ANALYSIS & PDE

Volume 13 No. 3 2020

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msp

ANALYSIS AND PDE
Vol. 13, No. 3, 2020
dx.doi.org/10.2140/apde.2020.13.901

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THE GRADIENT FLOW OF THE MÖBIUS ENERGY ε-REGULARITY AND CONSEQUENCES

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We study the gradient flow of the Möbius energy introduced by O'Hara (*Topology* **30**:2 (1991), 241–247). We will show a fundamental ε-regularity result that allows us to bound the infinity norm of all derivatives for some time if the energy is small on a certain scale. This result enables us to characterize the formation of a singularity in terms of concentrations of energy and allows us to construct a blow-up profile at a possible singularity. This solves one of the open problems listed by Zheng-Xu He (*Comm. Pure Appl. Math.* **53**:4 (2000), 399–431).

Ruling out blow-ups for planar curves, we will prove that the flow transforms every planar curve into a round circle.

1. Introduction

In their seminal paper, Freedman, He, and Wang [Freedman et al. 1994] suggested the study of the negative gradient flow of the Möbius energy introduced by O'Hara [1991]. For a closed curve $\gamma \in C^{0,1}(\mathbb{R}/l\mathbb{Z}, \mathbb{R}^n)$, $l > 0$, this energy is given by

$$E(\gamma) := \iint_{(\mathbb{R}/l\mathbb{Z})^2} \left(\frac{1}{|\gamma(x)-\gamma(y)|^2} - \frac{1}{d_\gamma(x,y)^2} \right) |\gamma'(x)||\gamma'(y)|\,dx\,dy, \tag{1-1}$$

where $d_\gamma(x,y)$ denotes the distance of the two points $\gamma(x)$, $\gamma(y)$ along γ. Among many other things, Freedman, He, and Wang showed that curves of finite energy are tame and that the Möbius energy can be minimized within every prime knot class. Abrams et al. [2003] proved that the circle minimizes the energy among all closed curves. It is an open problem whether these energies can be minimized within composite knot classes or not.

The evolution equation is governed by the law

$$\partial_t \gamma = -\mathcal{H}\gamma, \tag{1-2}$$

where

$$\mathcal{H}\gamma(x) := 2\,\text{p.v.}\int_{-l/2}^{l/2} \left(2\frac{P^{\perp}_{\gamma'}(\gamma(x+w)-\gamma(x))}{|\gamma(x+w)-\gamma(x)|^2} - \kappa_\gamma(x) \right) \frac{|\gamma'(x+w)|\,dw}{|\gamma(x+w)-\gamma(x)|^2}$$

and

$$P^{\perp}_{\gamma'} w := w - \left\langle w, \frac{\gamma'}{|\gamma'|} \right\rangle \frac{\gamma'}{|\gamma'|}$$

MSC2010: primary 53C44; secondary 35S10.
Keywords: Möbius energy, geometric evolution equations, gradient flow, long-time existence.

denotes the orthogonal projection onto the normal part along the curve γ [Freedman et al. 1994, Lemma 6.1]. Here, p.v.$\int_{-l/2}^{l/2}$ denotes Cauchy's principal value, i.e., is an abbreviation for $\lim_{\varepsilon\downarrow 0}\int_{I_{l,\varepsilon}}$, where $I_{l,\varepsilon}=[-l/2,l/2]\setminus(-\varepsilon,\varepsilon)$.

If γ is parametrized by arc-length, this further reduces to

$$\mathcal{H}\gamma(x):=2\,\text{p.v.}\int_{-l/2}^{l/2}\left(2\frac{P^{\perp}_{\gamma'}(\gamma(x+w)-\gamma(x))}{|\gamma(x+w)-\gamma(x)|^2}-\gamma''(x)\right)\frac{dw}{|\gamma(x+w)-\gamma(x)|^2}. \tag{1-3}$$

Zheng-Xu He [2000, Theorem 2.1] observed that (1-2) is a quasilinear equation of third order and stated a short-time existence result for smooth curves using the Nash–Moser implicit function theorem. Using refined estimates, in [Blatt 2012b] we proved short-time existence for embedded $C^{2+\alpha}$-curves by Banach's fixed-point theorem. Furthermore, we have shown, using a Łojasiewicz–Simon gradient estimate, that local minimizers of the energy are attractive in the sense that there is a $C^{2+\alpha}$-neighborhood of initial data for which the flow exists for all time and converges to a local minimizer. Lin and Schwetlick [2010] considered the elastic energy plus some positive multiple of the Möbius energy and the length. They could show long-time existence for the related negative gradient flow and convergence to critical points by essentially treating the flow as a perturbation of the elastic flow investigated in [Dziuk et al. 2002].

In this paper we derive an ε-regularity result for the evolution equation (1-2) that will be essential in the analysis of the long-time behavior of the flow. As for the Willmore flow [Kuwert and Schätzle 2002] or the biharmonic and polyharmonic heat flow in the critical dimension [Lamm 2004; Gastel 2006] a quantum of the energy has to concentrate whenever a singularity forms.

For any measurable subset $A\subset\mathbb{R}^n$ we define the localized energy

$$E_A(\gamma):=\iint_{(\gamma^{-1}(A))^2}\left(\frac{1}{|\gamma(x)-\gamma(y)|^2}-\frac{1}{d_\gamma(x,y)^2}\right)|\gamma'(x)||\gamma'(y)|\,dx\,dy. \tag{1-4}$$

Theorem 3.1 (ε-regularity). *There are constants $\varepsilon_0>0$ and $C_k<\infty$, $k\in\mathbb{N}$, depending only on n and $E(\gamma_0)$ such that the following holds: Let γ_t, $t\in[0,T)$, be a maximal smooth solution of (1-2) and let $t_0\in[0,T)$, $r>0$, be such that*

$$\sup_{x\in\mathbb{R}^n}E_{B_r(x)}(\gamma)\le\varepsilon_0.$$

Then $T>t_0+r^3$ and

$$\|\partial_s^k\gamma_{t_0+r^3}\|_{L^\infty}\le\frac{C_k}{(rt)^{(k-1)/3}}\quad\text{for all } t\in(t_0,t_0+r^3].$$

Though the structure of this result is similar to many well-known ε-regularity results for critical evolution equations, due to the nonlocality of the equation one has to develop new techniques in order to prove this theorem. These techniques will certainly be applicable to other nonlocal geometric partial differential equations. The main strategy is to consider the evolution of localized energies and derive differential inequalities. Due to the nonlocality of the equation, however, nonlocal terms appear in these inequalities which make it impossible to apply Gronwall's lemma. We will see that instead a "point-picking method" will help us out.

As a first consequence of this result we prove the following concentration compactness alternative for the flow.

Theorem 4.1 (characterization of singularities). *Let $\gamma \in C^\infty([0,T) \times \mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be a maximal smooth solution of (1-2). There is a constant $\varepsilon_0 > 0$ depending only on n and $E(\gamma_0)$ such that if $T < \infty$ there are times $t_k \uparrow T$, points $x_k \in \mathbb{R}^n$, and radii $r_k \downarrow 0$, with*

$$E_{B_{r_k}(x_k)}(\gamma_{t_k}) \geq \varepsilon_0.$$

If a singularity occurs, then, by choosing the points x_j in the last theorem more carefully, we can furthermore construct a so-called *blow-up profile*. It is simpler to formulate this theorem using the intrinsically defined local energies

$$E^{\text{int}}_{B_r(x_0)}(\gamma) := \int_{d_\gamma(y,x_0)\leq r} \int_{d_\gamma(x,x_0)\leq r} \left(\frac{1}{|\gamma(x)-\gamma(y)|^2} - \frac{1}{d_\gamma(x,y)^2} \right) |\gamma'(x)||\gamma'(y)|\,dx\,dy.$$

Theorem 4.2 (blow-up profile). *There is an $\varepsilon_0 > 0$ such that the following holds: Assume that γ_t is a solution to (1-2) that develops a singularity in finite time, i.e., $T < \infty$ and $r_j \to 0$. Then there are points x_j and times $t_j \to T$ such that*

$$E^{\text{int}}_{B_{r_j}(x_j)}(t_j) \geq \varepsilon_0.$$

Let us now choose the points $x_j \in \mathbb{R}$ and times $t_j \in [0,T)$ such that

$$\sup_{\tau\in[0,t_j],x\in\Gamma_\tau} E^{\text{int}}_{B_{r_j}(x)}(\gamma_{t_j}) \leq E^{\text{int}}_{B_{r_j}(x)}(\gamma_{t_j}) = \varepsilon_0,$$

and let $\tilde{\gamma}_j$ be reparametrizations by arc-length of the rescaled and translated curves

$$r_j^{-1}(\gamma_{t_j} - x_j)$$

such that $\tilde{\gamma}_j(0) \in B_2(0)$. Then these curves subconverge locally in C^∞ to an embedded closed or open curve $\tilde{\gamma}_\infty : I \to \mathbb{R}^n$, $I = \mathbb{R}/l\mathbb{Z}$ or $I = \mathbb{R}$ resp., parametrized by arc-length. This curve satisfies

$$\text{p.v.}\int_{-l/2}^{l/2} \left(2\frac{P^\perp_\tau(\tilde{\gamma}(y)-\tilde{\gamma}(x))}{|\gamma(y)-\gamma(x)|^2} - \kappa_\gamma(x) \right) \frac{dy}{|\gamma(y)-\gamma(x)|^2} = 0 \quad \textit{for all } x \in I, \tag{1-5}$$

and

$$E^{\text{int}}_{\bar{B}_1(0)}(\tilde{\gamma}_\infty) \geq \varepsilon_0.$$

This solves Problem 2 of the open problems list in [He 2000]. In the last part of this paper, we deduce a geometric interpretation of the Euler–Lagrange equation of the Möbius energy. In the case of codimension 1, he could show that the only closed critical curves of the Möbius energy are the circles. We will see that unfortunately the blow-up profiles are noncompact. Therefore we cannot apply this result of He in this context. Our new interpretation of the Euler–Lagrange equation allows us to show that the only planar solutions to the Euler–Lagrange equation (1-5) are straight lines and circles. Combining this result with a careful analysis of the asymptotic behavior of the flow, we can finally show:

Theorem 4.8 (planar curves). *Let $\gamma_0 \subset \mathbb{R}^2$ be a closed smoothly embedded curve. Then the negative gradient flow of the Möbius energy exists for all times and converges to a round circle as time goes to infinity.*

Though from the topological point of view the case of planar curves is of no interest, the techniques that lead to this last result reduce the study of the flow to the study of compact and noncompact smooth solutions of the Euler–Lagrange equation (1-5) in the very intuitive geometric form (4-3). Surprisingly, in the classification of planar blow-up profiles this equation is only used in one point, which gives hope that this geometric version of the equation might help to classify blow-up profiles in other situations.

2. Preliminaries and notation

As for most of our estimates the precise algebraic form of the terms does not matter, we will use the following notation to describe the essential structure of the terms.

For two Euclidean vectors v, w, we denote by $v * w$ a bilinear operator in v and w into another Euclidean vector space. For a regular curve γ, let $\partial_s = \partial_x/|\gamma'|$ denote the derivative with respect to arc-length. For $\mu, \nu \in \mathbb{N}$, a regular curve $\gamma \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, and a function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^k$, let $P^\mu_\nu(f)$ be a linear combination of terms of the form $\partial_s^{j_1} f * \cdots * \partial_s^{j,\nu} f$, where $j_1 + \cdots + j_\nu = \mu$. Furthermore, given a second function $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^k$, the expression $P^\mu_\nu(g, f)$ denotes a linear combination of terms of the form $\partial_s^{j_1} g * \partial_s^{j_2} f * \partial_s^{j_3} f * \cdots * \partial_s^{j,\nu} f$, where $j_1 + \cdots + j_\nu = \mu$.

2A. *Decomposition of the gradient and the operator Q.* We will always assume that our curve is parametrized by arc-length at the fixed time t we currently consider. Whenever we have to estimate $\mathcal{H}$ we will write it as

$$P^\perp_{\gamma'} \widetilde{\mathcal{H}}, \tag{2-1}$$

where

$$\widetilde{\mathcal{H}}\gamma(x) = 2\,\text{p.v.}\int_{-l/2}^{l/2} \left(2\frac{\gamma(u+w) - \gamma(u) - w\gamma'(u)}{|\gamma(u+w) - \gamma(u)|^2} - \gamma''(x) \right) \frac{dw}{|\gamma(u+w) - \gamma(u)|^2},$$

and take the decomposition

$$\widetilde{\mathcal{H}}\gamma = Q\gamma + R_1\gamma + R_2\gamma = Q\gamma + R\gamma, \tag{2-2}$$

where

$$\begin{aligned} Q\gamma(x) &= 2\,\text{p.v.}\int_{-l/2}^{l/2} \left(2\frac{\gamma(x+w) - \gamma(x) - w\gamma'(x)}{w^4} - \frac{\kappa(x)}{|w|^2} \right) dw \\ &= 4\,\text{p.v.}\int_{-l/2}^{l/2} \int_0^1 (1-s) \frac{\kappa(x+sw) - \kappa(x)}{|w|^2}\, ds\, dw = \widetilde{Q}\kappa(x), \\ R_1\gamma(x) &= 4 \int_{-l/2}^{l/2} (\gamma(x+w) - \gamma(x) - w\gamma'(x)) \left(\frac{1}{|\gamma(x+w) - \gamma(x)|^4} - \frac{1}{w^4} \right) dw, \\ R_2\gamma(x) &=: 2 \int_{-l/2}^{l/2} \kappa(x) \left(\frac{1}{w^2} - \frac{1}{|\gamma(x+w) - \gamma(x)|^2} \right) dw. \end{aligned}$$

He observed that the operator Q can be written as a multiple of the fractional Laplacian $(-\Delta)^{3/2}$ plus an operator of order 2 [He 2000]. Let us state the consequences of his result for the operator $\widetilde{Q}$ of order 1:

Lemma 2.1. *For every smooth function $f \in C^\infty(\mathbb{R}/l\mathbb{Z}, \mathbb{R}^n)$ we have*

$$\widetilde{Q} f = \frac{1}{l} \sum_{k\in\mathbb{Z}} \frac{\lambda_k}{k} \hat{f}(k),$$

where $\hat{f}(k)$ denotes the k-th Fourier coefficient and $\lambda_k = \pi/3 + O(1/k)$. Hence, for $l \geq 1$ we have

$$\left| \tfrac{1}{9}\pi^2 \|f'\|_{L^2}^2 - \|\widetilde{Q} f\|_{L^2}^2 \right| \leq C \|f\|_{L^2}^2.$$

Let us add another useful identity for the operator Q to the two identities we already have given above. For smooth f, g we observe, using first partial integration and then discrete partial integration,

$$\begin{aligned}
&\int_{\mathbb{R}/l\mathbb{Z}} \text{p.v.} \int_{-l/2}^{l/2} \int_0^1 (1-s) \frac{f''(x+sw) - f''(x)}{|w|^2} \, ds \, dw \, g(x) \, dx \\
&\quad = \int_{\mathbb{R}/l\mathbb{Z}} \text{p.v.} \int_{-l/2}^{l/2} \int_0^1 (1-s) \frac{f'(x+sw) - f'(x)}{|w|^2} g'(x) \, dw \, dx \\
&\quad = \frac{1}{2} \biggl(\int_{\mathbb{R}/l\mathbb{Z}} \text{p.v.} \int_{-l/2}^{l/2} \int_0^1 (1-s) \frac{f'(x+sw) - f'(x)}{|w|^2} g'(x) \, ds \, dw \, dx \\
&\qquad\qquad - \int_{\mathbb{R}/l\mathbb{Z}} \text{p.v.} \int_{-l/2}^{l/2} \int_0^1 (1-s) \frac{f'(x+sw) - f'(x)}{|w|^2} g'(x+sw) \, dw \, dx \biggr) \\
&\quad = \frac{1}{2} \int_{\mathbb{R}/l\mathbb{Z}} \text{p.v.} \int_{-l/2}^{l/2} \int_0^1 (1-s) \frac{(f'(x+sw) - f'(x))(g'(x+sw) - g'(x))}{w^2} \, ds \, dw \, dx.
\end{aligned}$$

Hence, as we do not need the principal value to make sense of the last expression we have

$$\int_{\mathbb{R}/l\mathbb{Z}} \langle Qf, g \rangle \, ds = 2 \int_{\mathbb{R}/l\mathbb{Z}} \int_{-l/2}^{l/2} \int_0^1 (1-s) \frac{(f'(x+sw) - f'(x))(g'(x+sw) - g'(x))}{w^2} \, ds \, dw \, dx. \tag{2-3}$$

2B. ***Coercivity of the Möbius energy and bi-Lipschitz estimates.*** Of fundamental importance in the following is the deep connection between the Möbius energy and fractional Sobolev spaces observed in [Blatt 2012a], which was sharpened in [Blatt 2018, Theorem 3.2]. We showed there that the Möbius energy of an embedded curve parametrized by arc-length is finite if and only if the curve is of class $w^{3/2,2}$. More precisely, we have

Theorem 2.2 (characterization of finite energy curves). *Let $\gamma \in C^1(\mathbb{R}/l\mathbb{Z}, \mathbb{R}^n)$ be a curve parametrized by arc-length. Then the energy $E(\gamma)$ is finite if and only if $\gamma \in W^{3/2,2}$. Moreover there exists a constant $C < \infty$ not depending on γ such that*

$$\|\gamma'\|_{W^{3/2,2}} \leq C(E(\gamma)). \tag{2-4}$$

So in particular, for a solution of the gradient flow (1-2), the $W^{3/2,2}$-norm of the gradient after reparametrizing the curve by arc-length is uniformly bounded in time. An essential ingredient of the proof of the theorem above and the analysis in this article is the following bi-Lipschitz estimate for curves of finite energy of [O'Hara 1991]. This bi-Lipschitz constant is also well known under the term *Gromov distortion*.

Lemma 2.3 (bi-Lipschitz estimate). *For an injective curve* $\gamma \in W^{3/2,2}(\mathbb{R}/l\mathbb{Z}, \mathbb{R}^n)$ *we get the following bound of the Gromov distortion*:

$$\beta = \beta(\gamma) = \sup_{x \neq y} \frac{d_\gamma(x, y)}{|\gamma(x) - \gamma(y)|} \leq 18 e^{E(\gamma)/4}.$$

If γ is parametrized by arc-length, we obtain

$$\frac{|w|}{|\gamma(x+w) - \gamma(x)|} \leq 18 e^{E(\gamma)/4} \quad \text{for all } x, w \in \mathbb{R}, \ |w| \leq \frac{l}{2}. \tag{2-5}$$

Let us sketch how this bi-Lipschitz estimate was used in [Blatt 2012a] to prove Theorem 2.2. For a curve $\gamma \in W^{3/2,2}(\mathbb{R}/l\mathbb{Z}, \mathbb{R}^n)$ parametrized by arc-length, $x \in \mathbb{R}/l\mathbb{Z}$, and $0 < |w| < l/2$, we deduce using this bi-Lipschitz estimate the following estimate for the integrand of the energy:

$$\begin{aligned} \frac{1}{|\gamma(x+w)| - \gamma(x)|^2} - \frac{1}{w^2} &= \frac{w^2}{|\gamma(x+w) - \gamma(x)|^2} \frac{1 - |\gamma(x+w) - \gamma(x)|^2/w^2}{|w^2|} \\ &\leq \frac{\beta}{2} \int_0^1 \int_0^1 \frac{|\gamma'(x+s_1 w) - \gamma'(x+s_2 w)|^2}{w^2} \, ds_1 \, ds_2 \\ &\leq 2\beta \int_0^1 \frac{|\gamma(x+sw) - \gamma(x)|^2}{w^2} \, ds. \end{aligned} \tag{2-6}$$

One then derives the statement of Theorem 2.2 by basically integrating this inequality over all x and w.

More generally, for $\alpha \geq 0$ and using that the function

$$x \to \frac{1 - x^{2+\alpha}}{1 - x^2}$$

is locally bounded on $(0, \infty)$ we get

$$\begin{aligned} \frac{|w|^\alpha}{|\gamma(x+w)| - \gamma(x)|^{\alpha+2}} - \frac{1}{w^2} &= \frac{|w|^{2+\alpha}}{|\gamma(x+w) - \gamma(x)|^{2+\alpha}} \frac{1 - |\gamma(x+w) - \gamma(x)|^{2+\alpha}/|w|^{2+\alpha}}{|w^2|} \\ &\leq C \frac{1 - |\gamma(x+w) - \gamma(x)|^2/|w|^2}{|w^2|} \\ &\leq C \int_0^1 \int_0^1 \frac{|\gamma'(x+s_1 w) - \gamma'(x+s_2 w)|^2}{w^2} \, ds_1 \, ds_2 \\ &\leq C \int_0^1 \frac{|\gamma(x+sw) - \gamma(x)|^2}{w^2} \, ds, \end{aligned} \tag{2-7}$$

where the constant C depends only on an upper bound for β like $E(\gamma)$ and α. Furthermore, we have

$$\frac{1}{|\gamma(x+w) - \gamma(x)|^2} - \frac{1}{w^2} = \frac{w^2/|\gamma(x+w) - \gamma(x)|^2 - 1}{w^2} \leq \frac{18^2 e^{E(\gamma)/2} - 1}{w^2}.$$

So we get the rough estimate

$$\int_{B_r(x)} \int_{l/2 \geq |w| \geq \Lambda r} \left(\frac{1}{|\gamma(x+w) - \gamma(x)|^2} - \frac{1}{w^2} \right) dw\, dx \leq C(\beta) \int_{\Lambda r}^{\infty} \frac{dw}{w^2} \leq C(\beta). \tag{2-8}$$

2C. ***Fractional Sobolev spaces and Besov spaces.*** In our calculation, fractional Sobolev spaces as well as Besov spaces naturally appear. For an introduction to Besov spaces we refer to [Triebel 1983; 1992]. Let $f \in L^1(\mathbb{R}/l\mathbb{Z})$. For $s \in (0,1)$ and $p, q \in [1, \infty)$ and for open subsets $\Omega \subset \mathbb{R}/l\mathbb{Z}$ we also consider the *Besov-type* seminorm

$$|f|_{B^s_{p,q}(B_R(x))} := \left(\int_{B_R(x)} \frac{\left(\int_{-R/2}^{R/2} |f'(u+w) - f'(u)|^p \, du \right)^{q/p}}{|w|^{1+qs}} \, dw \right)^{1/q}. \tag{2-9}$$

It is shown in the Appendix that

$$|f|_{B^s_{p,q}(B_R(x))} \leq C \|f\|_{B^s_{p,q}(B_{2R}(x))},$$
$$\|f\|_{B^s_{p,q}(B_R(x))} \leq C(|f|_{B^s_{p,q}(B_{2R}(x)) + \|f\|_{L^p(B_{2R}(x))}}).$$

3. An ε-regularity result

In this section we prove the main result of this article, an ε-regularity result for the flow (1-2):

Theorem 3.1 (ε-regularity). *There are constants $\varepsilon > 0$ and $C_k < \infty$, $k \in \mathbb{N}$, depending only on n and $E(\gamma_0)$ such that the following holds: Let γ_t, $t \in [0, T)$ be a maximal smooth solution of (1-2) and let $t_0 \in [0, T)$, r be such that*

$$\sup_{x \in \mathbb{R}^n} E_{B_r(x)}(\gamma_{t_0}) \leq \varepsilon. \tag{3-1}$$

Then $T > t_0 + r^3$ and

$$\|\partial_s^k \gamma_{t_0+r^3}\|_{L^\infty} \leq \frac{C_k}{(rt)^{(k-1)/3}} \quad \textit{for all } t \in (t_0, t_0 + r^3].$$

Remark 3.2. Note that the assumptions in the theorem are highly nonlocal. It is a very interesting and challenging question whether one can prove a local version of this regularity theorem.

Clearly, one only has to prove Theorem 3.1 for the special case $t_0 = 0$ and $r = 1$. Scaling and translation in time then give the full statement.

We will prove Theorem 3.1 in three steps using energy estimates for this special case. First we control the energy within a ball of radius 1 at later times, before we estimate the elastic energy, i.e., the L^2-norm of the curvature. In a last step we will then bound higher-order energies. The general strategy will always be to derive evolution equations for the quantities and use the quasilinear structure together with interpolation estimates in order to derive differential inequalities (see Lemmas 3.10, 3.19, and 3.20).

Due to the nonlocal structure of the inequalities, though we start with local quantities these differential inequalities are also nonlocal, which makes the usual application of Gronwall's lemma impossible. A kind of point-picking method will help us there.

3A. ***Estimates for the energy density.*** Let us fix a radial cutoff function $\phi(x) = \phi(|x|) \in C_c^\infty(\mathbb{R}^n)$ such that

$$\chi_{B_1(0)} \le \phi \le \chi_{B_2(0)}.$$

For $x_0 \in \mathbb{R}^n$ we set $\phi_{x_0}(x) := \phi(x - x_0)$ and define the localized energy

$$E^{\phi_{x_0}}(\gamma) := \iint_{(\mathbb{R}/\mathbb{Z})^2} \left(\frac{1}{|\gamma(x)-\gamma(y)|^2} - \frac{1}{d_\gamma(x,y)^2} \right) |\gamma'(x)|\,|\gamma'(y)| \phi_{x_0}(\gamma(x))\,dx\,dy. \tag{3-2}$$

A straightforward calculation leads to the following evolution equation for E^ϕ. We leave the proof to the reader.

Lemma 3.3 (evolution equation for local density). *Let γ_t be parametrized by arc-length and $(d/dt)\gamma_t = V$ be orthogonal to γ_t. Then we have*

$$\begin{aligned}
\frac{d}{dt} E^\phi(\gamma_t) &= 2\,\text{p.v.}\int_{-l/2}^{l/2} \int_{I_{l,\varepsilon}} \left\langle 2\frac{\gamma(x+w)-\gamma(x)}{|\gamma(x+w)-\gamma(x)|^4} - \frac{\kappa_\gamma(x)}{|\gamma(x+w)-\gamma(x)|^2}, V(x) \right\rangle \phi(\gamma(x))\,dw\,dx \\
&\quad + 2\int_{\mathbb{R}/l\mathbb{Z}} \int_{-l/2}^{l/2} \frac{\left\langle \gamma(x+w)-\gamma(x) - w\gamma'(x) - \frac{1}{2}|\gamma(x+w)-\gamma(x)|^2 \kappa(x), V(x)\right\rangle}{|\gamma(x+w)-\gamma(x)|^4} \\
&\qquad\qquad \times (\phi(\gamma(x+w)) - \phi(\gamma(x)))\,dw\,dx \\
&\quad + \int_{\mathbb{R}/l\mathbb{Z}} \int_{-l/2}^{l/2} \frac{\langle \kappa(x), V(x)\rangle}{|w|^2} \left(\phi(\gamma(x+w) + \phi(\gamma(x)) - 2\int_0^1 \phi(\gamma(x+\tau w)) d\tau \right) dw\,dx \\
&\quad + \int_{\mathbb{R}/l\mathbb{Z}} \int_{-l/2}^{l/2} \left(\frac{1}{|\gamma(x+w)-\gamma(x)|^2} - \frac{1}{|w|^2} \right) \langle V(x), \nabla\phi(\gamma(x))\rangle\,dw\,dx \\
&=: I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where l is the length of γ_t.

In the rest of this section we estimate these terms for the case

$$V = \mathcal{H}\gamma.$$

To make the calculations and formulas as simple as possible, we always assume that the curve γ_t is parametrized by arc-length at the current time t. We will use the intrinsically defined quantities

$$\begin{aligned}
M_{3/2} = M_{3/2}(t) &= \sup_{x \in \mathbb{R}/l\mathbb{Z}} \int_{-1/2}^{1/2} \int_{-l/2}^{l/2} \frac{|\gamma'(x+w) - \gamma'(x)|^2}{w^2}\,dw\,dx, \\
S_3(x) = S_3(x,t) &= \|\partial_s^3 \gamma_t\|^2_{L^2(B_\Lambda(x))} + \sum_{j=1}^\infty \frac{\|\partial_s^3\gamma_t\|^2_{L^2(B_{\Lambda+j}(x) \setminus B_{\Lambda+(j-1)}(x))}}{(\Lambda/2+j)^2}, \\
\widetilde{S}_3(x) = \widetilde{S}_3(x,t) &= \|\mathcal{H}\gamma_t(x)\|^2_{L^2(B_\Lambda(0))} + \sum_{j=1}^\infty \frac{\|\mathcal{H}\gamma_t\|^2_{L^2(B_{\Lambda+j}(x))}}{(\Lambda/2+j)^2}
\end{aligned}$$

for $\Lambda = 1000 \cdot 18 e^{E(\gamma_0)/4} < \infty$. Note, that due to Lemma 2.3 the quantity $18e^{E(\gamma_0)/4}$ bounds the Gromov distortion of the curves γ_t for all t. Hence, Λ is large compared with the Gromov distortion of γ.

To prevent complicated terms in the estimates, we will assume throughout this section that

$$M_{3/2} \leq 1.$$

Furthermore, we will assume that $\gamma(0) \in B_2(0)$ to get some preliminary estimates in terms of the intrinsically defined quantities above. In the final differential inequality we will use the extrinsic quantity

$$S_3^{\text{ext}}(x,t) := \|\mathcal{H}\gamma(x)\|^2_{L^2(\gamma^{-1}(B_1(0)))} + \sum_{j=1}^{\infty} \frac{\|\mathcal{H}\gamma\|^2_{L^2(\gamma^{-1}(B_{j+1}(x)\setminus B_j(x))}}{j^2}$$

in place of $S_3(0)$.

Let us start with the following easy, but useful lemma that will help us to control the part of the integrals defining I_i, $i = 1, \dots, 4$, for the case that $|w|$ is large.

Lemma 3.4. *For all $s \in [0,1]$, $p \in [1,\infty)$, and $x \in B_\beta(0)$ we have*

$$\int_{|w|\geq\Lambda} \frac{|f(x+sw)|^p}{w^2}\,dw \leq C\|f\|^p_{L^p(B_\Lambda(0))} + \sum_{j\in\mathbb{N}} \frac{\|f\|^p_{L^p(B_{\Lambda+j}(0)-B_{\Lambda+j-1}(0))}}{(\Lambda+j)^2}$$

Proof. The statement obviously holds for $s = 0$. For $s > 0$ we get substituting $\tilde{w} = sw$

$$\begin{aligned}\int_{|w|\geq\Lambda} \frac{|f(x+sw)|^p}{w^2}\,dw &= s\int_{\Lambda/2\geq|\tilde{w}|\geq s\Lambda/2} \frac{|f(x+\tilde{w})|^p}{\tilde{w}^2}\,d\tilde{w} + s\int_{|\tilde{w}|\geq\Lambda/2} \frac{|f(x+\tilde{w})|^p}{\tilde{w}^2}\,d\tilde{w}\\ &\leq C\|f\|^p_{L^p(B_\Lambda)} + s\int_{|\tilde{w}|\geq\Lambda/4} \frac{|f(\tilde{w})|^p}{\tilde{w}^2}\,d\tilde{w}\\ &\leq C\|f\|^p_{L^p(B_\Lambda)} + \sum_{j\in\mathbb{N}} \frac{\|f\|^p_{L^p(B_{\Lambda+j}(0)\setminus B_{\Lambda+j-1}(0))}}{(\Lambda+j)^2}. \qquad \square\end{aligned}$$

We start with estimating the term I_1, which contains the terms of highest order. The guideline for estimating the remainder terms will be throughout this section to distinguish between areas where $|w|$ is small and where $|w|$ is big. Combining this idea with the commutator estimates and interpolation inequalities in the Appendix (see Lemmas A.3 and A.4) we obtain the desired estimates.

Lemma 3.5 (estimate for I_1). *Let γ be parametrized by arc-length, $M_{3/2} \leq 1$, and $\gamma(0) \in B_2(0)$. Then there is a constant $\alpha > 0$*

$$I_1 = -\int_{\mathbb{R}/l\mathbb{Z}} |\mathcal{H}\gamma|^2 \phi(\gamma(x))\,dx = -\int_{\mathbb{R}/\mathbb{Z}} |Q\gamma(x)|^2\phi(\gamma(x))\,dx + R_I,$$

where for all $\varepsilon > 0$

$$R_I \leq (C_\varepsilon M^\alpha_{3/2} + \varepsilon)S_3(0) + C_\varepsilon$$

for some $C_\varepsilon < \infty$.

Proof. We have

$$\mathcal{H}\gamma(x) = P^\perp_{\gamma'(x)}(Q\gamma(x) + R_1\gamma(x) + R_2\gamma(x)),$$

where

$$\begin{aligned} Q\gamma(x) &= 2\lim_{\varepsilon\downarrow 0}\int_{I_{l,\varepsilon}}\left(2\frac{\gamma(x+w)-\gamma(x)-w\gamma'(x)}{w^4}-\frac{\kappa(x)}{|w|^2}\right)dw \\ &= 4\lim_{\varepsilon\downarrow 0}\int_{I_{l,\varepsilon}}\int_0^1(1-s)\frac{\kappa(x+sw)-\kappa(x)}{|w|^2}\,dw=\widetilde{Q}\kappa(x), \\ R_1\gamma(x) &= 4\int_{I_l}(\gamma(x+w)-\gamma(x)-w\gamma'(x))\left(\frac{1}{|\gamma(x+w)-\gamma(x)|^4}-\frac{1}{w^4}\right)dw, \\ R_2\gamma(x) &= 2\int_{I_l}\kappa(x)\left(\frac{1}{w^2}-\frac{1}{|\gamma(x+w)-\gamma(x)|^2}\right)dw. \end{aligned} \tag{3-3}$$

The bi-Lipschitz estimate together with $\gamma(0)\in B_2(0)$ tells us $\phi(\gamma(x))=0$ for all $x\notin B_{2\beta}(0)$. This yields

$$\begin{aligned} -\int_{\mathbb{R}/l\mathbb{Z}}|P_\gamma^\perp(Q\gamma(x))|^2\phi(\gamma(x))\,dx &= -\int_{\mathbb{R}/l\mathbb{Z}}|Q\gamma(x)|^2\phi(\gamma(x))\,dx+\int_{\mathbb{R}/l\mathbb{Z}}|\langle Q\gamma(x),\gamma'\rangle|^2\phi(\gamma(x))\,dx \\ &\le -\int_{\mathbb{R}/l\mathbb{Z}}|Q\gamma(x)|^2\phi(\gamma(x))\,dx+\int_{-2\beta}^{2\beta}|\langle Q\gamma(x),\gamma'\rangle|^2\,dx. \end{aligned} \tag{3-4}$$

Using that $\langle\kappa,\gamma'\rangle=0$ and that $\widetilde{Q}$ is a linear operator, we get

$$\begin{aligned} |\langle Q\gamma(x),\gamma'\rangle| &= |\langle\widetilde{Q}\kappa(x),\gamma'\rangle|=|\langle\widetilde{Q}\kappa(x),\gamma'\rangle-\widetilde{Q}[\langle\kappa,\gamma'\rangle](x)| \\ &= \left|\sum_{i=1}^n(\widetilde{Q}[\kappa_i](x)\gamma_i'(x)-\widetilde{Q}[\kappa_i\gamma_i'](x))\right|. \end{aligned}$$

Hence, applying first the commutator estimate (Lemma A.4) and then the interpolation estimates (Lemma A.3) we obtain

$$\begin{aligned} \|\langle Q\gamma,\gamma'\rangle\|_{L^2(B_{2\beta}(0))} &\le C(\|\kappa\|_{B^{1/2}_{4,2}(B_\Lambda(0))}\|\gamma'\|_{B^{1/2}_{4,2}(B_\Lambda(0))}+\|\kappa\|_{L^2(B_\Lambda(0))}(\|\gamma'\|_{C^{0,1}(B_\Lambda(0))}+1)) \\ &\le C(M_{3/2}^{1/2}S_3^{1/2}(0)+M_{3/2})\le C(M_{3/2}^{1/2}S_3^{1/2}(0)+1). \end{aligned} \tag{3-5}$$

Using Taylor's theorem and (2-6), we get

$$\begin{aligned} |R_1\gamma(x)| &= 4\left|\int_{-l/2}^{l/2}\int_0^1(1-s)\kappa(x+sw)\left(\frac{w^2}{|\gamma(x+w)-\gamma(x)|^2}-\frac{1}{w^2}\right)ds\,dw\right| \\ &\le C\int_{I_l}\iiint_{[0,1]^3}|\kappa(x+s_1w)|\frac{|\gamma'(x+s_2w)-\gamma'(x+s_3w)|^2}{|w|^2}\,ds_1\,ds_2\,ds_3\,dw \\ &\le C\int_{I_l}\iint_{[0,1]^2}|\kappa(x+s_1w)|\frac{|\gamma'(x+s_2w)-\gamma'(x)|^2}{|w|^2}\,ds_1\,ds_2\,dw \\ &= C(R_{11}\gamma(x)+R_{12}\gamma(x)), \end{aligned}$$

where

$$\begin{aligned} R_{11}\gamma(x) &= \int_{|w|\le\Lambda/2}\iint_{[0,1]^2}g_{w,s_1,s_2}(x)\,ds_1\,ds_2\,dw, \\ R_{12}\gamma(x) &= \int_{l/2\ge|w|\ge\Lambda/2}\iint_{[0,1]^2}g_{w,s_1,s_2}(x)\,ds_1\,ds_2\,dw, \end{aligned}$$

and

$$g_{w,s_1,s_2}(x) := |\kappa(x+s_1 w)| \frac{|\gamma'(x+s_2 w)-\gamma'(x)|^2}{|w|^2}.$$

Since

$$\int_{B_{2\beta}(0)} |g_{w,s_1,s_2}(x)|^2\,dx \le \|\kappa\|^2_{L^4(B_\Lambda(0))} \frac{\|\gamma'(\cdot+s_2 w)-\gamma'\|^4_{L^8(B_{2\beta}(0))}}{|w|^2},$$

we get

$$\begin{aligned}\|R_{11}\gamma(x)\|_{L^2(B_{2\beta}(0))} &\le C\|\kappa\|_{L^4(B_\Lambda(0))} \int_{|w|\le\Lambda/2} \iint_0^1 \frac{\|\gamma'(\cdot+s_2 w)-\gamma'\|^2_{L^8(B_{2\beta}(0))}}{|w|^2}\,ds\,dw \\ &\le C\|\kappa\|_{L^4(B_\Lambda(0))}\|\gamma'\|^2_{B^{1/2}_{8,2}(B_\Lambda(0))}.\end{aligned}$$

Furthermore, since $|\gamma'|\equiv 1$ we get by Cauchy's inequality and Lemma 3.4

$$\begin{aligned}|R_{12}\gamma(x)| &\le 4\int_{l/2\ge|w|\ge\Lambda/2}\int_0^1 \frac{|\kappa(x+sw)|}{|w|^2}\,ds\,dw \\ &\le C\Big(\int_0^1\int_{l/2\ge|w|\ge\Lambda/2} \frac{|\kappa(x+sw)|^2}{|w|^2}\,dw+1\Big) \\ &\le C\Big(\|\kappa\|^2_{L^2(B_\Lambda(0))}+\sum_{j=1}^\infty \frac{\|\kappa\|^2_{L^2(B_{\Lambda+j}-B_{\Lambda+j-1})}}{(\Lambda+j)^2}+1\Big).\end{aligned}$$

Thus

$$\|R_{12}\gamma(x)\|_{L^2(B_{2\beta}(0))} \le C\Big(\|\kappa\|^2_{L^2(B_\Lambda(0))}+\sum_{j=1}^\infty \frac{\|\kappa\|^2_{L^2(B_{\Lambda+j}-B_{\Lambda+j-1})}}{(\Lambda+j)^2}+1\Big).$$

Together with the interpolation inequalities from Lemma A.3 this leads to

$$\begin{aligned}\|R_1\gamma\|^2_{L^2(B_{2\beta}(0))} &\le C\Big(\|\kappa\|^2_{L^4(B_\Lambda(0))}\|\gamma'\|^4_{B^{1/2}_{8,2}(B_\Lambda(0))}+\|\kappa\|^2_{L^2(B_\Lambda(0))}+\sum_{j=1}^\infty \frac{\|\kappa\|^2_{L^2(B_{\Lambda+j}-B_{\Lambda+j-1})}}{(\Lambda+j)^2}+1\Big) \\ &\le C\Big(M^2_{3/2}S_3+\|\partial_s^3\gamma\|^{4/3}_{L^2(B_\Lambda)}+\sum_{j=1}^\infty \frac{\|\partial_s^3\gamma\|^{4/3}_{L^2(B_{\Lambda+j}-B_{\Lambda+j-1})}}{(\Lambda+j)^2}+1\Big) \\ &\le (CM^2_{3/2}+\varepsilon)S_3+C_\varepsilon,\end{aligned}$$

where we have used the Cauchy inequality in the last step.

In the same way, one deals with the term R_2 to get

$$\|P^\perp_{\gamma'}R\|^2_{L^2(B_{2\beta}(0))} \le \|R\|^2_{B_{2\beta}(0)} \le (CM^3_{3/2}+\varepsilon)S_3+C_\varepsilon. \tag{3-6}$$

From (3-4), (3-5), and (3-6) the assertion follows. □

Lemma 3.6 (estimate for I_2). *Let $M_{3/2}\le 1$ and $\gamma(0)\in B_2(0)$. For all $\varepsilon>0$*

$$|I_2| \le \varepsilon(S_3+\widetilde{S}_3)+C_\varepsilon$$

for some $C_\varepsilon<\infty$ depending only on ε and $E(\gamma_0)$.

Proof. We take the decomposition

$$I_2 = 2\int_{\mathbb{R}/l\mathbb{Z}}\int_{-l/2}^{l/2} \frac{\big\langle \gamma(x+w)-\gamma(x)-w\gamma'(x)-\frac{1}{2}|\gamma(x+w)-\gamma(x)|^2\kappa(x),\, V(x)\big\rangle}{|\gamma(x+w)-\gamma(x)|^4} \times (\phi(\gamma(x+w))-\phi(\gamma(x)))\,dw\,dx$$

$$= 2\int_{\mathbb{R}/l\mathbb{Z}}\int_{-l/2}^{l/2} \frac{\big\langle \gamma(x+w)-\gamma(x)-w\gamma'(x)-\frac{1}{2}|w|^2\kappa(x),\, V(x)\big\rangle}{|\gamma(x+w)-\gamma(x)|^4}(\phi(\gamma(x+w))-\phi(\gamma(x)))\,dw\,dx$$
$$-\frac{1}{2}\int_{\mathbb{R}/l\mathbb{Z}}\int_{-l/2}^{l/2} \frac{|\gamma(x+w)-\gamma(x)|^2-|w|^2}{|\gamma(x+w)-\gamma(x)|^4}\langle\kappa(x), V(x)\rangle(\phi(\gamma(x+w))-\phi(\gamma(x)))\,dw\,dx$$
$$=: I_{21}+I_{22}.$$

Using the bi-Lipschitz estimate (2-5) and Taylor's approximation up to the first order, we get

$$I_{21} \le C\int_{\mathbb{R}/l\mathbb{Z}}\int_{-l/2}^{l/2}\int_0^1 \frac{|\kappa(x+sw)-\kappa(x)|}{w^2}|V(x)|\,|\phi(\gamma(x+w)-\phi(\gamma(x))|\,ds\,dw\,dx.$$

Observing that

$$\phi(\gamma(x+w))-\phi(\gamma(x))=0$$

if both $|x|, |x+w| \ge 2\beta$, this can be estimated by

$$I_{21} \le C\int_{B_{\Lambda/2}(0)}\int_{-l/2}^{l/2}\int_0^1 \frac{|\kappa(x+sw)-\kappa(x)|}{|w|}|V(x)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,ds\,dw\,dx$$
$$+ C\int_{\mathbb{R}/l\mathbb{Z}}\int_{x+w\in B_{\Lambda/2}}\int_0^1 \frac{|\kappa(x+sw)-\kappa(x)|}{|w|^2}|V(x)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,ds\,dw\,dx$$
$$\le C\int_{B_{\Lambda/2}(0)}\int_{-l/2}^{l/2}\int_0^1 \frac{|\kappa(x+sw)-\kappa(x)|}{|w|}|V(x)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,ds\,dw\,dx$$
$$+ C\int_{\mathbb{R}/l\mathbb{Z}}\int_{x+w\in B_{\Lambda/2}}\int_0^1 \frac{|\kappa(x+sw)-\kappa(x+w)|}{|w|^2}|V(x)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,ds\,dw\,dx$$
$$+ C\int_{\mathbb{R}/l\mathbb{Z}}\int_{x+w\in B_{\Lambda/2}}\int_0^1 \frac{|\kappa(x+w)-\kappa(x)|}{|w|^2}|V(x)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,ds\,dw\,dx$$
$$\le C\int_{B_{\Lambda/2}(0)}\int_{-l/2}^{l/2}\int_0^1 \frac{|\kappa(x+sw)-\kappa(x)|}{|w|}|V(x)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,ds\,dw\,dx$$
$$+ C\int_{B_{\Lambda/2}}\int_{-l/2}^{l/2}\int_0^1 \frac{|\kappa(x+sw)-\kappa(x)|}{|w|^2}|V(x+w)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,ds\,dw\,dx$$
$$+ C\int_{B_{\Lambda/2}}\int_{-l/2}^{l/2} \frac{|\kappa(x+w)-\kappa(x)|}{|w|^2}|V(x+w)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,dw\,dx$$
$$\le C\sup_{s_1,s_2\in[0,1]}\int_{B_{\Lambda/2}}\int_{-l/2}^{l/2} \frac{|\kappa(x+s_1w)-\kappa(x)|}{|w|^2}|V(x+s_2w)|\,|\phi(\gamma(x+w))-\phi(\gamma(x))|\,dw\,dx.$$

To estimate this last supremum we decompose the integral into

$$\int_{B_{\Lambda/2}}\int_{-l/2}^{l/2}\frac{|\kappa(x+s_1w)-\kappa(x)|}{|w|^2}|V(x+s_2w)||\phi(\gamma(x+w))-\phi(\gamma(x))|\,dw\,dx$$
$$\leq\int_{B_{\Lambda/2}}\int_{-\Lambda/2}^{\Lambda/2}\frac{|\kappa(x+s_1w)-\kappa(x)|}{|w|}|V(x+s_2w)|\,dw\,dx$$
$$+\int_{B_{\Lambda/2}}\int_{|w|\geq\Lambda/2}\frac{|\kappa(x+s_1w)-\kappa(x)|}{|w|^2}|V(x+s_2w)|\,dw\,dx.$$

Then we can estimate the first term by

$$C\|\kappa\|_{B^{1/2}_{2,2}(B_\Lambda)}\|V\|_{L^2(B_{\Lambda/2})}\leq\varepsilon\widetilde{S}_3+C_\varepsilon\|\kappa\|^2_{B^{1/2}_{2,2}(B_\Lambda)}\leq\varepsilon(\widetilde{S}_3+S_3)+C_\varepsilon,$$

where we used the interpolation estimates in Lemma A.3 and $M_{3/2}\leq 1$. We estimate the second term using Lemma 3.4 and then again the interpolation estimates yield

$$C\int_{x\in B_{\Lambda/2}}\int_{w\geq\Lambda/2}\frac{|\kappa(x+s_1w)|^2+|V(x+s_2w)|^2}{|w|^2}\,ds$$
$$\leq\varepsilon S_3+C_\varepsilon\left(\|\kappa\|^2_{L^2(B_\Lambda(0))}+\sum_{j=1}^{\infty}\frac{\|\kappa\|^2_{L^2(B_{\Lambda+j}-B_{\Lambda+j-1})}}{(\Lambda+j)^2}\right)$$
$$\leq\varepsilon S_3+\varepsilon\|\partial_s^3\gamma\|^2_{B_\Lambda(0)}+\varepsilon\sum_{j=1}^{\infty}\frac{\|\partial_s^3\gamma\|^2_{L^2(B_{\Lambda+j}-B_{\Lambda+j-1})}}{(\Lambda+j)^2}+C_\varepsilon\left(1+\sum_{j=1}^{\infty}\frac{1}{(\Lambda+j)^2}\right)$$
$$\leq\varepsilon(S_3+\widetilde{S}_3)+C_\varepsilon.$$

Hence,

$$I_{21}\leq\varepsilon(S_3+\widetilde{S}_3)+C_\varepsilon.$$

Similarly, we get

$$I_{22}\leq C\int_{\mathbb{R}/l\mathbb{Z}}\int_{-l/2}^{l/2}\int_{[0,1]^2}\frac{|\gamma'(x+s_1w)-\gamma'(x+s_2w)|^2}{|w|^2}|\kappa(x)||V(x)||\phi(\gamma(x+w))-\phi(\gamma(x))|\,dw\,dx$$
$$\leq C\int_{\mathbb{R}/l\mathbb{Z}}\int_{-l/2}^{l/2}\int_0^1\frac{|\gamma'(x+s_1w)-\gamma'(x)|^2}{|w|^2}|\kappa(x)||V(x)||\phi(\gamma(x+w))-\phi(\gamma(x))|\,dw\,dx$$
$$\leq\sup_{s_1,s_2\in[0,1]}C\int_{B_{\Lambda/2}(0)}\int_{|w|\leq l/2}\int_0^1\frac{|\gamma'(x+s_1w)-\gamma'(x)|^2}{|w|^2}|\kappa(x+s_2w)|$$
$$\times|V(x+s_2w)||\phi(\gamma(x+w))-\phi(\gamma(x))|\,dw\,dx,$$

which as above can be estimated by

$$\varepsilon(S_3+\widetilde{S}_3)+C_\varepsilon.$$ □

Lemma 3.7 (estimate for I_3). *Let $M_{3/2}\leq 1$ and $\gamma(0)\in B_2(0)$. Given $\varepsilon>0$ we have*

$$|I_3|\leq\varepsilon(S_3+\widetilde{S}_3)+C_\varepsilon$$

for some $C_\varepsilon<\infty$.

Proof. We use that

$$\left|\phi(\gamma(x+w))+\phi(\gamma(x))-2\int_0^1 \phi(\gamma(x+\tau w))\,d\tau\right| = C|w|^2$$

and for $x \notin B_{\Lambda/2}(0)$

$$\left|\phi(\gamma(x+w))+\phi(\gamma(x))-2\int_0^1 \phi(\gamma(x+\tau w))\,d\tau\right| \leq \begin{cases} 0 & \text{for } |w| \leq |x|-2\beta, \\ 2 & \text{for } |x|-2\beta \leq |w| \leq |x|+2\beta, \\ 2/|w| & \text{for } |x|+2\beta \leq |w| \end{cases}$$

to get

$$\int_{-l/2}^{l/2} \frac{\left|\phi(\gamma(x+w))+\phi(\gamma(x))-2\int_0^1 \phi(\gamma(x+\tau w))\,d\tau\right|}{w^2}\,dw \leq \frac{C}{x^2}$$

if $|x| \geq \Lambda/2$ and

$$\int_{-l/2}^{l/2} \frac{\left|\phi(\gamma(x+w))+\phi(\gamma(x))-2\int_0^1 \phi(\gamma(x+\tau w))\,d\tau\right|}{w^2}\,dw \leq C$$

if $|x| \leq \Lambda/2$. These estimates then imply

$$I_3 \leq C\left(\int_{B_{\Lambda/2}(0)} |\kappa(x)||V(x)|\,dx + \int_{\mathbb{R}/l\mathbb{Z}-B_{\Lambda/2}(0)} \frac{|\kappa(x)||V(x)|}{|x|^2}\,dx\right).$$

From here again Hölder's inequality together with Lemma 3.4 and the interpolation inequalities of Lemma A.3 imply the assertion of the lemma as in the proof of Lemma 3.6. □

Lemma 3.8 (estimate for I_4). *Let $M_{3/2} \leq 1$ and $\gamma(0) \in B_2(0)$. For all $\varepsilon > 0$*

$$|I_4| \leq \varepsilon(S_3+\widetilde{S}_3)+C_\varepsilon$$

for some $C_\varepsilon < \infty$.

Proof. To estimate I_4 we use (2-6) to get

$$\begin{aligned}
|I_4| &= \left|\int_{\mathbb{R}/l\mathbb{Z}} \int_{-l/2}^{l/2} \left(\frac{1}{|\gamma(x+w)-\gamma(x)|^2} - \frac{1}{|w|^2}\right)\langle V(x), \nabla\phi(\gamma(x))\rangle\,dw\,dx\right| \\
&\leq C\int_{\mathbb{R}/l\mathbb{Z}} \int_{-l/2}^{l/2} \iint_{[0,1]^2} \frac{|\gamma'(x+s_1w)-\gamma'(x+s_2w)|^2}{|w|^2}|V(x)||\nabla\phi(\gamma(x))|\,ds_1\,ds_2\,dw\,dx \\
&\leq C\int_{B_{2\beta}(0)} \int_{-l/2}^{l/2} \int_0^1 \frac{|\gamma'(x+sw)-\gamma'(x)|^2}{|w|^2}|V(x)|\,ds\,dw\,dx \\
&\leq C\int_{-l/2}^{l/2} \int_0^1 \left(\int_{x\in B_{2\beta}(0)} \frac{|\gamma'(x+s_1w)-\gamma'(x)|^4}{|w|^2}\,dx\right)^{1/2} dw\,ds\,\|V\|_{L^2(B_2(0))}.
\end{aligned}$$

Since

$$\begin{aligned}\int_{-l/2}^{l/2}\int_0^1\Big(\int_{x\in B_{2\beta}(0)}\frac{|\gamma'(x+s_1w)-\gamma'(x)|^4}{|w|^2}\,dx\Big)^{1/2}dw\,ds \\ &\le \int_0^1\int_{-sl/2}^{sl/2}\Big(\int_{x\in B_{2\beta}(0)}\frac{|\gamma'(x+w)-\gamma'(x)|^4}{|w|^2}\,dx\Big)^{1/2}dw\,ds \\ &\le \int_0^1\int_{-\Lambda/2}^{\Lambda/2}\Big(\int_{x\in B_{2}(0)}\frac{|\gamma'(x+w)-\gamma'(x)|^4}{|w|^2}\,dx\Big)^{1/2}dw\,ds+C \\ &\le C(\|\gamma'\|^2_{B^{1/2}_{4,2}(B_\Lambda(0))}+1),\end{aligned}$$

again the interpolation Lemma A.3 gives the assertion. □

The final ingredient shows that the summands in $\widetilde{S}_3$ and S_3 are essentially the same.

Lemma 3.9. *Let $M_{3/2}\le 1$ and $\gamma(0)\in B_2(0)$. For all $\varepsilon>0$ we have*

$$\int_{B_1(0)}|\mathcal{H}|^2\,dx\le C\int_{B_{4\beta}(0)}|\partial_s^3\gamma|^2\,ds+(C_\varepsilon M^\alpha_{3/2}+\varepsilon)S_3+C_\varepsilon,$$

and hence in particular

$$\widetilde{S}_3\le C(S_3+1).$$

Furthermore, we have for all $\varepsilon>0$

$$\int_{B_1(0)}|\partial_s^3\gamma|^2\,dx\le C\int_{B_{4\beta}(0)}|\mathcal{H}\gamma|^2\,ds+(C_\varepsilon M^\alpha_{3/2}+\varepsilon)S_3+C_\varepsilon,$$

and hence in particular

$$S_3\le C\widetilde{S}_3+(C_\varepsilon M^\alpha_{3/2}+\varepsilon)S_3+C_\varepsilon$$

for some $C_\varepsilon<\infty$ depending ε and the bi-Lipschitz constant of γ. If $M_{3/2}$ is small enough, we have

$$S_3\le C(\widetilde{S}_3+1).$$

Proof. Lemma 3.5 tells us that

$$\Big|\int_{\mathbb{R}/\mathbb{Z}}|\mathcal{H}\gamma|^2\phi(\gamma)\,dx-\int_{\mathbb{R}/\mathbb{Z}}|Q\gamma|^2\phi(\gamma)\,dx\Big|\le C(M^\alpha_{3/2}+\varepsilon)S_3(x)+C_\varepsilon,$$

and hence in particular

$$\Big|\int_{B_1(x)}|\mathcal{H}\gamma|^2\,dx-\int_{B_1(0)}|Q\gamma|^2\,dx\Big|\le C(M^\alpha_{3/2}+\varepsilon)S_3(x)+C_\varepsilon. \tag{3-7}$$

Let $\psi\in C^\infty(\mathbb{R})$ be such that $\chi_{B_2(0)}\le\phi\le\chi_{B_4(0)}$. We get

$$\begin{aligned}\|\widetilde{Q}(\kappa)\|_{L^2(B_2(0))}&=\|\psi\widetilde{Q}(\kappa)\|_{L^2(B_2(0))}\\ &\le\|\widetilde{Q}[\psi\kappa]-\psi\widetilde{Q}[\kappa]-\kappa\widetilde{Q}[\psi]\|_{L^2(B_2(0))}+\|\widetilde{Q}[\psi\kappa]\|_{L^2(B_2(0))}+\|\kappa\widetilde{Q}[\psi]\|_{L^2(B_2(0))}.\end{aligned}$$

The commutator estimate (Lemma A.4) and the interpolation estimate (Lemma A.3) tell us that

$$\|\widetilde{Q}[\psi\kappa]-\psi\widetilde{Q}[\kappa]-\kappa Q[\psi]\|^2_{L^2(B_2(0))}\le C\|\kappa\|^2_{B^{1/2}_{4,2}(B_\Lambda(0))}\|\psi\|^2_{B^{1/2}_{4,2}B_\Lambda(0))}+C\sum_{j=1}^{\infty}\frac{\|\kappa\|^4_{L^4(B_{\Lambda+j}\setminus B_{\Lambda+j-1})}}{(\Lambda+j)^2}$$
$$\le \varepsilon S_3+C_\varepsilon.$$

As by Lemma 2.1

$$\|\widetilde{Q}[\psi\kappa]\|_{L^2(B_2(0))}\le C\|\psi\kappa\|_{W^{1,2}(\mathbb{R}/l\mathbb{Z})}\le C(\|\partial_s\psi\kappa\|_{L^2(\mathbb{R}/l\mathbb{Z})}+\|\psi\kappa\|_{L^2})$$
$$\le C(\|\partial^3\gamma\|_{L^2(B_4(0))}+\|\kappa\|_{L^2(B_2(0))})$$

and

$$\|\kappa\widetilde{Q}[\psi]\|_{L^2(B_2(0))}\le C\|\kappa\|_{L^2(B_2(0))},$$

we get using again the interpolation estimates

$$\|Q(\gamma)\|^2_{L^2(B_2(0))}=\|\widetilde{Q}(\kappa)\|^2_{L^2(B_2(0))}\le C\|\partial_s^k\gamma\|_{L^2(B_4(0)}+\varepsilon S_3+C_\varepsilon. \tag{3-8}$$

The estimates (3-7) and (3-8) imply the first inequality. Summing up yields the second.

On the other hand, for a cutoff function $\psi\in C^\infty(\mathbb{R})$ such that $\chi_{B_{1/2}(0)}\le\psi\le\chi_{B_1(0)}$ we have

$$\|Q\gamma\|_{L^2(B_1(0))}\ge\|\psi Q\gamma\|_{L^2(B_1(0))},$$

which implies as above

$$\|Q\gamma\|_{L^2(B_1(0))}\ge\|Q(\psi\kappa)\|_{L^2}-\varepsilon S_3+C_\varepsilon.$$

Using Lemma 2.1 we get

$$\|\nabla(\psi\kappa)\|^2_{L^2}\le\|Q(\psi\kappa)\|^2_{L^2}+\|\kappa\|^2_{L^2(B_1(0))}+\varepsilon S_3+C_\varepsilon$$

and hence using an interpolation estimate

$$\|\nabla\kappa\|^2_{L^2(B_{1/2}(0))}\le\|Q(\psi\kappa)\|^2_{L^2(B_1(0))}+C\|\kappa\|^2_{L^2(B_1(0))}+\varepsilon S_3+C_\varepsilon$$
$$\le C\|Q(\psi\kappa)\|^2_{L^2(B_1(0))}+\varepsilon S_3+C_\varepsilon.$$

Using (3-7) we obtain

$$\|\nabla\kappa\|^2_{L^2(B_{1/2}(0))}\le\|\mathcal{H}\gamma\|^2_{L^2(B_1(0))}+C(M^\alpha_{3/2}+\varepsilon)S_3(x)+C_\varepsilon,$$

and covering the ball $B_1(0)$ by balls of radius $\frac{1}{2}$ we get

$$\|\nabla\kappa\|^2_{L^2(B_1(0))}\le\|\mathcal{H}\gamma\|^2_{L^2(B_2(0))}+C(M^\alpha_{3/2}+\varepsilon)S_3(x)+C_\varepsilon.$$

This implies the remaining three inequalities of the lemma. □

Gathering all the estimates above, we can now show:

Lemma 3.10 (differential inequality). *For $1>\varepsilon>0$ there is a constant $C_\varepsilon<\infty$ such that*

$$\frac{d}{dt}E_\phi(\gamma_t)+\int_{\mathbb{R}/l\mathbb{Z}}|\mathcal{H}\gamma_t|^2\phi\le(C_\varepsilon M^\alpha_{3/2}(t)+\varepsilon)\widetilde{S}^{\text{ext}}_3(0,t)+C_\varepsilon$$

whenever $M_{3/2}$ is sufficiently small.

Proof. If $\gamma_t(\mathbb{R}/\mathbb{Z}) \cap B_2(0) = \varnothing$ we have

$$\frac{d}{dt} E_\phi(\gamma_t) = -\int_{\mathbb{R}/l\mathbb{Z}} |\mathcal{H}\gamma_t|^2 \phi,$$

since both sides of the equation are vanishing. Let us now assume that $\gamma(x_t, t) \in B_2(0)$ for some $x_t \in \mathbb{R}/\mathbb{Z}$. Then the Lemmas 3.5, 3.6, 3.7, 3.8, and 3.9 tell us that

$$\frac{d}{dt} E_\phi(\gamma_t) + \int_{\mathbb{R}/l\mathbb{Z}} |\mathcal{H}\gamma_t|^2 \phi \le (C_\varepsilon M^\alpha_{3/2}(t) + \varepsilon)\widetilde{S}_3(x_t, t) + C_\varepsilon.$$

It is an easy exercise to show using the bi-Lipschitz estimate that

$$\widetilde{S}_3(x_t, t) \le C \widetilde{S}^{\text{ext}}_3(0, t),$$

where the constant C depends on the bi-Lipschitz constant of γ. Hence,

$$\frac{d}{dt} E_\phi(\gamma_t) + \int_{\mathbb{R}/l\mathbb{Z}} |\mathcal{H}\gamma_t|^2 \phi \le (C_\varepsilon M^\alpha_{3/2}(t) + \varepsilon)\widetilde{S}^{\text{ext}}_3(0, t) + C_\varepsilon. \qquad \square$$

Exploiting this result, we get:

Proposition 3.11. *For every $\delta > 0$ there are constants $\varepsilon_0 > 0$ and $C < \infty$ such that* $\sup_{x \in \mathbb{R}^n} E_{B_1(x)}(\gamma_{t_0}) \le \varepsilon_0$ *for some $t_0 \in [0, T)$ implies*

$$\int_{t_0}^{t} \int_{\gamma_t^{-1}(B_1(x))} |\mathcal{H}\gamma_\tau|^2 \, ds \, d\tau \le C \quad \textit{and} \quad E_{B_1(x)}(\gamma_\tau) \le \delta$$

for all $\tau \in [t_0, \min\{T, t_0 + 1\})$ and $x \in \mathbb{R}^n$.

Proof. We assume without loss of generality that $t_0 = 0$. Clearly we only have to show the claim under the additional assumption that $\delta > 0$ is small. Furthermore, it is enough to show that

$$\int_{t_0}^{t} \int_{\gamma_t^{-1}(B_1(x))} |\partial_s^3 \gamma_\tau|^2 \, ds \, d\tau \le C \quad \text{and} \quad E_{B_1(x)}(\gamma_\tau) \le \delta$$

for all $\tau \in [t_0, \min\{T, t_0 + \varepsilon_2\})$ and $x \in \mathbb{R}^n$ for a sufficiently small ε_2. One then obtains the assertion in its original form by applying the preliminary result to the rescaled flow

$$\tilde{\gamma}(x, t) := \frac{1}{\sqrt[3]{\varepsilon_2}} \gamma\left(\frac{x}{\sqrt[3]{\varepsilon_2}}, \frac{t}{\varepsilon_2}\right),$$

which satisfies by a standard covering argument

$$E_{B_1(0)}(\tilde{\gamma}) \le C_n \frac{1}{(\varepsilon_2)^{n/3}} \varepsilon_1.$$

Lemma 3.10 tells us that

$$\frac{d}{dt} E_\phi(\gamma_t) + \int_{\mathbb{R}/l\mathbb{Z}} \int_{B_{1/4}(0)} |\mathcal{H}\gamma|^2 \phi \le (C_\varepsilon M^\alpha_{3/2} + \varepsilon) S^{\text{ext}}_3 + C_\varepsilon. \tag{3-9}$$

Let us assume that t_1 is the first time such that

$$\sup_{x\in\mathbb{R}^n} E_{B_1}(\gamma_{t_1}) \geq \delta.$$

We set

$$IM_3 := \sup_{x\in\mathbb{R}^n} \int_0^{t_1}\int_{\gamma^{-1}(B_{1/4}(x))} |\mathcal{H}\gamma|^2\, ds\, d\tau.$$

After a translation we can assume that

$$IM_3 = \int_0^{t_1}\int_{\gamma^{-1}(B_{1/4}(0))} |\mathcal{H}\gamma|^2\, ds\, d\tau.$$

Due to the definition of $\widetilde{S}_3(x)$ and Lemma 3.9 we know that

$$\int_{t_0}^{t_1} \widetilde{S}_3\, d\tau \leq C\int_0^{t_1} \widetilde{S}_3\, d\tau + C(M^{\alpha}_{3/2}+\varepsilon)\int_0^{t_1} \widetilde{S}_3\, d\tau + Ct_1 \leq C\,IM_3 + C(t_0-t_1).$$

Integrating (3-9) and using $\chi_{B_1(0)} \leq \phi \leq \chi_{B_2(0)}$ we hence get

$$E_\phi(\gamma_{t_1}) + c_0 IM_3 \leq E_\phi(\gamma_0) + C(\delta^\alpha+\varepsilon)IM_3 + C_\varepsilon t_1 \leq \varepsilon_0 + C(\delta^\alpha+\varepsilon)IM_3 + C_\varepsilon t_1. \tag{3-10}$$

If $C(\delta^\alpha+\varepsilon) \leq c_0/2$, this implies

$$\frac{c_0}{2} IM_3 \leq \varepsilon_0 + Ct.$$

Plugging this back into the inequality (3-10), we get for all $x \in \mathbb{R}^n$

$$E_{B_1(x)}(\gamma_{t_1}) \leq E_{\phi_x}(\gamma_{t_1}) \leq \varepsilon_0 + C(\delta^\alpha+\varepsilon)(\varepsilon_0+Ct) + C_\varepsilon t_1 < \delta$$

if we first choose $\varepsilon_0 > 0$ and then t small enough. □

3B. ***Estimating the elastic energy.*** In this section we derive estimates from the evolution equations of energies containing higher order terms. The following lemma was proven in [Blatt 2018]:

Lemma 3.12 (evolution of higher-order energies). *Let γ be a family of curves moving with normal speed V. Then*

$$\partial_t \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k\kappa|^2\phi\, ds = 2\int_{\mathbb{R}/\mathbb{Z}} \langle \partial_s^{k+2}V, \partial_s^k\kappa\rangle\phi\, ds + 2\int \langle P_2^k(V,\kappa)\tau, \partial_s^{k+1}\kappa\rangle\phi\, ds$$
$$+ 2\int \langle P_3^k(V,\kappa), \partial_s^k\kappa\rangle\phi\, ds - \int |\partial_s^k\kappa|^2\langle\kappa, V\rangle\phi\, ds + \int_{\mathbb{R}/l\mathbb{Z}} |\partial_s^k\kappa|\nabla_V\phi\, ds. \tag{3-11}$$

In this section, we will derive estimates for the right-hand side of (3-11) for the case that $V = -\mathcal{H}$. We use both the evolution equations from Lemma 3.12 and these estimates to bound the so-called *elastic energy* of the curve γ, i.e., the L^2-norm of its curvature.

Proposition 3.13 (estimate for the elastic energy). *Let $\gamma : [0,T)\times\mathbb{R}/\mathbb{R} \to \mathbb{R}^n$, $T > 1$, be a smooth solution of* (1-2). *There is an $\varepsilon_0 > 0$ depending only on n such that*

$$\sup_{(x,t)\in\mathbb{R}^n\times(0,1)} E_{B_1(x)}(\gamma(\cdot,t)) < \varepsilon_0$$

implies

$$\sup_{x\in\mathbb{R}/\mathbb{Z}} \int_{B_1(x)\cap\gamma_1} |\kappa_{\gamma_1}|^2\, ds \le C, \quad \textit{and} \quad \inf_{t\in[0,1]} \int_{B_1(x)} |\partial_s \kappa_t|^2\, ds \le C.$$

3B1. *Preliminary estimates.* To estimate the respective integrals appearing on the right-hand side of (3-11) we have to distinguish as before between $|w|$ big and $|w|$ small. The next lemma helps us to deal with the part where $|w|$ is big:

Lemma 3.14. *Let us assume that $p \in [1,\infty)$, $l_i \in \mathbb{N}$, $l_i \ge 2$, $p_i \in [1,\infty)$ for $i = 1,\dots,r$, and let $\in \mathbb{N}$ be chosen such that*

$$l_i \le m, \quad l_i - \frac{1}{p_i} \le m - \frac{1}{2}, \quad \textit{and} \quad \sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{p}.$$

For $\Lambda = 1000 \cdot 18 e^{E(\gamma_0)/4}$ we set

$$g(x) := \int_{l\ge|w|\ge\Lambda} \int_{s\in[0,1]^r} \frac{\prod_{i=1}^r |\partial^{l_i}\gamma(x+s_i w)|}{|w|^2}\, ds\, dw$$

and assume that $M_{3/2} \le 1$. Then there is a constant $\beta_1, \beta_2 > 0$ such that

$$\|g\|^p_{L^p(B_1(0))} \le C(M_{3/2}^{\beta_1} + M_{3/2}^{\beta_2}) \sum_{i=1}^r \biggl(\|\partial^m\gamma\|^{\theta_i}_{L^2(B_\Lambda(0))} + \sum_{j=1}^{\infty} \frac{\|\partial^m\gamma\|^{\theta_i}_{L^2(B_{\Lambda+j}(0)\setminus B_{\Lambda+j-1}(0))}}{(\Lambda+j)^2} \biggr) + C,$$

where

$$\theta_i = p\frac{l_i - \frac{1}{p_i}}{m - \frac{1}{2}}$$

and $C < \infty$ only depends on n and $E(\gamma_0)$.

Proof. Using Jensen's inequality, we obtain

$$\begin{aligned}\int_{B_1(0)} |g(x)|^p\, dx &= \int_{B_1(0)} \biggl(\int_{l\ge|w|\ge\Lambda} \int_{s\in[0,1]^r} \frac{\prod_{i=1}^r |\partial^{l_i}\gamma(x+s_i w)|}{|w|^2}\, ds\, dw \biggr)^p dx \\ &\le C \int_{B_1(0)} \int_{l\ge|w|\ge\Lambda} \int_{s\in[0,1]^r} \frac{\prod_{i=1}^r |\partial^{l_i}\gamma(x+s_i w)|^p}{|w|^2}\, ds\, dw\, dx.\end{aligned}$$

As $\sum_{i=0}^r 1/p_i = 1/p$, we get by Cauchy's inequality

$$\int_{B_1(0)} |g(x)|^p\, dx \le \frac{C}{\Lambda^{p-1}} \int_{B_1(0)} \int_{l\ge|w|\ge\Lambda} \int_{s\in[0,1]^r} \frac{\sum_{i=1}^r |\partial^{l_i}\gamma(x+s_i w)|^{p_i}}{|w|^2}\, ds\, dw\, dx. \tag{3-12}$$

We can estimate the summands further substituting $\tilde{w} = sw$ by

$$\begin{aligned}&\int_{B_1(0)} \int_{l\ge|w|\ge\Lambda} \int_0^1 \frac{|\partial^{l_i}\gamma(x+s_i w)|^{p_i}}{|w|^2}\, ds\, dw\, dx \\ &\quad \le \int_{B_1(0)} \int_0^1 s \int_{\Lambda\ge|\tilde{w}|\ge s\Lambda} \frac{|\partial^{l_i}\gamma(x+\tilde{w})|^{p_i}}{|\tilde{w}|^2}\, ds\, d\tilde{w}\, dx + \int_{B_1(0)} \int_0^1 \int_{l\ge|\tilde{w}|\ge\Lambda} \frac{|\partial^{l_i}\gamma(x+\tilde{w})|^{p_i}}{|\tilde{w}|^2}\, ds\, d\tilde{w}\, dx \\ &\quad \le C\|\partial^{l_i}\gamma\|^{p_i}_{L^{p_i}(B_\Lambda(0))} + \int_{l\ge|y|\ge\Lambda/2} \frac{|\partial^{l_i}\gamma(y)|^{p_i}}{|y|^2}\, dy.\end{aligned} \tag{3-13}$$

Applying a Gagliardo–Nirenberg-type inequality (Lemma A.3), we obtain for

$$\theta_i = p_i \frac{l_i - 1 - \frac{1}{p_i}}{m - \frac{1}{2}}$$

that

$$\|\partial^{l_i}\gamma\|^{p_i}_{L^{p_i}(B_1(0))} \leq C\|\partial^m\gamma\|^{\theta_i}_{L^2(B_1(0))} M_{3/2}^{(p_i-\theta_i)/2} + M_{3/2}^{p_i/2}.$$

Scaling this inequality, we get

$$\|\partial^{l_i}\gamma\|^{p_i}_{L^{p_i}(B_\Lambda(0))} \leq C(\|\partial^m\gamma\|^{\theta_i}_{L^2(B_\Lambda(0))} M_{3/2}^{(p_i-\theta)/2} + M_{3/2}^{p_i/2}).$$

Furthermore,

$$\int_{l\geq|y|\geq\Lambda/2} \frac{|\partial^{l_i}\gamma(y)|^{p_i}}{|y|^2}\,dy \leq \frac{C}{\Lambda^2}\|\partial^{l_i}\gamma\|^{p_i}_{L^{p_i}(B_\Lambda(0))} + C\sum_{j=1}^{\infty}\frac{\|\partial^{l_i}\gamma\|^{p_i}_{L^{p_i}(A_{\Lambda+j}(0))}}{(\Lambda+j)^2}$$
$$\leq CM_{3/2}^{(p_i-\theta)/2}\biggl(\|\partial^{l_i}\gamma\|^{\theta_i}_{L^2(B_\Lambda(0))} + \sum_{j=1}^{\infty}\|\partial^{l_i}\gamma\|^{\theta}_{L^2(B_\Lambda(0))}\biggr) + CM_{3/2}^{p_i/2}.$$

From (3-12) and (3-13) the assertion follows. □

The second ingredient is the following lemma, which helps to deal with small $|w|$.

Lemma 3.15. *Let*

$$g(x) := \int_{|w|\leq\Lambda}\int_{s\in[0,1]^r}\int_{\tau_1,\tau_2\in[0,1]}\prod_{i=1}^{r}|\partial^{l_i}\gamma(x+s_iw)|$$
$$\times\frac{|\partial^{l_{r+1}}\gamma(x+\tau_1w)-\partial^{l_{r+1}}\gamma(x+\tau_2w)|\,|\partial^{l_{r+2}}\gamma(x+\tau_1w)-\partial^{l_{r+2}}\gamma(x+\tau_2w)|}{w^2}\,ds\,d\tau_1\,d\tau_2\,dw.$$

Let $\tilde{l}_i = l_i$ for $i = 1,\dots,r$ and $\tilde{l}_i = l_i + \frac{1}{2}$ for $i = r+1, r+2$. If $\tilde{l}_i \leq m$ and $\tilde{l}_i - \frac{1}{p_i} < m - \frac{1}{2}$ for all $i = 1,\dots,r+2$, then there is a constant $\alpha > 0$ such that

$$\|g\|_{L^p(B_\Lambda)} \leq (C_\varepsilon M_{3/2}^{\alpha} + \varepsilon)(\|\partial^m\gamma\|^{\theta}_{L^2(B_{2\Lambda}} + C_\varepsilon),$$

where

$$\theta = \frac{\sum_{i=1}^{r+2}(\tilde{l}_i - 1) - \frac{1}{p}}{m - \frac{3}{2}}.$$

Proof. We write

$$g(x) = \int_{I_l}\int_{s\in[0,1]^r}\iint_{\tau_1,\tau_2\in[0,1]} g_{x,\tau_1,\tau_2}(x,w)\,ds\,d\tau_1\,d\tau_2\,dw,$$

where

$$g_{s,\tau_1,\tau_2}(x,w) := \prod_{i=1}^{r}|\partial^{l_i}\gamma(x+s_iw)|\frac{|\partial^{l_{r+1}}\gamma(x+\tau_1w)-\partial^{l_{r+1}}\gamma(x+\tau_2w)|\,|\partial^{l_{r+2}}\gamma(x+\tau_1w)-\partial^{l_{r+2}}\gamma(x+\tau_2w)|}{w^2}.$$

Using Hölder's inequality, we get for $|w| \leq \Lambda$

$$\|g_{s,\tau_1,\tau_2}(\,\cdot\,,w)\|_{L^p(B_\Lambda(0))} \leq \prod_{i=1}^{r}\|\partial^{l_i}\gamma\|_{L^{p_i}(B_{2\Lambda}(0))}\frac{\prod_{j=1,2}\|\partial^{l_{r+j}}\gamma(\,\cdot\,+(\tau_1-\tau_2)w)-\partial^{l_{r+j}}\gamma\|_{L^{p_{r+j}}(\Lambda)}}{w^2}.$$

Since

$$\iint_{\tau_1,\tau_2\in[0,1]}\int_{|w|\le\Lambda}\frac{\prod_{j=1,2}\|\partial^{l_{r+j}}\gamma(\,\cdot\,+(\tau_1-\tau_2)w)-\partial^{l_{r+j}}\gamma\|^2_{L^{p_{r+j}}(\Lambda)}}{w^2}\,dw\,d\tau_1\,d\tau_2$$

$$\le\iint_{\tau_1,\tau_2\in[0,1]}\prod_{j=1,2}\biggl(\frac{\int_{|w|\le\Lambda}\|\partial^{l_{r+j}}\gamma(\,\cdot\,+(\tau_1-\tau_2)w)-\partial^{l_{r+j}}\gamma\|^2_{L^{p_{r+j}}(\Lambda)}}{w^2}\,dw\biggr)^{1/2}d\tau_1\,d\tau_2$$

$$\le\iint_{\tau_1,\tau_2\in[0,1]}|\tau_1-\tau_2|^{1/2}\prod_{j=1,2}\biggl(\frac{\int_{|\tilde w|\le\Lambda}\|\partial^{l_{r+j}}\gamma(\,\cdot\,+\tilde w)-\partial^{l_{r+j}}\gamma\|^2_{L^{p_{r+j}}(\Lambda)}}{\tilde w^2}\,d\tilde w\biggr)^{1/2}d\tau_1\,d\tau_2$$

$$\le\prod_{j=1,2}\|\partial^{l_{r+j}}\gamma\|_{B(B_{3\Lambda}(p)},$$

we obtain

$$\|g\|_{L^p(B_\Lambda(0))}\le\prod_{j=1,2}\prod_{i=1}^{r}\|\partial^{l_i}\gamma\|_{L^{p_i}(B_{2\Lambda}(0))}\|\partial^{l_{r+j}}\gamma\|_{B^{p_{r+j},2}_{1/2}(B_{4\Lambda}(0))}.$$

As above, the assertion follows from the Gagliardo–Nirenberg interpolation estimates in Lemma A.3. □

3B2. *Estimating the derivatives of $\mathcal{H}$.* For $k\in\mathbb{N}_0$, $s\in(0,1)$, we define

$$S_{k+s}(x)=\iint_{B_\Lambda(x)}\frac{|\partial^k\gamma(y)-\partial^k\gamma(z)|^2}{|y-z|^{1+2s}}\,dz\,dy+\sum_{j=1}^{\infty}\frac{1}{(\Lambda+j)^2}\iint_{B_{\Lambda+j}(x)\setminus B_{\Lambda+j-1}(x)}\frac{|\partial^k\gamma(y)-\partial^k\gamma(z)|^2}{|y-z|^{1+2s}}\,dz\,dy,$$

$$S_k(x)=\|\partial_s^k\gamma\|^2_{L^2(B_\Lambda(0)}+\sum_{j=1}^{\infty}\frac{\|\partial_s^k\gamma\|^2_{L^2(B_{\Lambda+j}(x)\setminus B_{\Lambda+j-1}(x))}}{(\Lambda+j)^2},$$

and

$$\tilde M_{k+3/2}=\tilde M^{\phi}_{k+3/2}(x)=\int_{\mathbb{R}/l\mathbb{Z}}\int_{-1}^{1}\frac{|\partial^{k+1}\gamma(y+w)-\partial^{k+1}\gamma(y)|^2}{w^2}\phi_x(\gamma(y))\,dw\,dy.$$

As before, we will assume that $\gamma(0)\in B_2(0)$ to get some preliminary estimates in terms of the intrinsically defined quantities above. In the final differential inequality we will use the extrinsic quantity

$$S^{\mathrm{ext}}_{k+s}(x)=\iint_{\gamma^{-1}(B_1(x))}\frac{|\partial^k\gamma(y)-\partial^k\gamma(z)|^2}{|y-z|^{1+2s}}\,dz\,dy+\sum_{j=1}^{\infty}\frac{1}{j^2}\iint_{\gamma^{-1}(B_{j+1}(x)\setminus B_j(x))}\frac{|\partial^k\gamma(y)-\partial^k\gamma(z)|^2}{|y-z|^{1+2s}}\,dz\,dy$$

in place of $S_{k+s}(0)$.

We start with an estimate for $\widetilde{\mathcal{H}}$. Again we use the decomposition

$$\widetilde{\mathcal{H}}\gamma(x)=Q\gamma(x)+R_1\gamma(x)+R_2\gamma(x)=Q\gamma(x)+R\gamma(x),$$

where

$$Q\gamma(x)=2\lim_{\varepsilon\downarrow0}\int_{I_{l,\varepsilon}}\biggl(2\frac{\gamma(x+w)-\gamma(x)-w\gamma'(x)}{w^4}-\frac{\kappa(x)}{|w|^2}\biggr)dw$$

$$=4\lim_{\varepsilon\downarrow0}\int_{I_{l,\varepsilon}}\int_0^1(1-s)\frac{\kappa(x+sw)-\kappa(x)}{|w|^2}\,dw=\widetilde Q\kappa(x),$$

$$R_1\gamma(x) = 4\int_{I_l} (\gamma(x+w)-\gamma(x)-w\gamma'(x))\left(\frac{1}{|\gamma(x+w)-\gamma(x)|^4}-\frac{1}{w^4}\right)dw,$$

$$R_2\gamma(x) = 2\int_{I_l} \kappa(x)\left(\frac{1}{w^2}-\frac{1}{|\gamma(x+w)-\gamma(x)|^2}\right)dw,$$

and set $R = R_1 + R_2$.

Lemma 3.16. *Let $M_{3/2} \leq 1$ and $\gamma(0) \in B_2(0)$. For all $\tilde{k} \geq k$ there is a constant $\alpha > 0$ that for all $\varepsilon > 0$*

$$\|\partial_s^k P^{\perp}_{\gamma'}(R)\|_{L^2(B_1(0))} + \|\partial_s^k R\|^2_{L^2(B_1(0))} \leq (C_\varepsilon M_{3/2}^{\alpha_1} + \varepsilon) S^{\theta}_{\tilde{k}+2}(0) + C_\varepsilon$$

for some constant $C_\varepsilon < \infty$, where $\theta = (2k+3)/(2\tilde{k}+1)$. Hence, for every $\varepsilon > 0$ and $k_1 > k$ there is a $C_\varepsilon < \infty$ such that

$$\|\partial_s^k \widetilde{\mathcal{H}} - \partial_s^k Q\|^2_{L^2(B_1(0))} \leq \varepsilon S_{\tilde{k}+3} + C_\varepsilon.$$

Proof. First we will show that the two summands building R can be brought into a common form and can thus be dealt with simultaneously.

To this end we first use Taylor's theorem to rewrite

$$R_1\gamma(x) = 4\int_{I_l}\int_0^1 \kappa(x+sw)\left(\frac{1}{|\gamma(x+w)-\gamma(x)|^4}-\frac{1}{w^4}\right)ds\,dw.$$

For $\beta > 0$ we observe

$$\begin{aligned}\frac{1}{|\gamma(u+w)-\gamma(u)|^\beta}-\frac{1}{|w|^\beta} &= \frac{|w|^\beta}{|\gamma(u+w)-\gamma(u)|^\beta}\cdot\frac{1-|\gamma(u+w)-\gamma(u)|^\beta/|w|^\beta}{|w|^\beta}\\ &= G^{(\beta)}\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\frac{2-2|\gamma(u+w)-\gamma(u)|^2/w^2}{|w|^\beta}\\ &= \int_0^1\int_0^1 G^{(\beta)}\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\frac{|\gamma'(u+\tau_1 w)-\gamma'(u+\tau_2 w)|^2}{|w|^\beta}\,d\tau_1\,d\tau_2,\end{aligned}$$

where

$$G^{(\beta)}(z) := \frac{1}{2|z|^\beta}\cdot\frac{1-|z|^\beta}{1-|z|^2}$$

is an analytic function away from the origin. Defining

$$g^{(\alpha,\beta)}_{s_1,\tau_1,\tau_2}(u,w) := G^{(\beta)}\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\frac{|\gamma'(u+\tau_1 w)-\gamma'(u+\tau_2 w)|^2}{|w|^\alpha}\kappa(u+s_1 w)$$

we thus get

$$R\gamma(x) = 4\int_{w\in I_l}\iint_{[0,1]^2}\int_0^1 g^{4,2}_{s_1,\tau_1,\tau_2}(x,w)\,d\tau_1\,d\tau_2\,ds\,dw - 2\int_{w\in I_l}\iint_{[0,1]^2} g^{2,2}_{0,\tau_1,\tau_2}(x,w)\,d\tau_1\,d\tau_2\,dw. \tag{3-14}$$

We now give the details of the estimate for the first term. The second term can be estimated analogously.

We differentiate under the integral to get

$$\partial^k R_1\gamma(x) = 4\int_{w\in I_l}\iint_{[0,1]^2}\int_0^1 \partial_x^k g^{4,2}_{s_1,\tau_1,\tau_2}(x,w)\,d\tau_1\,d\tau_2\,ds\,dw$$
$$= 4\int_{|w|\geq\Lambda}\iint_{[0,1]^2}\int_0^1 \partial_x^k g^{4,2}_{s_1,\tau_1,\tau_2}(x,w)\,d\tau_1\,d\tau_2\,ds\,dw + \int_{|w|\leq\Lambda}\iint_{[0,1]^2}\int_0^1 \partial_x^k g^{4,2}_{s_1,\tau_1,\tau_2}(x,w)\,d\tau_1\,d\tau_2\,ds\,dw.$$

The product rule and Faà di Bruno's formula tell us that

$$\partial_x^k g^{2,2}_{s_1,\tau_1,\tau_2}(x,w) = \sum_{l_1+l_2+l_3+l_4=k}\Bigg(\sum_{\pi\in\Pi_{l_1}}(\partial^{|\pi|}G^\beta)\Big(\frac{\gamma(x+w)-\gamma(x)}{w^2}\Big)\prod_{B\in\pi}\frac{\partial^{|B|}\gamma(x+w)-\partial^{|B|}\gamma(x)}{w}\Bigg) \times\frac{(\partial^{l_2+1}\gamma(x+\tau_1 w)-\partial^{l_2+1}\gamma(x+\tau_2 w))(\partial^{l_3+1}\gamma(x+\tau_1 w)-\partial^{l_3+1}\gamma(x+\tau_2 w))}{w^2}\partial^{l_4}\kappa(x+s_1w),$$

where Π_{l_1} denotes the set of all partitions of the set $\{1,\dots,l_1\}$. Using the fundamental theorem of calculus this can be brought into the form

$$\partial_x^k g^{2,2}_{s_1,\tau_1,\tau_2}(x,w) = \sum_{l_1+l_2+l_3+l_4=k}\Bigg(\sum_{\pi\in\Pi_{l_1}}(\partial^{|\pi|}G^\beta)\Big(\frac{\gamma(x+w)-\gamma(x)}{w^2}\Big)\prod_{B\in\pi}\int_0^1\partial^{|B|+1}\gamma(x+s_Bw)\,ds_B\Bigg) \times\frac{(\partial^{l_2+1}\gamma(x+\tau_1 w)-\partial^{l_2+1}\gamma(x+\tau_2 w))(\partial^{l_3+1}\gamma(x+\tau_1 w)-\partial^{l_3+1}\gamma(x+\tau_2 w))}{w^2}\partial^{l_4}\kappa(x+s_1w).$$

We choose $p_B = p_i = (\#\pi+4)p$ and observe that

$$|B|+1-\tfrac{1}{p_i}\leq|B|+1\leq k+1\leq k+2-\tfrac12$$

and

$$l_i+\tfrac32-\tfrac12\leq l_i+\tfrac32\leq k+\tfrac32\leq k+2-\tfrac12.$$

Hence, we can apply Lemmas 3.14 and 3.15 to get an estimate as claimed for each of the summands with

$$\theta\leq\frac{(l_1+l_2+l_3+l_4+5-3)-\frac12}{\tilde{k}-\frac32}.$$

Using the identity

$$P^\perp_{\gamma'}(R) = R-\langle R,\gamma'\rangle\gamma'$$

and treating the second term in this difference in the same way as above, we get the second estimate in the assertion. □

Lemma 3.17. *We have*

$$\|\partial_s^k\mathcal{H}-\partial_s^k Q\|^2_{L^2(B_{2\beta}(0))}\leq(C_\varepsilon M^\alpha_{3/2}+\varepsilon)(S_{k+3}+C_\varepsilon)$$

for suitable constants $\alpha>0$ *and*

$$\|\partial_s^k\mathcal{H}\|_{L^2(B_{2\beta}(0))}\leq(C_\varepsilon M^\alpha_{3/2}+\varepsilon)S_{k+3}+C_\varepsilon(M_2^\beta+1).$$

Proof. We use

$$\mathcal{H}\gamma(x) = P^{\perp}_{\gamma'}\widetilde{\mathcal{H}}\gamma(x) = \widetilde{\mathcal{H}}\gamma(x) - \langle \widetilde{\mathcal{H}}\gamma(x), \gamma' \rangle \gamma'$$

together with the decomposition

$$\widetilde{\mathcal{H}} = Q + R$$

to write

$$\mathcal{H} = Q - P^{T}_{\gamma'} Q + P^{\perp}_{\gamma'} R,$$

where $P^{T}_{\gamma'}$ denotes the projection onto the tangential part.

Lemma 3.16 tells us that

$$\|\partial_s^k P^{\perp}_{\gamma'} R\|^2_{L^2} \leq C(M^{\alpha}_{3/2} + \varepsilon) S_{k+3} + C_{\varepsilon}.$$

To deal with the term containing Q we use $P^{T}_{\gamma'} Q = \langle Q\gamma, \gamma' \rangle \gamma'$. Leibniz's rule yields

$$\partial_s^k(\langle Q, \gamma' \rangle \gamma') = \langle \partial_s^k Q, \gamma' \rangle \gamma' + I_1,$$

where I_1 is a linear combination of terms

$$Q[\partial_s^{k_1}\gamma]\partial_s^{k_2}\gamma' \partial_s^{k_3}\gamma',$$

with $k_1, k_2, k_3 \in \mathbb{N}_0,\ k_1 + k_2 + k_3 = k$, and $k_2 + k_3 \geq 1$. By Hölder's inequality the L^2-norm over $B_{2\beta}(0)$ of all these terms can be estimated by

$$C\|Q\partial_s^{k_1}\gamma\|_{L^2(B_{2\beta})} \|\partial_s^{k}\gamma'\|_{L^4(B_{2\beta})} \|\partial_s^{k_3}\gamma'\|_{L^4(B_{2\beta})}.$$

As in the proof of Lemma 3.9 we see that

$$\|Q\partial_s^{k_1}\gamma\|_{L^2(B_{2\beta}(0)} \leq C\|\partial_s^{3+k_1}\gamma\|_{L^2(B_{4\beta}(0))} + \varepsilon S_{k_1+3} + C_{\varepsilon}.$$

Hence, the interpolation estimates give

$$I_1 \leq \varepsilon S_{k+7/2} + C_{\varepsilon}.$$

We now pick up the argument from the proof of Lemma 3.5 to estimate the term

$$\langle \partial_s^k Q\gamma, \gamma' \rangle \gamma'.$$

Using the linearity of Q, we can rewrite

$$\langle \partial_s^k Q, \gamma' \rangle = \sum_{i=1}^{n} \langle \widetilde{Q}[\partial_s^k \kappa_i]\gamma_i' - \widetilde{Q}[\partial_s^k \kappa_i \gamma_i'].$$

From Lemma A.4 we then get

$$\begin{aligned}
&\|\langle \partial_s^k Q\gamma, \gamma' \rangle \gamma'\|_{L^2(B^2(0))} \\
&\quad \leq C\left(\|\partial_s^k \kappa\|_{B^{1/2}_{4,2}(B_3(0))} \|\gamma'\|_{B^{1/2}_{4,2}(B_3(0))} + \sum_{j\in\mathbb{N}} \frac{\|\partial_s^k \kappa\|^2_{L^4}}{(\Lambda + j)^2} + 1 \right) + C\|\partial_s^k \kappa\|_{L^2(B_3(0))} (\|\gamma'\|_{C^{0,1}(B_3(0))} + 1) \\
&\quad \leq (C_{\varepsilon} M^{1/2}_{3/2} + \varepsilon) S^{1/2}_{k+3} + C_{\varepsilon}.
\end{aligned}$$

□

3B3. *Estimate of the highest-order term.*

Lemma 3.18. *If $M_{3/2} \leq 1$ and $\gamma(0) \in B_2(0)$, we have*

$$-\int_{\mathbb{R}/l\mathbb{Z}} \langle \partial_s^{k+2} \mathcal{H}\gamma, \partial_s^k \kappa \rangle \phi \, ds \leq -\tilde{M}_{k+7/2}(0) + C M_{3/2}^{\alpha} S_{k+7/2}(0).$$

Proof. The main strategy is to use partial integration to move $1 + \frac{1}{2}$ derivatives from the term $\partial_s^{k+2}\mathcal{H}$ to the term $\partial_s^k \kappa$. But first we want to get rid of the projection onto the normal part contained in the definition of $\mathcal{H}$. We have

$$\mathcal{H}\gamma(x) = P^{\perp}_{\gamma'(x)}(Q\gamma(x) + R\gamma(x)).$$

Let us first deal with the terms containing R. Integration by parts gives

$$\begin{aligned} -\int_{\mathbb{R}/l\mathbb{Z}} & \partial_s^{k+2}(P^{\perp}_{\gamma'(x)}(R\gamma(x)))\partial_s^k \kappa \phi \, ds \\ &= \int_{\mathbb{R}/l\mathbb{Z}} \partial_s^{k+1}(P^{\perp}_{\gamma'(x)}(R\gamma(x)))\partial_s^{k+1}\kappa\phi \, ds + \int_{\mathbb{R}/l\mathbb{Z}} \partial_s^{k+1}(P^{\perp}_{\gamma'(x)}(R\gamma(x)))\partial_s^k \kappa \phi' \, ds, \end{aligned}$$

which we can estimate using the product rule, Hölder's inequality, and Lemma 3.16 by

$$(CM_{3/2}^{\alpha} + \varepsilon) S_{k+7/2}^{(2(k+1)+3)/(4(k+2))} \|\partial_s^k \kappa\|_{L^2} \leq (CM_{3/2}^{\alpha} + \varepsilon) S_{k+7/2} + C_\varepsilon.$$

So we get

$$-\int_{\mathbb{R}/l\mathbb{Z}} \partial_s^{k+2}(P^{\perp}_{\gamma'(x)}(R\gamma(x)))\partial_s^k \kappa\phi \, ds \leq (CM_{3/2}^{\alpha} + \varepsilon) S_{k+7/2} + C_\varepsilon. \tag{3-15}$$

To estimate

$$\int_{\mathbb{R}/l\mathbb{Z}} \langle \partial_s^{k+2} P^{\perp}_{\gamma'} Q, \partial_s^k \kappa \rangle \phi \, ds$$

we write

$$P^{\perp}_{\gamma'(x)} Q\gamma(x) = Q\gamma(x) - \langle Q\gamma(x), \gamma'(x)\rangle \gamma'(x) = Q\gamma - P^T_{\gamma'} Q.$$

Using

$$\langle Q\gamma(x), \gamma'(x)\rangle = 2\int_{-l}^{l}\int_0^1 (1-s)\frac{(\gamma'(x+sw) - \gamma'(x))(\kappa(x+sw) - \kappa(x))}{w^2} \, ds \, dw \, dx$$

we get from Lemmas 3.14 and 3.15

$$\|\partial_s^{k+1} P^T_{\gamma'}(Q)\|^2_{L^2(B_{2\beta}(0))} \leq C(M_{3/2}^{\alpha} + \varepsilon) S_{k+7/2}^{(2k+3)/(2(2k+4))}.$$

Hence, Cauchy's inequality implies

$$-\int_{\mathbb{R}/l\mathbb{Z}} \partial_s^{k+1} P^T_{\gamma'} Q\gamma \, \partial_s^{k+1} \kappa \phi \, ds \leq \varepsilon S_{k+7/2} + C_\varepsilon. \tag{3-16}$$

The term

$$\int_{\mathbb{R}/l\mathbb{Z}} \langle \partial_s^{k+2} Q, \partial_s^k \kappa \rangle \, ds$$

can be rewritten using (2-3) as

$$\begin{aligned}
&\int_{\mathbb{R}/l\mathbb{Z}} \langle \partial_s^{k+2} Q\gamma, \partial^k \kappa \phi \rangle \, ds \\
&= \int_{\mathbb{R}/l\mathbb{Z}} \langle Q \partial^k \kappa, \partial^k \kappa \rangle \phi \, ds \\
&= 2 \int_{\mathbb{R}/l\mathbb{Z}} \int_{-l}^{l} \int_0^1 (1-s) \frac{|\partial_s^{k+1}\kappa(x+sw) - \partial_s^{k+1}\kappa(x)|^2}{w^2} \phi(\gamma(x)) \, ds \, dw \, dx \\
&\quad + 2 \int_{\mathbb{R}/l\mathbb{Z}} \int_{-l}^{l} \int_0^1 (1-s) \frac{(\partial_s^{k+1}\kappa(x+sw) - \partial_s^{k+1}\kappa(x))(\phi(\gamma(x+sw)) - \phi(\gamma(x)))}{w^2} \partial_x^{k+1}\kappa(x+sw) \, ds \, dw \, dx.
\end{aligned}$$

We observe that

$$\begin{aligned}
&\int_{\mathbb{R}/l\mathbb{Z}} \int_{-l}^{l} \int_0^1 (1-s) \frac{|\partial_s^{k+1}\kappa(x+sw) - \partial_s^{k+1}\kappa(x)|^2}{w^2} \phi(x) \, ds \, dw \, dx \\
&\qquad \geq \int_{B_1(0)} \int_{-l}^{l} \int_0^1 (1-s) \frac{|\partial_s^{k+1}\kappa(x+sw) - \partial_s^{k+1}\kappa(x)|^2}{w^2} \, ds \, dw \, dx \\
&\qquad \geq \int_{B_1(0)} \int_0^1 (1-s) s \int_{-sl}^{sl} \frac{|\partial_s^{k+1}\kappa(x+\tilde{w}) - \partial_s^{k+1}\kappa(x)|^2}{\tilde{w}^2} \, d\tilde{w} \, ds \, dx \\
&\qquad \geq c_0 \int_{B_1(0)} \int_{-l/2}^{l/2} \frac{|\partial_s^{k+1}\kappa(x+\tilde{w}) - \partial_s^{k+1}\kappa(x)|^2}{\tilde{w}^2} \, d\tilde{w} \, dx \geq c_0 \tilde{M}_{k+7/2}(0).
\end{aligned}$$

Furthermore, we take the decomposition

$$\begin{aligned}
&\left| \int_{\mathbb{R}/l\mathbb{Z}} \int_{-l}^{l} \int_0^1 (1-s) \frac{(\partial_s^{k+1}\kappa(x+sw) - \partial_s^{k+1}\kappa(x))(\phi(\gamma(x+sw)) - \phi(\gamma(x)))}{w^2} \partial_x^{k+1}\kappa(x+sw) \, ds \, dw \, dx \right| \\
&\qquad \leq \int_{B_\Lambda(0)} \int_{|w|\geq\Lambda} \int_0^1 (1-s) \frac{|\partial_s^{k+1}\kappa(x+sw) - \partial_s^{k+1}\kappa(x)|}{w^2} |\partial_s^{k+1}\kappa(x+sw)| \, ds \, dw \, dx \\
&\qquad\qquad + \int_{B_\Lambda(0)} \int_{|w|\leq\Lambda} \int_0^1 (1-s) \frac{|\partial_s^{k+1}\kappa(x+sw) - \partial_s^{k+1}\kappa(x)|}{|w|} |\partial_s^{k+1}\kappa(x+sw)| \, ds \, dw \, dx.
\end{aligned}$$

We use Lemmas 3.14 and 3.15 to estimate the first term and the second term by

$$S_{k+7/2}^{\theta} M_{3/2}^{1-\theta} + M_{3/2},$$

where

$$\theta = \frac{2k+3+\frac{1}{2}}{2k+4} < 1.$$

Hence, Cauchy's inequality yields

$$-\int_{\mathbb{R}/l\mathbb{Z}} \langle \partial_s^{k+2} Q\gamma, \partial^k \kappa \phi \rangle \, ds \leq -c_0 M_{k+7/2}(0) + \varepsilon S_{k+7/2} + C_\varepsilon. \tag{3-17}$$

The inequalities (3-15), (3-16), and (3-17) prove the statement of the lemma. □

Lemma 3.19 (differential inequality for energies of higher order). *For every $\varepsilon > 0$ there is a constant $C_\varepsilon < \infty$ depending only on ε, n, and k and $c_k > 0$ such that*

$$\partial_t \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa|^2 \phi \, ds + c_k \tilde{M}_{k+7/2} \leq C(M^\alpha_{3/2} + \varepsilon) S^{\text{ext}}_{k+7/2}(0) + C_\varepsilon.$$

Proof. From (3-11) we get

$$\partial_t \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa|^2 \phi \, ds = 2 \int_{\mathbb{R}/\mathbb{Z}} \langle \partial_s^{k+2} V, \partial_s^k \kappa \rangle \phi \, ds + 2 \int \langle P_2^k(V,\kappa)\tau, \partial_s^{k+1}\kappa \rangle \phi \, ds$$
$$+ 2 \int \langle P_3^k(V,\kappa), \partial_s^k \kappa \rangle \phi \, ds - \int |\partial_s^k \kappa|^2 \langle \kappa, V \rangle \phi \, ds + \int_{\mathbb{R}/l\mathbb{Z}} |\partial_s^k \kappa| \nabla_V \phi \, ds. \quad (3\text{-}18)$$

Lemma 3.18 gives

$$2 \int_{\mathbb{R}/\mathbb{Z}} \langle \partial_s^{k+2} V, \partial_s^k \kappa \rangle \phi \, ds \leq -\tilde{M}_{k+7/2}(0) + (M^\alpha_{3/2} + \varepsilon)\varepsilon S_{k+3/2} + C_\varepsilon.$$

Let $k_1 + k_2 = k$. Hölder's inequality, standard interpolation estimates and Lemma 3.17 give

$$\begin{aligned} \int_{\mathbb{R}/l\mathbb{Z}} \partial_s^{k_1} V * \partial_s^{k_2} \kappa * \tau * \partial_s^{k+1} \kappa \phi \, ds &\leq \|\partial_s^{k_1} V\|_{L^2} \|\partial_s^{k_2}\kappa\|_{L^4} \|\partial_s^{k+1}\kappa\|_{L^4} \\ &\leq C(M^\alpha_{3/2} + \varepsilon) S_{k+7/2} + C_\varepsilon, \end{aligned}$$

and hence

$$\int \langle P_2^k(V,\kappa)\tau, \partial_s^{k+1}\kappa \rangle \phi \, ds \leq C(M^\alpha_{3/2} + \varepsilon) S_{k+7/2} + C_\varepsilon.$$

Similarly we get the estimate

$$\int \langle P_3^k(V,\kappa), \partial_s^k \kappa \rangle \phi \, ds \leq C(M^\alpha_{3/2} + \varepsilon) S_{k+7/2} + C_\varepsilon,$$

and

$$\int_{\mathbb{R}/l\mathbb{Z}} |\partial_s^k \kappa|^2 \nabla_V \phi \, ds \leq \|\partial_s^k \kappa\|^2_{L^4} \|V\|_{L^2} \leq (C_\varepsilon M^\alpha_{3/2} + \varepsilon) S_{k+7/2} + C_\varepsilon.$$

Hence, we have

$$2 \int \langle P_2^k(V,\kappa)\tau, \partial_s^{k+1}\kappa \rangle \phi \, ds + 2 \int \langle P_3^k(V,\kappa), \partial_s^k \kappa \rangle \phi \, ds$$
$$- \int |\partial_s^k \kappa|^2 \langle \kappa, V \rangle \phi \, ds + \int_{\mathbb{R}/l\mathbb{Z}} |\partial_s^k \kappa| \nabla_V \phi \, ds \leq 5\varepsilon S_{k+7/2} + C_\varepsilon (M_2^\beta + 1).$$

Together, these estimates imply

$$\partial_t \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa|^2 \phi \, ds + \tilde{M}_{k+7/2}(0) \leq C(M^\alpha_{3/2} + \varepsilon) S_{k+7/2} + C_\varepsilon.$$

As $S_{k+7/2} \leq C S^{\text{ext}}_{k+7/2}$ we get the assertion. □

3B4. *Proof of Proposition 3.13.* We get from Lemma 3.19

$$\partial_t \int_{\mathbb{R}/\mathbb{Z}} |\kappa|^2 \phi \, ds + c_0 \tilde{M}_{7/2}(0) \le (C_\varepsilon M_{3/2}^\alpha + \varepsilon)(S_{7/2}^{\text{ext}}(x) + C_\varepsilon).$$

Integrating this inequality and using that $M_{3/2} \le \varepsilon_0 < 1$ we get

$$\begin{aligned}\int_{\mathbb{R}/l\mathbb{Z}} |\kappa_{\gamma_1}|^2 \phi \, ds + c_0 \int_\tau^1 \tilde{M}_{7/2}(0,t)\, dt &\le \int_{\mathbb{R}/l\mathbb{Z}} |\kappa_{\gamma_\tau}|^2 \phi \, ds + (C_\varepsilon \varepsilon_0^\alpha + \varepsilon) \int_\tau^1 S_{7/2}^{\text{ext}}(x,t)\, dt + C_\varepsilon (1-\tau) \\ &\le \int_{\mathbb{R}/l\mathbb{Z}} |\kappa_{\gamma_\tau}|^2 \, ds + (C_\varepsilon \varepsilon_0^\alpha + \varepsilon) \int_\tau^1 S_{7/2}^{\text{ext}}(x,t)\, dt + C_\varepsilon (1-\tau). \quad (3\text{-}19)\end{aligned}$$

Integrating again over $\tau \in \left[0, \frac{1}{2}\right]$ yields

$$\begin{aligned} c_0 \int_0^{1/2} \int_\tau^1 \tilde{M}_{7/2}(0,t)\, dt &\le \int_0^{1/2} \int_{\mathbb{R}/l\mathbb{Z}} |\kappa_{\gamma_\tau}|^2 \phi(\gamma) \, ds + (C_\varepsilon \varepsilon_0^\alpha + \varepsilon) \int_0^{1/2} \int_\tau^1 S_{7/2}^{\text{ext}}(x,t)\, dt\, d\tau + C_\varepsilon (1-\tau) \\ &\le \int_0^{1/2} \int_{\mathbb{R}/l\mathbb{Z}} |\kappa_{\gamma_\tau}|^2 \phi(\gamma) \, ds + (C_\varepsilon \varepsilon_0^\alpha + \varepsilon) \int_0^{1/2} \int_\tau^1 S_{7/2}^{\text{ext}}(x,t)\, dt + C_\varepsilon. \end{aligned}$$

We can estimate the first term, using interpolation estimates as in Section 3A, by

$$C\left(\int_0^{1/2} \widetilde{S}_3^{\text{ext}}\, dt + 1\right) \le C(IM_3 + 1),$$

which is bounded by Proposition 3.11. Assuming that

$$IM_{7/2} = \sup_{x \in \mathbb{R}^n} \int_0^{1/2} \int_{t_1}^1 \tilde{M}_{7/2}(x,t)\, d\tau\, dt = \int_0^{1/2} \int_\tau^1 \tilde{M}_{7/2}(0,t)\, dt,$$

and using that

$$\sup_{x \in \mathbb{R}/l\mathbb{Z}} \int_0^{1/2} \int_\tau^1 S_{7/2}^{\text{ext}}(x,t)\, d\tau\, dt = \int_0^{1/2} \int_\tau^1 \tilde{M}_{7/2}(0,t)\, d\tau\, dt \le C\, IM_{7/2},$$

we deduce

$$c_0 IM_{7/2} \le C + (C_\varepsilon \varepsilon_0^\alpha + \varepsilon) IM_{7/2} + C_\varepsilon.$$

Choose first $\varepsilon > 0$ and then $\varepsilon_0 > 0$ sufficiently small; then we get

$$IM_{7/2} \le C.$$

Plugging this back into (3-19) we get the assertion

3C. ***Estimates for higher-order energies.*** It is tempting to just iterate the above argument to get control of higher-order energies. Unfortunately, one would have to adapt ε_1 in each of the steps which would not yield to the desired result. Instead we improve the differential estimate from the end of the last subsection assuming that M_2 is finite. By literally the same argument as in the proof of the Lemmas in the last subsection but interpolating in all the arguments between $W^{k+7/2,2}$ and $W^{2,2}$ instead of $W^{k+7/2,2}$ and $W^{3/2,2}$ we get:

Lemma 3.20 (differential inequality for energies of higher order). *For every $\varepsilon > 0$ there is a constant C_ε depending only on ε, n, and k and a constant $c_k > 0$ such that*

$$\partial_t \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa|^2 \phi \, ds + c_k M_{k+7/2}(0) \le \varepsilon S^{\text{ext}}_{k+7/2} + C_\varepsilon M_2^\beta.$$

Now we are finally able to conclude the proof of the ε-regularity theorem. We prove inductively the following statement

Proposition 3.21. *There is an $\varepsilon_0 > 0$ and constant $C_k < \infty$ such that*

$$\sup_{x \in \mathbb{R}^n} E_{B_1(x)}(\gamma_0) \le \varepsilon_0$$

implies

$$\sup_{x \in \mathbb{R}^n} \|\partial_s^k \kappa_{\gamma(t)}\|_{L^2(B_1(x))} \le \frac{C}{t^{k/3-1/2}}.$$

Proof. We prove by induction on k that

$$\|\partial_s^k \kappa_{\gamma(t)}\|_{L^2(B_1(x))} \le \frac{C_k}{t^{k/3-1/2}}$$

and

$$\sup_{x \in \mathbb{R}^n} \int_{t/2}^t M_{k+7/2}(x,t)\, dt \le \frac{C_k}{t^{k/3-3/2}}.$$

Again by scaling properties of the solution it is enough to show these inequalities for $t = 1$. Let us fix $\varepsilon_0 > 0$ such that we can apply Propositions 3.11 and 3.13, i.e., such that the Möbius energies on balls of radius 1 are small and the elastic energy on unit balls is bounded for times larger than $t = \frac{1}{4}$. Hence, the statement is true for $k = 0$.

Let us assume that we have the bound claimed for $k-1$ and let $t = \frac{1}{2}$. By Lemma 3.20 for every $\varepsilon > 0$ we have

$$\partial_t \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa|^2 \phi \, ds + c_k \tilde{M}_{7/2+k}(x) \le \varepsilon S^{\text{ext}}_{k+7/2} + C_\varepsilon. \tag{3-20}$$

Integrating the inequality, we get for all $0 < \tau < 1$ that

$$\int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa_1|^2 \phi \, ds + c_k \int_\tau^1 \tilde{M}_{7/2+k}(x,t)\, dt \le \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa_\tau|^2 \phi \, ds + \varepsilon \int_\tau^1 S^{\text{ext}}_{k+7/2}(x,t)\, dt + C_\varepsilon (1-t).$$

We integrate this inequality for $\tau \in \left[\frac{1}{4}, \frac{1}{2}\right]$ to get

$$\frac{1}{4} \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa_1|^2 \phi \, ds + c_k \int_{1/4}^{1/2} \int_\tau^1 \tilde{M}_{7/2+k}(x,t)\, dt\, d\tau$$
$$\le \int_{1/4}^{1/2} \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa_\tau|^2 \phi \, ds\, dt\, d\tau + \varepsilon \int_{1/4}^{1/2} \int_\tau^1 S^{\text{ext}}_{k+7/2}(x,t)\, dt + C_\varepsilon.$$

Since interpolation estimates yield

$$\int_{1/4}^{1/2} \int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa_\tau|^2 \phi \, ds\, dt\, d\tau \le C \int_{1/4}^{1/2} (S^{\text{ext}}_{k+3/2} + 1)\, dt \le C \sup_{x \in \mathbb{R}^n} \int_{1/4}^{1/2} (M_{(k-1)+7/2}(x,t) + 1)\, dt \le C,$$

by the induction hypotheses, we deduce that

$$\frac{1}{4}\int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa_1|^2 \phi \, ds + c_k \int_{1/4}^{1/2}\int_\tau^1 \tilde{M}_{7/2+k}(x,t)\, dt\, d\tau \le C + \varepsilon \int_{1/4}^{1/2}\int_\tau^1 S^{\mathrm{ext}}_{k+7/2}(x,t)\, dt + C_\varepsilon. \tag{3-21}$$

Let us now assume that the supremum

$$IM_{k+7/2} = \sup_{x\in\mathbb{R}^n} \int_{1/4}^{1/2}\int_\tau^1 \tilde{M}_{k+7/2}(x,t)\, dt$$

is attained in the point $x = 0$. Since

$$\int_{1/4}^{1/2}\int_\tau^1 S^{\mathrm{ext}}_{k+7/2}(x,t)\, dt \le C\, IM_{k+7/2},$$

we deduce from (3-21)

$$IM_{k+7/2} \le \varepsilon IM_{k+7/2} + C_\varepsilon.$$

Choosing $\varepsilon > 0$ small enough and absorbing, we get

$$IM_{k+7/2} \le C.$$

Hence, in particular

$$\int_{1/2}^1 S^{\mathrm{ext}}_{k+7/2}(x,t)\, dt \le C \quad \text{for all } x \in \mathbb{R}^n.$$

Plugging this back into (3-21), we derive

$$\int_{\mathbb{R}/\mathbb{Z}} |\partial_s^k \kappa(s,1)|^2 \phi_x \, ds \le C \quad \text{for all } x \in \mathbb{R}^n.$$ □

Proof of Theorem 3.1. Using scaled Sobolev embeddings we get the claimed estimates from Proposition 3.21 as long as the flow exists. So the only thing left is to show that $T > 1$. But this follows by standard methods from the uniform estimates in Proposition 3.21. □

4. Applications

4A. *Blow-up profiles.* Using Theorem 3.1, we get the following classification of finite time blow-up.

Theorem 4.1 (characterization of singularities). *Let $\gamma \in C^\infty([0,T)\times\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ be a maximal smooth solution of (1-2). There is a constant $\varepsilon_0 > 0$ depending only on n and $E(\gamma_0)$ such that if $T < \infty$ there are times $t_k \uparrow T$, points $x_k \in \mathbb{R}^n$, and radii $r_k \downarrow 0$ with*

$$E_{B_{r_k}(x_k)}(\gamma_{t_k}) \ge \varepsilon_0.$$

Proof. Let us assume that $T < \infty$ and that there is an $r > 0$ such that for all $t \in [0,T)$ and all $x \in \mathbb{R}^n$ we have

$$E_{B_{r_j}(x_j)}(\gamma(t)) \le \varepsilon_0.$$

Then Theorem 3.1 would tell us that $T > t_j + r_j^3 \to T + r^3$. □

Picking the concentration times more carefully, we can construct a blow-up profile at a singularity. As mentioned in the Introduction, we localize the energy intrinsically for this purpose; i.e., we work with $E^{\text{int}}_{B_r(x)}$ instead of $E_{B_r(x)}$. We do this for the simple reason that $E^{\text{int}}_{B_r(x)}$ is continuous in r and x.

In the rest of this article we will express from time to time the integrals occurring as integrals over the image

$$\Gamma_t := \gamma(\mathbb{R}/l\mathbb{Z}, t).$$

Theorem 4.2 (blow-up profiles). *There is an $\varepsilon_0 > 0$ such that the following holds: Assume that γ_t is a solution of (1-2) that develops a singularity in finite time, i.e., $T < \infty$ and $r_j \to 0$. Then there are points x_j and times $t_j \to T$ such that*

$$E^{\text{int}}_{B_{r_j}(x_j)}(t_j) \geq \varepsilon_0.$$

Let us now choose the points $x_j \in \mathbb{R}$ and times $t_j \in [0, T)$ such that

$$\sup_{\tau\in[0,t_j],x\in\Gamma_\tau} E^{\text{int}}_{B_{r_j}(x)}(\gamma_{t_j}) \leq E^{\text{int}}_{B_{r_j}(x)}(\gamma_{t_j}) = \varepsilon_0,$$

and let $\tilde{\gamma}_j$ be reparametrizations by arc-length of the rescaled and translated curves

$$r_j^{-1}(\gamma_{t_j} - x_j)$$

such that $\tilde{\gamma}_j(0) \in B_2(0)$. Then these curves subconverge locally in C^∞ to an embedded closed or open curve $\tilde{\gamma}_\infty : I \to \mathbb{R}^n$, $I = \mathbb{R}/l\mathbb{Z}$ or $I = \mathbb{R}$ resp., parametrized by arc-length. This curve satisfies

$$\text{p.v.}\int_{-l/2}^{l/2}\left(2\frac{P^\perp_\tau(\tilde{\gamma}(y)-\tilde{\gamma}(x))}{|\gamma(y)-\gamma(x)|^2} - \kappa_\gamma(x)\right)\frac{dy}{|\gamma(y)-\gamma(x)|^2} = 0 \quad \textit{for all } x \in I, \tag{4-1}$$

and

$$E^{\text{int}}_{B_1(0)}(\tilde{\gamma}_\infty) \geq \varepsilon_0.$$

Proof. The first statement is an immediate consequence of Theorem 4.1 and the bi-Lipschitz estimate (2-5). We consider the rescaled flows

$$\tilde{\gamma}^{(j)}(x,t) := \frac{1}{r_j}(\gamma(x, r_j^3 t + t_j) - x_j)$$

for $t \in (-t_j/r_j^3, 0]$ which still solve (1-2). Under the assumptions of the theorem we get

$$E^{\text{int}}_{B_1(0)}(\tilde{\gamma}^{(j)}_t) \leq \varepsilon_0 \quad \text{for all } t \in \left[-\frac{t_j}{r_j^3}, 0\right],$$

and thus from the bi-Lipschitz estimate

$$E_{B_{\beta^{-1}}(0)}(\tilde{\gamma}^{(j)}_t) \leq \varepsilon_0 \quad \text{for all } t \in \left[-\frac{t_j}{r_j^3}, 0\right].$$

Hence we can apply Theorem 3.1 to find

$$\|\partial_s^k \tilde{\gamma}_t\|_{C^k} \leq C_k$$

for all $k \in \mathbb{N}$ and $t \in [-t_j/r_j^3 + 1, 0)$. As $-t_j/r_j^3 \to -\infty$, we can use the Arzelà–Ascoli theorem to get, after going to a subsequence,

$$\tilde{\gamma}_j \to \tilde{\gamma}$$

locally smoothly in time and space. Since all derivatives of γ_∞ are uniformly bounded we furthermore deduce that

$$\mathcal{H}\tilde{\gamma}_\infty(x) = \text{p.v.}\int_I \left(2\frac{P_\tau^\perp(\tilde{\gamma}(y) - \tilde{\gamma}(x))}{|\gamma(y) - \gamma(x)|^2} - \kappa_\gamma(x)\right)\frac{dy}{|\gamma(y) - \gamma(x)|^2}$$

is well-defined. Furthermore, we have

$$\int_{-\delta_0}^0 \int_{\mathbb{R}/l_t\mathbb{Z}} |\mathcal{H}(\gamma_t^{(j)})(x)|^2\, dx\, dt = E(\gamma_{t_j - r_j^3}) - E(\gamma_{t_j}) \to 0 \tag{4-2}$$

for some subsequence j and hence after going to a subsequence

$$\mathcal{H}\tilde{\gamma}^{(j)}(x) \to 0$$

pointwise almost everywhere. We now show that

$$\mathcal{H}\tilde{\gamma}_j \to \mathcal{H}\tilde{\gamma}_\infty$$

pointwise. For this purpose we again use the decomposition

$$\widetilde{\mathcal{H}\gamma} = Q\gamma + R_1\gamma + R_2\gamma.$$

As

$$\begin{aligned}\frac{|w|^\alpha}{|\gamma(x+w) - \gamma(x)|^{2+\alpha}} - \frac{1}{w^2} &\le C\frac{\int_0^1\int_0^1 |\gamma'(x+s_1 w) - \gamma'(x+s_2 w)|^2\, ds_1\, ds_2}{|w|^2} \\ &\le C\min\left\{\|\kappa\|_{L^\infty(B_R(x))}, \frac{1}{|w|^2}\right\},\end{aligned}$$

we get that the integrands of both $R_1(\gamma_j)$ and $R_2(\gamma_j)$ are uniformly bounded. As all the integrands also converge pointwise to the integrands of $R_1(\gamma_\infty)$ and $R_2(\gamma_\infty)$, the dominant convergence theorem yields

$$R(\gamma_j) \to R(\gamma_\infty).$$

For the integrand of Q we use Taylor's approximation up to order 2 to get

$$\frac{\gamma(x+w) - \gamma(x) - w\gamma'(x) - \frac{1}{2}w^2\gamma''(x)}{w^4} = \frac{\int_0^1 (1-s)^2\gamma'''(x+sw)\, ds}{w}$$

and write

$$Q\gamma = \int_{\mathbb{R}/l\mathbb{Z}} I\, dw,$$

where

$$\begin{cases} I(x,w) := \left(\int_0^1 (1-s)^2\gamma'''(x+sw) - \gamma'''(x)\, ds\right)/2w & \text{for } |w| \le 1, \\ I(x,w) := \left(\int_0^1 (1-s)\gamma''(x+sw) - \gamma''(x)\, ds\right)/|w|^2 & \text{else.}\end{cases}$$

The mean value theorem tells us that $|I(x,w)| \le C\|\gamma''''\|_{L^\infty(B_1(x))}$ if $|w|\le 1$, and $I(x,w)\le w^{-2}\|\gamma''\|_{L^\infty}\}$ else. We get using the dominated convergence theorem $Q\gamma_j \to Q\gamma_\infty$. This completes the proof of

$$\mathcal{H}\tilde{\gamma}^{(j)} \to \mathcal{H}\tilde{\gamma}_\infty$$

pointwise.

We get in view of (4-2)

$$\int_{-\delta_0}^{0}\int_\Gamma |\mathcal{H}\tilde{\gamma}_\infty|^2\, d\mathcal{H}^1(x)\, dt \le \lim_{j\to\infty}\int_{-\delta_0}^{0}\int_{\mathbb{R}/l_t\mathbb{Z}} |\mathcal{H}(\gamma_t^{(j)})(x)|^2\, dx\, dt = 0.$$

Since $\tilde{\gamma}_\infty$ is smooth, we obtain $\mathcal{H}\tilde{\gamma}_\infty \equiv 0$. Furthermore, the local smooth convergence together with $E^{\text{int}}_{B_1(0)}(\widetilde{\Gamma}_j) = \varepsilon_0$ implies

$$E^{\text{int}}_{\bar{B}_1(0)}(\widetilde{\Gamma}_\infty) \ge \varepsilon_0. \qquad \square$$

Using the evolutionary attractivity of critical points proven in [Blatt 2012b] we can further show that the blow-up profile cannot be compact.

Proposition 4.3 (blow-ups profiles are never compact). *The blow-up profile constructed in Theorem 4.2 cannot be compact.*

Proof. Let us assume that $\tilde{\gamma}_\infty$ is compact, i.e., that $\tilde{\gamma}_\infty \in C^\infty(\mathbb{R}/l\mathbb{Z}, \mathbb{R}^n)$ for suitable l. Then there would be a subsequence of $\tilde{\gamma}_j$ converging smoothly to the critical point $\tilde{\gamma}_\infty$ of E. Since furthermore $E(\gamma_t) \ge E(\tilde{\gamma}_\infty)$, we get from [Blatt 2012b, Theorem 1.5] for all $t \in [0,T)$, that for j large enough the flow $\tilde{\gamma}_t$ exists for all time and converges to a stationary point of E, which contradicts the assumption $T < \infty$. $\square$

4B. *Planar curves.* For a regular curve γ the *curvature vector* κ is given by

$$\kappa = \frac{\gamma''}{|\gamma'|^2} - \frac{\langle \gamma'', \gamma'\rangle}{|\gamma'|^4}\gamma',$$

which is equal to γ'' if γ is parametrized by arc-length.

Given two points $x, y \in I$ there is either a unique circle or a straight line — which we like to think of as a degenerate circle — going through $\gamma(x)$ and $\gamma(y)$ and being tangent to γ at x. See Figure 1. Note that this is the same circle used to define the integral tangent-point energies. We denote by $\kappa_\Gamma(x,y)$ the curvature vector of this circle in x and set $\kappa_\Gamma(x,y) = 0$ if the tangent on Γ in x is pointing in the direction of y — which is the curvature of the straight line.

Lemma 4.4. *We have*

$$\kappa_\gamma(x,y) = 2\frac{P^\perp_{\gamma'(x)}(\gamma(y)-\gamma(x))}{|\gamma(x)-\gamma(y)|^2}.$$

Proof. If the vectors γ' and $\gamma(x)-\gamma(y)$ are colinear, both sides of the identity obviously vanish. So we can assume that $P^\perp_{\gamma'(x)}(\gamma(y)-\gamma(x)) \ne 0$. The circle going through $\gamma(x)$ that is tangential to γ in the point x with curvature vector $\kappa = aP^\perp(\gamma(y)-\gamma(x))$ is the set of all points $z \in \mathbb{R}^n$ satisfying

$$\left| z - \gamma(x) - \frac{\kappa}{|\kappa|^2}\right|^2 = \frac{1}{|\kappa|^2}.$$

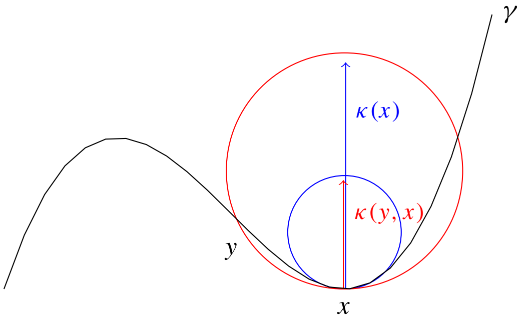

Figure 1. This picture shows the two circles playing a role in the geometric interpretation of the Euler–Lagrange equation of the Möbius energy: the inner circle, with curvature vector $\kappa(x)$, is the osculating circle at x, while the outer circle, with curvature vector $\kappa(x, y)$, is the circle going through x and y and tangent to Γ at x.

This circle contains $\gamma(y)$ if and only if

$$|\gamma(y)-\gamma(x)|^2 = 2\frac{\langle \kappa, \gamma(x)-\gamma(y)\rangle}{\kappa^2} = \frac{2}{a}.$$

Thus, $a = 2/|\gamma(x)-\gamma(y)|^2$ which proves the lemma. □

Using Lemma 4.4 we immediately get the following geometric interpretation of (4-1).

Lemma 4.5 (geometric interpretation of the Euler–Lagrange equation). *The curve γ parametrized by arc-length satisfies $\mathcal{H}\gamma \equiv 0$ if and only if*

$$\lim_{\varepsilon\downarrow 0}\int_{\Gamma\setminus B_\varepsilon(x)} \frac{\kappa_\gamma(x,y)-\kappa_\gamma(x)}{|x-y|^2}\, d\mathcal{H}^1(y) = 0 \tag{4-3}$$

for all $x \in I$.

In codimension 1, (4-3) is equivalent to

$$\lim_{\varepsilon\searrow 0}\int_{I/B_\varepsilon(x)} \frac{\langle \kappa_\gamma(x,y)-\kappa_\gamma(x), n(x)\rangle}{|x-y|^2}\, d\mathcal{H}^1(y) = 0, \tag{4-4}$$

where n is a unit normal along γ We are now looking for a situation that implies that the integrand on the left-hand side of (4-4) has a sign and thus must vanish identically. For $x \in I$ in which the curvature of γ does not vanish, we denote by $\mathrm{OB}(x)$ the open ball whose boundary is the osculating circle along γ in x; i.e.,

$$\mathrm{OB}(x) := B_{1/|\kappa(x)|}\left(\gamma(x)+\frac{\kappa}{|\kappa|^2}\right).$$

Lemma 4.6. *If there is a point $x \in I$ such that*

$$\mathrm{OB}(x)\cap\gamma(I)=\varnothing$$

or

$$\gamma(I)\subset\overline{\mathrm{OB}(x)}$$

then

$$\Gamma = \partial\,\mathrm{OB}(x),$$

i.e., Γ *is a circle.*

Proof. If $\Gamma \cap \mathrm{OB}(x) = \varnothing$, we get

$$\langle \kappa_\Gamma(x, y), n(x)\rangle \le \langle \kappa_\Gamma(x), n(x)\rangle,$$

and if $\Gamma \subset \overline{\mathrm{OB}(x)}$

$$\langle \kappa_\Gamma(x, y), n(x)\rangle \ge \langle \kappa_\Gamma(x), n(x)\rangle.$$

So in both cases

$$\langle \kappa_\Gamma(x, y), n(x)\rangle - \langle \kappa_\Gamma(x), n(x)\rangle$$

has a sign that is independent of $y \in \Gamma$.

Since $\mathcal{H}\gamma \equiv 0$ implies

$$\lim_{\varepsilon\searrow 0} \int_{\Gamma/B_\varepsilon(x)} \frac{\langle(\kappa_\gamma(x, y) - \kappa_\gamma(x)), n(x)\rangle}{|\gamma(y) - \gamma(x)|^2}\, d\mathcal{H}^1(y) = 0$$

and the integrand has a sign, we get

$$\langle(\kappa_\gamma(x, y) - \kappa_\gamma(x)), n(x)\rangle = 0$$

for all $y \in \Gamma$. But this implies

$$\kappa_\gamma(x, y) = \kappa_\gamma(x)$$

for all $y \in \Gamma$, which by the definition of $\kappa_\Gamma(x, y)$ implies

$$y \in \partial\,\mathrm{OB}(x). \qquad \square$$

Theorem 4.7. *Let* $\Gamma : I \to \mathbb{R}^2$ *be a properly embedded smooth curve parametrized by arc-length satisfying*

$$\mathrm{p.v.}\int_I \frac{\kappa_\gamma(x, y) - \kappa_\gamma(x)}{|\gamma(y) - \gamma(x)|^2}\, dy = 0.$$

Then γ *is either a straight line or a circle.*

Proof. Let us assume that γ is not a straight line. We will show that then there is a point $x \in I$ with $\kappa(x) \neq 0$ and

$$\mathrm{OB}(x) \cap \gamma(I) = \varnothing,$$

where $\mathrm{OB}(x)$ is the open ball surrounded by the osculating circle on γ at x; i.e.,

$$\mathrm{OB}(x) := \left\{ y \in \mathbb{R}^2 : \left| y - \left(\gamma(x) + \frac{\kappa(x)}{|\kappa(x)|^2}\right)\right| \le \frac{1}{|\kappa(x)|} \right\}.$$

Then the statement follows from Lemma 4.6. We construct this point as follows: As $\Gamma = \gamma(I)$ is not a straight line, we find a point $x_1 \in \Gamma$ with $\kappa_\Gamma(x_1) \neq 0$. Let n be the continuous unit normal field pointing in the direction of $\kappa_\Gamma(x_1)$ at the point x_1. Then if $\mathrm{OB}(x_1) \cap \Gamma = \varnothing$, we set $x = x_1$. If on the other hand $\mathrm{OB}(x_1) \cap \Gamma \neq \varnothing$, there is a ball $B_1 \subset \mathrm{OB}(x_1)$ touching Γ in x_1 and at least one other point. Let x_1' be one of these touching points nearest to x_1 and let Γ_1 denote the closed curve consisting of the arc of Γ between x_1 and x_1' and the part of the boundary of B_1 that makes this curve C^1 and let Ω_1 be the open set bounded by this curve.

We now start an iterative scheme in order to find the desired point x. So let $x_2 \in \Gamma$ be the point on the part of the curve between x_1 and x_1' which divides this arc into two parts of equal length. Note that $x_2 \notin \bar{B}_1$. We choose

$$r_2 := \sup\Big\{r : B_r\Big(x_2 + \frac{1}{r}n(x_2)\Big) \subset \Omega_1\Big\}$$

Then either $B_2 = B_{r_2}(x_2 + (1/r_2)n_2)$ touches Γ in x up to second order and we set $x = x_2$ and have found our point x, or we can choose $x_2' \in \Gamma_1$ to be one of the nearest points on Γ_1 touching $B_2 = B_{r_2}(x_2 + (1/r_2)n_2)$. But then x_2' must belong to the arc of Γ between x_1 and x_1' since else B_2 touches B_1 from within and hence $B_2 \subset B_1$, which is not possible, as $x_2 \in \bar{B}_2$ but $x_2 \notin \bar{B}_1$. Hence,

$$d_\Gamma(x_2, x_2') \leq \tfrac{1}{2} d_\Gamma(x_1, x_1'). \tag{4-5}$$

Then we repeat the construction above, and either get our point x in a finite number of steps, or get a sequence of points x_i, x_i' and balls $B_i \cap \Gamma = \varnothing$ such that B_i touches Γ in x_i, x_i', the intervals x_i, x_i' are nested, and the diameter of the balls B_i is bounded by the diameter of Γ_1 and from below by

$$\|\kappa_\Gamma|_{[x_1, x_1']}\|_{L^\infty}^{-1} > 0.$$

In the latter case, there is a point $x \in \Gamma$ with

$$x = \lim_{i\to\infty} x_i = \lim_{i\to\infty} x_i'$$

and it is well known that

$$r_i \to \frac{1}{|\kappa(x)|}.$$

We get for every $r < 1/|\kappa(x)|$ that

$$B_r\Big(x + \frac{1}{r}n(x)\Big) \subset B_n$$

for n large enough. Hence,

$$B_r \cap \Gamma = \varnothing \quad \text{for all } r < \frac{1}{\kappa(x)},$$

which implies

$$\mathrm{OB}(x) \cap \Gamma = \varnothing. \qquad \square$$

Using the characterization of the solutions to (1-2) we can now show:

Theorem 4.8 (the evolution of planar curves). *Let $\gamma_0 \subset \mathbb{R}^2$ be a closed smoothly embedded curve. Then the negative gradient flow of the Möbius energy exists for all times and converges to a round circle as time goes to infinity.*

Proof. Let us first prove the long-time existence of the flow. Assume that a singularity occurs after finite time. Then we construct a blow-up profile $\tilde{\gamma}_\infty$ as described in Theorem 3.1. But Theorem 4.7

implies that this blow-up must be a circle or straight line, which is not possible due to Proposition 4.3 and $E^{\text{int}}_{B_1(0)}(\tilde{\gamma}_\infty) \neq 0$.

To prove the statement about the asymptotic behavior of the flow, we let, for $t \in (0, \infty)$ and $\varepsilon_0 > \varepsilon > 0$ small enough, the radius $r_t > 0$ and $x_t \in \gamma_t$ be such that

$$E_{B^{\text{int}}_{r_t}(x_t)}(\gamma_t) = \sup_{x \in \gamma_t} E_{B^{\text{int}}_{r_t}(x)}(\gamma_t) = \varepsilon.$$

Let us assume that

$$M := \liminf_{t \in [0,\infty)} \frac{r_{t+r_t^3/2}}{r_t} < \infty. \tag{4-6}$$

Then we can choose a sequence $t_j \to \infty$ such that

$$r_{t_j + r_{t_j}^3/2} \leq 2M r_{t_j}.$$

As in Theorem 4.2, let $\tilde{\gamma}_j$ be reparametrizations of the rescaled curves

$$\frac{1}{r_j}\{\gamma_{t_j + r_{t_j}^3/2} - x_{t_j + r_{t_j}^3/2}\}$$

by arc-length such that $\tilde{\gamma}_j(0) = 0$. Then these curves γ_j subconverge locally smoothly to a curve γ_∞ satisfying $\mathcal{H}\gamma_\infty \equiv 0$, which is not a straight line. Hence, due to Theorem 4.7 γ_∞ is a circle. Since $\tilde{\gamma}_{t_j} \to \gamma_\infty$ smoothly we get that for j large enough, the flow starting with $\tilde{\gamma}_j$ converges smoothly to a circle as time goes to infinity. Hence, the same is true for γ_t.

Let us assume that (4-6) was wrong and let L_t denote the length of the curve γ_t. Then for every $\Lambda > 0$ there is a t_0 such that $r_{t+r_t^3/2} \geq \Lambda r_t$ for all $t \geq t_0$. We iteratively define $t_{j+1} := t_j + r_j^3/2$, where $r_j := r(t_j)$, and get

$$r_j \geq \Lambda^j r_{t_0}. \tag{4-7}$$

Scaling our a priori estimates in Theorem 3.1 we obtain

$$|\mathcal{H}\gamma_{\tilde{t}}| \leq \frac{C}{(\tilde{t} - t)^{2/3} r_t^2}$$

for all times $\tilde{t} \in t + (0, r_t^3)$ and hence

$$\left| \frac{d}{dt}\Big|_{t=\tilde{t}} L_t \right| \leq 2 \sup \left| \frac{d}{dt}\Big|_{t=\tilde{t}} \gamma \right| \leq \frac{C}{(\tilde{t} - t)^{2/3} r_t^2}.$$

Integrating this inequality we obtain

$$L_{t+r_t^3/2} \leq C \frac{r_t^3}{r_t^2} + L_t \leq C L_t.$$

Hence,

$$L_{t_j+1} \leq C L_{t_j}$$

and thus

$$r_j \leq L_{t_j} \leq C^j L_{t_0},$$

which contradicts (4-7) for $\Lambda > C$ and j large enough. □

Appendix: Besov spaces, commutator estimates and interpolation inequalities

For the convenience of the reader, let us gather some well-known and not so well-known facts about Besov spaces in this section. We will stick to the notation used in [Triebel 1983] and will assume that the reader is familiar with the definition of the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ on $\mathbb{R}^n$ and the respective spaces $B^s_{p,q}(\Omega)$ on smooth domains $\Omega \subset \mathbb{R}^n$ as defined in Sections 2.3.1 and 3.2.2 of [Triebel 1983].

Essential for our analysis is the following characterization of these spaces using finite differences. For an arbitrary function $f : \mathbb{R}^n \to \mathbb{R}$ these are inductively defined by

$$(\Delta^1_h f)(x) := f(x+h) - f(x), \quad (\Delta^l_h f) = \Delta^1_h \Delta^{l-1}_h f \quad \text{for } l = 2, 3, \dots.$$

Furthermore, for a set $\Omega \subset \mathbb{R}^n$ we set $\Omega_{h,l} = \bigcap_{j=0}^{l} \{x \in \Omega : x + jh \in \Omega\}$.

Lemma A.1 (equivalent norms, see [Triebel 1983, Sections 2.5.12, 3.4.2, and 2.5.10]). *The following estimates hold*:

(1) *For* $0 < p, q \leq \infty$, *we have* $s > \tilde{\sigma}_p := n(1/\min\{p, 1\} - 1)$. *If* $M > s$ *and* M *an integer*, *then*

$$\|f|B^s_{p,q}(\mathbb{R}^n)\|^{(2)}_M := \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} \frac{\|\Delta^M_h f|L^p(\mathbb{R}^n)\|^q}{|h|^{n+sq}}\,dh\right)^{1/q}$$

is an equivalent quasinorm on $B^s_{p,q}(\mathbb{R}^n)$.

(2) *If* $\Omega \subset \mathbb{R}^n$ *is a smooth domain* $1 < p < \infty$,. $s > 0$, *and* k, l *integers with* $0 \leq k < s$ *and* $s < l + k$, *then*

$$\|f|B^s_{p,q}(\Omega)\|^{(2)} := \|f|L^p(\Omega)\| + \sum_{|\alpha| \leq k} \left\| \left(\int_{\mathbb{R}^n} \left(\int_{\Omega_{h,l}} |\Delta^l_h \partial^\alpha f(x)|^q dx\right)^{q/p} \frac{dh}{|h|^{n+sq}}\right)^{1/q} \,\middle|\, L^p(\Omega)\right\|$$

is an equivalent quasinorm on $B^s_{p,q}(\Omega)$.

As an easy consequence, we get

Lemma A.2. *For* $1 < p, q < \infty$ *and* $1 > s > 0$ *we have*

$$\left(\int_{B_1(0)} \frac{\|\Delta^M_h|L^p(B_1(0))\|^q}{|h|^{n+sq}}\,dh\right)^{1/q} \leq C\|f|B^s_{p.q}(B_2(0))\|$$

and

$$\|f|B^s_{p.q}(B_1(0))\| \leq C\left(\int_{B_2(0)} \frac{\|\Delta^M_h|L^p(B_2(0))\|^q}{|h|^{n+sq}}\,dh\right)^{1/q}.$$

Proof. From the definition of the norm, we deduce that there is an extension $\tilde{f}$ of $f|B_2(0)$ such that

$$\|f\|_{B^s_{p,q}(B_1(0))} \leq \|\tilde{f}\|_{B^s_{p,q}(\mathbb{R}^n)} \leq 2\|f\|_{B^s_{p,q}(B_2(0))}.$$

Lemma A.1 gives

$$\left(\int_{B_1(0)} \frac{\|\Delta^M_h|L^p(B_1(0))\|^q}{|h|^{n+sq}}\,dh\right)^{1/q} \leq \|\tilde{f}|L^p(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} \frac{\|\Delta^M_h \tilde{f}|L^p(\mathbb{R}^n)\|^q}{|h|^{n+sq}}\,dh\right)^{1/q}$$
$$\leq C\|\tilde{f}|B^s_{p,q}(\mathbb{R}^n)\| \leq C\|f|B^s_{p,q}(B_2(0))\|.$$

To get the second estimate, we extend $f|B_1(0)$ to a function $\tilde{f}$ such that

$$\|f\|^M_{B^s_{p,q}(B_1(0))} \leq \|\tilde{f}\|_{B^s_{p,q}(\mathbb{R}^n)} \leq 2\|f\|^M_{B^s_{p,q}(B_2(0))}.$$

and argue as above. □

We will now state the following interpolation inequalities in Besov-space. Since it seems to be hard to find a proof of this result in the literature, we include a proof here for the sake of completeness.

Lemma A.3 (interpolation inequalities). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. If $0 \leq s_1 < s_2 < s_3$ and $p \in [2, \infty)$ satisfy $s_0 - n/2 < s_1 - n/p < s_w - n/2$ then*

$$\|f|B^{s_1}_{p_1,q}(\Omega)\| \leq C\|f|B^{s_0}_{2,2}(\Omega)\|^{1-\theta}\|f|B^{s_2}_{2,2}(\Omega)\|^{\theta}$$

for all $q \in [1, \infty]$ with $C = C(s, p, q, \Omega)$ and

$$\theta = \frac{(s_1 - n/p) - (s_0 - n/2)}{s_2 - s_0}.$$

Proof. By [Lunardi 1995, Proposition 1.3.2] we have to show that the real interpolation space

$$(B^{s_0}_{2,2}(\Omega), B^{s_0}_{2,2}(\Omega))_{\theta,1}$$

is continuously embedded in $B^{s_1}_{p,q}$. But this is indeed the case, as

$$(B^{s_0}_{2,2}(\Omega), B^{s_0}_{2,2}(\Omega))_{\theta,1} = B^{\tilde{s}}_{2,1}(\Omega),$$

with $\tilde{s} = (1-\theta)s_0 + \theta s_2 = s_1 + n/2 - n/p > s_1$ by [Triebel 1992, p. 204] and the Sobolev embedding for Besov spaces [Triebel 1992, p. 196] tells us that $B^{\tilde{s}}_{2,1}(\Omega)$ is continuously embedded into $B^{s_1}_{p_1,1}(\Omega) \subset B^{s_1}_{p_1,q}(\Omega)$ for all $q \in [1, \infty]$. □

One of the most important tools in this article is the following commutator estimate for our operator $\widetilde{Q}$. This is a very special case of well-known fractional Leibniz rule and known commutator estimates of Kato and Ponce, which we still decide to prove here in order to make the article as easily accessible as possible.

Lemma A.4 (commutator estimates). *For $f, g \in C^\infty([-\Lambda, \Lambda])$ we have*

$$\begin{aligned}&\|\widetilde{Q}[fg] - g\widetilde{Q}[f] - f\widetilde{Q}[g]\|_{L^r(B_{1/2}(0))}\\ &\quad\leq C\left(\|f\|_{B^{1/4}_{2p,2}(B_\Lambda(0))}\|g\|_{B^{1/4}_{2p,2}(B_\Lambda(0))} + \sum_{j=1}^{\infty}\frac{\|f\|^2_{L^{2p}(B_{\Lambda+j+1}(0)\setminus B_{\Lambda+j}(0))} + \|g\|^2_{L^{2p}(B_{\Lambda+j+1}(0)\setminus B_{\Lambda+j}(0))}}{(\Lambda+j)^2}\right).\end{aligned}$$

Proof. Remember that

$$\frac{\widetilde{Q}f(x)}{4} := \text{ p.v.}\int_{-l/2}^{l/2}\int_0^1 (1-s)\frac{\kappa(x+sw) - \kappa(x)}{|w|^2}\,dw = \widetilde{Q}\kappa(x).$$

Since

$$\frac{\widetilde{Q}^s[fg]-g\widetilde{Q}^s[f]-f\widetilde{Q}^s[g]}{4}$$
$$= \text{p.v.}\int_{-l/2}^{l/2}\int_0^1 (1-s)\frac{f(x+sw)g(x+sw)-f(x)g(x)-(f(x+sw)-f(x))g(x)-(g(x+sw)-g(x))f(x)}{|w|^2}\,dw$$
$$= \text{p.v.}\int_{-l/2}^{l/2}\int_0^1 (1-s)\frac{f(x+sw)g(x+sw)-f(x+sw)g(x)-f(x)g(x+sw)+f(x)g(x)}{|w|^2}\,dw$$
$$= \int_{-l/2}^{l/2}\int_0^1 (1-s)\frac{(f(x+sw)-f(x))(g(x+sw)-g(x))}{|w|^2}\,dw,$$

we get

$$\|\widetilde{Q}^s[fg]-g\widetilde{Q}^s[f]-f\widetilde{Q}^s[g]\|_{L^p(B_1(0))}$$
$$\leq C\int_{-l/2}^{l/2}\int_0^1 (1-s)\biggl(\int_{B_1(0)}\biggl(\frac{(f(x+sw)-f(x))(g(x+sw)-g(x))}{|w|^2}\biggr)^p dx\biggr)^{1/p} dw$$
$$\leq C\int_0^1 s(1-s)\biggl(\int_{B_1(0)}\int_{-l/2}^{l/2}\biggl(\frac{(f(x+w)-f(x))(g(x+w)-g(x))}{|w|^2}\biggr)^p dx\biggr)^{1/p} dw$$
$$\leq C\biggl(\int_{B_1(0)}\int_{-\Lambda/2}^{\Lambda/2}\biggl(\frac{(f(x+w)-f(x))(g(x+w)-g(x))}{|w|^2}\biggr)^p dx\biggr)^{1/p} dw$$
$$+C\biggl(\int_{B_1(0)}\int_{|w|\geq\Lambda/2}\biggl(\frac{(f(x+w)-f(x))(g(x+w)-g(x))}{|w|^2}\biggr)^p dx\biggr)^{1/p} dw$$
$$\leq \|f\|_{B^{1/2}_{4,2}(B_\Lambda(0))}\|g\|_{B^{1/2}_{4,2}(B_\Lambda(0))}+C\biggl(\int_{B_1(0)}\int_{|w|\geq\Lambda/2}\biggl(\frac{(f(x+w)-f(x))(g(x+w)-g(x))}{|w|^2}\biggr)^p dx\biggr)^{1/p} dw.$$

Factoring and using the Cauchy inequality, we can estimate it by

$$C\int_{|w|\geq\Lambda}\biggl(\int_{B_1(0)}\frac{|f(x)|^{2p}+|g(x)|^{2p}}{w^2}\,dx\biggr)^{1/p} dw$$
$$+\int_{|w|\geq\Lambda}\biggl(\int_{B_1(0)}\frac{|f(x+w)|^{2p}+|g(x+w)|^{2p}}{w^2}\,dx\biggr)^{1/p} dw$$
$$\leq C\|f\|^2_{L^p(B_\Lambda(0))}\|g\|^2_{L^p(B_\Lambda(0))}+\sum_{j\in\mathbb{N}}\frac{\|f\|^2_{L^{2p}(B_{\Lambda+j}\setminus B_{\Lambda+j-1})}+\|g\|_{L^{2p}(B_{\Lambda+j}\setminus B_{\Lambda+j-1})}}{(\Lambda+j)^2}. \qquad \square$$

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Received 30 Oct 2018. Accepted 7 Mar 2019.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 13 No. 3 2020