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REGULARITY ESTIMATES FOR ELLIPTIC NONLOCAL OPERATORS

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We study weak solutions to nonlocal equations governed by integrodifferential operators. Solutions are defined with the help of symmetric nonlocal bilinear forms. Throughout this work, our main emphasis is on operators with general, possibly singular, measurable kernels. We obtain regularity results which are robust with respect to the differentiability order of the equation. Furthermore, we provide a general tool for the derivation of Hölder a priori estimates from the weak Harnack inequality. This tool is applicable for several local and nonlocal, linear and nonlinear problems on metric spaces. Another aim of this work is to provide comparability results for nonlocal quadratic forms.

1. Introduction

The aim of this work is to develop a local regularity theory for general nonlocal operators. The main focus is on operators that are defined through families of measures, which might be singular. The main question that we ask is the following. Given a function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} (u(y)-u(x))\, \mu(x,\mathrm{d}y) = f(x) \quad (x \in D), \tag{1-1}$$

which properties of u can be deduced in the interior of D? Here $D \subset \mathbb{R}^d$ is a bounded open set and the family $(\mu(x,\cdot))_{x\in D}$ of measures satisfies some assumptions to be discussed later in detail. The measures $\mu(x,\cdot)$ are assumed to have a singularity for sets $A \subset \mathbb{R}^d$ with $x \in \bar{A}$. As a result, the operators of the

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form (1-1) are not bounded integral operators but integrodifferential operators. For this reason we are able to prove regularity results which resemble results for differential operators. One aim of this work is to establish the following result:

Theorem 1.1. *Assume $\mu(x, \mathrm{d}y)$ is uniformly (with respect to the variable x) comparable on small scales to $\nu^\alpha(\mathrm{d}y - \{x\})$ for some nondegenerate α-stable measure ν^α for some $\alpha \in (0,2)$. Then solutions to (1-1) satisfy uniform Hölder regularity estimates in the interior of D.*

Theorem 1.1 will be proved as a special case of Theorem 1.11, which we provide with all details in Section 1E. Special cases of Theorem 1.1 have received significant attention over the last years and we give a small overview of results below. Note that it is well known how to treat functions f in (1-1). For the sake of a clear presentation, we will sometimes restrict ourselves to the case $f = 0$.

In order to approach the question raised above, we need to establish the following results:

- weak Harnack inequality,
- implications of the weak Harnack inequality,
- comparability results for nonlocal quadratic forms.

The last topic needs to be included because our concept of solutions involves quadratic forms related to $\mu(x, \mathrm{d}y)$. We present the main results in Sections 1C–1E. The following two subsections are devoted to the set-up and our main assumptions.

1A. ***Function spaces.*** Before we can formulate the first result we need to set up quadratic forms and function spaces. Let $\mu = (\mu(x,\cdot))_{x\in\mathbb{R}^d}$ be a family of measures on $\mathbb{R}^d$ which is symmetric in the sense that for every set $A \times B \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}$

$$\int_A \int_B \mu(x, \mathrm{d}y)\,\mathrm{d}x = \int_B \int_A \mu(x, \mathrm{d}y)\,\mathrm{d}x. \tag{1-2}$$

We furthermore require

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \min(|x-y|^2, 1)\,\mu(x, \mathrm{d}y) < +\infty. \tag{1-3}$$

Example 1.2. An important example satisfying the above conditions is given by

$$\mu_\alpha(x, \mathrm{d}y) = (2-\alpha)|x-y|^{-d-\alpha}\,\mathrm{d}y \quad (0 < \alpha < 2). \tag{1-4}$$

The choice of the factor $(2-\alpha)$ will be discussed below in detail; see Sections 1B and 2.

For a given family μ and a real number $\alpha \in (0,2)$ we consider the following quadratic forms on $L^2(D) \times L^2(D)$, where $D \subset \mathbb{R}^d$ is some open set:

$$\mathcal{E}^\mu_D(u,u) = \int_D \int_D (u(y) - u(x))^2\,\mu(x, \mathrm{d}y)\,\mathrm{d}x. \tag{1-5}$$

We denote by $H^{\alpha/2}(\mathbb{R}^d)$ the usual Sobolev space of fractional order $\alpha/2 \in (0,1)$ with the norm

$$\|u\|_{H^{\alpha/2}(\mathbb{R}^d)} = (\|u\|^2_{L^2(\mathbb{R}^d)} + \mathcal{E}^{\mu_\alpha}_{\mathbb{R}^d}(u,u))^{\frac{1}{2}}. \tag{1-6}$$

If $D \subset \mathbb{R}^d$ is open and bounded, then by $H^{\alpha/2}_D = H^{\alpha/2}_D(\mathbb{R}^d)$ we denote the Banach space of functions from $H^{\alpha/2}(\mathbb{R}^d)$ which are zero almost everywhere on D^c. $H^{\alpha/2}(D)$ shall be the space of functions $u \in L^2(D)$ for which

$$\|u\|^2_{H^{\alpha/2}(D)} = \|u\|^2_{L^2(D)} + \int_D \int_D (u(y)-u(x))^2 \, \mu_\alpha(x, \mathrm{d}y) \, \mathrm{d}x$$

is finite. Note that for domains D with a Lipschitz boundary, $H^{\alpha/2}_D(\mathbb{R}^d)$ can be identified with the closure of $C^\infty_c(D)$ with respect to the norm of $H^{\alpha/2}(D)$. In general, these two objects might be different, though. By $V^{\alpha/2}(D \,|\, \mathbb{R}^d)$ we denote the space of all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ for which the quantity

$$\int_D \int_{\mathbb{R}^d} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \tag{1-7}$$

is finite, which implies finiteness of the quantity $\int_{\mathbb{R}^d} u(x)^2/(1+|x|)^{d+\alpha} \, \mathrm{d}x$. The function space $V^{\alpha/2}(D \,|\, \mathbb{R}^d)$ is a Hilbert space with the scalar product

$$(u, v) = \int_{\mathbb{R}^d} \frac{u(x)v(x)}{(1+|x|)^{d+\alpha}} \, \mathrm{d}x + \int_D \int_{\mathbb{R}^d} \frac{(u(y)-u(x))(u(v)-v(x))}{|x-y|^{d+\alpha}} \, \mathrm{d}x \, \mathrm{d}y. \tag{1-8}$$

The proof is similar to those of [Felsinger et al. 2015, Lemma 2.3] and [Dipierro et al. 2017a, Proposition 3.1]. If the scalar product (1-8) is defined with the expression $\int_{\mathbb{R}^d} u(x)v(x)/(1+|x|)^{d+\alpha}$ replaced by $\int_D u(x)v(x) \, \mathrm{d}x$, then the Hilbert space is identical. The following continuous embeddings trivially hold true:

$$H^{\frac{\alpha}{2}}_D(\mathbb{R}^d) \hookrightarrow H^{\frac{\alpha}{2}}(\mathbb{R}^d) \hookrightarrow V^{\frac{\alpha}{2}}(D \,|\, \mathbb{R}^d).$$

We make use of function spaces generated by general μ in the same way as above. Let $H^\mu(\mathbb{R}^d)$ be the vector space of functions $u \in L^2(\mathbb{R}^d)$ such that $\mathcal{E}^\mu(u,u) = \mathcal{E}^\mu_{\mathbb{R}^d}(u,u)$ is finite. If $D \subset \mathbb{R}^d$ is open and bounded, then by $H^\mu_D = H^\mu_D(\mathbb{R}^d)$ we denote the space of functions from $H^\mu(\mathbb{R}^d)$ which are zero almost everywhere on D^c. By $V^\mu_D = V^\mu(D \,|\, \mathbb{R}^d)$ we denote the space of all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ for which the quantity

$$\int_D \int_{\mathbb{R}^d} (u(y)-u(x))^2 \, \mu(x, \mathrm{d}y) \, \mathrm{d}x \tag{1-9}$$

is finite. Now we are in a position to present and discuss our main results.

1B. *Main assumptions.* Let us formulate our main assumptions on $(\mu(x,\cdot))_{x\in D}$. Given $\alpha \in (0,2)$ and $A \geq 1$, the following condition is an analog of (A′) for nonlocal energy forms:

$$\begin{gathered} \text{For every ball } B_\rho(x_0) \text{ with } \rho \in (0,1), x_0 \in B_1 \text{ and every } v \in H^{\alpha/2}(B_\rho(x_0)), \\ A^{-1} \, \mathcal{E}^\mu_{B_\rho(x_0)}(v,v) \leq \mathcal{E}^{\mu_\alpha}_{B_\rho(x_0)}(v,v) \leq A \, \mathcal{E}^\mu_{B_\rho(x_0)}(v,v). \end{gathered} \tag{A}$$

Condition (A) says that, locally in the unit ball, the energies $\mathcal{E}^\mu$ and $\mathcal{E}^{\mu_\alpha}$ are comparable on every scale. Note that this does not imply pointwise comparability of the densities of μ and μ_α. We also need to assume the existence of cut-off functions. Let $\alpha \in (0,2)$ and $B \geq 1$:

$$\begin{aligned} &\text{For } 0<\rho \leq R \leq 1 \text{ and } x_0 \in B_1 \text{ there is a nonnegative measurable function } \tau : \mathbb{R}^d \to \mathbb{R} \\ &\text{with } \operatorname{supp}(\tau) \subset \overline{B_{R+\rho}(x_0)},\ \tau(x) \equiv 1 \text{ on } B_R(x_0),\ \|\tau\|_\infty \leq 1, \text{ and} \\ &\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\, \mu(x,\mathrm{d}y) \leq B\rho^{-\alpha}. \end{aligned} \tag{B}$$

In most of the cases (B) does not impose an additional restriction because the standard cut-off function $\tau(x) = \max(0, 1+\min(0, (R-|x-x_0|)/\rho))$ is an appropriate choice. It is an interesting question whether, under assumptions (1-2), (1-3) and (A), condition (B) holds or whether it holds with this standard choice. Note that condition (B) becomes $|\nabla\tau|^2 \leq B\rho^{-2}$ when $\alpha \to 2-$ and $\mu(x,\mathrm{d}y)$ is as in Example 1.2.

For every $\alpha \in (0,2)$, the family of measures μ_α given in Example 1.2 satisfies the above conditions for some constants $A, B \geq 1$. The normalizing constant $2-\alpha$ in the definition of μ_α has the effect that the constants $A, B \geq 1$ can be chosen independently of α for $\alpha \to 2-$. Since in this work we do not care about the behavior of constants for $\alpha \to 0+$, in our examples we will use factors of the form $2-\alpha$. Let us look at more examples.

Example 1.3. Assume $0 < \beta \leq \alpha < 2$. Let $f, g : \mathbb{R}^d \to [1,2]$ be measurable and symmetric functions. Set

$$\mu(x,\mathrm{d}y) = f(x,y)\,\mu_\alpha(x,\mathrm{d}y) + g(x,y)\,\mu_\beta(x,\mathrm{d}y).$$

Then μ satisfies (1-2), (1-3), (A), and (B) with exponent α. This simply follows from

$$\frac{1}{|x-y|^{d+\alpha}} \leq \frac{1}{|x-y|^{d+\beta}} + \frac{1}{|x-y|^{d+\alpha}} \leq \frac{5}{|x-y|^{d+\alpha}} \quad (x,y \in B_1(x_0),\ x_0 \in \mathbb{R}^d).$$

For the verification of (B) we may choose the standard Lipschitz-continuous cut-off function.

Here is an example with some kernels which are not rotationally symmetric.

Example 1.4. Assume $\alpha_0 \in (0,2),\ 0 < \lambda < \Lambda,\ v \in S^{d-1}$, and $\theta \in [0,1)$. Set

$$M = \left\{ h \in \mathbb{R}^d : \left| \left\langle \frac{h}{|h|}, v \right\rangle \right| \geq \theta \right\}.$$

Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be any measurable function satisfying

$$\lambda \mathbb{1}_M(x-y) \frac{(2-\alpha)}{|x-y|^{d+\alpha}} \leq k(x,y) \leq \Lambda \frac{(2-\alpha)}{|x-y|^{d+\alpha}} \tag{1-10}$$

for some $\alpha \in [\alpha_0, 2)$ and for almost every $x, y \in \mathbb{R}^d$. Set $\mu(x,\mathrm{d}y) = k(x,y)\,\mathrm{d}y$. Then, as we will prove, there are $A \geq 1$, $B \geq 1$, independent of α, such that (A) and (B) hold.

The following example of a family of measures falls into our framework. Note that the measures do not possess a density with respect to the d-dimensional Lebesgue measure.

Example 1.5. Assume $\alpha_0 \in (0,2),\ \alpha_0 \leq \alpha < 2$. Set

$$\mu(x,dy) = (2-\alpha) \sum_{i=1}^{d} \left[|x_i - y_i|^{-1-\alpha} \mathrm{d}y_i \prod_{j\neq i} \delta_{\{x_j\}}(\mathrm{d}y_j) \right]. \tag{1-11}$$

Again, as we will prove, there are $A \geq 1$, $B \geq 1$, independent of α, such that (A) and (B) hold. Note that $\mu(x, A) = 0$ for every set A which has an empty intersection with any of the d lines $\{x + te_i : t \in \mathbb{R}\}$.

Let us now formulate our results.

1C. ***The weak Harnack inequality.*** Given functions $u, v : \mathbb{R}^d \to \mathbb{R}$ we define the quantity

$$\mathcal{E}^{\mu}(u, v) = \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(y) - u(x))(v(y) - v(x))\, \mu(x, \mathrm{d}y)\, \mathrm{d}x, \tag{1-12}$$

if it is finite. We write $\mathcal{E}$ instead of $\mathcal{E}^{\mu}$ when it is clear or irrelevant which measure μ is used. One aim of this work is to study properties of functions u satisfying $\mathcal{E}(u, \phi) \geq 0$ for every nonnegative test function ϕ. Note that $\mathcal{E}^{\mu}(u, \phi)$ is finite for $u \in V^{\mu}(D \mid \mathbb{R}^d)$, $\phi \in H^{\mu}_{D}(\mathbb{R}^d)$ for any open set $D \subset \mathbb{R}^d$. This follows from the definition of these function spaces, the Cauchy–Schwarz inequality and the decomposition

$$\mathcal{E}^{\mu}(u,\phi) = \iint_{DD} (u(y)-u(x))(\phi(y)-\phi(x))\, \mu(x,\mathrm{d}y)\, \mathrm{d}x + 2\iint_{DD^c} (u(y)-u(x))(\phi(y)-\phi(x))\, \mu(x,\mathrm{d}y)\, \mathrm{d}x.$$

Here is our first main result.

Theorem 1.6 (weak Harnack inequality). *Assume $0 < \alpha_0 < 2$ and $A \geq 1$, $B \geq 1$. Let μ satisfy* (A), (B) *for some $\alpha \in [\alpha_0, 2)$. Assume $f \in L^{q/\alpha}(B_1)$ for some $q > d$. Let $u \in V^{\mu}(B_1 \mid \mathbb{R}^d)$, $u \geq 0$ in B_1, satisfy $\mathcal{E}^{\mu}(u, \phi) \geq (f, \phi)$ for every nonnegative $\phi \in H^{\mu}_{B_1}(\mathbb{R}^d)$. Then*

$$\inf_{B_{1/4}} u \geq c \left(\fint_{B_{1/2}} u(x)^{p_0}\, \mathrm{d}x \right)^{\frac{1}{p_0}} - \sup_{x \in B_{15/16}} \int_{\mathbb{R}^d \setminus B_1} u^{-}(z)\, \mu(x, \mathrm{d}z) - \| f \|_{L^{q/\alpha}(B_{15/16})}, \tag{1-13}$$

with positive constants p_0 and c depending only on d, α_0, A, B. In particular, p_0 and c do not depend on α.

Note that below we explain a local counterpart to this result, which relates to the limit $\alpha \to 2-$; see Theorem 1.12.

Remark. It is remarkable that (A) and (B) do not imply a strong formulation of the Harnack inequality. Examples 1.4 and 1.5 provide cases in which the classical strong formulation fails. See the discussion in [Kassmann et al. 2014, Appendix A.1] and the concrete examples in [Bogdan and Sztonyk 2005, p. 148; Bass and Chen 2010, Section 3]. The nonlocal term, i.e., the integral of u^- in (1-13) is unavoidable since we do not assume nonnegativity of u on all of $\mathbb{R}^d$.

1D. ***Regularity estimates.*** A separate aim of our work is to provide consequences of the (weak) Harnack inequality. Before we explain this in a more abstract fashion let us formulate a regularity result, which will be derived from Theorem 1.6 and which is one of the main results of this work. We need an additional mild assumption on the decay of the kernels considered.

Given $\alpha \in (0, 2)$ we assume that for some constants $\chi > 1$, $C \geq 1$

$$\mu(x, \mathbb{R}^d \setminus B_{r2^j}(x)) \leq C r^{-\alpha} \chi^{-j} \quad (x \in B_1,\ 0 < r \leq 1,\ j \in \mathbb{N}_0). \tag{D}$$

Condition (D) rules out kernels with very heavy tails for large values of $|x-y|$. For example, μ given by $\mu(x,\mathrm{d}y) = k(x,y)\,\mathrm{d}y$ with $k(x,y) = |x-y|^{-d-1} + |x-y|^{-d}\ln(2+|x-y|)^{-2}$ does not satisfy (D).

Here is our main regularity result.

Theorem 1.7. *Let $\alpha_0 \in (0,2)$, $\chi > 0$, and $A \geq 1$, $B \geq 1$. Let μ satisfy (A), (B) and (D) for some $\alpha \in [\alpha_0, 2)$. Assume $u \in V^{\mu}(B_1 \mid \mathbb{R}^d)$ satisfies $\mathcal{E}(u,\phi) = 0$ for every $\phi \in H^{\mu}_{B_1}(\mathbb{R}^d)$. Then the following Hölder estimate holds for almost every $x, y \in B_{1/2}$:*

$$|u(x) - u(y)| \leq c\|u\|_\infty |x-y|^\beta, \tag{1-14}$$

where $c \geq 1$ and $\beta \in (0,1)$ are constants which depend only on $d, \alpha_0, A, B, C, \chi$. In particular, c and β do not depend on α.

This result contrasts the corresponding result for differential operators; see Theorem 1.13 below.

The main tool for the proof of Theorem 1.7 is the weak Harnack inequality, Theorem 1.6. The Harnack inequality itself is an interesting object of study for nonlocal operators. In Section 2 we explain different formulations of the Harnack inequality for nonlocal operators satisfying a maximum principle. A separate aim of this article is to prove a general tool that allows us to deduce regularity estimates from the Harnack inequality for nonlocal operators. This step was subject to discussion of many recent articles in the field. We choose the set-up of a metric measure space so that this tool can be of future use in different contexts.

In the first decades after publication, the Harnack inequality itself did not attract as much attention as the resulting convergence theorems. This changed when J. Moser in 1961 showed that the inequality itself leads to a priori estimates in Hölder spaces. His result can be formulated in a metric measure space (X, d, m) as follows. For $r > 0$, $x \in X$, set $B_r(x) = \{y \in X : d(y,x) < r\}$. For every $x \in X$ and $r > 0$ let $\mathcal{S}_{x,r}$ denote a family of measurable functions on X satisfying the conditions

$$\begin{aligned} r > 0,\ u \in \mathcal{S}_{x,r},\ a \in \mathbb{R} \quad &\Longrightarrow \quad au \in \mathcal{S}_{x,r},\ (u+1) \in \mathcal{S}_{x,r}, \\ B_r(x) \subset B_s(y) \quad &\Longrightarrow \quad \mathcal{S}_{y,s} \subset \mathcal{S}_{x,r}. \end{aligned}$$

An example for $\mathcal{S}_{x,r}$ is given by the set of all functions $u : \mathbb{R}^d \to \mathbb{R}$ satisfying some (possibly nonlinear) appropriate partial differential or integrodifferential equation in a ball $B_r(x)$.

Theorem 1.8 (compare [Moser 1961]). *Assume X is separable. Let $x_0 \in X$ and $\mathcal{S}_{x,r}$ be as above. Assume that there is $c \geq 1$ such that for $r > 0$*

$$(u \in \mathcal{S}_{x_0,r}) \wedge (u \geq 0 \text{ in } B_r(x_0)) \quad \Longrightarrow \quad \sup_{x \in B_{r/2}(x_0)} u \leq c \inf_{x \in B_{r/2}(x_0)} u. \tag{1-15}$$

Then there exist $\beta \in (0,1)$ such that for $r > 0$, $u \in \mathcal{S}_{x_0,r}$ and almost every $x \in B_r(x_0)$

$$|u(x) - u(x_0)| \leq 3\|u - u(x_0)\|_\infty \left(\frac{d(x,x_0)}{r}\right)^\beta.$$

Recall that "sup" denotes the essential supremum and "inf" the essential infimum. With the help of this theorem, regularity estimates can be established for various linear and nonlinear differential equations; see [Gilbarg and Trudinger 1998]. One aim of this article is to show that (1-15) can be relaxed significantly by allowing some global terms of u to show up in the Harnack inequality. Already in Section 2 we have seen that they naturally appear.

For $x \in X,\ r > 0$ let $\nu_{x,r}$ be a measure on $\mathcal{B}(X \setminus \{x\})$, which is finite on all sets M with $\operatorname{dist}(\{x\}, M) > 0$. We assume that for some $c \geq 1,\ \chi > 1$ and for every $j \in \mathbb{N}_0,\ x \in X$, and $0 < r \leq 1$

$$\nu_{x,r}(X \setminus B_{r2^j}(x)) \leq c\chi^{-j}. \tag{1-16}$$

We further assume that, given $K > 1$, there is $c \geq 1$ such that for $0 < r \leq R \leq Kr,\ x \in X,\ M \subset X \setminus B_r(x)$

$$\nu_{x,R}(M) \leq c\nu_{x,r}(M). \tag{1-17}$$

Conditions (1-16) and (1-17) will trivially hold true in the applications that are of importance to us. In Section 5 we discuss these conditions in detail. A standard case is provided in the following example.

Example 1.9. Let $\alpha \in (0, 2)$. For $x \in \mathbb{R}^d,\ r > 0$, and $A \in \mathcal{B}(\mathbb{R}^d \setminus \{x\})$ set

$$\nu_{x,r}(A) = r^\alpha \mu_\alpha(x, A) = r^\alpha \alpha(2-\alpha) \int_A |x-y|^{-d-\alpha}\, \mathrm{d}y. \tag{1-18}$$

Then $\nu_{x,r}$ satisfies conditions (1-16), (1-17).

The following result extends Theorem 1.8 to situations with nonlocal terms. It is an important tool in the theory of nonlocal operators.

Theorem 1.10. *Let $x_0 \in X,\ r_0 > 0$, and $\lambda > 1,\ \sigma > 1,\ \theta > 1$. Let $\mathcal{S}_{x,r}$ and $\nu_{x,r}$ be as above. Assume that conditions (1-16), (1-17) are satisfied. Assume that there is $c \geq 1$ and $p > 0$ such that for $0 < r \leq r_0$ the following holds*:

$(u \in x_0, r) \wedge (u \geq 0 \text{ in } B_r(x_0))$,

$$\implies \left(\fint_{B_{r/\lambda}(x_0)} u(x)^p\, m(\mathrm{d}x) \right)^{\frac{1}{p}} \leq c \inf_{x \in B_{r/\theta}(x_0)} u + c \sup_{x \in B_{r/\sigma}(x_0)} \int_X u^-(z)\, \nu_{x,r}(\mathrm{d}z). \tag{1-19}$$

Then there exists $\beta \in (0, 1)$ such that for $0 < r \leq r_0,\ u \in \mathcal{S}_{x_0,r}$

$$\operatorname{osc}_{B_\rho(x_0)} u \leq 2\theta^\beta \|u\|_\infty \left(\frac{\rho}{r}\right)^\beta \quad (0 < \rho \leq r), \tag{1-20}$$

where $\operatorname{osc}_M u := \sup_M u - \inf_M u$ for $M \subset X$.

Note that in Lemma 5.1 we provide several conditions that are equivalent to (1-16).

1E. *Comparability of nonlocal quadratic forms.* With regard to Theorem 1.7 one major problem is to provide conditions on μ which imply (A). Let us formulate our results in this direction.

Since $\mu = (\mu(x, \cdot))_{x \in \mathbb{R}^d}$ is a family of measures we need to impose a condition that fixes a uniform behavior of μ with respect to x. In our setup this condition implies that the integrodifferential operator from (1-1) is comparable to a translation-invariant operator — most often the generator of an α-stable process. We assume that there are measures ν_* and ν^* such that

$$\int f(x, x+z)\, \nu_*(\mathrm{d}z) \leq \int f(x, y)\, \mu(x, \mathrm{d}y) \leq \int f(x, x+z)\, \nu^*(\mathrm{d}z) \tag{T}$$

for every measurable function $f : \mathbb{R}^d \to [0,\infty]$ and every $x \in \mathbb{R}^d$. For a measure ν on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and a set $B \subset \mathbb{R}^d$ we define, abusing the previous notation slightly,

$$\mathcal{E}_B^\nu(u,v) = \int_B \int_{\mathbb{R}^d} (u(x)-u(x+z))(v(x)-v(x+z))\,\mathbb{1}_B(x+z)\,\nu(\mathrm{d}z)\,\mathrm{d}x. \tag{1-21}$$

Note that (T) implies for every $u \in L^2(B)$

$$\mathcal{E}_B^{\nu_*}(u,u) \le \mathcal{E}_B^{\mu}(u,u) \le \mathcal{E}_B^{\nu^*}(u,u).$$

Let $\bar\nu(A) = \nu(-A)$. It is easy to check that $\mathcal{E}^\nu = \mathcal{E}^{(\nu+\bar\nu)/2}$. Hence we may and do assume that the measures ν_*, ν^* are symmetric; i.e., $\nu_*(A) = \nu_*(-A)$ and $\nu^*(A) = \nu^*(-A)$.

We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the upper-bound assumption (U) if for some $C_U > 0$

$$\int_{\mathbb{R}^d} (r \wedge |z|)^2\,\nu(\mathrm{d}z) \le C_U r^{2-\alpha} \quad (0 < r \le 1). \tag{U}$$

We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the scaling assumption (S) if for some $a > 1$

$$\int_{\mathbb{R}^d} f(y)\,\nu(\mathrm{d}y) = a^{-\alpha} \int_{\mathbb{R}^d} f(ay)\,\nu(\mathrm{d}y) \tag{S}$$

for every measurable function $f : \mathbb{R}^d \to [0,\infty]$ with $\operatorname{supp} f \subset B_1$. For a linear subspace $E \subset \mathbb{R}^d$, let H_E denote the $\dim(E)$-dimensional Hausdorff measure supported on E.

We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the nondegeneracy assumption (ND) if for some $n \in \{1,\dots,d\}$

$$\begin{aligned}&\nu = \textstyle\sum_{k=1}^n f_k H_{E_k} \text{ for some linear subspaces } E_k \subset \mathbb{R}^d \text{ and densities } f_k\\ &\quad\text{with } \operatorname{lin}\big(\textstyle\bigcup_k E_k\big) = \mathbb{R}^d \text{ and } \textstyle\int_{B_1} f_k\,\mathrm{d}H_{E_k} > 0 \text{ for } k = 1,\dots,n.\end{aligned} \tag{ND}$$

Here is our result on local comparability of nonlocal energy forms. It contains Theorem 1.1 as a special case.

Theorem 1.11. *Let $\mu = (\mu(x,\cdot))_{x\in\mathbb{R}^d}$ be a family of measures on $\mathcal{B}(\mathbb{R}^d)$ satisfying (1-2). Assume that there exist measures ν_* and ν^* for which (T) and (U) hold with $\alpha_0 \in (0,2)$ and $C_U > 0$. Assume that*

(i) *ν_* is a nondegenerate α-stable measure (1-22), or*

(ii) *ν_* satisfies (ND) and for some $a > 1$ each measure $f_k H_{E_k}$ satisfies (S).*

Then there are $A \ge 1$, $B \ge 1$ such that (A) and (B) hold. One can choose $B = 4C_U$ but the constant A depends also on a, on the measure ν_* and on α_0.

The result is robust in the following sense: if $\mu^\alpha = (\mu^\alpha(x,\cdot))_{x\in\mathbb{R}^d}$ satisfies (1-2) and (T) with measures $(\nu_*)^\alpha$ and $(\nu^*)^\alpha$, $\alpha_0 \le \alpha < 2$, that are defined with the help of ν_* and ν^* as in Definition 6.9, then (A) holds with a constant A independent of $\alpha \in [\alpha_0, 2)$.

Recall that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ is a *nondegenerate α-stable measure* if for some $\alpha \in (0,2)$

$$\nu(E) = (2-\alpha) \int_{S^{d-1}} \int_0^\infty \mathbb{1}_E(r\theta) r^{-1-\alpha}\,\mathrm{d}r\,\pi(d\theta) \quad (E \in \mathcal{B}(\mathbb{R}^d)), \tag{1-22}$$

where π is some finite measure on S^{d-1} and $\operatorname{lin}(\operatorname{supp}\pi) = \mathbb{R}^d$.

1F. *Related results.* It is instructive to compare our results with two key results for differential operators in divergence form. Let $(A(x))_{x\in\mathbb{R}^d}$ be a family of $d\times d$-matrices. Given a subset $D\subset\mathbb{R}^d$ we introduce a bilinear form $\mathcal{A}_D$ by $\mathcal{A}_D(u,v)=\int_D(\nabla u(x), A(x)\nabla u(x))\,\mathrm{d}x$ for u and v from the Sobolev space $H^1(D)$. Instead of $\mathcal{A}_{\mathbb{R}^d}$ we write $\mathcal{A}$. The following theorem is at the heart of the theory named after E. De Giorgi, J. Moser and J. Nash; see [Gilbarg and Trudinger 1998, Chapters 8.8–8.9]:

Theorem 1.12 (weak Harnack inequality). *Let $\Lambda>1$. Assume that for all balls $B\subset B_1$ and all functions $v\in H^1(B)$*

$$\Lambda^{-1}\mathcal{A}_B(u,u)\le\int_B|\nabla u|^2\le\Lambda\mathcal{A}_B(u,u). \tag{A$'$}$$

Assume $f\in L^{q/2}(B_1)$ for some $q>d$. Let $u\in H^1(B_1)$ satisfy $u\ge 0$ in B_1 and $\mathcal{A}_{B_1}(u,\phi)\ge(f,\phi)$ for every nonnegative $\phi\in H^1_0(B_1)$. Then

$$c\inf_{B_{1/4}}u\ge\left(\fint_{B_{1/2}}u(x)^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}}-\|f\|_{L^{q/2}(B_{15/16})},$$

with constants $p_0, c\in(0,1)$ depending only on d and Λ.

Remark. This by now classical result can be seen as the limit case of Theorem 1.6 for $\alpha\to 2-$. Condition (A$'$) implies that the differential operator $\div(A(\cdot)\nabla u)$ is uniformly elliptic and obviously describes a limit situation of (A). One might object that the nonlocal term in (1-13) is unnatural but in fact, it is not. In Section 2 we explain this phenomenon in detail for the fractional Laplace operator.

If u is not only a supersolution but a solution in Theorem 1.12, then one obtains a classical Harnack inequality: $\sup_{B_{1/4}}u\le c\inf_{B_{1/4}}u$. Both the Harnack inequality and the weak Harnack inequality imply Hölder a priori regularity estimates:

Theorem 1.13. *Assume condition (A$'$) holds true. There exist $c\ge 1$, $\beta\in(0,1)$ such that for every $u\in H^1(B_1)$ satisfying $\mathcal{A}(u,\phi)=0$ for every $\phi\in H^1_0(B_1)$ the following Hölder estimate holds for almost every $x,y\in B_{1/2}$:*

$$|u(x)-u(y)|\le c\|u\|_\infty|x-y|^\beta. \tag{1-23}$$

The constants β, c depend only on d and Λ.

After having recalled corresponding results for local differential operators, let us review some related results for nonlocal problems. Note that we restrict ourselves to nonlocal equations related to bilinear forms and distributional solutions.

Theorem 1.7 has already been proved under additional assumptions. If $\mu(x,\cdot)$ has a density $k(x,\cdot)$ which satisfies some isotropic lower bound, e.g., for some $c_0>0$, $\alpha\in(0,2)$

$$\mu(x,\mathrm{d}y)=k(x,y)\,\mathrm{d}y,\quad k(x,y)\ge c_0|x-y|^{-d-\alpha}\quad(|x-y|\le 1),$$

then Theorem 1.7 is proved in and follows from [Komatsu 1995; Bass and Levin 2002; Chen and Kumagai 2003; Caffarelli et al. 2011]. In these works the constant c in (1-14) depends on $\alpha\in(0,2)$ with $c(\alpha)\to+\infty$ for $\alpha\to 2-$. The current work follows the strategy laid out in [Kassmann 2009], which, on the one hand, allows the constants to be independent of α for $\alpha\to 2-$ and, on the other hand, allows us to treat general measures. See [Felsinger and Kassmann 2013; Kassmann and Schwab 2014] for corresponding results in the parabolic case.

The articles [Di Castro et al. 2014; 2016] study Hölder regularity estimates and Harnack inequalities for nonlinear equations. Moreover, the results therein provide boundedness of weak solutions. In [Di Castro et al. 2014; 2016] the measures $\mu(x, \mathrm{d}y)$ are assumed to be absolutely continuous with respect to the Lebesgue measure. Another difference to the present article is that our local regularity estimates require only local conditions on the data and on the operator. Note that our study of implications of (weak) Harnack inequalities in Section 5 allows for nonlinear problems in metric measure spaces and could be used to deduce the regularity results of [Di Castro et al. 2016] from results in [Di Castro et al. 2014].

To our best knowledge there has been little research addressing the question of comparability of quadratic nonlocal forms; we note here [Dyda 2006; Husseini and Kassmann 2007; Prats and Saksman 2017]. This question becomes important when studying very irregular kernels as in [Silvestre 2016, Section 4].

Theorem 1.1 has recently been established in the translation-invariant case, i.e., when $\mu(x, \mathrm{d}y) = \nu^\alpha(\mathrm{d}y - \{x\})$ for some α-stable measure ν^α; see [Ros-Oton and Serra 2016]. The methods of that paper seem not to be applicable in the general case, though. Note that anisotropic translation-invariant integrodifferential operators allow for higher interior regularity; see [Ros-Oton and Valdinoci 2016].

Related questions on nonlocal Dirichlet forms on metric measure spaces are currently investigated by several groups. We refer to the exposition in [Grigor'yan et al. 2014; Chen et al. 2019] for a discussion of results regarding the fundamental solution.

1G. ***Notation.*** Throughout this article, "inf" denotes the essential infimum and "sup" the essential supremum. By $S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ we denote the unit sphere. We define the Fourier transform as an isometry of $L^2(\mathbb{R}^d)$ determined by

$$\hat{u}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} u(x) e^{-i\xi\cdot x} \,\mathrm{d}x, \quad u \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

1H. ***Structure of the article.*** The paper is organized as follows. In Section 2 we study the Harnack inequality for the Laplace and the fractional Laplace operators. We explain how one can formulate a Harnack inequality without assuming the functions under consideration to be nonnegative. In Section 3 we provide several auxiliary results and explain how the inequality $\mathcal{E}^\mu(u, \phi) \ge (f, \phi)$ is affected by rescaling the family of measures μ. In Section 4 we prove Theorem 1.6 under assumptions (A) and (B) adapting the approach by Moser to nonlocal bilinear forms. Section 5A provides the proof of Theorem 1.7. We first prove a general tool which allows us to deduce regularity results from weak Harnack inequalities; see Corollary 5.2. Then Theorem 1.7 follows immediately. Section 6 contains the proof of our main result on comparability, Theorem 1.11, in the two respective cases. We provide sufficient conditions on μ for (A) and (B) to hold true. In addition, we provide two examples of quite irregular kernels satisfying (A) and (B).

2. Harnack inequalities for the Laplace and the fractional Laplace operators

We establish a formulation of the Harnack inequality which does not require the functions to be nonnegative. This reformulation is especially interesting for nonlocal problems but our formulation seems to be new even for harmonic functions in the classical sense; see Theorem 2.5. For $\alpha \in (0,2)$ and $u \in C_c^2(\mathbb{R}^d)$ the fractional power of the Laplacian can be defined as

$$\Delta^{\frac{\alpha}{2}} u(x) = C_{\alpha,d} \lim_{\varepsilon \to 0+} \int_{|y-x|>\varepsilon} \frac{u(y)-u(x)}{|y-x|^{d+\alpha}} \, \mathrm{d}y = \frac{C_{\alpha,d}}{2} \int_{\mathbb{R}^d} \frac{u(x+h)-2u(x)+u(x-h)}{|h|^{d+\alpha}} \, \mathrm{d}h, \quad (2\text{-}1)$$

where

$$C_{\alpha,d} = \frac{\Gamma((d+\alpha)/2)}{2^{-\alpha}\pi^{\frac{d}{2}} |\Gamma(-\alpha/2)|}.$$

For later purposes we note that with some constant $c > 0$ for every $\alpha \in (0,2)$

$$c\alpha(2-\alpha) \leq C_{\alpha,d} \leq \frac{\alpha(2-\alpha)}{c}. \quad (2\text{-}2)$$

The use of the symbol $\Delta^{\alpha/2}$ and the term "fractional Laplacian" are justified because of $\widehat{(-\Delta)^{\alpha/2} u}(\xi) = |\xi|^\alpha \hat{u}(\xi)$ for $\xi \in \mathbb{R}^d$ and $u \in C_c^\infty(\mathbb{R}^d)$. Note that we write $\Delta^{\alpha/2} u$ instead of $-(-\Delta)^{\alpha/2} u$, which would be more appropriate. The potential theory of these operators was initiated in [Riesz 1938]. The following Harnack inequality can be easily established using the corresponding Poisson kernels.

Theorem 2.1. *There is a constant $c \geq 1$ such that for $\alpha \in (0,2)$ and $u \in C(\mathbb{R}^d)$ with*

$$\Delta^{\frac{\alpha}{2}} u(x) = 0 \quad (x \in B_1), \quad (2\text{-}3)$$

$$u(x) \geq 0 \quad (x \in \mathbb{R}^d) \quad (2\text{-}4)$$

the following inequality holds:

$$u(x) \leq c u(y) \quad (x, y \in B_{\frac{1}{2}}).$$

Note that $\Delta^{\alpha/2} u(x) = 0$ at a point $x \in \mathbb{R}^d$ requires that the integral in (2-1) converges. Thus some additional regularity of $u \in C(\mathbb{R}^d)$ is assumed implicitly. Since $\Delta^{\alpha/2}$ allows for shifting and scaling, the result holds true for $B_1, B_{1/2}$ replaced by $B_R(x_0), B_{R/2}(x_0)$ with the same constant c for arbitrary $x_0 \in \mathbb{R}^d$ and $R > 0$.

Theorem 2.1 formulates the Harnack inequality in the standard way for nonlocal operators. The function u is assumed to be nonnegative on all of $\mathbb{R}^d$. In the following we discuss the necessity of this assumption and possible alternatives. The following result proves that this assumption cannot be dropped completely.

Theorem 2.2. *Assume $\alpha \in (0,2)$. Then there exists a bounded function $u \in C(\mathbb{R}^d)$ which is infinitely many times differentiable in B_1 and satisfies*

$$\Delta^{\frac{\alpha}{2}} u(x) = 0 \quad (x \in B_1),$$

$$u(x) > 0 \quad (x \in B_1 \setminus \{0\}),$$

$$u(0) = 0.$$

Therefore, the classical local formulation of the Harnack inequality as well as the local maximum principle fail for the operator $\Delta^{\alpha/2}$.

A complicated and lengthy proof can be found in [Kassmann 2007a]. An elegant way to construct such a function u would be to mollify the function $v(x) = (1-|x/2|^2)_+^{-1+\alpha/2}$ and shift it such that $u(0) = 0$. Here we provide a short proof,[1] which includes a helpful observation on radial functions. See [Bucur and Valdinoci 2016; Dipierro et al. 2017b] for further alternatives.

For an open set $D \subset \mathbb{R}^d$, $x \in D$, $0 < \alpha \le 2$, and $v : \mathbb{R}^d \to \mathbb{R}$ $(0 < \alpha < 2)$ or $v : \overline{D} \to \mathbb{R}$ $(\alpha = 2)$ we write

$$H_\alpha(v \mid D)(x) = \int_{y \notin D} P_\alpha(x,y)\, v(y)\, \mathrm{d}y = \begin{cases} \int_{\mathbb{R}^d \setminus D} P_\alpha(x,y)\, v(y)\, \mathrm{d}y & (0<\alpha<2), \\ \int_{\partial D} P_2(x,y)\, v(y)\, \mathrm{d}y & (\alpha = 2). \end{cases} \tag{2-5}$$

In the case of a ball, the Poisson kernel is explicitly known; namely for $R > 0$ and $f : \mathbb{R}^d \setminus B_R(0) \to \mathbb{R}$

$$H_\alpha(f \mid B_R(0))(x) = \begin{cases} f(x) & (|x| \ge R), \\ c_\alpha (R^2 - |x|^2)^{\alpha/2} \int_{|y|>R} f(y)/((|y|^2 - R^2)^{\alpha/2} |x-y|^d)\, \mathrm{d}y & (|x| < R), \end{cases}$$

where $c_\alpha = \pi^{-d/2-1}\Gamma(d/2)\sin \pi\alpha/2$. For a function $\phi : [0,\infty) \to [0,\infty)$ we set

$$h_R^\phi := H_\alpha(\phi \circ |\cdot| B_R(0)).$$

Proposition 2.3. *For all* $0 < |x| < R$

$$h_R^\phi(x) = \frac{\sin \pi\alpha/2}{\pi} \int_0^\infty \phi\big(\sqrt{R^2 + s(R^2 - |x|^2)}\big) \frac{\mathrm{d}s}{(s+1)s^{\frac{\alpha}{2}}}.$$

Proof. Let us fix $R > 0$ and $x \in B_R(0)$. Using polar coordinates we obtain

$$h_R^\phi(x) = c_\alpha (R^2 - |x|^2)^{\frac{\alpha}{2}} \int_R^\infty \int_{\rho S^{d-1}} |x-y|^{-d} \sigma(\mathrm{d}y) \frac{\phi(\rho)\, \mathrm{d}\rho}{(\rho^2 - R^2)^{\frac{\alpha}{2}}}. \tag{2-6}$$

By the classical Poisson formula, see [Gilbarg and Trudinger 1998, formula (2.26)],

$$\int_{S^{d-1}} \frac{1-|w|^2}{|w-y|^d} \sigma(\mathrm{d}y) = |S^{d-1}| \quad (|w| < 1);$$

hence

$$\int_{\rho S^{d-1}} |x-y|^{-d} \sigma(\mathrm{d}y) = \rho^{-1} \int_{S^{d-1}} \left| \frac{x}{\rho} - y \right|^{-d} \sigma(\mathrm{d}y) = \rho^{-1} |S^{d-1}| \left(1 - \frac{|x|^2}{\rho^2}\right)^{-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} \frac{\rho}{\rho^2 - |x|^2}.$$

Plugging this into (2-6) yields

$$h_R^\phi(x) = \frac{c_\alpha \pi^{\frac{d}{2}}}{\Gamma(d/2)} (R^2 - |x|^2)^{\frac{\alpha}{2}} \int_R^\infty \frac{2\rho\phi(\rho)\, \mathrm{d}\rho}{(\rho^2 - |x|^2)(\rho^2 - R^2)^{\frac{\alpha}{2}}}.$$

The simple substitution $s = (\rho^2 - R^2)/(R^2 - |x|^2)$ leads to

$$\int_R^\infty \frac{2\rho\phi(\rho)\, \mathrm{d}\rho}{(\rho^2 - |x|^2)(\rho^2 - R^2)^{\frac{\alpha}{2}}} = \frac{1}{(R^2 - |x|^2)^{\frac{\alpha}{2}}} \int_0^\infty \phi\big(\sqrt{R^2 + s(R^2 - |x|^2)}\big) \frac{ds}{(s+1)s^{\frac{\alpha}{2}}}.$$

Thus the assertion follows. □

[1] We owe the idea of this proof to Wolfhard Hansen.

Theorem 2.2 now follows directly from the following corollary.

Corollary 2.4. *Let $R > 0$ and suppose that ϕ is decreasing on $[R, \infty)$ such that $\phi(s) < \phi(r)$ for some $R < r < s$. Then*

$$h_R^{\phi}(x) < h_R^{\phi}(y), \quad \text{whenever } 0 \le x < y < R.$$

In particular, $u := h_R^{\phi} - h_R^{\phi}(0)$ is a bounded function on $\mathbb{R}^d$ which is α-harmonic on $B_R(0)$ and satisfies $0 = u(0) < u(y)$ for every $y \in B_R(0)$.

In Theorem 2.1 the function u is assumed to be nonnegative on all of $\mathbb{R}^d$. It is not plausible that the assertion should be false for functions u with small negative values at points far from the origin. A similar question can be asked for classical harmonic functions. If u is positive and large on a large part of ∂B_1, it should not matter for the Harnack inequality on $B_{1/2}$ if u is negative with small absolute values on a small part of ∂B_1. Another motivation for a different formulation of the Harnack inequality is that Theorem 2.1 does not allow us to use Moser's approach to regularity estimates, like Theorem 1.8, in a straightforward manner.

Let us give a new formulation of the Harnack[2] inequality that does not need any sign assumption on u. It is surprising that this formulation seems not to have been established since Harnack's textbook in 1887. We treat the classical local case $\alpha = 2$ together with the nonlocal case $\alpha \in (0, 2)$.

Theorem 2.5. (*Harnack inequality for $\Delta^{\alpha/2}$, $0 < \alpha \le 2$*)

(1) *There is a constant $c \ge 1$ such that for $0 < \alpha \le 2$ and $u \in C(\mathbb{R}^d)$ satisfying*

$$\Delta^{\frac{\alpha}{2}} u(x) = 0 \quad (x \in B_1) \tag{2-7}$$

the following estimate holds for every $x, y \in B_{1/2}$:

$$c\big(u(y) - H_\alpha(u^+ \mid B_1)(y)\big) \le u(x) \le c\big(u(y) + H_\alpha(u^- \mid B_1)(y)\big). \tag{2-8}$$

(2) *There is a constant $c \ge 1$ such that for $0 < \alpha \le 2$ and every function $u \in C(\mathbb{R}^d)$ which satisfies (2-7) and is nonnegative in B_1 the following inequality holds for every $x, y \in B_{1/2}$:*

$$u(x) \le c\left(u(y) + \alpha(2-\alpha) \int_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d+\alpha}} \, \mathrm{d}z\right). \tag{2-9}$$

Proof of Theorem 2.5. The decomposition $u = u^+ - u^-$ and an application of Theorem 2.1 give

$$\begin{aligned} u(x) &= H_\alpha(u \mid B_1)(x) \le H(u^+ \mid B_1)(x) \le c H_\alpha(u^+ \mid B_1)(y) \\ &= c H_\alpha(u \mid B_1)(y) + c H_\alpha(u^- \mid B_1)(y) = c u(y) + c H_\alpha(u^- \mid B_1)(y), \end{aligned}$$

which proves the second inequality in (2-8). The first one is proved analogously.

[2] Kassmann would like to use the opportunity to correct an error in [Kassmann 2007b] concerning the name Harnack. The correct name of the mathematician Harnack is Carl Gustav Axel Harnack. His renowned twin brother Carl Gustav Adolf carried the last name "von Harnack" after being granted the honor.

Inequality (2-9) is proved as follows. Assume u is nonnegative in B_1. Using the same strategy as above we obtain for some $c_1, c_2 > 0$ and $c = \max(c_1, c_2)$

$$\begin{aligned} u(x) &\le c_1 H_\alpha(u \mid B_{\frac{3}{4}})(y) + c_1 H_\alpha(u^- \mid B_{\frac{3}{4}})(y) \\ &\le c_1 u(y) + c_2 \alpha(2-\alpha) \int_{\mathbb{R}^d \setminus B_1} \le cu(y) + c\alpha(2-\alpha) \int_{\mathbb{R}^d \setminus B_1} \frac{u^-(z)}{|z|^{d+\alpha}} \, \mathrm{d}z. \end{aligned}$$

The proof of the theorem is complete. Note that different versions of this result have been announced in [Kassmann 2011]. □

Let us make some observations:

(1) There is no assumption on the sign of u needed for (2-8). Inequality (2-8) does hold in the classical case $\alpha = 2$, too.

(2) If u is nonnegative on all of $\mathbb{R}^d$ ($\alpha \in (0,2)$) or nonnegative in B_1 ($\alpha = 2$), then the second inequality in (2-8) reduces to the well-known formulation of the Harnack inequality.

(3) If u is nonnegative in B_1, then (2-9) reduces for $\alpha \to 2$ to the original Harnack inequality.

(4) For the above results, one might want to impose regularity conditions on u such that $\Delta^{\alpha/2} u(x)$ exists at every point $x \in B_1$, e.g., $u|_{B_1} \in C^2(B_1)$ and $u(x)/(1+|x|^{d+\alpha}) \in L^1(\mathbb{R}^d)$. However, the assumption that the integral in (2-1) converges is sufficient.

The proof of Theorem 2.5 does not use the special structure of $\Delta^{\alpha/2}$. The proof only uses the decomposition $u = u^+ - u^-$ and the Harnack inequality for the Poisson kernel. Roughly speaking, it holds for every linear operator that satisfies a maximum principle. One more abstract way of formulating this result in a general framework is as follows:

Lemma 2.6. *Let* $(X, \mathcal{W})$ *be a balayage space* (see [Bliedtner and Hansen 1986]) *such that* $1 \in \mathcal{W}$. *Let* V, W *be open sets in* X *with* $\overline{V} \subset W$. *Let* $c > 0$. *Suppose that, for all* $x, y \in V$ *and* $h \in \mathcal{H}_b^+(V)$,

$$u(x) \le cu(y). \tag{2-10}$$

Then $\varepsilon_x^{V^c} \le c\varepsilon_y^{V^c}$ *and, for every* $u \in \mathcal{H}_b(W)$,

$$u(x) \le cu(y) + c \int u^- \, \mathrm{d}\varepsilon_y^{V^c}. \tag{2-11}$$

Here, $\mathcal{H}_b(A)$ denotes the set of bounded functions which are harmonic in the Borel set A. Functions in $\mathcal{H}_b^+(A)$, in addition, are nonnegative.

Proof. Since, for every positive continuous function f with compact support, the mapping $f \mapsto \varepsilon_z^{V^c}(f)$ belongs to $\mathcal{H}_b^+(V)$, the first statement follows. Let $u \in \mathcal{H}_b(W)$. Then $u(x) = \varepsilon_x^{V^c}(u)$, $u(y) = \varepsilon_y^{V^c}(u)$, and hence

$$u(x) \le \varepsilon_x^{V^c}(u^+) \le c\varepsilon_y^{V^c}(u^+) = c\varepsilon_y^{V^c}(u + u^-) = cu(y) + c \int u^- \, \mathrm{d}\varepsilon_y^{V^c}. \qquad \square$$

3. Functional inequalities and scaling property

In this section we collect several auxiliary results. In particular, we will need some properties of the Sobolev spaces $H^{\alpha/2}(D)$. The following fact about extensions has an elementary proof; see [Di Nezza et al. 2012]. However, one has to go through it and see that the constants do not depend on α, provided one has the factor $(2-\alpha)$ in front of the Gagliardo norm; see (1-4) and (1-6).

Fact 3.1 (extension). Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $0 < \alpha < 2$. Then there exists a constant $c = c(d, D)$, which is independent of α, and an extension operator $E : H^{\alpha/2}(D) \to H^{\alpha/2}(\mathbb{R}^d)$ with norm $\|E\| \le c$.

Furthermore, we will need the following Poincaré inequality; see [Ponce 2004].

Fact 3.2 (Poincaré I). Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let $0 < \alpha_0 \le \alpha < 2$. Then there exists a constant $c = c(d, \alpha_0, D)$, which is independent of α, such that

$$\left\| u - \frac{1}{|D|} \int_D u \,\mathrm{d}x \right\|^2_{L^2(D)} \le c \mathcal{E}^{\mu_\alpha}_D(u,u) \quad (u \in H^{\frac{\alpha}{2}}(D)). \tag{3-1}$$

The following results, Facts 3.3 and 3.4, are standard for fixed α. For $\alpha \to 2$ they follow from results in [Bourgain et al. 2001; Maz'ya and Shaposhnikova 2002; Ponce 2004]. They are established in the case when $B_r(x)$ denotes the cube of all $y \in \mathbb{R}^d$ such that $|y_i - x_i| < r$ for any $i \in \{1, \dots, d\}$. They hold true for balls likewise.

Fact 3.3 (Poincaré–Friedrichs). Assume $\alpha_0, \varepsilon > 0$ and $0 < \alpha_0 \le \alpha < 2$. There exists a constant c, which is independent of α, such that for $B_R = B_R(x_0)$

$$u \in H^{\frac{\alpha}{2}}(B_R), \quad |B_R \cap \{u = 0\}| \ge \varepsilon |B_R|$$

implies

$$\int_{B_R} (u(x))^2 \,\mathrm{d}x \le c R^\alpha \iint_{B_R B_R} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}} \,\mathrm{d}y \,\mathrm{d}x. \tag{3-2}$$

Fact 3.4 (Sobolev embedding). Assume $d \in \mathbb{N}$, $d \ge 2$, $R_0 > 0$, and $0 < \alpha_0 \le \alpha < 2$, $q \in [1, 2d/(d-\alpha)]$. Then there exists a constant c, which is independent of α, such that for $R \in (0, R_0)$ and $u \in H^{\alpha/2}(B_R)$

$$\left(\int_{B_R} |u(x)|^{\frac{2d}{d-\alpha}} \,\mathrm{d}x \right)^{\frac{d-\alpha}{d}} \le c \iint_{B_R B_R} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}} \,\mathrm{d}y \,\mathrm{d}x + c R^{-\alpha + \frac{d(q-2)}{q}} \left(\int_{B_R} |u(x)|^q \,\mathrm{d}x \right)^{\frac{2}{q}}.$$

We often make use of scaling and translations. Our main assumptions, conditions (A) and (B) assure a certain behavior of the family of measures μ with respect to the unit ball $B_1 \subset \mathbb{R}^d$. Let us formulate these conditions with respect to general balls $B_r(\xi) \subset \mathbb{R}^d$.

Given $\xi \in \mathbb{R}^d$, $r > 0$, $A \ge 1$, we say that μ satisfies (A; ξ, r) if:

For every ball $B_\rho(x_0)$ with $\rho \in (0, r)$, $x_0 \in B_r(\xi)$ and every $v \in H^{\alpha/2}(B_\rho(x_0))$,

$$A^{-1} \mathcal{E}^{\mu}_{B_\rho(x_0)}(v,v) \le \mathcal{E}^{\mu_\alpha}_{B_\rho(x_0)}(v,v) \le A \,\mathcal{E}^{\mu}_{B_\rho(x_0)}(v,v). \tag{A; ξ, r}$$

Given $\xi \in \mathbb{R}^d$, $r > 0$, $B \geq 1$, we say that μ satisfies (B; ξ, r) if:

$$\begin{aligned}&\text{For } 0 < \rho \leq R \leq r \text{ and } x_0 \in B_r(\xi) \text{ there is a nonnegative measurable function } \tau : \mathbb{R}^d \to \mathbb{R}\\ &\quad \text{with } \operatorname{supp}(\tau) \subset \overline{B_{R+\rho}(x_0)},\ \tau(x) \equiv 1 \text{ on } B_R(x_0),\ \|\tau\|_\infty \leq 1, \text{ and}\\ &\quad\quad \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \, \mu(x, dy) \leq B\rho^{-\alpha}.\end{aligned} \tag{B; ξ, r}$$

Let us explain how the operator under consideration behaves with respect to rescaled functions.

Lemma 3.5 (scaling property). *Assume $\xi \in \mathbb{R}^d$ and $r \in (0, 1)$. Let $u \in V^\mu(B_r(\xi) \mid \mathbb{R}^d)$ satisfy $\mathcal{E}^\mu(u, \phi) \geq (f, \phi)$ for every nonnegative $\phi \in H^\mu_{B_r(\xi)}(\mathbb{R}^d)$. Define a diffeomorphism J by $J(x) = rx + \xi$. Define rescaled versions $\tilde{f}, \tilde{u}$ of u and f by $\tilde{u}(x) = u(J(x))$ and $\tilde{f}$ by $\tilde{f}(x) = r^\alpha f(J(x))$.*

(1) *Then $\tilde{u}$ satisfies for all nonnegative $\phi \in H^{\tilde{\mu}}_{B_1}(\mathbb{R}^d)$*

$$\mathcal{E}^{\tilde{\mu}}(\tilde{u}, \phi) = \iint_{\mathbb{R}^d \mathbb{R}^d} (\tilde{u}(y) - \tilde{u}(x))(\phi(y) - \phi(x)) \, \tilde{\mu}(x, dy) \, dx \geq (\tilde{f}, \phi),$$

where

$$\tilde{\mu}(x, dy) = r^\alpha \mu_{J^{-1}}(J(x), dy) \quad \textit{and} \quad \mu_{J^{-1}}(z, A) = \mu(z, J(A)). \tag{3-3}$$

(2) *Assume μ satisfies conditions (A; ξ, r), (B; ξ, r) for some $\alpha \in (0, 2)$ and $A \geq 1$, $B \geq 1$, $\xi \in \mathbb{R}^d$, $r > 0$. Then the family of measures $\tilde{\mu} = \tilde{\mu}(\cdot, dy)$ satisfies assumptions (A) and (B) with the same constants.*

Remark. The condition (D) is affected by scaling in a noncritical way. We deal with this phenomenon further below in Section 4 and 5A.

Proof. For the proof of the first statement, let $\phi \in H^{\tilde{\mu}}_{B_1}(\mathbb{R}^d)$ be a nonnegative test function. Define $\phi_r \in H^\mu_{B_r(\xi)}(\mathbb{R}^d)$ by $\phi_r = \phi \circ J^{-1}$. Then

$$\begin{aligned}
&\iint (\tilde{u}(y) - \tilde{u}(x))(\phi(y) - \phi(x)) \, \tilde{\mu}(x, dy) \, dx \\
&\qquad = r^\alpha \iint (u(J(y)) - u(J(x)))(\phi_r(J(y)) - \phi_r(J(x))) \, \mu_{J^{-1}}(J(x), dy) \, dx \\
&\qquad = r^{\alpha - d} \iint (u(J(y)) - u(x))(\phi_r(J(y)) - \phi_r(x)) \, \mu_{J^{-1}}(x, dy) \, dx \\
&\qquad = r^{\alpha - d} \iint (u(y) - u(x))(\phi_r(y) - \phi_r(x)) \, \mu(x, dy) \, dx \\
&\qquad \geq r^{\alpha - d} \int f(x) \, \phi_r(x) \, dx = \int r^\alpha f(J(x)) \, \phi(x) \, dx = \int \tilde{f}(x) \phi(x) \, dx,
\end{aligned} \tag{3-4}$$

which is what we wanted to prove. Let us now prove that $\tilde{\mu}$ inherits properties (A), (B) from μ with the same constants A and B. Let us only consider the case $\xi = 0$. In order to verify condition (A) we need to consider an arbitrary ball $B_\rho(x_0)$ with $\rho \in (0, 1)$ and $x_0 \in B_1$. Let us simplify the situation further by assuming $x_0 = 0$. The general case can be proved analogously. Thus, we assume $r \in (0, 1)$ and $u \in H^{\alpha/2}(B_\rho)$. The estimate $\mathcal{E}^{\tilde{\mu}}_{B_\rho}(u, u) \leq A \mathcal{E}^{\mu_\alpha}_{B_\rho}(u, u)$ can be derived as follows. Define a function

$\hat{u} \in H^{\alpha/2}(B_{r\rho})$ by $\hat{u} = u \circ J^{-1}$. Then

$$\begin{aligned}
\mathcal{E}^{\tilde{\mu}}_{B_\rho}(u,u) &= \int_{B_\rho}\int_{B_\rho} (u(y)-u(x))^2\, \tilde{\mu}(x,\mathrm{d}y)\,\mathrm{d}x = r^\alpha \int_{B_\rho}\int_{B_\rho} (\hat{u}(J(y))-\hat{u}(J(x)))^2\, \mu_{J^{-1}}(J(x),\mathrm{d}y)\,\mathrm{d}x \\
&= r^{\alpha-d} \int_{B_{r\rho}}\int_{B_r} (\hat{u}(J(y))-\hat{u}(x))^2\, \mu_{J^{-1}}(x,\mathrm{d}y)\,\mathrm{d}x \\
&= r^{\alpha-d} \int_{B_{r\rho}}\int_{B_{r\rho}} (\hat{u}(y)-\hat{u}(x))^2\, \mu(x,\mathrm{d}y)\,\mathrm{d}x \le A r^{\alpha-d} \int_{B_{r\rho}}\int_{B_{r\rho}} \frac{(\hat{u}(y)-\hat{u}(x))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \\
&= A r^{-2d} \int_{B_{r\rho}}\int_{B_{r\rho}} \frac{(u(J^{-1}(y))-u(J^{-1}(x)))^2}{|J^{-1}(x)-J^{-1}(y)|^{d+\alpha}}\mathrm{d}y\,\mathrm{d}x = A \int_{B_\rho}\int_{B_\rho} \frac{(u(y)-u(x))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x,
\end{aligned}$$

which proves our claim. The estimate $\mathcal{E}^{\mu_\alpha}_{B_\rho}(u,u) \le A\,\mathcal{E}^{\tilde{\mu}}_{B_\rho}(u,u)$ follows in the same way.

In order to check condition (B) for $\tilde{\mu}$ we proceed as follows. Again, we assume $x_0 = 0$, $r \in (0,1)$. The general case can be proved analogously. Assume $R, \rho \in (0,1)$. Let $\hat{\tau} : \mathbb{R}^d \to \mathbb{R}$ satisfy $\mathrm{supp}(\hat{\tau}) \subset \overline{B}_{rR+r\rho}$, $\hat{\tau} \equiv 1$ on B_{rR} and

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y)-\hat{\tau}(x))^2\, \mu(x,\mathrm{d}y) \le B(r\rho)^{-\alpha} \quad \Longleftrightarrow \quad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y)-\hat{\tau}(J(x)))^2\, \mu(J(x),\mathrm{d}y) \le B(r\rho)^{-\alpha}.$$

Such a function $\hat{\tau}$ exists because, by assumption, μ satisfies (B; ξ, r). Next, define $\tau = \hat{\tau} \circ J$. Then τ satisfies $\mathrm{supp}(\tau) \subset \overline{B}_{R+\rho}$, $\tau \equiv 1$ on B_R, and, by a change of variables,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\, \tilde{\mu}(x,\mathrm{d}y) &= r^\alpha \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(J(y))-\hat{\tau}(J(x)))^2\, \mu_{J^{-1}}(J(x),\mathrm{d}y) \\
&= r^\alpha \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\hat{\tau}(y)-\hat{\tau}(J(x)))^2\, \mu(J(x),\mathrm{d}y) \le B\rho^{-\alpha},
\end{aligned}$$

which shows that $\tilde{\mu}$ satisfies (B) with the constant B. □

4. The weak Harnack inequality for nonlocal equations

The main aim of this section is to provide a proof of the weak Harnack inequality Theorem 1.6. The key result of this section is the corresponding result for supersolutions that are nonnegative in all of $\mathbb{R}^d$:

Theorem 4.1. *Assume $f : B_1 \to \mathbb{R}$ belongs to $L^{q/\alpha}(B_{15/16})$ for some $q \in (d,\infty]$, $\alpha \in [\alpha_0, 2)$. There are positive reals p_0, c such that for every $u \in V^\mu(B_1 \,|\, \mathbb{R}^d)$ with $u \ge 0$ in $\mathbb{R}^d$ satisfying*

$$\mathcal{E}(u,\phi) \ge (f,\phi) \quad \textit{for every nonnegative } \phi \in H^\mu_{B_1}(\mathbb{R}^d).$$

The following holds:

$$\inf_{B_{1/4}} u \ge c\left(\fint_{B_{1/2}} u(x)^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}} - \|f\|_{L^{q/\alpha}(B_{15/16})}.$$

The constants p_0, c depend only on d, α_0, A, B. They are independent of $\alpha \in [\alpha_0, 2)$.

Remark. All results in this section are robust with respect to $\alpha \in [\alpha_0, 2)$; i.e., constants do not depend on α.

The main application of this result is the following proof.

Proof of Theorem 1.6. Set $u = u^+ - u^-$. The assumptions imply for any nonnegative $\phi \in H^{\mu}_{B_1}(\mathbb{R}^d)$

$$\mathcal{E}(u^+, \phi) \geq \mathcal{E}(u^-, \phi) + (f, \phi) = \int_{B_1} \phi(x)\Big(f(x) - 2\int_{\mathbb{R}^d \setminus B_1} u^-(y)\,\mu(x, dy)\Big)\,dx;$$

i.e., u^+ satisfies all assumptions of Theorem 4.1 with $q = +\infty$ and $\tilde{f} : B_1 \to \mathbb{R}$ defined by

$$\tilde{f}(x) = f(x) - 2\int_{\mathbb{R}^d \setminus B_1} u^-(y)\,\mu(x, dy).$$

The assertion of the theorem is true if $\sup_{x \in B_{15/16}} \int_{\mathbb{R}^d \setminus B_1} u^-(y)\,\mu(x, dy)$ is infinite. Thus we can assume this quantity to be finite. Theorem 4.1 now implies

$$\inf_{B_{1/4}} u \geq c_1\Big(\fint_{B_{1/2}} u(x)^{p_0}\,dx\Big)^{\frac{1}{p_0}} - c_2 \sup_{x \in B_{15/16}}\Big(\int_{\mathbb{R}^d \setminus B_1} u^-(y)\,\mu(x, dy)\Big) - \|f\|_{L^{q/\alpha}(B_{15/16})}$$

for some positive constants c_1, c_2. □

By scaling and translation, we obtain the following corollary.

Corollary 4.2. *Let $x_0 \in \mathbb{R}^d$, $R \in (0, 1)$. Assume μ is a family of measures satisfying (A; ξ, r) and (B; ξ, r). Assume $u \in V^{\mu}(B_R(x_0) \,|\, \mathbb{R}^d)$ satisfies $u \geq 0$ in $B_R(x_0)$ and $\mathcal{E}(u, \phi) \geq 0$ for every nonnegative $\phi \in H^{\mu}_{B_R(x_0)}(\mathbb{R}^d)$. Then*

$$\inf_{B_{R/4}(x_0)} u \geq c\Big(\fint_{B_{R/2}(x_0)} u(x)^{p_0}\,dx\Big)^{\frac{1}{p_0}} - R^{\alpha} \sup_{x \in B_{15R/16}(x_0)} \int_{\mathbb{R}^d \setminus B_R(x_0)} u^-(y)\,\mu(x, dy),$$

with positive constants p_0, c which depend only on d, α_0, A, B. In particular, they are independent of $\alpha \in [\alpha_0, 2)$.

Let us proceed to the proof of Theorem 4.1.

Remark. Without further mentioning we assume that μ is a family of measures that satisfies (A) and (B) for some $A \geq 1$, $B \geq 1$, and $\alpha_0 \leq \alpha < 2$. The constants in the assertions below depend, among other things, on A, B, and α_0. They do not depend on α, though.

Let us first establish several auxiliary results. Our approach is closely related to the approach in [Kassmann 2009]. Instead of Lemma 2.5 in that paper, which would be sufficient for homogeneous equations,we will use the following auxiliary result.

Lemma 4.3. *There exist positive constants $c_1, c_2 > 0$ such that for every $a, b > 0$, $p > 1$, and $0 \leq \tau_1, \tau_2 \leq 1$ the following is true:*

$$(b - a)(\tau_1^2 a^{-p} - \tau_2^2 b^{-p}) \geq c_1(\tau_1 a^{\frac{-p+1}{2}} - \tau_2 b^{\frac{-p+1}{2}})^2 - \frac{c_2 p}{p-1}(\tau_1 - \tau_2)^2(b^{-p+1} + a^{-p+1}). \tag{4-1}$$

The above result is nothing but a discrete version of

$$(\nabla v, \nabla(\tau^2 v^{-p})) \geq c_1(p)|\nabla(\tau v^{\frac{-p+1}{2}})|^2 - c_2(p)|\nabla\tau|^2 v^{-p+1},$$

where v, τ are positive functions. We provide a detailed proof in the Appendix.

The next result is an extension of corresponding results in [Kassmann 2009; Barlow et al. 2009].

Lemma 4.4. *Assume $0 < \rho < r < 1$ and $z_0 \in B_1$. Set $B_r = B_r(z_0)$. Assume $f \in L^{q/\alpha}(B_{2r})$ for some $q > d$. Assume $u \in V^{\mu}(B_{2r} \mid \mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$\begin{aligned} \mathcal{E}(u,\phi) &\geq (f,\phi) \quad \text{for any nonnegative } \phi \in H^{\mu}_{B_{2r}}(\mathbb{R}^d), \\ u(x) &\geq \varepsilon \qquad\quad \text{for almost all } x \in B_{2r} \text{ and some } \varepsilon > 0. \end{aligned} \tag{4-2}$$

Then

$$\iint_{B_r B_r} \left(\sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\right) \mu(x, \mathrm{d}y)\,\mathrm{d}x \\ \leq c\rho^{-\alpha}|B_{r+\rho}| + \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{r+\rho})}\|\mathbb{1}\|_{L^{q/(q-\alpha)}(B_{r+\rho})}, \tag{4-3}$$

where $c > 0$ is independent of u, x_0, r, ρ, f, ε, α.

Note that for

$$\varepsilon \geq c_1(r+\rho)^{\delta}\|f\|_{L^{q/\alpha}(B_{r+\rho})},$$

with $\delta = \alpha((q-d)/q)$, one obtains

$$\iint_{B_r B_r} \left(\sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\right) \mu(x, \mathrm{d}y)\,\mathrm{d}x \leq c_2\rho^{-\alpha}|B_{r+\rho}|. \tag{4-4}$$

From the above lemma we will deduce $\log u \in \mathrm{BMO}(B_1)$, where $\mathrm{BMO}(B_1)$ contains all functions of bounded mean oscillations in B_1; see [John and Nirenberg 1961].

Proof. The proof uses several ideas developed in [Barlow et al. 2009]. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be a function according to (B); i.e., more precisely we assume

$$\operatorname{supp}(\tau) \subset \overline{B}_{r+\rho} \subset B_{2r}, \quad \|\tau\|_\infty \leq 1, \quad \tau \equiv 1 \text{ on } B_r, \quad \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y) \leq B\rho^{-\alpha}.$$

Then

$$\begin{aligned} \iint_{\mathbb{R}^d \mathbb{R}^d} &(\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &= \iint_{B_{r+\rho} B_{r+\rho}} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + 2\iint_{B_{r+\rho} B^c_{r+\rho}} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\leq 2\iint_{B_{r+\rho}\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\leq 2|B_{r+\rho}| \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y) \leq 2c\rho^{-\alpha}|B_{r+\rho}|. \end{aligned} \tag{4-5}$$

We choose $\phi(x) = -\tau^2(x)\, u^{-1}(x)$ as a test function. Denote $B_{r+\rho}$ by B. We obtain

$$
\begin{aligned}
(f,\phi) \ge& \iint_{\mathbb{R}^d\,\mathbb{R}^d} (u(y)-u(x))(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
=& \iint_{BB} \tau(x)\,\tau(y)\Big(\frac{\tau(x)\,u(y)}{\tau(y)\,u(x)}+\frac{\tau(y)\,u(x)}{\tau(x)\,u(y)}-\frac{\tau(y)}{\tau(x)}-\frac{\tau(x)}{\tau(y)}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&+2\iint_{BB^c} (u(y)-u(x))(\tau^2(x)\,u^{-1}(x)-\tau^2(y)\,u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&+\iint_{B^cB^c} (u(y)-u(x))(\tau^2(x)\,u^{-1}(x)-\tau^2(y)\,u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x. \qquad (4\text{-}6)
\end{aligned}
$$

Setting $A(x,y) = u(y)/u(x)$ and $B(x,y) = \tau(y)/\tau(x)$ we obtain

$$
\begin{aligned}
&\iint_{BB} \tau(x)\,\tau(y)\Big(\frac{A(x,y)}{B(x,y)}+\frac{B(x,y)}{A(x,y)}-B(x,y)-\frac{1}{B(x,y)}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad= \iint_{BB} \tau(x)\,\tau(y)\Big[\Big(\frac{A(x,y)}{B(x,y)}+\frac{B(x,y)}{A(x,y)}-2\Big)-\Big(\sqrt{B(x,y)}-\frac{1}{\sqrt{B(x,y)}}\Big)^2\Big]\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad= \iint_{BB} \tau(x)\,\tau(y)\Big(2\sum_{k=1}^{\infty}\frac{(\log A(x,y)-\log B(x,y))^{2k}}{(2k)!}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\qquad -\iint_{BB} \tau(x)\,\tau(y)\Big(\sqrt{B(x,y)}-\frac{1}{\sqrt{B(x,y)}}\Big)^2\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad= \iint_{BB} \tau(x)\,\tau(y)\Big(2\sum_{k=1}^{\infty}\frac{(\log(u(y)/\tau(y))-\log(u(x)/\tau(x)))^{2k}}{(2k)!}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\qquad\qquad -\iint_{BB} (\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad\ge \int_{B_r}\int_{B_r}\Big(2\sum_{k=1}^{\infty}\frac{(\log u(y)-\log u(x))^{2k}}{(2k)!}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x-\iint_{BB}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x,
\end{aligned}
$$

where we applied (4-5) and the fact that for positive real a, b

$$
\frac{(a-b)^2}{ab} = (a-b)(b^{-1}-a^{-1}) = 2\sum_{k=1}^{\infty}\frac{(\log a-\log b)^{2k}}{(2k)!}. \qquad (4\text{-}7)
$$

Altogether, we obtain

$$
\begin{aligned}
(f,\phi) \ge& \int_{B_r}\int_{B_r}\Big(2\sum_{k=1}^{\infty}\frac{(\log u(y)-\log u(x))^{2k}}{(2k)!}\Big)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x-\iint_{BB}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&+2\iint_{B_{r+\rho}B^c_{r+\rho}} (u(y)-u(x))(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x. \qquad (4\text{-}8)
\end{aligned}
$$

The third term on the right-hand side can be estimated as follows:

$$
\begin{aligned}
2 \iint_{B_{r+\rho} B^c_{r+\rho}} & (u(y)-u(x))(\tau^2(x)u^{-1}(x)-\tau^2(y)u^{-1}(y))\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= 2 \iint_{B_{r+\rho} B^c_{r+\rho}} (u(y)-u(x))\tau^2(x)u^{-1}(x)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&= 2 \int_{B_{r+\rho}} \int_{B^c_{r+\rho}} \frac{\tau^2(x)}{u(x)} u(y)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x - 2 \int_{B_{r+\rho}} \int_{B^c_{r+\rho}} \tau^2(x)\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq -2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y)-\tau(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x,
\end{aligned}
$$

where we used nonnegativity of u in $\mathbb{R}^d$. Therefore,

$$
\begin{aligned}
\int_{B_r} \int_{B_r} & \left(2 \sum_{k=1}^{\infty} \frac{(\log u(y) - \log u(x))^{2k}}{(2k)!}\right) \mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\leq 3 \iint_{\mathbb{R}^d \mathbb{R}^d} (\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + \|f\|_{L^{q/\alpha}(B_{r+\rho})} \|u^{-1}\|_{L^{q/(q-\alpha)}(B_{r+\rho})}. \qquad (4\text{-}9)
\end{aligned}
$$

The proof is complete after the trivial observation $|u^{-1}| \leq \varepsilon^{-1}$. □

Lemma 4.5. *Assume $0 < R < 1$ and $f \in L^{q/\alpha}(B_{5R/4})$ for some $q > d$. Assume $u \in V^{\mu}(B_{5R/4} \,|\, \mathbb{R}^d)$ is nonnegative in $\mathbb{R}^d$ and satisfies*

$$
\begin{aligned}
\mathcal{E}(u,\phi) &\geq (f,\phi) \quad \text{for any nonnegative } \phi \in H^{\mu}_{B_{5R/4}}(\mathbb{R}^d), \\
u(x) &\geq \varepsilon \quad \text{for almost all } x \in B_{\frac{5R}{4}} \text{ and some } \varepsilon > \tfrac{1}{4} R^{\delta} \|f\|_{L^{q/\alpha}(B_{9R/8})},
\end{aligned}
$$

where

$$
\delta = \alpha\left(\frac{q-d}{q}\right).
$$

Then there exist $\bar{p} \in (0,1)$ and $c > 0$ such that

$$
\left(\fint_{B_R} u(x)^{\bar{p}}\,\mathrm{d}x\right)^{\frac{1}{\bar{p}}} \mathrm{d}x \leq c \left(\fint_{B_R} u(x)^{-\bar{p}}\,\mathrm{d}x\right)^{-\frac{1}{\bar{p}}}, \qquad (4\text{-}10)
$$

where c and $\bar{p}$ are independent of x_0, R, u, ε, and α.

Proof. The main idea is to prove $\log u \in \mathrm{BMO}(B_R)$. Choose $z_0 \in B_R$ and $r > 0$ such that $B_{2r}(z_0) \subset B_{R/8}$. Set $\rho = r$. Lemma 4.4 and Assumption (A) imply

$$
\int_{B_r(z_0)} \int_{B_r(z_0)} \frac{(\log u(y) - \log u(x))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_{B_r(z_0)} \int_{B_r(z_0)} (\log u(y) - \log u(x))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \leq c_1 r^{d-\alpha}.
$$

Application of the Poincaré inequality, Fact 3.2, and the scaling property 3.3 leads to

$$
\int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}|^2\,\mathrm{d}x \leq c_2 r^d, \qquad (4\text{-}11)
$$

where

$$[\log u]_{B_r(z_0)} = |B_r(z_0)|^{-1} \int_{B_r(z_0)} \log u = \fint_{B_r(z_0)} \log u.$$

From here

$$\int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}| \,\mathrm{d}x \le \left(\int_{B_r(z_0)} |\log u(x) - [\log u]_{B_r(z_0)}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} |B_r(z_0)|^{\frac{1}{2}} \le c_3 r^d.$$

An application of the John–Nirenberg embedding, see [Gilbarg and Trudinger 1998, Chapter 7.8], then gives

$$\int_{B_R} e^{\bar{p}|\log u(y) - [\log u]_{B_r}|} \,\mathrm{d}y \le c_4 R^d,$$

where $\bar{p}$ and c_4 depend only on d and c_3. One obtains

$$\left(\int_{B_R} u(y)^{\bar{p}} \,\mathrm{d}y\right)\left(\int_{B_R} u(y)^{-\bar{p}} \,\mathrm{d}y\right) = \left(\int_{B_R} e^{\bar{p}(\log u(y) - [\log u]_{B_r})} \,\mathrm{d}y\right)\left(\int_{B_R} e^{-\bar{p}(\log u(y) - [\log u]_{B_r})} \,\mathrm{d}y\right)$$
$$\le c_4^2 R^{2d}.$$

The above inequality proves assertion (4-10). □

The next result allows us to apply Moser's iteration for negative exponents. It is a purely local result although the Dirichlet form is nonlocal.

Lemma 4.6. *Assume $x_0 \in B_1$ and $0 < 8\rho < R < 1 - \rho$. Set $B_R = B_R(x_0)$. Let $f \in L^{q/\alpha}(B_{5R/4})$ for some $q > d$. Assume $u \in V^\mu(B_{5R/4} \,|\, \mathbb{R}^d)$ is nonnegative on all of $\mathbb{R}^d$ and satisfies*

$$\begin{aligned} \mathcal{E}(u,\phi) &\ge (f,\phi) \quad \text{for any nonnegative } \phi \in H^\mu_{B_{5R/4}}(\mathbb{R}^d), \\ u(x) &\ge \varepsilon \quad \text{for almost all } x \in B_{\frac{9R}{8}} \text{ and some } \varepsilon > R^\delta \|f\|_{L^{q/\alpha}(B_{9R/8})}, \end{aligned}$$

where

$$\delta = \alpha\left(\frac{q-d}{q}\right).$$

Then for $p > 1$

$$\|u^{-1}\|^{p-1}_{L^{(p-1)d/(d-\alpha)}(B_R)} \le c\left(\frac{p}{p-1}\right)\rho^{-\alpha}\|u^{-1}\|^{p-1}_{L^{p-1}(B_{R+\rho})}, \tag{4-12}$$

where $c > 0$ is independent of u, x_0, R, ρ, p, ε, and α.

Proof. Let $\tau : \mathbb{R}^d \to \mathbb{R}$ be a function according to assumption (B); i.e.,

$$\begin{gathered} S := \operatorname{supp}(\tau) \subset \bar{B}_{R+\rho} \subset B_{\frac{9R}{8}}, \\ \|\tau\|_\infty \le 1 \quad \text{for all } x \in B_R \text{ such that } \tau(x) = 1, \\ \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\tau(y) - \tau(x))^2 \,\mu(x, \mathrm{d}y) \le B\rho^{-\alpha}. \end{gathered}$$

The assumptions of the lemma imply

$$\mathcal{E}(u, -\tau^2 u^{-p}) \le (f, -\tau^2 u^{-p}).$$

Let us observe the following:

$$\begin{aligned}\mathcal{E}(u,-\tau^2u^{-p}) &= \iint (u(y)-u(x))(\tau^2(x)\,u(x)^{-p}-\tau^2(y)\,u(y)^{-p})\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &= \int_S\int_S (u(y)-u(x))(\tau^2(x)\,u(x)^{-p}-\tau^2(y)\,u(y)^{-p})\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\qquad + 2\int_S\int_{\mathbb{R}^d\setminus S}(u(y)-u(x))(\tau(x)-\tau(y))^2\,u(x)^{-p}\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\geq \int_S\int_S (u(y)-u(x))(\tau^2(x)\,u(x)^{-p}-\tau^2(y)\,u(y)^{-p})\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\qquad - 2\int_S u(x)^{1-p}\int_{\mathbb{R}^d\setminus S}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x.\end{aligned}$$

The last term is finite because of our assumptions on τ. However, note that $\tau(y)=0$ for $y\in\mathbb{R}^d\setminus S$. Next, we choose $a=u(x)$, $b=u(y)$, $\tau_1=\tau(x)$, $\tau_2=\tau(y)$, and apply Lemma 4.3 to the integrand in the first term. Then

$$\begin{aligned}&\iint_{SS}(\tau(y)\,u(y)^{\frac{-p+1}{2}}-\tau(x)\,u(x)^{\frac{-p+1}{2}})^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\quad\leq \frac{cp}{p-1}\iint_{SS}(\tau(y)-\tau(x))^2(u(y)^{-p+1}+u(x)^{-p+1})\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\qquad\qquad + 2\int_S u(x)^{-p+1}\int_{\mathbb{R}^d\setminus S}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + c(f,-\tau^2u^{-p}) \\ &\quad\leq \left(\frac{2cp}{p-1}+2\right)\int_S u(x)^{-p+1}\int_{\mathbb{R}^d}(\tau(x)-\tau(y))^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x + c(f,-\tau^2u^{-p}) \\ &\quad\leq c_1(p)\rho^{-\alpha}\int_{B_{R+\rho}}u(x)^{-p+1}\,\mathrm{d}x + c(f,-\tau^2u^{-p}) \qquad (4\text{-}13)\end{aligned}$$

for some positive constant c, which is independent of p, R, f and u. It remains to estimate $|(f,-\tau^2u^{-p})|$ from above:

$$\begin{aligned}|(f,-\tau^2u^{-p})| &\leq \varepsilon^{-1}|(f,\tau^2u^{-p+1})| \leq \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{9R/8})}\|\tau^2u^{-p+1}\|_{L^{q/(q-\alpha)}(B_{9R/8})} \\ &= \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{9R/8})}\|\tau u^{\frac{-p+1}{2}}\|^2_{L^{2q/(q-\alpha)}(B_{9R/8})} \\ &\leq \varepsilon^{-1}\|f\|_{L^{q/\alpha}(B_{9R/8})}\{a\|\tau u^{\frac{-p+1}{2}}\|^2_{L^{2d/(d-\alpha)}(B_{9R/8})} + a^{-\frac{d}{q-d}}\|\tau u^{\frac{-p+1}{2}}\|^2_{L^2(B_{9R/8})}\} \\ &\leq R^{-\alpha\frac{q-d}{q}}a\|\tau^2u^{-p+1}\|_{L^{d/(d-\alpha)}(B_{9R/8})} + R^{-\alpha\frac{q-d}{q}}a^{-\frac{d}{q-d}}\|\tau^2u^{-p+1}\|_{L^1(B_{9R/8})},\end{aligned}$$

where $a>0$ is arbitrary. We choose $a=\omega R^{\alpha(q-d)/q}$ for some ω and obtain

$$|(f,-\tau^2u^{-p})| \leq \omega\|\tau^2u^{-p+1}\|_{L^{d/(d-\alpha)}(B_{9R/8})} + \omega^{-\frac{d}{q-d}}R^{-\alpha}\|\tau^2u^{-p+1}\|_{L^1(B_{9R/8})}.$$

Combining these estimates we obtain from (4-13) for every $p>1$ and every $\omega>0$

$$\begin{aligned}&\iint_{SS}(\tau(y)\,u(y)^{\frac{-p+1}{2}}-\tau(x)\,u(x)^{\frac{-p+1}{2}})^2\,\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &\qquad\qquad \leq c_3(p,\omega)\rho^{-\alpha}\int_{B_{R+\rho}}u(x)^{-p+1}\,\mathrm{d}x + c\omega\|\tau^2u^{-p+1}\|_{L^{d/(d-\alpha)}(B_{R+\rho})}.\end{aligned}$$

Next, we use Assumption (A) and apply the Sobolev inequality, Fact 3.4, to the left-hand side. Choosing ω small enough and subtracting the term $c\omega\|\tau^2 u^{-p+1}\|_{L^{d/(d-\alpha)}(B_{R+\rho})}$ from both sides, we prove the assertion of the lemma. □

Lemma 4.6 provides us with an estimate which can be iterated. As a result of this iteration we obtain the following corollary.

Corollary 4.7. *Assume $x_0 \in B_1$, $0 < R < \frac{1}{2}$, and $0 < \eta < 1 < \Theta$. Let $f \in L^{q/\alpha}(B_{\Theta R})$ for some $q > d$. Assume $u \in V^{\mu}(B_{\Theta R} \,|\, \mathbb{R}^d)$ satisfies*

$$\begin{aligned} \mathcal{E}(u,\phi) &\geq (f,\phi) && \textit{for any nonnegative } \phi \in H^{\mu}_{B_{\Theta R}}(\mathbb{R}^d), \\ u(x) &\geq \varepsilon && \textit{for almost all } x \in B_{\Theta R} \textit{ and some } \varepsilon > (\Theta R)^{\delta}\|f\|_{L^{q/\alpha}(B_{R(1+3\Theta)/4})}, \end{aligned}$$

where

$$\delta = \alpha\left(\frac{q-d}{q}\right).$$

Then for any $p_0 > 0$

$$\inf_{x \in B_{\eta R}} u(x) \geq c\left(\fint_{B_R} u(x)^{-p_0}\,\mathrm{d}x\right)^{-\frac{1}{p_0}}, \tag{4-14}$$

where $c > 0$ is independent of u, R, ε, and α.

Proof. The idea of the proof is to apply Lemma 4.6 to radii R_k, ρ_k with $R_k \searrow \eta R$ and $\rho_k \searrow 0$ for $k \to \infty$. For each k one chooses an exponent $p_k > 1$ with $p_k \to \infty$ for $k \to \infty$. Because of Assumption (A) we can apply the Sobolev inequality, Fact 3.4, to the left-hand side in (4-12). Next, one iterates the resulting inequality as in [Moser 1961]; see also Chapter 8.6 in [Gilbarg and Trudinger 1998]. The only difference to the proof in [Moser 1961] is that the factor $d/(d-2)$ now becomes $d/(d-\alpha)$. The assertion then follows from the fact

$$\left(\fint_{B_{R_k}(x_0)} u^{-p_k}\right)^{-\frac{1}{p_k}} \to \inf_{B_{\eta R}(x_0)} u \quad \text{for } k \to \infty.$$ □

Let us finally prove Theorem 4.1.

Proof of Theorem 4.1. Define $\bar{u} = u + \|f\|_{L^{q/\alpha}(B_{15/16})}$ and note that $\mathcal{E}(u,\phi) = \mathcal{E}(\bar{u},\phi)$ for every ϕ. We apply Lemma 4.5 for $R = \frac{3}{4}$ and obtain that there exist $\bar{p} \in (0,1)$ and $c > 0$ such that

$$\left(\fint_{B_{3/4}} \bar{u}(x)^{\bar{p}}\,\mathrm{d}x\right)^{\frac{1}{\bar{p}}} \mathrm{d}x \leq c\left(\fint_{B_{3/4}} u(x)^{-\bar{p}}\,\mathrm{d}x\right)^{-\frac{1}{\bar{p}}}.$$

Next, we apply Corollary 4.7 with $R = \frac{3}{4}$, $\eta = \frac{2}{3}$ and $\Theta = \frac{5}{4}$. Together with the estimate from above we obtain

$$\inf_{B_{1/2}} u \geq c\left(\frac{1}{|B_{\frac{3}{4}}|}\int_{B_{3/4}} \bar{u}(x)^{\bar{p}}\,\mathrm{d}x\right)^{\frac{1}{\bar{p}}}, \tag{4-15}$$

which, after recalling the definition of $\bar{u}$, proves Theorem 4.1. □

5. The weak Harnack inequality implies Hölder estimates

The aim of this section is to provide the proof of Theorem 1.10. As is explained in Section 1D it is well known that both the Harnack inequality and the weak Harnack inequality imply regularity estimates in Hölder spaces. Here we are going to establish such a result for quite general nonlocal operators in the framework of metric measure spaces.

We begin with a short study of condition (1-16). The standard example that we have in mind is given in Example 1.9. Let (X, d, m) be a metric measure space. For $R > r > 0,\ x \in X$, set

$$B_r(x) = \{y \in X : d(y, x) < r\}, \quad A_{r,R}(x) = B_R(x) \setminus B_r(x). \tag{5-1}$$

Lemma 5.1. *For $x \in X$, $r > 0$ let $\nu_{x,r}$ be a measure on $\mathcal{B}(X \setminus \{x\})$, which is finite on all sets M with* $\operatorname{dist}(\{x\}, M) > 0$. *Then the following conditions are equivalent*:

(1) *For some $\chi > 1,\ c \geq 1$ and all $x \in X,\ 0 < r \leq 1,\ j \in \mathbb{N}_0$*

$$\nu_{x,r}(X \setminus B_{r2^j}(x)) \leq c\chi^{-j}.$$

(2) *Given $\theta > 1$, there are $\chi > 1,\ c \geq 1$ such that for all $x \in X,\ 0 < r \leq 1,\ j \in \mathbb{N}_0$*

$$\nu_{x,r}(X \setminus B_{r\theta^j}(x)) \leq c\chi^{-j}.$$

(3) *Given $\theta > 1$, there are $\chi > 1,\ c \geq 1$ such that for all $x \in X,\ 0 < r \leq 1,\ j \in \mathbb{N}_0$*

$$\nu_{x,r}(A_{r\theta^j, r\theta^{j+1}}(x)) \leq c\chi^{-j}.$$

(4) *Given $\sigma > 1,\ \theta > 1$ there are $\chi > 1,\ c \geq 1$ such that for all $x \in X,\ 0 < r \leq 1,\ j \in \mathbb{N}_0$ and $y \in B_{r/\sigma}(x)$*

$$\nu_{y,r'}(A_{r\theta^j, r\theta^{j+1}}(x)) \leq c\chi^{-j}, \quad \text{where } r' = r\left(1 - \frac{1}{\sigma}\right). \tag{5-2}$$

If, in addition to any of the above conditions, (1-17) *holds, then* (5-2) *can be replaced by*

$$\nu_{y,r}(A_{r\theta^j, r\theta^{j+1}}(x)) \leq c\chi^{-j}. \tag{5-3}$$

Proof. If $\theta \geq 2$, the implication (1) $\Rightarrow$ (2) trivially holds true. For $\theta < 2$ it can be obtained by adjusting χ appropriately. The proof of (2) $\Rightarrow$ (1) is analogous. The implication (2) $\Rightarrow$ (3) trivially holds true. The implication (3) $\Rightarrow$ (2) follows from

$$\nu_{x,r}(X \setminus B_{r\theta^j}(x)) = \sum_{k=j}^{\infty} \nu_{x,r}(A_{r\theta^k, r\theta^{k+1}}(x)) \leq c \sum_{k=j}^{\infty} \chi^{-k} = c\left(\frac{\chi}{\chi - 1}\right)\chi^{-j}.$$

The implication (4) $\Rightarrow$ (3) trivially holds true. Instead of (3) $\Rightarrow$ (4) we explain the proof of (2) $\Rightarrow$ (4). Fix $\sigma > 1,\ \theta > 1,\ x \in X,\ r > 0,\ j \in \mathbb{N}_0$, and $y \in B_{r/\sigma}(x)$. Set $r' = r(1 - 1/\sigma)$. Then $X \setminus B_{r\theta^j}(x) \subset X \setminus B_{r'\theta^j}(y)$. Thus

$$\nu_{y,r'}(X \setminus B_{r\theta^j}(x)) \leq \nu_{y,r'}(X \setminus B_{r'\theta^j}(y)) \leq c\chi^{-j}. \qquad \square$$

Remark. Note that the conditions above imply that, given $j \in \mathbb{N}_0$ and $x \in X$, the quantity

$$\limsup_{r \to 0+} \nu_{x,r}(X \setminus B_{r2^j}(x))$$

is finite.

Remark. Let $x \in X$, $A \in \mathcal{B}(X \setminus \{x\})$, with $\operatorname{dist}(\{x\}, A) > 0$. In the applications that are of interest to us, the function $r \mapsto \nu_{x,r}(A)$ is strictly increasing with $\nu_{x,0}(A) = 0$.

Proof of Theorem 1.10. The proof follows closely the strategy of [Moser 1961]; see also [Silvestre 2006]. Throughout the proof, let us write B_t instead of $B_t(x_0)$ for $t > 0$. Fix $r \in (0, r_0)$ and $u \in \mathcal{S}_{x_0,r}$. Let $c_1 \geq 1$ be the constant in (1-19). Set $\kappa = (2c_1 2^{1/p})^{-1}$ and

$$\beta = \frac{\ln(2/(2-\kappa))}{\ln(\theta)} \quad \Longrightarrow \quad \left(1 - \tfrac{1}{2}\kappa\right) = \theta^{-\beta}.$$

Set $M_0 = \|u\|_\infty$, $m_0 = \inf_X u(x)$, and $M_{-n} = M_0$, $m_{-n} = m_0$ for $n \in \mathbb{N}$. We will construct an increasing sequence (m_n) and a decreasing sequence (M_n) such that for $n \in \mathbb{Z}$

$$\begin{aligned} m_n \leq u(z) \leq M_n \quad & \text{for almost all } z \in B_{r\theta^{-n}}, \\ M_n - m_n \leq K\theta^{-n\beta}&, \end{aligned} \tag{5-4}$$

where $K = M_0 - m_0 \in [0, 2\|u\|_\infty]$. Assume there is $k \in \mathbb{N}$ and there are M_n, m_n such that (5-4) holds for $n \leq k-1$. We need to choose m_k, M_k such that (5-4) still holds for $n = k$. Then the assertion of the lemma follows by complete induction. For $z \in X$ set

$$v(z) = \left(u(z) - \tfrac{1}{2}(M_{k-1} + m_{k-1})\right)\frac{2\theta^{(k-1)\beta}}{K}.$$

The definition of v implies $v \in \mathcal{S}_{x_0,r}$ and $|v(z)| \leq 1$ for almost any $z \in B_{r\theta^{-(k-1)}}$. Our next aim is to show that (1-19) implies that either $v \leq 1-\kappa$ or $v \geq -1+\kappa$ on $B_{r\theta^{-k}}$. Since our version of the Harnack inequality contains nonlocal terms we need to investigate the behavior of v outside of $B_{r\theta^{-(k-1)}}$. Given $z \in X$ with $d(z, x_0) \geq r\theta^{-(k-1)}$ there is $j \in \mathbb{N}$ such that

$$r\theta^{-k+j} \leq d(z, x_0) < r\theta^{-k+j+1}.$$

For such z and j we conclude

$$\begin{aligned} \frac{K}{2\theta^{(k-1)\beta}} v(z) &= \left(u(z) - \tfrac{1}{2}(M_{k-1} + m_{k-1})\right) \\ &\leq \left(M_{k-j-1} - m_{k-j-1} + m_{k-j-1} - \tfrac{1}{2}(M_{k-1} + m_{k-1})\right) \\ &\leq \left(M_{k-j-1} - m_{k-j-1} - \tfrac{1}{2}(M_{k-1} - m_{k-1})\right) \leq \left(K\theta^{-(k-j-1)\beta} - \tfrac{1}{2}K\theta^{-(k-1)\beta}\right), \end{aligned}$$

that is,

$$v(z) \leq 2\theta^{j\beta} - 1 \leq 2\left(\theta \frac{d(z, x_0)}{r\theta^{-(k-1)}}\right)^\beta - 1, \tag{5-5}$$

and

$$\begin{aligned}\frac{K}{2\theta^{(k-1)\beta}}v(z) &= \big(u(z)-\tfrac{1}{2}(M_{k-1}+m_{k-1})\big)\\ &\geq \big(m_{k-j-1}-M_{k-j-1}+M_{k-j-1}-\tfrac{1}{2}(M_{k-1}+m_{k-1})\big)\\ &\geq \big(-(M_{k-j-1}-m_{k-j-1})+\tfrac{1}{2}(M_{k-1}-m_{k-1})\big)\geq \big(-K\theta^{-(k-j-1)\beta}+\tfrac{1}{2}K\theta^{-(k-1)\beta}\big),\end{aligned}$$

that is,

$$v(z)\geq 1-2\theta^{j\beta}\geq 1-2\bigg(\theta\frac{d(z,x_0)}{r\theta^{-(k-1)}}\bigg)^{\beta}.$$

Now there are two cases:

Case 1: $m(\{x\in B_{r\theta^{-k+1}/\lambda}:v(x)\leq 0\})\geq \frac{1}{2}m(B_{r\theta^{-k+1}/\lambda})$.

Case 2: $m(\{x\in B_{r\theta^{-k+1}/\lambda}:v(x)> 0\})\geq \frac{1}{2}m(B_{r\theta^{-k+1}/\lambda})$.

We work out details for Case 1 and comment afterwards on Case 2. In Case 1 our aim is to show $v(z)\leq 1-\kappa$ for almost every $z\in B_{r\theta^{-k}}$ and some $\kappa\in(0,1)$. Because then for almost any $z\in B_{r\theta^{-k}}$

$$\begin{aligned}u(z)&\leq \tfrac{1}{2}(1-\kappa)K\theta^{-(k-1)\beta}+\tfrac{1}{2}(M_{k-1}+m_{k-1})\\ &=\tfrac{1}{2}(1-\kappa)K\theta^{-(k-1)\beta}+\tfrac{1}{2}(M_{k-1}-m_{k-1})+m_{k-1}\\ &=m_{k-1}+\tfrac{1}{2}(1-\kappa)K\theta^{-(k-1)\beta}+\tfrac{1}{2}K\theta^{-(k-1)\beta}\\ &\leq m_{k-1}+K\theta^{-k\beta}.\end{aligned}\tag{5-6}$$

We then set $m_k=m_{k-1}$ and $M_k=m_k+K\theta^{-k\beta}$ and obtain, using (5-6), $m_k\leq u(z)\leq M_k$ for almost every $z\in B_{r\theta^{-k}}$, which is what needs to be proved.

Consider $w=1-v$ and note $w\in\mathcal{S}_{x_0,r\theta^{-(k-1)}}$ and $w\geq 0$ in $B_{r\theta^{-(k-1)}}$. We apply (1-19) and obtain

$$\bigg(\fint_{B_{r\theta^{-k+1}/\lambda}(x_0)} w^p\,\mathrm{d}m\bigg)^{\frac{1}{p}}\leq c_1\inf_{B_{r\theta^{-k}}} w+c_1\sup_{x\in B_{r\theta^{-k+1}/\sigma}}\int_X w^-(z)\nu_{x,r\theta^{-(k-1)}}(\mathrm{d}z),\tag{5-7}$$

In Case 1 the left-hand side of (5-7) is bounded from below by $\left(\frac{1}{2}\right)^{1/p}$. This, together with the estimate (5-5) on v from above, leads to

$$\begin{aligned}\inf_{B_{r\theta^{-k}}} w&\geq (c_1 2^{\frac{1}{p}})^{-1}-\sup_{x\in B_{r\theta^{-k+1}/\sigma}}\int_X w^-(z)\nu_{x,r\theta^{-(k-1)}}(\mathrm{d}z)\\ &\geq (c_1 2^{\frac{1}{p}})^{-1}-\sum_{j=1}^{\infty}\sup_{x\in B_{r\theta^{-k+1}/\sigma}}\int \mathbb{1}_{A_{r\theta^{-k+j},r\theta^{-k+j+1}}(x_0)}(1-v(z))^-\nu_{x,r\theta^{-(k-1)}}(\mathrm{d}z)\\ &\geq (c_1 2^{\frac{1}{p}})^{-1}-\sum_{j=1}^{\infty}(2\theta^{j\beta}-2)\eta_{x_0,r,\theta,j,k},\end{aligned}$$

where

$$\eta_{x_0,r,\theta,j,k}=\sup_{x\in B_{r\theta^{-k+1}/\sigma}}\nu_{x,r\theta^{-(k-1)}}(A_{r\theta^{-k+j},r\theta^{-k+j+1}}(x_0)).$$

Now, (5-3) implies that $\eta_{x_0,r,\theta,j,k} \le c\chi^{-j-1}$. Thus we obtain

$$\inf_{B_{r\theta^{-k}}} w \ge (c_1 2^{\frac{1}{p}})^{-1} - 2c \sum_{j=1}^{\infty} (\theta^{j\beta} - 1)\chi^{-j-1}. \tag{5-8}$$

Note that $\sum_{j=1}^{\infty} \theta^{j\beta} \chi^{-j-1} < \infty$ for $\beta > 0$ small enough; i.e., there is $l \in \mathbb{N}$ with

$$\sum_{j=l+1}^{\infty} (\theta^{j\beta} - 1)\chi^{-j-1} \le \sum_{j=l+1}^{\infty} \theta^{j\beta}\chi^{-j-1} \le (16c_1)^{-1}.$$

Given l we choose $\beta > 0$ smaller (if needed) in order to ensure

$$\sum_{j=1}^{l} (\theta^{j\beta} - 1)\chi^{-j-1} \le (16c_1)^{-1}.$$

The number β depends only on c_1, c, χ from (5-3) and on θ. Thus we have shown that $w \ge \kappa$ on $B_{r\theta^{-k}}$ or, equivalently, $v \le 1 - \kappa$ on $B_{r\theta^{-k}}$.

In Case 2 our aim is to show $v(x) \ge -1 + \kappa$. This time, set $w = 1 + v$. Following the strategy above one sets $M_k = M_{k-1}$ and $m_k = M_k - K\theta^{-k\beta}$ leading to the desired result.

Let us show how (5-4) proves the assertion of the lemma. Given $\rho \le r$, there exists $j \in \mathbb{N}_0$ such that

$$r\theta^{-j-1} \le \rho \le r\theta^{-j}.$$

From (5-4) we conclude

$$\operatorname{osc}_{B_\rho} u \le \operatorname{osc}_{B_{r\theta^{-j}}} u \le M_j - m_j \le 2\theta^{\beta}\|u\|_\infty \left(\frac{\rho}{r}\right)^{\beta}. \qquad \square$$

Corollary 5.2. *Let $\Omega = B_{r_0}(x_0) \subset X$ and let $\sigma, \theta, \lambda > 1$. Let $\mathcal{S}_{x,r}$ and $\nu_{x,r}$ be as above. Assume that conditions (1-16), (1-17) are satisfied. Assume that there is $c \ge 1$ such that for $0 < r \le r_0$*

$$(B_r(x) \subset \Omega) \wedge (u \in x, r) \wedge (u \ge 0 \text{ in } B_r(x))$$

$$\Longrightarrow \left(\fint_{B_{r/\lambda}(x)} u(\xi)^p \, m(d\xi) \right)^{\frac{1}{p}} \le c \inf_{B_{r/\theta}(x)} u + c \sup_{\xi \in B_{r/\sigma}(x)} \int_X u^-(z)\, \nu_{\xi,r}(dz). \tag{5-9}$$

Then there exist $\beta \in (0,1)$ such that for every $u \in \mathcal{S}_{x_0,r_0}$ and almost every $x, y \in \Omega$

$$|u(x) - u(y)| \le 16\theta^{\beta}\|u\|_\infty \left(\frac{d(x,y)}{d(x,\Omega^c) \vee d(y,\Omega^c)} \right)^{\beta}. \tag{5-10}$$

Proof. By symmetry, we may assume that $r := d(y,\Omega^c) \ge d(x,\Omega^c)$. Furthermore, it is enough to prove (5-10) for pairs x, y such that $d(x,y) < r/8$, as in the opposite case the assertion is obvious.

We fix a number $\rho \in (0, r_0/4)$ and consider all pairs of $x, y \in \Omega$ such that

$$\tfrac{1}{2}\rho \le d(x,y) \le \rho. \tag{5-11}$$

We cover the ball $B_{r_0-4\rho}(x_0)$ by a countable family of balls $\widetilde{B}_i$ with radii ρ. Without loss of generality, we may assume that $\widetilde{B}_i \cap B_{r_0-4\rho}(x_0) \ne \varnothing$. Let B_i denote the ball with the same center as the ball $\widetilde{B}_i$

and radius 2ρ and let B_i^* denote the ball with the same center as the ball $\widetilde{B}_i$ with radius the maximal radius that allows $B_i^* \subset \Omega$.

Let $x, y \in \Omega$ satisfy (5-11). From $r > 8d(x,y) \geq 4\rho$ it follows that $y \in B_{r_0-4\rho}(x_0)$; therefore $y \in \widetilde{B}_i$ for some index i. We observe that both x and y belong to B_i. We apply Theorem 1.10 to x_0 and r_0 being the center and radius of B_i^*, respectively, and obtain

$$\begin{aligned}\operatorname{osc}_{B_i} u &\leq 2\theta^\beta \|u\|_\infty \left(\frac{\operatorname{radius}(B_i)}{\operatorname{radius}(B_i^*)}\right)^\beta \\ &\leq 2\theta^\beta \|u\|_\infty \left(\frac{\rho}{r-\rho}\right)^\beta \leq \frac{16}{3}\theta^\beta \|u\|_\infty \left(\frac{d(x,y)}{r}\right)^\beta.\end{aligned}$$

Hence (5-10) holds, provided x and y are such that $|u(x)-u(y)| \leq \operatorname{osc}_{B_i} u$.

By considering $\rho = r_0 2^{-j}$ for $j = 3, 4, \ldots,$ we prove (5-10) for almost all x and y such that $d(x,y) \leq r_0/8$; hence the proof is finished. □

5A. *Proof of Theorem 1.7.* We are now going to use the above results and prove one of our main results.

Proof of Theorem 1.7. The proof of Theorem 1.7 follows from Corollaries 4.2 and 5.2. The proof is complete once we can apply Corollary 5.2 for $x_0 = 0$ and $r_0 = \frac{1}{2}$. Assume $0 < r \leq r_0$ and $B_r(x) \subset B_{1/2}$. Let $\mathcal{S}_{x,r}$ be the set of all functions $u \in V^\mu(B_r(x) \,|\, \mathbb{R}^d)$ satisfying $\mathcal{E}(u,\phi) = 0$ for every $\phi \in H^\mu_{B_r(x)}(\mathbb{R}^d)$. Assume $u \in \mathcal{S}_{x,r}$ and $u \geq 0$ in $B_r(x)$. Then Corollary 4.2 implies

$$\inf_{B_{r/4}(x)} u \geq c\left(\fint_{B_{r/2}(x)} u(x)^{p_0}\,\mathrm{d}x\right)^{\frac{1}{p_0}} - r^\alpha \sup_{y \in B_{15R/16}(x)} \int_{\mathbb{R}^d \setminus B_r(x)} u^-(z)\,\mu(y,\mathrm{d}z),$$

with positive constants p_0, c which depend only on d, α_0, A, B. Set $\theta = 4$, $\lambda = 2$, $\sigma = \frac{16}{15}$. Let $\nu_{x,r}$ be the measure on $\mathbb{R}^d \setminus B_r(x)$ defined by

$$\nu_{x,r}(A) = r^\alpha \mu(x,A).$$

The condition (1-17) obviously holds true. The condition (1-16) follows from (D). Thus we can apply Corollary 5.2 for $x_0 = 0$ and $r_0 = \frac{1}{2}$ and obtain the assertion of Theorem 1.7. □

6. Local comparability results for nonlocal quadratic forms

The aim of this section is to provide the proof of Theorem 1.11. First, we show that (T) and (U) imply (B). Then we establish the upper bound in (A) in the two cases (i) and (ii). The lower bound in (A) is more challenging. We prove it for the two cases in separate subsections. The last subsection contains two examples, which are not covered by cases (i) and (ii).

6A. (T) *and* (U) *imply* (B). It is easy to prove that (T) and (U) imply (B) with a constant $B \geq 1$ independent of $\alpha \in (\alpha_0, 2)$: Let $\tau \in C^\infty(\mathbb{R}^d)$ be a function satisfying $\operatorname{supp}(\tau) = \overline{B}_{R+\rho}$, $\tau \equiv 1$ on B_R, $0 \leq \tau \leq 1$ on $\mathbb{R}^d$, and $|\tau(x)-\tau(y)| \leq 2\rho^{-1}|x-y|$ for all $x, y \in \mathbb{R}^d$. In particular, we have then

$|\tau(x)-\tau(y)| \le (2\rho^{-1}|x-y|) \wedge 1$. For every $x \in \mathbb{R}^d$ we obtain

$$\int_{\mathbb{R}^d} (\tau(x)-\tau(y))^2\, \mu(x,\mathrm{d}y) \le \int_{\mathbb{R}^d} ((4\rho^{-2}|z|^2) \wedge 1)\, \nu^*(\mathrm{d}z)$$
$$= 4\rho^{-2} \int_{\mathbb{R}^d} \left(|z|^2 \wedge \tfrac{1}{4}\rho^2\right) \nu^*(\mathrm{d}z) \le 2^\alpha C_U \rho^{-\alpha} \le 4C_U\rho^{-\alpha}.$$

Thus we only need to concentrate on proving (A).

6B. *Upper bound in* (A). Let us formulate and prove the following comparability result.

Proposition 6.1. *Assume that ν satisfies (U) with the constant C_U and let $0 < \alpha_0 \le \alpha < 2$. If $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then there exists a constant $c = c(\alpha_0, d, C_U, D)$ such that*

$$\mathcal{E}_D^\nu(u,u) \le c\mathcal{E}_D^{\mu_\alpha}(u,u), \quad u \in H^{\frac{\alpha}{2}}(D). \tag{6-1}$$

The constant c may be chosen such that (6-1) holds for all balls $D = B_r$ of radius $r < 1$, and for all $\alpha \in [\alpha_0, 2)$.

Proof. By E we denote the extension operator from $H^{\alpha/2}(D)$ to $H^{\alpha/2}(\mathbb{R}^d)$; see Fact 3.1. By subtracting a constant, we may and do assume that $\int_D u\,\mathrm{d}x = 0$. We have by Plancherel's formula and Fubini's theorem

$$\begin{aligned}
\mathcal{E}_D^\nu(u,u) &= \int_D \int_{D-y} (u(y+z)-u(y))^2\, \nu(\mathrm{d}z)\,\mathrm{d}y && (6\text{-}2)\\
&\le \int_D \int_{B_{\operatorname{diam} D}(0)} (Eu(y+z)-Eu(y))^2\, \nu(\mathrm{d}z)\,\mathrm{d}y\\
&\le \int_{B_{\operatorname{diam} D}(0)} \int_{\mathbb{R}^d} (Eu(y+z)-Eu(y))^2\, \mathrm{d}y\, \nu(\mathrm{d}z)\\
&= \int_{\mathbb{R}^d} \left(\int_{B_{\operatorname{diam} D}(0)} |e^{i\xi\cdot z}-1|^2 \nu(\mathrm{d}z)\right) |\widehat{Eu}(\xi)|^2\, d\xi\\
&= \int_{\mathbb{R}^d} \left(\int_{B_{\operatorname{diam} D}(0)} 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu(\mathrm{d}z)\right) |\widehat{Eu}(\xi)|^2\, d\xi. && (6\text{-}3)
\end{aligned}$$

For $|\xi| > 2$ we obtain, using (U)

$$\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu(\mathrm{d}z) \le |\xi|^2 \int (|z|^2 \wedge 4|\xi|^{-2})\, \nu(\mathrm{d}z) \le 4C_U|\xi|^\alpha, \tag{6-4}$$

and for $|\xi| \le 2$

$$\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu(\mathrm{d}z) \le 4 \int \left(\left|\frac{\xi\cdot z}{2}\right|^2 \wedge 1\right) \nu(\mathrm{d}z) \le 4C_U.$$

Thus

$$\begin{aligned}
\mathcal{E}_D^\nu(u,u) &\le c' \int_{\mathbb{R}^d} (|\xi|^\alpha + 1)|\widehat{Eu}(\xi)|^2\, \mathrm{d}\xi\\
&\le c'\|Eu\|^2_{H^{\alpha/2}(\mathbb{R}^d)} \le c\|u\|^2_{H^{\alpha/2}(D)} = c(\mathcal{E}_D^{\mu_\alpha}(u,u) + \|u\|^2_{L^2(D)}), && (6\text{-}5)
\end{aligned}$$

with $c = c(d, C_U, D)$. Since $\int_D u \,\mathrm{d}x = 0$, we have by Fact 3.2

$$\mathcal{E}_D^{\mu_\alpha}(u,u) \ge c(\alpha_0, d, D) \int_D u^2(x)\,\mathrm{d}x$$

and this together with (6-5) proves (6-1).

By scaling, the last assertion of the theorem is satisfied with a constant $c = c(\alpha_0, d, C_U, B_1)$. □

Proof of Theorem 1.11: upper bound in (A). The second inequality in (A) follows from Proposition 6.1. We note that the constant in this inequality is robust under the mere assumption that α is bounded away from zero. □

6C. *Lower bound in (A), case* (i). The aim of this subsection is to complete the proof of Theorem 1.11 in the case (i). The strategy[3] is as follows. We will begin with a simple specific case. We set $e^k = (0,\dots,0,1,0,\dots,0) \in \mathbb{R}^d$; i.e., e^k is the k-th standard unit vector in $\mathbb{R}^d$.

Theorem 6.2. *Let $d \ge 2$, $0 < \alpha < 2$, and let μ be as in (1-11), i.e.,*

$$\mu(x,dy) = (2-\alpha)\sum_{i=1}^{d}\Big[|x_i - y_i|^{-1-\alpha}\,\mathrm{d}y_i \prod_{j\ne i}\delta_{\{x_j\}}(\mathrm{d}y_j)\Big] =: \sum_{i=1}^{d}\mu_i(x,\mathrm{d}y). \tag{6-6}$$

Then there exists a constant $A = A(d)$ such that

$$\begin{gathered}\text{for every ball } B_\rho(x_0) \text{ with } \rho\in(0,1),\ x_0\in B_1, \text{ and every } v \in H^{\frac{\alpha}{2}}(B_\rho(x_0)),\\ \mathcal{E}^{\mu_\alpha}_{B_\rho(x_0)}(v,v) \le A\,\mathcal{E}^{\mu}_{B_\rho(x_0)}(v,v).\end{gathered} \tag{6-7}$$

Proof. Let us fix $B = B_\rho(x_0)$ as in the theorem. We may assume that $x_0 = 0$, because the measures considered are translation invariant. For a permutation σ of $\{1,2,\dots,d\}$ and $x, y \in B$ we define

$$p_k^\sigma(x,y) = (a_1,\dots,a_d), \quad \text{where } a_j = \begin{cases} y_j & \text{if } \sigma^{-1}(j) \le k,\\ x_j & \text{if } \sigma^{-1}(j) > k.\end{cases}$$

For example,

$$p_0^\sigma(x,y) = x, \quad p_1^\sigma(x,y) = (x_1,\dots,x_{\sigma(1)-1}, y_{\sigma(1)}, x_{\sigma(1)+1},\dots,x_d), \quad p_d^\sigma(x,y) = y.$$

That is, $p_k(x,y)$ are vertices of a polygonal chain joining x and y whose consecutive line segments are parallel to the coordinate axes; more precisely, the j-th line segment is parallel to $\sigma(j)$-th axis. Furthermore, let

$$E^\sigma(B,x) = \{y \in B : p_k^\sigma(x,y) \in B \text{ for each } k = 1,\dots,d\} \tag{6-8}$$

be the set of all points y which may be connected with x by such a polygonal chain lying completely in B. We obtain

$$\begin{aligned} I^\sigma &:= \int_B\int_{E^\sigma(B,x)} (u(x)-u(y))^2\,\mu_\alpha(x,\mathrm{d}y)\,\mathrm{d}x\\ &\le d\sum_{k=1}^{d}\int_B\int_{E^\sigma(B,x)} (u(p_{k-1}^\sigma(x,y)) - u(p_k^\sigma(x,y)))^2\,\mu_\alpha(x,\mathrm{d}y)\,\mathrm{d}x =: d\sum_{k=1}^{d} I_k^\sigma. \end{aligned} \tag{6-9}$$

[3]The authors thank an anonymous referee for the idea of the proof.

We will bound I_k^σ appearing on the right of (6-9), assuming for notational simplicity that σ is the identity permutation, i.e., $\sigma(k)=k$. Then

$$\begin{aligned} I_k^\sigma &= \int_B \int_{E^\sigma(B,x)} (u(p_{k-1}^\sigma(x,y)) - u(p_k^\sigma(x,y)))^2 \, \mu_\alpha(x,\mathrm{d}y)\,\mathrm{d}x \\ &= (2-\alpha)\int_B \int_{E^\sigma(B,x)} \frac{(u(y_1,\dots,y_{k-1},x_k,\dots,x_d) - u(y_1,\dots,y_k,x_{k+1},\dots,x_d))^2}{|x-y|^{d+\alpha}}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

We would like to change the order of integration, so that we integrate outside with respect to

$$w := p_{k-1}^\sigma(x,y) = (y_1,\dots,y_{k-1},x_k,\dots,x_d),$$

and inside with respect to

$$z := x+y-w = (x_1,\dots,x_{k-1},y_k,\dots,y_d).$$

Then $|x-y| = |z-w|$ and $p_k^\sigma(x,y) = w + (z_k - w_k)e^k$. Let

$$\begin{aligned} F(B,w) &:= \{z \in \mathbb{R}^d : w + (z_k - w_k)e^k \in B\}, \\ F_0(B,w) &:= \{t \in \mathbb{R} : w + (t - w_k)e^k \in B\}. \end{aligned}$$

We note that if $x \in B$ and $y \in E^\sigma(B,x)$, then $w \in B$ and $p_k(x,y) \in B$; hence $z \in F(B,w)$. Therefore

$$\begin{aligned} I_k^\sigma &\le (2-\alpha)\int_B \int_{F(B,w)} \frac{(u(w) - u(w+(z_k-w_k)e^k))^2}{|w-z|^{d+\alpha}}\,\mathrm{d}z\,\mathrm{d}w \\ &= (2-\alpha)\int_B \int_{F_0(B,w)} \left[(u(w) - u(w+(z_k-w_k)e^k))^2 \int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}z_1\cdots \mathrm{d}z_{k-1}\,\mathrm{d}z_{k+1}\cdots \mathrm{d}z_d}{|w-z|^{d+\alpha}} \right] \mathrm{d}z_k\,\mathrm{d}w. \end{aligned}$$

The inner integral over $\mathbb{R}^{d-1}$ is simple to calculate using scaling; it gives

$$\int_{\mathbb{R}^{d-1}} \frac{\mathrm{d}z_1\cdots \mathrm{d}z_{k-1}\,\mathrm{d}z_{k+1}\cdots \mathrm{d}z_d}{|w-z|^{d+\alpha}} = |w_k - z_k|^{-\alpha-1} c(d) \int_0^\infty (1+t^2)^{\frac{-d-\alpha}{2}} t^{d-2}\,\mathrm{d}t \le C(d)|w_k - z_k|^{-\alpha-1}.$$

Thus

$$\begin{aligned} I_k^\sigma &\le C(d)(2-\alpha)\int_B \int_{F_0(B,w)} [(u(w) - u(w+(z_k-w_k)e^k))^2 |w_k - z_k|^{-\alpha-1}]\,\mathrm{d}z_k\,\mathrm{d}w \\ &= C(d)\int_B \int_B (u(w) - u(z))^2 \, \mu_k(w,\mathrm{d}z)\,\mathrm{d}w. \end{aligned}$$

The same inequality as above holds for I_k^σ with an arbitrary permutation σ. We obtain

$$\sum_\sigma I^\sigma = \sum_\sigma \sum_{k=1}^d I_k^\sigma \le C(d)d!\,\mathcal{E}_B^\mu(u,u), \tag{6-10}$$

where the sum runs over the set of all permutations on $\{1,2,\dots,d\}$. On the other hand, for each pair $(x,y) \in B \times B$, there exists a permutation σ such that $y \in E^\sigma(B,x)$. Indeed, if $M = \#\{j : |y_j| < |x_j|\}$, then as σ we may take any permutation satisfying $\sigma(\{1,\dots,M\}) = \{j : |y_j| < |x_j|\}$. If $1 \le j \le M$, then

$|p_j(x,y)| < |x|$, and if $j > M$, then $|p_j(x,y)| \le |y|$; therefore $p_j(x,y) \in B$ for all j; i.e., $y \in E^\sigma(B,x)$ as claimed. Thus

$$\sum_\sigma I^\sigma = \sum_\sigma \int_B \int_{E^\sigma(B,x)} (u(x)-u(y))^2 \, \mu_\alpha(x,\mathrm{d}y)\,\mathrm{d}x \ge \mathcal{E}_B^{\mu_\alpha}(u,u),$$

which together with (6-10) gives the assertion of the theorem. □

Next, we consider linear transformations of μ. Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transform. For a measure μ on $\mathbb{R}^d$ we define the measure $\mu \circ L$ by

$$(\mu \circ L)(E) = \mu(L(E)), \quad E \subset \mathbb{R}^d, \quad \text{where } E \text{ is a Borel set,}$$

or, equivalently, by

$$\int f(x)(\mu \circ L)(\mathrm{d}x) = \int f(L^{-1}(x))\, \mu(\mathrm{d}x), \tag{6-11}$$

for all Borel measurable functions $f : \mathbb{R}^d \to [0,\infty)$.

Lemma 6.3. *Let $0 < \alpha_0 \le \alpha < 2$ and let a measure μ on $\mathbb{R}^d$ satisfy condition (6-7) with some constant A, with $\mathcal{E}^\mu$ defined by (1-21). Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear transform. Then $\mu \circ L$ also satisfies condition (6-7) with the constant depending only on A, d, α_0 and the norms $\|L\|$ and $\|L^{-1}\|$.*

Proof. Let u be a Borel measurable function on $\mathbb{R}^d$; let $B = B_\rho(x_0)$ with $x_0 \in B_1$ and $\rho \in (0,1)$. Let

$$v(x) = u(L^{-1}(x-x_0)+x_0), \quad x \in \mathbb{R}^d.$$

By a linear change of variable and (6-11) we obtain

$$\mathcal{E}_B^{\mu\circ L}(u,u) = \frac{1}{\det L} \int_{x_0+L(B(0,\rho))} \int_{\mathbb{R}^d} (v(x)-v(x+z))^2 \, \mathbb{1}_{x_0+L(B(0,\rho))}(x+z)\, \mu(\mathrm{d}z)\,\mathrm{d}t. \tag{6-12}$$

We observe that $B(0,s\rho) \subset L(B(0,\rho))$, where $s = \|L^{-1}\|^{-1} \wedge 1$; therefore

$$\begin{aligned}\mathcal{E}_B^{\mu\circ L}(u,u) &\ge \frac{1}{\det L} \int_{B(x_0,s\rho)} \int_{\mathbb{R}^d} (v(x)-v(x+z))^2 \, \mathbb{1}_{B(x_0,s\rho)}(x+z)\, \mu(\mathrm{d}z)\,\mathrm{d}t \\ &= \frac{1}{\det L} \mathcal{E}^\mu_{B(x_0,s\rho)}(v,v) \ge \frac{A^{-1}}{\det L} \mathcal{E}^{\mu_\alpha}_{B(x_0,s\rho)}(v,v),\end{aligned}$$

by the assumption and the fact that $s \le 1$. Since $L(B(0,st\rho)) \subset B(0,s\rho)$, where $t = \|L\|^{-1} \wedge 1$, we get

$$\begin{aligned}\mathcal{E}_B^{\mu\circ L}(u,u) &\ge \frac{A^{-1}}{\det L} \int_{x_0+L(B(0,st\rho))} \int_{\mathbb{R}^d} (v(x)-v(x+z))^2 \, \mathbb{1}_{x_0+L(B(0,st\rho))}(x+z) \mu_\alpha(dz)\,\mathrm{d}t \\ &= A^{-1} \mathcal{E}^{\mu_\alpha \circ L}_{B(0,st\rho)}(u,u),\end{aligned} \tag{6-13}$$

where in the last line we used (6-12) with μ_α in place of μ.

However,

$$(\mu_\alpha \circ L)(E) = (2-\alpha)\int_{L(E)} |x|^{-d-\alpha}\,\mathrm{d}x = (2-\alpha)\det L \int_E |L(x)|^{-d-\alpha}\,\mathrm{d}x$$
$$\geq (2-\alpha)\det L\|L\|^{-d-\alpha}\int_E |x|^{-d-\alpha}\,\mathrm{d}x = \det L\|L\|^{-d-\alpha}\mu_\alpha(E).$$

Plugging this into (6-13) we obtain

$$\mathcal{E}_B^{\mu\circ L}(u,u) \geq \det L\|L\|^{-d-\alpha}A^{-1}\mathcal{E}^{\mu_\alpha}_{B(0,st\rho)}(u,u).$$

The theorem follows now from Lemma 6.13; since $\det L \geq \|L^{-1}\|$ and $st = (\|L^{-1}\|^{-1}\wedge 1)(\|L\|^{-1}\wedge 1)$, the constants depend only on A, d, α_0, $\|L\|$ and $\|L^{-1}\|$. We note here that the proof of this lemma, although presented later, does not use any previous results, i.e., there is no circular reasoning. □

With the help of Lemma 6.3 we are able to prove the following generalization of Theorem 6.2.

Corollary 6.4. *Let $0 < \alpha_0 \leq \alpha < 2$. Let $f^1, \dots, f^d \in S^{d-1}$ be linearly independent. Assume that $L : \mathbb{R}^d \to \mathbb{R}^d$ is the linear transform that maps e^j to f^j. Then the measure*

$$\mu(E) = (2-\alpha)\sum_{j=1}^{d}\int_0^\infty \delta_{rf_j}(E)r^{-1-\alpha}\,\mathrm{d}r \quad (E \in \mathcal{B}(\mathbb{R}^d)) \tag{6-14}$$

satisfies condition (6-7) with $\mathcal{E}^\mu$ defined by (1-21) and the constant depending only on d, α_0 and the norm $\|L^{-1}\|$.

Proof. Since $\|L\| \leq \sqrt{d}$, the result follows from Theorem 6.2 and Lemma 6.3. □

In order to prove comparability for all nondegenerate α-stable measures, we need to study combinations of measures as in (6-14). To this end, we will apply the following lemma, which essentially is contained in [Krickeberg 1968].

Lemma 6.5. *If π is a finite Borel measure on S^{d-1}, then there exists a Borel function $\phi : [0, \pi(X)] \to S^{d-1}$ such that $|\phi^{-1}(A)| = \pi(A)$ for every Borel set $A \subset S^{d-1}$ and $\phi([0,\pi(X)]) \subset \operatorname{supp}\pi$.*

Proof. It is enough to prove the result for measures π which are either purely atomic, or nonatomic. In the first case the construction of such ϕ is straightforward: if $\{a_j : 0 \leq j < N\}$ are all the atoms of π (where $N \in \mathbb{N}\cup\{\infty\}$), then we put

$$\phi(t) = \begin{cases} a_j & \text{for } t \in \left[\sum_{0\leq i<j}\pi(\{a_i\}), \sum_{0\leq i\leq j}\pi(\{a_i\})\right) \text{ and } 0 \leq j < N,\\ a_0 & \text{for } t = \pi(X).\end{cases}$$

In the nonatomic case, since π is Radon, the result follows from [Krickeberg 1968, Hilfssatz, page 64; Oxtoby 1970, Theorem 2]. □

We finally provide the proof of the comparability result for general α-stable measures.

Proof of Theorem 1.11(i). Assume that π is a measure on S^{d-1} as in (1-22). Let $x_1, \dots, x_d \in \operatorname{supp}\pi$ be a basis of $\mathbb{R}^d$. Then for $\varepsilon > 0$ small enough and some $M > 0$, any $y_j \in \overline{B(x_j,\varepsilon)} \cap S^{d-1} =: B_j$ also span $\mathbb{R}^d$. If L is the linear operator mapping $e^j = (0,\dots,0,1,0,\dots,0)$ to y_j, then the norms $\|L\|$ and

$\|L^{-1}\|$ are bounded from above by M. The number $m = \min\{\pi(B_1), \dots, \pi(B_d)\}$ is strictly positive, because x_j belong to the support of π. Let

$$\pi_j = \frac{m}{\pi(B_j)} \pi(\cdot \cap B_j), \quad j = 1, \dots, d.$$

Then π_j are Borel measures on S^{d-1} with mass m and $\operatorname{supp} \pi_j \subset \bar{B}_j$. Let $\phi_j : [0, m] \to S^{d-1}$ be the Borel functions from Lemma 6.5 corresponding to π_j. For every Borel $E \subset S^{d-1}$

$$\pi(E) \geq \sum_{j=1}^{d} \pi_j(E) = \sum_{j=1}^{d} |\phi_j^{-1}(E)| = \int_0^m \sum_{j=1}^{d} \delta_{\phi_j(t)}(E) \, \mathrm{d}t.$$

Therefore

$$\nu(E) = (2-\alpha) \int_0^\infty \pi(r^{-1}E) r^{-1-\alpha} \, \mathrm{d}r \geq \int_0^m \left((2-\alpha) \sum_{j=1}^{d} \int_0^\infty \delta_{r\phi_j(t)}(E) r^{-1-\alpha} \, \mathrm{d}r \right) \mathrm{d}t.$$

By Corollary 6.4, the measure in the parentheses in the line above satisfies condition (6-7) with the constant depending on d, α_0 and M, but independent of t. Therefore also ν satisfies condition (6-7) with the constant depending on d, α_0 and M, i.e., on α_0 and π. □

6D. ***Lower bound in* (A)*, case* (ii).** The aim of this subsection is to complete the proof of Theorem 1.11 in the case (ii).

The main difficulty in establishing the lower bound in (A) is that the measures might be singular. We will introduce a new convolution-type operation that, on the one hand, smooths the support of the measures and, on the other hand, interacts nicely with our quadratic forms. The main result of this subsection is Proposition 6.14.

For $\lambda < 1 \leq \eta$ and $\alpha \in (0, 2)$ let

$$g_\lambda^\eta(y, z) = \frac{1}{2-\alpha} |y+z|^\alpha \, \mathbb{1}_{A_{|y+z|}}(y) \, \mathbb{1}_{A_{|y+z|}}(z), \quad y, z \in \mathbb{R}^d, \tag{6-15}$$

where

$$A_r = B(0, \eta r) \setminus B(0, \lambda r).$$

Definition 6.6. For measures ν_1, ν_2 on $\mathcal{B}(\mathbb{R}^d)$ satisfying (U) with some $\alpha \in (0, 2)$, define a new measure $\nu_1 \heartsuit \nu_2$ on $\mathcal{B}(\mathbb{R}^d)$ by

$$\nu_1 \heartsuit \nu_2(E) = \iint \mathbb{1}_{E \cap B_2}(\eta(y+z)) \, g_\lambda^\eta(y, z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z),$$

i.e.,

$$\int f(x) \nu_1 \heartsuit \nu_2(\mathrm{d}x) = \iint (f \cdot \mathbb{1}_{B_2})(\eta(y+z)) g_\lambda^\eta(y, z) \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z),$$

for every measurable function $f : \mathbb{R}^d \to [0, \infty]$.

This definition is tailored for our applications and needs some explanations. We consider $\nu_1 \heartsuit \nu_2$ only for measures ν_j which satisfy (U) with some $\alpha \in (0, 2)$ for $j \in \{1, 2\}$. This α equals the exponent α in the definition of g_λ^η. The above definition does not require ν_j to satisfy (S) but most often this will be the case. Note that Definition 6.6 is valid for any choice $\lambda < 1 \le \eta$. However, it will be important to choose λ small enough and η large enough. The precise bounds depend on the number a from (S); see Proposition 6.14. Before we explain and prove the rather technical details, let us treat an example.

Let us study Example 1.5 in $\mathbb{R}^2$. Assume $\alpha \in (0,2)$ and

$$\nu_1(\mathrm{d}h) = (2-\alpha)|h_1|^{-1-\alpha}\,\mathrm{d}h_1\,\delta_{\{0\}}(\mathrm{d}h_2),$$
$$\nu_2(\mathrm{d}h) = (2-\alpha)|h_2|^{-1-\alpha}\,\mathrm{d}h_2\,\delta_{\{0\}}(\mathrm{d}h_1).$$

Both measures are one-dimensional α-stable measures which are orthogonal to each other. The factor $(2-\alpha)$ ensures that for $\alpha \to 2-$ the measures do not explode. Let us show that $\nu_1 \heartsuit \nu_2$ is already absolutely continuous with respect to the two-dimensional Lebesgue measure. For $E \subset B_2$, by the Definition 6.6 and the Fubini theorem,

$$\begin{aligned}
&\nu_1 \heartsuit \nu_2(E)\\
&\quad= (2-\alpha)\iiiint |y+z|^{\alpha}\,\mathbb{1}_E(\eta(y+z))\,\mathbb{1}_{A_{|y+z|}}(y)\,\mathbb{1}_{A_{|y+z|}}(z)|y_1|^{-1-\alpha}|z_2|^{-1-\alpha}\\
&\qquad\qquad\cdots\delta_{\{0\}}(\mathrm{d}y_2)\,\delta_{\{0\}}(\mathrm{d}z_1)\,\mathrm{d}y_1\,\mathrm{d}z_2\\
&\quad= (2-\alpha)\iint |(y_1,z_2)|^{\alpha}\,\mathbb{1}_E(\eta(y_1,z_2))\,\mathbb{1}_{A_{|(y_1,z_2)|}}(y_1,0)\,\mathbb{1}_{A_{|(y_1,z_2)|}}(0,z_2)|y_1|^{-1-\alpha}|z_2|^{-1-\alpha}\,\mathrm{d}y_1\,\mathrm{d}z_2\\
&\quad= (2-\alpha)\iint \mathbb{1}_E(\eta x)\,\mathbb{1}_{A_{|x|}}(x_1,0)\,\mathbb{1}_{A_{|x|}}(0,x_2)|x|^{\alpha}|x_1|^{-1-\alpha}|x_2|^{-1-\alpha}\,\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

The above computation shows that the measure $\nu_1 \heartsuit \nu_2$ is absolutely continuous with respect to the two-dimensional Lebesgue measure, because $\nu_1 \heartsuit \nu_2(\mathbb{R}^d \setminus B_2) = 0$. Let us look at the density more closely.

So far, we have not specified λ and η in the definition of g_λ^η. If $\lambda < 1$ is too large (in this particular case, if $\lambda > 1/\sqrt{2}$), then $\mathbb{1}_{A_{|x|}}(x_1,0)\,\mathbb{1}_{A_{|x|}}(0,x_2) = 0$ for all $x \in \mathbb{R}^2$. If λ is sufficiently small, then the support of the function $\mathbb{1}_{A_{|x|}}(x_1,0)\,\mathbb{1}_{A_{|x|}}(0,x_2)$ is a double-cone centered around the diagonals $\{x \in \mathbb{R}^2 : |x_1| = |x_2|\}$. Let us denote this support by M. Note that on M the function $|x|^{\alpha}|x_1|^{-1-\alpha}|x_2|^{-1-\alpha}$ is comparable to $|x|^{-2-\alpha}$. Thus indeed the quantity $\nu_1 \heartsuit \nu_2$ is comparable to an α-stable measure in $\mathbb{R}^2$. If we continue the procedure and define

$$\tilde{\nu} = (\nu_1 \heartsuit \nu_2) \heartsuit (\nu_1 \heartsuit \nu_2),$$

then we can make use of the fact that $(\nu_1 \heartsuit \nu_2)$ is already absolutely continuous with respect to the two-dimensional Lebesgue measure. Note that, if $\mu_j = h_j\,\mathrm{d}x$, then $\mu_1 \heartsuit \mu_2$ has a density $h_1 \heartsuit h_2$ with respect to the Lebesgue measure given by

$$h_1 \heartsuit h_2(\eta y) = \frac{\eta^{-d}|y|^{\alpha}}{2-\alpha}\int \mathbb{1}_{A_{|y|}}(y-z)\,\mathbb{1}_{A_{|y|}}(z)\,h_1(y-z)\,h_2(z)\,\mathrm{d}z, \quad \eta y \in B_2. \tag{6-16}$$

In this way we conclude that $\tilde{\nu}$ has full support and is comparable to a rotationally symmetric α-stable measure in $\mathbb{R}^2$. With this observation we end our study of Definition 6.6 in light of Example 1.5.

Before we proceed to the proofs, let us informally explain the idea behind Definition 6.6 and our strategy. In the inner integral defining

$$\mathcal{E}_B^{\nu}(u,u) = \int_B \int_{\mathbb{R}^d} (u(x) - u(x+h))^2 \, \mathbb{1}_B(x+h) \, \nu(\mathrm{d}h) \, \mathrm{d}x$$

we take into account squared increments $(u(x) - u(x+h))^2$ in these directions h, which are charged by the measure ν and such that $x+h$ is still in B. By changing the variables, we see that we also have squared increments $(u(x+h) - u(x+h+z))^2$, again in directions z, which are charged by the measure ν and such that $x+h+z$ is still in B. This allows us to estimate the integral $\mathcal{E}_B^{\nu}(u,u)$ from below by a similar integral with ν replaced by some kind of a convolution of ν with itself. Measure $\nu \heartsuit \nu$ turns out to be the right convolution for this purpose; see Lemma 6.12.

In the definition of $\nu \heartsuit \nu$, the function g_λ^η vanishes if $|y|$ or $|z|$ is bigger than $\eta|y+z|$ or smaller than $\lambda|y+z|$. This means, in our interpretation, that we consider only those pairs of jumps which are comparable to the size of the whole two-step jump (and in particular, the jumps must be comparable to each other).

To conclude these informal remarks on the definition of $\nu_1 \heartsuit \nu_2$ let us note that if ν_1 and ν_2 have "good properties", then so has $\nu_1 \heartsuit \nu_2$ (see Lemmas 6.7 and 6.11) and that $\mathcal{E}_B^{\nu_1 \heartsuit \nu_2}(u,u)$ can be estimated from above by $\mathcal{E}_B^{\nu_j}(u,u)$ (see Lemma 6.12). This allows us to reduce the problem of estimating $\mathcal{E}_B^{\nu}(u,u)$ from below to estimating $\mathcal{E}_B^{\nu \heartsuit \nu}(u,u)$ from below, and this turns out to be easier, since the $\heartsuit$-convolution makes the measure more "smooth"; see Proposition 6.14.

Lemma 6.7. *If two measures ν_j for $j \in \{1,2\}$ satisfy the scaling assumption (S) for some $a > 1$, then so does the measure $\nu_1 \heartsuit \nu_2$ for the same constant a.*

Proof. If $\operatorname{supp} f \subset B_1$, then

$$\begin{aligned}\int f(ax) \nu_1 \heartsuit \nu_2(\mathrm{d}x) &= \iint f(\eta a(y+z)) \, \mathbb{1}_{B_2}(\eta(y+z)) \, g_\lambda^\eta(y,z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) \\ &= a^{-\alpha} \iint f(\eta(ay+az)) \, g_\lambda^\eta(ay,az) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z),\end{aligned}$$

because $g_\lambda^\eta(y,z) = a^{-\alpha} g_\lambda^\eta(ay,az)$. We observe that the function $(y,z) \mapsto f(\eta(y+z)) g_\lambda^\eta(y,z)$ vanishes outside $B_1 \times B_1$. Hence we may apply (S) twice to obtain

$$\int f(ax) \, \nu_1 \heartsuit \nu_2(\mathrm{d}x) = a^{\alpha} \iint f(\eta(y+z)) \, g_\lambda^\eta(y,z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) = a^{\alpha} \int f(x) \, \nu_1 \heartsuit \nu_2(\mathrm{d}x). \qquad \square$$

Next, we establish conditions which are equivalent to (U). We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the upper-bound assumption (U0) if for some $C_0 > 0$

$$\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, \nu(\mathrm{d}z) \le C_0. \tag{U0}$$

We say that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ satisfies the upper-bound assumption (U1) if there exists $C_1 > 0$ such that for every $r \in (0,1)$

$$\int_{B_r(0)} |z|^2 \nu(\mathrm{d}z) \le C_1 r^{2-\alpha}. \tag{U1}$$

Lemma 6.8. (U) $\iff$ (U0) $\wedge$ (U1).

If the constants C_0, C_1 are independent of $\alpha \in [\alpha_0, 2)$, then so is C_U, and vice versa.

Proof. The implications (U) $\Rightarrow$ (U1) and (U) $\Rightarrow$ (U0) are obvious; we may take $C_0 = C_1 := C_U$. Let us now assume that (U1) and (U0) hold true. Fix $0 < r \le 1$. We consider $n = 0, 1, 2, \ldots$ such that $2^{n+1} r \le 1$ (the set of such n's is empty if $r > \frac{1}{2}$). We have by (U1)

$$\int_{2^n r \le |z| < 2^{n+1} r} \nu(\mathrm{d}z) \le 2^{-2n} r^{-2} \int_{2^n r \le |z| < 2^{n+1} r} |z|^2 \nu(\mathrm{d}z)$$
$$\le 2^{-2n} r^{-2} C_1 \, 2^{(n+1)(2-\alpha)} r^{2-\alpha} = 2^{-n\alpha} 2^{2-\alpha} C_1 r^{-\alpha}.$$

After summing over all such n we obtain

$$\int_{r \le |z| < \frac{1}{2}} \nu(\mathrm{d}z) \le \frac{2^{2-\alpha} C_1}{1 - 2^{-\alpha}} r^{-\alpha}.$$

Finally,

$$\int_{\frac{1}{2} \le |z|} \nu(\mathrm{d}z) \le 4 \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, \nu(\mathrm{d}z) \le 4C_0 \le 4C_0 r^{-\alpha}.$$

Combining the two inequalities above and (U1) we get (U) with

$$C_U = \left(\frac{2^{2-\alpha}}{1 - 2^{-\alpha}} + 1 \right) C_1 + 4C_0. \qquad \square$$

The following definition interpolates between measures ν which are related to different values of $\alpha \in (0, 2)$. Such a construction is important for us because we want to prove comparability results which are robust in the sense that constants stay bounded when $\alpha \to 2^-$.

Definition 6.9. Assume ν^{α_0} is a measure on $\mathcal{B}(\mathbb{R}^d)$ satisfying (U) or (S) for some $\alpha_0 \in (0, 2)$. For $\alpha_0 \le \alpha < 2$ we define a new measure ν^{α,α_0} by

$$\nu^{\alpha,\alpha_0} = \frac{2-\alpha}{2-\alpha_0} |x|^{\alpha_0 - \alpha} \nu^{\alpha_0}(dx) \quad \text{if } \alpha > \alpha_0 \qquad \text{and} \qquad \text{by } \nu^{\alpha_0,\alpha_0} = \nu^{\alpha_0}. \tag{6-17}$$

To shorten notation we write ν^α instead of ν^{α,α_0} whenever there is no ambiguity.

The above definition is consistent in the following ways. On the one hand, the first part of (6-17) holds true for $\alpha = \alpha_0$. On the other hand, for $0 < \alpha_0 < \alpha < \beta < 2$, the following is true: $\nu^{\beta,\alpha_0} = (\nu^{\alpha,\alpha_0})^{\beta,\alpha}$. This requires that ν^{α,α_0} itself satisfies (U) or (S) which is established in the following lemma.

Lemma 6.10. *Assume ν^{α_0} satisfies (U) with some $\alpha_0 \in (0, 2)$, $C_U > 0$ or condition (S) with some $\alpha_0 \in (0, 2)$, $a > 1$. Assume $\alpha_0 \le \alpha < 2$ and ν^α as in Definition 6.9.*

(a) *If ν^{α_0} satisfies (U), then for every $0 < b < 1$, $0 < r \le 1$*

$$\int_{br \le |z| < r} |z|^2 \nu^\alpha(\mathrm{d}z) \le \frac{2-\alpha}{2-\alpha_0} C_U b^{\alpha_0 - \alpha} r^{2-\alpha}, \tag{6-18}$$

$$\int_{B_r^c} \nu^\alpha(\mathrm{d}z) \le \frac{2-\alpha}{2-\alpha_0} C_U r^{-\alpha}. \tag{6-19}$$

(b) *If ν^{α_0} satisfies (U), then ν^{α} satisfies (U) with exponent α and constant $13C_U(2-\alpha_0)^{-1}$. In particular, the constant does not depend on α.*

(c) *If ν^{α_0} satisfies (S), then ν^{α} satisfies (S) with exponent α.*

Proof. Let $0<r\le 1$ and $0<b<1$. To prove (a)enumi, we derive,

$$\begin{aligned}\int_{br\le|z|<r}|z|^2\nu^{\alpha}(\mathrm{d}z)&=\frac{2-\alpha}{2-\alpha_0}\int_{br\le|z|<r}|z|^{2+\alpha_0-\alpha}\nu^{\alpha_0}(\mathrm{d}z)\\&\le\frac{2-\alpha}{2-\alpha_0}(br)^{\alpha_0-\alpha}\int_{B_r}|z|^2\nu^{\alpha_0}(\mathrm{d}z)\le\frac{2-\alpha}{2-\alpha_0}b^{\alpha_0-\alpha}C_Ur^{2-\alpha},\end{aligned}$$

which proves (6-18). Furthermore,

$$\int_{B_r^c}\nu^{\alpha}(\mathrm{d}z)=\frac{2-\alpha}{2-\alpha_0}\int_{B_r^c}|z|^{\alpha_0-\alpha}\nu^{\alpha_0}(\mathrm{d}z)\le\frac{2-\alpha}{2-\alpha_0}r^{\alpha_0-\alpha}C_Ur^{-\alpha_0}$$

and (6-19) follows. To prove part (b)enumi, we use (6-18) and conclude

$$\begin{aligned}\int_{B_r}|z|^2\nu^{\alpha}(\mathrm{d}z)&=\sum_{n=0}^{\infty}\int_{\frac{r}{2^{n+1}}\le|z|<\frac{r}{2^n}}|z|^2\nu^{\alpha_0}(\mathrm{d}z)\le\frac{2-\alpha}{2-\alpha_0}C_U2^{\alpha-\alpha_0}r^{2-\alpha}\sum_{n=0}^{\infty}2^{n(\alpha-2)}\\&=\frac{C_U2^{\alpha-\alpha_0}r^{2-\alpha}}{2-\alpha_0}\frac{2-\alpha}{1-2^{\alpha-2}}\le\frac{32C_U}{3(2-\alpha_0)}r^{2-\alpha},\end{aligned}\tag{6-20}$$

since the function $x\mapsto x/(1-2^{-x})$ is increasing. Furthermore, by (6-19),

$$\int_{B_r^c}r^2\nu^{\alpha}(\mathrm{d}z)\le\frac{2C_U}{2-\alpha_0}r^{2-\alpha},\tag{6-21}$$

and therefore (b)enumi follows. Finally, part (c)enumi is obvious. □

Lemma 6.11. *Assume $\nu_j^{\alpha_0}$ for $j\in\{1,2\}$ satisfies (U) with some $\alpha_0\in(0,2)$, $C_U>0$. Assume $\alpha_0\le\alpha<2$ and ν_j^{α} as in Definition 6.9. Then the measure $\nu_1^{\alpha}\heartsuit\nu_2^{\alpha}$ satisfies (U) with the same exponent α and a constant depending only on α_0, C_U, λ and η.*

Proof. By Lemma 6.8, it suffices to show that $\nu_1^{\alpha}\heartsuit\nu_2^{\alpha}$ satisfies (U0) and (U1). For $0<r\le1$ we derive

$$\begin{aligned}\int_{B_r}|x|^2\,\nu_1^{\alpha}\heartsuit\nu_2^{\alpha}(dx)&\le\frac{1}{2-\alpha}\iint_{\lambda|y+z|\le|y|,|z|\le\eta|y+z|}|\eta(y+z)|^2\mathbb{1}_{B_r}(\eta(y+z))|y+z|^{\alpha}\,\nu_1^{\alpha}(\mathrm{d}y)\,\nu_2^{\alpha}(\mathrm{d}z)\\&\le\frac{1}{2-\alpha}\iint_{\lambda|y+z|\le|y|,|z|\le\eta|y+z|<r}\frac{\eta^2|y|^2}{\lambda^2}\frac{|z|^{\alpha}}{\lambda^{\alpha}}\nu_1^{\alpha}(\mathrm{d}y)\,\nu_2^{\alpha}(\mathrm{d}z)\\&\le\frac{1}{2-\alpha}\frac{\eta^2}{\lambda^{2+\alpha}}\int_{B_r}|z|^{\alpha}\int_{\frac{\lambda|z|}{\eta}\le|y|\le\frac{\eta|z|}{\lambda}}|y|^2\,\nu_1^{\alpha}(\mathrm{d}y)\,\nu_2^{\alpha}(\mathrm{d}z)\\&\le\frac{\eta^4(C_U)^2}{\lambda^4}\frac{13}{(2-\alpha_0)^2}r^{2-\alpha},\end{aligned}$$

where in the last passage we used parts (b)enumi and (a)enumi of Lemma 6.10. Furthermore, by (6-19),

$$\int_{\mathbb{R}^d\setminus B_1} \nu_1^\alpha \heartsuit \nu_2^\alpha(\mathrm{d}x) \le \frac{1}{2-\alpha} \iint_{\lambda|y+z|\le|y|,|z|<\eta|y+z|} \mathbb{1}_{B_2\setminus B_1}(\eta(y+z))|y+z|^\alpha \nu_1^\alpha(y)\nu_2^\alpha(\mathrm{d}z)$$
$$\le \frac{2^\alpha}{2-\alpha} \iint_{\frac{\lambda}{\eta}\le|y|,|z|} \nu_1^\alpha(y)\nu_2^\alpha(\mathrm{d}z) \le \frac{8(C_U)^2\eta^4}{\lambda^4(2-\alpha_0)^2}. \qquad \square$$

The following lemma shows that the quadratic form with respect to $\nu_1 \heartsuit \nu_2$ is dominated by the sum of the quadratic forms with respect to ν_1 and ν_2. Some enlargement of the domain is needed which is taken care of in Lemma 6.13 by a covering argument.

Lemma 6.12. *Assume $\nu_j^{\alpha_0}$ for $j \in \{1,2\}$ satisfies (U) and (S) with some $\alpha_0 \in (0,2)$, $a > 1$, and $C_U > 0$. Assume $\alpha_0 \le \alpha < 2$ and ν_j^α as in Definition 6.9. Let $\eta = a^k > 1$ for some $k \in \mathbb{Z}$. For $B = B_r(x_0)$ let us set $B^* = B_{3\eta r}(x_0)$. Then with $c = 4C_U\eta^6\lambda^{-4}$ it holds that*

$$\mathcal{E}_B^{\nu_1\heartsuit\nu_2}(u,u) \le c(\mathcal{E}_{B^*}^{\nu_1}(u,u) + \mathcal{E}_{B^*}^{\nu_2}(u,u)) \tag{6-22}$$

for any measurable function u on B_1 and any B such that $B^* \subset B_1$.

Proof. Let $B = B_r(x_0)$ be such that $B^* \subset B_1$. In particular, this means that $r \le 1/(3\eta)$. By definition, we obtain

$$\begin{aligned}
\mathcal{E}_B^{\nu_1\heartsuit\nu_2}(u,u) &= \iint (u(x)-u(x+z))^2\, \mathbb{1}_B(x)\, \mathbb{1}_B(x+z)\, \nu_1 \heartsuit\, \nu_2(\mathrm{d}z)\,\mathrm{d}x \\
&\le \iiint (u(x)-u(x+\eta(y+z)))^2\, \mathbb{1}_B(x)\, \mathbb{1}_B(x+\eta(y+z))\, g_\lambda^\eta(y,z)\, \nu_1(\mathrm{d}y)\, \nu_2(\mathrm{d}z)\,\mathrm{d}x \\
&\le 2\iiint [(u(x)-u(x+\eta y))^2 + (u(x+\eta y)-u(x+\eta(y+z)))^2] \\
&\qquad\qquad \times \mathbb{1}_B(x)\, \mathbb{1}_B(x+\eta(y+z))\, g_\lambda^\eta(y,z)\, \nu_1(\mathrm{d}y)\, \nu_2(\mathrm{d}z)\,\mathrm{d}x \\
&= 2[I_1+I_2].
\end{aligned} \tag{6-23}$$

We may assume that

$$\lambda|y+z| \le |z| < \eta|y+z| \le 2r,$$
$$\lambda|y+z| \le |y| < \eta|y+z| \le 2r,$$

as otherwise the expression $\mathbb{1}_B(x)\,\mathbb{1}_B(x+\eta(y+z))g_\lambda^\eta(y,z)$ would be zero. Since $2r \le 1$, it follows that $\lambda|y|/\eta < |z| \le \eta|y|/\lambda \wedge 1$. Therefore, by changing the order of integration,

$$I_1 \le \int_B \int_{B_{2r}} \int_{\frac{\lambda|y|}{\eta}\vee\lambda|y+z|\le|z|\le\frac{\eta|y|}{\lambda}\wedge 1} (u(x)-u(x+\eta y))^2|y+z|^\alpha\, \nu_2(\mathrm{d}z)\, \nu_1(\mathrm{d}y)\,\mathrm{d}x.$$

We estimate the inner integral above:

$$J := \int_{\frac{\lambda|y|}{\eta}\vee\lambda|y+z|\le|z|\le\frac{\eta|y|}{\lambda}\wedge 1} |y+z|^\alpha \nu_2(\mathrm{d}z) \le \int_{|z|\le\frac{\eta|y|}{\lambda}\wedge 1} \frac{|z|^\alpha}{\lambda^\alpha} \frac{|z|^{2-\alpha}}{(\lambda|y|/\eta)^{2-\alpha}} \nu_2(\mathrm{d}z) \le \frac{\eta^4 C_U}{\lambda^4}.$$

Coming back to I_1 we obtain,

$$I_1 \le \frac{\eta^4 C_U}{\lambda^4} \int_B \int_{B_{2r}} (u(x) - u(x+\eta y))^2 \, \nu_1(\mathrm{d}y) \, \mathrm{d}x$$
$$= \frac{\eta^4 C_U}{\lambda^4} \eta^\alpha \int_B \int_{B_{2\eta r}} (u(x) - u(x+y))^2 \, \nu_1(\mathrm{d}y) \, \mathrm{d}x \le \frac{\eta^6 C_U}{\lambda^4} \mathcal{E}^{\nu_1}_{B^*}(u,u),$$

where we used (S) and the fact that $B_{2\eta r} \subset B_1$.

Finally, in order to estimate I_2, we first change variables $x = w - \eta y$,

$$I_2 \le \int_B \int_{B_{2r}} \int_{B_{2r}} (u(x+\eta y) - u(x+\eta(y+z)))^2 \, \mathbb{1}_B(x+\eta(y+z)) \, g^\eta_\lambda(y,z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) \, \mathrm{d}x$$
$$\le \int_{B^*} \int_{B_{2r}} (u(w) - u(w+\eta z))^2 \, \mathbb{1}_B(w+\eta z) \int_{B_{2r}} g^\eta_\lambda(y,z) \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) \, \mathrm{d}w$$
$$\le \int_{B^*} \int_{B_{2r}} (u(w) - u(w+\eta z))^2 \, \mathbb{1}_B(w+\eta z) \int_{\frac{\lambda|z|}{\eta} \vee \lambda|y+z| \le |y| \le \frac{\eta|z|}{\lambda} \wedge 1} |y+z|^\alpha \, \nu_1(\mathrm{d}y) \, \nu_2(\mathrm{d}z) \, \mathrm{d}w.$$

By symmetry, the following integral may be estimated exactly like J before:

$$\int_{\frac{\lambda|z|}{\eta} \vee \lambda|y+z| \le |y| \le \frac{\eta|z|}{\lambda} \wedge 1} |y+z|^\alpha \, \nu_1(\mathrm{d}y) \le \frac{\eta^4 C_U}{\lambda^4}.$$

This leads to an estimate

$$I_2 \le \frac{\eta^4 C_U}{\lambda^4} \int_{B^*} \int_{B_{2r}} (u(w) - u(w+\eta z))^2 \, \mathbb{1}_B(w+\eta z) \, \nu_2(\mathrm{d}z) \, \mathrm{d}w$$
$$= \frac{\eta^4 C_U}{\lambda^4} \eta^\alpha \int_{B^*} \int_{B_{2\eta r}} (u(w) - u(w+t))^2 \, \mathbb{1}_B(w+t) \, \nu_2(\mathrm{d}t) \, \mathrm{d}w \le \frac{\eta^6 C_U}{\lambda^4} \mathcal{E}^{\nu_2}_{B^*}(u,u),$$

where we used (S) and the fact that $B_{2\eta r} \subset B_1$. The result follows from (6-23) and the obtained estimates of I_1 and I_2. □

Lemma 6.13. *Let $0 < \alpha_0 < \alpha < 2$, $r_0 > 0$, $\kappa \in (0,1)$, and ν be a measure on $\mathcal{B}(\mathbb{R}^d)$. For $B = B_r(x)$, $x \in \mathbb{R}^d$, $r > 0$, we set $B^* = B_{r/\kappa}(x)$. Suppose that for some $c_\nu > 0$*

$$\mathcal{E}^\nu_{B^*}(u,u) \ge c_\nu \mathcal{E}^{\mu_\alpha}_B(u,u)$$

for every $0 < r \le r_0$, for every $u \in L^2(B_{r_0})$, and for every ball $B \subset B_{r_0}$ of radius κr. Then there exists a constant $c = c(d, \alpha_0, \kappa)$, such that for every ball $B \subset B_{r_0}$ of radius $r \le r_0$ and every $u \in L^2(B_{r_0})$

$$\mathcal{E}^\nu_B(u,u) \ge c c_\nu \mathcal{E}^{\mu_\alpha}_B(u,u).$$

Proof. Fix some $0 < r \le r_0$ and a ball D of radius r. We take $\mathcal{B}$ to be a family of balls with the following properties:

(i) For some $c = c(d)$ and any $x, y \in D$, if $|x-y| < c \operatorname{dist}(x, D^c)$, then there exists $B \in \mathcal{B}$ such that $x, y \in B$.

(ii) For every $B \in \mathcal{B}$, we have $B^* \subset D$.

(iii) Family $\{B^*\}_{B\in\mathcal{B}}$ has the finite overlapping property; that is, each point of D belongs to at most $M = M(d)$ balls B^*, where $B \in \mathcal{B}$.

Such a family $\mathcal{B}$ may be constructed by considering Whitney decomposition of D into cubes and then covering each Whitney cube by an appropriate family of balls.

We have

$$\begin{aligned}\mathcal{E}_D^{\nu}(u,u) &\geq \frac{1}{M^2}\sum_{B\in\mathcal{B}}\int_{B^*}\int_{B^*}(u(x)-u(x+y))^2\,\nu(\mathrm{d}y)\,\mathrm{d}x\\ &\geq \frac{c_\nu}{M^2}(2-\alpha)\sum_{B\in\mathcal{B}}\int_B\int_B(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\\ &\geq \frac{c_\nu}{M^2}(2-\alpha)\int_D\int_{|x-y|<c\,\mathrm{dist}(x,D^c)}(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}\tag{6-24}$$

By [Dyda 2006, Proposition 5 and proof of Theorem 1], we may estimate

$$\begin{aligned}\int_D\int_{|x-y|<c\,\mathrm{dist}(x,D^c)}&(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x\\ &\geq c(\alpha,d)\int_D\int_D(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}\tag{6-25}$$

with some constant $c(\alpha,d)$. We note that in [Dyda 2006, proof of Theorem 1] the constant depends on the domain in question, but in our case, by scaling, we can take the same constant independent of the choice of the ball D. One may also check that $c(\alpha,d)$ stays bounded when $\alpha\in[\alpha_0,2)$. By (6-24) and (6-25) the lemma follows. □

For a linear subspace $E \subset \mathbb{R}^d$, we denote by H_E the $(\dim E)$-dimensional Hausdorff measure on $\mathbb{R}^d$ with the support restricted to E. In particular, $H_{\{0\}} = \delta_{\{0\}}$, the Dirac delta measure at 0.

Proposition 6.14. *Let $E_1, E_2 \subset \mathbb{R}^d$ be two linear subspaces with $E_1, E_2 \neq \{0\}$. Assume that ν_j, $j \in \{1,2\}$, are measures on $\mathcal{B}(\mathbb{R}^d)$ of the form $\nu_j = f_j H_{E_j}$ satisfying $\nu_j(B_1) > 0$, (U), and (S) with $\alpha_0 \in (0,2)$, $C_U > 0$, and $a > 1$. Then the following are true:*

(1) *$\nu_1 \heartsuit \nu_2$ is absolutely continuous with respect to $H_{E_1+E_2}$ and satisfies (U) and (S).*

(2) *If $\eta \geq a^2/(a-1)$ and $\lambda \leq 1/(a^3+1)$, then $\nu_1 \heartsuit \nu_2(B_1) > 0$.*

(3) *If $\nu_j^{\alpha_0} = \nu_j$ and ν_j^{α} is defined as in Definition 6.9 for $\alpha_0 \leq \alpha < 2$, then*

$$\nu_1^{\alpha}\heartsuit\nu_2^{\alpha} \geq \eta^{-2}(\nu_1^{\alpha_0}\heartsuit\nu_2^{\alpha_0})^{\alpha}.\tag{6-26}$$

Proof. Properties (U) and (S) follow from Lemmas 6.11 and 6.7, respectively. Let $E = E_1 \cap E_2$ and let F_j be linear subspaces such that $E_j = E \oplus F_j$, where $j = 1,2$. For $y \in E_1$ let us write $y = Y + \tilde{y}$, where $Y \in E$ and $\tilde{y} \in F_1$; similarly, for $z \in E_2$ we write $z = Z + \hat{z}$, where $Z \in E$ and $\hat{z} \in F_2$. Then for $A \subset B_2$

$$\begin{aligned}\nu_1 \heartsuit \nu_2(A) &= \iiiint \mathbb{1}_A(\eta(Y+\tilde{y}+Z+\hat{z}))\, g_\lambda^\eta(Y+\tilde{y}, Z+\hat{z}) \\ &\qquad\qquad \times f_1(Y+\tilde{y})\, f_2(Z+\hat{z})\, H_E(\mathrm{d}Y)\, H_E(\mathrm{d}Z)\, H_{F_1}(\mathrm{d}\tilde{y})\, H_{F_2}(\mathrm{d}\hat{z}) \\ &= \iiint \mathbb{1}_A(\eta(W+\tilde{y}+\hat{z}))\Big(\int g_\lambda^\eta(Y+\tilde{y}, W-Y+\hat{z})\, f_1(Y+\tilde{y})\, f_2(W-Y+\hat{z})\, H_E(\mathrm{d}Y)\Big) \\ &\qquad\qquad \times H_E(\mathrm{d}W)\, H_{F_1}(\mathrm{d}\tilde{y})\, H_{F_2}(\mathrm{d}\hat{z}) \end{aligned} \tag{6-27}$$

and since $\nu_1 \heartsuit \nu_2(\mathbb{R}^d \setminus B_2) = 0$, the desired absolute continuity follows.

To show nondegeneracy, let $G_n := B_{a^{-n}} \setminus B_{a^{-n-1}}$. By scaling property (S) it follows that $\nu_j(G_{n+1}) = a^\alpha \nu_j(G_n)$; therefore $\nu_j(G_n) > 0$ for each $n = 0, 1, \ldots$. Hence

$$\nu_1 \heartsuit \nu_2(B_1) \geq \frac{1}{2-\alpha_0} \int_{G_n} \int_{G_{n+2}} \mathbb{1}_{B_1}(\eta(y+z))\, \mathbb{1}_{A_{|y+z|}}(y)\, \mathbb{1}_{A_{|y+z|}}(z)\, |y+z|^\alpha\, \nu_1(\mathrm{d}y)\, \nu_2(\mathrm{d}z).$$

For $(y, z) \in G_{n+2} \times G_n$ it holds that

$$\frac{a-1}{a^2}(|y| \vee |z|) \leq |y+z| \leq (a^3+1)(|y| \wedge |z|)$$

and also $\eta(y+z) \in B_1$, provided n is large enough. Therefore $\nu_1 \heartsuit \nu_2(B_1) > 0$, if $\eta \geq a^2/(a-1)$ and $\lambda \leq 1/(a^3+1)$.

To prove the last part of the lemma, we calculate first the most inner integral in (6-27) corresponding to $\nu_1^\alpha \heartsuit \nu_2^\alpha$; it equals

$$\begin{aligned} L &:= \int g_\lambda^\eta(Y+\tilde{y}, W-Y+\hat{z})\, f_1^\alpha(Y+\tilde{y})\, f_2^\alpha(W-Y+\hat{z})\, H_E(\mathrm{d}Y) \\ &= \frac{2-\alpha}{(2-\alpha_0)^2} \int |W+\tilde{y}+\hat{z}|^\alpha\, |Y+\tilde{y}|^{\alpha_0-\alpha}\, |W-Y+\hat{z}|^{\alpha_0-\alpha}\, \mathbb{1}(\cdots)\, f_1^{\alpha_0}(Y+\tilde{y})\, f_2^{\alpha_0}(W-Y+\hat{z})\, H_E(\mathrm{d}Y), \end{aligned}$$

where we used an abbreviation

$$\mathbb{1}(\cdots) := \mathbb{1}_{A_{|W+\tilde{y}+\hat{z}|}}(Y+\tilde{y})\, \mathbb{1}_{A_{|W+\tilde{y}+\hat{z}|}}(W-Y+\hat{z}).$$

On the other hand, the most inner integral in (6-27) corresponding to $(\nu_1^{\alpha_0} \heartsuit \nu_2^{\alpha_0})^\alpha$ is

$$\begin{aligned} R &:= \frac{2-\alpha}{2-\alpha_0}(\eta|W+\tilde{y}+\hat{z}|)^{\alpha_0-\alpha} \int g_\lambda^\eta(Y+\tilde{y}, W-Y+\hat{z})\, f_1^{\alpha_0}(Y+\tilde{y})\, f_2^{\alpha_0}(W-Y+\hat{z})\, H_E(\mathrm{d}Y) \\ &= \frac{(2-\alpha)\eta^{\alpha_0-\alpha}}{(2-\alpha_0)^2} \int |W+\tilde{y}+\hat{z}|^{2\alpha_0-\alpha}\, \mathbb{1}(\cdots)\, f_1^{\alpha_0}(Y+\tilde{y})\, f_2^{\alpha_0}(W-Y+\hat{z})\, H_E(\mathrm{d}Y). \end{aligned}$$

Inequality (6-26) follows now from the estimate

$$|Y+\tilde{y}|^{\alpha_0-\alpha}|W-Y+\hat{z}|^{\alpha_0-\alpha}\mathbb{1}(\cdots) \geq (\eta|W+\tilde{y}+\hat{z}|)^{2(\alpha_0-\alpha)}\mathbb{1}(\cdots)$$

and the fact that both sides of (6-26) are zero on $\mathbb{R}^d \setminus B_2$. □

Proof of Theorem 1.11: lower bound in (A). We recall from Section 1E that we may and do assume that f_k are symmetric, i.e., $f_k(x) = f_k(-x)$ for all x. By Proposition 6.14 it follows that the measure

$$\nu := (f_1 H_{E_1}) \heartsuit (f_2 H_{E_2}) \heartsuit \cdots \heartsuit (f_n H_{E_n})$$

satisfies (U) and (S) and has density h with respect to the Lebesgue measure on $\mathcal{B}(\mathbb{R}^d)$ with $\int_{B_1} h(x)\,\mathrm{d}x > 0$, if η is large enough and λ small enough. We will show that the measure $\nu \heartsuit \nu$ possesses a density $h^{\heartsuit}$ with $h^{\heartsuit}(x) \geq c|x|^{-d-\alpha_0}$ for all $x \in B_1 \setminus \{0\}$ and some positive constant c to be specified. This, together with the preliminary results, will establish the assertion.

Condition (S) for ν implies that $h(ax) = a^{-d-\alpha_0} h(x)$ if $x \in B_{1/a}$. Therefore $\int_{G_0} h(x)\,\mathrm{d}x > 0$, where $G_0 = B_1 \setminus B_{1/a}$. Define $h^{G_0}(x) = h(x)\,\mathbb{1}_{G_0}(x) \wedge 1$. The function

$$x \mapsto h^{G_0} * h^{G_0}(x) = \int h^{G_0}(y-x)\, h^{G_0}(y)\,\mathrm{d}y$$

is continuous and strictly positive at 0. Thus there exists $\delta \in (0, (2a)^{-1})$ and $\varepsilon > 0$ such that

$$h^{G_0} * h^{G_0}(x) \geq \varepsilon \quad \text{for } x \in B_\delta.$$

We consider the measure $\nu \heartsuit \nu$; it has density $h^{\heartsuit}$ with respect to the Lebesgue measure on $\mathcal{B}(B_2)$ given by formula, see (6-16),

$$\begin{aligned} h^{\heartsuit}(x) &= \eta^{-2d} \int g_\lambda^\eta\Big(\frac{w}{\eta}, \frac{x-w}{\eta}\Big) h\Big(\frac{w}{\eta}\Big) h\Big(\frac{x-w}{\eta}\Big)\,\mathrm{d}w \\ &\geq \eta^{2\alpha_0} \int_{G_0} g_\lambda^\eta\Big(\frac{w}{\eta}, \frac{x-w}{\eta}\Big)\, \mathbb{1}_{G_0}(x-w)\, h(w) h(x-w)\,\mathrm{d}w \\ &= \frac{\eta^{\alpha_0}}{2-\alpha_0} \int_{G_0} |x|^{\alpha_0}\, \mathbb{1}_{A_{|x|}}(w)\, \mathbb{1}_{A_{|x|}}(x-w)\, \mathbb{1}_{G_0}(x-w)\, h(w)\, h(x-w)\,\mathrm{d}w. \end{aligned}$$

Suppose $\eta \geq a^2/\delta$ and $\lambda \leq 1/(a\delta)$. Then for $x \in B_\delta \setminus B_{\delta/a^2}$ and $w \in G_0$ such that $x - w \in G_0$ it holds that

$$\mathbb{1}_{A_{|x|}}(w)\, \mathbb{1}_{A_{|x|}}(x-w) = 1.$$

This leads to the estimate

$$h^{\heartsuit}(x) \geq \frac{\eta^{\alpha_0}\delta^{\alpha_0} a^{-2\alpha_0}}{2-\alpha_0} h^{G_0} * h^{G_0}(x) \geq \frac{\varepsilon}{2-\alpha_0} \qquad \text{for } x \in B_\delta \setminus B_{\delta/a^2}.$$

For $x \in B_1 \setminus \{0\}$ let $k \in \mathbb{Z}$ be such that $\delta/a^2 < |x|a^k < \delta < |x|a^{k+1}$. Then, by scaling (S),

$$h^{\heartsuit}(x) = a^{k(d+\alpha_0)}\, h^{\heartsuit}(xa^k) \geq \frac{a^{k(d+\alpha_0)}\varepsilon}{2-\alpha_0} \geq \frac{\delta^{d+\alpha_0}\varepsilon}{a^{2d+2\alpha_0}(2-\alpha_0)} |x|^{-d-\alpha_0}.$$

Now from Lemmas 6.12 and 6.13 it follows that for any $B \subset B_1$

$$\mathcal{E}_B^{\mu_{\alpha_0}}(u,u) \leq c\,\mathcal{E}_B^{\nu_*}(u,u), \tag{6-28}$$

with $c = c((f_j),(E_j))$.

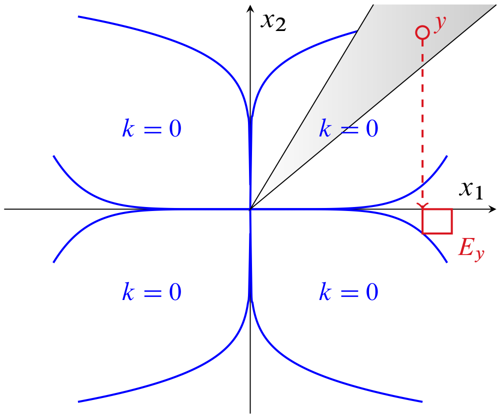

Figure 1. Support of the kernel k $\left(\text{with } b=\frac{1}{6}\right)$ consisting of four thorns. The set P from the proof below is shown, too.

Finally, to obtain a robust result, we observe that by (6-26)

$$\underbrace{(\nu_*)^\alpha\heartsuit\cdots\heartsuit(\nu_*)^\alpha}_{2n \text{ factors}} \geq \eta^{-2(2n-1)}(\underbrace{\nu_*\heartsuit\cdots\heartsuit\nu_*}_{2n\text{ factors}})^\alpha \geq \eta^{-2(2n-1)}\frac{2-\alpha}{2-\alpha_0}|x|^{\alpha_0-\alpha}\frac{\delta^{d+\alpha_0}\varepsilon}{a^{2d+2\alpha_0}}|x|^{-d-\alpha_0}\mathbb{1}_{B_1}(x)\,\mathrm{d}x.$$

This together with Lemmas 6.12 and 6.13 gives us

$$\mathcal{E}_B^\alpha(u,u) \leq c\,\mathcal{E}_B^{(\nu_*)^\alpha}(u,u),$$

with the constant c *not depending* on $\alpha \in [\alpha_0, 2)$. □

6E. ***Examples.*** In this subsection, we provide two examples showing that the assumptions of Theorem 1.11 are not necessary for (A) and (B). Note that condition (A) relates to integrated quantities but does not require pointwise bounds on the density of $\mu(x,\mathrm{d}y)$.

Example 6.15. Let $b \in (0,1)$ and

$$\Gamma = \{(x_1,x_2) \in \mathbb{R}^2 : |x_2| \geq |x_1|^b \text{ or } |x_1| \geq |x_2|^b\}.$$

We consider the following function

$$k(z) = (2-\alpha)\,\mathbb{1}_{\Gamma\cap B_1}(z)|z|^{-2-\beta}, \quad z \in \mathbb{R}^2, \tag{6-29}$$

where $\beta = \alpha - 1 + 1/b$; see Figure 1. Let us show that conditions (A) and (B) are satisfied in this case.

We have, for $0 < r < 1$,

$$\begin{aligned}\int_{B_r} |z|^2 k(z)\,\mathrm{d}z &\leq 8(2-\alpha)\int_0^r\int_0^{x^{1/b}} (x^2+y^2)^{-\frac{\beta}{2}}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq 8(2-\alpha)\int_0^r\int_0^{x^{1/b}} x^{-\beta}\,\mathrm{d}y\,\mathrm{d}x = 8r^{2-\alpha};\end{aligned} \tag{6-30}$$

hence k satisfies (U1) with $C_1 = 8$. Since (U0) is clear, from Lemma 6.8 we conclude that k satisfies (U).

Let

$$P = \{x \in B_{\frac{1}{4}} : 0 < x_1 < x_2 < 2x_1\}$$

and, for $y = (x_1, x_2) \in P$, let

$$E_y = [x_1, x_1 + x_1^{\frac{1}{b}}] \times [-x_1^{\frac{1}{b}}, 0].$$

It is easy to check that if $y \in P$ and $z \in E_y$, then

$$\frac{|y|}{3} \le |z| \le 4|y|, \quad \frac{|y|}{3} \le |y-z| \le 4|y|, \quad \text{and} \quad z, y-z \in \Gamma \cap B_1.$$

Let $\eta = 4$ and $\lambda = \frac{1}{3}$. Then for $y \in P$

$$\begin{aligned} k \heartsuit k(\eta y) &= \frac{|y|^\alpha}{2-\alpha} \int \mathbb{1}_{A_{|y|}}(z)\, \mathbb{1}_{A_{|y|}}(y-z)(2-\alpha)^2\, \mathbb{1}_{\Gamma\cap B_1}(z)\, \mathbb{1}_{\Gamma\cap B_1}(y-z)\, |z|^{-2-\beta}\, |y-z|^{-2-\beta}\, \mathrm{d}z \\ &\ge (2-\alpha)|y|^\alpha \int_{E_y} |z|^{-2-\beta}\, |y-z|^{-2-\beta}\, \mathrm{d}z \\ &\ge (2-\alpha)|y|^\alpha (4|y|)^{2(-2-\beta)} x_1^{\frac{2}{b}} \ge (2-\alpha)3^{-\frac{2}{b}} 4^{-4-2\beta} |y|^{-2-\alpha} \ge 4^{-6} 12^{-\frac{2}{b}} (2-\alpha)|y|^{-2-\alpha}. \end{aligned}$$

In the following example, we provide a condition that cannot be handled by Theorem 1.11 but still implies comparability of corresponding quadratic forms.

Example 6.16. For a measure ν on $\mathcal{B}(\mathbb{R}^d)$ with a density k with respect to the Lebesgue measure we formulate the following condition:

There exist $a > 1$ and $C_2, C_3 > 0$ such that every annulus $B_{a^{-n+1}} \setminus B_{a^{-n}}$ $(n = 0, 1, \dots)$ contains a ball B_n with radius $C_2 a^{-n}$ such that $k(z) \ge C_3(2-\alpha)|z|^{-d-\alpha}$, $z \in B_n$. (6-31)

The following proposition provides a substitute for Theorem 1.11.

Proposition 6.17. *Let $a > 1$, $\alpha_0 \in (0,2)$, $\alpha \in [\alpha_0, 2)$, and $C_U, C_2, C_3 > 0$. Let $\mu = (\mu(x,\cdot))_{x\in\mathbb{R}^d}$ be a family of measures on $\mathbb{R}^d$ which satisfies (1-2). Furthermore, we assume that there exist measures ν_* and ν^* with property (T) such that (U) and (6-31) hold with exponent α and the constants C_U, C_2, C_3. Then there is $A = A(a, \alpha_0, C_U, C_2, C_3) \ge 1$ not depending on α such that (A) hold.*

Proof. We fix $\lambda < 2/C_2 \wedge 1$ and $\eta \ge 2a^2/C_2 \vee 1$. For some $n \in \{0, 1, \dots\}$, let

$$\frac{C_2}{2} a^{-n-1} \le |y| \le \frac{C_2}{2} a^{-n},$$

and assume that $\eta y \in B_2$. By formula (6-16), we obtain

$$k \heartsuit k(\eta y) \ge \frac{\eta^{-d}|y|^\alpha}{2-\alpha} \int \mathbb{1}_{A_{|y|}}(y-z)\, \mathbb{1}_{A_{|y|}}(z)\, k(y-z)\, k(z)\, \mathrm{d}z.$$

Let us denote by B_n^o the ball concentric with B_n, but with radius $C_2 a^{-n}/2$ (that is, B_n^o is twice smaller than B_n). We observe that if $z \in B_n^o$, then $y - z \in B_n$. Furthermore, by our choice of λ and η it follows that

$$\lambda|y| \le |y-z| < \eta|y|, \quad \lambda|y| \le |z| < \eta|y| \quad \text{if } z \in B_n^o;$$

that is, $y-z,\, z \in A_{|y|}$ for $z \in B_n^o$. Hence

$$\begin{aligned} k \heartsuit k(\eta y) &\geq \frac{\eta^{-d}|y|^\alpha}{2-\alpha} C_3^2 (2-\alpha)^2 \int_{B_n^0} |y-z|^{-d-\alpha} |z|^{-d-\alpha}\,\mathrm{d}z \\ &\geq \frac{C_3^2 \eta^{-d}(2-\alpha) C_2^{2d+2\alpha}}{2^{2d+2\alpha} a^{3d+4\alpha}} |y|^{-d-\alpha} \\ &\geq C(\alpha_0, d, C_2, C_3, \eta, a)(2-\alpha)|y|^{-d-\alpha}, \end{aligned}$$

or, equivalently, for $w \in B_2$

$$k \heartsuit k(w) \geq C'(\alpha_0, d, C_2, C_3, \eta, a)(2-\alpha)|w|^{-d-\alpha}.$$

By Lemmas 6.12 and 6.13 we conclude that the lower estimate in (A) holds. The upper estimate is in turn a consequence of Proposition 6.1. □

7. Global comparability results for nonlocal quadratic forms

In this section we provide a global comparability result; i.e., we study comparability in the whole $\mathbb{R}^d$. This result is not needed for the other results in this article; however it contains an interesting and useful observation.

Proposition 7.1. *Assume (U) holds. Then there exists a constant $c = c(\alpha, d, C_U)$ such that*

$$\mathcal{E}^\mu(u,u) \leq c(\mathcal{E}^{\mu_\alpha}(u,u) + \|u\|^2_{L^2(\mathbb{R}^d)}) \quad \textit{for every } u \in L^2(\mathbb{R}^d). \tag{7-1}$$

Furthermore, if (U) is satisfied for all $r > 0$, *then for every $u \in L^2(\mathbb{R}^d)$*

$$\mathcal{E}^\mu(u,u) \leq c\mathcal{E}^{\mu_\alpha}(u,u). \tag{7-2}$$

If the constant C_U in (U) is independent of $\alpha \in (\alpha_0, 2)$, where $\alpha_0 > 0$, then so are the constants in (7-1) and (7-2).

Proof. By E we denote the identity operator from $H^{\alpha/2}(\mathbb{R}^d)$ to itself. One easily checks that the proof of Proposition 6.1 from (6-2) until (6-5) works also in the present case of $D = \mathbb{R}^d$. Hence (7-1) follows.

To prove (7-2) we observe that if (U) holds for all $r > 0$, then also (6-4) holds for *all* $\xi \neq 0$; we plug it into (6-3) and we are done. □

We consider the following condition.

(K2,r_0) There exists $c_0 > 0$ such that for all $h \in S^{d-1}$ and all $0 < r < r_0$

$$\int_{\mathbb{R}^d} r^2 \sin^2\left(\frac{h \cdot z}{r}\right) \nu_*(\mathrm{d}z) \geq c_0 r^{2-\alpha}. \tag{7-3}$$

Clearly (6-31) implies (K2,r_0)2,r_0 for $r_0 = 1$, and if C_3 is independent of $\alpha \in (\alpha_0, 2)$, where $\alpha_0 > 0$, then so is c_0. Condition (K2,r_0)2,r_0 is also satisfied if for all $h \in S^{d-1}$ and all $0 < r < r_0$

$$\int_{B_r(0)} |h \cdot z|^2 \nu_*(\mathrm{d}z) \geq c_2 r^{2-\alpha}. \tag{7-4}$$

We note that (7-5) under condition (7-4) has been proved in [Abels and Husseini 2010]. The following theorem extends their result by giving a *characterization* of kernels ν_* admitting comparability (7-5). We stress that $r_0 = \infty$ is allowed, and in such a case we put $1/r_0^\alpha = 0$.

Theorem 7.2. *Let* $0 < r_0 \le \infty$. *If* (K2,r_0)2,r_0 *holds, then*

$$\mathcal{E}^{\mu_\alpha}(u,u) \le \frac{1}{c_0}\mathcal{E}^{\mu}(u,u) + \frac{2^\alpha}{r_0^\alpha}\|u\|_{L^2}^2, \quad u \in C_c^1(\mathbb{R}^d). \tag{7-5}$$

Conversely, if for some $c < \infty$

$$\mathcal{E}^{\mu_\alpha}(u,u) \le c\mathcal{E}^{\nu_*}(u,u) + \frac{2^\alpha}{r_0^\alpha}\|u\|_{L^2}^2, \quad u \in \mathcal{S}(\mathbb{R}^d), \tag{7-6}$$

then (K2,r_0)2,r_0 *holds.*

Proof. Recalling that $(u(\cdot + z))\hat{}\,(\xi) = e^{i\xi\cdot z}\hat{u}(\xi)$ and using Plancherel's formula we obtain

$$\begin{aligned} \mathcal{E}^{\mu}(u,u) &\ge \iint (u(x) - u(x+z))^2 \,\mathrm{d}x\, \nu_*(\mathrm{d}z) \\ &= \iint |e^{i\xi\cdot z} - 1|^2 |\hat{u}(\xi)|^2 \,\mathrm{d}\xi\, \nu_*(\mathrm{d}z) \\ &= \int \left(\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu_*(\mathrm{d}z) \right) |\hat{u}(\xi)|^2 \,\mathrm{d}\xi. \end{aligned} \tag{7-7}$$

If (K2,r_0)2,r_0 holds, then for all $|\xi| > 2/r_0$

$$\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu_*(\mathrm{d}z) \ge \frac{4c_0}{2^\alpha}|\xi|^\alpha \ge c_0|\xi|^\alpha.$$

For $|\xi| \le 2/r_0$ we have $|\xi|^\alpha \le (2/r_0)^\alpha$. Inequality (7-5) follows from

$$\frac{\mathcal{A}_{d,-\alpha}}{2\alpha(2-\alpha)}\mathcal{E}^\alpha_{\mathbb{R}^d}(u,u) = \int_{\mathbb{R}^d} |\xi|^\alpha |\hat{u}(\xi)|^2 \,\mathrm{d}\xi. \tag{7-8}$$

Now we prove the converse. Assume (7-6). By (7-7), the right-hand side of (7-6) equals

$$\int \left(c\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu_*(\mathrm{d}z) + \frac{2^\alpha}{r_0^\alpha} \right) |\hat{u}(\xi)|^2 \,\mathrm{d}\xi;$$

hence by (7-8) and (7-6) we obtain that

$$c\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu_*(\mathrm{d}z) + \frac{2^\alpha}{r_0^\alpha} \ge |\xi|^\alpha \quad \text{for a.e. } \xi \in \mathbb{R}^d. \tag{7-9}$$

By continuity of the function

$$\mathbb{R}^d \setminus \{0\} \ni \xi \mapsto \int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu_*(\mathrm{d}z),$$

(7-9) holds for all $\xi \in \mathbb{R}^d$. For $|\xi| \ge 2^{1+1/\alpha} r_0^{-1}$ we have by (7-9)

$$c\int 4\sin^2\left(\frac{\xi\cdot z}{2}\right) \nu_*(\mathrm{d}z) \ge \frac{|\xi|^\alpha}{2},$$

and hence (K2,$2^{-1/\alpha}r_0$) holds with $c_0 = 2^{\alpha-3}c^{-1}$. Since

$$\sin^2\left(\frac{h\cdot z}{2r}\right) \geq \frac{1}{4}\sin^2\left(\frac{h\cdot z}{r}\right),$$

also (K2,r_0)2,r_0 holds with *some* constant c_0. □

Appendix

We give the proof of Lemma 4.3. It only uses basic observations.

Lemma A.1. *Assume* $\tau_1, \tau_2 \geq 0$ *and* $\tau_1/\tau_2 \in \left[\frac{1}{2}, 2\right]$. *Then*

$$\frac{\tau_1^2+\tau_2^2}{|\tau_1^2-\tau_2^2|} \geq \frac{5}{3}.$$

Proof. Note that

$$\frac{\tau_1^2+\tau_2^2}{|\tau_1^2-\tau_2^2|} = \frac{\tau_1^2/\tau_2^2+1}{|\tau_1^2/\tau_2^2-1|} = \frac{t+1}{|t-1|},$$

where $t = \tau_1^2/\tau_2^2$. There are three cases:

(1) If $t = 1$, then

$$\frac{t+1}{t-1} = +\infty$$

and the assertion is true.

(2) If $t > 1$, then

$$\frac{t+1}{|t-1|} = \frac{t+1}{t-1}.$$

Note that $(t+1)/(t-1) \geq \frac{5}{3}$ holds true if and only if

$$t+1 \geq \frac{5}{3}t - \frac{5}{3} \iff t \leq 4 \iff \frac{\tau_1}{\tau_2} \leq 2.$$

(3) If $t < 1$, then

$$\frac{t+1}{|t-1|} = \frac{t+1}{-t+1}.$$

Note that $(t+1)/(-t+1) \geq \frac{5}{3}$ holds true if and only if

$$t+1 \geq -\frac{5}{3}t + \frac{5}{3} \iff t \geq \frac{1}{4} \iff \frac{\tau_1}{\tau_2} \geq \frac{1}{2}.$$ □

Lemma A.2. *Assume* $p > 1$ *and* $\eta \in \left(1, \frac{5}{3}\right)$. *Set* $\lambda = ((\eta-1)/(1+\eta))^{1/p}$. *Assume* $a, b > 0$ *and* $b/a \notin (\lambda, 1/\lambda)$. *Then*

$$\frac{a^{-p}+b^{-p}}{|a^{-p}-b^{-p}|} \leq \eta.$$

Proof. Set $t = (b/a)^p$. Then

$$\frac{a^{-p}+b^{-p}}{|a^{-p}-b^{-p}|} = \frac{(a/b)^{-p}+1}{|(a/b)^{-p}-1|} = \frac{t+1}{|t-1|}.$$

Now there are two cases:

Case 1: $t > 1$.

$$\frac{t+1}{|t-1|} \le \eta \iff \frac{t+1}{t-1} \le \eta \iff t \ge \frac{1+\eta}{\eta-1} \iff \frac{b}{a} \ge \left(\frac{1+\eta}{\eta-1}\right)^{1/p}.$$

Case 2: $t < 1$.

$$\frac{t+1}{|t-1|} \le \eta \iff \frac{t+1}{-t+1} \le \eta \iff t \le \frac{\eta-1}{1+\eta} \iff \frac{b}{a} \le \left(\frac{\eta-1}{1+\eta}\right)^{1/p}. \qquad \square$$

Lemma A.3. *There is $c_1 > 0$ such that for $p > 1$, $\lambda = \left(\frac{1}{7}\right)^{1/p}$, and $a, b > 0$ with $b/a \in (\lambda, 1/\lambda)$ the following is true*:

$$\frac{|b-a|(a^{-p}+b^{-p})^2}{|a^{-p}-b^{-p}|} \le \frac{c_1}{p}(b^{-p+1}+a^{-p+1}).$$

Proof. Set $b/a = \xi \in (\lambda, 1/\lambda)$. Then

$$\begin{aligned}\frac{|b-a|(a^{-p}+b^{-p})^2}{|a^{-p}-b^{-p}|} \le \frac{c_1}{p}(b^{-p+1}+a^{-p+1}) &\iff \frac{|a||\xi-1|a^{-2p}(1+\xi^{-p})^2}{|\xi^{-p}-1|a^{-p}} \le \frac{c_1}{p}a^{-p+1}(\xi^{-p+1}+1)\\ &\iff \frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|} \le \frac{c_1}{p}(\xi^{-p+1}+1)\\ &\iff \frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \le \frac{c_1}{p}. \qquad \text{(A-1)}\end{aligned}$$

Let us prove (A-1). Note that

$$\frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \le \frac{|\xi-1|(1+7)^2}{|\xi^{-p}-1|} = 64\frac{|\xi-1|}{|\xi^{-p}-1|}.$$

We want to apply the mean value theorem. Set $\xi \mapsto g(\xi) = \xi^{-p}$. Then $g'(\xi) = (-p)\xi^{-(p+1)}$. The mean value theorem implies

$$\frac{|\xi^{-p}-1|}{|\xi-1|} = \frac{|g(\xi)-1|}{|\xi-1|} = |g(x)| = px^{-(p+1)} \quad \text{for some } x \in (\xi,1) \cup (1,\xi).$$

Thus,

$$\frac{|\xi^{-p}-1|}{|\xi-1|} \ge p\left(\frac{1}{\lambda}\right)^{-(p+1)} = p(7^{1/p})^{-(p+1)} = p7^{-1-\frac{1}{p}},$$

from which we deduce

$$\frac{|\xi-1|(1+\xi^{-p})^2}{|\xi^{-p}-1|(\xi^{-p+1}+1)} \le 64\frac{7^{1+\frac{1}{p}}}{p} \le \frac{64\cdot 49}{p} = \frac{c_1}{p}. \qquad \square$$

Lemma A.4. *For $p>1$ and $a,b>0$ the following is true*:

$$(b-a)(a^{-p}-b^{-p}) \geq \frac{2}{p-1}(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}})^2.$$

The proof of the above lemma is simple and can be found in several places, e.g., in [Kassmann 2009].

Lemma A.5. *Assume $p>1$, $a,b>0$, and $\tau_1,\tau_2\geq 0$. Then*

$$(\tau_1+\tau_2)^2(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}})^2 \geq 2(\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}})^2-2(\tau_1-\tau_2)^2(a^{-p+1}+b^{-p+1}).$$

Proof. Note

$$2(\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}}) = (\tau_1-\tau_2)(a^{\frac{-p+1}{2}}+b^{\frac{-p+1}{2}})+(\tau_1+\tau_2)(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}}).$$

From this equality we obtain the assertion as follows:

$$\begin{aligned}4(\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}})^2 &\leq 2(\tau_1-\tau_2)^2(a^{\frac{-p+1}{2}}+b^{\frac{-p+1}{2}})^2+2(\tau_1+\tau_2)^2(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}})^2\\ &\leq 4(\tau_1-\tau_2)^2(a^{-p+1}+b^{-p+1})+2(\tau_1+\tau_2)^2(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}})^2. \qquad \square\end{aligned}$$

Finally, we can give the proof of Lemma 4.3.

Proof of Lemma 4.3. Let us first consider the case $\tau_1/\tau_2 \notin \left(\frac{1}{2},2\right)$. Note that, in this case

$$\max\{\tau_1,\tau_2\} \leq 2|\tau_1-\tau_2| \tag{A-2}$$

and

$$-(\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}})^2 = -\tau_1^2 a^{-p+1}-\tau_2^2 b^{-p+1}+2\tau_1 a^{\frac{-p+1}{2}}\tau_2 b^{\frac{-p+1}{2}} \geq -\tau_1^2 a^{-p+1}-\tau_2^2 b^{-p+1}.$$

Thus, we obtain

$$\begin{aligned}(b-a)(\tau_1^2 a^{-p}-\tau_2^2 b^{-p}) &\geq -\tau_1^2 a^{-p+1}-\tau_2^2 b^{-p+1}+(\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}})^2-(\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}})^2\\ &\geq (\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}})^2-2\tau_1^2 a^{-p+1}-2\tau_2^2 b^{-p+1}\\ &\geq (\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}})^2-2\max\{\tau_1,\tau_2\}^2 a^{-p+1}-2\max\{\tau_1,\tau_2\}^2 b^{-p+1}\\ &\geq (\tau_1 a^{\frac{-p+1}{2}}-\tau_2 b^{\frac{-p+1}{2}})^2-8(\tau_1-\tau_2)^2(a^{-p+1}+b^{-p+1}).\end{aligned}$$

The proof in the case $\tau_1/\tau_2 \notin \left(\frac{1}{2},2\right)$ is complete.

Let us now assume $\tau_1/\tau_2 \in \left[\frac{1}{2},2\right]$. A general observation is

$$(b-a)(\tau_1^2 a^{-p}-\tau_2^2 b^{-p}) = \underbrace{\tfrac{1}{2}(b-a)(\tau_1^2-\tau_2^2)(a^{-p}+b^{-p})}_{:=P}+\underbrace{\tfrac{1}{2}(b-a)(\tau_1^2+\tau_2^2)(a^{-p}-b^{-p})}_{:=G}.$$

Recall that Lemma A.4 implies

$$\frac{1}{2}(b-a)(a^{-p}-b^{-p}) \geq \frac{1}{p-1}(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}})^2. \tag{A-3}$$

Choose $\eta=\frac{4}{3}$ and $\lambda=\left(\frac{1}{7}\right)^{1/p}$. Let us consider two subcases.

Case 1: $b/a \in (\lambda, 1/\lambda)$, $\tau_1/\tau_2 \in \left[\frac{1}{2}, 2\right]$. In this case

$$\begin{aligned}|P| &= \left[\frac{1}{4}(\tau_1+\tau_2)|b-a|^{\frac12}|a^{-p}-b^{-p}|^{\frac12}\right][2|\tau_1-\tau_2|\,|a^{-p}-b^{-p}|^{-\frac12}|b-a|^{\frac12}(a^{-p}+b^{-p})] \\ &\le \frac{1}{16}(\tau_1+\tau_2)^2(b-a)(a^{-p}-b^{-p}) + 4(\tau_1-\tau_2)^2\underbrace{\frac{(b-a)(a^{-p}+b^{-p})^2}{(a^{-p}-b^{-p})}}_{:=F}.\end{aligned}$$

Because of Lemma A.3, we know that there is $c_5 > 0$ such that $|F| \le (c_5/p)(b^{-p+1}+a^{-p+1})$. Altogether, we obtain

$$\begin{aligned}&(b-a)(\tau_1^2a^{-p}-\tau_2^2b^{-p}) \\ &\quad= \frac12(b-a)(\tau_1^2-\tau_2^2)(a^{-p}+b^{-p}) + \frac12(b-a)(\tau_1^2+\tau_2^2)(a^{-p}-b^{-p}) \\ &\quad\ge \frac12(b-a)(\tau_1^2-\tau_2^2)(a^{-p}+b^{-p}) + \frac14(b-a)(\tau_1+\tau_2)^2(a^{-p}-b^{-p}) \\ &\quad\ge -\frac{1}{16}(\tau_1+\tau_2)^2(b-a)(a^{-p}-b^{-p}) - 4(\tau_1-\tau_2)^2\frac{(b-a)(a^{-p}+b^{-p})^2}{(a^{-p}-b^{-p})} + \frac14(b-a)(\tau_1+\tau_2)^2(a^{-p}-b^{-p}) \\ &\quad= \frac{3}{16}(\tau_1+\tau_2)^2(b-a)(a^{-p}-b^{-p}) - 4(\tau_1-\tau_2)^2\frac{(b-a)(a^{-p}+b^{-p})^2}{(a^{-p}-b^{-p})} \\ &\quad\ge \frac{3}{16(p-1)}(\tau_1+\tau_2)^2(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}})^2 - \frac{4c_5}{p}(\tau_1-\tau_2)^2(b^{-p+1}+a^{-p+1}) \\ &\quad\ge \frac{6}{16(p-1)}(\tau_1a^{\frac{-p+1}{2}}-\tau_2b^{\frac{-p+1}{2}})^2 - \left(\frac{4c_5}{p}+\frac{6}{16(p-1)}\right)(\tau_1-\tau_2)^2(b^{-p+1}+a^{-p+1}),\end{aligned}$$

where we applied Lemma A.5. The first case has been completed.

Case 2: $b/a \notin (\lambda, 1/\lambda)$, $\tau_1/\tau_2 \in \left[\frac12, 2\right]$. Then Lemmas A.1 and A.2 imply

$$\begin{aligned}P \ge -|P| &= -\frac12|b-a|\,|\tau_1^2-\tau_2^2|(a^{-p}+b^{-p}) \ge -\frac{3}{10}|b-a|(\tau_1^2+\tau_2^2)(a^{-p}+b^{-p}) \\ &\ge -\frac{3}{10}\cdot\frac43|b-a|(\tau_1^2+\tau_2^2)|a^{-p}-b^{-p}| = -\frac25(b-a)(\tau_1^2+\tau_2^2)(a^{-p}-b^{-p}) = -\frac45 G.\end{aligned}$$

Thus, due to Lemma A.4, we obtain

$$\begin{aligned}(b-a)(\tau_1^2a^{-p}-\tau_2^2b^{-p}) = P+G \ge \tfrac15 G &\ge \frac{1}{5(p-1)}(\tau_1^2+\tau_2^2)(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}})^2 \\ &\ge \frac{1}{10(p-1)}(\tau_1+\tau_2)^2(a^{\frac{-p+1}{2}}-b^{\frac{-p+1}{2}})^2 \\ &\ge \frac{1}{5(p-1)}(\tau_1a^{\frac{-p+1}{2}}-\tau_2b^{\frac{-p+1}{2}})^2 - \frac{1}{5(p-1)}(\tau_1-\tau_2)^2(b^{-p+1}+a^{-p+1}).\end{aligned}$$

The proof in the case $\tau_1/\tau_2 \in \left[\frac12, 2\right]$ is complete. The proof of Lemma 4.3 is complete if we choose c_1 and c_2 appropriately. □

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Note in proof

Since the submission of the article, interesting related contributions have appeared, e.g., [Chen et al. 2019].

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