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ON THE EXISTENCE OF TRANSLATING SOLUTIONS OF MEAN CURVATURE FLOW IN SLAB REGIONS

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We prove, in all dimensions $n \geq 2$, that there exists a convex translator lying in a slab of width $\pi \sec\theta$ in $\mathbb{R}^{n+1}$ (and in no smaller slab) if and only if $\theta \in \left[0, \frac{\pi}{2}\right]$. We also obtain convexity and regularity results for translators which admit appropriate symmetries and study the asymptotics and reflection symmetry of translators lying in slab regions.

1. Introduction

A solution of mean curvature flow is a smooth one-parameter family $\{\Sigma_t\}_{t\in\mathbb{R}}$ of hypersurfaces Σ_t in $\mathbb{R}^{n+1}$ with normal velocity equal to the mean curvature vector. A translating solution of mean curvature flow is one which evolves purely by translation: $\Sigma_{t+s} = \Sigma_t + se$ for some $e \in \mathbb{R}^{n+1} \setminus \{0\}$ and each $s, t \in (-\infty, \infty)$. In that case, the time slices are all congruent and satisfy

$$H = -\langle \nu, e\rangle, \tag{1}$$

where ν is a choice of unit normal field and $H = \operatorname{div}\nu$ is the corresponding mean curvature. Conversely, if a hypersurface satisfies (1) then the one-parameter family of translated hypersurfaces $\Sigma_t := \Sigma + te$ satisfies mean curvature flow. We shall eliminate the scaling invariance and isotropy of (1) by restricting attention to translating solutions which move with unit speed in the "upwards" direction. That is, we henceforth assume that $e = e_{n+1}$. We will refer to a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ satisfying (1) with $e = e_{n+1}$ as a *translator*.

The most prominent example of a translator is the Grim Reaper curve, $\Gamma^1 \subset \mathbb{R}^2$, defined by

$$\Gamma^1 := \left\{(x, -\log\cos x) : |x| < \tfrac{\pi}{2}\right\}.$$

Taking products with lines then yields the Grim hyperplanes

$$\Gamma^n := \left\{(x_1, \dots, x_n, -\log\cos x_1) : |x_1| < \tfrac{\pi}{2}\right\}.$$

The Grim hyperplane Γ^n lies in the slab $\left\{(x_1, \dots, x_n) : |x_1| < \frac{\pi}{2}\right\}$ (and in no smaller slab). More generally, if Σ^{n-k} is a translator in $\mathbb{R}^{n-k+1}$ then $\Sigma^{n-k} \times \mathbb{R}^k$ is a translator in $\mathbb{R}^{n-k+1} \times \mathbb{R}^k \cong \mathbb{R}^{n+1}$.

There is also a family of "oblique" Grim planes $\Gamma^n_{\theta,\phi}$ parametrized by $(\theta, \phi) \in \left[0, \frac{\pi}{2}\right) \times S^{n-2}$. These are obtained by rotating the "standard" Grim plane Γ^n through the angle $\theta \in \left[0, \frac{\pi}{2}\right)$ in the plane $\operatorname{span}\{\phi, e_{n+1}\}$

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for some unit vector $\phi \in \operatorname{span}\{e_2, \dots, e_n\}$ and then scaling by the factor $\sec\theta$. To see that the result is indeed a translator, we need only check that

$$-H_\theta = -\cos\theta H = \cos\theta\langle \nu, e_{n+1}\rangle = \langle \cos\theta\nu + \sin\theta\phi, e_{n+1}\rangle = \langle \nu_\theta, e_{n+1}\rangle,$$

where H_θ and ν_θ are the mean curvature and outward unit normal of $\Gamma^n_{\theta,\phi}$ respectively. The oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ lies in the slab $S^{n+1}_\theta := \big\{(x_1, \dots, x_n) : |x_1| < \frac{\pi}{2}\sec\theta\big\}$ (and in no smaller slab). More generally, if Σ^{n-k} is a translator in $\mathbb{R}^{n-k+1}$ then the hypersurface $\Sigma^n_{\theta,\phi}$ obtained by rotating $\Sigma^{n-k} \times \mathbb{R}^k$ counterclockwise through angle θ in the plane $\phi \wedge e_{n+1}$ and then scaling by $\sec\theta$ is a translator in $\mathbb{R}^{n+1}$, so long as ϕ is a nonzero vector in $\operatorname{span}\{e_{n-k+1}, \dots, e_n\}$. The oblique Grim hyperplanes will play an important role in our analysis.

A convex entire translator asymptotic to a paraboloid was constructed in [Altschuler and Wu 1994]; see also [Clutterbuck et al. 2007]. White conjectured [2003, Conjecture 2] that the bowl is the only strictly convex translator of dimension $n \ge 2$. X.-J. Wang [2011] proved that it is the only convex entire translator in $\mathbb{R}^3$ and constructed further convex entire examples in higher dimensions. This disproves White's conjecture; however, White [2003, unnumbered remark on page 133] also stated that, even if the conjecture is false, it may be true for translating limit flows to an embedded mean-convex flow. Since limit flows to mean convex, embedded flows are noncollapsing (and hence entire) [Andrews 2012; Sheng and Wang 2009; White 2003], Wang's result proves the modified conjecture when the dimension is 2. More recently, Haslhofer [2015] proved that the bowl is the only noncollapsing translator of dimension $n \ge 2$ which is uniformly two-convex, confirming White's modified conjecture for two-convex, embedded mean curvature flows. The first two authors removed the embeddedness requirement when $n \ge 3$ [Bourni and Langford 2016].

Wang also proved the existence of strictly convex translating solutions which lie in slab regions in $\mathbb{R}^{n+1}$ for all $n \ge 2$. Since convexity of solutions of the Dirichlet problem for the graphical translator equation remains open,[1] this was achieved by exploiting the Legendre transform and the existence of convex solutions of certain fully nonlinear equations [Wang 2011]. Unfortunately, this method loses track of the precise geometry of the domain on which the solution is defined and so it remained unclear exactly which slabs admit translators; see [Spruck and Xiao 2017, Remark 1.6]. Our main result resolves this problem.

Recall that the slab region $S^{n+1}_\theta \subset \mathbb{R}^{n+1}$ is defined by

$$S^{n+1}_\theta := \Big\{(x, y, z) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : |x| < \tfrac{\pi}{2}\sec\theta\Big\} \subset \mathbb{R}^{n+1}.$$

Theorem 1 (existence of convex translators in slab regions). *For every $n \ge 2$ and every $\theta \in \big(0, \frac{\pi}{2}\big)$ there exists a strictly convex translator Σ^n_θ which lies in S^{n+1}_θ and in no smaller slab.*

The solutions we construct are reflection symmetric across the midplane of the slab, rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2, \dots, e_n\}$ and asymptotic to the "correct" oblique Grim

[1] Recently, Spruck and Xiao [2017] proved that complete mean convex translators in $\mathbb{R}^3$ are necessarily convex. We extend their result to higher dimensions in Section 3, assuming the translator has at most two distinct principal curvatures.

hyperplanes $\Gamma^n_{\theta,\phi}$ in the following sense: if ϕ is any unit vector in $\mathbb{E}^{n-1}$ then the curve $\{\sin\omega\phi - \cos\omega e_{n+1} : \omega \in (0,\theta)\}$ lies in the normal image of Σ^n_θ and the translators

$$\Sigma^n_{\theta,\omega} := \Sigma^n_\theta - P(\sin\omega\phi - \cos\omega e_{n+1})$$

converge locally uniformly in the smooth topology to the oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ as $\omega \to \theta$, where $P : S^n \to \Sigma^n_\theta$ is the inverse of the Gauss map.

Spruck and Xiao [2017, Theorem 1.1] recently proved that every mean convex translator is actually convex and Wang [2011, Corollary 2.2] proved that any convex translator which is not an entire graph must lie in a slab region. The bowl translator of Altschuler and Wu and the Grim hyperplane provide examples in the limiting cases $\theta \in \{0, \frac{\pi}{2}\}$ and there can exist no convex translator inside a slab of width less than π (the Grim hyperplane is a barrier); Theorem 1 provides the existence of a convex translator in all remaining cases, so we obtain the following corollary.

Corollary 2. *Let Ω be an open subset of $\mathbb{R}^n$ for some $n \geq 2$. There exists a convex translator in the cylinder $\Omega \times \mathbb{R}$ (and in no smaller cylinder) if and only if Ω is a slab of width $\pi \sec\theta$ for some $\theta \in [0, \frac{\pi}{2}]$.*

A systematic classification of translators lying in slab regions remains an open problem. As a first step towards addressing it, we show that the asymptotics of the solutions described in Theorem 1 are universal.

Theorem 3 (unique asymptotics modulo translation). *Given $n \geq 2$ and $\theta \in (0, \frac{\pi}{2})$ let Σ^n_θ be a convex translator which lies in S^{n+1}_θ and in no smaller slab. If $n \geq 3$, assume in addition that Σ^n_θ is rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2, \dots, e_n\}$. Given any unit vector $\phi \in \mathbb{E}^{n-1}$ the curve $\{\sin\omega\phi - \cos\omega e_{n+1} : \omega \in [0,\theta)\}$ lies in the normal image of Σ^n_θ and the translators*

$$\Sigma^n_{\theta,\omega} := \Sigma^n_\theta - P(\sin\omega\phi - \cos\omega e_{n+1})$$

converge locally uniformly in the smooth topology to the oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ as $\omega \to \theta$, where $P : S^n \to \Sigma^n_\theta$ is the inverse of the Gauss map.

We note that, in the important special case $n = 2$, this result was already obtained in [Spruck and Xiao 2017] using different methods.

The rotational symmetry hypothesis — which is not required when $n = 2$ — may be necessary in higher dimensions: it is conceivable that there exist convex translators in the slab $S^4_\theta \subset \mathbb{R}^4$, for example, which are asymptotic to an "oblique" $\Sigma^2_\theta \times \mathbb{R}$, where $\Sigma^2_\theta \subset \mathbb{R}^3$ is the translator from Theorem 1.

Using the Alexandrov reflection principle, we deduce that such solutions are reflection symmetric.

Corollary 4. *Given $\theta \in (0, \frac{\pi}{2})$, let Σ be a strictly convex translator which lies in S^{n+1}_θ and in no smaller slab. If $n \geq 3$ assume in addition that Σ is rotationally symmetric with respect to $\mathbb{E}^{n-1}$. Then Σ is reflection symmetric across the hyperplane $\{0\} \times \mathbb{R}^n$.*

This result was also obtained in [Spruck and Xiao 2017] when $n = 2$.

Remark. After this work was completed, Hoffman, Ilmanen, Martín and White [Hoffman et al. 2019] provided a different approach to the problem of existence of graphical translators over strip regions in $\mathbb{R}^3$ and moreover proved uniqueness of such translators.

2. Compactness

Recall that, given a smooth function u over a domain $\Omega \subset \mathbb{R}^n$, the downward-pointing unit normal ν and the mean curvature $H[u]$ of $\operatorname{graph} u$ are given by

$$\nu = \frac{(Du, -1)}{\sqrt{1+|Du|^2}} \quad \text{and} \quad H[u] = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right),$$

respectively. So $\operatorname{graph} u$ is a translator (possibly with boundary) when

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}. \tag{2}$$

We will derive uniform $C^{1,\alpha}$ estimates for hypersurfaces that are given by the graphs of rotationally symmetric solutions of the Dirichlet problem

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}} \quad \text{in } \Omega, \qquad u = \psi \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is a bounded open subset of $\mathbb{R}^{n+1}$ with $C^{1,\alpha}$ boundary and $\psi : \partial\Omega \to \mathbb{R}$ is a $C^{1,\alpha}$ function for some $\alpha \in (0, 1]$.

By Allard's regularity theorems [1972; 1975], see also [Bourni 2016], the desired estimates are a consequence of the following lemma. We remark that the usual dimension restriction is circumvented here due to the rotational symmetry of the solutions; see Remark 2.4 below.

Lemma 2.1. *Given any $\varepsilon, K > 0$ there exists $\lambda_0 = \lambda_0(\varepsilon, K)$ with the following property: Let u be a solution of* (3), *with $\partial\Omega$ and ψ being rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2, \dots, e_n\}$ and having $C^{1,\alpha}$ norms bounded by K. For any $p \in \operatorname{graph} u$ and $\lambda \le \lambda_0$*

$$\lambda^{-1} \sup_{y \in \operatorname{graph} u \cap B^{n+1}_\lambda(p)} \operatorname{dist}(y - p, P) < \varepsilon \tag{4}$$

for some n-dimensional linear subspace $P = P(p, \varepsilon, \lambda)$. If $B^{n+1}_\lambda(p) \cap \operatorname{graph} \psi = \varnothing$ then

$$\omega_n^{-1} \lambda^{-n} \mathcal{H}^n(\operatorname{graph} u \cap B^{n+1}_\lambda(p)) \le 1 + \varepsilon. \tag{5}$$

If $p \in \operatorname{graph} \psi$ then (4) *holds with P replaced by an n-dimensional half-hyperplane $P_+ = P_+(p, \varepsilon, \lambda)$ such that $0 \in \partial P_+$,*

$$\lambda^{-1} \sup_{\operatorname{graph} \psi \cap B^{n+1}_\lambda(p)} \operatorname{dist}(y - p, \partial P_+) < \varepsilon \tag{6}$$

and (5) *holds with the bound $1 + \varepsilon$ replaced by $\frac{1}{2} + \varepsilon$.*

Proof. We assume that the conclusion is not true. Then there exist $\varepsilon_0 > 0$ and $K_0 > 0$, sequences of rotationally symmetric domains Ω_i and boundary data $\psi_i : \partial\Omega_i \to \mathbb{R}$ bounded in $C^{1,\alpha}$ by K_0, corresponding solutions u_i of the Dirichlet problem (3), points $p_i \in \operatorname{graph} u_i$ and scales $\lambda_i \downarrow 0$ such that either (4) or (5) (or (6) in the case $p_i \in \operatorname{graph} \psi_i$), with this ε_0 and with $u = u_i$, $p = p_i$ and $\lambda = \lambda_i$, fails for all i.

Set $\widetilde{\Omega}_i = \eta_{p_i,\lambda_i}(\Omega_i)$, $\Psi_i :=$ graph ψ_i and $\widetilde{\Psi}_i = \eta_{p_i,\lambda_i}(\Psi_i)$, where $\eta_{p,\lambda}(y) = \lambda^{-1}(y-p)$. We define the current $\widetilde{T}_i = \eta_{p_i,\lambda_i\#}(T_i)$, where $T_i = [\![\operatorname{graph} u_i]\!]$ and note that $\widetilde{T}_i = [\![\operatorname{graph} \tilde{u}_i]\!]$, where $\tilde{u}_i \in C^{1,\alpha}(\widetilde{\Omega}_i)$ is defined by $\tilde{u}_i(p) = \eta_{p_i,\lambda_i}(u_i(\lambda_i p + p_i))$ with mean curvature satisfying

$$\widetilde{H}_i(p) = \lambda_i H_i(x_i + \lambda_i p) \le \lambda_i \quad \Longrightarrow \quad \|\widetilde{H}_i\|_{0,\widetilde{\Omega}_i} \xrightarrow{i\to\infty} 0.$$

It follows, after passing to a subsequence, that [Bourni 2011, Lemma 2.15], see also [Simon 1983, Theorem 34.5]:

(i) $\widetilde{T}_i \to T$ in the weak sense of currents, where T is area-minimizing.

(ii) $\mu_{\widetilde{T}_i} \to \mu_T$ as Radon measures, where $\mu_{\widetilde{T}_i}$ and μ_T denote the total variation measures of $\widetilde{T}_i$ and T respectively.

(iii) For any $\varepsilon > 0$ and any compact subset $W \subset \mathbb{R}^{n+1}$ such that $W \cap \operatorname{spt} T \neq \varnothing$ there exists i_0 such that, for all $i \ge i_0$,

$$\operatorname{spt} T_i \cap W \subset \varepsilon\text{-neighborhood of } \operatorname{spt} T.$$

By the measure convergence (ii), for every $\varepsilon > 0$ there exists i_0 such that, for all $i \ge i_0$,

$$\lambda_i^{-n} \mu_{T_i}(B^{n+1}_{\lambda_i}(p_i)) = \mu_{\widetilde{T}_i}(B_1^{n+1}(0)) \le |\operatorname{spt} T \cap B_1^{n+1}(0)| + \varepsilon.$$

By the Hausdorff convergence (3), for any $\varepsilon > 0$ there exists i_0 such that, for all $i \ge i_0$,

$$\frac{1}{\lambda_i} \sup_{y \in B^{n+1}_{\lambda_i}(p_i) \cap \operatorname{spt} T_i} \operatorname{dist}(y - x_i, \operatorname{spt} T) = \sup_{y \in B_1^{n+1}(0) \cap \operatorname{spt} \widetilde{T}_i} \operatorname{dist}(y, \operatorname{spt} T) < \varepsilon.$$

So it remains to prove that $\operatorname{spt} T$ is either a hyperplane or a half-hyperplane.

It suffices to consider the following three cases for the sequence of points p_i:

Case 1: $p_i \in \Psi_i = \partial \operatorname{graph} u_i$.

Case 2a: $p_i = (x_i, y_i, u(x_i, y_i)) \notin \Psi_i$, $y_i \in \mathbb{R}^{n-1}$ with $|y_i| = 0$ for all i and $\liminf_i \operatorname{dist}(p_i, \Psi_i) \neq 0$.

Case 2b: $p_i = (x_i, y_i, u(x_i, y_i)) \notin \Psi_i$, $y_i \in \mathbb{R}^{n-1}$ with $\liminf_i |y_i| \neq 0$ and $\liminf_i \operatorname{dist}(p_i, \Psi_i) \neq 0$.

We will show that in Case 1 $\operatorname{spt} T$ is half-hyperplane and in Cases 2a and 2b it is a hyperplane.

We need the following fact, which is a consequence of the divergence theorem applied to the normals of the graphs (extended to be independent of the e_{n+1}-direction) in two appropriately chosen domains. For a proof see [Bourni 2011, Lemmas 2.10, 2.12].

Claim 2.1.1. *There exists a constant c such that for any i, $p \in \overline{\Omega}_i \times \mathbb{R}$ and $\rho > 0$ the following hold:*

(i) *Let H_i denote the mean curvature of* graph u_i; *then*

$$\mathcal{H}^n(\operatorname{graph} u_i \cap B^{n+1}_\rho(p)) \le c(1 + \rho\|H_i\|_0)\omega_n \rho^n.$$

(ii) *Let $\sigma \in (0, \rho)$, $Q_{\rho,\sigma} = [-\sigma\rho, \sigma\rho] \times B^n_\rho(0)$ and q be an orthogonal transformation of $\mathbb{R}^{n+1}$ such that $q(0) = p$. Then*

$$\mathcal{H}^n(\operatorname{graph} u_i \cap q(Q_{\rho,\sigma})) \le \omega_n \rho^n (1 + c\sigma(n + \rho\|H_i\|_0)).$$

In Case 1, by [Bourni 2011, Lemma 2.15], $\operatorname{spt} T$ is an n-dimensional half-space and $\partial T = [\![\Psi]\!]$ with Ψ being the limit of $\widetilde{\Psi}_i$ and where the convergence $\widetilde{\Psi}_i \to \Psi$ is with respect to the $C^{1,\beta}$ topology for any $\beta < \alpha$, which implies that

$$\frac{1}{\lambda_i} \sup_{y \in B^{n+1}_{\lambda_i}(x_i) \cap \Psi_i} \operatorname{dist}(y - x_i, \operatorname{spt} \partial T) = \sup_{y \in B^{n+1}_1(0) \cap \widetilde{\Psi}_i} \operatorname{dist}(y, \Psi) < \varepsilon.$$

Hence taking $P_+ = \operatorname{spt} T$ we get a contradiction for Case 1.

Having proven the boundary case, we will now proceed with the interior. We will first consider Case 2a, that is when $p_i = (x_i, y_i, u(x_i, y_i)) \notin \Psi_i$ with $y_i = 0 \in \mathbb{R}^{n-1}$ and $\liminf_i \operatorname{dist}(p_i, \Psi_i) \neq 0$. In this case the support of the area-minimizing current T is rotationally symmetric in the y-space. Using the uniform area ratio bounds, Claim 2.1.1 and the interior monotonicity formula [Allard 1972], see also [Simon 1983, Section 17], we have

$$\begin{aligned} 1 \leq \omega_n^{-1} r^{-n} \mu_T(B^{n+1}_r(p)) &= \omega_n^{-1} r^{-n} \lim_i \mu_{\widetilde{T}_i}(B^{n+1}_r(p)) \\ &= \omega_n^{-1} (\lambda_i r)^{-n} \lim_i \mu_{T_i}(B^{n+1}_{\lambda_i r}(p)) \leq c \end{aligned} \tag{7}$$

for all $p \in \operatorname{spt} T$ and any $r > 0$, where c is a constant which is independent of i. Thus, for a sequence $\{\Lambda_k\} \uparrow \infty$, we can apply the Federer–Fleming compactness theorem [1960], see also [Simon 1983, Theorem 32.2], to the sequence $T_{0,\Lambda_k} = \eta_{0,\Lambda_k \#} T$; after passing to a subsequence, this yields $T_{0,\Lambda_k} \to C$ in the weak sense of currents, where C is an area-minimizing cone, and $\mu_{T_{p,\Lambda_k}} \to \mu_C$ as radon measures. Note that C is rotationally symmetric in the y-space, $\mathbb{E}^{n-1}$. Since $\operatorname{spt} C \cap S^n$ is an embedded minimal surface in S^n which is rotationally symmetric with respect to $\mathbb{E}^{n-1}$, it must be congruent to either the equator S^{n-1} or the Clifford torus $S^1_{\sqrt{1/(n-1)}} \times S^{n-2}_{\sqrt{(n-2)/(n-1)}}$ [Brito and Leite 1990; Ôtsuki 1970; 1972]. Since the cone over the Clifford torus cannot arise as a limit of graphs, we conclude that $C = m[\![\mathbb{R}^n \times \{0\}]\!]$. We claim that in fact $m = 1$.

For $\sigma \in (0, 1)$, let $Q_{1,\sigma} = B^n_1(0) \times [-\sigma, \sigma]$. Then $\mu_C(Q_{1,\sigma}) = m\omega_n$. By the measure convergence $\mu_{T_{0,\Lambda_k}} \to \mu_C$ and $\mu_{\widetilde{T}_i} \to \mu_T$, we have that for any $\sigma_0 \in (0, 1)$ and any $\delta > 0$ there exists some $\Lambda > 0$ and k_0 such that for all $k \geq k_0$ and $\sigma \leq \sigma_0$

$$m - \delta \leq \frac{1}{\Lambda^n \omega_n} \mu_{\widetilde{T}_i}(p + \Lambda Q_{1,\sigma}).$$

Using Claim 2.1.1, the right-hand side of the above inequality is less than $1 + c\sigma\Lambda$ and hence taking σ small enough we conclude that m has to be 1. Hence, recalling (7), we obtain

$$\omega_n^{-1} r^{-n} \mu_T(B^{n+1}_r(0)) = 1 \quad \text{for all } r > 0,$$

which implies that $\operatorname{spt} T$ itself is a hyperplane and the multiplicity is 1. This provides a contradiction for Case 2a.

We are left with Case 2b. So suppose that $\liminf_i |y_i| \neq 0$ and $\liminf_i \operatorname{dist}(p_i, \Psi_i) \neq 0$. After passing to a subsequence we can assume that $\lim |y_i| = |y_\infty|$ exists, with $|y_\infty| \in (0, \infty]$. Rotational symmetry of $\operatorname{graph} u_i$ in the y-space then implies that $T = [\![\mathbb{R}^{n-2}]\!] \times T_0$, where T_0 is an area-minimizing 2-current in $\mathbb{R}^3$.

Since any such current is regular, T_0, and hence also T, is regular. We conclude that spt T_0 must be a plane [do Carmo and Peng 1979; Pogorelov 1981; Schoen 1983] with (arguing as in Case 2a) multiplicity 1. This provides a contradiction for Case 2b. □

Lemma 2.1 allows us to apply Allard's interior and boundary regularity theorems [1972; 1975] to obtain uniform $C^{1,\alpha}$ estimates for the graphs of solutions u to (3) with boundary data that satisfy the hypotheses of Lemma 2.1. Assuming higher-regularity of the boundary data we can apply Schauder theory to obtain higher-regularity estimates for these graphs.

Corollary 2.2. *Given any $K > 0$ and $\ell_0 \in \mathbb{N}$, there exists a constant C with the following property: Let u be a solution of* (3) *with $\partial\Omega$ and ψ bounded in $C^{\ell_0+2,\alpha}$ by K for some $\alpha \in (0, 1]$ and rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_2, \dots, e_n\}$. Then*

$$\sup_{p\in \operatorname{graph} u} |\nabla^\ell A(p)| \le C \quad \text{for all } \ell \in \{0, \dots, \ell_0\},$$

where A is the second fundamental form of graph u *and $\nabla^0 A := A$.*

Remark 2.3. If we allow $\ell_0 = -1$ in the hypotheses of Corollary 2.2 then we obtain uniform $C^{1,\alpha}$ estimates for the graphs of solutions u to (3) with boundary data that satisfy the hypotheses of Lemma 2.1 using the results of [Bourni 2016].

Remark 2.4. If $n \le 6$ then Lemma 2.1, and hence Corollary 2.2 and Remark 2.3, still hold without the rotational symmetry hypothesis on the boundary data. To see this, note that the proof of the boundary case (Case 1) of Lemma 2.1 does not make use of the rotational symmetry hypothesis and hence holds in all dimensions without this restriction. To show interior regularity in the case $n+1 \le 7$ we can refer to known results on regularity of almost-minimizing surfaces; see for example [Duzaar and Steffen 1993; Massari and Miranda 1984]. One can alternatively see this from Cases 2a and 2b in the proof of Lemma 2.1, since there are no stable nonplanar minimal cones in low dimensions [Simons 1968]; see also [Schoen et al. 1975] or [Simon 1983, Appendix B].

3. Convexity

We need to extend the convexity result [Spruck and Xiao 2017, Theorem 1.1] to higher dimensions. Our proof is a straightforward modification of theirs.

We make use of the following lemma.

Lemma 3.1. *Let Σ^n be a connected translator in $\mathbb{R}^{n+1}$. Suppose that Σ^n has constant mean curvature H_0. Then $H_0 = 0$ and Σ^n lies in a vertical minimal cylinder. In particular, if $n = 2$ or, more generally, if Σ^n has at most two principal curvatures at each point, then Σ^n lies in a vertical hyperplane.*

Proof. The mean curvature of Σ^n satisfies

$$-(\Delta + \nabla_V)H = |A|^2 H,$$

where $V := e_{n+1}^\top$. Thus,

$$\langle \nu, e_{n+1}\rangle = -H \equiv 0,$$

so e_{n+1} is tangential and hence $V \equiv e_{n+1}$. It follows that the integral curves of V are vertical lines, which completes the proof. □

Theorem 3.2. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a strictly mean convex translator with at most two distinct principal curvatures at each point and bounded second fundamental form. Then Σ is convex.*

Proof. Denote the principal curvatures of Σ by $\kappa \le \mu$. Note that κ is smooth and has constant multiplicity $m \in \{1, \dots, n-1\}$ in the open set $U := \{X \in \Sigma : \kappa(X) < 0\}$. Recall that

$$-(\nabla_V + \Delta)A = |A|^2 A,$$

where $V := e_{n+1}^{\top}$ is the tangential part of e_{n+1}. Computing locally in a principal frame $\{\tau_1, \dots, \tau_n\}$ with $\kappa_i = A_{ii} = \kappa$ when $i \le m$ and $\kappa_i = A_{ii} = \mu$ when $i \ge m+1$, we obtain

$$-(\nabla_V + \Delta)\kappa = |A|^2\kappa + 2\sum_{\ell=1}^{n}\sum_{p=m+1}^{n}\frac{(\nabla_\ell A_{1p})^2}{\mu - \kappa} \quad \text{in } U.$$

Since the mean curvature satisfies

$$-(\nabla_V + \Delta)H = |A|^2 H,$$

straightforward manipulations then yield

$$-(\nabla_V + \Delta)\frac{\kappa}{\mu} = -(\nabla_V + \Delta)\frac{(n-m)\kappa}{H - m\kappa} = \frac{2}{n-m}\frac{H}{\mu^2}\sum_{\ell=1}^{n}\sum_{p=m+1}^{n}\frac{(\nabla_\ell A_{1p})^2}{\mu-\kappa} + 2\left\langle \nabla\frac{\kappa}{\mu}, \frac{\nabla\mu}{\mu}\right\rangle. \tag{8}$$

Suppose that

$$-\varepsilon_0 := \inf_{\Sigma}\frac{\kappa}{\mu} < 0.$$

If the infimum is attained at some point $X_0 \in \Sigma$ then $\kappa(X_0) < 0$ and the strong maximum principle yields $\kappa/\mu \equiv -\varepsilon_0 < 0$. In particular,

$$0 \equiv \nabla_\ell \frac{\kappa}{\mu} = \frac{\nabla_\ell A_{pp}}{\kappa} - \frac{\kappa}{\mu}\frac{\nabla_\ell A_{qq}}{\mu}$$

when $p \le m < q$. It is a general observation that

$$0 = \tau_\ell A_{ij} = \nabla_\ell A_{ij} + (\kappa_j - \kappa_i)\Gamma_{\ell ij} = \nabla_\ell A_{ij} \tag{9}$$

for each ℓ whenever $\kappa_i = \kappa_j$ and $i \ne j$, where $\Gamma_{\ell ij} := \langle \nabla_\ell \tau_i, \tau_j\rangle$. Thus,[2]

$$0 = \nabla_\ell A_{11} \quad \text{when } \ell = 2, \dots, m,$$
$$0 = \nabla_\ell A_{nn} \quad \text{when } \ell = m+1, \dots, n-1.$$

Recalling (8), we also find that

$$0 \equiv \sum_{\ell=1}^{n}\sum_{p=m+1}^{n}(\nabla_\ell A_{1p})^2.$$

[2]Here, and elsewhere, we freely make use of the Codazzi identity.

It follows that the components $\nabla_1 A_{nn}$, $\nabla_1 A_{11}$, $\nabla_n A_{11}$ and $\nabla_n A_{nn}$ are all identically zero and hence, by the translator equation (1),

$$0 \equiv m\nabla_\ell A_{11} + (n-m)\nabla_\ell A_{nn} = \nabla_\ell H$$

for each $\ell = 1, \dots, n$. Lemma 3.1 now implies that Σ^n is a vertical hyperplane, contradicting strict mean convexity.

Suppose then that the infimum is not attained. Since

$$\frac{\kappa}{\mu} \geq -\frac{n-m}{m}$$

and the sectional curvatures of Σ are bounded, the Omori–Yau maximum principle may be applied. This yields a sequence of points $X_i \to \infty$ such that

$$\frac{\kappa}{\mu}(X_i) \to -\varepsilon_0, \qquad \left|\nabla\frac{\kappa}{\mu}(X_i)\right| \leq \frac{1}{i} \quad \text{and} \quad -\Delta\frac{\kappa}{\mu}(X_i) \leq \frac{1}{i}. \tag{10}$$

Consider the sequence of translators $\Sigma_i := \Sigma - X_i$. By Corollary 2.2, the translators Σ_i converge locally uniformly in C^∞, after passing to a subsequence, to a limit translator Σ_∞. Note that, whenever $\kappa < 0 < \mu$,

$$\nabla_\ell\frac{m\kappa}{\mu} = \frac{m\nabla_\ell A_{11}}{\mu} - \frac{m\kappa}{\mu^2}\nabla_\ell A_{nn} = \frac{\nabla_\ell H}{\mu} - m\left(\frac{n-m}{m} + \frac{\kappa}{\mu}\right)\frac{\nabla_\ell A_{nn}}{\mu}. \tag{11}$$

We claim that

$$\left(\frac{n-m}{m} + \frac{\kappa}{\mu}(X_i)\right)\frac{\nabla_k A_{nn}}{\mu}(X_i) \to 0 \quad \text{as } i \to \infty \tag{12}$$

for each $\ell = 1, \dots, n$. Suppose that this is not the case. Then there exists $i_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that

$$\left(\frac{n-m}{m} + \frac{\kappa}{\mu}(X_i)\right)\frac{|\nabla_k A_{nn}|}{\mu}(X_i) > \delta_0 \tag{13}$$

for all $i > i_0$ and some $k \in \{1, \dots, n\}$. By (10),

$$\left(\frac{\nabla_\ell A_{11}}{\mu} - \frac{\kappa}{\mu}\frac{\nabla_\ell A_{nn}}{\mu}\right)(X_i) \to 0$$

for each $\ell = 1, \dots, n$ as $i \to \infty$ so that, replacing δ_0 and i_0 if necessary,

$$\left(\frac{n-m}{m} + \frac{\kappa}{\mu}(X_i)\right)\frac{|\nabla_k A_{11}|}{\mu}(X_i) > \delta_0 \tag{14}$$

for all $i > i_0$. Moreover, by (9),

$$\frac{\nabla_\ell A_{nn}}{\mu}(X_i) \to 0$$

as $i \to \infty$ for all $\ell = 2, \dots, n-1$. So (13) (and hence also (14)) holds with $k \in \{1, n\}$. Combining (10) and (8) we obtain, at the point X_i,

$$\begin{aligned}
\frac{1}{i} &\geq \frac{1}{n-m}\frac{H}{\mu-\kappa}\sum_{\ell=1}^{n}\sum_{p=m+1}^{n}\frac{(\nabla_\ell A_{1p})^2}{\mu^2} + \left\langle \nabla\frac{\kappa}{\mu}, \frac{\nabla\mu}{\mu}\right\rangle \\
&= \frac{1}{n-m}\frac{H}{\mu-\kappa}\left(\sum_{p=m+1}^{n}\frac{(\nabla_p\kappa)^2}{\mu^2} + (n-m)\frac{(\nabla_1\mu)^2}{\mu^2}\right) + \nabla_1\frac{\kappa}{\mu}\frac{\nabla_1\mu}{\mu} \\
&\qquad\qquad + \sum_{\ell=2}^{m}\nabla_\ell\frac{\kappa}{\mu}\frac{\nabla_\ell\mu}{\mu} + \sum_{\ell=m+1}^{n}\nabla_\ell\frac{\kappa}{\mu}\left(\frac{\nabla_\ell\kappa}{\kappa} - \frac{\mu}{\kappa}\nabla_\ell\frac{\kappa}{\mu}\right) \\
&= -\frac{\mu}{\kappa}\sum_{\ell=m+1}^{n}\left(\nabla_\ell\frac{\kappa}{\mu}\right)^2 + \sum_{\ell=2}^{m}\nabla_\ell\frac{\kappa}{\mu}\frac{\nabla_\ell\mu}{\mu} + \nabla_1\frac{\kappa}{\mu}\frac{\nabla_1\mu}{\mu} + \frac{H}{\mu-\kappa}\frac{(\nabla_1\mu)^2}{\mu^2} \\
&\qquad\qquad + \frac{\mu}{\kappa}\sum_{\ell=m+1}^{n}\nabla_\ell\frac{\kappa}{\mu}\frac{\nabla_\ell\kappa}{\mu} + \frac{1}{n-m}\frac{H}{\mu-\kappa}\sum_{\ell=m+1}^{n}\frac{(\nabla_\ell\kappa)^2}{\mu^2} \\
&\geq -\frac{\mu}{\kappa}\sum_{\ell=m+1}^{n}\left(\nabla_\ell\frac{\kappa}{\mu}\right)^2 + \sum_{\ell=2}^{m}\nabla_\ell\frac{\kappa}{\mu}\frac{\nabla_\ell\mu}{\mu} + \left(m\frac{(n-m)/m+\kappa/\mu}{1-\kappa/\mu}\frac{|\nabla_1\mu|}{\mu} - \left|\nabla_1\frac{\kappa}{\mu}\right|\right)\frac{|\nabla_1\mu|}{\mu} \\
&\qquad\qquad + \sum_{\ell=m+1}^{n}\left(\frac{m}{n-m}\frac{(n-m)/m+\kappa/\mu}{1-\kappa/\mu}\frac{|\nabla_\ell\kappa|}{\mu} + \frac{\mu}{\kappa}\left|\nabla_\ell\frac{\kappa}{\mu}\right|\right)\frac{|\nabla_\ell\kappa|}{\mu}.
\end{aligned}$$

Suppose that $k = 1$ in (13). If

$$\left(\frac{n-m}{m} + \frac{\kappa}{\mu}(X_i)\right)\frac{|\nabla_n\kappa|}{\mu}(X_i) \not\to 0 \quad \text{as } i \to \infty$$

then, taking $i \to \infty$, we find $(|\nabla_1\mu|/\mu)(X_i) \to 0$ as $i \to \infty$, contradicting (13). Else,

$$\frac{|\nabla_n\kappa|}{\mu}(X_i) \leq \frac{|\nabla_1\mu|}{\mu}(X_i)$$

for i sufficiently large and we again obtain $(|\nabla_1\mu|/\mu)(X_i) \to 0$ as $i \to \infty$, contradicting (13). If $k = n$ in (13) we may argue similarly, using (14).

So (12) does indeed hold. Applying (10) and (12) to (11) yields

$$\frac{\nabla_\ell H}{\mu}(X_i) \to 0$$

for each $\ell = 1, \dots, n$. On the other hand, by the translator equation,

$$\frac{\nabla_\ell H}{\mu} = -\frac{\kappa_\ell\langle\tau_\ell, e_{n+1}\rangle}{\mu}.$$

Since $(\kappa/\mu)(X_i) \to -\varepsilon_0 \neq 0$, we conclude that $\nu(X_i) \to -e_{n+1}$ and hence $H(X_i) \to 1$. Since the infimum of κ/μ is attained at the origin on Σ_∞, we deduce as before that Σ_∞ has constant mean curvature, which must be 1 since $H(X_i) \to 1$. But this contradicts Lemma 3.1. □

4. Barriers

Next, we introduce appropriate barriers. When $n = 2$, the outer barrier is obtained by (nonisotropically) "stretching" the level set function corresponding to the Angenent oval so that it lies in the correct slab and is asymptotic to the correct oblique Grim planes. The higher-dimensional barrier is then obtained by rotating in the $(n-1)$-dimensional complimentary subspace.

Lemma 4.1. *The function* $\underline{u} : \left\{(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |x| < \frac{\pi}{2} \sec\theta\right\} \to \mathbb{R}$ *defined by*

$$\underline{u}(x, y) := -\sec^2\theta \log\cos\left(\frac{x}{\sec\theta}\right) + \tan^2\theta \log\cosh\left(\frac{|y|}{\tan\theta}\right)$$

is a subsolution of the graphical translator equation (2).

In particular, given any $R > 0$, *the surface*

$$\underline{\Sigma}_R := \operatorname{graph} \underline{u}_R$$

is a subsolution of the translator equation (1), *where*

$$\underline{u}_R := \underline{u} - \tan^2\theta \log\cosh\left(\frac{R}{\tan\theta}\right).$$

Proof. The relevant derivatives of $\underline{u}$ are given by

$$D\underline{u} = \left(\sec\theta \tan\left(\frac{x}{\sec\theta}\right), \tan\theta \tanh\left(\frac{|y|}{\tan\theta}\right)\frac{y}{|y|}\right)$$

and

$$D^2\underline{u} = \begin{pmatrix} \sec^2\left(\frac{x}{\sec\theta}\right) & \cdots \; 0 \; \cdots \\ \vdots & \\ 0 & \operatorname{sech}^2\left(\frac{|y|}{\tan\theta}\right)\frac{y_i y_j}{|y|^2} + \tan\theta \tanh\left(\frac{|y|}{\tan\theta}\right)\left(\frac{\delta_{ij}}{|y|} - \frac{y_i y_j}{|y|^3}\right) \\ \vdots & \end{pmatrix}.$$

So

$$\begin{aligned} 1 + |D\underline{u}|^2 &= 1 + \sec^2\theta \tan^2\left(\frac{x}{\sec\theta}\right) + \tan^2\theta \tanh^2\left(\frac{|y|}{\tan\theta}\right) \\ &= \sec^2\theta \sec^2\left(\frac{x}{\sec\theta}\right) - \tan^2\theta \operatorname{sech}^2\left(\frac{|y|}{\tan\theta}\right). \end{aligned}$$

Estimating

$$\Delta\underline{u} \geq \sec^2\left(\frac{x}{\sec\theta}\right) + \operatorname{sech}^2\left(\frac{|y|}{\tan\theta}\right),$$

we find

$$\begin{aligned} (1 + |D\underline{u}|^2)^{3/2} H[\underline{u}] &= (1 + |D\underline{u}|^2)\Delta\underline{u} - D^2\underline{u}(D\underline{u}, D\underline{u}) \\ &\geq 1 + |D\underline{u}|^2 + \sec^2\left(\frac{x}{\sec\theta}\right)\operatorname{sech}^2\left(\frac{|y|}{\tan\theta}\right) \geq 1 + |D\underline{u}|^2. \end{aligned}$$

□

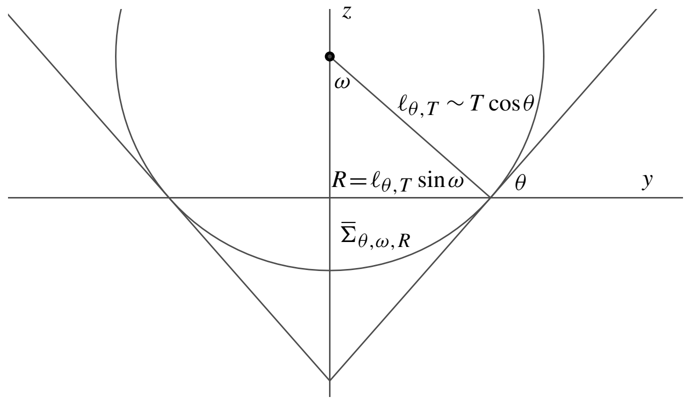

Figure 1. Given any $\varepsilon \in (0, \varepsilon_0(n, \theta))$, the portion of Π_R (the rotated time-T-slice of the Angenent oval of width $\pi \sec\theta$, where $T = \sec^2\theta \cosh(R/\tan\theta)$) lying below height $z = -R\cos(\theta-\varepsilon)/\sin\theta$ is a supersolution of the translator equation when $R > R_\varepsilon := 2(n-1)/\varepsilon$. The surface $\overline{\Sigma}_{R,\varepsilon}$ is obtained by translating this piece upward so that its boundary lies in $\mathbb{R}^n \times \{0\}$.

Consider the "outer" domain

$$\underline{\Omega}_R := \{(x, y) \in S^{n-1}_\theta : \underline{u}_R(x, y) < 0\} = \left\{(x, y) \in S^{n-1}_\theta : \cos\left(\frac{x}{\sec\theta}\right) < \left[\frac{\cosh(|y|/\tan\theta)}{\cosh(R/\tan\theta)}\right]^{\sin^2\theta}\right\},$$

where $S^n_\theta := \left(-\frac{\pi}{2}\sec\theta, \frac{\pi}{2}\sec\theta\right) \times \mathbb{R}^{n-1}$. Note that

$$\partial\underline{\Omega}_R = \partial(\underline{\Sigma}_R \cap \mathbb{R}^n \times (-\infty, 0]).$$

The inner barrier is obtained by rotating the Angenent oval of width $\pi\sec\theta$ and cutting off at an appropriate height (see Figure 1).

Lemma 4.2. *Given $R > 0$, let $\Pi_R \subset \mathbb{R}^{n+1}$ be the surface formed by rotating about the x-axis the time-T-slice of the Angenent oval of width $\pi\sec\theta$, where*

$$T := -\sec^2\theta\cosh\left(\frac{R}{\tan\theta}\right).$$

That is,

$$\Pi_R := \{(x, y, z) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : v(x, y, z) = T\},$$

where

$$v := \sec^2\theta\left[\log\left(\cosh\left(\frac{\sqrt{|y|^2+z^2}}{\sec\theta}\right)\right) - \log\left(\cos\left(\frac{x}{\sec\theta}\right)\right)\right].$$

There exists $\varepsilon_0 = \varepsilon_0(n, \theta) > 0$ such that the sublevel set

$$\overline{\Sigma}_{R,\varepsilon} := \Pi_R \cap \left\{z \le -R\frac{\cos(\theta-\varepsilon)}{\sin\theta}\right\} + R\frac{\cos(\theta-\varepsilon)}{\sin\theta}e_{n+1}$$

is a supersolution of the translator equation (1) *whenever $\varepsilon < \varepsilon_0$ and $R > R_\varepsilon := 2(n-1)/\varepsilon$.*

Proof. Set $w=(y,z)$. Then

$$Dv=\sec\theta\left(\tan\left(\frac{x}{\sec\theta}\right),\tanh\left(\frac{|w|}{\sec\theta}\right)\frac{w}{|w|}\right)$$

and

$$D^2v=\begin{pmatrix}\sec^2\left(\frac{x}{\sec\theta}\right) & \cdots\ 0\ \cdots \\ \vdots & \\ 0 & \operatorname{sech}^2\left(\frac{|w|}{\sec\theta}\right)\frac{w_iw_j}{|w|^2}+\sec\theta\tanh\left(\frac{|w|}{\sec\theta}\right)\left(\frac{\delta_{ij}}{|w|}-\frac{w_iw_j}{|w|^3}\right) \\ \vdots & \end{pmatrix}.$$

Tedious computations then yield, on the one hand,

$$-\langle\nu,e_{n+1}\rangle=\left\langle\frac{Dv}{|Dv|},e_{n+1}\right\rangle=\frac{\tanh(\sqrt{|y|^2+z^2}/\sec\theta)(|z|/\sqrt{|y|^2+z^2})}{\sqrt{\tan^2(x/\sec\theta)+\tanh^2(\sqrt{|y|^2+z^2}/\sec\theta)}}$$

and, on the other hand,

$$H=\operatorname{div}\left(\frac{Dv}{|Dv|}\right)=\frac{1/\sec\theta+((n-1)/|w|)\tanh(|w|/\sec\theta)}{\sqrt{\tan^2(x/\sec\theta)+\tanh^2(|w|/\sec\theta)}}.$$

It follows that Π_R is a supersolution in the region where

$$\frac{|z|-(n-1)}{\sqrt{|y|^2+z^2}}\tanh\left(\frac{\sqrt{|y|^2+z^2}}{\sec\theta}\right)\ge\cos\theta.$$

Note that

$$|y|\le\frac{\sin(\theta-\varepsilon)}{\sin\theta}R$$

wherever

$$|z|\ge\frac{\cos(\theta-\varepsilon)}{\sin\theta}R.$$

Thus, whenever

$$R>R_\varepsilon:=\frac{2(n-1)}{\varepsilon}\quad\text{and}\quad z\le-\frac{\cos(\theta-\varepsilon)}{\sin\theta}R,$$

we have

$$\begin{aligned}\frac{|z|-(n-1)}{\sqrt{|y|^2+z^2}}\tanh\left(\frac{\sqrt{|y|^2+z^2}}{\sec\theta}\right)&\ge\left(\cos(\theta-\varepsilon)-\frac{(n-1)}{R}\sin\theta\right)\tanh\left(\frac{\cos(\theta-\varepsilon)}{\tan\theta}R\right)\\&\ge\cos\theta\big(1+\tfrac{1}{2}\varepsilon\tan\theta+o(\varepsilon)\big)\sqrt{1-4\mathrm{e}^{-2(n-1)\cos^2\theta\sin\theta/\varepsilon}}.\end{aligned}$$

This is no less than $\cos\theta$ when $\varepsilon<\varepsilon_0(n,\theta)$. □

Consider the "inner" domain

$$\overline{\Omega}_{R,\varepsilon}:=\left\{(x,y)\in S^n_\theta:\cos\left(\frac{x}{\sec\theta}\right)<\frac{\cosh(\sqrt{|y|^2\sin^2\theta+R^2\cos^2(\theta-\varepsilon)}/\tan\theta)}{\cosh(R/\tan\theta)}\right\}.$$

Note that $\partial\overline{\Omega}_{R,\varepsilon}=\partial\overline{\Sigma}_{R,\varepsilon}$.

The following lemma implies that the inner barrier which touches the outer barrier at Re_2 lies above it, so long as R is sufficiently large.

Lemma 4.3. *Given any $R > 0$, we have $\overline{\Omega}_{\rho_\varepsilon,\varepsilon} \subset \underline{\Omega}_R$, where*

$$\rho_\varepsilon := \frac{\sin\theta}{\sin(\theta-\varepsilon)} R.$$

Proof. It suffices to show that the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(\zeta) := \frac{\cosh(\sqrt{\zeta^2 \sin^2\theta + \rho_\varepsilon^2 \cos^2(\theta-\varepsilon)}/\tan\theta)}{\cosh(\rho_\varepsilon/\tan\theta)} - \left[\frac{\cosh(\zeta/\tan\theta)}{\cosh(R/\tan\theta)}\right]^{\sin^2\theta}$$

is nonpositive. This follows from log-concavity of the function

$$g(w) := \cosh\left(\frac{\sqrt{w}}{\tan\theta}\right).$$

Indeed, given any $s \in (0, 1)$ and $w > 0$, log-concavity of g implies that the function

$$G(z) := \frac{g(sz + (1-s)w)}{g(z)^s}$$

is monotone nondecreasing for $z < w$. Since

$$\zeta < R < \frac{\tan\theta}{\tan(\theta-\varepsilon)} R = \frac{\cos(\theta-\varepsilon)}{\cos\theta} \rho_\varepsilon,$$

this implies

$$\frac{g(\zeta^2 \sin^2\theta + \rho_\varepsilon^2 \cos^2(\theta-\varepsilon))}{g(\zeta^2)^{\sin^2\theta}} \le \frac{g(R^2 \sin^2\theta + \rho_\varepsilon^2 \cos^2(\theta-\varepsilon))}{g(R^2)^{\sin^2\theta}} = \frac{g(\rho_\varepsilon^2)}{g(R^2)^{\sin^2\theta}}.$$

The claim follows. □

Corollary 4.4. *Set $\varepsilon_R := 2(n-1)/R$, $\overline{\Sigma}_R := \overline{\Sigma}_{\rho_{\varepsilon_R},\varepsilon_R}$ and $\overline{\Omega}_R := \overline{\Omega}_{\rho_{\varepsilon_R},\varepsilon_R}$. Then, for $R > R_0 := 2(n-1)/\varepsilon_0$, $\overline{\Sigma}_R$ is a supersolution of the translator equation with boundary $\partial\overline{\Sigma}_R = \partial\overline{\Omega}_R$.*

5. Existence

We are ready to prove the existence theorem, which we now recall.

Theorem (existence of convex translators in slab regions). *For every $n \ge 2$ and every $\theta \in \left(0, \frac{\pi}{2}\right)$ there exists a strictly convex translator Σ_θ^n which lies in S_θ^{n+1} and in no smaller slab.*

Proof. Given $R > 0$, let u_R be the solution of

$$H[u_R] = \frac{1}{\sqrt{1+|Du_R|^2}} \quad \text{in } \Omega_R, \qquad u_R = 0 \quad \text{on } \partial\Omega_R,$$

where $\Omega_R := \underline{\Omega}_R$. Since the equation admits upper and lower barriers (0 and $\underline{u}_R$, respectively), existence and uniqueness of a smooth solution follows from well-known methods; see, for example, [Gilbarg and Trudinger 1983, Chapter 15]. Uniqueness implies that u_R is rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} = \operatorname{span}\{e_2, \dots, e_n\}$. Since $\underline{u}_R$ is a subsolution, its graph lies below graph u_R. Since the two surfaces coincide on the boundary $\partial\Omega_R$,

$$H[u_R] = -\langle \nu_R, e_{n+1}\rangle \geq -\langle \underline{\nu}_R, e_{n+1}\rangle \geq \cos\theta\cos(x\cos\theta) \geq \cos\theta\left(1 - \frac{x}{\frac{\pi}{2}\sec\theta}\right) \tag{15}$$

on $\partial\Omega_R$, where $\underline{\nu}_R$ is the downward-pointing unit normal to graph $\underline{u}_R$. By Corollary 4.4, we also find, for $R > R_0$, that

$$-u_R(0) \geq \frac{1-\cos\theta}{\sin\theta} R \to \infty \quad \text{as } R \to \infty. \tag{16}$$

Let $R_i \to \infty$ be a diverging sequence and consider the translators-with-boundary

$$\Sigma_i := \operatorname{graph} u_{R_i} - u_{R_i}(0)e_{n+1}.$$

By Corollary 2.2 and the height estimate (16) some subsequence converges locally uniformly in the smooth topology to some limiting translator, Σ, with bounded second fundamental form. By Theorem 3.2, Σ is convex.

Certainly Σ lies in the slab S_θ, so it remains only to prove that it lies in no smaller slab (strict convexity will then follow from the splitting theorem and uniqueness of the Grim Reaper). Set

$$v := 1 - \frac{x}{\frac{\pi}{2}\sec\theta},$$

where $x(X) := \langle X, e_1\rangle$. We claim that

$$\inf_{\Sigma\cap\{x>0\}} \frac{H}{v} > 0. \tag{17}$$

Since $\inf_\Sigma H = 0$, we conclude that $\sup_\Sigma x = \frac{\pi}{2}\sec\theta$ as desired. To prove (17), first observe that

$$-(\Delta + \nabla_V)v = 0$$

and hence

$$-(\Delta + \nabla_V)\frac{H}{v} = |A|^2\frac{H}{v} + 2\left\langle \nabla\frac{H}{v}, \frac{\nabla v}{v}\right\rangle,$$

where V is the tangential projection of e_{n+1}. The maximum principle then yields

$$\min_{\Sigma_i\cap\{x>0\}} \frac{H}{v} \geq \min\left\{\min_{\partial\Sigma_i\cap\{x>0\}}\frac{H}{v}, \min_{\Sigma_i\cap\{x=0\}}\frac{H}{v}\right\} = \min\left\{\cos\theta, \min_{\Sigma_i\cap\{x=0\}} H\right\}.$$

If $\liminf_{i\to\infty}\min_{\Sigma_i\cap\{x=0\}} H > 0$ then we are done. So suppose that $\liminf_{i\to\infty} H(X_i) = 0$ along some sequence of points $X_i \in \Sigma_i \cap \{x = 0\}$. Then, by Corollary 2.2, after passing to a subsequence, the translators-with-boundary

$$\widehat{\Sigma}_i := \Sigma_i - X_i$$

converge locally uniformly in C^∞ to a translator (possibly with boundary) $\widehat{\Sigma}$ which lies in S_θ and satisfies $H \geq 0$ with equality at the origin. By Corollary 2.2 the origin must be an interior point since, recalling (15), $H > \cos\theta$ on $\partial\Sigma_i \cap \{x = 0\}$ for all i. The strong maximum principle then implies that $H \equiv 0$ on $\widehat{\Sigma}$ and we conclude that $\widehat{\Sigma}$ is either a hyperplane or half-hyperplane. Since, by the reflection symmetry, the limit cannot be parallel to $\{0\}\times\mathbb{R}^{n-1}\times\mathbb{R}$, neither option can be reconciled with the fact that $\widehat{\Sigma}$ lies in S_θ. □

6. Asymptotics and reflection symmetry

We next prove that, after translation, our translators have the correct asymptotics (Theorem 3).

Theorem (unique asymptotics modulo translation). *Given $n \geq 2$ and $\theta \in \left(0, \frac{\pi}{2}\right)$ let Σ^n_θ be a convex translator which lies in S^{n+1}_θ and in no smaller slab. If $n \geq 3$, assume in addition that Σ^n_θ is rotationally symmetric with respect to the subspace $\mathbb{E}^{n-1} := \operatorname{span}\{e_1, \dots, e_n\}$. Given any unit vector $\phi \in \mathbb{E}^{n-1}$ the curve $\{\sin\omega\phi - \cos\omega e_{n+1} : \omega \in [0, \theta)\}$ lies in the normal image of Σ^n_θ and the translators*

$$\Sigma^n_{\theta,\omega} := \Sigma^n_\theta - P(\sin\omega\phi - \cos\omega e_{n+1})$$

converge locally uniformly in the smooth topology to the oblique Grim hyperplane $\Gamma^n_{\theta,\phi}$ as $\omega \to \theta$, where $P : S^n \to \Sigma^n_\theta$ is the inverse of the Gauss map.

Fix a unit vector $\phi \in \operatorname{span}\{e_2, \dots, e_n\}$ and define

$$\bar{\omega} := \sup\{\omega \in [0, \infty) : \sin\omega\phi - \cos\omega e_{n+1} \in \nu(\Sigma)\}.$$

Let ω_i be a sequence of points converging to $\bar{\omega}$. Then the translators

$$\Sigma_{i,\phi} := \Sigma - P_\phi(\omega_i)$$

have uniformly bounded curvature and pass through the origin. After passing to a subsequence, they must therefore converge locally uniformly to a limit translator. The limit must be the oblique Grim hyperplane $\Gamma^n_{\bar{\omega},\phi}$ since it contains the ray $\{r(\cos\bar{\omega}\phi + \sin\bar{\omega}e_{n+1}) : r > 0\}$ and lies in a slab parallel to S_θ (and, when $n \geq 3$, splits off an additional $n-2$ lines due to the rotational symmetry). In fact, since the components of the normal are monotone along the curve $\gamma(\omega) := P(\sin\omega\phi - \cos\omega e_{n+1})$, the normal must actually converge (to $\sin\bar{\omega}\phi - \cos\bar{\omega}e_{n+1}$) along γ. It follows that the limit is independent of the subsequence and we conclude that the translators

$$\Sigma_{\omega,\phi} := \Sigma - P_\phi(\omega)$$

converge locally uniformly in C^∞ to $\Gamma^n_{\bar{\omega},\phi}$ as $\omega \to \bar{\omega}$. Note that $\bar{\omega} \leq \theta$ since the limit $\Gamma^n_{\bar{\omega},\phi}$ must lie in S_θ. It remains to show that $\bar{\omega} \geq \theta$.

Suppose, to the contrary, that $\bar{\omega} < \theta$. Given $\omega \in \left[0, \frac{\pi}{2}\right)$, let $\Pi^\omega_t = \sec\omega \Pi_{\cos^2\omega t}$ be the rotationally symmetric ancient pancake which lies in the slab Ω_ω (and no smaller slab) and becomes extinct at the origin at time zero. The "radius" $\ell_\omega(t)$ of the pancake satisfies [Bourni et al. 2017]

$$\ell_\omega(t) := \max_{p \in \Pi^\omega_t} \langle p, e_2\rangle = \sec\omega\, \ell_0(\cos^2\omega t) = -t\cos\omega + (n-1)\sec\omega\log(-t) + c + o(1) \tag{18}$$

as $t \to -\infty$, where the constant c and the remainder term depend on ω and n. Observe that the ray $L_\omega = \{r(\cos\omega\phi + \sin\omega e_{n+1}) : r > 0\}$ is tangent to the circle in the plane $\operatorname{span}\{\phi, e_{n+1}\}$ of radius $-\cos\omega t$ centered at $-te_{n+1}$. Indeed, a point $r(\cos\omega\phi + \sin\omega e_{n+1})$ lies on this circle if and only if

$$|r\cos\omega\phi + (r\sin\omega + t)e_{n+1}|^2 = \cos^2\omega t^2 \iff (r - \sin\omega(-t))^2 = 0.$$

So there exists a unique point with this property, as claimed. Since, by hypothesis, $\theta < \bar{\omega}$, we conclude from (18) that the circle of radius $\ell_\theta(-t)$ lies above the line $L_{\bar{\omega}}$ for $-t$ sufficiently large (see Figure 2).

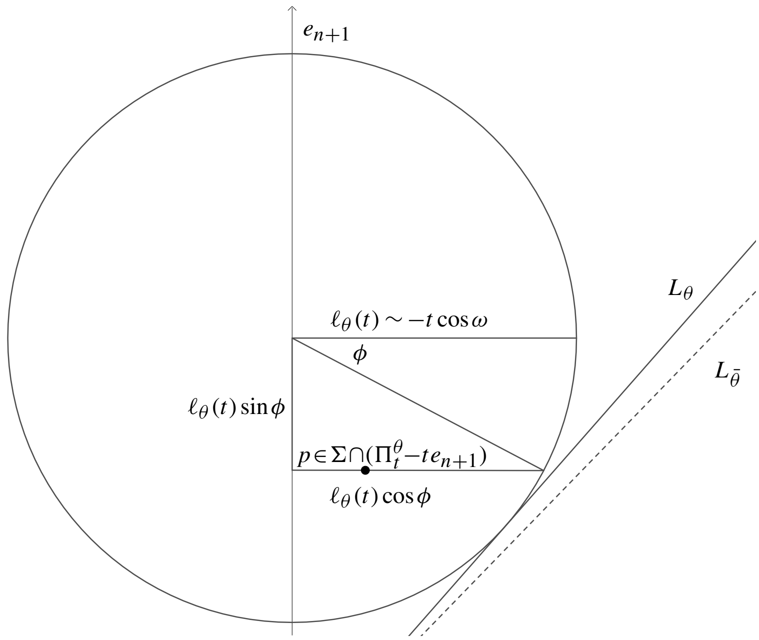

Figure 2. If $\bar\omega < \theta$ then the pancake lies above the translator for $-t$ sufficiently large.

We will show that, in fact, $\Pi_t^\theta - te_{n+1}$ lies above Σ for $-t$ sufficiently large (and hence Π_t^θ lies above $\Sigma_t := \Sigma + te_{n+1}$ for $-t$ sufficiently large). But Π_t^θ and Σ_t both reach the origin at time zero, so this contradicts the avoidance principle.

We will need an estimate for the "width" of Σ. Given $p \in \Sigma$ set

$$x(p) := \langle p, e_1\rangle, \quad y(p) := \langle p, \phi\rangle \quad \text{and} \quad z(p) := \langle p, e_{n+1}\rangle$$

and, given $h > 0$, set

$$\ell(h) := \max_{p\in\Sigma_h} y(p),$$

where Σ_h is the level set $\Sigma_h := \{p \in \Sigma : z(p) = h\}$. We know that, near its "edge region", Σ looks like a Grim hyperplane of width $\sec\bar\omega$, whereas, in its "middle region", it looks like two parallel planes of width $\sec\theta$. By convexity, it must lie outside the linearly interpolating region in between (see Figure 3). The following estimate quantifies this elementary observation.

Lemma 6.1 (width estimate). *Set*

$$\beta := \sec\theta - \sec\bar\omega > 0 \quad \textit{and} \quad x_0 := \lim_{\omega\to\bar\omega} x(P_\phi(\omega)).$$

For any $\varepsilon > 0$ there exist $K_\varepsilon < \infty$ and $h_\varepsilon < \infty$ with the following property: Given $h > h_\varepsilon$, $p \in \Sigma_h$ and $s \in [0, 1]$, suppose that

$$0 \le y(p) \le s(\ell(h) - K_\varepsilon).$$

Then

$$|x(p) - x_0| \ge \tfrac{\pi}{2}\big(\sec\bar\omega + (1-s)\big(\beta - \tfrac{2}{\pi}x_0\big) - \varepsilon\big).$$

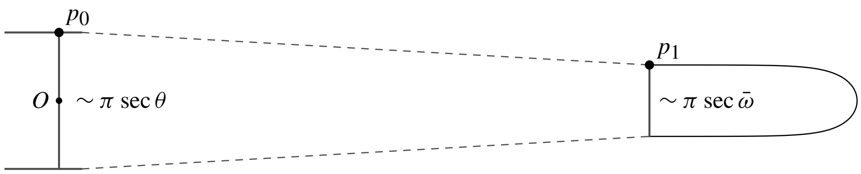

Figure 3. Linearly interpolating between the "middle" and "edge" regions in the level set Σ_h. The horizontal axis is compressed.

Proof of Lemma 6.1. Choose $\varepsilon > 0$. Because Σ converges to the oblique Grim hyperplane $\Gamma_{\bar{\omega},\phi}$ after translating the "tips", $P_\phi(\omega)$, we can find some h_ε and K_ε such that

$$|x(p) - x_0| \geq \tfrac{\pi}{2}\sec\bar{\omega} - \varepsilon$$

for all $p \in \Sigma_h$ satisfying

$$0 \leq y(p) \leq \ell(h) - K_\varepsilon$$

so long as $h \geq h_\varepsilon$. Choose some $h \geq h_\varepsilon$ and consider the point $p_1 \in \Sigma_h \cap \{e_1, \phi, e_{n+1}\}$ satisfying $x(p_1) \geq x_0$ and $0 \leq y(p_1) = \ell(h) - K_\varepsilon$. (If there is no such point then the claim is vacuously true, else p_1 is uniquely determined.) Then

$$x(p_1) - x_0 \geq \tfrac{\pi}{2}\sec\omega - \varepsilon.$$

On the other hand, because Σ converges to the boundary of S_θ after translating vertically, we can assume that h_ε is so large that

$$x(p_0) \geq \tfrac{\pi}{2}\sec\theta - \varepsilon$$

at the point $p_0 \in \Sigma_h \cap \operatorname{span}\{e_1, \phi, e_{n+1}\}$ satisfying $y(p_0) = 0$ and $x(p_0) \geq x_0$. Since Σ_h is convex, we conclude that any point $p \in \Sigma_h \cap \operatorname{span}\{e_1, \phi, e_{n+1}\}$ satisfying $0 \leq y(p) \leq \ell(h) - K_\varepsilon$ and $x(p) \geq x_0$ lies beyond the segment joining p_0 and p_1. In particular, if $y(p) \leq s(\ell(h) - K_\varepsilon)$ then

$$\begin{aligned} x(p) &\geq s x(p_1) + (1-s)x(p_0) \\ &\geq s\big(x_0 + \tfrac{\pi}{2}\sec\bar{\omega} - \varepsilon\big) + (1-s)\big(\tfrac{\pi}{2}\sec\theta - \varepsilon\big) \\ &= x_0 + \tfrac{\pi}{2}\big(\sec\bar{\omega} + (1-s)\big(\beta - \tfrac{2}{\pi}x_0\big)\big) - \varepsilon. \end{aligned}$$

The other inequality is proved in much the same way (simply choose the points p_0 and p_1 on the other side of the $\{x = x_0\}$-plane). □

Reflecting Σ^n through the $\{x = 0\}$-hyperplane if necessary, we may assume in what follows that $x_0 \geq 0$.

Given $\varepsilon > 0$, choose h_ε and K_ε as in Lemma 6.1 and consider $h \geq h_\varepsilon$. Then, given any $p \in \Sigma$ satisfying $0 \leq y(p) < (\ell(h) - K_\varepsilon)$ and $x(p) \geq x_0$, we can choose

$$s = s(p) := \frac{|y(p)|}{\ell(h) - K_\varepsilon} \in [0, 1]$$

and hence estimate

$$x(p) \geq \frac{\pi}{2}\left(\sec\theta - \beta\frac{|y(p)|}{\ell(h) - K_\varepsilon} - \varepsilon\right).$$

Choosing h_ε larger if necessary, we may assume that $\ell(h_\varepsilon) \geq 2K_\varepsilon$ and hence

$$x(p) \geq \frac{\pi}{2}\left(\sec\theta - \beta\frac{|y(p)|}{\ell(h)}\left(1+\frac{2K_\varepsilon}{\ell(h)}\right) - \varepsilon\right). \tag{19}$$

Note also that, by convexity,

$$\tan\bar{\omega} \geq \frac{h}{\ell(h)} \to \tan\bar{\omega} \quad \text{as } h \to \infty.$$

Assume now that, given $t < 0$ and $\omega \in (\bar{\omega}, \theta)$, there is some point $p \in (\Pi_t^\omega - te_{n+1}) \cap \Sigma \cap \{X : \langle X, \phi\rangle > 0\}$. Then there is some $\phi \in \left[0, \frac{\pi}{2}\right]$ such that

$$h := z(p) = -t - \ell_\omega(t)\sin\phi, \quad |y(p)| \leq \ell_\omega(t)\cos\phi \quad \text{and} \quad |x(p)| < \tfrac{\pi}{2}\sec\omega,$$

where ℓ_ω is defined by (18) (see Figure 2). Suppose further that $h \geq h_\varepsilon$. Recalling (19), we find

$$\begin{aligned}\sec\omega &\geq \sec\theta - \beta\frac{|y(p)|}{h}\frac{h}{\ell(h)}\left(1+\frac{2K_\varepsilon}{h}\frac{h}{\ell(h)}\right) - \varepsilon \\ &\geq \sec\theta - \beta\frac{\ell_\omega(t)\cos\phi}{h}\tan\bar{\omega}\left(1+\frac{2K_\varepsilon}{h}\tan\bar{\omega}\right) - \varepsilon.\end{aligned}$$

That is,

$$\begin{aligned}\frac{\sec\theta - \sec\omega}{\sec\theta - \sec\bar{\omega}} &\leq \frac{\ell_\omega(t)\cos\phi}{h}\tan\bar{\omega}\left(1+\frac{2K_\varepsilon}{h}\tan\bar{\omega}\right) + \frac{\varepsilon}{\beta} \\ &= \frac{\ell_\omega(t)\cos\phi\tan\bar{\omega}}{-t-\ell_\omega(t)\sin\phi}\left(1+\frac{2K_\varepsilon\tan\bar{\omega}}{-t-\ell_\omega(t)\sin\phi}\right) + \frac{\varepsilon}{\beta}.\end{aligned}$$

Since the right-hand side is nonincreasing with respect to ϕ for $\phi \in \left[0, \frac{\pi}{2}\right]$, we may estimate

$$\frac{\sec\theta - \sec\omega}{\sec\theta - \sec\bar{\omega}} \leq \frac{\ell_\omega(t)}{-t}\tan\bar{\omega}\left(1+\frac{2K_\varepsilon}{-t}\tan\bar{\omega}\right) + \frac{\varepsilon}{\beta}.$$

But $\ell_\omega(t)/-t \to \cos\omega$ as $t \to -\infty$, so we conclude, for $-t \geq -t_\varepsilon$ sufficiently large, that

$$\frac{\sec\theta - \sec\omega}{\sec\theta - \sec\bar{\omega}} \leq \cos\omega\tan\bar{\omega} + \frac{2\varepsilon}{\beta} \leq \sin\bar{\omega} + \frac{2\varepsilon}{\beta}.$$

Choosing ω sufficiently close to $\bar{\omega}$ and ε sufficiently small results in a contradiction. This completes the proof of Theorem 3 in the case $n \geq 3$. It remains to consider the case that $n = 2$ and Σ is asymptotic to the correct oblique Grim plane in one direction, say $-e_2$, but not the other, e_2. This can be achieved with a similar argument by centering the ancient pancake not on the z-axis but rather on the axis bisecting the two asymptotic lines, i.e., the ray

$$\left\{r\left(\cos\frac{\theta-\bar{\omega}}{2}e_3 + \sin\frac{\theta-\bar{\omega}}{2}e_2\right) : r > 0\right\}.$$

We omit the details since the result in this case was already proved in [Spruck and Xiao 2017].

Combining the unique asymptotics with the Alexandrov reflection principle, we may now prove Corollary 4.

Corollary. *Given $\theta \in \left(0, \frac{\pi}{2}\right)$, let Σ be a strictly convex translator which lies in S_θ^{n+1} and in no smaller slab. If $n \geq 3$, assume in addition that Σ is rotationally symmetric with respect to $\mathbb{E}^{n-1}$. Then Σ is reflection symmetric across the hyperplane $\{0\} \times \mathbb{R}^n$.*

We proceed much as in [Bourni et al. 2017, Theorem 6.2]. Let us begin by introducing some notation. Given a unit vector $e \in S^n$ and some $\alpha \in \mathbb{R}$, denote by $\mathrm{H}_{e,\alpha}$ the half-space $\{p \in \mathbb{R}^{n+1} : \langle p, e\rangle < \alpha\}$ and by $R_{e,\alpha} \cdot \Sigma := \{p - 2(\langle p, e\rangle - \alpha)e : p \in \Sigma\}$ the reflection of Σ across the hyperplane $\partial \mathrm{H}_{e,\alpha}$. We say that Σ *can be reflected strictly about* $\mathrm{H}_{e,\alpha}$ if $(R_{e,\alpha} \cdot \Sigma) \cap \mathrm{H}_{e,\alpha} \subset \Omega \cap \mathrm{H}_{e,\alpha}$.

Lemma 6.2 (Alexandrov reflection principle). *Let Σ be a convex translator. If*

$$\Sigma_h := \Sigma \cap \{(x, y, z) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : z > h\}$$

can be reflected strictly about $\mathrm{H}_{e,\alpha}$ for some $e \in \{e_{n+1}\}^\perp$ then Σ can be reflected strictly about $\mathrm{H}_{e,\alpha}$.

Proof. This is a consequence of the strong maximum principle and the boundary point lemma; see [Gilbarg and Trudinger 1983, Chapter 10]. □

Claim 6.2.1. *For every $\alpha \in \left(0, \frac{\pi}{4}\right)$ there exists $h_\alpha < \infty$ such that*

$$\Sigma_{h_\alpha} := \Sigma \cap \{(x, y, z) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : z > h_\alpha\}$$

can be reflected strictly about $\mathrm{H}_\alpha := \mathrm{H}_{e_1,\alpha}$.

Proof. Suppose that the claim does not hold. Then there must be some $\alpha \in \left(0, \frac{\pi}{4}\right)$ and a sequence of heights $h_i \to \infty$ such that $(R_\alpha \cdot \Sigma_{h_i}) \cap \mathrm{H}_\alpha \cap \Sigma_{h_i} \neq \varnothing$. Choose a sequence of points $p_i = x_i e_1 + y_i e_2 \in \Sigma_{h_i}$ whose reflection about the hyperplane H_α satisfies

$$(2\alpha - x_i)e_1 + y_i e_2 \in (R_\alpha \cdot \Sigma_{h_i}) \cap \Sigma_{h_i} \cap \mathrm{H}_\alpha$$

and set $p_i' = x_i' e_1 + y_i' e_2 := (2\alpha - x_i)e_1 + y_i e_2$. Without loss of generality, we may assume that $y_i' = y_i \geq 0$. Since $\alpha \leq x_i < \frac{\pi}{2}$, the point p_i' satisfies $\alpha \geq x_i' > -\frac{\pi}{2} + 2\alpha$ so that, after passing to a subsequence, $\lim_{i\to\infty} x_i' \in \left[-\frac{\pi}{2} + 2\alpha, \alpha\right]$. But since Σ is convex and converges, after translating in the plane $\operatorname{span}\{e_2, e_{n+1}\}$, to the Grim hyperplane $\Gamma_{e_2,\theta}$, we conclude that

$$0 = \lim_{i\to\infty} (x_i + x_i') = 2\alpha.$$

So $\alpha = 0$, a contradiction. □

It now follows from Lemma 6.2 that Σ can be reflected across H_α for all $\alpha \in \left(0, \frac{\pi}{2}\right)$. The same argument applies when the half-space H_α is replaced by $-\mathrm{H}_\alpha = \{(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-1} : x > -\alpha\}$. Now take $\alpha \to 0$.

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