{C_1V_1}, \frac{u/w}{q^{1+\alpha}}\biggr)\mathcal{L}\biggl(E_2 \otimes \chi_{C_1V_1}, \frac{v/w}{q^{1+\beta}}\biggr)\biggr|$$
$$\ll g^{10}\tau_{10}(AB)\sqrt{AB}\biggl(\biggl|\mathcal{L}\biggl(E_1 \otimes \chi_{C_1V_1}, \frac{u/w}{q^{1+\alpha}}\biggr)\biggr|^2 + \biggl|\mathcal{L}\biggl(E_2 \otimes \chi_{C_1V_1}, \frac{v/w}{q^{1+\beta}}\biggr)\biggr|^2\biggr).$$

So

$$S(V \neq 0)$$
$$\ll q^{2g-(X+Y)/2}g^{10} \sum_{\substack{c_2 \le Z \\ c_1+2c_2 \le 2g+1}} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_2 \in \mathcal{M}_{c_2} \\ C_1 \mid \Delta}} \frac{\tau(C)\tau_{10}(C)\tau(C_1)}{\sqrt{|C|}|C_1||C_2|^2}$$
$$\times \oint_{|u|=q^{1/2-1/g}} \oint_{|v|=q^{1/2-1/g}} \oint_{|w|=r} \sum_{V_1 \in \mathcal{H}} \biggl|\mathcal{L}\biggl(E_1 \otimes \chi_{C_1V_1}, \frac{u}{q^{1+\alpha}}\biggr)\biggr|^2 \frac{du\,dv\,dw}{|w|^{X+Y+\deg(N)-\deg(V_1)+c_1+2c_2-2g+1}}.$$

If $\deg(V_1) \le X + Y + c_1 + 2c_2 - 2g$, then we move the w contour to $|w| = q^{-3/4}$, otherwise we move the w contour to $|w| = q^{-5/4}$. Using the upper bounds for moments as in Theorem 4.1 (see Remark 4.2) we find that

$$S(V\neq 0)\ll q^{(X+Y)/2}g^{11}\sum_{\substack{c_2\le Z\\ c_1+2c_2\le 2g+1}}\sum_{\substack{C_1\in\mathcal{M}_{c_1}\\ C_2\in\mathcal{M}_{c_2}\\ C_1\mid\Delta}}\frac{\tau(C)\tau_{10}(C)}{\sqrt{|C|}}$$

$$\ll q^{(X+Y)/2}g^{11}\sum_{C_2\in\mathcal{M}_{\le Z}}\frac{\tau(C_2)\tau_{10}(C_2)}{\sqrt{|g(C_2)|}}\ll q^{(X+Y+Z)/2}g^{30},$$

which finishes the proof of Lemma 3.4. □

Proposition 3.1 follows upon combining the estimates and choosing $Z=g-(X+Y)/4$.

4. Upper bounds for moments

The aim of this section is to bound the tails of the Dirichlet series in the approximate functional equations in Lemmas 2.2 and 2.3. We start with the following upper bounds for moments.

Theorem 4.1. *Let $k>0$, $u=e^{i\theta}$, $v=e^{i\gamma}$ with $\theta,\gamma\in[0,2\pi]$ and let $m=\deg(\mathcal{L}(E\otimes\chi_D,w))$. Then*

$$\sum_{D\in\mathcal{H}^*_{2g+1}}\left|\mathcal{L}\left(E\otimes\chi_D,\frac{u}{q^{1/2+\alpha}}\right)\mathcal{L}\left(E\otimes\chi_D,\frac{v}{q^{1/2+\beta}}\right)\right|^k\ll_\varepsilon q^{2g}g^\varepsilon\exp\left(k\mathcal{M}(u,v,m)+\frac{k^2}{2}\mathcal{V}(u,v,m)\right)$$

uniformly for $|\alpha|,|\beta|\le 1/g$, where $\mathcal{M}(u,v,m)$ and $\mathcal{V}(u,v,m)$ are given by (4-9) *and* (4-11) *respectively.*

Remark 4.2. Note that the same upper bound as above holds if we replace $\mathcal{L}(E\otimes\chi_D,w)$ with $\mathcal{L}(E\otimes\chi_{D\ell},w)$ for a fixed polynomial ℓ with $(\ell,\Delta)=1$. Since the proof of the upper bound for this twisted moment is the same as the proof of Theorem 4.1, we only focus on $\ell=1$.

We first need the following proposition, whose proof is similar to the proof of Theorem 3.3 in [Altuğ and Tsimerman 2014].

Proposition 4.3. *Let $D\in\mathcal{H}^*_{2g+1}$ and let $m=\deg(\mathcal{L}(E\otimes\chi_D,w))$. Then for $h\le m$ and z with $\Re(z)\ge 0$ we have*

$$\log\left|L\left(E\otimes\chi_D,\tfrac12+z\right)\right|\le\frac{m}{h}+\frac1h\Re\left(\sum_{\substack{j\ge1\\ \deg(P^j)\le h}}\frac{\chi_D(P^j)(\alpha(P)^j+\beta(P)^j)\log q^{h-j\deg(P)}}{|P|^{j(1/2+z+1/(h\log q))}\log q^j}\right).$$

Proof. We write

$$L(E\otimes\chi_D,s)=\prod_{j=1}^m(1-\alpha_jq^{1/2-s}),$$

where $|\alpha_j|=1$ (see [Hall et al. 2017]). Then

$$\frac{L'}{L}(E\otimes\chi_D,s)=\log q\left(-\frac{m}{2}+\sum_{j=1}^m\left(\frac{1}{1-\alpha_jq^{1/2-s}}-\frac12\right)\right).\tag{4-1}$$

We put $s=\sigma+z$ and integrate (4-1) with respect to σ from $\frac{1}{2}$ to σ_0, where $\sigma_0>\frac{1}{2}$. Taking real parts gives

$$\begin{aligned}\log\bigl|L\bigl(E\otimes\chi_D,\tfrac{1}{2}+z\bigr)\bigr|&-\log|L(E\otimes\chi_D,\sigma_0+z)|\\&=\frac{m\log q}{2}\left(\sigma_0-\frac{1}{2}\right)-\frac{1}{2\log q}\sum_{j=1}^{m}\log\frac{q^{\sigma_0-1/2}-2q^{-\Re(z)}\cos(\theta_j-\log q\Im(z))+q^{1/2-\sigma_0-2\Re(z)}}{1+q^{-2\Re(z)}-2q^{-\Re(z)}\cos(\theta_j-\log q\Im(z))},\end{aligned}$$

where $\alpha_j=e^{i\theta_j}$. We use the inequality $\log(1+x)\geq x/(1+x)$ for $x>0$ and get that

$$\begin{aligned}\log\bigl|L\bigl(E\otimes\chi_D,\tfrac{1}{2}+z\bigr)\bigr|&-\log|L(E\otimes\chi_D,\sigma_0+z)|\\&\leq\frac{m\log q}{2}\left(\sigma_0-\frac{1}{2}\right)-\frac{1}{2\log q}\sum_{j=1}^{m}\frac{(1-q^{1/2-\sigma_0-2\Re(z)})(1-q^{1/2-\sigma_0})}{1-2q^{1/2-\sigma_0-\Re(z)}\cos(\theta_j-\log q\Im(z))+q^{1-2\sigma_0-2\Re(z)}}\\&=\frac{m\log q}{2}\left(\sigma_0-\frac{1}{2}\right)-\frac{G_z(\sigma_0)(1-q^{1/2-\sigma_0-2\Re(z)})(1-q^{1/2-\sigma_0})}{(1-q^{1-2\sigma_0-2\Re(z)})\log q},\end{aligned}\tag{4-2}$$

where

$$G_z(\sigma)=\frac{1-q^{1-2\sigma-2\Re(z)}}{2}\sum_{j=1}^{m}\frac{1}{1-2q^{1/2-\sigma-\Re(z)}\cos(\theta_j-\log q\Im(z))+q^{1-2\sigma-2\Re(z)}}.\tag{4-3}$$

Now similarly as in [Altuğ and Tsimerman 2014] we compute the integral

$$\frac{1}{2\pi i}\int_{2}^{2+2\pi i/\log q}-\frac{L'}{L}(E\otimes\chi_D,\sigma+z+w)\frac{q^{hw}q^{-w}}{(1-q^{-w})^2}\,dw$$

in two different ways. First we write

$$-\frac{L'}{L}(E\otimes\chi_D,s)=\sum_{n\geq 0}\frac{\lambda_D(n)}{q^{ns}},$$

and integrate term by term. Secondly we continue analytically to the left and pick up the residues. We integrate with respect to σ from σ_0 to ∞ and take real parts, which gives

$$\begin{aligned}-\frac{1}{(\log q)^2}\Re\biggl(\sum_{n\leq h}\frac{\lambda_D(n)\log q^{h-n}}{q^{n(\sigma_0+z)}\log q^n}\biggr)&\\=-\frac{h}{\log q}\log|L(E\otimes\chi_D,\sigma_0+z)|&-\frac{1}{(\log q)^2}\Re\left(\frac{L'}{L}(E\otimes\chi_D,\sigma_0+z)\right)\\&+\sum_{j=1}^{m}\Re\left(\int_{\sigma_0}^{\infty}\frac{(\alpha_jq^{1/2-\sigma_0-z})^h\alpha_j^{-1}q^{\sigma+z-1/2}}{(1-\alpha_j^{-1}q^{\sigma+z-1/2})^2}\,d\sigma\right).\end{aligned}\tag{4-4}$$

Now we have

$$\Re\left(\int_{\sigma_0}^{\infty}\frac{(\alpha_j q^{1/2-\sigma_0-z})^h\alpha_j^{-1}q^{\sigma+z-1/2}}{(1-\alpha_j^{-1}q^{\sigma+z-1/2})^2}\,d\sigma\right)\le\int_{\sigma_0}^{\infty}\frac{q^{(1/2-\sigma-\Re(z))h}q^{\sigma+\Re(z)-1/2}}{|1-\alpha_j^{-1}q^{\sigma+z-1/2}|^2}\,d\sigma$$
$$=\int_{\sigma_0}^{\infty}\frac{q^{(1/2-\sigma-\Re(z))h}q^{1/2-\sigma-\Re(z)}}{1-2q^{1/2-\sigma-\Re(z)}\cos(\theta_j-\log q\,\Im(z))+q^{1-2\sigma-2\Re(z)}}\,d\sigma.$$

By taking the derivative of

$$f(x)=\frac{q^{-x}}{1-2q^{1/2-x-\Re(z)}\cos(\theta_j-\log q\,\Im(z))+q^{1-2x-2\Re(z)}}$$

we can see that f is decreasing on $[\sigma_0,\infty)$. Hence

$$\sum_{j=1}^{m}\Re\left(\int_{\sigma_0}^{\infty}\frac{(\alpha_j q^{1/2-\sigma_0-z})^h\alpha_j^{-1}q^{\sigma+z-1/2}}{(1-\alpha_j^{-1}q^{\sigma+z-1/2})^2}\,d\sigma\right)$$
$$\le\sum_{j=1}^{m}\frac{q^{1/2-\sigma_0-\Re(z)}}{1-2q^{1/2-\sigma_0-\Re(z)}\cos(\theta_j-\log q\,\Im(z))+q^{1-2\sigma_0-2\Re(z)}}\int_{\sigma_0}^{\infty}q^{(1/2-\sigma-\Re(z))h}\,d\sigma$$
$$=\frac{2G_z(\sigma_0)q^{(1/2-\sigma_0-\Re(z))h}q^{1/2-\sigma_0-\Re(z)}}{(1-q^{1-2\sigma_0-2\Re(z)})\log q^h}$$
$$\le\frac{2G_z(\sigma_0)q^{(1/2-\sigma_0-\Re(z))h}}{(\sigma_0+\Re(z)-1/2)\log q\,\log q^h},$$

with $G_z(\sigma_0)$ as in (4-3). Now from (4-1) note that

$$\Re\left(\frac{L'}{L}(E\otimes\chi_D,\sigma_0+z)\right)=\log q\left(-\frac{m}{2}+G_z(\sigma_0)\right).$$

Combining the equations above and (4-4) gives

$$\log|L(E\otimes\chi_D,\sigma_0+z)|\le\frac{1}{\log q^h}\Re\left(\sum_{n\le h}\frac{\lambda_D(n)\log q^{h-n}}{q^{n(\sigma_0+z)}\log q^n}\right)+\frac{1}{h}\left(\frac{m}{2}-G_z(\sigma_0)\right)+\frac{2G_z(\sigma_0)q^{(1/2-\sigma_0-\Re(z))h}}{h(\sigma_0+\Re(z)-1/2)\log q^h}.$$

This and (4-2) lead to

$$\log\left|L\left(E\otimes\chi_D,\tfrac{1}{2}+z\right)\right|\le\frac{m}{2h}+\frac{m\log q}{2}\left(\sigma_0-\frac{1}{2}\right)+\frac{1}{\log q^h}\Re\left(\sum_{n\le h}\frac{\lambda_D(n)\log q^{h-n}}{q^{n(\sigma_0+z)}\log q^n}\right)$$
$$+G_z(\sigma_0)\left(-\frac{1}{h}+\frac{2q^{(1/2-\sigma_0-\Re(z))h}}{h(\sigma_0+\Re(z)-1/2)\log q^h}-\frac{(1-q^{1/2-\sigma_0-2\Re(z)})(1-q^{1/2-\sigma_0})}{(1-q^{1-2\sigma_0-2\Re(z)})\log q}\right).$$

Choosing $\sigma_0 = \frac{1}{2} + 1/\log q^h$ ensures that the coefficient of $G_z(\sigma_0)$ is negative. Since

$$\lambda_D(n) = \log q \sum_{j \mid n} \sum_{\deg(P^j)=n} \deg(P)\chi_D(P)^j(\alpha(P)^j + \beta(P)^j),$$

the conclusion follows. □

Before proving Theorem 4.1, we also need the following lemma (see Lemma 8.4 in [Florea 2017a]).

Lemma 4.4. *Let h, l be integers such that $2hl \le 2g+1$. For any complex numbers $a(P)$ we have*

$$\sum_{D \in \mathcal{H}^*_{2g+1}} \bigg| \sum_{\deg(P) \le h} \frac{\chi_D(P)a(P)}{|P|^{1/2}} \bigg|^{2l} \ll q^{2g} \frac{(2l)!}{l!2^l} \bigg(\sum_{\deg(P) \le h} \frac{|a(P)|^2}{|P|} \bigg)^l.$$

Let

$$N(V, u, v) = \bigg| \bigg\{ D \in \mathcal{H}^*_{2g+1} : \log \bigg| \mathcal{L}\Big(E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}}\Big) \mathcal{L}\Big(E \otimes \chi_D, \frac{v}{q^{1/2+\beta}}\Big) \bigg| \ge V + \mathcal{M}(u, v, m) \bigg\} \bigg|.$$

We will prove the following lemma.

Lemma 4.5. *If $\sqrt{\log m} \le V \le \mathcal{V}(u, v, m)$, then*

$$N(V, u, v) \ll q^{2g+1} \exp\bigg(-\frac{V^2}{2\mathcal{V}(u, v, m)} \bigg(1 - \frac{8}{\log\log m} \bigg) \bigg);$$

if $\mathcal{V}(u, v, m) < V \le \frac{1}{13} \log\log m \mathcal{V}(u, v, m)$, then

$$N(V, u, v) \ll q^{2g+1} \exp\bigg(-\frac{V^2}{2\mathcal{V}(u, v, m)} \bigg(1 - \frac{8V}{\mathcal{V}(u, v, m) \log\log m} \bigg)^2 \bigg);$$

if $V > \frac{1}{13} \log\log m \mathcal{V}(u, v, m)$, then

$$N(V, u, v) \ll q^{2g+1} \exp\bigg(-\frac{V \log V}{4500} \bigg).$$

Using Lemma 4.5 above we can prove Theorem 4.1 as follows.

Proof of Theorem 4.1. We have the following.

$$\begin{aligned} \sum_{D \in \mathcal{H}^*_{2g+1}} \bigg| \mathcal{L}\Big(E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}}\Big) \mathcal{L}\Big(E \otimes \chi_D, \frac{v}{q^{1/2+\beta}}\Big) \bigg|^k &= -\int_{-\infty}^{\infty} \exp(kV + k\mathcal{M}(u, v, m))\, dN(V, u, v) \\ &= k \int_{-\infty}^{\infty} \exp(kV + k\mathcal{M}(u, v, m)) N(V, u, v)\, dV. \end{aligned}$$

In the equation above we use Lemma 4.5 in the form

$$N(V, u, v) \ll \begin{cases} q^{2g+1} m^{o(1)} \exp(-V^2/2\mathcal{V}(u, v, m)) & \text{if } V \le 8k\mathcal{V}(u, v, m), \\ q^{2g+1} m^{o(1)} \exp(-4kV) & \text{if } V > 8k\mathcal{V}(u, v, m), \end{cases}$$

which finishes the proof of Theorem 4.1. □

Proof of Lemma 4.5. We assume without loss of generality that α, β are positive and real. Indeed, notice that if $\alpha \in \mathbb{C}$, since $|\alpha| \leq 1/g$, we have $|\Re(\alpha)| \leq 1/g$ and $|\Im(\alpha)| \leq 1/g$. The proof that follows goes through in exactly the same way, with α replaced by $\Re(\alpha)$ and θ replaced by $\theta - \Im(\alpha)\log q$. Once we assume α is real, we can also assume that α is positive, since by the functional equation we have

$$\left|\mathcal{L}\left(E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}}\right)\right| = q^{-\alpha(\mathfrak{n}+2\deg(D))}\left|\mathcal{L}\left(E \otimes \chi_D, \frac{1}{q^{1/2-\alpha}u}\right)\right|.$$

Let

$$\frac{m}{h} = \frac{V}{A} \quad \text{and} \quad h_0 = \frac{h}{\log m},$$

where

$$A = \begin{cases} \frac{1}{2}\log\log m & \text{if } V \leq \mathcal{V}(u, v, m), \\ \frac{1}{2V}(\log\log m)\mathcal{V}(u, v, m) & \text{if } \mathcal{V}(u, v, m) < V \leq \frac{1}{16}\log\log m\mathcal{V}(u, v, m), \\ 8 & \text{if } V > \frac{1}{16}\log\log m\mathcal{V}(u, v, m). \end{cases}$$

Using Proposition 4.3 gives

$$\log\left|\mathcal{L}\left(E\otimes\chi_D, \frac{u}{q^{1/2+\alpha}}\right)\mathcal{L}\left(E\otimes\chi_D, \frac{v}{q^{1/2+\beta}}\right)\right|$$
$$\leq \frac{2m}{h} + \frac{1}{h}\Re\left(\sum_{\substack{j\geq 1 \\ \deg(P^j)\leq h}} \frac{\chi_D(P^j)(\alpha(P)^j+\beta(P)^j)\log q^{h-j\deg(P)}}{|P|^{(1/2+1/h\log q)j}\log q^j}(|P|^{-(\alpha-i\theta/\log q)j}+|P|^{-(\beta-i\gamma/\log q)j})\right). \quad (4\text{-}5)$$

Note that the contribution of the terms with $j \geq 3$ is bounded by $O(1)$.

The terms with $j = 2$ in (4-5) will contribute, up to a term of size $O(\log\log m)$ coming from those P with $P \mid D$,

$$\frac{1}{2}\sum_{\deg(P)\leq h/2} \frac{(h-2\deg(P))(\lambda(P^2)-1)}{h|P|^{1+2/h\log q}}\left(\frac{\cos(2\theta\deg(P))}{|P|^{2\alpha}} + \frac{\cos(2\gamma\deg(P))}{|P|^{2\beta}}\right).$$

Let

$$F_\alpha(h, \theta) = \sum_{n=1}^{h} \frac{\cos(2n\theta)}{nq^{2n\alpha+n/h\log q}}.$$

Similarly as in [Florea 2017a, Lemma 9.1], we can show that

$$F_\alpha(h, \theta) = \log\min\left\{h, \frac{1}{2\bar{\theta}}\right\} + O(1), \quad (4\text{-}6)$$

where for $\theta \in [0, 2\pi]$ we denote $\bar{\theta} = \min\{\theta, 2\pi - \theta\}$. Now using the fact that

$$\sum_{\deg(P)\leq h} \frac{\lambda(P^2)\cos(2\theta\deg(P))}{|P|^{1+1/h\log q}} = O(\log\log h), \quad (4\text{-}7)$$

it follows that the contribution from $j = 2$ will be equal to

$$
\begin{aligned}
-\frac{1}{2} \sum_{\deg(P) \le \frac{h}{2}} & \frac{h - 2\deg(P)}{h|P|^{1+2/h \log q}} \left(\frac{\cos(2\theta \deg(P))}{|P|^{2\alpha}} + \frac{\cos(2\gamma \deg(P))}{|P|^{2\beta}} \right) + O(\log\log m) \\
&= -\frac{1}{2}\left(F_\alpha\left(\frac{h}{2}, \theta\right) + F_\beta\left(\frac{h}{2}, \gamma\right)\right) + O(\log\log m) \\
&\le -\frac{1}{2}(F_\alpha(m,\theta) + F_\beta(m,\gamma)) + \frac{2m}{h} + O(\log\log m) = \mathcal{M}(u,v,m) + \frac{2m}{h} + O(\log\log m), \qquad (4\text{-}8)
\end{aligned}
$$

where

$$
\mathcal{M}(u,v,m) = -\frac{1}{2}\left(\log\min\left\{m, \frac{1}{\overline{2\theta}}\right\} + \log\min\left\{m, \frac{1}{\overline{2\gamma}}\right\}\right) \qquad (4\text{-}9)
$$

by formula (4-6). Note that in the second line of the equation above we used the fact that

$$
F_\alpha(m,\theta) - F_\alpha\left(\frac{h}{2}, \theta\right) = \sum_{n=1}^{m} \frac{\cos(2n\theta)}{nq^{2n\alpha}e^{n/m}} - \sum_{n=1}^{h/2} \frac{\cos(2n\theta)}{nq^{2n\alpha}e^{2n/h}},
$$

and since $e^{-x} = 1 + O(x)$, it follows that

$$
F_\alpha(m,\theta) - F_\alpha\left(\frac{h}{2}, \theta\right) = \sum_{n=h/2+1}^{m} \frac{\cos(2n\theta)}{nq^{2n\alpha}} + O(1) \le \frac{2m}{h} + O(1).
$$

Hence, using (4-8) in (4-5) we get

$$
\begin{aligned}
\log & \left| \mathcal{L}\left(E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}}\right) \mathcal{L}\left(E \otimes \chi_D, \frac{v}{q^{1/2+\beta}}\right) \right| \\
&\le \mathcal{M}(u,v,m) + \frac{5m}{h} + \sum_{\deg(P) \le h} \frac{(h - \deg(P))\chi_D(P)\lambda(P)}{h|P|^{1/2+1/(h\log q)}} \left(\frac{\cos(\theta \deg(P))}{|P|^{\alpha}} + \frac{\cos(\gamma \deg(P))}{|P|^{\beta}} \right).
\end{aligned}
$$

Let S_1 be the sum above truncated at $\deg(P) \le h_0$ and S_2 be the sum over primes with $h_0 < \deg(P) \le h$. If D is such that

$$
\log \left| \mathcal{L}\left(E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}}\right) \mathcal{L}\left(E \otimes \chi_D, \frac{v}{q^{1/2+\beta}}\right) \right| \ge \mathcal{M}(u,v,m) + V,
$$

then

$$
S_1 \ge V_1 := V\left(1 - \frac{6}{A}\right) \quad \text{or} \quad S_2 \ge \frac{V}{A}.
$$

Let

$$
\mathcal{F}_1 = \{D \in \mathcal{H}^*_{2g+1} : S_1 \ge V_1\} \quad \text{and} \quad \mathcal{F}_2 = \left\{ D \in \mathcal{H}^*_{2g+1} : S_2 \ge \frac{V}{A} \right\}.
$$

If $D \in \mathcal{F}_2$, then by Markov's inequality and Lemma 4.4 it follows that

$$
|\mathcal{F}_2| \ll q^{2g} \frac{(2l)!}{l!2^l} \left(\frac{A}{V}\right)^{2l} \left(\sum_{h_0 < \deg(P) \le h} \frac{|a(P)|^2}{|P|} \right)^{l},
$$

for any $l \le g/h$ where

$$a(P) = \frac{(h - \deg(P))\lambda(P)}{h|P|^{1/(h \log q)}}\left(\frac{\cos(\theta \deg(P))}{|P|^{\alpha}} + \frac{\cos(\gamma \deg(P))}{|P|^{\beta}}\right).$$

Picking $l = [g/h]$ and noting that $a(P) \ll 1$ and $m = 4g + O(1)$, we get that

$$|\mathcal{F}_2| \ll q^{2g}\left(\frac{2l}{e}\right)^{l}\left(\frac{A}{V}\right)^{2l}(\log\log m)^{l} \ll q^{2g}\exp\left(-\frac{V\log V}{8A}\right). \tag{4-10}$$

If $D \in \mathcal{F}_1$ then for any $l \le g/h_0$, we have

$$|\mathcal{F}_1| \ll q^{2g}\frac{(2l)!}{l!2^l}\frac{1}{V_1^{2l}}\left(\sum_{\deg(P)\le h_0}\frac{|a(P)|^2}{|P|}\right)^{l}.$$

Using the expression for $a(P)$ and (4-7) we get that

$$|\mathcal{F}_1| \ll q^{2g}\frac{(2l)!}{l!2^l V_1^{2l}}(\mathcal{V}(u, v, m) + O(\log\log m))^{l},$$

where

$$\begin{aligned}\mathcal{V}(u, v, m) &\\ = \log m + \tfrac{1}{2}(F_\alpha(m, \theta) + F_\beta(m, \gamma)) &+ F_{(\alpha+\beta)/2}\left(m, \frac{\overline{\theta+\gamma}}{2}\right) + F_{(\alpha+\beta)/2}\left(m, \frac{\overline{\theta-\gamma}}{2}\right)\\ = \log m + \tfrac{1}{2}\left(\log\min\left\{m, \frac{1}{\overline{\overline{2\theta}}}\right\} + \log\min\left\{m, \frac{1}{\overline{\overline{2\gamma}}}\right\}\right) &+ \log\min\left\{m, \frac{1}{\overline{\overline{\theta+\gamma}}}\right\} + \log\min\left\{m, \frac{1}{\overline{\overline{\theta-\gamma}}}\right\},\end{aligned} \tag{4-11}$$

and the last line of the equation above follows from (4-6). Then

$$|\mathcal{F}_1| \ll q^{2g}\left(\frac{2l}{eV_1^2}(\mathcal{V}(u, v, m) + O(\log\log m))\right)^{l}.$$

If $V \le \mathcal{V}(u, v, m)^2$, then we pick $l = [V_1^2/2\mathcal{V}(u, v, m)]$, and if $V > \mathcal{V}(u, v, m)^2$, then we pick $l = [10V]$. In doing so we get

$$|\mathcal{F}_1| \ll q^{2g}\exp\left(-\frac{V_1^2}{2\mathcal{V}(u, v, m)}\right) + q^{2g}\exp(-V\log V). \tag{4-12}$$

Combining the bounds (4-12) and (4-10) finishes the proof of Lemma 4.5. □

The following is an immediate corollary of Theorem 4.1.

Corollary 4.6. *Let $u = e^{i\theta}$ and $v = e^{i\gamma}$ with $\theta, \gamma \in [0, 2\pi]$. Then*

$$\sum_{D\in\mathcal{H}^*_{2g+1}}\left|\mathcal{L}\left(E\otimes\chi_D, \frac{u}{\sqrt{q}}\right)\mathcal{L}\left(E\otimes\chi_D, \frac{v}{\sqrt{q}}\right)\right| \\ \ll_\varepsilon q^{2g}g^{1/2+\varepsilon}\min\left\{g, \frac{1}{\overline{\overline{2\theta}}}\right\}^{-1/4}\min\left\{g, \frac{1}{\overline{\overline{2\gamma}}}\right\}^{-1/4}\min\left\{g, \frac{1}{\overline{\overline{(\theta-\gamma)}}}\right\}^{1/2}\min\left\{g, \frac{1}{\overline{\overline{(\theta+\gamma)}}}\right\}^{1/2}.$$

For $N \mid \Delta^\infty$ and fixed $n \in \mathbb{N}$, we define the truncated sums

$$\mathcal{E}_{1,E}(N, X, n) := \sum_{X < \deg(f) \le n + \deg(D)} \frac{\lambda(f)\chi_D(Nf)}{\sqrt{|f|}} \tag{4-13}$$

and

$$\mathcal{E}_{2,E}(N, X, n) := \sum_{X < \deg(f) \le n + \deg(D)} \frac{(n + \deg(D) - \deg(f))\lambda(f)\chi_D(Nf)}{\sqrt{|f|}}. \tag{4-14}$$

We are now ready to prove the following upper bounds for $\mathcal{E}_{i,E}(N, X, n)$.

Proposition 4.7. *For $i = 1, 2$ and any fixed $n \in \mathbb{N}$ we have*

$$\sum_{D \in \mathcal{H}^*_{2g+1}} |\mathcal{E}_{i,E}(N, X, n)|^2 \ll_\varepsilon q^{2g} g^{1/2+\varepsilon}(2g - X)^{2i}.$$

Proof. Using the Perron formula for the sum over f in (4-13), we get that

$$\mathcal{E}_{1,E}(N, X, n) = \frac{\chi_D(N)}{2\pi i} \oint_{|u|=1} \mathcal{L}\left(E \otimes \chi_D, \frac{u}{\sqrt{q}}\right)\left(\frac{1}{u^{n+\deg(D)}} - \frac{1}{u^X}\right) \frac{du}{u(1-u)}.$$

Note that there is no pole at $u = 1$. So we need to bound the following expression

$$\frac{1}{(2\pi i)^2} \oint_{|u|=1} \oint_{|v|=1} \sum_{D \in \mathcal{H}^*_{2g+1}} \mathcal{L}\left(E \otimes \chi_D, \frac{u}{\sqrt{q}}\right) \mathcal{L}\left(E \otimes \chi_D, \frac{v}{\sqrt{q}}\right) \times \left(\frac{1}{u^{n+\deg(D)}} - \frac{1}{u^X}\right)\left(\frac{1}{v^{n+\deg(D)}} - \frac{1}{v^X}\right) \frac{du\,dv}{uv(1-u)(1-v)}. \tag{4-15}$$

We use Corollary 4.6 to bound the integral above and consider θ and γ on different arcs on the unit circle. We bound the integral on these arcs and notice that we obtain the biggest upper bound when θ, γ are not close to 0 (i.e., u and v are not close to 1) and when θ is not close to γ or to $2\pi - \gamma$ (by close we mean on an arc of length on the scale of $1/g$).

For example, if θ and γ are both on an arc C_1 of length on the scale of $1/g$ around 0, then the double integral in (4-15) over the arcs C_1 is $O_\varepsilon(q^{2g} g^{-1+\varepsilon})$ (since from the corollary we get a power of g which gets multiplied by g^{-2}, the product of the sizes of the arcs.)

If θ, γ are both on the complement of C_1, but close to each other (i.e., θ is within $1/g$ of γ), we get that the corresponding integral over the two arcs is $O_\varepsilon(q^{2g} g^\varepsilon)$. We get a similar bound if θ is close to $2\pi - \gamma$, under the same conditions.

We are left with the case when θ, γ are on the complement of C_1 and θ is far from γ and from $2\pi - \gamma$. In this case the corresponding integral will be $O_\varepsilon(q^{2g} g^{1/2+\varepsilon})$. This finishes the proof of the upper bound when $i = 1$.

When $i = 2$, using the Perron formula for the sum over f in (4-14), we have

$$\mathcal{E}_{2,E}(N, X, n) = \frac{\chi_D(N)}{2\pi i} \oint_{|u|=1} \mathcal{L}\left(E \otimes \chi_D, \frac{u}{\sqrt{q}}\right)\left(\frac{1}{u^{n+\deg(D)}} - \frac{(n + \deg(D) - X)(1-u) + u}{u^{X+1}}\right) \frac{du}{(1-u)^2}.$$

Hence

$$\mathcal{E}_{2,E}(N, X, n)^2 = \frac{1}{(2\pi i)^2} \oint_{|u|=1} \oint_{|v|=1} \mathcal{L}\Big(E \otimes \chi_D, \frac{u}{\sqrt{q}}\Big) \mathcal{L}\Big(E \otimes \chi_D, \frac{v}{\sqrt{q}}\Big) \times \left(\frac{1}{u^{n+\deg(D)}} - \frac{(n+\deg(D)-X)(1-u)+u}{u^{X+1}}\right) \times \left(\frac{1}{v^{n+\deg(D)}} - \frac{(n+\deg(D)-X)(1-v)+v}{v^{X+1}}\right) \frac{du dv}{(1-u)^2(1-v)^2}.$$

We proceed as before and keeping in mind that $|u| = |v| = 1$, it follows that

$$\sum_{D \in \mathcal{H}^*_{2g+1}} |\mathcal{E}_{2,E}(N, X, n)|^2 \ll_\varepsilon q^{2g} g^{1/2+\varepsilon} (2g - X)^4,$$

as required. □

5. Proof of Theorem 1.1

For $N \mid \Delta^\infty$, let

$$R_E(N, X) := \sum_{D \in \mathcal{H}^*_{2g+1}} \sum_{f \in \mathcal{M}_{\le X}} \frac{\lambda(f) \chi_D(Nf)}{\sqrt{|f|}}.$$

We will prove the following lemma.

Lemma 5.1. *We have*

$$R_E(N, X) = |\mathcal{H}_{2g+1}| \mathcal{C}_E(N; 1) L(\mathrm{Sym}^2 E, 1) + O_\varepsilon(q^{2g - X/2 + \varepsilon g}) + O_\varepsilon(q^{X/2 + \epsilon g}),$$

where the value $\mathcal{C}_E(N; 1)$ is defined in (5-2)*.*

Proof. Note that

$$R_E(N, X) = S_{E,E}(N, X, 0; 0, 0),$$

where $S_{E_1,E_2}(N, X, Y; \alpha, \beta)$ is defined as in (3-1). We proceed as in Section 3, see (3-2), (3-3) and (3-7), and write

$$R_E(N, X) = S_{E,E}(N, X, 0, g; 0, 0) = R_E(N, X; V = 0) + R_E(N, X; V \ne 0),$$

where $R_E(N, X; V \ne 0) = S_{E,E}(N, X, 0, g; 0, 0; V \ne 0)$ and

$$\begin{aligned} R_E(N, X; V = 0) &= S_{E,E}(N, X, 0, g; 0, 0; V = 0) \\ &= |\mathcal{H}_{2g+1}| \sum_{\substack{f \in \mathcal{M}_{\le X} \\ Nf = \square}} \frac{\lambda(f)}{\sqrt{|f|}} \prod_{P \mid \Delta f} \left(1 - \frac{1}{|P|}\right) \sum_{\substack{C_2 \mid (\Delta f)^\infty \\ \deg(C_2) \le g}} \frac{1}{|C_2|^2} + O_\varepsilon(q^{\varepsilon g}). \end{aligned}$$

We first evaluate $R_E(N,X;V=0)$. From (3-6) we have

$$\sum_{\substack{C_2\mid(\Delta f)^\infty\\ \deg(C_2)\le g}}\frac{1}{|C_2|^2}=\prod_{P\mid\Delta f}\left(1-\frac{1}{|P|^2}\right)^{-1}+O_\varepsilon(q^{-2g+\varepsilon g}).$$

The contribution of the error term to $R_E(N,X;V=0)$ is

$$\ll_\varepsilon q^{\varepsilon g}\sum_{l\in\mathcal{M}_{\le X/2}}\frac{|\lambda(Nl^2)|}{|l|}\ll_\varepsilon q^{\varepsilon g}.$$

Hence

$$R_E(N,X;V=0)=|\mathcal{H}_{2g+1}|\sum_{\substack{f\in\mathcal{M}_{\le X}\\ Nf=\square}}\frac{\lambda(f)}{\sqrt{|f|}}\prod_{P\mid\Delta f}\left(1+\frac{1}{|P|}\right)^{-1}+O_\varepsilon(q^{\varepsilon g}).$$

Applying the Perron formula to the sum over f yields

$$R_E(N,X;V=0)=\frac{|\mathcal{H}_{2g+1}|}{2\pi i}\oint_{|u|=r}\mathcal{B}_E(N;u)\frac{du}{u^{X+1}(1-u)}+O_\varepsilon(q^{\varepsilon g})\tag{5-1}$$

for any $r<1$, where

$$\mathcal{B}_E(N;u)=\sum_{\substack{f\in\mathcal{M}\\ Nf=\square}}\frac{\lambda(f)u^{\deg(f)}}{\sqrt{|f|}}\prod_{P\mid\Delta f}\left(1+\frac{1}{|P|}\right)^{-1}.$$

We can write $\mathcal{B}_E(N;u)$ in terms of its Euler product,

$$\begin{aligned}\mathcal{B}_E(N;u)&=\prod_{P\nmid\Delta}\left(1+\left(1+\frac{1}{|P|}\right)^{-1}\sum_{i\ge1}\frac{\lambda(P^{2i})u^{2i\deg(P)}}{|P|^i}\right)\prod_{P\mid\Delta}\left(\left(1+\frac{1}{|P|}\right)^{-1}\sum_{i+\mathrm{ord}_P(N)\text{ even}}\frac{\lambda(P^i)u^{i\deg(P)}}{|P|^{i/2}}\right)\\&=\mathcal{C}_E(N;u)\mathcal{L}\left(\mathrm{Sym}^2E,\frac{u^2}{q}\right),\end{aligned}\tag{5-2}$$

where $\mathcal{C}_E(N;u)$ is some Euler product which is uniformly bounded for $|u|\le q^{1/2-\varepsilon}$. We shift the contour in (5-1) to $|u|=q^{1/2-\varepsilon}$, encountering a simple pole at $u=1$. Then

$$R_E(N,X;V=0)=|\mathcal{H}_{2g+1}|\mathcal{C}_E(N;1)L(\mathrm{Sym}^2E,1)+O_\varepsilon(q^{2g-X/2+\varepsilon g}).\tag{5-3}$$

Now we will bound $R_E(N,X;V\ne0)$. As in Section 3D, see (3-12), it suffices to bound the term

$$R(V\ne0)=q^{2g+1}\overline{\tau(q)}\sum_{\substack{f\in\mathcal{M}_{\le X}\\ \deg(Nf)\text{ odd}}}\frac{\lambda(f)}{|N||f|^{3/2}}\sum_{\substack{C_1\mid\Delta\\ C_2\mid(\Delta f)^\infty\\ \deg(C_1)+2\deg(C_2)\le 2g+1}}\frac{\mu(C_1)\chi_{C_1}(Nf)}{|C_1||C_2|^2}\sum_{V\in\mathcal{M}_{\deg(Nf)+\deg(C_1)+2\deg(C_2)-2g-2}}G(V,Nf).$$

Using the fact that

$$\sum_{\substack{C_2 \in \mathcal{M}_{c_2} \\ C_2 \mid (\Delta f)^\infty}} \frac{1}{|C_2|^2} = \frac{q^{-2c_2}}{2\pi i} \oint_{|w|=r} \prod_{P \mid \Delta f} (1 - w^{\deg(P)})^{-1} \frac{dw}{w^{c_2+1}}$$

for $r < 1$, and writing $V = V_1 V_2^2$ with V_1 a square-free polynomial, we have

$$\begin{aligned} R(V \neq 0) = \frac{q^{2g+1}\overline{\tau(q)}}{|N|} & \sum_{c_1+2c_2 \le 2g+1} q^{-2c_2} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_1 \mid \Delta}} \frac{\mu(C_1)\chi_{C_1}(N)}{|C_1|} \\ & \times \sum_{\substack{n \le X \\ n+\deg(N) \text{ odd}}} \sum_{\substack{j \le n+\deg(N)+c_1+2c_2-2g-2 \\ j+c_1 \text{ odd}}} \sum_{V_1 \in \mathcal{H}_j} \sum_{V_2 \in \mathcal{M}_{(n+\deg(N)+c_1-j)/2+c_2-g-1}} \\ & \times \frac{1}{2\pi i} \oint_{|w|=r} \sum_{f \in \mathcal{M}_n} \frac{\chi_{C_1}(f)\lambda(f)G(V_1V_2^2, Nf)}{|f|^{3/2}} \prod_{P \mid \Delta f} (1 - w^{\deg(P)})^{-1} \frac{dw}{w^{c_2+1}}. \end{aligned}$$

Now

$$\begin{aligned} \sum_{f \in \mathcal{M}} \frac{\chi_{C_1}(f)\lambda(f)G(V_1V_2^2, Nf)}{|f|^{3/2}} & \prod_{P \mid \Delta f} (1 - w^{\deg(P)})^{-1} u^{\deg(f)} \\ & = \mathcal{H}(V_1; u, w)\mathcal{I}(V_1V_2^2, N; u, w)\mathcal{J}(V_1V_2^2; u, w), \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}(V_1; u, w) &= \prod_{P \nmid V_1} \left(1 + \frac{\chi_{C_1V_1}(P)\lambda(P)u^{\deg(P)}}{|P|}(1 - w^{\deg(P)})^{-1}\right), \\ \mathcal{I}(V_1V_2^2, N; u, w) &= \prod_{P \mid \Delta} \left(\sum_j \frac{\chi_{C_1}(P^j)\lambda(P^j)G(V_1V_2^2, P^{j+n_P})u^{j\deg(P)}}{|P|^{3j/2}}\right)(1 - w^{\deg(P)})^{-1} \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}(V_1V_2^2; u, w) = \prod_{\substack{P \mid V_1V_2 \\ P \nmid \Delta}} & \left(1 + \sum_{j \ge 1} \frac{\chi_{C_1}(P^j)\lambda(P^j)G(V_1V_2^2, P^j)u^{j\deg(P)}}{|P|^{3j/2}}(1 - w^{\deg(P)})^{-1}\right) \\ & \times \prod_{\substack{P \nmid V_1 \\ P \mid \Delta V_2}} \left(1 + \frac{\chi_{C_1V_1}(P)\lambda(P)u^{\deg(P)}}{|P|}(1 - w^{\deg(P)})^{-1}\right)^{-1}. \end{aligned}$$

We use the Perron formula for the sum over f and obtain

$$R(V\neq 0)=\frac{q^{2g+1}\overline{\tau(q)}}{|N|}\sum_{c_1+2c_2\leq 2g+1}q^{-2c_2}\sum_{\substack{C_1\in\mathcal{M}_{c_1}\\ C_1\mid\Delta}}\frac{\mu(C_1)\chi_{C_1}(N)}{|C_1|}$$
$$\times\sum_{\substack{n\leq X\\ n+\deg(N)\text{ odd}}}\sum_{\substack{j\leq n+\deg(N)+c_1+2c_2-2g-2\\ j+c_1\text{ odd}}}\sum_{V_1\in\mathcal{H}_j}\sum_{V_2\in\mathcal{M}_{(n+\deg(N)+c_1-j)/2+c_2-g-1}}$$
$$\times\frac{1}{(2\pi i)^2}\oint_{|u|=r}\oint_{|w|=r}\mathcal{H}(V_1;u,w)\mathcal{I}(V_1V_2^2,N;u,w)\mathcal{J}(V_1V_2^2;u,w)\,\frac{du}{u^{n+1}}\frac{dw}{w^{c_2+1}}. \tag{5-4}$$

Let $r_1=q^{1/2-\varepsilon}$, $r_2=q^{-\varepsilon}$, and let k_0 be minimal such that $|r_1r_2^{k_0}|<1$. Then we can write

$$\mathcal{H}(V_1;u,w)=\mathcal{L}\Big(E\otimes\chi_{C_1V_1},\frac{u}{q}\Big)\mathcal{L}\Big(E\otimes\chi_{C_1V_1},\frac{uw}{q}\Big)\cdots\mathcal{L}\Big(E\otimes\chi_{C_1V_1},\frac{uw^{k_0-1}}{q}\Big)\mathcal{K}(V_1;u,w), \tag{5-5}$$

where

$$\mathcal{K}(V_1;u,w)\ll_\varepsilon|C_1|^\varepsilon$$

uniformly for $|u|\leq r_1$ and $|w|\leq r_2$. We also have

$$\mathcal{I}(V_1V_2^2,N;u,w)\ll_\varepsilon|V_1V_2|^\varepsilon\quad\text{and}\quad\mathcal{J}(V_1V_2^2;u,w)\ll_\varepsilon|V_2|^\varepsilon$$

in this region. We now move the contours in (5-4) to $|u|=r_1$ and $|w|=r_2$. We then use the Lindelöf bound for each L-function and trivially bound the rest of the expression to obtain that

$$R(V\neq 0)\ll_\varepsilon q^{X/2+\varepsilon g}.$$

Combining this with (5-3) finishes the proof of Lemma 5.1. □

To prove Theorem 1.1, note that from Lemma 2.2 we have

$$\sum_{D\in\mathcal{H}^*_{2g+1}}L\big(E\otimes\chi_D,\tfrac12\big)=R_E(1,[\mathfrak{n}/2]+2g+1)+\epsilon_{2g+1}\epsilon(E)R_E(M,[(\mathfrak{n}+1)/2]+2g).$$

Using Lemma 5.1 and choosing $X=g$ we have that

$$\frac{1}{|\mathcal{H}^*_{2g+1}|}\sum_{D\in\mathcal{H}^*_{2g+1}}L\big(E\otimes\chi_D,\tfrac12\big)=c_1(M)L(\operatorname{Sym}^2E,1)+O_\varepsilon(q^{-g+\varepsilon g}),$$

where

$$c_1(M)=(\mathcal{C}_E(1;1)+\epsilon_{2g+1}\epsilon(E)\mathcal{C}_E(M;1))\prod_{P\mid\Delta}\frac{|P|+1}{|P|}. \tag{5-6}$$

This finishes the proof of the theorem.

6. Proof of Theorem 1.2

For $N \mid \Delta^\infty$, let

$$R_E(N, X; \alpha) := \sum_{D \in \mathcal{H}^*_{2g+1}} \sum_{f \in \mathcal{M}_{\le X}} \frac{\lambda(f)\chi_D(Nf)}{|f|^{1/2+\alpha}}.$$

Note that $R_E(N, X; 0) = R_E(N, X)$ with $R_E(N, X)$ as in Section 5. Similarly as in Section 5 and for $|\alpha| \le 1/g$ we get that

$$R_E(N, X; \alpha) = |\mathcal{H}_{2g+1}|\mathcal{C}_E(N; q^{-\alpha})L(\mathrm{Sym}^2 E, 1+2\alpha) + O_\varepsilon(q^{2g-X/2+\epsilon g}) + O_\varepsilon(q^{X/2+\epsilon g}). \tag{6-1}$$

Using Lemma 2.3 we obtain

$$\begin{aligned}\sum_{D \in \mathcal{H}^*_{2g+1}} &\epsilon^- L'\bigl(E \otimes \chi_D, \tfrac{1}{2}\bigr) \\ &= (\log q)([\mathfrak{n}/2] + 2g + 1)(R_E(1, [\mathfrak{n}/2] + 2g + 1) - \epsilon_{2g+1}\epsilon(E)R_E(M, [\mathfrak{n}/2] + 2g + 1)) \\ &\quad + \frac{\partial}{\partial\alpha}(R_E(1, [\mathfrak{n}/2] + 2g + 1; \alpha) - \epsilon_{2g+1}\epsilon(E)R_E(M, [\mathfrak{n}/2] + 2g + 1; \alpha))\Big|_{\alpha=0}.\end{aligned}$$

From Lemma 5.1 and (6-1) it follows that

$$\frac{1}{|\mathcal{H}^*_{2g+1}|} \sum_{D \in \mathcal{H}^*_{2g+1}} \epsilon^- L'\bigl(E \otimes \chi_D, \tfrac{1}{2}\bigr) = c_2(M)L(\mathrm{Sym}^2 E, 1)g + c_3(M) + O_\varepsilon(q^{-g+\epsilon g}),$$

where

$$c_2(M) = 2(\log q)(\mathcal{C}_E(1; 1) - \epsilon_{2g+1}\epsilon(E)\mathcal{C}_E(M; 1)) \prod_{P \mid \Delta} \frac{|P|+1}{|P|} \tag{6-2}$$

and

$$\begin{aligned}c_3(M) = (((\log q)([\mathfrak{n}/2] + 1)L(\mathrm{Sym}^2 E, 1) + 2L'(\mathrm{Sym}^2 E, 1))(\mathcal{C}_E(1; 1) - \epsilon_{2g+1}\epsilon(E)\mathcal{C}_E(M; 1)) \\ + L(\mathrm{Sym}^2 E, 1)(\mathcal{C}'_E(1; 1) - \epsilon_{2g+1}\epsilon(E)\mathcal{C}'_E(M; 1))) \prod_{P \mid \Delta} \frac{|P|+1}{|P|}.\end{aligned} \tag{6-3}$$

7. Proof of Theorem 1.3

Following Lemma 2.2, for $X < 2g$, we define

$$\mathcal{M}_{1,E}(X) := (1+\epsilon) \sum_{f \in \mathcal{M}_{\le X}} \frac{\lambda(f)\chi_D(f)}{\sqrt{|f|}},$$

so that

$$L\bigl(E \otimes \chi_D, \tfrac{1}{2}\bigr) = \mathcal{M}_{1,E}(X) + \mathcal{E}_{1,E}(1, X, [\mathfrak{n}/2]) + \epsilon_{2g+1}\epsilon(E)\mathcal{E}_{1,E}(M, X, [(\mathfrak{n}-1)/2]), \tag{7-1}$$

where recall expression (4-13) for $\mathcal{E}_{1,E}(N, X, n)$. Hence

$$\begin{aligned}L&\bigl(E \otimes \chi_D, \tfrac{1}{2}\bigr)^2 \\ &= 2L(E \otimes \chi_D, \tfrac{1}{2})\mathcal{M}_{1,E}(X) - \mathcal{M}_{1,E}(X)^2 + (\mathcal{E}_{1,E}(1, X, [\mathfrak{n}/2]) + \epsilon_{2g+1}\epsilon(E)\mathcal{E}_{1,E}(M, X, [(\mathfrak{n}-1)/2]))^2.\end{aligned}$$

By Cauchy's inequality and Proposition 4.7 we get

$$\sum_{D\in\mathcal{H}^*_{2g+1}} L\big(E\otimes\chi_D,\tfrac{1}{2}\big)^2 = 2\sum_{D\in\mathcal{H}^*_{2g+1}} L\big(E\otimes\chi_D,\tfrac{1}{2}\big)\mathcal{M}_{1,E}(X) - \sum_{D\in\mathcal{H}^*_{2g+1}} \mathcal{M}_{1,E}(X)^2 + O_\varepsilon(q^{2g}g^{1/2+\varepsilon}(2g-X)^2).$$

Now using Lemma 2.2 again and expanding $\mathcal{M}_{1,E}(X)^2$, the first line of the equation above is

$$2(S_E(1,[\mathfrak{n}/2]+2g+1,X)+S_E(1,[(\mathfrak{n}+1)/2]+2g,X)-S_E(1,X,X)) + 2\epsilon_{2g+1}\epsilon(E)(S_E(M,[\mathfrak{n}/2]+2g+1,X)+S_E(M,[(\mathfrak{n}+1)/2]+2g,X)-S_E(M,X,X)),$$

where recall the definition of $S_E(M,X,Y)$ in Section 3. Using Proposition 3.1, this is equal to

$$|\mathcal{H}^*_{2g+1}|c_2(M)L(\operatorname{Sym}^2 E,1)^3X + O(q^{2g}) + O(q^{2g-X/5}g^3) + O(q^{5g/4+3X/8}g^{30}),$$

where

$$c_2(M) = 2(\mathcal{C}_E(1;1,1,1)+\epsilon_{2g+1}\epsilon(E)\mathcal{C}_E(M;1,1,1))\prod_{P\mid\Delta}\frac{|P|+1}{|P|}, \tag{7-2}$$

and $\mathcal{C}_E(N;1,1,1)$ is defined in (3-9). Thus

$$\frac{1}{|\mathcal{H}^*_{2g+1}|}\sum_{D\in\mathcal{H}^*_{2g+1}} L\big(E\otimes\chi_D,\tfrac{1}{2}\big)^2 = c_4(M)L(\operatorname{Sym}^2 E,1)^3X + O(q^{-X/5}g^3) + O(q^{-3g/4+3X/8}g^{30}) + O_\varepsilon(g^{1/2+\varepsilon}(2g-X)^2).$$

Choosing $X = 2g - 100\log g$ we obtain the theorem.

8. Proof of Theorem 1.4

Following Lemma 2.3, for $X < 2g$ and fixed $n\in\mathbb{N}$, we define

$$\mathcal{M}_{2,E}(X,n) := (1-\epsilon)\sum_{f\in\mathcal{M}_{\le X}} \frac{(n+\deg(D)-\deg(f))\lambda(f)\chi_D(f)}{\sqrt{|f|}},$$

so that

$$\epsilon^- L'\big(E\otimes\chi_D,\tfrac{1}{2}\big) = (\log q)\mathcal{M}_{2,E}(X,[\mathfrak{n}/2]) + (\log q)(\mathcal{E}_{2,E}(1,X,[\mathfrak{n}/2]) - \epsilon_{2g+1}\epsilon(E)\mathcal{E}_{2,E}(M,X,[\mathfrak{n}/2])), \tag{8-1}$$

where recall that $\mathcal{E}_{2,E}(M,X,n)$ is given in (4-14). Then we get that

$$\epsilon^- L'\big(E\otimes\chi_D,\tfrac{1}{2}\big)^2 = 2(\log q)\epsilon^- L'\big(E\otimes\chi_D,\tfrac{1}{2}\big)\mathcal{M}_{2,E}(X,[\mathfrak{n}/2]) - (\log q)^2\mathcal{M}^2_{2,E}(X,[\mathfrak{n}/2]) + (\log q)^2(\mathcal{E}_{2,E}(1,X,[\mathfrak{n}/2]) - \epsilon_{2g+1}\epsilon(E)\mathcal{E}_{2,E}(M,X,[\mathfrak{n}/2]))^2.$$

By Cauchy's inequality and Proposition 4.7 we get that

$$\sum_{D\in\mathcal{H}^*_{2g+1}} \epsilon^- L'\big(E\otimes\chi_D, \tfrac{1}{2}\big)^2 = 2(\log q) \sum_{D\in\mathcal{H}^*_{2g+1}} \epsilon^- L'\big(E\otimes\chi_D, \tfrac{1}{2}\big)\mathcal{M}_{2,E}(X, [\mathfrak{n}/2]) \\ - (\log q)^2 \sum_{D\in\mathcal{H}^*_{2g+1}} \mathcal{M}_{2,E}(X, [\mathfrak{n}/2])^2 + O_\varepsilon(q^{2g} g^{1/2+\epsilon}(2g-X)^4).$$

Now by Lemma 2.3 we have

$$\begin{aligned}
&\sum_{D\in\mathcal{H}^*_{2g+1}} \epsilon^- L'\big(E\otimes\chi_D, \tfrac{1}{2}\big)\mathcal{M}_{2,E}(X, [\mathfrak{n}/2]) \\
&\quad = 2(\log q)([\mathfrak{n}/2]+2g+1)^2(S_E(1, [\mathfrak{n}/2]+2g+1, X) - \epsilon_{2g+1}\epsilon(E)S_E(M, [\mathfrak{n}/2]+2g+1, X)) \\
&\qquad +2([\mathfrak{n}/2]+2g+1)\frac{\partial}{\partial\beta}(S_E(1, [\mathfrak{n}/2]+2g+1, X; 0, \beta) - \epsilon_{2g+1}\epsilon(E)S_E(M, [\mathfrak{n}/2]+2g+1, X; 0, \beta))\big|_{\beta=0} \\
&\qquad +2([\mathfrak{n}/2]+2g+1)\frac{\partial}{\partial\alpha}(S_E(1, [\mathfrak{n}/2]+2g+1, X; \alpha, 0) - \epsilon_{2g+1}\epsilon(E)S_E(M, [\mathfrak{n}/2]+2g+1, X; \alpha, 0))\big|_{\alpha=0} \\
&\qquad +\frac{2}{\log q}\frac{\partial^2}{\partial\alpha\partial\beta}(S_E(1, [\mathfrak{n}/2]+2g+1, X; \alpha, \beta) - \epsilon_{2g+1}\epsilon(E)S_E(M, [\mathfrak{n}/2]+2g+1, X; \alpha, \beta))\big|_{\alpha=\beta=0},
\end{aligned}$$

and similarly

$$\begin{aligned}
&\sum_{D\in\mathcal{H}^*_{2g+1}} \mathcal{M}_{2,E}(X, [\mathfrak{n}/2])^2 \\
&\qquad = 2([\mathfrak{n}/2]+2g+1)^2(S_E(1, X, X) - \epsilon_{2g+1}\epsilon(E)S_E(M, X, X)) \\
&\qquad\quad + \frac{2}{\log q}([\mathfrak{n}/2]+2g+1)\frac{\partial}{\partial\beta}(S_E(1, X, X; 0, \beta) - \epsilon_{2g+1}\epsilon(E)S_E(M, X, X; 0, \beta))\big|_{\beta=0} \\
&\qquad\quad + \frac{2}{\log q}([\mathfrak{n}/2]+2g+1)\frac{\partial}{\partial\alpha}(S_E(1, X, X; \alpha, 0) - \epsilon_{2g+1}\epsilon(E)S_E(M, X, X; \alpha, 0))\big|_{\alpha=0} \\
&\qquad\quad + \frac{2}{(\log q)^2}\frac{\partial^2}{\partial\alpha\partial\beta}(S_E(1, X, X; \alpha, \beta) - \epsilon_{2g+1}\epsilon(E)S_E(M, X, X; \alpha, \beta))\big|_{\alpha=\beta=0}.
\end{aligned}$$

Choosing $X = 2g - 100\log g$ and using Proposition 3.1, we obtain that

$$\frac{1}{|\mathcal{H}^*_{2g+1}|}\sum_{D\in\mathcal{H}^*_{2g+1}} \epsilon^- L'\big(E\otimes\chi_D, \tfrac{1}{2}\big)^2 = c_5(M)L(\mathrm{Sym}^2 E, 1)^3 g^3 + O_\varepsilon(g^{2+\epsilon}),$$

where

$$c_5(M) = 16(\log q)^2(\mathcal{C}_E(1; 1, 1, 1) - \epsilon_{2g+1}\epsilon(E)\mathcal{C}_E(M; 1, 1, 1))\prod_{P\mid\Delta}\frac{|P|+1}{|P|}. \tag{8-2}$$

9. Proof of Theorem 1.5

By combining (7-1) and (8-1),

$$\begin{aligned}\epsilon_2^- L\big(E_1\otimes\chi_D,\tfrac12\big)L'\big(E_2\otimes\chi_D,\tfrac12\big)&\\ =\epsilon_2^-\mathcal{M}_{1,E_1}(X)L'\big(E_2\otimes\chi_D,\tfrac12\big)&+(\log q)L\big(E_1\otimes\chi_D,\tfrac12\big)\mathcal{M}_{2,E_2}(X,[\mathfrak{n}_2/2])\\ &-(\log q)\mathcal{M}_{1,E_1}(X)\mathcal{M}_{2,E_2}(X,[\mathfrak{n}_2/2])\\ &+(\log q)(\mathcal{E}_{1,E_1}(1,X,[\mathfrak{n}_1/2])+\epsilon_{2g+1}\epsilon(E_1)\mathcal{E}_{1,E_1}(M_1,X,[(\mathfrak{n}_1-1)/2]))\\ &\qquad\times(\mathcal{E}_{2,E_2}(1,X,[\mathfrak{n}_2/2])-\epsilon_{2g+1}\epsilon(E_2)\mathcal{E}_{2,E_2}(M_2,X,[\mathfrak{n}_2/2])).\end{aligned}$$

We bound the last term above using Cauchy's inequality and Proposition 4.7. In doing so we get

$$\begin{aligned}\sum_{D\in\mathcal{H}^*_{2g+1}}&\epsilon_2^- L\big(E_1\otimes\chi_D,\tfrac12\big)L'\big(E_2\otimes\chi_D,\tfrac12\big)\\ &=\sum_{D\in\mathcal{H}^*_{2g+1}}\epsilon_2^-\mathcal{M}_{1,E_1}(X)L'\big(E_2\otimes\chi_D,\tfrac12\big)+(\log q)\sum_{D\in\mathcal{H}^*_{2g+1}}L\big(E_1\otimes\chi_D,\tfrac12\big)\mathcal{M}_{2,E_2}(X,[\mathfrak{n}_2/2])\\ &\quad-(\log q)\sum_{D\in\mathcal{H}^*_{2g+1}}\mathcal{M}_{1,E_1}(X)\mathcal{M}_{2,E_2}(X,[\mathfrak{n}_2/2])+O_\varepsilon(q^{2g}g^{1/2+\varepsilon}(2g-X)^3).\end{aligned}\tag{9-1}$$

We shall estimate the remaining three terms using Proposition 3.1. They all have similar forms. For the first term, by Lemma 2.3 again we have

$$\begin{aligned}\sum_{D\in\mathcal{H}^*_{2g+1}}&\epsilon_2^-\mathcal{M}_{1,E_1}(X)L'\big(E_2\otimes\chi_D,\tfrac12\big)\\ &=(\log q)\sum_{D\in\mathcal{H}^*_{2g+1}}(1+\epsilon_1)(1-\epsilon_2)\sum_{\substack{f\in\mathcal{M}_{\le X}\\ h\in\mathcal{M}_{\le[\mathfrak{n}_2/2]+2g+1}}}\frac{([\mathfrak{n}_2/2]+2g+1-\deg(h))\lambda_1(f)\lambda_2(h)\chi_D(fh)}{\sqrt{|fh|}}.\end{aligned}$$

By expanding out, this equals

$$\begin{aligned}(\log q)([\mathfrak{n}_2/2]+2g+1)\big(S_{E_1,E_2}&(1,[\mathfrak{n}_2/2]+2g+1,X;0,0)\\ &+\epsilon_{2g+1}\epsilon(E_1)S_{E_1,E_2}(M_1,[\mathfrak{n}_2/2]+2g+1,X;0,0)\\ &-\epsilon_{2g+1}\epsilon(E_2)S_{E_1,E_2}(M_2,[\mathfrak{n}_2/2]+2g+1,X;0,0)\\ &-\epsilon(E_1)\epsilon(E_2)S_{E_1,E_2}(M_1M_2,[\mathfrak{n}_2/2]+2g+1,X;0,0)\big)\\ +\frac{\partial}{\partial\beta}\big(S_{E_1,E_2}&(1,[\mathfrak{n}_2/2]+2g+1,X;0,\beta)\\ &+\epsilon_{2g+1}\epsilon(E_1)S_{E_1,E_2}(M_1,[\mathfrak{n}_2/2]+2g+1,X;0,\beta)\\ &-\epsilon_{2g+1}\epsilon(E_2)S_{E_1,E_2}(M_2,[\mathfrak{n}_2/2]+2g+1,X;0,\beta)\\ &-\epsilon(E_1)\epsilon(E_2)S_{E_1,E_2}(M_1M_2,[\mathfrak{n}_2/2]+2g+1,X;0,\beta)\big)\big|_{\beta=0},\end{aligned}$$

which is, by Proposition 3.1 and Cauchy's residue theorem,

$$|\mathcal{H}^*_{2g+1}| c_6(M_1, M_2) L(\mathrm{Sym}^2 E_1, 1) L(\mathrm{Sym}^2 E_2, 1) L(E_1 \otimes E_2, 1) g + O(q^{2g}) + O(q^{5g/4+3X/8} g^{31}),$$

where

$$\begin{aligned} c_6(M_1, M_2) = 2(\log q)\big(&\mathcal{C}_{E_1,E_2}(1; 1,1,1,0,0) + \epsilon_{2g+1}\epsilon(E_1)\mathcal{C}_{E_1,E_2}(M_1; 1,1,1,0,0) \\ &- \epsilon_{2g+1}\epsilon(E_2)\mathcal{C}_{E_1,E_2}(M_2; 1,1,1,0,0) \\ &- \epsilon(E_1)\epsilon(E_2)\mathcal{C}_{E_1,E_2}(M_1M_2; 1,1,1,0,0)\big) \prod_{P \mid \Delta} \frac{|P|+1}{|P|}. \end{aligned} \tag{9-2}$$

The other two terms in (9-1) have the same asymptotics so we obtain

$$\begin{aligned} &\frac{1}{|\mathcal{H}^*_{2g+1}|} \sum_{D \in \mathcal{H}^*_{2g+1}} \epsilon_2^- L\big(E_1 \otimes \chi_D, \tfrac12\big) L'\big(E_2 \otimes \chi_D, \tfrac12\big) \\ &= c_6(M_1, M_2) L(\mathrm{Sym}^2 E_1, 1) L(\mathrm{Sym}^2 E_2, 1) L(E_1 \otimes E_2, 1) g + O(q^{-3g/4+3X/8} g^{31}) + O_\varepsilon(g^{1/2+\varepsilon}(2g-X)^3). \end{aligned}$$

Choosing $X = 2g - 100 \log g$ we obtain the theorem.

10. Proof of Theorem 1.6

We argue as in the previous section. From (8-1) we have

$$\begin{aligned} \epsilon_1^-\epsilon_2^- L'\big(E_1 \otimes \chi_D, \tfrac12\big) L'\big(E_2 \otimes \chi_D, \tfrac12\big) &= (\log q)\big(\epsilon_2^- \mathcal{M}_{2,E_1}(X, [\mathfrak{n}_1/2]) L'\big(E_2 \otimes \chi_D, \tfrac12\big) + \epsilon_1^- L'\big(E_1 \otimes \chi_D, \tfrac12\big) \mathcal{M}_{2,E_2}(X, [\mathfrak{n}_2/2])\big) \\ &\quad - (\log q)^2 \mathcal{M}_{2,E_1}(X, [\mathfrak{n}_1/2]) \mathcal{M}_{2,E_2}(X, [\mathfrak{n}_2/2]) \\ &\quad + (\log q)^2 (\mathcal{E}_{2,E_1}(1, X, [\mathfrak{n}_1/2]) - \epsilon_{2g+1}\epsilon(E_1)\mathcal{E}_{2,E_1}(M_1, X, [\mathfrak{n}_1/2])) \\ &\qquad \times (\mathcal{E}_{2,E_2}(1, X, [\mathfrak{n}_2/2]) - \epsilon_{2g+1}\epsilon(E_2)\mathcal{E}_{2,E_2}(M_2, X, [\mathfrak{n}_2/2])). \end{aligned}$$

Bounding the last term above using Cauchy's inequality and Proposition 4.7 leads to

$$\begin{aligned} \sum_{D \in \mathcal{H}^*_{2g+1}} &\epsilon_1^-\epsilon_2^- L'\big(E_1 \otimes \chi_D, \tfrac12\big) L'\big(E_2 \otimes \chi_D, \tfrac12\big) \\ &= (\log q) \sum_{D \in \mathcal{H}^*_{2g+1}} \epsilon_2^- \mathcal{M}_{2,E_1}(X, [\mathfrak{n}_1/2]) L'\big(E_2 \otimes \chi_D, \tfrac12\big) \\ &\quad + (\log q) \sum_{D \in \mathcal{H}^*_{2g+1}} \epsilon_1^- L'\big(E_1 \otimes \chi_D, \tfrac12\big) \mathcal{M}_{2,E_2}(X, [\mathfrak{n}_2/2]) \\ &\quad - (\log q)^2 \sum_{D \in \mathcal{H}^*_{2g+1}} \mathcal{M}_{2,E_1}(X, [\mathfrak{n}_1/2]) \mathcal{M}_{2,E_2}(X, [\mathfrak{n}_2/2]) + O_\varepsilon(q^{2g} g^{1/2+\varepsilon}(2g-X)^4). \end{aligned} \tag{10-1}$$

We shall illustrate the evaluation of the third term using Proposition 3.1. The first two terms can be treated in the same way, and in fact they all have the same asymptotics. We have

$$(\log q)^2 \sum_{D\in\mathcal{H}^*_{2g+1}} \mathcal{M}_{2,E_1}(X,[\mathfrak{n}_1/2])\mathcal{M}_{2,E_2}(X,[\mathfrak{n}_2/2]) = (\log q)^2 \sum_{D\in\mathcal{H}^*_{2g+1}} (1-\epsilon_1)(1-\epsilon_2) \sum_{f,h\in\mathcal{M}_{\le X}} \frac{([\mathfrak{n}_1/2]+2g+1-\deg(f))([\mathfrak{n}_2/2]+2g+1-\deg(h))\lambda_1(f)\lambda_2(h)\chi_D(fh)}{\sqrt{|fh|}}.$$

By expanding out, this equals

$$\begin{aligned}
&(\log q)^2([\mathfrak{n}_1/2]+2g+1)([\mathfrak{n}_2/2]+2g+1)\big(S_{E_1,E_2}(1,X,X;0,0)-\epsilon_{2g+1}\epsilon(E_1)S_{E_1,E_2}(M_1,X,X;0,0)\\
&\qquad -\epsilon_{2g+1}\epsilon(E_2)S_{E_1,E_2}(M_2,X,X;0,0)+\epsilon(E_1)\epsilon(E_2)S_{E_1,E_2}(M_1M_2,X,X;0,0)\big)\\
&+(\log q)([\mathfrak{n}_1/2]+2g+1)\frac{\partial}{\partial\beta}\big(S_{E_1,E_2}(1,X,X;0,\beta)-\epsilon_{2g+1}\epsilon(E_1)S_{E_1,E_2}(M_1,X,X;0,\beta)\\
&\qquad -\epsilon_{2g+1}\epsilon(E_2)S_{E_1,E_2}(M_2,X,X;0,\beta)+\epsilon(E_1)\epsilon(E_2)S_{E_1,E_2}(M_1M_2,X,X;0,\beta)\big)\big|_{\beta=0}\\
&+(\log q)([\mathfrak{n}_2/2]+2g+1)\frac{\partial}{\partial\alpha}\big(S_{E_1,E_2}(1,X,X;\alpha,0)-\epsilon_{2g+1}\epsilon(E_1)S_{E_1,E_2}(M_1,X,X;\alpha,0)\\
&\qquad -\epsilon_{2g+1}\epsilon(E_2)S_{E_1,E_2}(M_2,X,X;\alpha,0)+\epsilon(E_1)\epsilon(E_2)S_{E_1,E_2}(M_1M_2,X,X;\alpha,0)\big)\big|_{\alpha=0}\\
&+\frac{\partial^2}{\partial\alpha\partial\beta}\big(S_{E_1,E_2}(1,X,X;\alpha,\beta)-\epsilon_{2g+1}\epsilon(E_1)S_{E_1,E_2}(M_1,X,X;\alpha,\beta)\\
&\qquad -\epsilon_{2g+1}\epsilon(E_2)S_{E_1,E_2}(M_2,X,X;\alpha,\beta)+\epsilon(E_1)\epsilon(E_2)S_{E_1,E_2}(M_1M_2,X,X;\alpha,\beta)\big)\big|_{\alpha=\beta=0}.
\end{aligned}$$

In view of Proposition 3.1 and Cauchy's residue theorem, this is

$$|\mathcal{H}^*_{2g+1}|c_7(M_1,M_2)L(\mathrm{Sym}^2 E_1,1)L(\mathrm{Sym}^2 E_2,1)L(E_1\otimes E_2,1)g^2+O(q^{2g}g)+O(q^{g/2+3X/4}g^{32}),$$

where

$$\begin{aligned}
c_7(M_1,M_2)=4(\log q)^2\big(&\mathcal{C}_{E_1,E_2}(1;1,1,1,0,0)-\epsilon_{2g+1}\epsilon(E_1)\mathcal{C}_{E_1,E_2}(M_1;1,1,1,0,0)\\
&-\epsilon_{2g+1}\epsilon(E_2)\mathcal{C}_{E_1,E_2}(M_2;1,1,1,0,0)\\
&+\epsilon(E_1)\epsilon(E_2)\mathcal{C}_{E_1,E_2}(M_1M_2;1,1,1,0,0)\big)\prod_{P\mid\Delta}\frac{|P|+1}{|P|}.
\end{aligned}\tag{10-2}$$

The other two terms in (10-1) have the same asymptotics so we obtain

$$\begin{aligned}
\frac{1}{|\mathcal{H}^*_{2g+1}|}\sum_{D\in\mathcal{H}^*_{2g+1}} \epsilon_1^-\epsilon_2^- L'\big(E_1\otimes\chi_D,\tfrac12\big)L'\big(E_2\otimes\chi_D,\tfrac12\big)
&=c_7(M_1,M_2)L(\mathrm{Sym}^2 E_1,1)L(\mathrm{Sym}^2 E_2,1)L(E_1E_2,1)g^2\\
&\quad+O(g)+O(q^{-3g/2+3X/4}g^{32})+O_\varepsilon(g^{1/2+\varepsilon}(2g-X)^4).
\end{aligned}$$

Choosing $X=2g-100\log g$ we obtain the theorem.

11. Proof of Corollary 1.7

The results in Section 4 imply that

$$\sum_{D\in\mathcal{H}^*_{2g+1}} L(E\otimes\chi_D,\tfrac{1}{2})^4 \ll_\varepsilon q^{2g}g^{6+\varepsilon}. \tag{11-1}$$

We next obtain some upper bounds for moments of the derivatives. We have

$$L^{(l)}\big(E\otimes\chi_D,\tfrac{1}{2}\big)^k = \left(\frac{l!}{2\pi i}\right)^k \oint\cdots\oint \frac{L(E\otimes\chi_D,1/2+\alpha_1)\cdots L(E\otimes\chi_D,1/2+\alpha_k)}{\alpha_1^{l+1}\cdots\alpha_k^{l+1}}d\alpha_1\cdots d\alpha_k,$$

where we are integrating along small circles of radii r around the origin. Then using Hölder's inequality leads to

$$\sum_{D\in\mathcal{H}^*_{2g+1}} \big|L^{(l)}\big(E\otimes\chi_D,\tfrac{1}{2}\big)\big|^k \ll \frac{(l!)^k}{r^{lk+1}}\oint_{|\alpha|=r}\sum_{D\in\mathcal{H}^*_{2g+1}}\big|L\big(E\otimes\chi_D,\tfrac{1}{2}+\alpha\big)\big|^k d\alpha.$$

Choosing $r=1/g$ and using upper bounds for moments of L-functions we get that

$$\sum_{D\in\mathcal{H}^*_{2g+1}} \big|L^{(l)}\big(E\otimes\chi_D,\tfrac{1}{2}\big)\big|^k \ll_\varepsilon q^{2g}(l!)^k g^{lk+k(k-1)/2+\varepsilon}.$$

In particular, with $l=1$ and $k=4$, we have

$$\sum_{D\in\mathcal{H}^*_{2g+1}} L'\big(E\otimes\chi_D,\tfrac{1}{2}\big)^4 \ll_\varepsilon q^{2g}g^{10+\varepsilon}. \tag{11-2}$$

Now from Hölder's inequality we have

$$\Bigg(\sum_{\substack{D\in\mathcal{H}_{2g+1}\\(D,\Delta_1)=1}} L\big(E_1\otimes\chi_D,\tfrac{1}{2}\big)^4\Bigg)\Bigg(\sum_{\substack{D\in\mathcal{H}_{2g+1}\\(D,\Delta_2)=1}} L'\big(E_2\otimes\chi_D,\tfrac{1}{2}\big)^4\Bigg)\Bigg(\sum_{\substack{D\in\mathcal{H}_{2g+1},\ (D,\Delta_1\Delta_2)=1\\ \epsilon_2^- L(E_1\otimes\chi_D,1/2)L'(E_2\otimes\chi_D,1/2)\neq 0}} 1\Bigg)^2$$
$$\geq \Bigg(\sum_{\substack{D\in\mathcal{H}_{2g+1}\\(D,\Delta_1\Delta_2)=1}} \epsilon_2^- L\big(E_1\otimes\chi_D,\tfrac{1}{2}\big)L'\big(E_2\otimes\chi_D,\tfrac{1}{2}\big)\Bigg)^4.$$

Combining (11-1) and (11-2) with Theorem 1.5 we get

$$\#\big\{D\in\mathcal{H}^*_{2g+1}:\epsilon_2^- L\big(E_1\otimes\chi_D,\tfrac{1}{2}\big)L'\big(E_2\otimes\chi_D,\tfrac{1}{2}\big)\neq 0\big\} \gg_\varepsilon \frac{q^{2g}}{g^{6+\varepsilon}},$$

which implies the first statement. The second statement can be obtained similarly.

Acknowledgements

A. Florea gratefully acknowledges the support of an NSF Postdoctoral Fellowship during part of the research which led to this paper. J.P. Keating is supported by a Royal Society Wolfson Research Merit Award, EPSRC Programme Grant EP/K034383/1 LMF: L-Functions and Modular Forms, and by ERC

Advanced Grant 740900 (LogCorRM). The authors would also like to thank Chantal David, Matilde Lalin and Zeev Rudnick and the anonymous referee for useful comments on the paper.

References

[Altuğ and Tsimerman 2014] S. A. Altuğ and J. Tsimerman, "Metaplectic Ramanujan conjecture over function fields with applications to quadratic forms", *Int. Math. Res. Not.* **2014**:13 (2014), 3465–3558. MR Zbl

[Andrade and Keating 2012] J. C. Andrade and J. P. Keating, "The mean value of $L(\frac{1}{2}, \chi)$ in the hyperelliptic ensemble", *J. Number Theory* **132**:12 (2012), 2793–2816. MR Zbl

[Andrade and Keating 2014] J. C. Andrade and J. P. Keating, "Conjectures for the integral moments and ratios of L-functions over function fields", *J. Number Theory* **142** (2014), 102–148. MR Zbl

[Baig and Hall 2012] S. Baig and C. Hall, "Experimental data for Goldfeld's conjecture over function fields", *Exp. Math.* **21**:4 (2012), 362–374. MR Zbl

[Bump et al. 1990a] D. Bump, S. Friedberg, and J. Hoffstein, "Eisenstein series on the metaplectic group and nonvanishing theorems for automorphic L-functions and their derivatives", *Ann. of Math.* (2) **131**:1 (1990), 53–127. MR Zbl

[Bump et al. 1990b] D. Bump, S. Friedberg, and J. Hoffstein, "Nonvanishing theorems for L-functions of modular forms and their derivatives", *Invent. Math.* **102**:3 (1990), 543–618. MR Zbl

[Conrey and Keating 2015a] B. Conrey and J. P. Keating, "Moments of zeta and correlations of divisor-sums, I", *Philos. Trans. Roy. Soc. A* **373**:2040 (2015), art. id. 20140313. MR Zbl

[Conrey and Keating 2015b] B. Conrey and J. P. Keating, "Moments of zeta and correlations of divisor-sums, II", pp. 75–85 in *Advances in the theory of numbers*, edited by A. Alaca et al., Fields Inst. Commun. **77**, Fields Inst. Res. Math. Sci., Toronto, 2015. MR Zbl

[Conrey and Keating 2015c] B. Conrey and J. P. Keating, "Moments of zeta and correlations of divisor-sums, III", *Indag. Math. (N.S.)* **26**:5 (2015), 736–747. MR Zbl

[Conrey and Keating 2016] B. Conrey and J. P. Keating, "Moments of zeta and correlations of divisor-sums, IV", *Res. Number Theory* **2** (2016), art. id. 24. MR Zbl

[Conrey and Keating 2019] B. Conrey and J. P. Keating, "Moments of zeta and correlations of divisor-sums, V", *Proc. Lond. Math. Soc.* (3) **118**:4 (2019), 729–752. MR Zbl

[Conrey et al. 2003] B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, "Autocorrelation of random matrix polynomials", *Comm. Math. Phys.* **237**:3 (2003), 365–395. MR Zbl

[Conrey et al. 2005] B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, "Integral moments of L-functions", *Proc. Lond. Math. Soc.* (3) **91**:1 (2005), 33–104. MR Zbl

[Diaconu 2019] A. Diaconu, "On the third moment of $L(\frac{1}{2}, \chi_d)$, I: The rational function field case", *J. Number Theory* **198** (2019), 1–42. MR Zbl

[Diaconu et al. 2003] A. Diaconu, D. Goldfeld, and J. Hoffstein, "Multiple Dirichlet series and moments of zeta and L-functions", *Compos. Math.* **139**:3 (2003), 297–360. MR Zbl

[Florea 2017a] A. Florea, "The fourth moment of quadratic Dirichlet L-functions over function fields", *Geom. Funct. Anal.* **27**:3 (2017), 541–595. MR Zbl

[Florea 2017b] A. Florea, "The second and third moment of $L(\frac{1}{2}, \chi)$ in the hyperelliptic ensemble", *Forum Math.* **29**:4 (2017), 873–892. MR Zbl

[Florea 2017c] A. M. Florea, "Improving the error term in the mean value of $L(\frac{1}{2}, \chi)$ in the hyperelliptic ensemble", *Int. Math. Res. Not.* **2017**:20 (2017), 6119–6148. MR Zbl

[Hall et al. 2017] C. Hall, J. P. Keating, and E. Roditty-Gershon, "Variance of sums in arithmetic progressions of arithmetic functions associated with higher degree L-functions in $\mathbb{F}_q[t]$", preprint, 2017. arXiv

[Harper 2013] A. J. Harper, "Sharp conditional bounds for moments of the Riemann zeta function", preprint, 2013. arXiv

[Hayes 1966] D. R. Hayes, "The expression of a polynomial as a sum of three irreducibles", *Acta Arith.* **11** (1966), 461–488. MR Zbl

[Heath-Brown 2004] D. R. Heath-Brown, "The average analytic rank of elliptic curves", *Duke Math. J.* **122**:3 (2004), 591–623. MR Zbl

[Hoffstein and Rosen 1992] J. Hoffstein and M. Rosen, "Average values of L-series in function fields", *J. Reine Angew. Math.* **426** (1992), 117–150. MR Zbl

[Iwaniec 1990] H. Iwaniec, "On the order of vanishing of modular L-functions at the critical point", *Sém. Théor. Nombres Bordeaux* (2) **2**:2 (1990), 365–376. MR Zbl

[Katz and Sarnak 1999] N. M. Katz and P. Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, Amer. Math. Soc. Colloq. Publ. **45**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Keating and Snaith 2000a] J. P. Keating and N. C. Snaith, "Random matrix theory and L-functions at $s = 1/2$", *Comm. Math. Phys.* **214**:1 (2000), 91–110. MR Zbl

[Keating and Snaith 2000b] J. P. Keating and N. C. Snaith, "Random matrix theory and $\zeta(1/2+it)$", *Comm. Math. Phys.* **214**:1 (2000), 57–89. MR Zbl

[Murty and Murty 1991] M. R. Murty and V. K. Murty, "Mean values of derivatives of modular L-series", *Ann. of Math.* (2) **133**:3 (1991), 447–475. MR Zbl

[Petrow 2014] I. Petrow, "Moments of $L'\left(\frac{1}{2}\right)$ in the family of quadratic twists", *Int. Math. Res. Not.* **2014**:6 (2014), 1576–1612. MR Zbl

[Radziwiłł and Soundararajan 2015] M. Radziwiłł and K. Soundararajan, "Moments and distribution of central L-values of quadratic twists of elliptic curves", *Invent. Math.* **202**:3 (2015), 1029–1068. MR Zbl

[Rudnick and Soundararajan 2005] Z. Rudnick and K. Soundararajan, "Lower bounds for moments of L-functions", *Proc. Natl. Acad. Sci. USA* **102**:19 (2005), 6837–6838. MR Zbl

[Rudnick and Soundararajan 2006] Z. Rudnick and K. Soundararajan, "Lower bounds for moments of L-functions: symplectic and orthogonal examples", pp. 293–303 in *Multiple Dirichlet series, automorphic forms, and analytic number theory*, edited by S. Friedberg et al., Proc. Sympos. Pure Math. **75**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Soundararajan 2009] K. Soundararajan, "Moments of the Riemann zeta function", *Ann. of Math.* (2) **170**:2 (2009), 981–993. MR Zbl

[Soundararajan and Young 2010] K. Soundararajan and M. P. Young, "The second moment of quadratic twists of modular L-functions", *J. Eur. Math. Soc.* **12**:5 (2010), 1097–1116. MR Zbl

Communicated by Roger Heath-Brown
Received 2019-02-01 Revised 2019-06-24 Accepted 2019-09-02

hung.bui@manchester.ac.uk *Department of Mathematics, University of Manchester, United Kingdom*

aflorea@math.columbia.edu *Department of Mathematics, Columbia University, New York, NY, United States*

keating@maths.ox.ac.uk *Mathematical Institute, University of Oxford, United Kingdom*

edvar@hit.ac.il *Department of Applied Mathematics, Holon Institute of Technology, Holon, Israel*

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

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nonprofit scientific publishing

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Algebra & Number Theory

Volume 14 No. 7 2020

1937-0652(2020)14:7;1-V