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A SPIRAL INTERFACE WITH POSITIVE ALT–CAFFARELLI–FRIEDMAN LIMIT AT THE ORIGIN

MARK ALLEN AND DENNIS KRIVENTSOV

We give an example of a pair of nonnegative subharmonic functions with disjoint support for which the Alt–Caffarelli–Friedman monotonicity formula has strictly positive limit at the origin, and yet the interface between their supports lacks a (unique) tangent there. This clarifies a remark of Caffarelli and Salsa (*A geometric approach to free boundary problems*, 2005) that the positivity of the limit of the ACF formula implies unique tangents; this is true under some additional assumptions, but false in general. In our example, blow-ups converge to the expected piecewise linear two-plane function along subsequences, but the limiting function depends on the subsequence due to the spiraling nature of the interface.

1. Introduction

The Alt–Caffarelli–Friedman monotonicity formula (hereafter denoted ACF formula) has been and continues to be a powerful tool in the study of free boundary problems. It was introduced in [Alt et al. 1984] in order to prove that the solutions to a two-phase Bernoulli free boundary problem are Lipschitz continuous. The formula was then adapted to treat more general two-phase problems, and a discussion of the formula, its proof, and its applications to two-phase free boundary problems may be found in [Caffarelli and Salsa 2005]. The ACF formula has also been effective in studying obstacle-type problems, and applications of the formula for obstacle-type problems are found in [Petrosyan et al. 2012]. Further applications also include the study of segregation problems in [Caffarelli et al. 2009]. While the most typical use of the formula is to prove the optimal regularity of solutions or flatness of the free boundary, it can also be used for other purposes, such as to show the separation of phases in free boundary problems; see [Allen and Petrosyan 2012; Allen et al. 2015; Allen and Shi 2016].

The key property of the ACF formula (1-1) is given in the following proposition:

Proposition 1.1. *Let $u_1, u_2 \geq 0$ be two continuous subharmonic functions in B_R with $u_1 \cdot u_2 = 0$ and $u_1(0) = u_2(0) = 0$. Then*

$$\Phi(r, u_1, u_2) := \frac{1}{r^4} \int_{B_r(0)} \frac{|\nabla u_1|^2}{|x|^{n-2}} \int_{B_r(0)} \frac{|\nabla u_2|^2}{|x|^{n-2}} \tag{1-1}$$

is nondecreasing for $0 < r < R$. Consequently, the limit

$$\Phi(0+, u_1, u_2) := \lim_{r \searrow 0} \Phi(r, u_1, u_2)$$

is well defined.

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Our paper is motivated by the following claim, which appears as Lemma 12.9 in [Caffarelli and Salsa 2005]:

Claim 1.2. *Let $u \geq 0$ be continuous in B_1 and harmonic in $\{u > 0\}$. Let Ω_1 be a connected component of $\{u > 0\}$ and let $0 \in \partial\Omega_1$. If $u_1 = u|_{\Omega_1}$ and $u_2 = u - u_1$, then if $\Phi(0+, u_1, u_2) > 0$, exactly two connected components Ω_1 and Ω_2 of $\{u > 0\}$ are tangent at 0, and in a suitable system of coordinates,*

$$u(x) = \alpha x_1^+ + \beta x_1^- + o(|x|), \tag{1-2}$$

with $\alpha, \beta > 0$.

As no proof of this Lemma 12.9 is provided in [Caffarelli and Salsa 2005] (it is followed only by some general remarks), it is not entirely clear whether it is meant to be taken at face value. We note, for example, that if u is also assumed to satisfy a two-phase free boundary problem of the type treated in [Caffarelli and Salsa 2005], then the claim is valid, but requires heavy use of the free boundary relation to prove.

Claim 1.2, and in particular the question of whether it is true in the generality stated, drew the authors' interest when the second author was tempted to use it while working on certain eigenvalue optimization problems [Kriventsov and Lin 2019] but was unable to write down a proof. Typically, a monotonicity formula is applied together with other tools making explicit use of the free boundary relation in order to prove regularity of an interface; however, Claim 1.2 would imply that the ACF monotonicity formula, on its own, yields some regularity of the interface. This makes the claim very powerful and useful, especially in problems where the free boundary condition is difficult to exploit, such as the vector-valued free boundary problems arising from spectral optimization [Kriventsov and Lin 2018; 2019].

Unfortunately, it is also not true: the main result of this paper is to provide a counterexample to Claim 1.2.

Theorem 1.3. *For any dimension $n \geq 2$, there exist two continuous subharmonic functions $u, \tilde{u} \geq 0$ with $u, \tilde{u}$ both harmonic in their respective positivity sets and $u \cdot \tilde{u} = 0$. Furthermore, $\Phi(0+, u, \tilde{u}) > 0$. However, $\partial\{u > 0\}$ and $\partial\{\tilde{u} > 0\}$ (which are given by a piecewise smooth, connected hypersurface when restricted to any annulus $B_1 \backslash B_r$) do not admit tangents (or approximate tangents) at the origin, nor do there exist numbers $\alpha, \beta > 0$ and a change of coordinates such that $u + \tilde{u} = \alpha x_1^+ + \beta x_1^- + o(|x|)$.*

In the above, the boundary of a measurable set A is said to *admit a tangent* (*plane*) at the origin if

$$0 < \liminf_{r \searrow 0} \frac{|B_r \cap A|}{|B_r|} \leq \limsup_{r \searrow 0} \frac{|B_r \cap A|}{|B_r|} < 1 \tag{1-3}$$

and there is a unit vector ν such that

$$\lim_{r \searrow 0} \frac{1}{r} \max_{x \in \partial A \cap B_r} |x \cdot \nu| = 0.$$

The boundary is said to admit an *approximate tangent* (*plane*) if (1-3) holds and

$$\lim_{r \searrow 0} \frac{1}{r^{n+1}} \int_{B_r \cap \partial A} |x \cdot \nu|^2 \, d\mathcal{H}^{n-1} = 0.$$

Here $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure. Note that if $u, \tilde{u}$ are as in Claim 1.2 and A is either $\{u > 0\}$ or $\{\tilde{u} > 0\}$, then (1-3) holds; see Corollary 12.4 in [Caffarelli and Salsa 2005].

It seems that the notion of approximate tangent above (or another similar measure-theoretic notion) is the more meaningful one in this context. Indeed, there are simpler constructions which produce functions $u, \tilde{u}$ as in Claim 1.2 for which $\partial\{u>0\}$ does not admit a tangent at 0 but does admit an approximate tangent.

If one only considers functions u for which $\partial\{u>0\}$ is, say, given by a 1-Lipschitz graph over some plane π_r on every annulus $B_{2r}\setminus B_r$, these two notions of tangent plane are equivalent. This property holds for the example constructed in the proof of Theorem 1.3.

The functions $u, \tilde{u}$ we construct in proving the theorem have $\partial\{u>0\}$ a spiral: while $u+\tilde{u}$ looks more and more like $\alpha(x\cdot\nu)_+ + \beta(x\cdot\nu)_-$ on progressively smaller balls B_r, the choice of ν cannot be made uniformly in r, and the optimal ν rotates (slowly) as r decreases. Some free boundary problems are known to exhibit spiraling patterns for the interface; see [Blank 2001; Terracini et al. 2019] for examples, although the spirals produced there have different properties from ours. We also remark that an example of nonunique tangents for an energy minimization problem is given in [White 1992].

Further questions. Before turning to the proof of Theorem 1.3 we would like to offer some discussion of the further questions raised by this theorem and speculate on what "optimal" results, both positive and negative, might look like.

A standard argument with the ACF formula shows that if $u, \tilde{u}$ are as in Claim 1.2, then for every sequence $r_k \to 0$, there is a subsequence r_{k_j} such that

$$\lim_{j\to\infty} \frac{1}{r_{k_j}^{n+2}} \int_{B_{r_{k_j}}} |u(x) - \alpha(x\cdot\nu)_+ - \beta(x\cdot\nu)_-|^2 = 0,$$

where α, β, ν depend on the subsequence. Let us refer to any such subsequence r_{k_j} as a *blow-up subsequence*. We are interested in whether or not these parameters may be chosen independent of the blow-up subsequence.

In the example constructed below, the functions u and $\tilde{u}$ are rotations of one another around the origin; in particular, this means that for all of the blow-up subsequences, $\alpha = \beta = c\sqrt{\Phi(0+, u, \tilde{u})}$ are the same, while ν depends on the particular subsequence.

This example gives one way for (1-2) to fail. There could, in principle, be another way: say that $\partial\{u>0\} = \partial\{\tilde{u}>0\}$ is given by a C^1 hypersurface (including up to the origin, so that it admits a tangent there), and that $u, \tilde{u}$ are as in Claim 1.2. Can one find a pair $u, \tilde{u}$ like this for which (1-2) fails? This would mean that between the various blow-up subsequences, ν would remain fixed, while α and β would vary. Note that if the hypersurface is more regular near the origin (in particular, if it is a Lyapunov–Dini surface), then this is impossible.

Another set of questions is related to optimality in Theorem 1.3. To clarify the discussion, define, for each r, $\nu(r)$ to be the best approximating normal vector:

$$\int_{B_r\cap\partial\{u>0\}} |x\cdot\nu(r)|^2 \, d\mathcal{H}^{n-1} = \min_{\nu\in S^{n-1}} \int_{B_r\cap\partial\{u>0\}} |x\cdot\nu|^2 \, d\mathcal{H}^{n-1}.$$

It may be verified that $\nu(r)$ is uniquely determined from this relation and depends in a Lipschitz manner on r. The property of having an approximate tangent, then, can be reformulated as saying that $\nu(r)$ has a limit as $r \to 0$, while Theorem 1.3 gives an example where

$$\int_0^1 \left| \frac{d\nu(r)}{dr} \right| = \infty. \tag{1-4}$$

What restrictions on the change in $\nu(r)$, one may ask then, are implied by the conditions in Claim 1.2? We conjecture that under those conditions, one must have

$$\int_0^1 r \left| \frac{d\nu(r)}{dr} \right|^2 < \infty; \tag{1-5}$$

on the other hand, for any $\nu_0(r)$ satisfying (1-4) and (1-5), there is a pair of functions u, $\tilde{u}$ as in Claim 1.2 with $\nu_0(r)$ with

$$\left| \frac{d\nu(r)}{dr} \right| \geq \left| \frac{d\nu_0(r)}{dr} \right|.$$

To explain the source of (1-5), let us point out that in Section 2, we construct a pair of functions u, $\tilde{u}$ for which

$$\int_0^\infty \left| \frac{d\nu(r)}{dr} \right| = \theta \quad \text{and} \quad \frac{\Phi(0+, u, \tilde{u})}{\Phi(\infty, u, \tilde{u})} \geq 1 - \theta^2$$

(and this dependence on θ seems to be sharp up to constants). By gluing truncated and scaled versions of this construction, one might hope to attain functions u, $\tilde{u}$ satisfying the hypotheses of Claim 1.2, and with

$$\int_{2^j}^{2^{j+1}} \left| \frac{d\nu(r)}{dr} \right| \approx \theta_i$$

for any sequence θ_i for which $\prod_i (1-\theta_i^2) > 0$. This restriction is equivalent to (1-5) for such a construction. In the actual proof of Theorem 1.3, we are unable to perform the truncation and gluing steps uniformly in θ, and so do not obtain such a quantitative result.

Finally, over the past two decades enormous progress has been made in understanding the relationship between the behavior of positive harmonic functions with zero Dirichlet condition near the boundaries of domains and the geometric measure-theoretic properties of the boundary; we do not attempt to provide a summary here, but refer the reader to the introduction and references in [Azzam et al. 2016]. We suggest that the questions above can be thought of as a continuation, or extension, of this program, with the goal of relating (finer) geometric properties of a boundary to the simultaneous behavior of positive harmonic functions on a domain and its compliment, using the ACF formula as a crucial tool.

Outline of proof. To prove Theorem 1.3 we will construct a subharmonic function $u \geq 0$ in $\mathbb{R}^2$ such that u is harmonic in its positivity set and $u(0) = 0$. Furthermore, $\partial\{u > 0\}$ will be invariant under a rotation of π. Consequently, if $\tilde{u}(z) := u(-z)$, then the pair u, $\tilde{u}$ will satisfy the assumptions of the ACF formula in Proposition 1.1. Before explaining the construction of u and the outline of the paper, we first give two definitions.

We define the class of functions in $\mathbb{R}^2$

$$\mathcal{K} := \big\{u \in C(B_1) : u \geq 0 \text{ in } B_1,\ \Delta u = 0 \text{ in } \{u>0\},\\ u(0)=0,\ u(z)\cdot u(-z)=0,\ \text{and } \partial\{u(z)>0\} = \partial\{u(-z)>0\}\big\}.$$

By working in the class $\mathcal{K}$, we may consider using a one-sided rescaled version of the ACF formula. If $u \in \mathcal{K}$, then

$$J(r,u) := \left(\frac{2}{\pi r^2}\int_{B_r} |\nabla u|^2\right)^{1/2}$$

is monotonically nondecreasing in r since $J(r,u) = \left(\frac{2}{\pi}\right)^2\sqrt{\Phi(r,u(z),u(-z))}$. Furthermore, if u is C^1 up to $\partial\{u>0\}$ near the origin, then $J(0+,u) = |\nabla u(0)|$.

In order to prove Theorem 1.3 we first show in Section 2, working on unbounded domains, that it is possible to turn $\partial\{u>0\}$ so that its asymptotic behavior at infinity differs from its tangent near the origin by an angle of θ while arranging that $J(\infty,u) - J(0+,u) < 1-\theta^2$ (for small θ). In Section 3 we transfer this result to a bounded domain. In Section 4 we inductively construct a sequence of functions in $\mathcal{K}$ and take a limit to obtain the u in Theorem 1.3. Heuristically, the value of $J(0+,u)$ should be $\prod(1-\theta_i^2)$, and this is strictly positive if, say, $\theta_i = i^{-1}$. On successively smaller balls, the interface $\{u=0\}$ will have turned a total amount of $\sum i^{-1} \to \infty$, which implies that the interface spirals towards the origin and therefore lacks a unique tangent there. We make these heuristic ideas rigorous, and then we show how the pair u, $\tilde{u}$ also provide a counterexample in higher dimensions.

2. Conformal mapping

We utilize the Schwarz–Christoffel formula to obtain a conformal mapping. For a fixed angle $0 < \theta < \frac{\pi}{2}$, we map the upper half-plane to the domain Ω_θ (see Figure 1) by the conformal mapping f_θ with derivative

$$f_\theta'(z) = (z-(-1))^{(\pi+\theta)/\pi-1}(z-1)^{(\pi-\theta)/\pi-1} = \left(\frac{z+1}{z-1}\right)^{\theta/\pi}. \tag{2-1}$$

We translate f_θ by a constant z_0, so that the midpoint of the line segment in the image is the origin $0+0i$. We define $t_\theta \in (-1,1) \subset \mathbb{R}$ to be $t_\theta = f_\theta^{-1}(0+0i)$. Clearly, $t_\theta \to 0$ as $\theta \to 0$. What is of importance is how quickly $t_\theta \to 0$. In order to determine this decay rate we use the following result:

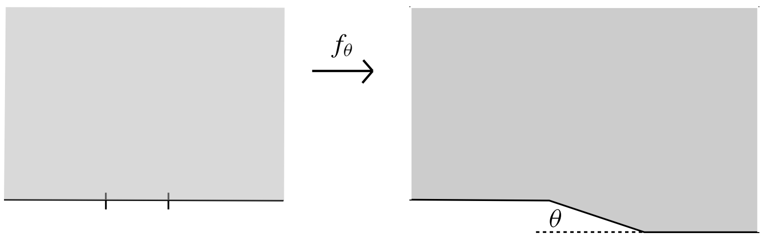

Figure 1. Conformal map.

Lemma 2.1. *Let $f, g > 0$ be integrable functions on an interval I. If f/g is an increasing function, then for any $x_1 < x_2 < x_3 < x_4$ with each $x_i \in I$, we have*

$$\frac{\int_{x_1}^{x_2} f}{\int_{x_1}^{x_2} g} \le \frac{\int_{x_3}^{x_4} f}{\int_{x_3}^{x_4} g}.$$

Proof. Since f/g is increasing we have

$$\int_{x_1}^{x_2} f(x)\,dx \le \int_{x_1}^{x_2} \frac{f(x_2)}{g(x_2)} g(x)\,dx.$$

Consequently, we have

$$\frac{\int_{x_1}^{x_2} f(x)\,dx}{\int_{x_1}^{x_2} g(x)\,dx} \le \frac{f(x_2)}{g(x_2)}.$$

By the same argument, we have

$$\frac{f(x_3)}{g(x_3)} \le \frac{\int_{x_3}^{x_4} f(x)\,dx}{\int_{x_3}^{x_4} g(x)\,dx},$$

and so the conclusion follows. □

We will also need the following:

Lemma 2.2. *Let $f \ge g > 0$ be integrable and continuous on $[0, 1)$ with $f \ge g$ and f/g increasing, and*

$$\int_0^1 f > M \quad \textit{and} \quad \int_0^1 g > M.$$

Let x_1, x_2 be the unique values such that

$$M + \int_0^{x_1} g = \int_{x_1}^1 g \quad \textit{and} \quad M + \int_0^{x_2} f = \int_{x_2}^1 f. \tag{2-2}$$

Then $x_1 \le x_2$.

Proof. We have

$$\frac{M + \int_0^{x_1} f}{M + \int_0^{x_1} g} \le \frac{\int_0^{x_1} f}{\int_0^{x_1} g} \le \frac{\int_{x_1}^1 f}{\int_{x_1}^1 g},$$

where the second inequality is due to Lemma 2.1. Since x_1 is chosen so that (2-2) holds, we have that the denominator in the inequality above is the same so that

$$M + \int_0^{x_1} f \le \int_{x_1}^1 f.$$

Then $x_1 \le x_2$. □

The two lemmas above allow us to prove:

Lemma 2.3. *Let f_θ be defined as in (2-1) and let $t_\theta = f_\theta^{-1}(0 + 0i)$. Then there exists $\theta_0 > 0$ such that $0 < t_\theta \le 2\theta/\pi$ as long as $0 < \theta \le \theta_0$.*

Proof. To determine the midpoint of a line segment it suffices to find the x-value. Consequently, we focus on the real part of the mapping f_θ. If $t \in (-1, 1)$, then

$$f'(t) = \left((-1)\frac{1+t}{1-t}\right)^{\theta/\pi} = \left(\frac{1+t}{1-t}\right)^{\theta/\pi} e^{i\theta}.$$

Thus, t_θ is the unique value in $(-1, 1)$ such that

$$\int_{-1}^{t_\theta} \left(\frac{1+t}{1-t}\right)^{\theta/\pi} dt = \int_{t_\theta}^{1} \left(\frac{1+t}{1-t}\right)^{\theta/\pi} dt.$$

We now note that

$$\left(\frac{1+t}{1-t}\right)^{\theta/\pi} \geq \left(\frac{1+t}{2}\right)^{\theta/\pi} \quad \text{if } -1 \leq t \leq 0.$$

Then $t_\theta \leq \xi_\theta$, where ξ_θ is the unique value such that

$$\int_{-1}^{0} \left(\frac{1+t}{2}\right)^{\theta/\pi} dt + \int_{0}^{\xi_\theta} \left(\frac{1+t}{1-t}\right)^{\theta/\pi} dt = \int_{\xi_\theta}^{1} \left(\frac{1+t}{1-t}\right)^{\theta/\pi} dt.$$

We also have

$$\left(\frac{1+t}{1-t}\right)^{\theta/\pi} \leq \left(\frac{1}{1-t}\right)^{2\theta/\pi} \quad \text{if } 0 \leq t \leq 1,$$

and

$$\frac{(1/(1-t))^{2\theta/\pi}}{((1+t)/(1-t))^{\theta/\pi}} = \left(\frac{1}{1-t^2}\right)^{\theta/\pi}$$

is an increasing function on $(0, 1)$. If we let

$$M = \int_{-1}^{0} \left(\frac{1+t}{2}\right)^{\theta/\pi} dt,$$

then we may apply Lemma 2.2 and conclude that $t_\theta \leq \xi_\theta \leq \tau_\theta$, where τ_θ is given by

$$\int_{-1}^{0} \left(\frac{1+t}{2}\right)^{\theta/\pi} dt + \int_{0}^{\tau_\theta} \left(\frac{1}{1-t}\right)^{2\theta/\pi} dt = \int_{\tau_\theta}^{1} \left(\frac{1}{1-t}\right)^{2\theta/\pi} dt.$$

The integrals above have elementary antiderivatives. In order to show that $\tau_\theta \leq 2\theta/\pi$ for small θ, we choose $2\theta/\pi$ as the point of integration. By taking explicit antiderivatives and simplifying, it suffices to show that for small enough θ,

$$\frac{\left(\frac{1}{2}\right)^{\theta/\pi}}{1+\theta/\pi} + \frac{1-2(1-2\theta/\pi)^{1-2\theta/\pi}}{1-2\theta/\pi} \geq 0. \tag{2-3}$$

The expression on the left of (2-3) approaches zero as $\theta \to 0$. If we take the derivative of the left side of (2-3) with respect to θ and let $\theta \to 0$ we obtain $\left(1+\ln\left(\frac{1}{2}\right)\right)/\pi > 0$. Then (2-3) is true as long as $0 < \theta \leq \theta_0$ for $\theta_0 > 0$ chosen small enough. Hence we conclude that $t_\theta \leq \tau_\theta \leq 2\theta/\pi$ for any $0 < \theta \leq \theta_0$. □

From (2-1) we have $|f'_\theta(z)| \to 1$ as $|z| \to \infty$. We let ϕ_θ be the harmonic function in Ω_θ defined by

$$y^+ = \phi_\theta(u, v),$$

where $f_\theta = u + iv$. Since $1 = |\nabla\phi_\theta||f'(z)|$, we have $|\nabla\phi_\theta| \to 1$ as $|z| \to \infty$. By a rotation of $\frac{\pi}{2}$ of ϕ_θ we have a complementary harmonic function $\tilde{\phi}_\theta$ and can thus apply the ACF monotonicity formula. We have $J(\infty, \phi_\theta, \tilde{\phi}_\theta) = 1$. To find $J(0+, \phi_\theta, \tilde{\phi}_\theta)$ we find $|\nabla\phi_\theta(0)|$. This is given by

$$1 = |\nabla\phi_\theta(0)||f'(t_\theta)|.$$

Thus

$$1 \ge |\nabla\phi_\theta(0)| = \left(\frac{1-t_\theta}{1+t_\theta}\right)^{\theta/\pi},$$

so $|\nabla\phi_\theta(0)|$ is an increasing function of θ, and

$$1 \ge |\nabla\phi_\theta(0)| \ge \left(\frac{1-2\theta/\pi}{1+2\theta/\pi}\right)^{\theta/\pi}.$$

Using L'Hospital's rule we conclude that

$$\lim_{\theta\to 0} \frac{1-((1-2\theta/\pi)/(1+2\theta/\pi))^{\theta/\pi}}{(\theta/\pi)^2} = 4 > 0.$$

As a consequence we have the following result:

Lemma 2.4. *There exists θ_0 such that if $0 < \theta \le \theta_0$, then*

$$0 < 1-\theta^2 < |\nabla\phi_\theta(0)| \le 1. \tag{2-4}$$

Since $J(\infty, \phi_\theta) = 1$ and $J(0+, \phi_\theta) = |\nabla\phi_\theta(0)|$, Lemma 2.4 shows that

$$J(\infty, \phi_\theta) - J(0+, \phi_\theta) < 1-\theta^2.$$

3. Bounded domain

The aim of this section is to transfer the inequality in (2-4) to a harmonic function on a bounded domain. We approximate Ω_θ with domains $\Omega_{\theta,M}$; see Figure 2. If $f_{\theta,M}$ is the conformal mapping of the upper half-plane onto $\Omega_{\theta,M}$, then

$$f'_{\theta,M}(z) = \left(\frac{z+1}{z-1}\right)^{\theta/\pi}\left(\frac{z-z_2}{z+z_2}\right)^{1/2}\left(\frac{z+z_1}{z-z_1}\right)^{1/2}, \tag{3-1}$$

where $z_1, z_2 \in \mathbb{R}$ and $1 < z_1 < z_2$. We again translate $f_{\theta,M}$ by a constant so that the domain is centered on the origin as in Figure 2. The points z_1, z_2 are chosen so that $f_{\theta,M}(z_2) = M + 0i$. We point out that $|f'_{\theta,M}| \to 1$ as $|z| \to \infty$. We define $\phi_{\theta,M}(u, v) = y^+$, where $f_{\theta,M} = u + iv$.

Lemma 3.1. *Fix $\theta \le \theta_0$. There exists $M > 0$, possibly depending on θ, such that $J(\infty, \phi_{\theta,M}) = 1$ and $J(0+, \phi_{\theta,M}) > 1-\theta^2$.*

Proof. That $J(\infty, \phi_{\theta,M}) = 1$ follows from the definition of $\phi_{\theta,M}$ and (3-1). Now from the explicit formulas given for $f'_\theta(z)$ and $f_{\theta,M}$ in (2-1) and (3-1) respectively, we have $\phi_{\theta,M} \to \phi_\theta$ in C^1 up to the boundary in a neighborhood of the origin. Since $|\nabla\phi_\theta(0)| > 1-\theta^2$, the conclusion follows. □

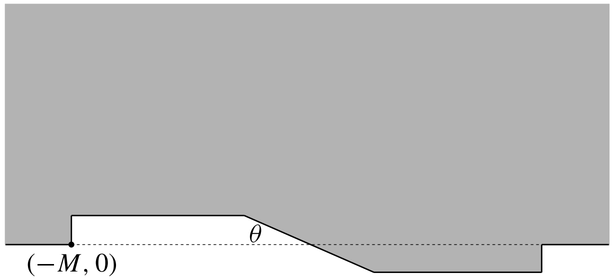

Figure 2. Domain $\Omega_{\theta,M}$.

Remark 3.2. Since $J(r, \phi_{\theta,M})$ is monotonically increasing in r, it follows from Lemma 3.1 that

$$J(\infty, \phi_{\theta,M}) - J(0+, \phi_{\theta,M}) < 1 - \theta^2.$$

For any $\theta \le \theta_0$, we fix an M that satisfies Lemma 3.1. We now transfer the decrease in energy to a finite domain.

Lemma 3.3. *Let θ and $\phi_{\theta,M}$ be as in Lemma 3.1. Let $\Omega_{\theta,M}$ be defined as before. If we define w_R to be such that*

$$\begin{cases} \Delta w_R = 0 & \text{in } B_R \cap \Omega_{\theta,M}, \\ w_R = 0 & \text{on } \partial\Omega_{\theta,M} \cap B_R, \\ w_R = y & \text{on } (\partial B_R)^+, \end{cases}$$

then $w_R \to \phi_{\theta,M}$ locally uniformly in $\Omega_{\theta,M}$ and in C^1 in $B_\rho \cap \Omega_{\theta,M}$ for small enough ρ.

Proof. Using the rescaling

$$\phi_R := \frac{\phi_{\theta,M}(Rx, Ry)}{R},$$

we have $\phi_R \to y^+$ in C^1 on $(\partial B_1)^+$. Thus, for any $\eta > 0$, there exists $R_0 > 0$ such that if $R \ge R_0$, then

$$(1-\eta)y^+ \le \phi_R \le (1+\eta)y^+ \quad \text{on } (\partial B_1)^+.$$

Then rescaling back we obtain that $(1-\eta)y^+ \le \phi_{\theta,M} \le (1+\eta)y^+$ on $(\partial B_R)^+$ if $R \ge R_0$. From the maximum principle we then have

$$(1-\eta)w_R \le \phi_{\theta,M} \le (1+\eta)w_R \quad \text{for any } R \ge R_0.$$

Then as $R \to \infty$, we have $w_R \to w_\infty$ locally uniformly in $\Omega_{\theta,M}$ and in C^1 in a neighborhood of the origin. Furthermore, we have $(1-\eta)w_\infty \le \phi_{\theta,M} \le (1+\eta)w_\infty$. Since η can be taken to be arbitrarily small, we conclude that $w_\infty = \phi_{\theta,M}$. □

We end this section by defining a θ-turn. If $u \in \mathcal{K}$ and for some $\rho > 0$ we have $\partial\{u > 0\} \cap B_\rho$ is a line segment with inward unit normal ν, then a θ-turn in B_ρ gives a new function v with

(i) $v \in \mathcal{K}$,

(ii) $v = u$ on ∂B_1,

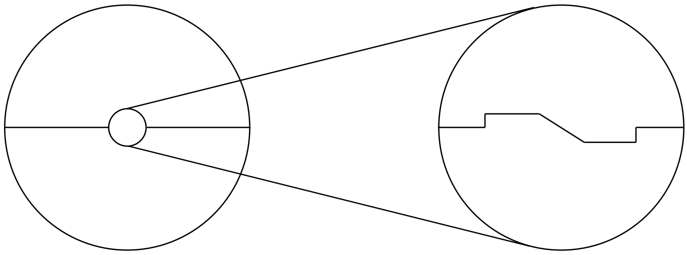

Figure 3. θ-turn when $\nu = i$.

(iii) $\partial\{v > 0\} \cap (B_1 \setminus \bar{B}_\rho) = \partial\{u > 0\} \cap (B_1 \setminus \bar{B}_\rho)$,

(iv) $\partial\{v > 0\} \cap B_\rho = \partial\{\phi_{\theta,M}(e^{i(\nu-\theta)}(2M/\rho)z) > 0\} \cap B_\rho$.

The idea of property (iv) is to shrink $\phi_{\theta,M}$ on B_{2M} to B_ρ and give v the same positivity set; see Figure 3 for when $\nu = i$.

4. Construction of counterexample

As before we let θ_0 be as in Lemma 2.4. This next lemma shows how to apply a θ-turn to a function that is almost linear at the origin.

Lemma 4.1. *Fix $\epsilon > 0$. Assume $u \in \mathcal{K}$, and that there is $s < r_0 < 1$ with*

(1) $B_s \cap \partial\{u > 0\} = B_s \cap \{y_n = 0\}$,

(2) $|u| < 2J(1, u)r_0$ *on* B_{r_0}.

If $\theta \le \theta_0$, then there exists r, ρ with $s > r > \rho > 0$ with a θ-turn in B_ρ such that if v is the redefined function, then v satisfies

(A) $|v| < 2J(1, v)r$ *on* B_r,

(B) $|v| \le (1 + \theta^2) \sup_{B_t} |u|$ *on* B_t *for* $t \in [r_0, 1]$,

(C) $J(1, v) \le (1 + \theta^2) J(1, u)$,

(D) $J(0+, v) > (1 - \theta^2)^2 J(0+, u)$.

Proof. We choose $r < s$ small enough so that

$$\left\| \frac{u(rx)}{r} - J(0+, u)y^+ \right\|_{C^1((\partial B_1)^+)} < \delta, \tag{4-1}$$

and so that $|u| < 2J(1, u)r$. We now apply a θ-turn in B_ρ with $0 < \rho < r$. As $\rho \to 0$, we have $v \to u$ uniformly away from the origin, so that by choosing ρ small enough, v satisfies (B).

We now let $\eta > 0$ be small and use a cut-off function and obtain in the standard way the Caccioppoli inequality

$$\int_{B_1 \setminus B_\eta} |\nabla v - u|^2 \le C(\eta) \int_{B_1 \setminus B_{\eta/2}} |v - u|^2.$$

Then as $\rho \to 0$, we have $v \to u$ in $H^1(B_1 \setminus B_\eta)$ for any $\eta > 0$. We now use the monotonicity of $J(r, v)$ to prove that $v \to u$ in $H^1(B_1)$ as $\rho \to 0$. We have

$$\int_{B_\eta} |\nabla v|^2 \le \eta^2 \int_{B_1} |\nabla v|^2 = \delta^2 \int_{B_1 \setminus B_\eta} |\nabla v|^2 + \int_{B_\eta} |\nabla v|^2,$$

so that

$$\int_{B_\eta} |\nabla v|^2 \le \frac{\eta^2}{1-\eta^2} \int_{B_1 \setminus B_\eta} |\nabla v|^2,$$

and we conclude that

$$\int_{B_1} |\nabla v|^2 \le \frac{1}{1-\eta^2} \int_{B_1 \setminus B_\eta} |\nabla v|^2.$$

Then $\|v\|_{H^1(B_1)}$ is bounded as $\rho \to 0$, so that $v \rightharpoonup u$ in $H^1(B_1)$ as $\rho \to 0$. We now have

$$\int_{B_1} |\nabla u|^2 \le \lim_{\rho \to 0} \int_{B_1} |\nabla v|^2 \le \lim_{\rho \to 0} \frac{1}{1-\eta^2} \int_{B_1 \setminus B_\eta} |\nabla v|^2 = \frac{1}{1-\eta^2} \int_{B_1 \setminus B_\eta} |\nabla u|^2.$$

Since η can be chosen arbitrarily small, we have $\nabla v \to \nabla u$ in $L^2(B_1)$ and thus conclude that $v \to u$ in $H^1(B_1)$ as $\rho \to 0$. Consequently, we may choose ρ even smaller so that properties (A) and (C) hold.

From (4-1), if ρ is chosen small enough we have

$$\left\| \frac{v(rz)}{r} - J(0+, u) y^+ \right\|_{C^1((\partial B_1)^+)} < \delta,$$

so that $(1-\delta) J(0+, u) y^+ \le v(rz)/r$ on $(\partial B_1)^+$. We now define w to be the solution to

$$\begin{cases} \Delta w = 0 & \text{in } \{v(rz)/r > 0\} \cap B_1, \\ w = 0 & \text{on } \partial\{v(rz)/r > 0\} \cap B_1, \\ w = (1-\delta) J(0+) y^+ & \text{on } (\partial B_1)^+. \end{cases}$$

We have $w \le v$ in B_1, so that $|\nabla w(0)| \le |\nabla v(0)|$ or $J(0+, w) \le J(0+, v)$. We may rescale w and apply Lemma 3.3 to obtain that for small enough ρ, we have

$$J(0+, w) > (1-\theta^2)(1-\delta) J(0+, u).$$

By choosing $\delta < \theta^2$ we obtain (D). □

Proof of Theorem 1.3 in dimension $n = 2$. We now use Lemma 4.1 to construct a sequence $u_k \in \mathcal{K}$ with $\lim u_k \to u$. The pair u and $\tilde{u}(z) := u(-z)$ will be a counterexample to Claim 1.2. The sequence u_k is constructed inductively as follows. We choose $\theta_k = 1/(k + N_0)$, where $N_0 \in \mathbb{N}$ is chosen large enough so that $\theta_k \le \theta_0$. We then let $u_0 = y^+$ on B_1. By Lemma 4.1 there exists $\rho_1 < r_1$ such that if a θ_1-turn is applied in B_{ρ_1} to obtain u_1, then u_1 will satisfy properties (A)–(D). We now suppose that u_k has been constructed for some $k \ge 1$. By rotating u_k it will satisfy assumption (1) of Lemma 4.1. Assumption (2) will also be satisfied because u_k satisfies (A) for $r = r_k$. By Lemma 4.1 there exists $\rho_{k+1} < r_{k+1}$ with $r_{k+1} < \rho_k$ so that if we apply a θ_{k+1}-turn to u_k to obtain u_{k+1} we have

(i) $|u_{k+1}| < 2J(1, u_{k+1}) r$ on B_r,

(ii) $|u_{k+1}| \le \prod_{j=1}^{k}(1+\theta_j^2)\sup_{B_t}|u_0|$ on B_t for $t \in [r_k, 1]$,

(iii) $J(1, u_{k+1}) \le \prod_{j=1}^{k}(1+\theta_j^2) J(1, u_0) = \prod_{j=1}^{k}(1+\theta_j^2)$,

(iv) $J(0+, u_{k+1}) > \prod_{j=1}^{k}(1-\theta_j^2)^2 J(0+, u_0) = \prod_{j=1}^{k}(1-\theta_j^2)^2$.

From the same arguments involving the Caccioppoli inequality as in the proof of Lemma 4.1, there exists u such that $u_k \to u$ in $H^1(B_1)$ and locally uniformly away from the origin. Then u is continuous away from the origin. From (i) we obtain that $|u| \le Cr$ on B_r for $0 < r \le 1$, so that u is continuous up to the origin, and $u(0) = 0$.

Now $0 < \prod_{k=1}^{\infty}(1-\theta_k^2)^2$ if and only if $0 < \prod_{k=1}^{\infty}(1-\theta_k^2)$ if and only if

$$\sum_{k=1}^{\infty}(k+N_0)^{-2} = \sum_{k=1}^{\infty}\theta_k^2 < \infty.$$

Since the inequality above is true, we conclude that

$$0 < \prod_{k=1}^{\infty}(1-\theta_k^2)^2 < \prod_{k=1}^{\infty}(1-\theta_k^2) < 1.$$

The last inequality above is due to the fact that all the terms are less than 1. Since $u_k \to u$ in $H^1(B_1)$ and from properties (ii) and (iii), we conclude that

$$0 < \prod_{k=1}^{\infty}(1-\theta_k^2)^2 \le J(r, u) \le CJ(1, u) < \infty \quad \text{for all } 0 < r \le 1,$$

so that $J(0+, u) > 0$.

If we let $\tilde{u}_k(z) = u_k(-z)$, then $\tilde{u}_k \to \tilde{u}$, where $\tilde{u}(z) = u(-z)$. Furthermore, $u \cdot \tilde{u} = 0$ in B_1. Since also $u, \tilde{u}$ are nonnegative, continuous, and harmonic when positive, they satisfy the assumptions of the ACF monotonicity formula in Proposition 1.1. We now show that $u, \tilde{u}$ are a counterexample to Claim 1.2. We assume by way of contradiction that $\{u > 0\}$ and $\{\tilde{u} > 0\}$ are tangent at the origin and after a rotation $u(z) + \tilde{u}(z) = \alpha x_1^+ + \beta x_1^- + o(|z|)$. Then for any small $\delta > 0$, there exists r_0 such that if $r \le r_0$ and $|z| > \frac{1}{2}$ and $|\mathrm{Arg}(z)| < \delta$, then

$$\frac{u(rz) + \tilde{u}(rz)}{r} > \frac{\alpha x_1^+}{2} > 0. \tag{4-2}$$

We now recall that from the construction

$$\partial\{u > 0\} \cap (B_{r_k} \setminus B_{\rho_k}) = \{z : z = te^{-i\sum_{j=1}^{k}\theta_j} \text{ and } \rho_k \le |t| < r_k\}. \tag{4-3}$$

Since $\sum \theta_k = \infty$ and $\theta_k \to 0$, we obtain from (4-3) there exist infinitely many z_k with $|z_k| \to 0$ and $|\mathrm{Arg}(z)| < \delta$ such that $u(z_k) + \tilde{u}(z_k) = 0$. This contradicts (4-2), and so Claim 1.2 is not true. □

We now show that the pair u and $\tilde{u}$ are also a counterexample in higher dimensions.

Proof of Theorem 1.3 in dimension $n > 2$. For u as in the proof for dimension 2, we let $w_n(x_1, x_2, \ldots, x_n) = u(x_1, x_2)$. Since in dimension $n = 2$ we have

$$\frac{1}{r^2}\int_{B_r} |\nabla u|^2 \geq C > 0,$$

it follows that in dimension n,

$$\frac{1}{r^n}\int_{B_r} |\nabla w|^2 \geq C.$$

Then

$$\frac{1}{r^2}\int_{B_r} \frac{|\nabla w|^2}{|x|^{n-2}} \geq \frac{1}{r^2}\int_{B_r} \frac{|\nabla w|^2}{r^{n-2}} = \frac{1}{r^n}\int_{B_r} |\nabla w|^2 \geq C > 0,$$

so that $\Phi(r, w, \tilde{w}) > 0$. We have already shown that $u + \tilde{u}$ cannot satisfy the conclusions in Claim 1.2; consequently, $w + \tilde{w}$ also do not satisfy those conclusions. □

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MARK ALLEN: allen@mathematics.byu.edu
Department of Mathematics, Brigham Young University, Provo, UT, United States

DENNIS KRIVENTSOV: dennisk@cims.nyu.edu
Courant Institute of Mathematical Sciences, New York University, New York, NY, United States

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