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ASYMPTOTIC EXPANSIONS OF FUNDAMENTAL SOLUTIONS IN PARABOLIC HOMOGENIZATION

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For a family of second-order parabolic systems with rapidly oscillating and time-dependent periodic coefficients, we investigate the asymptotic behavior of fundamental solutions and establish sharp estimates for the remainders.

1. Introduction

In this paper we study the asymptotic behavior of fundamental solutions $\Gamma_\varepsilon(x, t; y, s)$ for a family of second-order parabolic operators $\partial_t + \mathcal{L}_\varepsilon$ with rapidly oscillating and time-dependent periodic coefficients. Specifically, we consider

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon, t/\varepsilon^2)\nabla) \tag{1-1}$$

in $\mathbb{R}^d \times \mathbb{R}$, where $\varepsilon > 0$ and $A(y, s) = (a_{ij}^{\alpha\beta}(y, s))$ with $1 \le i, j \le d$ and $1 \le \alpha, \beta \le m$. Throughout the paper we will assume that the coefficient matrix $A = A(y, s)$ is real, bounded measurable and satisfies the ellipticity condition

$$\|A\|_\infty \le \mu^{-1} \quad \text{and} \quad \mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y, s)\xi_i^\alpha \xi_j^\beta \tag{1-2}$$

for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{m\times d}$ and a.e. $(y, s) \in \mathbb{R}^{d+1}$, where $\mu > 0$. We also assume that A is 1-periodic; i.e.,

$$A(y + z, s + t) = A(y, s) \quad \text{for } (z, t) \in \mathbb{Z}^{d+1} \text{ and a.e. } (y, s) \in \mathbb{R}^{d+1}. \tag{1-3}$$

Under these assumptions it is known that as $\varepsilon \to 0$, the operator $\partial_t + \mathcal{L}_\varepsilon$ G-converges to a parabolic operator $\partial_t + \mathcal{L}_0$ with constant coefficients [Bensoussan et al. 1978].

In the scalar case $m = 1$, it follows from a celebrated theorem of John Nash [1958] that local solutions of $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0$ are Hölder continuous. More precisely, if $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0$ in $Q_{2r} = Q_{2r}(x_0, t_0)$ for some $(x_0, t_0) \in \mathbb{R}^{d+1}$ and $0 < r < \infty$, where

$$Q_r(x_0, t_0) = B(x_0, r) \times (t_0 - r^2, t_0), \tag{1-4}$$

then there exists some $\sigma \in (0, 1)$, depending only on d and μ, such that

$$\|u_\varepsilon\|_{C^{\sigma,\sigma/2}(Q_r)} \le C r^{-\sigma} \left(\frac{1}{|Q_{2r}|} \int_{Q_{2r}} |u_\varepsilon|^2 \right)^{1/2}, \tag{1-5}$$

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where $C > 0$ depends only on d and μ. In particular, C and σ are independent of $\varepsilon > 0$. The periodicity assumption (1-3) is not needed here. It follows that the fundamental solution $\Gamma_\varepsilon(x, t; y, s)$ for $\partial_t + \mathcal{L}_\varepsilon$ exists and satisfies the Gaussian estimate

$$|\Gamma_\varepsilon(x, t; y, s)| \le \frac{C}{(t-s)^{d/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \tag{1-6}$$

for any $x, y \in \mathbb{R}^d$ and $-\infty < s < t < \infty$, where $\kappa > 0$ depends only on μ and $C > 0$ depends on d and μ (also see [Aronson 1967; Fabes and Stroock 1986] for lower bounds).

If $m \ge 2$, the global Hölder estimate (1-5) for $1 < r < \infty$ was established recently in [Geng and Shen 2015] for any $\sigma \in (0, 1)$ under the assumptions that A is elliptic, periodic, and $A \in \mathrm{VMO}_x$ (see (2-4) for the definition of VMO_x). We mention that the local Hölder estimate for $0 < r < \varepsilon$ without the periodicity condition was obtained earlier in [Byun 2007; Krylov 2007]. Consequently, by [Hofmann and Kim 2004; Cho et al. 2008], the matrix of fundamental solutions $\Gamma_\varepsilon(x, t; y, s) = (\Gamma_\varepsilon^{\alpha\beta}(x, t; y, s))$, with $1 \le \alpha, \beta \le m$, exists and satisfies the estimate (1-6), where $\kappa > 0$ depends only on μ. The constant $C > 0$ in (1-6) depends on d, m, μ and the function $A^\#(r)$ in (2-5), but not on $\varepsilon > 0$.

The primary purpose of this paper is to study the asymptotic behavior, as $\varepsilon \to 0$, of $\Gamma_\varepsilon(x, t; y, s)$, $\nabla_x \Gamma_\varepsilon(x, t; y, s)$, $\nabla_y \Gamma_\varepsilon(x, t; y, s)$, and $\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)$. Our main results extend the analogous estimates for elliptic operators $-\operatorname{div}(A(x/\varepsilon)\nabla)$ in [Avellaneda and Lin 1991; Kenig et al. 2014] to the parabolic setting. As demonstrated in the elliptic case [Kenig and Shen 2011], the estimates in this paper open the doors for the use of layer potentials in solving initial-boundary value problems for the parabolic operators $\partial_t + \mathcal{L}_\varepsilon$ with sharp estimates that are uniform in $\varepsilon > 0$.

Let $\Gamma_0(x, t; y, s)$ denote the matrix of fundamental solutions for the homogenized operator $\partial_t + \mathcal{L}_0$, where $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$ and $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$ is given by (2-7). Since $\hat{A}$ is constant and satisfies the ellipticity condition (2-8), it is well known that $\Gamma_0(x, t; y, s) = \Gamma_0(x - y, t - s; 0, 0)$ and for any $x, y \in \mathbb{R}^d$ and $-\infty < s < t < \infty$,

$$|\nabla_x^M \partial_t^N \Gamma_0(x, t; y, s)| \le \frac{C}{(t-s)^{(d+M+2N)/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \tag{1-7}$$

for any $M, N \ge 0$, where $\kappa > 0$ depends only on μ, and C depends on d, m, M, N, and μ.

Our first result provides the sharp estimate for $\Gamma_\varepsilon - \Gamma_0$.

Theorem 1.1. *Suppose that the coefficient matrix A satisfies conditions* (1-2) *and* (1-3). *If $m \ge 2$, we also assume that $A \in \mathrm{VMO}_x$. Then*

$$|\Gamma_\varepsilon(x, t; y, s) - \Gamma_0(x, t; y, s)| \le \frac{C\varepsilon}{(t-s)^{(d+1)/2}} \exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\} \tag{1-8}$$

for any $x, y \in \mathbb{R}^d$ and $-\infty < s < t < \infty$, where $\kappa > 0$ depends only on μ. The constant C depends on d, m, μ, and $A^\#$ (if $m \ge 2$).

Let $\chi(y, s) = (\chi_j^{\alpha\beta}(y, s))$, where $1 \le j \le d$ and $1 \le \alpha, \beta \le m$, denote the matrix of correctors for $\partial_t + \mathcal{L}_\varepsilon$ (see Section 2 for its definition). The next theorem gives an asymptotic expansion for $\nabla_x \Gamma_\varepsilon(x, t; y, s)$.

Theorem 1.2. *Suppose that the coefficient matrix A satisfies conditions* (1-2) *and* (1-3). *Also assume that A is Hölder continuous*,

$$|A(x,t)-A(y,s)| \le \tau(|x-y|+|t-s|^{1/2})^{\lambda} \tag{1-9}$$

for any $(x,t),(y,s)\in\mathbb{R}^{d+1}$, *where* $\tau\ge 0$ *and* $\lambda\in(0,1)$. *Then*

$$\begin{aligned}|\nabla_x\Gamma_\varepsilon(x,t;y,s)-(I+\nabla\chi(x/\varepsilon,t/\varepsilon^2))\nabla_x\Gamma_0(x,t;y,s)|&\\ \le \frac{C\varepsilon}{(t-s)^{(d+2)/2}}\log(2+\varepsilon^{-1}|t-s|^{1/2})\exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\}&\end{aligned} \tag{1-10}$$

for any $x,y\in\mathbb{R}^d$ *and* $-\infty<s<t<\infty$, *where* $\kappa>0$ *depends only on* μ. *The constant C depends on* d, m, μ, *and* (λ,τ) *in* (1-9).

With the summation convention this means that for $1\le i\le d$ and $1\le\alpha,\beta\le m$

$$\left|\frac{\partial\Gamma_\varepsilon^{\alpha\beta}}{\partial x_i}(x,t;y,s)-\frac{\partial\Gamma_0^{\alpha\beta}}{\partial x_i}(x,t;y,s)-\frac{\partial\chi_j^{\alpha\gamma}}{\partial x_i}(x/\varepsilon,t/\varepsilon^2)\frac{\partial\Gamma_0^{\gamma\beta}}{\partial x_j}(x,t;y,s)\right| \tag{1-11}$$

is bounded by the right-hand side of (1-10). Let $\tilde{A}(y,s)=(\tilde{a}_{ij}^{\alpha\beta}(y,s))$, where $\tilde{a}_{ij}^{\alpha\beta}(y,s)=a_{ji}^{\beta\alpha}(y,-s)$. Let $\widetilde{\Gamma}_\varepsilon(x,t;y,s)=(\widetilde{\Gamma}_\varepsilon^{\alpha\beta}(x,t;y,s))$ denote the matrix of fundamental solutions for the operator $\partial_t+\widetilde{\mathcal{L}}_\varepsilon$, where $\widetilde{\mathcal{L}}_\varepsilon=-\operatorname{div}(\tilde{A}(x/\varepsilon,t/\varepsilon^2)\nabla)$. Then

$$\Gamma_\varepsilon^{\alpha\beta}(x,t;y,s)=\widetilde{\Gamma}_\varepsilon^{\beta\alpha}(y,-s;x,-t). \tag{1-12}$$

Since $\tilde{A}$ satisfies the same conditions as A, it follows from (1-10), (1-11) and (1-12) that

$$\left|\frac{\partial\Gamma_\varepsilon^{\beta\alpha}}{\partial y_i}(x,t;y,s)-\frac{\partial\Gamma_0^{\beta\alpha}}{\partial y_i}(x,t;y,s)-\frac{\partial\tilde{\chi}_j^{\alpha\gamma}}{\partial y_i}(y/\varepsilon,-s/\varepsilon^2)\frac{\partial\Gamma_0^{\beta\gamma}}{\partial y_j}(x,t;y,s)\right| \tag{1-13}$$

is bounded by the right-hand side of (1-10), where $\tilde{\chi}(y,s)=(\tilde{\chi}_j^{\alpha\beta}(y,s))$ denotes the correctors for $\partial_t+\widetilde{\mathcal{L}}_\varepsilon$. That is,

$$\begin{aligned}|\nabla_y\Gamma_\varepsilon^T(x,t;y,s)-(I+\nabla\tilde{\chi}(y/\varepsilon,-s/\varepsilon^2))\nabla_y\Gamma_0^T(x,t;y,s)|&\\ \le \frac{C\varepsilon}{(t-s)^{(d+2)/2}}\log(2+\varepsilon^{-1}|t-s|^{1/2})\exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\},&\end{aligned} \tag{1-14}$$

where Γ_ε^T denotes the transpose of the matrix Γ_ε.

We also obtain an asymptotic expansion for $\nabla_x\nabla_y\Gamma_\varepsilon(x,t;y,s)$.

Theorem 1.3. *Under the same assumptions on A as in Theorem 1.2, the estimate*

$$\begin{aligned}&\bigg|\frac{\partial}{\partial x_i\partial y_j}\{\Gamma_\varepsilon^{\alpha\beta}(x,t;y,s)\}\\ &\quad-\frac{\partial}{\partial x_i}\{\delta^{\alpha\gamma}x_k+\varepsilon\chi_k^{\alpha\gamma}(x/\varepsilon,t/\varepsilon^2)\}\frac{\partial^2}{\partial x_k\partial y_\ell}\{\Gamma_0^{\gamma\sigma}(x,t;y,s)\}\frac{\partial}{\partial y_j}\{\delta^{\beta\sigma}y_\ell+\varepsilon\tilde{\chi}_\ell^{\beta\sigma}(y/\varepsilon,-s/\varepsilon^2)\}\bigg|\\ &\qquad\le\frac{C\varepsilon}{(t-s)^{(d+3)/2}}\log(2+\varepsilon^{-1}|t-s|^{1/2})\exp\left\{-\frac{\kappa|x-y|^2}{t-s}\right\}\end{aligned} \tag{1-15}$$

holds for $x, y \in \mathbb{R}^d$ *and* $-\infty < s < t < \infty$, *where* κ *depends only on* μ. *The constant* C *depends on* d, m, μ, *and* (λ, τ) *in* (1-9).

Remark 1.4. The estimates (1-10), (1-14) and (1-15) are sharp, up to the logarithmic factor $\log(2 + \varepsilon^{-1}|t-s|^{1/2})$, which is probably not necessary. It may be possible to remove the logarithmic factor by using higher-order correctors in the proof. However, we will not pursue this idea in the present paper.

In the scale case $m = 1$, the estimate (1-8), *without* the exponential factor, is known under the conditions that A is elliptic, periodic, symmetric, and time-independent; see [Jikov et al. 1994, p. 77]. This was proved by using the Floquet–Bloch decomposition of the fundamental solutions and by studying the spectral properties of elliptic operators

$$-(\nabla + ik) \cdot A(\nabla + ik)$$

in a periodic cell, where $i = \sqrt{-1}$ and $k \in \mathbb{R}^d$. Such an approach is not available when the coefficient matrix A is time-dependent. To the best of authors' knowledge, the Gaussian bound in Theorem 1.1 as well as our estimates in Theorems 1.2 and 1.3 are new even in the case that $m = 1$ and A is time-independent.

As a corollary of Theorems 1.1 and 1.2, we establish an interesting result on equistabilization for time-dependent coefficients; cf. [Jikov et al. 1994, p. 77].

Corollary 1.5. *Assume that* A *satisfies the same conditions as in Theorem 1.1. Let* $f \in L^\infty(\mathbb{R}^d)$ *and* u_ε *be the bounded solution of the Cauchy problem*,

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon) u_\varepsilon = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u_\varepsilon = f & \text{on } \mathbb{R}^d \times \{t = 0\}, \end{cases} \tag{1-16}$$

with $\varepsilon = 1$ *or* 0. *Then for any* $x \in \mathbb{R}^d$ *and* $t \geq 1$,

$$|u_1(x,t) - u_0(x,t)| \leq C t^{-1/2} \|f\|_\infty. \tag{1-17}$$

Furthermore, *if* A *is Hölder continuous*,

$$\left| \nabla u_1^\alpha(x,t) - \nabla u_0^\alpha(x,t) - \nabla \chi_j^{\alpha\beta}(x,t) \frac{\partial u_0^\beta}{\partial x_j}(x,t) \right| \leq C t^{-1} \log(2+t) \|f\|_\infty \tag{1-18}$$

for any $x \in \mathbb{R}^d$ *and* $t \geq 1$.

We now describe some of the key ideas in the proof of Theorems 1.1, 1.2, and 1.3. As indicated earlier, our main results extend the analogous results in [Avellaneda and Lin 1991; Kenig et al. 2014] for the elliptic operators $-\operatorname{div}(A(x/\varepsilon)\nabla)$, where $A = A(y)$ is elliptic and periodic. Our general approach is inspired by [Kenig et al. 2014], which uses a two-scale expansion and relies on regularity estimates that are uniform in $\varepsilon > 0$. Following [Geng and Shen 2017], we consider the two-scale expansion

$$w_\varepsilon = u_\varepsilon(x,t) - u_0(x,t) - \varepsilon \chi(x/\varepsilon, t/\varepsilon^2) S_\varepsilon(\nabla u_0) - \varepsilon^2 \phi(x/\varepsilon, t/\varepsilon^2) \nabla S_\varepsilon(\nabla u_0), \tag{1-19}$$

where $\chi(y,s)$ and $\phi(y,s)$ are correctors and dual correctors respectively for $\partial_t + \mathcal{L}_\varepsilon$ (see Section 2 for their definitions). In (1-19) the operator S_ε is a parabolic smoothing operator at scale ε. In comparison with the elliptic case, an extra term is added in the right-hand side of (1-19). This modification allows us to show that if $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0$, then

$$(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = \varepsilon \operatorname{div}(F_\varepsilon) \tag{1-20}$$

for some function F_ε, which depends only on u_0. As a consequence, we may apply the uniform interior L^∞ estimates established in [Geng and Shen 2015] to the function w_ε. To fully utilize the ideas above, we will consider the functions

$$\begin{aligned} u_\varepsilon(x,t) &= \int_{-\infty}^{t}\int_{\mathbb{R}^d} \Gamma_\varepsilon(x,t;y,s) f(y,s) e^{-\psi(y)}\,dy\,ds,\\ u_0(x,t) &= \int_{-\infty}^{t}\int_{\mathbb{R}^d} \Gamma_0(x,t;y,s) f(y,s) e^{-\psi(y)}\,dy\,ds, \end{aligned} \tag{1-21}$$

where ψ is a Lipschitz function in $\mathbb{R}^d$ and $f \in C_0^\infty(Q_r(y_0,s_0);\mathbb{R}^m)$. The main technical step in proving Theorem 1.1 involves bounding the L^∞ norm

$$\|e^{\psi}(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))} \tag{1-22}$$

by $\|f\|_{L^2(Q_r(y_0,s_0))}$, where $0<\varepsilon<r=c\sqrt{t_0-s_0}$. We remark that the use of weighted inequalities with weight e^{ψ} to generate the exponential factor in the Gaussian bound is more or less well known. Our approach may be regarded as a variation of the standard one found in [Hofmann and Kim 2004; Cho et al. 2008]; also see earlier work in [Fabes and Stroock 1986; Davies 1987a; Davies 1987b].

The proof of Theorem 1.2 uses the estimate in Theorem 1.1. The stronger assumption that A is Hölder continuous allows us to apply the uniform interior Lipschitz estimate obtained in [Geng and Shen 2015] to the function w_ε in (1-19). To see Theorem 1.3, one uses the fact that as a function of (x,t), $\nabla_y \Gamma_\varepsilon(x,t;y,s)$ is a solution of $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0$, away from the pole (y,s).

We end this section with some notation that will be used throughout the paper. A function $h=h(y,s)$ in $\mathbb{R}^{d+1}$ is said to be 1-periodic if h is periodic with respect to $\mathbb{Z}^{d+1}$. We will use the notation

$$⨍_E f = \frac{1}{|E|}\int_E f \quad \text{and} \quad h^\varepsilon(x,t) = h(x/\varepsilon, t/\varepsilon^2)$$

for $\varepsilon>0$, as well as the summation convention that the repeated indices are summed. Finally, we shall use κ to denote positive constants that depend only on μ, and C constants that depend at most on d, m, μ and the smoothness of A, but never on ε.

2. Preliminaries

Let $\mathcal{L}_\varepsilon = -\operatorname{div}(A^\varepsilon(x,t)\nabla)$, where $A^\varepsilon(x,t) = A(x/\varepsilon, t/\varepsilon^2)$. Assume that $A(y,s)$ is 1-periodic in (y,s) and satisfies the ellipticity condition (1-2). For $1\le j\le d$ and $1\le\beta\le m$, the corrector $\chi_j^\beta = \chi_j^\beta(y,s) =$

$(\chi_j^{\alpha\beta}(y,s))$ is defined as the weak solution of the cell problem

$$\begin{cases} (\partial_s + \mathcal{L}_1)(\chi_j^\beta) = -\mathcal{L}_1(P_j^\beta) & \text{in } Y, \\ \chi_j^\beta = \chi_j^\beta(y,s) \text{ is 1-periodic in } (y,s), \\ \int_Y \chi_j^\beta = 0, \end{cases} \tag{2-1}$$

where $Y = [0,1)^{d+1}$, $P_j^\beta(y) = y_j e^\beta$, and $e^\beta = (0, \dots, 1, \dots, 0)$ with 1 in the β-th position. Note that

$$(\partial_s + \mathcal{L}_1)(\chi_j^\beta + P_j^\beta) = 0 \quad \text{in } \mathbb{R}^{d+1}. \tag{2-2}$$

By the rescaling property of $\partial_t + \mathcal{L}_\varepsilon$, one obtains

$$(\partial_t + \mathcal{L}_\varepsilon)\{\varepsilon \chi_j^\beta(x/\varepsilon, t/\varepsilon^2) + P_j^\beta(x)\} = 0 \quad \text{in } \mathbb{R}^{d+1}. \tag{2-3}$$

We say $A \in \mathrm{VMO}_x$ if

$$\lim_{r\to 0} A^\#(r) = 0, \tag{2-4}$$

where

$$A^\#(r) = \sup_{\substack{0<\rho<r \\ (x,t)\in\mathbb{R}^{d+1}}} \fint_{t-\rho^2}^{t} \fint_{y\in B(x,\rho)} \fint_{z\in B(x,\rho)} |A(y,s) - A(z,s)|\, dz\, dy\, ds. \tag{2-5}$$

Observe that if $A(y,s)$ is continuous in the variable y, uniformly in (y,s), then $A \in \mathrm{VMO}_x$.

Lemma 2.1. *Assume that $A(y,s)$ is 1-periodic in (y,s) and satisfies* (1-2). *If $m \geq 2$, we also assume $A \in \mathrm{VMO}_x$. Then $\chi_j^\beta \in L^\infty(Y; \mathbb{R}^m)$.*

Proof. In the scalar case $m = 1$, this follows from (2-2) by Nash's classical estimate. Moreover, the estimate

$$\left(\fint_{Q_r(x,t)} |\nabla \chi_j^\beta|^2 \right)^{1/2} \leq C r^{\sigma-1} \tag{2-6}$$

holds for any $0 < r < 1$ and $(x,t) \in \mathbb{R}^{d+1}$, where $Q_r(x,t) = B(x,r) \times (t-r^2, t)$, and $C > 0$ and $\sigma \in (0,1)$ depend on d and μ. If $m \geq 2$ and $A \in \mathrm{VMO}_x$, the boundedness of χ_j^β follows from the interior $W^{1,p}$ estimates for local solutions of $(\partial_t + \mathcal{L}_1)(u) = \mathrm{div}(f)$ [Byun 2007; Krylov 2007]. In this case the estimate (2-6) holds for any $\sigma \in (0,1)$. □

Let $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$, where $1 \leq i, j \leq d$, $1 \leq \alpha, \beta \leq m$, and

$$\hat{a}_{ij}^{\alpha\beta} = \fint_Y \left[a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} \chi_j^{\gamma\beta} \right]; \tag{2-7}$$

that is

$$\hat{A} = \fint_Y \{A + A\nabla\chi\}.$$

It is known that the constant matrix $\hat{A}$ satisfies the ellipticity condition

$$\mu|\xi|^2 \leq \hat{a}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \quad \text{for any } \xi = (\xi_j^\beta) \in \mathbb{R}^{m\times d}, \tag{2-8}$$

and $|\hat{a}_{ij}^{\alpha\beta}| \le \mu_1$, where $\mu_1 > 0$ depends only on d, m and μ [Bensoussan et al. 1978]. Define

$$\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla).$$

Then $\partial_t + \mathcal{L}_0$ is the homogenized operator for the family of parabolic operators $\partial_t + \mathcal{L}_\varepsilon,\ \varepsilon > 0$.

To introduce the dual correctors, we consider the 1-periodic matrix-valued function

$$B = A + A\nabla\chi - \hat{A}. \tag{2-9}$$

More precisely, $B = B(y, s) = (b_{ij}^{\alpha\beta})$, where $1 \le i, j \le d,\ 1 \le \alpha, \beta \le m$, and

$$b_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma}\frac{\partial \chi_j^{\gamma\beta}}{\partial y_k} - \hat{a}_{ij}^{\alpha\beta}. \tag{2-10}$$

Lemma 2.2. *Let $1 \le j \le d$ and $1 \le \alpha, \beta \le m$. Then there exist 1-periodic functions $\phi_{kij}^{\alpha\beta}(y, s)$ in $\mathbb{R}^{d+1}$ such that $\phi_{kij}^{\alpha\beta} \in H^1(Y)$,*

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}(\phi_{kij}^{\alpha\beta}) \quad \text{and} \quad \phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}, \tag{2-11}$$

where $1 \le k, i \le d+1$, $b_{ij}^{\alpha\beta}$ is defined by (2-10) for $1 \le i \le d$, $b_{(d+1)j}^{\alpha\beta} = -\chi_j^{\alpha\beta}$, and we have used the notation $y_{d+1} = s$.

Proof. This lemma was proved in [Geng and Shen 2015]. We give a proof here for reader's convenience. By (2-1) and (2-7), $b_{ij}^{\alpha\beta} \in L^2(Y)$ and

$$\int_Y b_{ij}^{\alpha\beta} = 0 \tag{2-12}$$

for $1 \le i \le d+1$. It follows that there exist $f_{ij}^{\alpha\beta} \in H^2(Y)$ such that

$$\begin{aligned} &\Delta_{d+1} f_{ij}^{\alpha\beta} = b_{ij}^{\alpha\beta} \quad \text{in } \mathbb{R}^{d+1},\\ &f_{ij}^{\alpha\beta} \text{ is 1-periodic in } \mathbb{R}^{d+1}, \end{aligned} \tag{2-13}$$

where Δ_{d+1} denotes the Laplacian in $\mathbb{R}^{d+1}$. Write

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}\left\{\frac{\partial}{\partial y_k} f_{ij}^{\alpha\beta} - \frac{\partial}{\partial y_i} f_{kj}^{\alpha\beta}\right\} + \frac{\partial}{\partial y_i}\left\{\frac{\partial}{\partial y_k} f_{kj}^{\alpha\beta}\right\}, \tag{2-14}$$

where the index k is summed from 1 to $d+1$. Note that by (2-1),

$$\sum_{i=1}^{d+1} \frac{\partial b_{ij}^{\alpha\beta}}{\partial y_i} = \sum_{i=1}^{d} \frac{\partial}{\partial y_i} b_{ij}^{\alpha\beta} - \frac{\partial}{\partial s}\chi_j^{\alpha\beta} = 0. \tag{2-15}$$

In view of (2-13) this implies

$$\sum_{i=1}^{d+1} \frac{\partial}{\partial y_i} f_{ij}^{\alpha\beta}$$

is harmonic in $\mathbb{R}^{d+1}$. Since it is 1-periodic, it must be constant. Consequently, by (2-14), we obtain

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}(\phi_{kij}^{\alpha\beta}), \tag{2-16}$$

where

$$\phi_{kij}^{\alpha\beta} = \frac{\partial}{\partial y_k} f_{ij}^{\alpha\beta} - \frac{\partial}{\partial y_i} f_{kj}^{\alpha\beta} \tag{2-17}$$

is 1-periodic and belongs to $H^1(Y)$. It is easy to see that $\phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}$. □

The 1-periodic functions $(\phi_{kij}^{\alpha\beta})$, given by Lemma 2.2, are called dual correctors for the family of parabolic operators $\partial_t + \mathcal{L}_\varepsilon$, $\varepsilon > 0$.

Lemma 2.3. *Let $\phi = (\phi_{kij}^{\alpha\beta})$ be the dual correctors, given by Lemma 2.2. Under the same assumptions as in Lemma 2.1, one has $\phi_{kij}^{\alpha\beta} \in L^\infty(Y)$.*

Proof. It follows from (2-6) that if $(x, t) \in \mathbb{R}^{d+1}$ and $0 < r < 1$,

$$\int_{Q_r(x,t)} |b_{ij}^{\alpha\beta}|^2 \le C r^{d+2\sigma} \tag{2-18}$$

for some $\sigma \in (0, 1)$. By covering the interval $(t - r, t)$ with intervals of length r^2, we obtain

$$\int_{B_r(x,t)} |b_{ij}^{\alpha\beta}|^2 \le C r^{d-1+2\sigma},$$

where $B_r(x, t) = B(x, r) \times (t - r, t)$. Hence, by Hölder's inequality,

$$\int_{B_r(x,t)} |b_{ij}^{\alpha\beta}| \le C r^{d+\sigma}.$$

Thus, for any $(x, t) \in Y$,

$$\int_Y \frac{|b_{ij}^{\alpha\beta}(y, s)|}{(|x - y| + |t - s|)^d}\, dy\, ds \le C \sum_{j=1}^{\infty} 2^{jd} \int_{|y-x|+|t-s|\sim 2^{-j}} |b_{ij}^{\alpha\beta}(y, s)|\, dy\, ds \le C. \tag{2-19}$$

In view of (2-13), by using the fundamental solution for Δ_{d+1} in $\mathbb{R}^{d+1}$, we may show that

$$\|\nabla_{y,s} f_{ij}^{\alpha\beta}\|_{L^\infty(Y)} \le C \|\nabla_{y,s} f_{ij}^{\alpha\beta}\|_{L^2(Y)} + \sup_{(x,t)\in Y} \int_Y \frac{|b_{ij}^{\alpha\beta}(y, s)|}{(|x - y| + |t - s|)^d}\, dy\, ds,$$

where $\nabla_{y,s}$ denotes the gradient in $\mathbb{R}^{d+1}$. This, together with (2-19), shows that $|\nabla_{y,s} f_{ij}^{\alpha\beta}| \in L^\infty(Y)$. By (2-17) we obtain $\phi_{kij}^{\alpha\beta} \in L^\infty(Y)$. □

Remark 2.4. Suppose $A = A(y, s)$ is Hölder continuous in (y, s). By (2-2) and the standard regularity theory for $\partial_s + \mathcal{L}_1$, we have $\nabla \chi(y, s)$ is Hölder continuous in (y, s). It follows that $b_{ij}^{\alpha\beta}(y, s)$ is Hölder continuous in (y, s). In view of (2-13) and (2-17) one may deduce that $\nabla_{y,s} \phi_{kij}^{\alpha\beta}$ is Hölder continuous in (y, s). This will be used in the proof of Theorems 1.2 and 1.3.

Theorem 2.5. *Suppose that A satisfies the conditions* (1-2) *and* (1-3). *If $m \ge 2$, we also assume $A \in \mathrm{VMO}_x$. Let u_ε be a weak solution of $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = \operatorname{div}(f)$ in $Q_{2r} = Q_{2r}(x_0, t_0)$ for some $0 < r < \infty$, where $f = (f_i^\alpha) \in L^p(Q_{2r}; \mathbb{R}^{m\times d})$ for some $p > d + 2$. Then*

$$\|u_\varepsilon\|_{L^\infty(Q_r)} \le C \left\{ \left(\fint_{Q_{2r}} |u_\varepsilon|^2 \right)^{1/2} + r \left(\fint_{Q_{2r}} |f|^p \right)^{1/p} \right\}, \tag{2-20}$$

where C depends only on d, m, p, μ, and $A^\#$ in (2-5) (*if $m \ge 2$*).

Proof. If $m = 1$, this follows from the well-known Nash's estimate. The periodicity is not needed. If $m \geq 2$, (2-20) follows from the uniform interior Hölder estimate in [Geng and Shen 2015, Theorem 1.1]. □

Under the assumptions on A in Theorem 2.5, the matrix of fundamental solutions for $\partial_t + \mathcal{L}_\varepsilon$ in $\mathbb{R}^{d+1}$ exists and satisfies the Gaussian estimate (1-6). This follows from the L^∞ estimate (2-20) by a general result in [Hofmann and Kim 2004]; also see [Auscher 1996; Cho et al. 2008].

Theorem 2.6. *Suppose that A satisfies conditions* (1-2) *and* (1-3). *Also assume that A satisfies the Hölder condition* (1-9). *Let u_ε be a weak solution of $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = F$ in $Q_{2r} = Q_{2r}(x_0, t_0)$ for some $0 < r < \infty$, where $F \in L^p(Q_{2r}; \mathbb{R}^m)$ for some $p > d + 2$. Then*

$$\|\nabla u_\varepsilon\|_{L^\infty(Q_r)} \leq C\left\{\frac{1}{r}\left(\fint_{Q_{2r}} |u_\varepsilon|^2\right)^{1/2} + r\left(\fint_{Q_{2r}} |F|^p\right)^{1/p}\right\}, \tag{2-21}$$

where C depends only on d, m, p, μ, and (λ, τ) in (1-9).

Proof. This was proved in [Geng and Shen 2015, Theorem 1.2]. □

The Lipschitz estimate (2-21) allows us to bound $\nabla_x \Gamma_\varepsilon(x, t; y, s)$, $\nabla_y \Gamma_\varepsilon(x, t; y, s)$ and $\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)$.

Theorem 2.7. *Assume that A satisfies the same conditions as in Theorem 2.6. Then*

$$|\nabla_x \Gamma_\varepsilon(x, t; y, s)| + |\nabla_y \Gamma_\varepsilon(x, t; y, s)| \leq \frac{C}{(t - s)^{(d+1)/2}} \exp\left\{-\frac{\kappa|x - y|^2}{t - s}\right\}, \tag{2-22}$$

$$|\nabla_x \nabla_y \Gamma_\varepsilon(x, t; y, s)| \leq \frac{C}{(t - s)^{(d+2)/2}} \exp\left\{-\frac{\kappa|x - y|^2}{t - s}\right\} \tag{2-23}$$

for any $x, y \in \mathbb{R}^d$ and $-\infty < s < t < \infty$, where $\kappa > 0$ depends only on μ. The constant C depends on d, m, μ, and (λ, τ) in (1-9).

Proof. Fix $x_0, y_0 \in \mathbb{R}^d$ and $s_0 < t_0$. Let $u_\varepsilon(x, t) = \Gamma_\varepsilon(x, t; y_0, s_0)$. Then $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = 0$ in $Q_{2r}(x_0, t_0)$, where $r = \sqrt{t_0 - s_0}/8$. The estimate for $|\nabla_x \Gamma(x_0, t_0; y_0, s_0)|$ now follows from (2-21) and (1-6) (with a different κ). In view of (1-12) this also gives the estimate for $|\nabla_y \Gamma_\varepsilon(x_0, t_0; y_0, s_0)|$. Finally, to see (2-23), we let $v_\varepsilon(x, t) = \nabla_y \Gamma_\varepsilon(x, t; y_0, s_0)$. Then $(\partial_t + \mathcal{L}_\varepsilon)v_\varepsilon = 0$ in $Q_{2r}(x_0, t_0)$. By applying (2-21) to v_ε and using the estimate in (2-22) for $\nabla_y \Gamma_\varepsilon(x, t; y, s)$, we obtain the desired estimate for $|\nabla_x \nabla_y \Gamma_\varepsilon(x_0, t_0; y_0, s_0)|$. □

3. A two-scale expansion

Suppose that

$$(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 \tag{3-1}$$

in $\Omega \times (T_0, T_1)$, where $\Omega \subset \mathbb{R}^d$. Let S_ε be a linear operator to be chosen later. Following [Geng and Shen 2017], we consider the two-scale expansion $w_\varepsilon = (w_\varepsilon^\alpha)$, where

$$w_\varepsilon^\alpha(x, t) = u_\varepsilon^\alpha(x, t) - u_0^\alpha(x, t) - \varepsilon \chi_j^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) S_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\right) - \varepsilon^2 \phi_{(d+1)ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_i} S_\varepsilon\left(\frac{\partial u_0^\beta}{\partial x_j}\right), \tag{3-2}$$

and $\chi_j^{\alpha\beta}$, $\phi_{(d+1)ij}^{\alpha\beta}$ are the correctors and dual correctors introduced in the last section. The repeated indices i, j in (3-2) are summed from 1 to d.

Proposition 3.1. *Let $u_\varepsilon \in L^2(T_0, T_1; H^1(\Omega))$ and $u_0 \in L^2(T_0, T_1; H^2(\Omega))$. Let w_ε be defined by* (3-2). *Assume* (3-1) *holds in $\Omega \times (T_0, T_1)$. Then*

$$(\partial_t + \mathcal{L}_\varepsilon) w_\varepsilon = \varepsilon \operatorname{div}(F_\varepsilon) \tag{3-3}$$

in $\Omega \times (T_0, T_1)$, where $F_\varepsilon = (F^\alpha_{\varepsilon,i})$ and

$$\begin{aligned} F^\alpha_{\varepsilon,i}(x,t) &= \varepsilon^{-1}(a^{\alpha\beta}_{ij}(x/\varepsilon, t/\varepsilon^2) - \hat{a}^{\alpha\beta}_{ij}) \biggl(\frac{\partial u^\beta_0}{\partial x_j} - S_\varepsilon\biggl(\frac{\partial u^\beta_0}{\partial x_j}\biggr)\biggr) \\ &\quad + a^{\alpha\beta}_{ij}(x/\varepsilon, t/\varepsilon^2) \chi^{\beta\gamma}_k(x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_j} S_\varepsilon\biggl(\frac{\partial u^\gamma_0}{\partial x_k}\biggr) \\ &\quad + \phi^{\alpha\beta}_{ikj}(x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_k} S_\varepsilon\biggl(\frac{\partial u^\beta_0}{\partial x_j}\biggr) + \varepsilon \phi^{\alpha\beta}_{i(d+1)j}(x/\varepsilon, t/\varepsilon^2) \partial_t S_\varepsilon\biggl(\frac{\partial u^\beta_0}{\partial x_j}\biggr) \\ &\quad - a^{\alpha\beta}_{ij}(x/\varepsilon, t/\varepsilon^2) \biggl(\frac{\partial}{\partial x_j}(\phi^{\beta\gamma}_{(d+1)\ell k})\biggr)(x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_\ell} S_\varepsilon\biggl(\frac{\partial u^\gamma_0}{\partial x_k}\biggr) \\ &\quad - \varepsilon a^{\alpha\beta}_{ij}(x/\varepsilon, t/\varepsilon^2) \phi^{\beta\gamma}_{(d+1)\ell k}(x/\varepsilon, t/\varepsilon^2) \frac{\partial^2}{\partial x_j \partial x_\ell} S_\varepsilon\biggl(\frac{\partial u^\gamma_0}{\partial x_k}\biggr). \end{aligned} \tag{3-4}$$

The repeated indices i, j, k, ℓ are summed from 1 to d.

Proof. This proposition was proved in [Geng and Shen 2017, Theorem 2.2]. □

We now introduce a parabolic smoothing operator. Let

$$\mathcal{O} = \{(x,t) \in \mathbb{R}^{d+1} : |x|^2 + |t| \le 1\}.$$

Fix a nonnegative function $\theta = \theta(x,t) \in C^\infty_0(\mathcal{O})$ such that $\int_{\mathbb{R}^{d+1}} \theta = 1$. Let $\theta_\varepsilon(x,t) = \varepsilon^{-d-2}\theta(x/\varepsilon, t/\varepsilon^2)$. Define

$$S_\varepsilon(f)(x,t) = f * \theta_\varepsilon(x,t) = \int_{\mathbb{R}^{d+1}} f(x-y, t-s)\theta_\varepsilon(y,s)\, dy\, ds. \tag{3-5}$$

Lemma 3.2. *Let $g = g(x,t)$ be a 1-periodic function in (x,t) and $\psi = \psi(x)$ a bounded Lipschitz function in $\mathbb{R}^d$. Then*

$$\|e^\psi g^\varepsilon S_\varepsilon(f)\|_{L^p(\mathbb{R}^{d+1})} \le C\, e^{\varepsilon \|\nabla \psi\|_\infty} \|g\|_{L^p(Y)} \|e^\psi f\|_{L^p(\mathbb{R}^{d+1})} \tag{3-6}$$

for any $1 \le p < \infty$, where $g^\varepsilon(x,t) = g(x/\varepsilon, t/\varepsilon^2)$ and C depends only on d and p.

Proof. Using $\int_{\mathbb{R}^{d+1}} \theta_\varepsilon = 1$ and Hölder's inequality, we obtain

$$|S_\varepsilon(e^{-\psi} f)(x,t)|^p \le \int_{\mathbb{R}^{d+1}} |e^{-\psi(y)} f(y,s)|^p\, \theta_\varepsilon(x-y, t-s)\, dy\, ds.$$

It follows that

$$\begin{aligned} |e^{\psi(x)} S_\varepsilon(e^{-\psi} f)(x,t)|^p &\le \int_{\mathbb{R}^{d+1}} |e^{\psi(x)-\psi(y)} f(y,s)|^p\, \theta_\varepsilon(x-y, t-s)\, dy\, ds \\ &\le e^{\varepsilon p \|\nabla \psi\|_\infty} \int_{\mathbb{R}^{d+1}} |f(y,s)|^p\, \theta_\varepsilon(x-y, t-s)\, dy\, ds, \end{aligned}$$

where we have used the facts that $|\psi(x)-\psi(y)|\le\|\nabla\psi\|_\infty|x-y|$ and $\theta_\varepsilon(x-y,t-s)=0$ if $|x-y|>\varepsilon$, for the last step. Hence, by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^{d+1}}&|g^\varepsilon(x,t)|^p|e^\psi S_\varepsilon(e^{-\psi}f)(x,t)|^p\,dx\,dt\\&\le e^{\varepsilon p\|\nabla\psi\|_\infty}\sup_{(y,s)\in\mathbb{R}^{d+1}}\int_{\mathbb{R}^{d+1}}|g^\varepsilon(x,t)|^p\theta_\varepsilon(x-y,t-s)\,dx\,dt\int_{\mathbb{R}^{d+1}}|f(y,s)|^p\,dy\,ds\\&\le C\,e^{\varepsilon p\|\nabla\psi\|_\infty}\|g\|^p_{L^p(Y)}\|f\|^p_{L^p(\mathbb{R}^{d+1})},\end{aligned}$$

where C depends only on d. This gives (3-6). ☐

Remark 3.3. Let $\Omega\subset\mathbb{R}^d$ and $(T_0,T_1)\subset\mathbb{R}$. Define

$$\Omega_\varepsilon=\{x\in\mathbb{R}^d:\operatorname{dist}(x,\Omega)<\varepsilon\}. \tag{3-7}$$

Observe that for $(x,t)\in\Omega\times(T_0,T_1)$, we have $S_\varepsilon(f)(x,t)=S_\varepsilon(f\eta_\varepsilon)(x,t)$, where $\eta_\varepsilon=\eta_\varepsilon(x,t)$ is the characteristic function of $\Omega_\varepsilon\times(T_0-\varepsilon^2,T_1+\varepsilon^2)$. By applying (3-6) to the function $f\eta_\varepsilon$, one may deduce that

$$\int_{T_0}^{T_1}\int_\Omega|e^\psi g^\varepsilon S_\varepsilon(f)|^p\,dx\,dt\le Ce^{\varepsilon p\|\nabla\psi\|_\infty}\|g\|^p_{L^p(Y)}\int_{T_0-\varepsilon^2}^{T_1+\varepsilon^2}\int_{\Omega_\varepsilon}|e^\psi f|^p\,dx\,dt. \tag{3-8}$$

Using $\int_{\mathbb{R}^{d+1}}|\nabla\theta_\varepsilon|\,dx\,dt\le C\varepsilon^{-1}$, the same argument as in the proof of Lemma 3.2 also shows that

$$\int_{T_0}^{T_1}\int_\Omega|e^\psi g^\varepsilon\nabla S_\varepsilon(f)|^p\,dx\,dt\le C\varepsilon^{-p}e^{\varepsilon p\|\nabla\psi\|_\infty}\|g\|^p_{L^p(Y)}\int_{T_0-\varepsilon^2}^{T_1+\varepsilon^2}\int_{\Omega_\varepsilon}|e^\psi f|^p\,dx\,dt \tag{3-9}$$

for $1\le p<\infty$, where C depends only on d and p.

Lemma 3.4. *Let S_ε be defined as in* (3-5)*. Let $1\le p<\infty$ and ψ be a bounded Lipschitz function in $\mathbb{R}^d$. Then for $\Omega\subset\mathbb{R}^d$ and $(T_0,T_1)\subset\mathbb{R}$,*

$$\int_{T_0}^{T_1}\int_\Omega|e^\psi(S_\varepsilon(\nabla f)-\nabla f)|^p\,dx\,dt\le C\varepsilon^pe^{\varepsilon p\|\nabla\psi\|_\infty}\int_{T_0-\varepsilon^2}^{T_1+\varepsilon^2}\int_{\Omega_\varepsilon}|e^\psi(|\nabla^2f|+|\partial_tf|)|^p\,dx\,dt, \tag{3-10}$$

where Ω_ε is given by (3-7) *and C depends only on d and p.*

Proof. Write

$$S_\varepsilon(\nabla f)(x,t)-\nabla f(x,t)=J_1(x,t)+J_2(x,t),$$

where

$$\begin{aligned}J_1(x,t)&=\int_{\mathbb{R}^{d+1}}\theta_\varepsilon(y,s)(\nabla f(x-y,t-s)-\nabla f(x-y,t))\,dy\,ds,\\J_2(x,t)&=\int_{\mathbb{R}^{d+1}}\theta_\varepsilon(y,s)(\nabla f(x-y,t)-\nabla f(x,t))\,dy,\,ds.\end{aligned}$$

To estimate J_2, we observe that by Hölder's inequality and the fact $\int_{\mathbb{R}^{d+1}}\theta_\varepsilon\,dy\,ds=1$,

$$|J_2(x,t)|^p\le\int_{\mathbb{R}^{d+1}}\theta_\varepsilon(y,s)|\nabla f(x-y,t)-\nabla f(x,t)|^p\,dy\,ds,$$

and

$$|\nabla f(x-y,t)-\nabla f(x,t)| = \left|\int_0^1 \frac{\partial}{\partial\tau}\nabla f(x-\tau y,t)\,d\tau\right| \le |y|\int_0^1 |\nabla^2 f(x-\tau y,t)|\,d\tau \le |y|\left(\int_0^1 |\nabla^2 f(x-\tau y,t)|^p\,d\tau\right)^{1/p}.$$

It follows by Fubini's theorem that

$$\begin{aligned}\int_{T_0}^{T_1}\int_\Omega |e^{\psi(x)}J_2(x,t)|^p\,dx\,dt &\le \int_{T_0}^{T_1}\int_\Omega\int_{\mathbb{R}^{d+1}}\int_0^1 e^{p\psi(x)}\theta_\varepsilon(y,s)|y|^p|\nabla^2 f(x-\tau y,t)|^p\,d\tau\,dy\,ds\,dx\,dt\\ &\le \varepsilon^p e^{\varepsilon p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_\Omega\int_{\mathbb{R}^{d+1}}\int_0^1 e^{p\psi(x-\tau y)}\theta_\varepsilon(y,s)|\nabla^2 f(x-\tau y,t)|^p\,d\tau\,dy\,ds\,dx\,dt\\ &\le \varepsilon^p e^{\varepsilon p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_{\Omega_\varepsilon}|e^{\psi}\nabla^2 f|^p\,dx\,dt,\end{aligned}$$

where we have used the facts that $|\psi(x)-\psi(x-\tau y)| \le |\tau|\,|y|\|\nabla\psi\|_\infty$ and $\theta_\varepsilon(y,s)=0$ if $|y|>\varepsilon$.

Finally, to estimate J_1, we first use integration by parts to obtain

$$|J_1(x,t)| \le \int_{\mathbb{R}^{d+1}} |\nabla\theta_\varepsilon(y,s)|\,|f(x-y,t-s)-f(x-y,t)|\,dy\,ds.$$

By Hölder's inequality,

$$|J_1(x,t)|^p \le C\varepsilon^{1-p}\int_{\mathbb{R}^{d+1}} |\nabla\theta_\varepsilon(y,s)|\,|f(x-y,t-s)-f(x-y,t)|^p\,dy\,ds,$$

where we have also used the fact $\int_{\mathbb{R}^{d+1}}|\nabla\theta_\varepsilon|\,dy\,ds \le C\varepsilon^{-1}$. Using

$$|f(x-y,t-s)-f(x-y,t)| \le \left|\int_0^1 \frac{\partial}{\partial\tau} f(x-y,t-\tau s)\,d\tau\right| \le |s|\left(\int_0^1 |\partial_t f(x-y,t-\tau s)|^p\,d\tau\right)^{1/p},$$

we see that by Fubini's theorem,

$$\begin{aligned}\int_{T_0}^{T_1}\int_\Omega |e^{\psi(x)}J_1(x,t)|^p\,dx\,dt &\le C\varepsilon^{1-p}\int_{T_0}^{T_1}\int_\Omega\int_{\mathbb{R}^{d+1}}\int_0^1 e^{p\psi(x)}|\nabla\theta_\varepsilon(y,s)|\,|s|^p|\partial_t f(x-y,t-\tau s)|^p\,d\tau\,dy\,ds\,dx\,dt\\ &\le C\varepsilon^{1+p}e^{\varepsilon p\|\nabla\psi\|_\infty}\int_{T_0}^{T_1}\int_\Omega\int_{\mathbb{R}^{d+1}}\int_0^1 e^{p\psi(x-y)}|\nabla\theta_\varepsilon(y,s)|\,|\partial_t f(x-y,t-\tau s)|^p\,d\tau\,dy\,ds\,dx\,dt\\ &\le C\varepsilon^p e^{\varepsilon p\|\nabla\psi\|_\infty}\int_{T_0-\varepsilon^2}^{T_1+\varepsilon^2}\int_{\Omega_\varepsilon}|e^{\psi}\partial_t f|^p\,dx\,dt,\end{aligned}$$

where we have used the facts that $|\psi(x)-\psi(x-y)| \le \|\nabla\psi\|_\infty|y|$ and $\theta_\varepsilon(y,s)=0$ if $|y|>\varepsilon$ or $|s|>\varepsilon^2$. This, together with the estimate for J_2, completes the proof. □

Theorem 3.5. *Let $F_\varepsilon = (F^\alpha_{\varepsilon,i})$ be given by* (3-4) *and $1 \le p < \infty$. Then for any $\Omega \subset \mathbb{R}^d$ and $(T_0, T_1) \subset \mathbb{R}$,*

$$\int_{T_0}^{T_1}\int_\Omega |e^\psi F_\varepsilon|^p \, dx\, dt \le C e^{\varepsilon p \|\nabla\psi\|_\infty} \int_{T_0-\varepsilon^2}^{T_1+\varepsilon^2}\int_{\Omega_\varepsilon} \{|e^\psi \nabla^2 u_0|^p + |e^\psi \partial_t u_0|^p\}\, dx\, dt, \tag{3-11}$$

where Ω_ε is given by (3-7) *and C depends only on d, m, p and μ.*

Proof. Observe that

$$\begin{aligned}
&\int_{T_0}^{T_1}\int_\Omega |e^\psi F_\varepsilon|^p \, dx\, dt \\
&\quad\le C\varepsilon^{-p}\int_{T_0}^{T_1}\int_\Omega |\nabla u_0 - S_\varepsilon(\nabla u_0)|^p e^{p\psi}\, dx\, dt + C\int_{T_0}^{T_1}\int_\Omega |\chi^\varepsilon|^p |S_\varepsilon(\nabla^2 u_0)|^p e^{p\psi}\, dx\, dt \\
&\qquad + C\int_{T_0}^{T_1}\int_\Omega |\phi^\varepsilon|^p |S_\varepsilon(\nabla^2 u_0)|^p e^{p\psi}\, dx\, dt + C\varepsilon^p\int_{T_0}^{T_1}\int_\Omega |\phi^\varepsilon|^p |\nabla S_\varepsilon(\partial_t u_0)|^p e^{p\psi}\, dx\, dt \\
&\qquad + C\int_{T_0}^{T_1}\int_\Omega |(\nabla\phi)^\varepsilon|^p |S_\varepsilon(\nabla^2 u_0)|^p e^{p\psi}\, dx\, dt + C\varepsilon^p\int_{T_0}^{T_1}\int_\Omega |\phi^\varepsilon|^p |\nabla S_\varepsilon(\nabla^2 u_0)|^p e^{p\psi}\, dx\, dt,
\end{aligned} \tag{3-12}$$

where C depends only on d and μ. In (3-12) we have also used the observation that $\partial_t S_\varepsilon(\nabla u_0) = \nabla S_\varepsilon(\partial_t u_0)$ and $\nabla S_\varepsilon(\nabla u_0) = S_\varepsilon(\nabla^2 u_0)$.

We now proceed to bound each term in the right-hand side of (3-12), using Lemma 3.4 and Remark 3.3. By Lemma 3.4, the first term in the right-hand side of (3-12) is bounded by

$$C e^{p\varepsilon\|\nabla\psi\|_\infty} \int_{T_0-\varepsilon^2}^{T_1+\varepsilon^2}\int_{\Omega_\varepsilon} |e^\psi (|\nabla^2 u_0| + |\partial_t u_0|)|^p \, dx\, dt. \tag{3-13}$$

Using (3-8) we may bound the second, third, fifth terms in the right-hand side of (3-12) by

$$C e^{p\varepsilon\|\nabla\psi\|_\infty} \int_{T_0-\varepsilon^2}^{T_1+\varepsilon^2}\int_{\Omega_\varepsilon} |e^\psi \nabla^2 u_0|^p \, dx\, dt. \tag{3-14}$$

Finally, by (3-9), the fourth and sixth terms in the right-hand side of (3-12) are bounded by (3-13). □

4. Weighted estimates for $\partial_t + \mathcal{L}_0$

Recall that $\Gamma_0(x, t; y, s)$ denotes the matrix of fundamental solutions for the homogenized operator $\partial_t + \mathcal{L}_0$ in $\mathbb{R}^{d+1}$. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a bounded Lipschitz function and

$$u_0(x, t) = \int_{-\infty}^{t}\int_{\mathbb{R}^d} \Gamma_0(x, t; y, s) f(y, s) e^{-\psi(y)}\, dy\, ds, \tag{4-1}$$

where $f \in C_0^\infty(\mathbb{R}^{d+1}; \mathbb{R}^m)$. Then

$$(\partial_t + \mathcal{L}_0) u_0 = e^{-\psi} f \quad \text{in } \mathbb{R}^{d+1}. \tag{4-2}$$

The goal of this section is to prove the following.

Theorem 4.1. *Let u_0 be defined by* (4-1). *Suppose that $f(x,t)=0$ for $t\le s_0$. Then*

$$\int_{s_0}^{t}\int_{\mathbb{R}^d} |e^{\psi}(|\nabla^2 u_0|+|\partial_t u_0|)|^2\,dx\,dt \le Ce^{\kappa(t-s_0)\|\nabla\psi\|_\infty^2}\int_{s_0}^{t}\int_{\mathbb{R}^d}|f|^2\,dx\,dt \tag{4-3}$$

for any $s_0<t<\infty$, where $\kappa>0$ depends only on μ and C depends only on d and μ.

We start with an estimate on a lower-order term.

Lemma 4.2. *Let u_0 be defined by* (4-1). *Suppose that $f(x,t)=0$ for $t<s_0$. Then*

$$\int_{s_0}^{t}\int_{\mathbb{R}^d} |e^{\psi}\nabla u_0|^2\,dx\,dt \le C(t-s_0)e^{\kappa_1(t-s_0)\|\nabla\psi\|_\infty^2}\int_{s_0}^{t}\int_{\mathbb{R}^d}|f|^2\,dx\,dt \tag{4-4}$$

for any $s_0<t<\infty$, where $\kappa_1>0$ depends only on μ and C depends only on d and μ.

Proof. It follows from (1-7) that for $x,y\in\mathbb{R}^d$ and $t>s$,

$$\begin{aligned} e^{\psi(x)-\psi(y)}|\nabla_x\Gamma_0(x,t;y,s)| &\le \frac{C}{(t-s)^{(d+1)/2}}\exp\left\{\psi(x)-\psi(y)-\frac{\kappa|x-y|^2}{t-s}\right\}\\ &\le \frac{C}{(t-s)^{(d+1)/2}}\exp\left\{\|\nabla\psi\|_\infty|x-y|-\frac{\kappa|x-y|^2}{t-s}\right\}. \end{aligned}$$

This, together with the inequality

$$\|\nabla\psi\|_\infty|x-y|\le\frac{(t-s)\|\nabla\psi\|_\infty^2}{2\kappa}+\frac{\kappa|x-y|^2}{2(t-s)}, \tag{4-5}$$

yields

$$e^{\psi(x)-\psi(y)}|\nabla_x\Gamma_0(x,t;y,s)|\le Ce^{(t-s)\|\nabla\psi\|_\infty^2/(2\kappa)}\cdot\frac{1}{(t-s)^{(d+1)/2}}e^{-\kappa|x-y|^2/(2(t-s))}. \tag{4-6}$$

It follows that

$$\begin{aligned} |e^{\psi(x)}\nabla u_0(x,t)| &\le \int_{s_0}^{t}\int_{\mathbb{R}^d} e^{\psi(x)-\psi(y)}|\nabla_x\Gamma_0(x,t;y,s)|\,|f(y,s)|\,dy\,ds\\ &\le Ce^{(t-s_0)\|\nabla\psi\|_\infty^2/(2\kappa)}\int_{s_0}^{t}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{(d+1)/2}}e^{-\kappa|x-y|^2/(2(t-s))}|f(y,s)|\,dy\,ds\\ &\le Ce^{(t-s_0)\|\nabla\psi\|_\infty^2/(2\kappa)}(t-s_0)^{1/4}\left(\int_{s_0}^{t}\int_{\mathbb{R}^d}\frac{1}{(t-s)^{(d+1)/2}}e^{-\kappa|x-y|^2/(2(t-s))}|f(y,s)|^2\,dy\,ds\right)^{1/2}, \end{aligned}$$

where we have used Hölder's inequality for the last step. The estimate (4-4) now follows by Fubini's theorem. □

Proof of Theorem 4.1. In view of (4-2) we have

$$(\partial_t+\mathcal{L}_0)\frac{\partial u_0}{\partial x_k}=\frac{\partial}{\partial x_k}(e^{-\psi}f)$$

in $\mathbb{R}^{d+1}$. It follows that

$$\int_{\mathbb{R}^d} \partial_t \nabla u_0 \cdot (\nabla u_0) e^{2\psi}\,dx - \int_{\mathbb{R}^d} \hat{a}_{ij}^{\alpha\beta} \frac{\partial^3 u_0^\beta}{\partial x_i \partial x_j \partial x_k} \cdot \frac{\partial u_0^\alpha}{\partial x_k} e^{2\psi}\,dx = \int_{\mathbb{R}^d} \frac{\partial}{\partial x_k}(e^{-\psi} f^\alpha) \frac{\partial u_0^\alpha}{\partial x_k} e^{2\psi}\,dx.$$

Using integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi}\,dx + \int_{\mathbb{R}^d} \hat{a}_{ij}^{\alpha\beta} \frac{\partial^2 u_0^\beta}{\partial x_j \partial x_k} \cdot \frac{\partial^2 u_0^\alpha}{\partial x_i \partial x_k} e^{2\psi}\,dx \\ = -\int_{\mathbb{R}^d} f\cdot(\Delta u_0) e^{\psi}\,dx - \int_{\mathbb{R}^d} e^{-\psi} f^\alpha \frac{\partial u_0^\alpha}{\partial x_k}\frac{\partial e^{2\psi}}{\partial x_k}\,dx - \int_{\mathbb{R}^d} \hat{a}_{ij}^{\alpha\beta} \frac{\partial^2 u_0^\beta}{\partial x_j \partial x_k} \cdot \frac{\partial u_0^\alpha}{\partial x_k}\frac{\partial e^{2\psi}}{\partial x_i}\,dx.\end{aligned}$$

By the ellipticity of $\mathcal{L}_0$, this yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi}\,dx + \mu\int_{\mathbb{R}^d} |\nabla^2 u_0|^2 e^{2\psi}\,dx \\ &\le C\int_{\mathbb{R}^d} |f||\nabla^2 u_0| e^{\psi}\,dx + C\int_{\mathbb{R}^d} |f|^2\,dx + C\|\nabla\psi\|_\infty^2 \int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi}\,dx + C\|\nabla\psi\|_\infty \int_{\mathbb{R}^d} |\nabla^2 u_0||\nabla u_0| e^{2\psi}\,dx,\end{aligned}$$

where C depends only on d and μ. Using the Cauchy inequality, we may further deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi}\,dx + \frac{\mu}{2}\int_{\mathbb{R}^d} |\nabla^2 u_0|^2 e^{2\psi}\,dx \le C\int_{\mathbb{R}^d} |f|^2\,dx + C\|\nabla\psi\|_\infty^2 \int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi}\,dx.$$

We now integrate the inequality above in t over the interval (s_0, s_1). This leads to

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_0(x, s_1)|^2 e^{2\psi}\,dx + \frac{\mu}{2}\int_{s_0}^{s_1}\int_{\mathbb{R}^d} |\nabla^2 u_0|^2 e^{2\psi}\,dx\,dt \\ \le C\int_{s_0}^{s_1}\int_{\mathbb{R}^d} |f|^2\,dx\,dt + C\|\nabla\psi\|_\infty^2 \int_{s_0}^{s_1}\int_{\mathbb{R}^d} |\nabla u_0|^2 e^{2\psi}\,dx\,dt \\ \le Ce^{\kappa(s_1-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{s_1}\int_{\mathbb{R}^d} |f|^2\,dx\,dt,\end{aligned} \tag{4-7}$$

where we have used (4-4) for the last inequality. Estimate (4-3) follows readily from (4-7). □

5. Proof of Theorem 1.1

We start with some weighted estimates.

Lemma 5.1. *Suppose that*

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon) w_\varepsilon = \varepsilon\,\mathrm{div}(F_\varepsilon) & \text{in } \mathbb{R}^d \times (s_0, \infty), \\ w_\varepsilon = 0 & \text{on } \mathbb{R}^d \times \{t = s_0\}. \end{cases} \tag{5-1}$$

Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a bounded Lipschitz function. Then for any $t > s_0$

$$\int_{\mathbb{R}^d} |w_\varepsilon(x, t)|^2 e^{2\psi(x)}\,dx \le C\varepsilon^2 e^{\kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{t}\int_{\mathbb{R}^d} |F_\varepsilon(x, s)|^2 e^{2\psi(x)}\,dx\,ds, \tag{5-2}$$

where $\kappa > 0$ and $C > 0$ depends only on μ.

Proof. Let

$$I(t)=\int_{\mathbb{R}^d}|w_\varepsilon(x,t)|^2e^{2\psi(x)}\,dx. \tag{5-3}$$

Note that

$$\begin{aligned}
I'(t)&=2\int_{\mathbb{R}^d}\langle\partial_t w_\varepsilon, e^{2\psi}w_\varepsilon\rangle\,dx\\
&=-2\int_{\mathbb{R}^d}\langle\mathcal{L}_\varepsilon(w_\varepsilon), e^{2\psi}w_\varepsilon\rangle\,dx+2\varepsilon\int_{\mathbb{R}^d}\langle\operatorname{div}(F_\varepsilon), e^{2\psi}w_\varepsilon\rangle\,dx\\
&=-2\int_{\mathbb{R}^d}A^\varepsilon\nabla w_\varepsilon\cdot\nabla(e^{2\psi}w_\varepsilon)\,dx-2\varepsilon\int_{\mathbb{R}^d}F_\varepsilon\cdot\nabla(e^{2\psi}w_\varepsilon)\,dx\\
&=-2\int_{\mathbb{R}^d}A^\varepsilon\nabla w_\varepsilon\cdot(\nabla w_\varepsilon)e^{2\psi}\,dx-2\int_{\mathbb{R}^d}A^\varepsilon\nabla w_\varepsilon\cdot\nabla(e^{2\psi})w_\varepsilon\,dx\\
&\qquad\qquad-2\varepsilon\int_{\mathbb{R}^d}F_\varepsilon\cdot(\nabla w_\varepsilon)e^{2\psi}\,dx-2\varepsilon\int_{\mathbb{R}^d}F_\varepsilon\cdot\nabla(e^{2\psi})w_\varepsilon\,dx,
\end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the pairing in $H^{-1}(\mathbb{R}^d;\mathbb{R}^m)\times H^1(\mathbb{R}^d;\mathbb{R}^m)$. It follows that

$$\begin{aligned}
I'(t)\le-2\mu\int_{\mathbb{R}^d}|\nabla w_\varepsilon|^2e^{2\psi}\,dx+\kappa\|\nabla\psi\|_\infty\int_{\mathbb{R}^d}|\nabla w_\varepsilon||w_\varepsilon|e^{2\psi}\,dx\\
+2\varepsilon\int_{\mathbb{R}^d}|\nabla w_\varepsilon||F_\varepsilon|e^{2\psi}\,dx+4\varepsilon\|\nabla\psi\|_\infty\int_{\mathbb{R}^d}|w_\varepsilon||F_\varepsilon|e^{2\psi}\,dx,
\end{aligned}$$

where $\kappa>0$ depends only on μ. By the Cauchy inequality this implies

$$I'(t)\le\kappa\|\nabla\psi\|_\infty^2 I(t)+\kappa\varepsilon^2\int_{\mathbb{R}^d}|F_\varepsilon(x,t)|^2e^{2\psi}\,dx, \tag{5-4}$$

where $\kappa>0$ depends only on μ. Hence,

$$\frac{d}{dt}\{I(t)e^{-\kappa(t-s_0)\|\nabla\psi\|_\infty^2}\}\le C\varepsilon^2e^{-\kappa(t-s_0)\|\nabla\psi\|_\infty^2}\int_{\mathbb{R}^d}|F_\varepsilon(x,t)|^2e^{2\psi}\,dx.$$

Since $I(s_0)=0$, it follows that

$$\begin{aligned}
I(t)&\le C\varepsilon^2\int_{s_0}^t\int_{\mathbb{R}^d}e^{\kappa(t-s)\|\nabla\psi\|_\infty^2}|F_\varepsilon(x,s)|^2e^{2\psi}\,dx\,ds\\
&\le C\varepsilon^2e^{\kappa(t-s_0)\|\nabla\psi\|_\infty^2}\int_{s_0}^t\int_{\mathbb{R}^d}|F_\varepsilon(x,s)|^2e^{2\psi}\,dx\,ds.
\end{aligned}$$ □

Lemma 5.2. *Suppose that $u_\varepsilon\in L^2((-\infty,T);H^1(\mathbb{R}^d))$ and $u_0\in L^2((-\infty,T);H^2(\mathbb{R}^d))$ for any $T\in\mathbb{R}$, and that*

$$\begin{cases}(\partial_t+\mathcal{L}_\varepsilon)u_\varepsilon=(\partial_t+\mathcal{L}_0)u_0 & \text{in } \mathbb{R}^{d+1},\\ u_\varepsilon(x,t)=u_0(x,t)=0 & \text{for } t\le s_0.\end{cases}$$

Let w_ε be defined by (3-2), *where the operator S_ε is given by* (3-5). *Then for any $t > s_0$,*

$$\int_{\mathbb{R}^d} |w_\varepsilon(x,t)|^2 e^{2\psi(x)}\,dx \le C\varepsilon^2 e^{2\varepsilon\|\nabla\psi\|_\infty+\kappa(t-s_0)\|\nabla\psi\|_\infty^2} \int_{s_0}^{t+\varepsilon^2}\int_{\mathbb{R}^d} \{|\nabla^2 u_0(x,s)|^2 + |\partial_s u_0(x,s)|^2\} e^{2\psi(x)}\,dx\,ds, \quad (5\text{-}5)$$

where ψ is a bounded Lipschitz function in $\mathbb{R}^d$, κ depends only on μ, and C depends only on d, m and μ.

Proof. This follows readily from Lemma 5.1 and Theorem 3.5 with $p = 2$. ☐

The next theorem gives a weighted L^∞ estimate.

Theorem 5.3. *Assume that A is* 1*-periodic and satisfies* (1-2). *If $m \ge 2$, we also assume that $A \in \mathrm{VMO}_x$. Suppose that $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0$ in $B(x_0, 3r) \times (t_0 - 5r^2, t_0 + r^2)$ for some $(x_0, t_0) \in \mathbb{R}^{d+1}$ and $\varepsilon \le r < \infty$. Then*

$$\begin{aligned}\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))} &\le Ce^{3r\|\nabla\psi\|_\infty}\left(\fint_{Q_{2r}(x_0,t_0)} |e^\psi(u_\varepsilon - u_0)|^2\right)^{1/2}\\ &\quad + C\varepsilon r e^{3r\|\nabla\psi\|_\infty}\|e^\psi(|\nabla^2 u_0| + |\partial_t u_0|)\|_{L^\infty(B(x_0,3r)\times(t_0-5r^2,t_0+r^2))}\\ &\quad + C\varepsilon e^{3r\|\nabla\psi\|_\infty}\|e^\psi\nabla u_0\|_{L^\infty(B(x_0,3r)\times(t_0-5r^2,t_0+r^2))},\end{aligned} \quad (5\text{-}6)$$

where ψ is a Lipschitz function in $\mathbb{R}^d$ and C depends only on d, m, μ and $A^\#$ (if $m \ge 2$).

Proof. Let w_ε be defined by (3-2). Then $(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = \varepsilon\,\mathrm{div}(F_\varepsilon)$ in $Q_{2r}(x_0, t_0)$, where F_ε is given by (3-4). It follows by Theorem 2.5 that

$$\|w_\varepsilon\|_{L^\infty(Q_r(x_0,t_0))} \le C\left\{\left(\fint_{Q_{2r}(x_0,t_0)} |w_\varepsilon|^2\right)^{1/2} + \varepsilon r\left(\fint_{Q_{2r}(x_0,t_0)} |F_\varepsilon|^p\right)^{1/p}\right\}, \quad (5\text{-}7)$$

where $p > d + 2$. This leads to

$$\begin{aligned}\|u_\varepsilon - u_0\|_{L^\infty(Q_r(x_0,t_0))} &\le C\left(\fint_{Q_{2r}(x_0,t_0)} |u_\varepsilon - u_0|^2\right)^{1/2} + C\varepsilon r\left(\fint_{Q_{2r}(x_0,t_0)} |F_\varepsilon|^p\right)^{1/p}\\ &\quad + C\varepsilon\|S_\varepsilon(\nabla u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))} + C\varepsilon^2\|S_\varepsilon(\nabla^2 u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))},\end{aligned}$$

where we have used the boundedness of χ and ϕ in Lemmas 2.1 and 2.3. Hence, using $|\psi(x) - \psi(y)| \le 2r\|\nabla\psi\|_\infty$ for $x, y \in B(x_0, 2r)$, we obtain

$$\begin{aligned}&\|e^\psi(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))}\\ &\quad\le Ce^{2r\|\nabla\psi\|_\infty}\left(\fint_{Q_{2r}(x_0,t_0)} |e^\psi(u_\varepsilon - u_0)|^2\right)^{1/2} + C\varepsilon r e^{2r\|\nabla\psi\|_\infty}\left(\fint_{Q_{2r}(x_0,t_0)} |e^\psi F_\varepsilon|^p\right)^{1/p}\\ &\qquad + C\varepsilon e^{2r\|\nabla\psi\|_\infty}\|e^\psi S_\varepsilon(\nabla u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))} + C\varepsilon^2 e^{2r\|\nabla\psi\|_\infty}\|e^\psi S_\varepsilon(\nabla^2 u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))}.\end{aligned} \quad (5\text{-}8)$$

Finally, we use Theorem 3.5 to bound the second term in the right-hand side of (5-8). This yields

$$
\begin{aligned}
&\|e^{\psi}(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))}\\
&\le Ce^{2r\|\nabla\psi\|_\infty}\left(\fint_{Q_{2r}(x_0,t_0)} |e^{\psi}(u_\varepsilon - u_0)|^2\right)^{1/2} + C\varepsilon r e^{3r\|\nabla\psi\|_\infty}\left(\fint_{t_0-5r^2}^{t_0+r^2}\fint_{B(x_0,3r)} \{|e^{\psi}\nabla^2 u_0|^p + |e^{\psi}\partial_t u_0|^p\}\right)^{1/p}\\
&\qquad + C\varepsilon e^{2r\|\nabla\psi\|_\infty}\|e^{\psi} S_\varepsilon(\nabla u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))} + C\varepsilon^2 e^{2r\|\nabla\psi\|_\infty}\|e^{\psi} S_\varepsilon(\nabla^2 u_0)\|_{L^\infty(Q_{2r}(x_0,t_0))},
\end{aligned}
$$

where $p > d+2$ and we also used the assumption $\varepsilon \le r$. Estimate (5-6) now follows. □

We are now in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. We begin by fixing $x_0, y_0 \in \mathbb{R}^{d+1}$ and $s_0 < t_0$. We may assume that

$$\varepsilon < r = (t_0 - s_0)^{1/2}/100.$$

For otherwise the desired estimate (1-8) follows directly from (1-6).

For $f \in C_0^\infty(Q_r(y_0, s_0); \mathbb{R}^m)$, define

$$
\begin{aligned}
u_\varepsilon(x,t) &= \int_{-\infty}^{t}\int_{\mathbb{R}^d} e^{-\psi(y)}\Gamma_\varepsilon(x,t;y,s) f(y,s)\,dy\,ds,\\
u_0(x,t) &= \int_{-\infty}^{t}\int_{\mathbb{R}^d} e^{-\psi(y)}\Gamma_0(x,t;y,s) f(y,s)\,dy\,ds,
\end{aligned}
$$

where ψ is a bounded Lipschitz function in $\mathbb{R}^d$ to be chosen later. Then

$$(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 = e^{-\psi} f \quad \text{in } \mathbb{R}^{d+1}$$

and $u_\varepsilon(x,t) = u_0(x,t) = 0$ if $t \le s_0$. Let w_ε be defined by (3-2). It follows from Lemma 5.2 and Theorem 4.1 that

$$\int_{\mathbb{R}^d} |w_\varepsilon(x,t)|^2 e^{2\psi(x)}\,dx \le C\varepsilon^2 e^{2\varepsilon\|\nabla\psi\|_\infty + \kappa(t-s_0+\varepsilon^2)\|\nabla\psi\|_\infty^2}\int_{s_0}^{t+\varepsilon^2}\int_{\mathbb{R}^d} |f|^2\,dx\,ds \tag{5-9}$$

for any $t > s_0$.

Next, we use (5-6) to obtain

$$
\begin{aligned}
\|e^{\psi}(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))} \le Ce^{3r\|\nabla\psi\|_\infty}&\left(\fint_{Q_{2r}(x_0,t_0)} |e^{\psi} w_\varepsilon|^2\right)^{1/2}\\
&+ C\varepsilon r e^{3r\|\nabla\psi\|_\infty}\|e^{\psi}(|\nabla^2 u_0| + |\partial_t u_0|)\|_{L^\infty(B(x_0,3r)\times(t_0-5r^2, t_0+r^2))}\\
&\quad + C\varepsilon e^{3r\|\nabla\psi\|_\infty}\|e^{\psi}\nabla u_0\|_{L^\infty(B(x_0,3r)\times(t_0-5r^2,t_0+r^2))}.
\end{aligned} \tag{5-10}
$$

Since $\operatorname{supp}(f) \subset Q_r(y_0, s_0)$, it follows from the estimate (1-7) for $\Gamma_0(x,t;y,s)$ that

$$|\nabla^2 u_0(x,t)| + |\partial_t u_0(x,t)| + r^{-1}|\nabla u_0(x,t)| \le C\exp\left\{-\frac{\kappa|x_0 - y_0|^2}{t_0 - s_0}\right\}\fint_{Q_r(y_0,s_0)} |fe^{-\psi}|\,dy\,ds \tag{5-11}$$

for any $x \in B(x_0, 3r)$ and $|t - t_0| \le 5r^2$, where $\kappa > 0$ depends only on μ. Thus, by (5-10), we obtain

$$\begin{aligned}
&\|e^{\psi}(u_\varepsilon - u_0)\|_{L^\infty(Q_r(x_0,t_0))} \\
&\quad\le Ce^{3r\|\nabla\psi\|_\infty}\left(\fint_{Q_{2r}(x_0,t_0)} |e^{\psi} w_\varepsilon|^2\right)^{1/2} + \varepsilon r e^{c(|x_0-y_0|+r)\|\nabla\psi\|_\infty} \exp\left\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\right\} \fint_{Q_r(y_0,s_0)} |f|\,dy\,ds \\
&\quad\le C\varepsilon r e^{cr\|\nabla\psi\|_\infty}\left\{e^{cr^2\|\nabla\psi\|_\infty^2} + e^{c|x_0-y_0|\|\nabla\psi\|_\infty} \exp\left\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\right\}\right\} \cdot \left(\fint_{Q_r(y_0,s_0)} |f|^2\right)^{1/2}, \qquad (5\text{-}12)
\end{aligned}$$

where we have used (5-9) for the last step. By duality this implies

$$\begin{aligned}
&\left(\fint_{Q_r(y_0,s_0)} |e^{\psi(x)-\psi(y)}(\Gamma_\varepsilon(x,t;y,s) - \Gamma_0(x,t;y,s))|^2\,dy\,ds\right)^{1/2} \\
&\qquad\qquad \le C\varepsilon r^{-d-1} e^{cr\|\nabla\psi\|_\infty}\left\{e^{cr^2\|\nabla\psi\|_\infty^2} + e^{c|x_0-y_0|\|\nabla\psi\|_\infty} \exp\left\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\right\}\right\} \qquad (5\text{-}13)
\end{aligned}$$

for any $(x, t) \in Q_r(x_0, t_0)$.

To deduce the L^∞ bound for

$$e^{\psi(x)-\psi(y)}(\Gamma_\varepsilon(x,t;y,s) - \Gamma_0(x,t;y,s))$$

from its L^2 bound in (5-13), we apply Theorem 5.3 (with ψ replaced by $-\psi$ and A replaced by $\tilde{A} = \tilde{A}(y,s) = A^*(y,-s)$) to the functions

$$v_\varepsilon(y,s) = \Gamma_\varepsilon(x_0,t_0;y,-s) \quad \text{and} \quad v_0(y,s) = \Gamma_0(x_0,t_0;y,-s).$$

Note that $(\partial_t + \widetilde{\mathcal{L}}_\varepsilon)v_\varepsilon = (\partial_t + \widetilde{\mathcal{L}}_0)v_0 = 0$ in $B(y_0, 3r) \times (-s_0 - 5r^2, -s_0 + r^2)$. Since $\tilde{A}$ satisfies the same conditions as A, we obtain

$$\begin{aligned}
&|e^{\psi(x_0)-\psi(y_0)}(v_\varepsilon(y_0,-s_0) - v_0(y_0,-s_0))| \\
&\quad\le Ce^{3r\|\nabla\psi\|_\infty}\left(\fint_{Q_r(y_0,-s_0)} |e^{\psi(x_0)-\psi(y)}(v_\varepsilon - v_0)|^2\,dy\,ds\right)^{1/2} \\
&\qquad\qquad + C\varepsilon r^{-d-1} e^{cr\|\nabla\psi\|_\infty} e^{\psi(x_0)-\psi(y_0)} \exp\left\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\right\} \\
&\quad= Ce^{3r\|\nabla\psi\|_\infty}\left(\fint_{Q_r(y_0,s_0+r^2)} |e^{\psi(x_0)-\psi(y)}(\Gamma_\varepsilon(x_0,t_0;y,s) - \Gamma_0(x_0,t_0;y,s))|^2\,dy\,ds\right)^{1/2} \\
&\qquad\qquad + C\varepsilon r^{-d-1} e^{cr\|\nabla\psi\|_\infty} e^{\psi(x_0)-\psi(y_0)} \exp\left\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\right\} \\
&\quad\le C\varepsilon r^{-d-1} e^{cr\|\nabla\psi\|_\infty}\left\{e^{cr^2\|\nabla\psi\|_\infty^2} + e^{c|x_0-y_0|\|\nabla\psi\|_\infty} \exp\left\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\right\}\right\}, \qquad (5\text{-}14)
\end{aligned}$$

where we have used (5-13) for the last inequality.

Finally, as in [Hofmann and Kim 2004; Cho et al. 2008], we let $\psi(y) = \gamma\psi_0(|y - y_0|)$, where $\gamma \ge 0$ is to be chosen, $\psi_0(\rho) = \rho$ if $\rho \le |x_0 - y_0|$, and $\psi_0(\rho) = |x_0 - y_0|$ if $\rho > |x_0 - y_0|$. Note that $\|\nabla\psi\|_\infty = \gamma$

and $\psi(x_0)-\psi(y_0)=\gamma|x_0-y_0|$. It follows from (5-14) that

$$|\Gamma_\varepsilon(x_0,t_0;y_0,s_0)-\Gamma_0(x_0,t_0;y_0,s_0)| \le C\varepsilon r^{-d-1}e^{-\gamma|x_0-y_0|+c\gamma\sqrt{t_0-s_0}}\left\{e^{c\gamma^2(t_0-s_0)}+e^{c\gamma|x_0-y_0|}\exp\left\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\right\}\right\}, \quad (5\text{-}15)$$

where $c>0$ depends at most on μ. If $|x_0-y_0|\le 2c\sqrt{t_0-s_0}$, we may simply choose $\gamma=0$. This gives

$$|\Gamma_\varepsilon(x_0,t_0;y_0,s_0)-\Gamma_0(x_0,t_0;y_0,s_0)|\le C\varepsilon r^{-d-1}\le C\varepsilon(t_0-s_0)^{-(d+1)/2}\exp\left\{-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\right\}.$$

If $|x_0-y_0|>2c\sqrt{t_0-s_0}$, we choose

$$\gamma=\frac{\delta|x_0-y_0|}{t_0-s_0}.$$

Note that

$$\begin{aligned}-\gamma|x_0-y_0|+c\gamma\sqrt{t_0-s_0}+c\gamma^2(t_0-s_0)&=-\delta(1-c\delta)\frac{|x_0-y_0|^2}{t_0-s_0}+c\delta\frac{|x_0-y_0|}{\sqrt{t_0-s_0}}\\&\le\left\{-\delta(1-c\delta)+\tfrac{1}{2}\delta\right\}\frac{|x_0-y_0|^2}{t_0-s_0}\le\frac{-\delta|x_0-y_0|^2}{4(t_0-s_0)}\end{aligned}$$

if $\delta\le\frac{1}{4}c^{-1}$. Also, observe that

$$c\gamma\sqrt{t_0-s_0}+c\gamma|x_0-y_0|-\frac{\kappa|x_0-y_0|^2}{t_0-s_0}\le\left\{\tfrac{1}{2}\delta+c\delta-\kappa\right\}\frac{|x_0-y_0|^2}{t_0-s_0}\le-\frac{\kappa|x_0-y_0|^2}{2(t_0-s_0)},$$

if $\delta\le\frac{1}{2}\left(c+\frac{1}{2}\right)^{-1}\kappa$. Recall that $r=(100)^{-1}\sqrt{t_0-s_0}$. As a result, we have proved that there exists $\kappa_1>0$, depending only on μ, such that

$$|\Gamma_\varepsilon(x_0,t_0;y_0,s_0)-\Gamma_0(x_0,t_0;y_0,s_0)|\le\frac{C\varepsilon}{(t_0-s_0)^{(d+1)/2}}\exp\left\{-\frac{\kappa_1|x_0-y_0|^2}{t_0-s_0}\right\}.$$

This completes the proof of Theorem 1.1. □

6. Proof of Theorem 1.2

Define

$$\|F\|_{C^{\lambda,0}(K)}=\sup\left\{\frac{|F(x,t)-F(y,t)|}{|x-y|^\lambda}:(x,t),(y,t)\in K \text{ and } x\ne y\right\}.$$

The proof of Theorem 1.2 relies on the following Lipschitz estimate.

Theorem 6.1. *Assume that A satisfies conditions* (1-2), (1-3) *and* (1-9). *Suppose that*

$$(\partial_t+\mathcal{L}_\varepsilon)u_\varepsilon=(\partial_t+\mathcal{L}_0)u_0$$

in $Q_{2r}(x_0,t_0)$ for some $(x_0,t_0)\in\mathbb{R}^{d+1}$ and $\varepsilon\le r<\infty$. Then

$$\begin{aligned}\|\nabla u_\varepsilon-\nabla u_0-(\nabla\chi)^\varepsilon\nabla u_0\|_{L^\infty(Q_r(x_0,t_0))}\le Cr^{-1}\biggl(\fint_{Q_{2r}(x_0,t_0)}|u_\varepsilon-u_0|^2\biggr)^{1/2}+C\varepsilon r^{-1}\|\nabla u_0\|_{L^\infty(Q_{2r}(x_0,t_0))}\\ +C\varepsilon\ln[\varepsilon^{-1}r+2]\||\nabla^2u_0|+\varepsilon|\partial_t\nabla u_0|+\varepsilon|\nabla^3u_0|\|_{L^\infty(Q_{2r}(x_0,t_0))}\\ +C\varepsilon^{1+\lambda}\||\nabla^2u_0|+\varepsilon|\partial_t\nabla u_0|+\varepsilon|\nabla^3u_0|\|_{C^{\lambda,0}(Q_{2r}(x_0,t_0))},\end{aligned}\tag{6-1}$$

where C depends only on d, m, μ and (λ,τ) in (1-9).

Proof. Let

$$w_\varepsilon=u_\varepsilon-u_0-\varepsilon\chi_j^\varepsilon\frac{\partial u_0}{\partial x_j}-\varepsilon^2\phi^\varepsilon_{(d+1)ij}\frac{\partial^2u_0}{\partial x_i\partial x_j},\tag{6-2}$$

where $\chi_j^\varepsilon(x,t)=\chi_j(x/\varepsilon,t/\varepsilon^2)$ and $\phi^\varepsilon_{(d+1)ij}(x,t)=\phi_{(d+1)ij}(x/\varepsilon,t/\varepsilon^2)$. It follows by Proposition 3.1 that $(\partial_t+\mathcal{L}_\varepsilon)w_\varepsilon=\varepsilon\operatorname{div}(F_\varepsilon)$ in $Q_{2r}(x_0,t_0)$, where F_ε is given by (3-4) with S_ε being the identity operator. Choose a cut-off function $\varphi\in C_0^\infty(\mathbb{R}^{d+1})$ such that

$$\begin{gathered}0\le\varphi\le1,\qquad\varphi=1\quad\text{in }Q_{3r/2}(x_0,t_0),\\ \varphi(x,t)=0\quad\text{if }|x-x_0|\ge\tfrac{7}{4}r\text{ or }t<t_0-\bigl(\tfrac{7}{4}r\bigr)^2,\\ |\nabla\varphi|\le Cr^{-1},\quad|\nabla^2\varphi|+|\partial_t\varphi|\le Cr^{-2}.\end{gathered}$$

Using

$$(\partial_t+\mathcal{L}_\varepsilon)(\varphi w_\varepsilon)=(\partial_t\varphi)w_\varepsilon+\varepsilon\operatorname{div}(\varphi F_\varepsilon)-\varepsilon F_\varepsilon(\nabla\varphi)-\operatorname{div}(A^\varepsilon(\nabla\varphi)w_\varepsilon)-A^\varepsilon\nabla w_\varepsilon(\nabla\varphi),$$

where $A^\varepsilon(x,t)=A(x/\varepsilon,t/\varepsilon^2)$, we may deduce that for any $(x,t)\in Q_r(x_0,t_0)$,

$$\begin{aligned}w_\varepsilon(x,t)&=\int_{-\infty}^t\int_{\mathbb{R}^d}\Gamma_\varepsilon(x,t;y,s)\{(\partial_s\varphi)w_\varepsilon-\varepsilon F_\varepsilon(\nabla\varphi)-A^\varepsilon\nabla w_\varepsilon(\nabla\varphi)\}\,dy\,ds\\ &\qquad-\int_{-\infty}^t\int_{\mathbb{R}^d}\nabla_y\Gamma_\varepsilon(x,t;y,s)\{\varepsilon\varphi F_\varepsilon-A^\varepsilon(\nabla\varphi)w_\varepsilon\}\,dy\,ds\\ &=I(x,t)+J(x,t),\end{aligned}$$

where

$$J(x,t)=-\varepsilon\int_{-\infty}^t\int_{\mathbb{R}^d}\nabla_y\Gamma_\varepsilon(x,t;y,s)\varphi(y,s)F_\varepsilon(y,s)\,dy\,ds.$$

Since $\varphi=1$ in $Q_{3r/2}(x_0,t_0)$, we see that for $(x,t)\in Q_r(x_0,t_0)$,

$$\begin{aligned}|\nabla I(x,t)|&\le C\int_{-\infty}^t\int_{\mathbb{R}^d}|\nabla_x\Gamma_\varepsilon(x,t;y,s)|\{|\partial_s\varphi||w_\varepsilon|+\varepsilon|F_\varepsilon||\nabla\varphi|+|\nabla w_\varepsilon||\nabla\varphi|\}\,dy\,ds\\ &\qquad+C\int_{-\infty}^t\int_{\mathbb{R}^d}|\nabla_x\nabla_y\Gamma_\varepsilon(x,t;y,s)||\nabla\varphi||w_\varepsilon|\,dy\,ds\\ &\le C\biggl\{\frac{1}{r}\fint_{Q_{2r}(x_0,t_0)}|w_\varepsilon|+\varepsilon\fint_{Q_{2r}(x_0,t_0)}|F_\varepsilon|+\fint_{Q_{7r/4}(x_0,t_0)}|\nabla w_\varepsilon|\biggr\}\\ &\le C\biggl\{\frac{1}{r}\biggl(\fint_{Q_{2r}(x_0,t_0)}|w_\varepsilon|^2\biggr)^{1/2}+\varepsilon\biggl(\fint_{Q_{2r}(x_0,t_0)}|F_\varepsilon|^2\biggr)^{1/2}\biggr\},\end{aligned}$$

where we have used (parabolic) Caccioppoli's inequality for the last step. In view of (3-4) with S_ε being the identity operator,

$$|F_\varepsilon| \le C\{|\nabla^2 u_0| + \varepsilon|\partial_t \nabla u_0| + \varepsilon|\nabla^3 u_0|\},$$

where we have used the boundedness of $\nabla\phi$ (see Remark 2.4). It follows that $\|\nabla I\|_{L^\infty(Q_r(x_0,t_0))}$ is bounded by the right-hand side of (6-1).

Finally, to estimate $J(x,t)$, we write

$$J(x,t) = -\varepsilon \int_{-\infty}^{t} \int_{\mathbb{R}^d} \nabla_y \{\Gamma_\varepsilon(x,t;y,s)\varphi(y,s)\}(F_\varepsilon(y,s) - F_\varepsilon(x,s))\,dy\,ds + \varepsilon \int_{-\infty}^{t} \int_{\mathbb{R}^d} \Gamma_\varepsilon(x,t;y,s)(\nabla\varphi)(y,s)F_\varepsilon(y,s)\,dy\,ds.$$

It follows that for $(x,t) \in Q_r(x_0,t_0)$

$$\begin{aligned} |\nabla J(x,t)| &\le \varepsilon \int_{Q_{2r}(x_0,t_0)} |\nabla_x \nabla_y \{\Gamma_\varepsilon(x,t;y,s)\varphi(y,s)\}|\,|F_\varepsilon(y,s) - F_\varepsilon(x,s)|\,dy\,ds \\ &\qquad + \varepsilon \int_{Q_{2r}(x_0,t_0)} |\nabla_x \Gamma_\varepsilon(x,t;y,s)|\,|\nabla\varphi(y,s)|\,|F_\varepsilon(y,s)|\,dy\,ds \\ &\le C\varepsilon \int_{Q_{2r}(x_0,t_0)} \frac{|F_\varepsilon(y,s) - F_\varepsilon(x,s)|}{(|x-y| + |t-s|^{1/2})^{d+2}}\,dy\,ds + C\varepsilon \fint_{Q_{2r}(x_0,t_0)} |F_\varepsilon|. \end{aligned} \tag{6-3}$$

To bound the first integral in the right-hand side of (6-3), we subdivide the domain $Q_{2r}(x_0,t_0)$ into $Q_\varepsilon(x,t)$ and $Q_{2r}(x_0,t_0) \setminus Q_\varepsilon(x,t)$. On $Q_{2r}(x_0,t_0) \setminus Q_\varepsilon(x,t)$, we use the bound

$$|F_\varepsilon(y,s) - F_\varepsilon(x,s)| \le 2\|F_\varepsilon\|_{L^\infty(Q_{2r}(x_0,t_0))},$$

while for $Q_\varepsilon(x,t)$, we use

$$|F_\varepsilon(y,s) - F_\varepsilon(x,s)| \le |x-y|^\lambda \|F\|_{C^{\lambda,0}(Q_{2r}(x_0,t_0))}.$$

This leads to

$$\begin{aligned} |\nabla J(x,t)| &\le C\varepsilon \ln[\varepsilon^{-1} r + 1]\|F_\varepsilon\|_{L^\infty(Q_{2r}(x_0,t_0))} + C\varepsilon^{1+\lambda}\|F_\varepsilon\|_{C^{\lambda,0}(Q_{2r}(x_0,t_0))} \\ &\le C\varepsilon \ln[\varepsilon^{-1} r + 1]\||\nabla^2 u_0| + \varepsilon|\partial_t \nabla u_0| + \varepsilon|\nabla^3 u_0|\|_{L^\infty(Q_{2r}(x_0,t_0))} \\ &\qquad + C\varepsilon^{1+\lambda}\||\nabla^2 u_0| + \varepsilon|\partial_t \nabla u_0| + \varepsilon|\nabla^3 u_0|\|_{C^{\lambda,0}(Q_{2r}(x_0,t_0))}. \end{aligned}$$

Thus, in view of the estimate for $\nabla I(x,t)$, we have proved that $\|\nabla w_\varepsilon\|_{L^\infty(Q_r(x_0,t_0))}$ is bounded by the right-hand side of (6-1). Since

$$\|\nabla w_\varepsilon - \{\nabla u_\varepsilon - \nabla u_0 - (\nabla\chi)^\varepsilon \nabla u_0\}\|_{L^\infty(Q_r(x_0,t_0))} \le C\varepsilon\||\nabla^2 u_0| + \varepsilon|\nabla^3 u_0|\|_{L^\infty(Q_r(x_0,t_0))},$$

the estimate (6-1) follows. □

To prove Theorem 1.2, we fix $x_0, y_0 \in \mathbb{R}^d$ and $s_0 < t_0$. We may assume that $\varepsilon < (t_0 - s_0)/8$. For otherwise the estimate (1-10) follows directly from (2-22). We apply Theorem 6.1 to the functions

$$u_\varepsilon(x,t) = \Gamma_\varepsilon(x,t;y_0,s_0) \quad \text{and} \quad u_0(x,t) = \Gamma_0(x,t;y_0,s_0)$$

in $Q_{2r}(x_0, t_0)$, where $r = (t_0 - s_0)/8$. Note that $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 = 0$ in $Q_{4r}(x_0, t_0)$. To bound the first term in the right-hand side of (6-1), we use the estimate (1-8) in Theorem 1.1. All other terms in the right-hand side of (6-1) may be handled easily by using the estimates (1-7) for $\Gamma_0(x, t; y, s)$. We leave the details to the reader.

7. Proof of Theorem 1.3

To prove Theorem 1.3, we fix $x_0, y_0 \in \mathbb{R}^d$ and $s_0 < t_0$. As before, we may assume that $\varepsilon < (t_0 - s_0)/8$, for otherwise the estimate (1-15) follows directly from (2-23).

Let $r = (t_0 - s_0)/8$. Fix $1 \le j \le d$ and $1 \le \beta \le m$. We apply Theorem 6.1 to the functions $u_\varepsilon = (u_\varepsilon^\alpha)$ and $u_0 = (u_0^\alpha)$ in $Q_{2r}(x_0, t_0)$, where

$$
\begin{aligned}
u_\varepsilon^\alpha(x, t) &= \frac{\partial}{\partial y_j}\{\Gamma_\varepsilon^{\alpha\beta}\}(x, t; y_0, s_0),\\
u_0^\alpha(x, t) &= \frac{\partial}{\partial y_\ell}\{\Gamma_0^{\alpha\sigma}\}(x, t; y_0, s_0) \cdot \Big\{\delta^{\beta\sigma}\delta_{j\ell} + \frac{\partial}{\partial y_j}(\tilde{\chi}_\ell^{\beta\sigma})(y_0/\varepsilon, -s_0/\varepsilon^2)\Big\},
\end{aligned}
$$

where $\tilde{\chi}$ denotes the correctors for $\partial_t + \widetilde{\mathcal{L}}_\varepsilon$. Observe that $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = (\partial_t + \mathcal{L}_0)u_0 = 0$ in $Q_{4r}(x_0, t_0)$. To bound the first term in the right-hand side of (6-1), we use the estimate (1-14). As in the proof of Theorem 1.1, all other terms in the right-hand side of (6-1) may be handled readily by using estimate (1-7) for $\Gamma_0(x, t; y, s)$.

References

[Aronson 1967] D. G. Aronson, "Bounds for the fundamental solution of a parabolic equation", *Bull. Amer. Math. Soc.* **73** (1967), 890–896. MR Zbl

[Auscher 1996] P. Auscher, "Regularity theorems and heat kernel for elliptic operators", *J. London Math. Soc.* (2) **54**:2 (1996), 284–296. MR Zbl

[Avellaneda and Lin 1991] M. Avellaneda and F.-H. Lin, "L^p bounds on singular integrals in homogenization", *Comm. Pure Appl. Math.* **44**:8-9 (1991), 897–910. MR Zbl

[Bensoussan et al. 1978] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, Studies in Math. Appl. **5**, North-Holland, Amsterdam, 1978. MR Zbl

[Byun 2007] S.-S. Byun, "Optimal $W^{1,p}$ regularity theory for parabolic equations in divergence form", *J. Evol. Equ.* **7**:3 (2007), 415–428. MR Zbl

[Cho et al. 2008] S. Cho, H. Dong, and S. Kim, "On the Green's matrices of strongly parabolic systems of second order", *Indiana Univ. Math. J.* **57**:4 (2008), 1633–1677. MR Zbl

[Davies 1987a] E. B. Davies, "Explicit constants for Gaussian upper bounds on heat kernels", *Amer. J. Math.* **109**:2 (1987), 319–333. MR Zbl

[Davies 1987b] E. B. Davies, "Heat kernel bounds for second order elliptic operators on Riemannian manifolds", *Amer. J. Math.* **109**:3 (1987), 545–569. MR Zbl

[Fabes and Stroock 1986] E. B. Fabes and D. W. Stroock, "A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash", *Arch. Rational Mech. Anal.* **96**:4 (1986), 327–338. MR Zbl

[Geng and Shen 2015] J. Geng and Z. Shen, "Uniform regularity estimates in parabolic homogenization", *Indiana Univ. Math. J.* **64**:3 (2015), 697–733. MR Zbl

[Geng and Shen 2017] J. Geng and Z. Shen, "Convergence rates in parabolic homogenization with time-dependent periodic coefficients", *J. Funct. Anal.* **272**:5 (2017), 2092–2113. MR Zbl

[Hofmann and Kim 2004] S. Hofmann and S. Kim, "Gaussian estimates for fundamental solutions to certain parabolic systems", *Publ. Mat.* **48**:2 (2004), 481–496. MR Zbl

[Jikov et al. 1994] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer, 1994. MR Zbl

[Kenig and Shen 2011] C. E. Kenig and Z. Shen, "Layer potential methods for elliptic homogenization problems", *Comm. Pure Appl. Math.* **64**:1 (2011), 1–44. MR Zbl

[Kenig et al. 2014] C. E. Kenig, F. Lin, and Z. Shen, "Periodic homogenization of Green and Neumann functions", *Comm. Pure Appl. Math.* **67**:8 (2014), 1219–1262. MR Zbl

[Krylov 2007] N. V. Krylov, "Parabolic and elliptic equations with VMO coefficients", *Comm. Partial Differential Equations* **32**:1-3 (2007), 453–475. MR Zbl

[Nash 1958] J. Nash, "Continuity of solutions of parabolic and elliptic equations", *Amer. J. Math.* **80** (1958), 931–954. MR Zbl

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