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# Remarks on Landau–Siegel zeros

Debmalya Basak, Jesse Thorner and Alexandru Zaharescu

*Dedicated to Dorian Goldfeld on the fiftieth anniversary of his elegant proof of Siegel's theorem*

For certain families of $L$-functions, we prove that if each $L$-function in the family has only real zeros in a fixed yet arbitrarily small neighborhood of $s = 1$, then one may considerably improve upon the known results on Landau–Siegel zeros. Sarnak and the third author proved a similar result under much more restrictive hypotheses.

## 1. Introduction

Building on the seminal work of Hadamard and de la Vallée Poussin for the Riemann zeta function $\zeta(s)$, it is known that if $\chi \pmod{q_\chi}$ is a primitive Dirichlet character, then there exists an absolute and effectively computable constant $c_1 > 0$ such that the Dirichlet $L$-function $L(s, \chi)$ has at most one zero $\beta$ (necessarily real and simple) in the region

$$\operatorname{Re} s \geq 1 - c_1/\log(q_\chi(|\operatorname{Im} s| + 3)).$$

If $\beta$ exists, then $\chi$ is real and nontrivial. It follows from Siegel's lower bound on $L(1, \chi)$ [9] that for any $\varepsilon > 0$, there exists a constant $c_2(\varepsilon) > 0$ such that if $\beta$ exists, then

$$\beta \leq 1 - c_2(\varepsilon) q_\chi^{-\varepsilon}.$$

See Goldfeld [3] for a concise proof of Siegel's lower bound on $L(1, \chi)$. Unfortunately, no known proof provides an effective determination of $c_2(\varepsilon)$ in terms of $\varepsilon$. Define

$$\mathcal{S} = \{\chi \pmod{q_\chi} : \chi \text{ primitive and real}\}.$$

It follows from work of Tatuzawa [10] that Siegel's result can be refined as follows: For all $\varepsilon > 0$, there exists an *effectively computable* constant $c_3(\varepsilon) > 0$ such that

$$\#\{\chi \in \mathcal{S} : q_\chi \geq c_3(\varepsilon) \text{ and } L(s, \chi) \text{ has a real zero in } [1 - q_\chi^{-\varepsilon}, 1)\} \leq 1. \tag{1-1}$$

Let Hypothesis H denote the hypothesis that if $\chi \in \mathcal{S}$, then all zeros of $L(s, \chi)$ lie on $\operatorname{Re} s = \frac{1}{2}$ or $\operatorname{Im} s = 0$. In other words, the generalized Riemann hypothesis is assumed to hold only for the nonreal zeros, so Landau–Siegel zeros are permitted to exist. Under Hypothesis H, the work of Sarnak and

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Zaharescu [8, Proof of Theorem 1] implies the following improvement over (1-1): For all $\varepsilon > 0$, there exists an effectively computable constant $c_4(\varepsilon) > 0$ such that

$$\#\{\chi \in \mathcal{S} : q_\chi \geq c_4(\varepsilon) \text{ and } L(s,\chi) \text{ has a real zero in } [1-(\log q_\chi)^{-\varepsilon}, 1)\} \leq 1. \tag{1-2}$$

This "exponentiates" the quality of the zero-free region at the cost of a strong hypothesis for the nonreal zeros of Dirichlet $L$-functions.

Here, we will use Turán's power sum method to prove that (1-2) holds under a much weaker hypothesis. In order to state our hypothesis, we fix $0 < \delta < \frac{1}{10}$.

**Hypothesis $H_\delta$.** If $\chi \in \mathcal{S}$, then all the zeros of $L(s,\chi)$ in the disk $|z-1| < \delta$ are real.

**Theorem 1.1.** *Fix $0 < \delta < \frac{1}{10}$. If $H_\delta$ is true, then for all $\varepsilon > 0$, there exists an effectively computable constant $q_0 = q_0(\delta, \varepsilon) > 0$ such that*

$$\#\{\chi \in \mathcal{S} : q_\chi \geq q_0 \text{ and } L(s,\chi) \text{ has a real zero in } [1-(\log q_\chi)^{-\varepsilon}, 1)\} \leq 1.$$

**Remark 1.2.** Our proof provides an explicit permissible expression for $q_0$. See (2-1).

**Remark 1.3.** It follows from Heath-Brown's zero density estimate in [4, Theorem 3] that if the $\chi \in \mathcal{S}$ are ordered by conductor $q_\chi$, then a density 1 subset of $\chi \in \mathcal{S}$ satisfy $H_\delta$. In contrast, Hypothesis H in [8] has not been verified for any nontrivial $\chi \in \mathcal{S}$ yet.

Our next result, which is ineffective, is an immediate corollary of Theorem 1.1.

**Corollary 1.4.** *Fix $0 < \delta < \frac{1}{10}$, and assume that $H_\delta$ is true. For all $\varepsilon > 0$, there exists an ineffective constant $c(\delta, \varepsilon) > 0$ such that if $\chi \in \mathcal{S}$ and $\sigma \geq 1 - c(\delta,\varepsilon)(\log q_\chi)^{-\varepsilon}$, then $L(\sigma, \chi) \neq 0$.*

The difference between our proof of Theorem 1.1 and the proof of [8, Theorem 1] is subtle. We contrast our work with [8] in Remark 3.5 below.

Variants of the hypothesis $H_\delta$ for other families of $L$-functions lead to results for those families similar to Theorem 1.1. After proving Theorem 1.1, we will sketch some examples.

## 2. Preliminaries

Fix $0 < \delta < \frac{1}{10}$. It suffices to let $0 < \varepsilon < 1$. We define

$$q_0 = \exp\Big(\exp \frac{10\,000}{\delta^3 \varepsilon^2}\Big). \tag{2-1}$$

Let $q_1$ and $q_2$ be positive integers such that

$$q_0 \leq q_2 \leq q_1. \tag{2-2}$$

For $j \in \{1,2\}$, let $\chi_j \pmod{q_j}$ be distinct primitive real Dirichlet characters. Let $\psi$ be the primitive Dirichlet character that induces $\chi_1\chi_2$. It follows that $\psi$ is real, with conductor at most $q_1^2$. We assume

$H_\delta$, which implies that $L(s,\chi_1)$, $L(s,\chi_2)$, and $L(s,\psi)$ have no nonreal zeros in the disk $|z-1|<\delta$. The Riemann zeta function $\zeta(s)$ provably has no zero in this disk, so it follows that

$$D(s)=\zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\psi) \tag{2-3}$$

has no nonreal zeros in the disk $|z-1|<\delta$.

Let $\beta_j$ be the greatest real zero of $L(s,\chi_j)$. We suppose to the contrary that

$$\beta_1 \geq 1-(\log q_1)^{-\varepsilon} \quad \text{and} \quad \beta_2 \geq 1-(\log q_2)^{-\varepsilon}. \tag{2-4}$$

It follows from (2-1), (2-2), and (2-4) that

$$\max\{1-\beta_1, 1-\beta_2\} \leq (\log q_0)^{-\varepsilon} \leq \frac{\delta}{10}. \tag{2-5}$$

We define

$$\eta=\frac{\delta}{e}+\beta_2-1. \tag{2-6}$$

It follows from (2-1), (2-2), and (2-4) that

$$\frac{\delta}{2e} \leq \eta \leq \frac{\delta}{e} \tag{2-7}$$

## 3. Proof of Theorem 1.1

We begin with a relation for $\beta_1$, $\beta_2$, and all nonreal zeros of $D(s)$.

**Lemma 3.1.** *Let $k \geq 1$ and $\ell \geq 2$ be integers. If $\omega$ denotes a zero of $D(s)$ and $\eta$ is as in* (2-6), *then*

$$\frac{1}{\eta^{k\ell}}-\frac{1}{(1+\eta-\beta_1)^{k\ell}} \geq \frac{1}{(1+\eta-\beta_2)^{k\ell}}+\operatorname{Re}\sum_{\operatorname{Im}\omega\neq 0}\frac{1}{(1+\eta-\omega)^{k\ell}}.$$

*Proof.* If $\operatorname{Re} s>1$, then we have the Dirichlet series expansion

$$-\frac{D'}{D}(s)=\sum_{p \text{ prime}}\sum_{m=1}^{\infty}\frac{(1+\chi_1(p^m)+\chi_2(p^m)+\psi(p^m))\log p}{p^{ms}}=\sum_{n=1}^{\infty}\frac{a_D(n)}{n^s}.$$

Since $D(s)$ is the Dedekind zeta function of a biquadratic extension of $\mathbb{Q}$, we have $a_D(n)\geq 0$ for all $n \geq 1$. On the other hand, $D(s)$ has a Hadamard product factorization. In particular, there exist $a_D, b_D \in \mathbb{C}$ such that

$$(s-1)D(s)=s^{\operatorname{ord}_{s=0} D(s)}e^{a_D+b_D s}\prod_{\substack{\omega\neq 0\\ D(\omega)=0}}\Big(1-\frac{s}{\omega}\Big)e^{s/\omega}.$$

We emphasize that $\omega$ ranges over the trivial and nontrivial zeros of each factor of $D(s)$.

When $\operatorname{Re} s > 1$, we equate the Dirichlet series expansion of $-(D'/D)(s)$ with the logarithmic derivative of the Hadamard product of $D(s)$, thus obtaining

$$\sum_{n=1}^{\infty} \frac{a_D(n)}{n^s} = \frac{1}{s-1} - b_D - \frac{\operatorname{ord}_{s=0} D(s)}{s} - \sum_{\omega \neq 0} \Big( \frac{1}{s-\omega} + \frac{1}{\omega} \Big).$$

Let $k \geq 1$ and $\ell \geq 2$ be integers. We take the real part of the $(k\ell - 1)$-th derivative of both sides, arriving at

$$\frac{1}{(k\ell-1)!} \operatorname{Re} \sum_{n=1}^{\infty} \frac{a_D(n)(\log n)^{k\ell-1}}{n^s} = \operatorname{Re}\Big( \frac{1}{(s-1)^{k\ell}} - \sum_{\omega} \frac{1}{(s-\omega)^{k\ell}} \Big).$$

Now, the possible trivial zero $\omega = 0$ is included in the sum over $\omega$.

We let $s = 1 + \eta$. Since $a_D(n) \geq 0$ uniformly, it follows that

$$\operatorname{Re} \sum_{\omega} \frac{1}{(1+\eta-\omega)^{k\ell}} < \frac{1}{\eta^{k\ell}}.$$

Note that if $\omega$ is a real zero of $D(s)$, then $\omega < 1$ and $(1+\eta-\omega)^{k\ell} \geq 0$. Since $\beta_1$ and $\beta_2$ are real zeros of $D(s)$, we conclude via nonnegativity that

$$\frac{1}{(1+\eta-\beta_1)^{k\ell}} + \frac{1}{(1+\eta-\beta_2)^{k\ell}} + \operatorname{Re} \sum_{\substack{\omega \\ \operatorname{Im} \omega \neq 0}} \frac{1}{(1+\eta-\omega)^{k\ell}} \leq \operatorname{Re} \sum_{\omega} \frac{1}{(1+\eta-\omega)^{k\ell}} < \frac{1}{\eta^{k\ell}}.$$

The desired result follows. □

We define a sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ as follows. Let

$$z_1 = (1+\eta-\beta_2)^{-\ell}. \tag{3-1}$$

Choose an ordering $\{\omega_j\}_{j=2}^{\infty}$ of the zeros $\omega$ with $\operatorname{Im} \omega \neq 0$ such that if

$$z_j = (1+\eta-\omega_j)^{-\ell},$$

then $|z_2| \geq |z_3| \geq \cdots \geq |z_j| \geq |z_{j+1}| \geq \cdots$. It follows from (2-5), (2-7), and our assumption of $H_\delta$ that if $\omega$ is a nonreal zero of $D(s)$, then

$$|1+\eta-\omega| \geq \sqrt{\eta^2+\delta^2} \geq \eta + \frac{\delta}{2} \geq 1+\eta-\beta_2.$$

In particular, $|z_1| \geq |z_2| \geq |z_3| \geq \cdots$.

**Lemma 3.2.** *Fix* $0 < \delta < \frac{1}{10}$. *Let* $\eta$ *be as in* (2-6), *and let* $\ell = \lceil \log\log q_1 \rceil$. *If* $H_\delta$ *is true*, *then*

$$\sum_{j=1}^{\infty} \frac{|z_j|}{|z_1|} \leq 5.$$

*Proof.* Write $\omega_j = \beta_j + i\gamma_j$. Recall that our assumption of $H_\delta$ implies that $D(s)$ has no nonreal zeros in the disk $|z-1| < \delta$. Consequently, $H_\delta$ implies that if $j \geq 2$ and $|\gamma_j| \leq 1$, then $|z_j| \leq \delta^{-\ell}$. Per (2-6) and (3-1), we have $|z_1| = (\delta/e)^{-\ell}$. Therefore, if

$$N_D(T) = \#\big\{\omega : D(\omega) = 0,\ |\mathrm{Im}\,\omega| \leq T,\ 0 \leq \mathrm{Re}\,\omega \leq 1\big\},$$

then the sum to be estimated is at most

$$1 + \frac{\delta^{-\ell}}{|z_1|} \sum_{|\gamma_j| \leq 1} 1 + \frac{1}{|z_1|} \int_1^\infty \frac{dN_D(t)}{t^\ell} \leq 1 + \frac{\delta^{-\ell}}{(\delta/e)^{-\ell}} N_D(1) + \frac{\ell}{(\delta/e)^{-\ell}} \int_1^\infty \frac{N_D(t)}{t^{\ell+1}} dt. \tag{3-2}$$

It follows from [1, Corollary 1.2] that if $T \geq \frac{5}{7}$, then

$$\left| N_D(T) - \frac{T}{\pi} \log\Big( q_1 q_2 q_\psi \Big(\frac{T}{2\pi e}\Big)^4 \Big) \right| \leq \log(q_1 q_2 q_\psi T^4) + 28.$$

Recall that $q_2 \leq q_1$ and $q_\psi \leq q_1 q_2$. Thus, if $T \geq 1$, then

$$N_D(T) \leq \frac{T}{\pi} \log\Big( q_1^3 \Big(\frac{T}{2\pi e}\Big)^4 \Big) + \log(q_1^3 T^4) + 28.$$

Since $\ell \geq 3$, it follows that (3-2) is

$$\leq 1 + e^{-\ell}\Big(28 - \frac{4\log(2\pi e)}{\pi} + 3\Big(1 + \frac{1}{\pi}\Big)\log q_1\Big) + 3(\delta/e)^\ell \Big(\Big(1 + \frac{1}{(1-\ell^{-1})\pi}\Big)\log q_1 - 1\Big).$$

The desired result now follows from our choice of $\ell$, our range of $\delta$, and the bounds (2-1) and (2-2). □

**Lemma 3.3.** *Fix* $0 < \delta < \frac{1}{10}$. *If* $H_\delta$ *is true, then there exists an integer* $1 \leq k \leq 120$ *such that*

$$\frac{1}{(1+\eta-\beta_2)^{k\ell}} + \mathrm{Re} \sum_{\mathrm{Im}\,\omega \neq 0} \frac{1}{(1+\eta-\omega)^{k\ell}} \geq \frac{1}{8(1+\eta-\beta_2)^{k\ell}}.$$

*Proof.* Let $\{y_j\}_{j=1}^\infty$ be a sequence of complex numbers such that $|y_1| \geq |y_2| \geq |y_3| \geq \cdots$. Set

$$K = \sum_{j \geq 1} \frac{|y_j|}{|y_1|}.$$

Turán proved that there exists $1 \leq k \leq 24K$ such that

$$\mathrm{Re} \sum_{j \geq 1} y_j^k \geq \frac{1}{8} |y_1|^k$$

(see [7, Chapter 9, Lemma 2]). We apply Turán's result to $\{z_j\}_{j=1}^\infty$, bounding $K$ from above using Lemma 3.2. □

It follows from Lemmata 3.1 and 3.3 that

$$\frac{1}{\eta^{k\ell}} - \frac{1}{(1+\eta-\beta_1)^{k\ell}} \geq \frac{1}{8(1+\eta-\beta_2)^{k\ell}}.$$

Multiplying through by $\eta^{k\ell}$, and observing that $\lceil \log\log q_1 \rceil \le 2\log\log q_1$, we find that

$$1-\Big(1-\frac{1-\beta_1}{1+\eta-\beta_1}\Big)^{2k\log\log q_1} \ge \frac{1}{8}\Big(1-\frac{1-\beta_2}{1+\eta-\beta_2}\Big)^{2k\log\log q_1}. \tag{3-3}$$

If $\beta_1, \beta_2 \in (0, 1)$ and $\eta > 0$, then

$$\frac{1-\beta_1}{1+\eta-\beta_1}, \frac{1-\beta_2}{1+\eta-\beta_2} \in (0, 1).$$

**Lemma 3.4** (Bernoulli's inequality). *If $0 < a < 1$ and $b > 1$, then $ab > 1-(1-a)^b$.*

*Proof of Theorem 1.1.* Applying Lemma 3.4 to the left-hand side of (3-3) with

$$a = \frac{1-\beta_1}{1+\eta-\beta_1}, \quad b = 2k\log\log q_1,$$

we obtain the bound

$$\frac{2k(1-\beta_1)\log\log q_1}{1+\eta-\beta_1} \ge \frac{1}{8}\Big(1-\frac{1-\beta_2}{1+\eta-\beta_2}\Big)^{2k\log\log q_1} = \tfrac{1}{8}(\log q_1)^{2k\log\left(1-\frac{1-\beta_2}{1+\eta-\beta_2}\right)}.$$

(We have $b \ge 2$ by (2-1) and (2-2).) Dividing through by $(2k\log\log q_1)/(1+\eta-\beta_1)$, we arrive at

$$1-\beta_1 \ge \frac{1+\eta-\beta_1}{16k\log\log q_1}(\log q_1)^{2k\log\left(1-\frac{1-\beta_2}{1+\eta-\beta_2}\right)} \ge \frac{\eta}{1920\log\log q_1}(\log q_1)^{240\log\left(1-\frac{1-\beta_2}{1+\eta-\beta_2}\right)}. \tag{3-4}$$

It follows from (2-1), (2-2), and (2-7) that

$$\varepsilon > 2\frac{\log(1920\eta^{-1}\log\log q_1)}{\log\log q_1},$$

so

$$\frac{\eta}{1920\log\log q_1}(\log q_1)^{240\log\left(1-\frac{1-\beta_2}{1+\eta-\beta_2}\right)} \ge (\log q_1)^{240\log\left(1-\frac{1-\beta_2}{1+\eta-\beta_2}\right)-\varepsilon/2}. \tag{3-5}$$

Therefore, by (2-4), (3-4), and (3-5), we find that

$$(\log q_1)^{-\varepsilon} \ge (\log q_1)^{240\log\left(1-\frac{1-\beta_2}{1+\eta-\beta_2}\right)-\varepsilon/2}. \tag{3-6}$$

Recalling the definition of $\eta$ in (2-6), we solve for $\beta_2$ in (3-6), thus obtaining

$$\beta_2 \le 1-\frac{\delta}{e}(1-e^{-\varepsilon/480}). \tag{3-7}$$

However, by (2-1) and (2-2), the bound (3-7) contradicts (2-4), as desired. □

**Remark 3.5.** In [8, Proof of Theorem 1], Sarnak and Zaharescu begin with the Guinand–Weil explicit formula for $D(s)$: If $B > 0$ and

$$\phi(x) = \Big(\frac{\sin(2\pi x)}{2\pi x}\Big)^2, \quad \hat{\phi}(y) = \int_{-\infty}^{\infty}\phi(x)e^{-2\pi ixy}\,dx = \begin{cases}\frac{1}{2}\left(1-\frac{1}{2}|y|\right) & \text{if } |y| \le 2,\\ 0 & \text{otherwise,}\end{cases}$$

and $\rho=\beta+i\gamma$ runs through the nontrivial zeros of $D(s)$, then

$$\sum_{\rho}\phi\Big(\frac{B}{2\pi i}\big(\rho-\tfrac{1}{2}\big)\Big)+\frac{2}{B}\sum_{n=1}^{\infty}\frac{(1+\chi_1(n)+\chi_2(n)+\psi(n))\Lambda(n)}{\sqrt{n}}\hat{\phi}\Big(\frac{\log n}{B}\Big)$$
$$=\frac{\log(q_\psi q_{\chi_1}q_{\chi_2})+O(1)}{2B}+2\phi\Big(\frac{B}{4\pi i}\Big)+2\phi\Big(-\frac{B}{4\pi i}\Big). \quad (3\text{-}8)$$

Note that $\phi(0)=1$, $\phi(y)\geq 0$ for $y\in\mathbb{R}\cup i\mathbb{R}$, and $\hat{\phi}(y)\geq 0$ for $y\in\mathbb{R}$. Therefore, they can discard the sum over $n$ by the nonnegativity of the Dirichlet coefficients. Assuming that $\beta=\frac{1}{2}$ or $\gamma=0$ always, they discard the contribution from each $\rho$ except for $\rho=\beta_1$ and $\rho=\beta_2$. The conclusion (1-2) now follows; otherwise, they would obtain a contradiction by choosing $B=(1+3\varepsilon/4)\log\log q_1$. See also the discussion in [6, Section 5], especially the remark at the end of the section.

In our proof of Theorem 1.1, we cannot afford to discard all of the terms in the sum over zeros. At the same time, it is unclear how to obtain a strong lower bound on the sum over nonreal zeros in (3-8). We circumvent this problem by taking the $(k\ell-1)$-th derivative of $-(D'/D)(s)$, expressed both as a Dirichlet series and in terms of its Hadamard factorization, and bounding the sum over zeros from below using Turán's power sum method. However, the power sum method will fail us if $k\ell-1$ is larger than $O(\log\log q_1)$; we handle this using $H_\delta$.

## 4. Extensions of Theorem 1.1

We briefly describe a way to extend Theorem 1.1 to Dirichlet characters whose order exceeds 2. Let $m\geq 2$ be an integer, let $j\in\{1,2\}$, let $q_j\geq 3$ be an integer, and let $\chi_j \pmod{q_j}$ be a primitive Dirichlet character of order dividing $m$. Note that $L(s,\chi_1)$ and $L(s,\overline{\chi}_1)$ have the same real zeros, so we assume that $\chi_2\notin\{\chi_1,\overline{\chi}_1\}$. Let $G=\langle\chi_1,\chi_2\rangle$ be the group of Dirichlet characters generated by $\chi_1$ and $\chi_2$, and let $G^*$ be the set of primitive Dirichlet characters that induce the characters in $G$. The Dirichlet series $D(s)=\prod_{\chi\in G^*}L(s,\chi)$ is the Dedekind zeta function of the compositum of two cyclic extensions of degree $m$ over $\mathbb{Q}$. In particular, the Dirichlet coefficients of $D(s)$ are nonnegative. Also, the conductor of $D(s)$ is bounded by $(q_1q_2)^{m^2}$. The identity element of a group is unique, so $D(s)$ has a simple pole at $s=1$ arising from the factor of $\zeta(s)$. Note that if $\chi_2\in\langle\chi_1\rangle-\{\chi_1,\overline{\chi}_1\}$, then $D(s)$ is the product of $L$-functions of primitive characters that induce the characters in $\{\chi_1^j : 0\leq j\leq m-1\}$. In order to ensure that all zeros of $D(s)$ in the disk $|z-1|<\delta$ are real, it suffices to assume for some fixed $0<\delta<\frac{1}{10}$ that for all primitive Dirichlet characters $\nu$ of order dividing $m$, all zeros of $L(s,\nu)$ in the disk $|z-1|<\delta$ are real. Small modifications to our proof of Theorem 1.1 yield the following result.

**Theorem 4.1.** *Fix an integer $m\geq 2$, and let $\mathcal{S}_m$ be the set of primitive Dirichlet characters of order dividing $m$. Let $\sim$ be the equivalence relation on $\mathcal{S}_m$ defined by $\chi_1\sim\chi_2$ if $\chi_2\in\{\chi_1,\overline{\chi}_1\}$, and let $[\chi]$ be the equivalence class of $\chi\in\mathcal{S}_m$ in $\mathcal{S}_m/\sim$. Fix $0<\delta<\frac{1}{10}$. Assume that if $\nu\in\mathcal{S}_m$, then all zeros of $L(s,\nu)$ in the disk $|z-1|<\delta$ are real. For all $\varepsilon>0$, there exists an effectively computable constant*

$q_0 = q_0(\delta, \varepsilon, m) > 0$ *such that*

$$\#\big\{[\chi] \in \mathcal{S}_m/\sim\,:\, q_\chi \geq q_0 \textit{ and } L(s,\chi) \textit{ has a real zero in } [1-(\log q_\chi)^{-\varepsilon},\ 1)\ \big\} \leq 1.$$

**Remark 4.2.** Theorem 4.1 recovers Theorem 1.1 when $m = 2$.

Let $\phi(x+iy)$ be a Hecke–Maaß cusp form on the modular surface $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ with Laplace eigenvalue $\lambda_\phi > 0$, and let $\mathrm{Sym}^2\,\phi$ be its symmetric square lift. Gelbart and Jacquet [2] proved that there exists a cuspidal automorphic representation of $\mathrm{GL}_3$ whose $L$-function is $L(s, \mathrm{Sym}^2\,\phi)$. In [5, Appendix], Goldfeld, Hoffstein, and Lieman proved that there exists an absolute and effectively computable constant $c_5 > 0$ such that

$$L(\sigma, \mathrm{Sym}^2\,\phi) \neq 0, \quad \sigma \geq 1 - c_5/(\log\lambda_\phi).$$

By modifying our proof of Theorem 1.1 in a manner similar to the work of Hoffstein and Lockhart in [5], one can prove the following result.

**Theorem 4.3.** *Fix* $0 < \delta < \frac{1}{10}$. *Let* $\mathfrak{S}$ *be the set of Hecke–Maaß cusp forms on* $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. *Suppose that for all* $\phi_1, \phi_2 \in \mathfrak{S}$ *satisfying* $\phi_1 \neq \phi_2$, *all zeros of*

$$\zeta(s)L(s,\mathrm{Sym}^2\,\phi_1)L(s,\mathrm{Sym}^2\,\phi_2)L(s,\mathrm{Sym}^2\,\phi_1\times\mathrm{Sym}^2\,\phi_2)$$

*in the disk* $|z-1| < \delta$ *are real. For all* $\varepsilon > 0$, *there exists an effectively computable constant* $\lambda_0 = \lambda_0(\delta, \varepsilon)$ *such that*

$$\#\big\{\phi \in \mathfrak{S} : \lambda_\phi \geq \lambda_0 \textit{ and } L(s,\mathrm{Sym}^2\,\phi) \textit{ has a real zero in } [1-(\log\lambda_\phi)^{-\varepsilon},\ 1)\big\} \leq 1.$$


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