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PARABOLIC L^p DIRICHLET BOUNDARY VALUE PROBLEM AND VMO-TYPE TIME-VARYING DOMAINS

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We prove the solvability of the parabolic L^p Dirichlet boundary value problem for $1 < p \leq \infty$ for a PDE of the form $u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u$ on time-varying domains where the coefficients $A = [a_{ij}(X, t)]$ and $B = [b_i]$ satisfy a certain natural small Carleson condition. This result brings the state of affairs in the parabolic setting up to the elliptic standard.

Furthermore, we establish that if the coefficients A, B of the operator satisfy a vanishing Carleson condition and the time-varying domain is of VMO type then the parabolic L^p Dirichlet boundary value problem is solvable for all $1 < p \leq \infty$. This result is related to results in papers by Maz'ya, Mitrea and Shaposhnikova, and Hofmann, Mitrea and Taylor, where the fact that the boundary of the domain has a normal in VMO or near VMO implies invertibility of certain boundary operators in L^p for all $1 < p \leq \infty$, which then (using the method of layer potentials) implies solvability of the L^p boundary value problem in the same range for certain elliptic PDEs.

Our result does not use the method of layer potentials since the coefficients we consider are too rough to use this technique, but remarkably we recover L^p solvability in the full range of p 's as in the two papers mentioned above.

1. Introduction

Let us consider a parabolic differential equation on a time-varying domain Ω of the form

$$\begin{cases} u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ u = f & \text{on } \partial\Omega, \end{cases} \quad (1-1)$$

where $A = [a_{ij}(X, t)]$ is an $n \times n$ matrix satisfying the uniform ellipticity condition with $X \in \mathbb{R}^n$, $t \in \mathbb{R}$. That is, there exist positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(X, t)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad (1-2)$$

for almost every $(X, t) \in \Omega$ and all $\xi \in \mathbb{R}^n$. In addition, we assume that the coefficients of A and B satisfy a natural, minimal smoothness condition (1-6) and we do not assume any symmetry on A .

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It has been observed via the method of layer potentials that when the domain on which we consider certain boundary value problems for elliptic or parabolic PDEs is sufficiently smooth, the question of L^p invertibility of a certain boundary operator can be resolved using Fredholm theory since this operator is just a compact perturbation of the identity. This observation then implies invertibility of this boundary operator for all $1 < p \leq \infty$ and hence solvability of the corresponding L^p boundary value problem in this range.

The notion of how smooth the domain has to be for the above observation to hold has evolved. Initial results for constant-coefficient elliptic PDEs required domains of at least $C^{1,\alpha}$ type. This was reduced to C^1 domains by Fabes, Jodeit, and Rivière [Fabes et al. 1978]. Later the method of layer potentials was adapted to variable coefficient settings, and the results were extended to elliptic PDEs with variable coefficients [Dindoš 2008] on C^1 domains.

Further progress was made after advancements in singular integrals theory on sets that are not necessary of graph type [Semmes 1991; Hofmann et al. 2010]. It turns out that compactness of the mentioned boundary operator only requires that the normal (which must be well-defined at almost every boundary point) belongs to VMO.

This observation for the Stokes system was made in [Mazya et al. 2009], where boundary value problems for domains whose normal belongs to VMO (or is near to VMO in the BMO norm) were considered. In [Hofmann et al. 2015] symbol calculus for operators of layer potential type on surfaces with VMO normals was developed and applied to various elliptic PDEs including elliptic systems.

So far we have only mentioned elliptic results. One of the first results for the heat equation in Lipschitz cylinders is by Brown [1989]. Here the domain considered is time-independent and Fourier methods in the time variable are used. Domains of a time-varying type for the heat operator were first considered in [Lewis and Murray 1995; Hofmann and Lewis 1996] and again the method of layer potentials was used to establish L^2 solvability. The question of solvability of various boundary value problems for parabolic PDEs on time-varying domains has a long history. Recall that in the elliptic setting [Dahlberg 1977] has shown in a Lipschitz domain that the harmonic measure and surface measure are mutually absolutely continuous and that the elliptic Dirichlet problem is solvable with data in L^2 with respect to the surface measure. R. Hunt then asked whether Dahlberg's result holds for the heat equation in domains whose boundaries are given locally as functions $\phi(x, t)$, Lipschitz in the spatial variable. It was conjectured (due to the natural parabolic scaling) that the correct regularity of $\phi(x, t)$ should be a Hölder condition of order $\frac{1}{2}$ in the time variable t and Lipschitz in x . It turns out that under this assumption the parabolic measure associated with (1-1) is doubling [Nyström 1997].

However, to answer R. Hunt's question positively, one has to consider more regular classes of domains than the one just described above. This follows from the counterexample of [Kaufman and Wu 1988], where it was shown that under just the $\text{Lip}(1, \frac{1}{2})$ condition on the domain Ω the associated caloric measure (that is, the measure associated with the operator $\partial_t - \Delta$) might not be mutually absolutely continuous with the natural surface measure. The issue was resolved in [Lewis and Murray 1995], where it was established that mutual absolute continuity of caloric measure and a certain parabolic analogue of the surface measure holds when ϕ has $\frac{1}{2}$ of a time derivative in the parabolic $\text{BMO}(\mathbb{R}^n)$ space, which is

a slightly stronger condition than $\text{Lip}(1, \frac{1}{2})$. We shall say such domains are of Lewis–Murray type. Hofmann and Lewis [1996] subsequently showed that this condition is sharp. We thoroughly discuss these domains in Section 2A.

Further work was done in [Hofmann and Lewis 2001; Rivera-Noriega 2003; 2014] in graph domains and time-varying cylinders satisfying the Lewis–Murray condition, where they proved the L^p Dirichlet problem was solvable for all $p > p'$ for some potentially very large p' (due to the technique used, there is no control on the size of p'). Finally, [Dindoš and Hwang 2018] established L^p solvability $2 \leq p \leq \infty$ in domains that are local of Lewis–Murray type under a small Carleson condition.

While researching literature on domains of Lewis–Murray type and ways this concept can be localized (in the time variable the half-derivative is a nonlocal operator, and hence any condition imposed on it is difficult to localize), we have realized that important results we have planned to rely on have issues (either in their proofs or even worse are simply false; see in particular Remark 2.7 in the next section). This has prompted us to write Section 2A on parabolic domains in substantially more detail than we originally intended to. This sets the literature record straight and more importantly in detail explains the concept of localized domains of Lewis–Murray type. For readability of the paper and this section, we have moved long proofs into an Appendix.

We establish L^p solvability results for parabolic PDEs on time-varying cylinders satisfying locally the Lewis–Murray condition in the full range $1 < p \leq \infty$, improving the solvability range from [Dindoš and Hwang 2018] as well as older results such as [Hofmann and Lewis 1996], where only $p = 2$ was considered. The coefficients we consider are very rough and, in particular, the method of layer potentials cannot be used. Despite this, we recover (in the parabolic setting) an analogue of [Mazya et al. 2009; Hofmann et al. 2015]. When the domain Ω , on which the parabolic PDE is considered, is of VMO type (that is, certain derivatives both in temporal and spatial variables will be in VMO) and the coefficients of the operator satisfy a vanishing Carleson condition the L^p solvability can be established for all $1 < p \leq \infty$. Remarkably this is the full range of solvability that holds for smooth coefficients (via the layer potential method).

Our proof is however completely different from the layer potential method; for example at no point is compactness used. The proof is also substantially different than the case $2 \leq p \leq \infty$ of [Dindoš and Hwang 2018] in the following way. We were inspired by [Dindoš et al. 2007] and have used a so-called p -adapted square function to prove L^p solvability. However, due to the presence of the parabolic term, a second-square-function-type object will arise, namely

$$\int_{\Omega} |u_t(X, t)|^2 |u(X, t)|^{p-2} \delta(X, t)^3 \, dX \, dt, \tag{1-3}$$

where $\delta(X, t)$ is the parabolic distance to the boundary defined as

$$\delta(X, t) = \inf_{(Y, \tau) \in \partial\Omega} (|X - Y|^2 + |t - \tau|)^{1/2}.$$

When $p = 2$ such an object was called the “area function” and in [Dindoš and Hwang 2018] it was shown that it the usual square function can dominate it. It turns out however that the case $1 < p < 2$ is substantially more complicated and we were only able to establish required bounds for (1-3) for nonnegative u after a substantial effort.

There is also an issue of whether the p -adapted square function is actually well-defined and locally finite (as the exponent on $|u|$ is negative). We prove that when u is a solution of a parabolic PDE the p -adapted square function is indeed well-defined by adapting a recent regularity result [Dindoš and Pipher 2019]. That paper deals with complex-coefficient elliptic PDEs but the method used there can be adapted to the parabolic setting; see Theorem 4.1 for details.

Many results in the parabolic setting are motivated by previous results in the elliptic setting and ours is not different. Let us, therefore, give an overview of the major elliptic results related to our main theorem.

The papers [Kenig et al. 2000] and [Kenig and Pipher 2001] started the study of nonsymmetric divergence elliptic operators with bounded and measurable coefficients. Kenig and Pipher [2001] used [Kenig et al. 2000] to show that the elliptic measure of operators satisfying a type of Carleson measure condition is in A_∞ and hence the L^p Dirichlet problem is solvable for some, potentially large, p . In [Dindoš et al. 2007], the authors improved the result of [Kenig and Pipher 2001] in the following way. They showed that if

$$\delta(X)^{-1} \left(\operatorname{osc}_{B_{\delta(X)/2}(X)} a_{ij} \right)^2 \quad \text{and} \quad \delta(X) \left(\sup_{B_{\delta(X)/2}(X)} b_i \right)^2 \tag{1-4}$$

are densities of Carleson measures with vanishing Carleson norms then the L^p Dirichlet problem is solvable for all $1 < p \leq \infty$. A similar result for the elliptic Neumann and regularity boundary value problem was established in [Dindoš et al. 2017].

The parabolic analogue of the elliptic Carleson condition (1-4) is that

$$\delta(X, t)^{-1} \sup_{i,j} \left(\operatorname{osc}_{B_{\delta(X,t)/2}(X,t)} a_{ij} \right)^2 + \delta(X, t) \left(\sup_{B_{\delta(X,t)/2}(X,t)} b_i \right)^2 \tag{1-5}$$

is the density of a Carleson measure on Ω with a small Carleson norm and $\delta(X, t)$ is the parabolic distance of a point (X, t) to the boundary $\partial\Omega$.

The condition (1-5) arises naturally as follows. Let $\Omega = \{(x_0, x, t) : x_0 > \phi(x, t)\}$ for a function ϕ which satisfies the Lewis–Murray condition above. Let $\rho : U \rightarrow \Omega$ be a mapping from the upper half-space U to Ω . Consider $v = u \circ \rho$. It will follow that if u solves (1-1) in Ω then v will be a solution to a parabolic PDE similar to (1-1) in U . In particular, if ρ is chosen to be the mapping in (2-26) then the coefficients of the new PDE for v will satisfy a Carleson condition like (1-5), see Lemma 2.18, provided the original coefficients (for u) were either smooth or constant.

Furthermore, if we do not insist on control over the size of the Carleson norm, then we can still infer solvability of the L^p Dirichlet problem for large p , as in [Hofmann and Lewis 2001; Rivera-Noriega 2003; 2014].

Finally, we ready to state our main result; some notions used here are defined in detail in Section 2.

Theorem 1.1. *Let Ω be a domain as in Definition 2.10 with character (ℓ, η, N, d) and let A be bounded and elliptic as in (1-2), and B be measurable. Consider any $1 < p \leq \infty$ and assume that either*

$$(1) \quad d\mu_1 = \left[\delta(X, t)^{-1} \sup_{i,j} \left(\operatorname{osc}_{B_{\delta(X,t)/2}(X,t)} a_{ij} \right)^2 + \delta(X, t) \sup_{B_{\delta(X,t)/2}(X,t)} |B|^2 \right] dX dt \tag{1-6}$$

is a Carleson measure on Ω with Carleson norm $\|\mu_1\|_C$,

(2) or assume in addition that $\nabla A, \partial_t A$ are well-defined at a.e. point in Ω and

$$d\mu_2 = (\delta(X, t)|\nabla A|^2 + \delta(X, t)^3|\partial_t A|^2 + \delta(X, t)|B|^2) dX dt \tag{1-7}$$

is a Carleson measure on Ω with Carleson norm $\|\mu_2\|_C$ and

$$\delta(X, t)|\nabla A| + \delta(X, t)^2|\partial_t A| + \delta(X, t)|B| \leq \|\mu_2\|_C^{1/2}. \tag{1-8}$$

Then there exists $K = K(\lambda, \Lambda, \ell, n, p) > 0$ such that if for some $r_0 > 0$

$$\max\{\eta, \|\mu_1\|_{C, r_0}\} < K \quad \text{or} \quad \max\{\eta, \|\mu_2\|_{C, r_0}\} < K$$

the L^p Dirichlet boundary value problem (1-1) is solvable (see Definition 2.26) and the following estimate holds for all continuous boundary data $f \in C_0(\partial\Omega)$:

$$\|N(u)\|_{L^p(\partial\Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega, d\sigma)},$$

where the implied constant depends only on the operator, n, p and character (ℓ, η, N, d) , and $N(u)$ is the nontangential maximal function of u .

Corollary 1.2. In particular, if Ω is of VMO type (η in the character (ℓ, η, N, d) can be taken arbitrary small), and the Carleson measure μ from Theorem 1.1 is a vanishing Carleson measure then the L^p Dirichlet boundary value problem (1-1) is solvable for all $1 < p \leq \infty$.

2. Preliminaries

Here and throughout we consistently use ∇u to denote the gradient in the spatial variables and u_t or $\partial_t u$ to denote the gradient in the time variable.

2A. Parabolic domains. In this subsection, we define a class of time-varying domains whose boundaries are given locally as functions $\phi(x, t)$, Lipschitz in the spatial variable and satisfying the Lewis–Murray condition in the time variable. At each time $\tau \in \mathbb{R}$ the set of points in Ω with fixed time $t = \tau$, that is, $\Omega_\tau = \Omega \cap \{t = \tau\}$, is a nonempty bounded Lipschitz domain in \mathbb{R}^n . We start with a discussion of the Lewis–Murray condition, give a summary and clarification of the results in the literature, and introduce some new equivalent definitions.

We define a *parabolic cube* in $\mathbb{R}^{n-1} \times \mathbb{R}$, for a constant $r > 0$, as

$$Q_r(x, t) = \{(y, s) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_i - y_i| < r \text{ for all } 1 \leq i \leq n - 1, |t - s|^{1/2} < r\}.$$

Let $J_r \subset \mathbb{R}^{n-1}$ be a *spatial cube* of radius r . For a given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ let

$$f_{Q_r} = \frac{1}{|Q_r|} \int_{Q_r} f(x, t) dx dt.$$

When we write $f \in \text{BMO}(\mathbb{R}^n)$ we mean that f belongs to the parabolic version of the usual BMO space with the norm $\|f\|_*$, where

$$\|f\|_* = \sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} |f - f_{Q_r}| dx dt < \infty. \tag{2-1}$$

Recall that the Lewis–Murray condition imposed that a half-derivative in time of $\phi(x, t)$ belongs to parabolic BMO. There are a few different ways one can define half-derivatives and BMO-Sobolev spaces, and there are also some erroneous results in the literature which we correct here. To bring clarity, we start by discussing the various definitions in the global setting of a graph domain $\Omega = \{(x_0, x, t) : x_0 > \phi(x, t)\}$, where $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$. We follow the standard notation of [Hofmann and Lewis 1996].

If $g \in C_0^\infty(\mathbb{R})$ and $0 < \alpha < 2$ then the one-dimensional fractional differentiation operators D_α are defined on the Fourier side by

$$\widehat{D_\alpha g}(\tau) = |\tau|^\alpha \widehat{g}(\tau).$$

If $0 < \alpha < 1$ then by standard results

$$D_\alpha g(t) = c \int_{\mathbb{R}} \frac{g(t) - g(s)}{|t - s|^{1+\alpha}} ds.$$

Therefore, we define the *pointwise half-derivative in time* of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ to be

$$D_{1/2}^t \phi(x, t) = c_n \int_{\mathbb{R}} \frac{\phi(x, s) - \phi(x, t)}{|s - t|^{3/2}} ds \tag{2-2}$$

for a properly chosen constant c_n ; see [Hofmann and Lewis 1996]. In order for the Fourier transform to be well-defined, ϕ should be a tempered distribution modulo first-degree polynomials in x ; see [Hofmann 1995; Strichartz 1980].

However, this definition ignores the spatial coordinates. Instead by following [Fabes and Rivière 1967] we may define the *parabolic half-derivative in time* of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ to be

$$\widehat{\mathbb{D}_n \phi}(\xi, \tau) = \frac{\tau}{\|(\xi, \tau)\|} \widehat{\phi}(\xi, \tau), \tag{2-3}$$

where ξ and τ denote the spatial and temporal variables on the Fourier side respectively, and $\|(x, t)\| = |x| + |t|^{1/2}$ denotes the parabolic norm. In addition we define the *parabolic derivative* (in space and time) of $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ to be

$$\widehat{\mathbb{D} \phi}(\xi, \tau) = \|(\xi, \tau)\| \widehat{\phi}(\xi, \tau). \tag{2-4}$$

\mathbb{D}^{-1} is the parabolic Riesz potential. Again, we assume here that ϕ is a tempered distribution modulo first-degree polynomials in x . One can also represent \mathbb{D} as

$$\mathbb{D} = \sum_{j=1}^n R_j \mathbb{D}_j, \tag{2-5}$$

where $\mathbb{D}_j = \partial_j$ for $1 \leq j \leq n - 1$, \mathbb{D}_n is defined above and R_j are the parabolic Riesz transforms defined on the Fourier side as

$$\begin{aligned} \widehat{R}_j(\xi, \tau) &= \frac{i\xi_j}{\|(\xi, \tau)\|} \quad \text{for } 1 \leq j \leq n - 1, \\ \widehat{R}_n(\xi, \tau) &= \frac{\tau}{\|(\xi, \tau)\|^2}. \end{aligned} \tag{2-6}$$

Furthermore the kernels of R_j have average zero on (parabolically weighted) spheres around the origin, obey the standard Calderón–Zygmund kernel and therefore by standard Calderón–Zygmund theory each

R_j defines a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and is bounded on $BMO(\mathbb{R}^n)$ [Peetre 1966; Fabes and Rivière 1966; 1967; Hofmann and Lewis 1996].

We say that $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ is $\text{Lip}(1, \frac{1}{2})$ with Lipschitz constant ℓ if ϕ is Lipschitz in the spatial variables and Hölder continuous of order $\frac{1}{2}$ in the temporal variable. That is,

$$|\phi_j(x, t) - \phi_j(y, t)| \leq \ell(|x - y| + |t - s|^{1/2}). \tag{2-7}$$

The *Lewis–Murray condition* on the domain Ω , for which they proved the mutual absolute continuity of the caloric measure and the natural surface measure, is $\phi \in \text{Lip}(1, \frac{1}{2})$ and $\|D_{1/2}^t \phi\|_* \leq \eta$; note this BMO norm is taken over \mathbb{R}^n .

It is worth remarking that none of the operators $D_{1/2}^t, \mathbb{D}_n$ or \mathbb{D} easily lend themselves to being localized to a function $\phi : Q_d \rightarrow \mathbb{R}$ due to their nonlocal natures. However, our goal is to provide a theory where the domain is locally given by graphs satisfying the Lewis–Murray condition. The parabolic nature of the PDE (especially time irreversibility and exponential decay of solutions with vanishing boundary data) suggests we should expect to need only local conditions on the functions describing the boundary.

To this end, we state the following theorems, where we show some statements equivalent to the Lewis–Murray condition for a global function $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$. Furthermore, the final conditions admit themselves to being localized easily as well as amiable to an extension; see Theorem 2.8 for details on an extension.

The equivalence of (1) and (2) below is shown in [Hofmann and Lewis 1996] with an equivalence of norms in the small and large sense; see (2.10) and Theorem 7.4 in that work for precise details, and see (2-5) and (2-6).

Theorem 2.1. *Let $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi \in \text{Lip}(1, \frac{1}{2})$. Then the following conditions are equivalent:*

- (1) $D_{1/2}^t \phi \in BMO(\mathbb{R}^n)$.
- (2) $\mathbb{D}_n \phi \in BMO(\mathbb{R}^n)$.
- (3) $\mathbb{D} \phi \in BMO(\mathbb{R}^n)$.

We also note that since $\mathbb{D}_n \phi = R_n \mathbb{D} \phi$ we have $\|\mathbb{D}_n \phi\|_* \lesssim \|\mathbb{D} \phi\|_*$ by the boundedness of R_n on $BMO(\mathbb{R}^n)$.

We now extend this theorem by adding three more equivalent statements. To motivate (6) of Theorem 2.3 below we first recall a characterisation of BMO from [Strichartz 1980, p. 546]. Let $M(f, Q) = (1/|Q|) \int_Q f$ denote the average of f over a cube Q , and let $\tilde{Q}_\rho(x)$ be the cube of radius ρ with x in the upper-right corner.

Lemma 2.2 [Strichartz 1980]. *We have $f \in BMO(\mathbb{R}^n)$ is equivalent to*

$$\sup_{Q_r} \sum_{k=1}^n \frac{1}{|Q_r|} \int_{Q_r} \int_0^r |M(f, \tilde{Q}_\rho(x)) - M(f, \tilde{Q}_\rho(x - \rho e_k))|^2 \frac{d\rho}{\rho} dx = B < \infty, \tag{2-8}$$

where e_k are the usual unit vectors in \mathbb{R}^n , and $\|f\|_*^2 \sim B$.

The equivalence of (3) and (4) in the theorem below is a generalisation of [Strichartz 1980] to the parabolic setting that is stated in [Rivera-Noriega 2003]; see also [Fefferman and Stein 1972; Calderón and Torchinsky 1975; 1977]. We have some questions about the proof given in [Rivera-Noriega 2003]; however, the argument we give for (5) also works for (4) and hence the claim in that paper is correct.

Theorem 2.3. *Let $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi \in \text{Lip}(1, \frac{1}{2})$. Then the following conditions are equivalent:*

(3) $\mathbb{D}\phi \in \text{BMO}(\mathbb{R}^n)$.

(4)
$$\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{\|(y,s)\| \leq r} \frac{|\phi(x+y, t+s) - 2\phi(x, t) + \phi(x-y, t-s)|^2}{\|(y, s)\|^{n+3}} dy ds dx dt = B_{(4)} < \infty. \quad (2-9)$$

(5) (a)
$$\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{|y| < r} \frac{|\phi(x+y, t) - 2\phi(x, t) + \phi(x-y, t)|^2}{|y|^{n+1}} dy dx dt = B_{(5.a)} < \infty. \quad (2-10)$$

(b)
$$\sup_{Q_r=J_r \times I_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x, t) - \phi(x, s)|^2}{|t-s|^2} ds dt dx = B_{(5.b)} < \infty. \quad (2-11)$$

(6) *Let $u = (u', u_n) \in \mathbb{S}^{n-1}$ and let e_n be the unit vector in the time direction. For $k = 1, \dots, n-1$ let*

$$A_k = \int_0^1 \rho u' \cdot (M(\nabla\phi, \tilde{Q}_\rho(x + \lambda\rho u', t)) - M(\nabla\phi, \tilde{Q}_\rho(x + \lambda\rho u' - \rho e_k, t))) d\lambda,$$

$$A_n = \int_0^1 \rho u' \cdot (M(\nabla\phi, \tilde{Q}_\rho(x + \lambda\rho u', t)) - M(\nabla\phi, \tilde{Q}_\rho(x + \lambda\rho u', t - \rho^2))) d\lambda.$$

Then

(a)
$$\sup_{Q_r} \sum_{k=1}^n \frac{1}{|Q_r|} \int_{Q_r} \int_{u \in \mathbb{S}^{n-1}} \int_0^r \frac{|A_k|^2}{\rho^3} d\rho du dx dt = B_{(6.a)} < \infty, \quad (2-12)$$

(b)
$$\sup_{Q_r=J_r \times I_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x, t) - \phi(x, s)|^2}{|t-s|^2} ds dt dx = B_{(6.b)} < \infty. \quad (2-11)$$

Furthermore we have equivalence of the norms

$$\|\mathbb{D}\phi\|_*^2 \sim B_{(4)} \sim B_{(5.a)} + B_{(5.b)} \sim B_{(6.a)} + B_{(6.b)}. \quad (2-13)$$

We give a proof of this result in the [Appendix](#) at the end of the paper.

Remark 2.4. Condition (6.a) does not immediately look too similar to its supposed motivation, (2-8) in Lemma 2.2. However, if we move back into Cartesian coordinates and undo the mean value theorem, then we obtain something very similar to a combination of (2-8) and an endpoint version of [Strichartz 1980, (3.1)]. The reason why we can obtain the endpoint, whereas [Strichartz 1980, (3.1)] can only be used for a fractional derivative smaller than 1, is due to additional integrability and cancellation coming from (A-1). Consider

$$A'_k = M(\phi, \tilde{Q}_{\|(y,s)\|}(x+y, t)) - M(\phi, \tilde{Q}_{\|(y,s)\|}(x, t)) \\ - M(\phi, \tilde{Q}_{\|(y,s)\|}(x+y - \|(y, s)\|e_k, t)) + M(\phi, \tilde{Q}_{\|(y,s)\|}(x - \|(y, s)\|e_k, t)),$$

$$A'_n = M(\phi, \tilde{Q}_{\|(y,s)\|}(x+y, t)) - M(\phi, \tilde{Q}_{\|(y,s)\|}(x, t)) \\ - M(\phi, \tilde{Q}_{\|(y,s)\|}(x+y, t - \|(y, s)\|^2)) + M(\phi, \tilde{Q}_{\|(y,s)\|}(x, t - \|(y, s)\|^2)).$$

Then (6.a) is equivalent to

$$\sup_{Q_r} \sum_{k=1}^n \frac{1}{|Q_r|} \int_{Q_r} \int_{\|(y,s)\| < r} \frac{|A'_k|^2}{\|(y,s)\|^{n+3}} dy ds dx dt = \tilde{B}_{(6.a)} < \infty. \tag{2-14}$$

Proposition 2.5. *Let $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi \in \text{Lip}(1, \frac{1}{2})$. Let $B_{(5.a)}$ and $B_{(6.a)}$ be as in Theorem 2.3. Then we have*

$$B_{(6.a)} \lesssim \|\nabla\phi\|_*^2, \quad B_{(5.a)} \lesssim \sup_t \|\nabla\phi(\cdot, t)\|_{\text{BMO}(\mathbb{R}^{n-1})}^2,$$

where $\text{BMO}(\mathbb{R}^{n-1})$ denotes the BMO norm in the spatial variables only.

Proof. The statement $\nabla\phi \in \text{BMO}(\mathbb{R}^{n-1})$ implies (5.a) follows from [Strichartz 1980, Theorem 3.3]. In order to establish the second claim, for the ease of notation let us fix Q_r and k in $1 \leq k \leq n - 1$. Then since $|u'| \leq 1$ after changing the order of integration (and the substitution $y = x + \lambda\rho u' \in Q_{2r}$) we get that $B_{(6.a)}$ defined by (2-12) is bounded by

$$\int_0^1 \int_{\mathbb{S}^{n-1}} \int_0^r \frac{1}{|Q_r|} \int_{Q_{2r}} |(M(\nabla u, \tilde{Q}_\rho(y, t)) - M(\nabla u, \tilde{Q}_\rho(y - \rho e_k, t)))|^2 dy dt \frac{d\rho}{\rho} du d\lambda.$$

Then by Lemma 2.2 the two interior integrals are bounded by $C\|\nabla\phi\|_*^2$. Therefore $B_{(6.a)}$ is controlled by $C\|\nabla\phi\|_*^2$. □

It is not immediately obvious whether the opposite implication is true or false due to the highly singular nature of Riesz potentials; see (2-5) and (2-6).

Corollary 2.6. *Let $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi \in \text{Lip}(1, \frac{1}{2})$. If $\|\nabla\phi\|_* \lesssim \eta$ and $B_{(5.b)} \lesssim \eta^2$ then $\|\mathbb{D}\phi\|_* \lesssim \eta$.*

Here we have replaced conditions (5.a) and (6.a) by the slightly stronger but easier to verify condition $\|\nabla\phi\|_* \lesssim \eta$. We believe that, without too much extra work, one could formulate our main theorem and associated lemmas with a local version of (5.a) in place of $\|\nabla\phi\|_*$.

Remark 2.7. In [Rivera-Noriega 2003, Lemma 2.1], it is stated that another condition is equivalent to those given in Theorems 2.1 and 2.3; however this claim is not correct and only one of the stated implications holds.

Theorem 3.3 in [Strichartz 1980] states that in the one-dimensional setting $D_{1/2}^t\phi(t) \in \text{BMO}(\mathbb{R})$ is equivalent to the one-dimensional version of (5.b) and (6.b)

$$\sup_{I' \subset \mathbb{R}} \left(\frac{1}{|I'|} \int_{I'} \int_{I'} \frac{|\phi(t) - \phi(s)|^2}{|t - s|^2} dt ds \right)^{1/2} \leq B, \tag{2-15}$$

with $B \sim \|D_{1/2}^t\phi(\cdot)\|_{\text{BMO}(\mathbb{R})}$.

In [Rivera-Noriega 2003, Lemma 2.1] it is claimed that given $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi \in \text{Lip}(1, \frac{1}{2})$ the pointwise n -dimensional analogue of (2-15)

$$\sup_{x \in \mathbb{R}^{n-1}} \sup_{I' \subset \mathbb{R}} \left(\frac{1}{|I'|} \int_{I'} \int_{I'} \frac{|\phi(x, t) - \phi(x, s)|^2}{|t - s|^2} dt ds \right)^{1/2} \leq B \tag{2-16}$$

is equivalent to $\mathbb{D}\phi \in \text{BMO}(\mathbb{R}^n)$ with $B \sim \|\mathbb{D}\phi\|_{\text{BMO}(\mathbb{R}^n)}$. This does not appear to be correct. The paper [Rivera-Noriega 2003] does not give a proof and provides instead a reference to [Strichartz 1980] that is irrelevant for the claim. By [Strichartz 1980] (2-16) is equivalent to $D_{1/2}^t \phi(x, \cdot) \in \text{BMO}(\mathbb{R})$ pointwise for a.e. x . After some tedious and technical calculations we were able to show $\sup_x D_{1/2}^t \phi(x, \cdot) \in \text{BMO}(\mathbb{R})$ implies $D_{1/2}^t \phi \in \text{BMO}(\mathbb{R}^n)$ and hence $\mathbb{D}\phi \in \text{BMO}(\mathbb{R}^n)$ via (4) of Theorem 2.3. However, whether the converse holds is not clear even if we assume more structure for the function $\phi(x, t)$. This is due to the fact that there is “no reasonable Fubini theorem relating $\text{BMO}(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R})$ ” [Strichartz 1980, p. 558].

Fortunately, the lack of a converse implication does not cast doubt over the subsequent results of [Rivera-Noriega 2003] since the author only uses the claimed equivalence in the correct direction — that (2-16) implies $\mathbb{D}\phi \in \text{BMO}(\mathbb{R}^n)$.

2B. Localisation. After the comprehensive review of the Lewis–Murray condition for a graph domain Ω we continue in our aim to construct a time-varying domain which is locally described by local graphs ϕ_j .

For a vector $x \in \mathbb{R}^{n-1}$ we consider the norm $|x|_\infty = \sup_i |x_i|$.

Consider $\phi : Q_{8d} \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}$. The localised version of (2-11) from Theorem 2.3 is simply

$$\sup_{\substack{Q_r = J_r \times I_r \\ Q_r \subset Q_{8d}}} \frac{1}{|Q_r|} \int_{Q_r} \int_{I_r} \frac{|\phi(x, t) - \phi(x, s)|^2}{|t - s|^2} ds dt dx < \infty. \tag{2-17}$$

We denote by $\|f\|_{*,d}$ the BMO norm of f where the supremum in the BMO norm, see (2-1), is taken over all cubes Q_r with $r \leq d$. For a function $f : J \times I \rightarrow \mathbb{R}$, where $J \subset \mathbb{R}^{n-1}$ and $I \subset \mathbb{R}$ are closed bounded cubes, we consider the norm $\|f\|_{*,J \times I}$ defined as above where the supremum is taken over all parabolic cubes Q_r contained in $J \times I$. The norm $\|f\|_{*,J \times I,d}$ is where the supremum is taken over all parabolic cubes Q_r with $r \leq d$ contained in $J \times I$. If the context is clear we suppress the $J \times I$ and write $\|f\|_*$ or $\|f\|_{*,d}$.

Recall that $\text{VMO}(\mathbb{R}^n)$ is defined as the closure of all bounded uniformly continuous functions (which we denote by $C_{b,u}(\mathbb{R}^n)$) in the BMO norm or equivalently BMO functions f such that $\|f\|_{*,d} \rightarrow 0$ as $d \rightarrow 0$. Alternatively, if we define

$$d(f, \text{VMO}) := \inf_{h \in C_{b,u}(\mathbb{R}^n)} \|f - h\|_*$$

then $f \in \text{VMO}$ if and only if $d(f, \text{VMO}) = 0$; for $f \in \text{BMO}$ this measures the distance of f to VMO . In our case, the boundary of the parabolic domains we consider can be locally described as a graph of a continuous function. However, as our domain is unbounded in time, we may potentially require an infinite family of local graphs $\{\phi_j\}$. Therefore we need to measure the distance to VMO uniformly across this infinite family.

Let $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\delta(0) = 0$ and δ be continuous at 0. Then we define C_δ to be the set of bounded continuous functions with the same modulus of continuity δ . That is,

$$C_\delta = \{g : \mathbb{R}^n \rightarrow \mathbb{R} : |g(x) - g(y)| \leq \delta(|x - y|) \text{ for all } x, y, \text{ and } g \text{ is bounded}\}. \tag{2-18}$$

Note that every family of bounded equicontinuous functions is a subset of C_δ for some modulus of continuity δ . Also $C_{b,u} = \bigcup_\delta C_\delta$. For $f : Q_{8d} \rightarrow \mathbb{R}$ we define $d(f, C_\delta)$ as

$$d(f, C_\delta) = \inf_{h \in C_\delta} \|f - h\|_{*, Q_{8d}}.$$

We are now ready to state and prove the result on the extensibility of $\phi : Q_{8d} \rightarrow \mathbb{R}$ to a global function.

Theorem 2.8. *Let $\phi : Q_{8d} \subset \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ be $\text{Lip}(1, \frac{1}{2})$ with Lipschitz constant ℓ . If there exist a scale r_1 , a constant $\eta > 0$ and a modulus of continuity δ such that*

$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{8d}, s \leq r_1}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi(x, t) - \phi(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \leq \eta^2 \tag{2-19}$$

and

$$d(\nabla\phi, C_\delta) \leq \eta \tag{2-20}$$

then there exists a scale $d' \leq d$ that only depends on d, δ, η , and r_1 and not ϕ such that for all $Q_r \subset Q_{4d}$ with $r \leq d'$ there exists a global $\text{Lip}(1, \frac{1}{2})$ function $\Phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ with the following properties for all $0 < \varepsilon < 1$:

- (i) $\Phi|_{Q_r} = \phi|_{Q_r}$.
- (ii) The $\text{Lip}(1, \frac{1}{2})$ constant of Φ is ℓ .
- (iii) $\|\nabla\Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.
- (iv) $\sup_{Q_s = J_s \times I_s} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\Phi(x, t) - \Phi(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \lesssim \eta^2$.

Therefore by [Corollary 2.6](#), $\|\mathbb{D}\Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.

We again give the proof of this result in the [Appendix](#). We are now ready to define the class of parabolic domains on which we will work. Motivated by the usual definition of a Lipschitz domain we have:

Definition 2.9. $\mathbb{Z} \subset \mathbb{R}^n \times \mathbb{R}$ is an ℓ -cylinder of diameter d if there exists a coordinate system $(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$ obtained from the original coordinate system by translation in spatial and time variables and rotation only in the spatial variables such that

$$\mathbb{Z} = \{(x_0, x, t) : |x| \leq d, |t|^{1/2} \leq d, |x_0| \leq (\ell + 1)d\}$$

and for $s > 0$

$$s\mathbb{Z} := \{(x_0, x, t) : |x| < sd, |t|^{1/2} \leq sd, |x_0| \leq (\ell + 1)sd\}.$$

Definition 2.10. $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is an *admissible parabolic domain* with character (ℓ, η, N, d) if there exists a positive scale r_1 , and a modulus of continuity δ such that for any time $\tau \in \mathbb{R}$ there are at most N ℓ -cylinders $\{\mathbb{Z}_j\}_{j=1}^N$ of diameter d satisfying the following conditions:

- (1) $\partial\Omega \cap \{|t - \tau| \leq d^2\} = \bigcup_j (\mathbb{Z}_j \cap \partial\Omega)$.
- (2) In the coordinate system (x_0, x, t) of the ℓ -cylinder \mathbb{Z}_j

$$\mathbb{Z}_j \cap \Omega \supset \{(x_0, x, t) \in \Omega : |x| < d, |t| < d^2, \delta(x_0, x, t) \leq d/2\}.$$

(3) $8\mathbb{Z}_j \cap \partial\Omega$ is the graph $\{x_0 = \phi_j(x, t)\}$ of a function $\phi_j : Q_{8d} \rightarrow \mathbb{R}$, with $Q_{8d} \subset \mathbb{R}^{n-1} \times \mathbb{R}$, such that

$$|\phi_j(x, t) - \phi_j(y, s)| \leq \ell(|x - y| + |t - s|^{1/2}) \quad \text{and} \quad \phi_j(0, 0) = 0. \tag{2-21}$$

(4)
$$d(\nabla\phi_j, C_\delta) \leq \eta \tag{2-22}$$

and

$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{8d}, s \leq r_1}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi_j(x, t) - \phi_j(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \leq \eta^2. \tag{2-23}$$

Here and throughout $\delta(x_0, x, t) := \text{dist}((x_0, x, t), \partial\Omega)$, and by dist we denote the parabolic distance $\text{dist}[(X, t), (Y, s)] = |X - Y| + |t - s|^{1/2}$.

We say that Ω is of VMO type if η in the character (ℓ, η, N, d) can be taken arbitrarily small (at the expense of a potentially smaller d and r_1 , and larger N).

Remark 2.11. When (2-22) holds for small or vanishing η it follows that for a fixed time τ the normal ν to the fixed-time spatial domain $\Omega_\tau = \Omega \cap \{t = \tau\}$ can be written in local coordinates as

$$\nu = \frac{1}{|(-1, \nabla\phi_j)|} (-1, \nabla\phi_j)$$

and hence $d(\nu, \text{VMO}) \lesssim \eta$. Therefore Ω_τ is similar to the domains considered in [Mazya et al. 2009; Hofmann et al. 2015], which dealt with the elliptic problems on domains with normal in or near VMO.

Remark 2.12. It follows from this definition that for each $\tau \in \mathbb{R}$ the time-slice Ω_τ of an admissible parabolic domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ is a bounded Lipschitz domain in \mathbb{R}^n and they all have a uniformly bounded diameter. That is,

$$\inf_{\tau \in \mathbb{R}} \text{diam}(\Omega_\tau) \sim d \sim \sup_{\tau \in \mathbb{R}} \text{diam}(\Omega_\tau),$$

where d is the scale from Definition 2.10 and the implied constants only depend on N . In particular, if $\mathcal{O} \subset \mathbb{R}^n$ is a bounded Lipschitz domain then the parabolic cylinder $\Omega = \mathcal{O} \times \mathbb{R}$ is an example of a domain satisfying Definition 2.10.

Definition 2.13. Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ be an admissible parabolic domain with character (ℓ, η, N, d) . The *measure* σ defined on sets $A \subset \partial\Omega$ is

$$\sigma(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(A \cap \{(X, t) \in \partial\Omega\}) dt, \tag{2-24}$$

where \mathcal{H}^{n-1} is the $(n-1)$ -dimensional Hausdorff measure on the Lipschitz boundary $\partial\Omega_\tau$.

We consider solvability of the L^p Dirichlet boundary value problem with respect to this measure σ . The measure σ may not be comparable to the usual surface measure on $\partial\Omega$: in the t -direction the functions ϕ_j from Definition 2.10 are only $\frac{1}{2}$ -Lipschitz and hence the standard surface measure might not be locally finite. Our definition assures that for any $A \subset 8\mathbb{Z}_j$, where \mathbb{Z}_j is an ℓ -cylinder, we have

$$\mathcal{H}^n(A) \sim \sigma(\{(\phi_j(x, t), x, t) : (x, t) \in A\}), \tag{2-25}$$

where the constants in (2-25), by which these measures are comparable, only depend on ℓ of the character (ℓ, η, N, d) of the domain Ω . If Ω has a smoother boundary, such as Lipschitz (in all variables) or better, then the measure σ is comparable to the usual n -dimensional Hausdorff measure \mathcal{H}^n . In particular, this holds for a parabolic cylinder $\Omega = \mathcal{O} \times \mathbb{R}$.

Corollary 2.14. *Let Ω be defined as in Definition 2.10 by a family of functions $\{\phi_j\}$, $\phi_j : Q_{8d} \rightarrow \mathbb{R}$. Then there exists an extended family $\{\Phi_j\}$, $\Phi_j : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$, such that*

- (i) $\{\Phi_j|_{Q_{8r}}\}$ still describes Ω , as in Definition 2.10, but with character $(\ell, \eta, \tilde{N}, r)$ instead of (ℓ, η, N, d) , where $\tilde{N} \geq N$ and $r \leq r_1 \leq d$ as by Theorem 2.8,
- (ii) $\|\nabla \Phi_j\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$, and
- (iii) $\|\mathbb{D} \Phi_j\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell$.

Here, \tilde{N}, r depend on the original character variables ℓ, η, N, d , the modulus of continuity δ and the dimension n .

Proof. This follows from Theorem 2.8 and by tiling the support of each ϕ_j into parabolic cubes of size $8r$ with enough overlap. □

Corollary 2.15. *If Ω is a VMO-type domain then we may take η arbitrarily small in Corollary 2.14, or in (2-22) and (2-23) of Definition 2.10, by reducing r .*

2C. Pullback transformation and Carleson condition. We now briefly recall the pullback mapping of Dahlberg, Kenig, Nečas and Stein on the upper half-space $U \rho : U \rightarrow \Omega$, see [Hofmann and Lewis 1996; 2001], in the setting of parabolic equations defined by

$$\rho(x_0, x, t) = (x_0 + P_{\gamma x_0} \phi(x, t), x, t). \tag{2-26}$$

For simplicity, assume

$$\Omega = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > \phi(x, t)\}, \tag{2-27}$$

where $\phi(x, t) : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ and satisfies (4) and (3) of Definition 2.10. This transformation maps the upper half-space

$$U = \{(x_0, x, t) : x_0 > 0, x \in \mathbb{R}^{n-1}, t \in \mathbb{R}\} \tag{2-28}$$

into Ω and allows us to consider the L^p solvability of the PDE (1-1) in the upper half-space instead of in the original domain Ω .

To complete the definition of the mapping ρ we define a parabolic approximation to the identity P to be an even nonnegative function $P(x, t) \in C_0^\infty(Q_1(0, 0))$ for $(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, with $\int P(x, t) dx dt = 1$ and set

$$P_\lambda(x, t) := \lambda^{-(n+1)} P\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right).$$

Let $P_\lambda \phi$ be the convolution operator

$$P_\lambda \phi(x, t) := \int_{\mathbb{R}^{n-1} \times \mathbb{R}} P_\lambda(x - y, t - s) \phi(y, s) dy ds.$$

Then P satisfies for constants γ

$$\lim_{(y_0, y, s) \rightarrow (0, x, t)} P_{\gamma y_0} \phi(y, s) = \phi(x, t)$$

and ρ defined in (2-26) extends continuously to $\rho : \bar{U} \rightarrow \bar{\Omega}$. The usual surface measure on ∂U is comparable with the measure σ defined by (2-24) on $\partial \Omega$.

Suppose that $v = u \circ \rho$ and $f^v = f \circ \rho$. Then (1-1) transforms to a new PDE for the variable v

$$\begin{cases} v_t = \operatorname{div}(A^v \nabla v) + B^v \cdot \nabla v & \text{in } U, \\ v = f^v & \text{on } \partial U, \end{cases} \tag{2-29}$$

where $A^v = [a_{ij}^v(X, t)]$, $B^v = [b_i^v(X, t)]$ are $(n \times n)$ and $(1 \times n)$ matrices.

The precise relations between the original coefficients A and B and the new coefficients A^v and B^v are detailed in [Rivera-Noriega 2014, p. 448]. We note that if the constant $\gamma > 0$ is chosen small enough then the coefficients $a_{ij}^v, b_i^v : U \rightarrow \mathbb{R}$ are Lebesgue measurable and A^v satisfies the standard uniform ellipticity condition with constants λ^v and Λ^v , since the original matrix A did.

Definition 2.16. Let Ω be a parabolic domain from Definition 2.10. For $(Y, s) \in \partial \Omega$, $(X, t), (Z, \tau) \in \Omega$ and $r > 0$ we write

$$\begin{aligned} B_r(X, t) &= \{(Z, \tau) \in \mathbb{R}^n \times \mathbb{R} : \operatorname{dist}[(X, t), (Z, \tau)] < r\}, \\ Q_r(X, t) &= \{(Z, \tau) \in \mathbb{R}^n \times \mathbb{R} : |x_i - z_i| < r \text{ for all } 0 \leq i \leq n - 1, |t - \tau|^{1/2} < r\}, \\ \Delta_r(Y, s) &= \partial \Omega \cap B_r(Y, s), \\ T(\Delta_r) &= \Omega \cap B_r(Y, s), \\ \delta(X, t) &= \inf_{(Y, s) \in \partial \Omega} \operatorname{dist}[(X, t), (Y, s)]. \end{aligned}$$

Definition 2.17 (Carleson measure). A measure $\mu : \Omega \rightarrow \mathbb{R}^+$ is a *Carleson measure* if there exists a constant $C = C(d)$ such that for all $r \leq d$ and all surface balls Δ_r

$$\mu(T(\Delta_r)) \leq C \sigma(\Delta_r). \tag{2-30}$$

The best possible constant C is called the *Carleson norm* and is denoted by $\|\mu\|_{C,d}$. Occasionally, for brevity, we drop the d and just write $\|\mu\|_C$ if the context is clear. We say that μ is a vanishing Carleson measure if $\|\mu\|_{C,d} \rightarrow 0$ as $d \rightarrow 0+$.

When $\partial \Omega$ is locally given as a graph of a function $x_0 = \phi(x, t)$ in the coordinate system (x_0, x, t) and μ is a measure supported on $\{x_0 > \phi(x, t)\}$, we can reformulate the Carleson condition locally using the parabolic boundary cubes Q_r and corresponding Carleson regions $T(Q_r)$. The Carleson condition (2-30) then becomes

$$\mu(T(Q_r)) \leq C |Q_r| = Cr^{n+1}. \tag{2-31}$$

Note that the Carleson norms induced from (2-30) and (2-31) are not equal but are comparable.

We now return to the pullback transformation and investigate the Carleson condition on the coefficients of A and B . The following result comes directly from a careful reading of the proofs of Lemma 2.8 and Theorem 7.4 in [Hofmann and Lewis 1996] combined with Theorems 2.1 and 2.3.

Lemma 2.18. *Let σ and θ be nonnegative integers, $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ a multi-index with $l = \sigma + |\alpha| + \theta$, d a scale and fix γ . If $\phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies for all $x, y \in \mathbb{R}^{n-1}$, $t, s \in \mathbb{R}$, and for some positive constants ℓ and η*

$$\begin{aligned} |\phi(x, t) - \phi(y, s)| &\leq \ell(|x - y| + |t - s|^{1/2}), \\ \|\mathbb{D}\phi\|_* &\leq \eta \end{aligned} \tag{2-32}$$

then the measure ν defined at (x_0, x, t) by

$$d\nu = \left(\frac{\partial^l P_{\gamma x_0} \phi}{\partial x_0^\sigma \partial x^\alpha \partial t^\theta} \right)^2 x_0^{2l+2\theta-3} dx dt dx_0$$

is a Carleson measure on cubes of diameter $\leq d/4$ whenever either $\sigma + \theta \geq 1$ or $|\alpha| \geq 2$, with

$$\nu[(0, r) \times Q_r(x, t)] \lesssim \eta |Q_r(x, t)|,$$

where $r \leq d/4$. Moreover, if $l \geq 1$ then at (x_0, x, t) , with $x_0 \leq d/4$,

$$\left| \frac{\partial^l P_{\gamma x_0} \phi}{\partial x_0^\sigma \partial x^\alpha \partial t^\theta} \right| \lesssim \eta(1 + \ell)x_0^{1-l-\theta}, \tag{2-33}$$

where the implicit constants depend on d, l, n .

The drift term B^v from the pullback transformation in (2-29) includes the term

$$\frac{\partial}{\partial t} P_{\gamma x_0} \phi u_{x_0}.$$

From Lemma 2.18 with $\sigma = |\alpha| = 0$ and $\theta = 1$, we see that

$$x_0 \left[\frac{\partial}{\partial t} P_{\gamma x_0} \phi(x, t) \right]^2 dX dt$$

is a Carleson measure in U . Thus it is natural to expect that

$$d\mu_1(X, t) = x_0 |B^v|^2(X, t) dX dt \tag{2-34}$$

is a Carleson measure in U and B^v satisfies

$$x_0 |B^v|(X, t) \leq \Lambda_B < \|\mu_1\|_C^{1/2}. \tag{2-35}$$

Indeed, this is the case provided the original vector B satisfies the assumption that

$$d\mu(X, t) = \delta(X, t) \left[\sup_{B_{\delta(X,t)/2}(X,t)} |B| \right]^2 dX dt \tag{2-36}$$

is a Carleson measure in Ω . Here $\|\mu_1\|_C$ depends on η and the Carleson norm of (2-36).

Similarly, for the matrix A^v if we apply Lemma 2.18 and use the calculations in [Rivera-Noriega 2014, §6] then

$$d\mu_2(X, t) = (x_0 |\nabla A^v|^2 + x_0^3 |A_t^v|^2)(X, t) dX dt \tag{2-37}$$

is a Carleson measure in U and A^v satisfies

$$(x_0 |\nabla A^v| + x_0^2 |A_t^v|)(X, t) \leq \|\mu_2\|_C^{1/2} \tag{2-38}$$

for almost every $(X, t) \in U$ provided the original matrix A satisfies that

$$d\mu(X, t) = \left(\delta(X, t) \left[\sup_{B_{\delta(X,t)/2}(X,t)} |\nabla A| \right]^2 + \delta(X, t)^3 \left[\sup_{B_{\delta(X,t)/2}(X,t)} |\partial_t A| \right]^2 \right) dX dt \tag{2-39}$$

is a Carleson measure in Ω .

We note that if both $\|\mu\|_{C,r}$ and η are small then so too are the Carleson norms $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ of the matrix A^v and vector B^v , at least if we restrict ourselves to small Carleson regions $r \leq d$; this comes from [Theorem 2.8](#), [Corollary 2.14](#) and [Corollary 2.15](#). Then by [Lemma 2.18](#) we see that $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ only depend on η and $\|\mu\|_{C,r}$ on Carleson regions of size $r \leq d$. In particular, they are small if both η and $\|\mu\|_{C,r}$ are small. It further follows by [Corollary 2.15](#) that we can make $\|\mu_1\|_{C,r}$ and $\|\mu_2\|_{C,r}$ as small as we like if μ is a vanishing Carleson norm and the domain Ω is of VMO type.

Observe that condition (2-39) is slightly stronger than (1-6), which we claimed to assume in [Theorem 1.1](#). We replace condition (2-39) by the weaker condition (1-6) later via perturbation results of [[Sweezy 1998](#)].

Definition 2.19. We define $\rho_j : U \rightarrow 8\mathbb{Z}_j$ to be the local pullback mapping in $8\mathbb{Z}_j$ associated to the function Φ_j in [Theorem 2.8](#), the extension of ϕ_j from [Definition 2.10](#).

Remark 2.20. By [[Ball and Zarnescu 2017](#)] and its adaptation to the setting of admissible domains in [[Dindoš and Hwang 2018](#), §2.3], one may construct a “proper generalised distance” globally when η in the character of the domain is small. The smallness of η in the character of the domain is used to guarantee that overlapping coordinate charts, generated by a local construction, are almost parallel. We may then use the result of [[Ball and Zarnescu 2017](#), Theorem 5.1] to show there exists a domain Ω^ε of class C^∞ , a homeomorphism $f^\varepsilon : \bar{\Omega} \rightarrow \bar{\Omega}^\varepsilon$ such that $f^\varepsilon(\partial\Omega) = \partial\Omega^\varepsilon$ and $f^\varepsilon : \Omega \rightarrow \Omega^\varepsilon$ is a C^∞ diffeomorphism.

2D. Parabolic nontangential cones, maximal functions and p -adapted square and area functions. We proceed with the definition of parabolic nontangential cones and define the cones in a (local) coordinate system where $\Omega = \{(x_0, x, t) : x_0 > \phi(x, t)\}$, which also applies to the upper half-space U .

Definition 2.21. For a constant $a > 0$, we define the *parabolic nontangential cone* at a point $(x_0, x, t) \in \partial\Omega$ by

$$\Gamma_a(x_0, x, t) = \{(y_0, y, s) \in \Omega : |y - x| + |s - t|^{1/2} < a(y_0 - x_0), x_0 < y_0\}.$$

We occasionally truncate the cone Γ at the height r :

$$\Gamma_a^r(x_0, x, t) = \{(y_0, y, s) \in \Omega : |y - x| + |s - t|^{1/2} < a(y_0 - x_0), x_0 < y_0 < x_0 + r\}.$$

Definition 2.22 (nontangential maximal function). For a function $u : \Omega \rightarrow \mathbb{R}$, the *nontangential maximal function* $N_a(u) : \partial\Omega \rightarrow \mathbb{R}$ and its truncated version at a height r are defined as

$$\begin{aligned} N_a(u)(x_0, x, t) &= \sup_{(y_0, y, s) \in \Gamma_a(x_0, x, t)} |u(y_0, y, s)|, \\ N_a^r(u)(x_0, x, t) &= \sup_{(y_0, y, s) \in \Gamma_a^r(x_0, x, t)} |u(y_0, y, s)| \quad \text{for } (x_0, x, t) \in \partial\Omega. \end{aligned} \tag{2-40}$$

The following p -adapted square function was introduced in [[Dindoš et al. 2007](#)] and has been modified appropriately for the parabolic setting. It is used to control the spatial derivatives of the solution. When

$p = 2$ it is equivalent to the usual square function and when $p < 2$ we use the convention that the expression $|\nabla u|^2 |u|^{p-2}$ is zero whenever ∇u vanishes.

Definition 2.23 (p -adapted square function). For a function $u : \Omega \rightarrow \mathbb{R}$, the p -adapted square function $S_{p,a}(u) : \partial\Omega \rightarrow \mathbb{R}$ and its truncated version at a height r are defined as

$$\begin{aligned} S_{p,a}(u)(Y, s) &= \left(\int_{\Gamma_a(Y,s)} |\nabla u(X, t)|^2 |u(X, t)|^{p-2} \delta(X, t)^{-n} \, dX \, dt \right)^{1/p}, \\ S_{p,a}^r(u)(Y, s) &= \left(\int_{\Gamma_a^r(Y,s)} |\nabla u(X, t)|^2 |u(X, t)|^{p-2} \delta(X, t)^{-n} \, dX \, dt \right)^{1/p}. \end{aligned} \tag{2-41}$$

By applying Fubini we also have

$$\|S_{p,a}(u)\|_{L^p(\partial U)}^p \sim \int_U |\nabla u|^2 |u|^{p-2} x_0 \, dx_0 \, dx \, dt. \tag{2-42}$$

It is not known a priori if these integrals are locally integrable even for $p > 2$. However, [Theorem 4.1](#) shows that these expressions make sense and are finite for solutions to (1-1).

We also need a p -adapted version of an object called the area function, which was introduced in [\[Dindoš and Hwang 2018\]](#) and is used to control the solution in the time variable. Again when $p = 2$ this is just the area function of that work.

Definition 2.24 (p -adapted area function). For a function $u : \Omega \rightarrow \mathbb{R}$, the p -adapted area function $A_{p,a}(u) : \partial\Omega \rightarrow \mathbb{R}$ and its truncated version at a height r are defined as

$$\begin{aligned} A_{p,a}(u)(Y, s) &= \left(\int_{\Gamma_a(Y,s)} |u_t|^2 |u(X, t)|^{p-2} \delta(X, t)^{2-n} \, dX \, dt \right)^{1/p}, \\ A_{p,a}^r(u)(Y, s) &= \left(\int_{\Gamma_a^r(Y,s)} |u_t|^2 |u(X, t)|^{p-2} \delta(X, t)^{2-n} \, dX \, dt \right)^{1/p}. \end{aligned} \tag{2-43}$$

Also by Fubini

$$\|A_{p,a}(u)\|_{L^p(\partial U)}^p \sim \int_U |u_t|^2 |u|^{p-2} x_0^3 \, dx_0 \, dx \, dt. \tag{2-44}$$

As before, it is not known a priori if these expressions are finite for solutions to (1-1) but in [Lemma 4.5](#) we establish control of $A_{p,a}$ by $S_{p,2a}$ and use the finiteness of $S_{p,a}$ from [Theorem 4.1](#).

2E. The L^p solvability of the Dirichlet boundary value problem. We are now in the position to define the L^p Dirichlet boundary value problem and our main results.

Definition 2.25 [\[Aronson 1968\]](#). We say that u is a *weak solution* to a parabolic operator of the form (1-1) in Ω if $u, \nabla u \in L^2_{\text{loc}}(\Omega)$, $\sup_t \|u(\cdot, t)\|_{L^2_{\text{loc}}(\Omega_t)} < \infty$ and

$$\int_{\Omega} (-u\phi_t + A\nabla u \cdot \nabla\phi - \phi B \cdot \nabla u) \, dX \, dt = 0$$

for all $\phi \in C_0^\infty(\Omega)$.

Definition 2.26. We say that the L^p Dirichlet problem with boundary data in $L^p(\partial\Omega, d\sigma)$ is solvable if the unique solution u to (1-1) for any continuous boundary data f decaying to 0 as $t \rightarrow \pm\infty$ satisfies the nontangential maximum function estimate

$$\|N(u)\|_{L^p(\partial\Omega, d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega, d\sigma)}, \tag{2-45}$$

with the implied constant depending only on the operator, n , p and Ω .

Remark 2.27. Since the space $C_0(\Omega)$ is dense in $L^p(\Omega)$ for $p < \infty$ it follows that the solution operator $f \mapsto u$ has a unique extension onto the whole space $L^p(\Omega)$ with the bound (2-45) being satisfied for such u . Hence we can assign to every boundary datum $f \in L^p(\Omega)$ a unique solution u such that (2-45) holds.

3. Basic results and interior estimates

In this section, we now recall some foundational estimates that will be used later. First, we state interior estimates of a weak solution of the parabolic operator

$$u_t = \operatorname{div}(A\nabla u) + B \cdot \nabla u. \tag{3-1}$$

Definition 3.1 [Aronson 1968]. We say that u is a *weak solution* to a parabolic operator of the form (3-1) in Ω if $u, \nabla u \in L^2_{\text{loc}}(\Omega)$, $\sup_t \|u(\cdot, t)\|_{L^2_{\text{loc}}(\Omega_t)} < \infty$ and

$$\int_{\Omega} (-u\phi_t + A\nabla u \cdot \nabla\phi - \phi B \cdot \nabla u) \, dX \, dt = 0$$

for all $\phi \in C_0^\infty(\Omega)$.

Lemma 3.2 (a Caccioppoli inequality, see [Aronson 1968]). *Let A and B satisfy (1-2) and (2-35) and suppose that u is a weak solution of (3-1) in $Q_{4r}(X, t)$ with $0 < r < \delta(X, t)/8$. Then there exists a constant $C = C(\lambda, \Lambda, n)$ such that*

$$\begin{aligned} r^n \left(\sup_{Q_{r/2}(X, t)} u \right)^2 &\leq C \sup_{t-r^2 \leq s \leq t+r^2} \int_{Q_r(X, t) \cap \{t=s\}} u^2(Y, s) \, dY + C \int_{Q_r(X, t)} |\nabla u|^2 \, dY \, ds \\ &\leq \frac{C^2}{r^2} \int_{Q_{2r}(X, t)} u^2(Y, s) \, dY \, ds. \end{aligned}$$

Lemmas 3.4 and 3.5 in [Hofmann and Lewis 2001] give the following estimates for weak solutions of (3-1).

Lemma 3.3 (interior Hölder continuity). *Let A and B satisfy (1-2) and (2-35) and suppose that u is a weak solution of (3-1) in $Q_{4r}(X, t)$ with $0 < r < \delta(X, t)/8$. Then for any $(Y, s), (Z, \tau) \in Q_{2r}(X, t)$*

$$|u(Y, s) - u(Z, \tau)| \leq C \left(\frac{|Y - Z| + |s - \tau|^{1/2}}{r} \right)^\alpha \sup_{Q_{4r}(X, t)} |u|,$$

where $C = C(\lambda, \Lambda, n)$, $\alpha = \alpha(\lambda, \Lambda, n)$, and $0 < \alpha < 1$.

Lemma 3.4 (Harnack inequality). *Let A and B satisfy (1-2) and (2-35) and suppose that u is a weak nonnegative solution of (3-1) in $Q_{4r}(X, t)$, with $0 < r < \delta(X, t)/8$. Suppose that $(Y, s), (Z, \tau) \in Q_{2r}(X, t)$. Then there exists $C = C(\lambda, \Lambda, n)$ such that, for $\tau < s$,*

$$u(Z, \tau) \leq u(Y, s) \exp \left[C \left(\frac{|Y - Z|^2}{|s - \tau|} + 1 \right) \right].$$

We state a version of the maximum principle from [Dindoš and Hwang 2018] that is a modification of [Hofmann and Lewis 2001, Lemma 3.38].

Lemma 3.5 (maximum principle). *Let A and B satisfy (1-2) and (2-35), and let u and v be bounded continuous weak solutions to (3-1) in Ω . If $|u|, |v| \rightarrow 0$ uniformly as $t \rightarrow -\infty$ and*

$$\limsup_{(Y,s) \rightarrow (X,t)} (u - v)(Y, s) \leq 0$$

for all $(X, t) \in \partial\Omega$, then $u \leq v$ in Ω .

Remark 3.6 [Dindoš and Hwang 2018]. The proof of Lemma 3.38 from [Hofmann and Lewis 2001] works given the assumption that $|u|, |v| \rightarrow 0$ uniformly as $t \rightarrow -\infty$. Even with this additional assumption, the lemma as stated is sufficient for our purposes. We shall mostly use it when $u \leq v$ on the boundary of $\Omega \cap \{t \geq \tau\}$ for a given time τ . Obviously then the assumption that $|u|, |v| \rightarrow 0$ uniformly as $t \rightarrow -\infty$ is not necessary. Another case when the lemma as stated here applies is when $u|_{\partial\Omega}, v|_{\partial\Omega} \in C_0(\partial\Omega)$, where $C_0(\partial\Omega)$ denotes the class of continuous functions decaying to zero as $t \rightarrow \pm\infty$. This class is dense in any $L^p(\partial\Omega, d\sigma)$, $1 < p < \infty$, allowing us to consider an extension of the solution operator from $C_0(\partial\Omega)$ to L^p .

The following result is from [Dindoš and Hwang 2018], which was adapted from the elliptic result in [Dindoš 2002].

Lemma 3.7. *Let $r > 0$ and $0 < a < b$. Consider the nontangential maximal functions defined using two set of cones Γ_a^r and Γ_b^r . Then for any $p > 0$ there exists a constant $C_p > 0$ such that for all $u : U \rightarrow \mathbb{R}$*

$$N_a^r(u) \leq N_b^r(u) \quad \text{and} \quad \|N_b^r(u)\|_{L^p(\partial U)} \leq C_p \|N_a^r(u)\|_{L^p(\partial U)}.$$

4. Improved regularity for p -adapted square function

Here we extend recent work of [Dindoš and Pipher 2019] for complex-coefficient elliptic equations to the real parabolic setting. The goal is to obtain an improved regularity result for weak solutions of (1-1) implying that $|\nabla u|^2 |u|^{p-2}$ belongs to $L^1_{\text{loc}}(\Omega)$ when $1 < p < 2$. Having this it follows that the p -adapted square function $S_{p,a}$ is well-defined at almost every boundary point.

Theorem 4.1 (see [Dindoš and Pipher 2019, Theorem 1.1]). *Suppose $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to $\mathcal{L}u = u_t$, where $\mathcal{L}u = \text{div}(A\nabla u) + B\nabla u$, A is bounded and elliptic and B is locally bounded and satisfies*

$$\delta(X, t) |B(X, t)| \leq K \tag{4-1}$$

for some uniform constant $K > 0$. Then for any parabolic ball $B_{4r}(X, t) \subset \Omega$ and $p, q \in (1, \infty)$ we have the following improvement in regularity:

$$\left(\int_{B_r(X,t)} |u|^p \right)^{1/p} \leq C_\varepsilon \left(\int_{B_{2r}(X,t)} |u|^q \right)^{1/q} + \varepsilon \left(\int_{B_{2r}(X,t)} |u|^2 \right)^{1/2}. \tag{4-2}$$

Here the constant C_ε only depends on $p, q, \varepsilon, n, \lambda, \Lambda$, and K but not on $u, (X, t)$ or r . In addition, for all $1 < p < \infty$

$$r^2 \int_{B_r(X,t)} |\nabla u|^2 |u|^{p-2} \leq C_\varepsilon \int_{B_{2r}(X,t)} |u|^p + \varepsilon \left(\int_{B_{2r}(X,t)} |u|^2 \right)^{p/2}, \tag{4-3}$$

where again the constant only depends on ε, p, n , the ellipticity constants of A , and K . This also shows that $|u|^{(p-2)/2} \nabla u \in L^2_{\text{loc}}(\Omega)$.

Remark 4.2. If $q \geq 2$ in (4-2) or if $p \geq 2$ in (4-3) then one can take $\varepsilon = 0$ because the L^2 averages can be controlled by the first term on the right-hand side of these inequalities.

We focus only on the case $1 < p < 2$ as the $p \geq 2$ result above follows from the Caccioppoli inequality, Lemma 3.2. We shall establish the following lemma for the $1 < p < 2$ case, which concludes the proof of Theorem 4.1.

Lemma 4.3 (see [Dindoš and Pipher 2019, Lemma 2.7]). *Let u be a weak solution to $\mathcal{L}u = u_t$ in Ω for A elliptic and bounded, and B bounded satisfying (4-1). Then for any $p < 2$, any ball $B_r(X, t)$ with $r < \delta(X, t)/4$, and any $\varepsilon > 0$*

$$r^2 \int_{B_r(X,t)} |\nabla u|^2 |u|^{p-2} \leq C_\varepsilon \int_{B_{2r}(X,t)} |u|^p + \varepsilon \left(\int_{B_{2r}(X,t)} |u|^2 \right)^{p/2}, \tag{4-4}$$

$$\left(\int_{B_r(X,t)} |u|^2 \right)^{1/2} \leq C_\varepsilon \left(\int_{B_{2r}(X,t)} |u|^p \right)^{1/p} + \varepsilon \left(\int_{B_{2r}(X,t)} |u|^2 \right)^{1/2}, \tag{4-5}$$

where the constants only depend on $n, \varepsilon, \lambda, \Lambda$ and K . In particular, $|u|^{(p-2)/2} \nabla u \in L^2_{\text{loc}}(\Omega)$.

Proof. We start by assuming that A and B are smooth. Then the solution u to $\mathcal{L}u = u_t$ is smooth. We prove the above inequalities have constants that do not depend on the smoothness of A or B . It follows then that the smoothness assumption can be removed by a limiting argument; that is, A and B are approximated by sequences of smooth functions for which (4-4) and (4-5) hold uniformly. This is done in detail in the elliptic setting in [Dindoš and Pipher 2019, Lemma 2.7] and a similar argument in the parabolic case is shown in [Hofmann and Lewis 2001]. We skip further details as the argument is fairly standard.

To simplify notation, we suppress the argument of the ball $B_r(X, t)$. Let

$$\rho_\delta(s) = \begin{cases} \delta^{(p-2)/2} & 0 \leq s \leq \delta, \\ s^{(p-2)/2} & s > \delta. \end{cases} \tag{4-6}$$

The choice of cut-off function ρ_δ in this proof is inspired by [Langer 1999, p. 311; Cialdea and Mazya 2005, p. 1088]. We multiply $\mathcal{L}u = u_t$ by $\rho_\delta^2(|u|)u$ and integrate by parts to obtain

$$\int_{B_r} \nabla(\rho_\delta^2(|u|)u)A \nabla u = \int_{B_r} \rho_\delta^2(|u|)uu_t + \int_{B_r} \rho_\delta^2(|u|)B \cdot \nabla u + \int_{\partial B_r} (\rho_\delta^2(|u|))\nu \cdot A \nabla u \, d\sigma(y, s), \tag{4-7}$$

where ν is the outer unit normal to B_r . Consider $E_\delta = \{u > \delta\}$. Then the left-hand side of (4-7) is

$$\int_{B_r} \nabla(\rho_\delta^2(|u|)u)A\nabla u = \delta^{p-2} \int_{B_r \setminus E_\delta} \nabla u \cdot A\nabla u + \int_{B_r \cap E_\delta} A\nabla u \cdot \nabla(|u|^{p-2}u) \tag{4-8}$$

and by the ellipticity of A on the open set $B_r \cap E_\delta$ we have for some $\lambda' > 0$

$$\lambda' \int_{B_r \cap E_\delta} |u|^{p-2}|\nabla u|^2 \leq \int_{B_r \cap E_\delta} A\nabla u \cdot \nabla(|u|^{p-2}u). \tag{4-9}$$

Our strategy is to let $\delta \rightarrow 0$ and show all the integrals involving $B_r \setminus E_\delta$ tend to 0.

First, we use the following result from [Langer 1999]. They proved if $u \in C^2(\bar{B}_r)$ and $u = 0$ on ∂B_r then for $q > -1$

$$\lim_{\delta \rightarrow 0} \delta^q \int_{B_r \setminus E_\delta} |\nabla u|^2 = 0. \tag{4-10}$$

To deal with the boundary integral in (4-7) we note that (4-7) to (4-9) remain valid for any enlarged ball $B_{\alpha r}$ for $1 \leq \alpha \leq \frac{5}{4}$. We write (4-7) for every $B_{\alpha r}$ and then average in α over the interval $[1, \frac{5}{4}]$. The last term in (4-7) then turns into a solid integral over $B_{5r/4} \setminus B_r$. Therefore,

$$\begin{aligned} & \lambda' \int_{B_r \cap E_\delta} |u|^{p-2}|\nabla u|^2 \\ & \leq \sup_{\alpha \in [1, 5/4]} \left| \int_{B_{\alpha r}} \rho_\delta^2(|u|)uu_t \right| + \sup_{\alpha \in [1, 5/4]} \left| \int_{B_{\alpha r}} \rho_\delta^2(|u|)uB \cdot \nabla u \right| + \left| r^{-1} \int_{B_{5\alpha r/4} \setminus B_r} \rho_\delta^2(|u|)u\nu \cdot A\nabla u \right| + o(1) \\ & = I + II + III + o(1), \end{aligned}$$

where $o(1)$ contains the integral over $B_{\alpha r} \setminus E_\delta$, which tends to 0 as $\delta \rightarrow 0$. We bound II and III as [Dindoš and Pipher 2019]

$$II + III \leq C_\varepsilon r^{-2} \int_{B_{5r/4}} |u|^p + \varepsilon r^{p-2} \int_{B_{5r/4}} |\nabla u|^p + o(1).$$

Now we turn to I and use the same idea as the proof of (4-10) in [Langer 1999, (3.3)] to show I converges as expected. By splitting the integral with the set E_δ , using the fact $\delta^{p-2} \leq |u|^{p-2}$ on $B_{\alpha r} \setminus E_\delta$ (since $p < 2$), and the smoothness of u , which implies $|u|^{p-2}uu_t \in L^1(B_{\alpha r})$, we obtain

$$\begin{aligned} \int_{B_{\alpha r}} \rho_\delta^2(|u|)uu_t &= \int_{B_{\alpha r} \cap E_\delta} |u|^{p-2}uu_t + \delta^{p-2} \int_{B_{\alpha r} \setminus E_\delta} uu_t \\ &\leq \int_{B_{\alpha r} \cap E_\delta} |u|^{p-2}uu_t + \int_{B_{\alpha r} \setminus E_\delta} |u|^{p-2}uu_t \\ &\leq \int_{B_{\alpha r}} |u|^{p-1}|u_t| < \infty. \end{aligned}$$

Therefore by the dominated convergence theorem

$$\int_{B_{\alpha r}} \rho_\delta^2(|u|)uu_t \rightarrow \int_{B_{\alpha r}} |u|^{p-2}uu_t. \tag{4-11}$$

We change from working with balls to integrating over parabolic cubes $Q_{\alpha r}$ and denote by $Q_{\alpha r}|_s$ the cube $Q_{\alpha r}$ restricted to the hypersurface $\{t = s\}$. Using the fundamental theorem of calculus, we obtain in the limit that

$$\begin{aligned} \int_{B_{\alpha r}} |u|^{p-2} u u_t &\sim \int_{B_{\alpha r}} \frac{\partial}{\partial t} (|u|^p) dt dX \\ &\leq \int_{Q_{\alpha r}} \frac{\partial}{\partial t} (|u|^p) dt dX = \int_{t_0 - (\alpha r)^2}^{t_0 + (\alpha r)^2} \frac{d}{dt} \int_{Q_{\alpha r}|_s} |u|^p dX ds \\ &\leq \|u\|_{L^p_X(Q_{\alpha r}|_{t_0 + (\alpha r)^2})}^p + \|u\|_{L^p_X(Q_{\alpha r}|_{t_0 - (\alpha r)^2})}^p. \end{aligned} \tag{4-12}$$

Observe that (4-12) holds for all time-restricted cubes $Q_{\alpha r}|_{t_0 \pm (\alpha r)^2}$ with $\alpha \in [1, 1.1]$. Once again we average over these cubes to show

$$\int_{B_{\alpha r}} |u|^{p-2} u u_t \lesssim \frac{1}{r^2} \int_{Q_{1.1\alpha r}} |u|^p dX dt.$$

Since $Q_{1.1\alpha r} \subset B_{2r}$, in the limit as $\delta \rightarrow 0$

$$I \lesssim \frac{1}{r^2} \int_{B_{2r}} |u|^p dX dt.$$

Therefore grouping the estimates we have the bound

$$\lambda' \int_{B_r \cap E_\delta} |u|^{p-2} |\nabla u|^2 \lesssim C_\varepsilon r^{-2} \int_{B_{2r}} |u|^p + \varepsilon r^{p-2} \int_{B_{5r/4}} |\nabla u|^p + o(1). \tag{4-13}$$

We let $\delta \rightarrow 0$ and proceed as [Dindoš and Pipher 2019] to obtain (4-4) and (4-5) for smooth A and B . Finally, since no constants depend on the smoothness of A or B , we can remove the smoothness assumption by the same argument as in [Hofmann and Lewis 2001]. We suppose A is just elliptic and bounded, and B satisfies (4-1). Then we approximate A and B by smooth matrices and vectors respectively. For each smooth approximation, we have (4-4) and (4-5) and then passing to the limit we obtain analogous estimates for $W_{loc}^{1,2}$ solutions u of $\mathcal{L}u = u_t$, with the constants having the same dependence as before. \square

It follows that the p -adapted square function $S_{p,a}$ is well-defined. The paper [Dindoš and Hwang 2018] also considered an area function and established in its Lemma 5.2 that the usual square function can control this area function. The case $1 < p < 2$ is significantly more complicated so for this reason we focus only on nonnegative solutions u .

We fix a boundary point $(Y, s) \in \partial\Omega$ and consider $A_{p,a}(Y, s)$. Clearly, the nontangential cone $\Gamma_a(Y, s)$ can be covered by a nonoverlapping collection of Whitney cubes $\{Q_i\}$ with the properties:

$$\Gamma_a(Y, s) \subset \bigcup_i Q_i \subset \Gamma_{2a}(Y, s), \quad r_i := \text{diam}(Q_i) \sim \text{dist}(Q_i, \partial\Omega), \quad 4Q_i \subset \Omega, \tag{4-14}$$

and the cubes $\{2Q_i\}$ having only finite overlap. It follows that

$$\begin{aligned} [A_{p,a}(Y, s)]^p &\lesssim \sum_i (r_i)^{2-n} \int_{Q_i} |u_t|^2 u^{p-2} dX dt \\ &\lesssim \sum_i (r_i)^{2-n} \int_{Q_i} |\nabla^2 u|^2 u^{p-2} + (|\nabla A|^2 + |B|^2) |\nabla u|^2 u^{p-2} dX dt. \end{aligned} \tag{4-15}$$

We need the following estimate on each Q_i .

Lemma 4.4. *Assume the ellipticity condition (1-2) and that the coefficients A and B of (1-1) satisfy the conditions*

$$|\nabla A(X, t)| \leq K/\delta(X, t) \quad \text{and} \quad |B(X, t)| \leq K/\delta(X, t),$$

for some uniform constant $K > 0$. Then for all nonnegative solutions u of (1-1) and any parabolic cube Q such that $4Q \subset \Omega$ we have the estimate

$$\int_Q |\nabla^2 u|^2 u^{p-2} \, dX \, dt \lesssim r^{-2} \int_{2Q} |\nabla u|^2 u^{p-2} \, dX \, dt, \quad (4-16)$$

where $r = \text{diam}(Q)$.

Proof. Since we assume differentiability of the matrix A in the spatial variables, we may also assume that A is symmetric. Let us set $W = (w_k)$, where $w_k = \partial_k u$ for $k = 0, 1, \dots, n-1$. Differentiating (1-1) we obtain the following PDE for each w_k :

$$(w_k)_t - \text{div}(A \nabla w_k) = \text{div}((\partial_k A)W) + \partial_k(B \cdot W). \quad (4-17)$$

We multiply (4-17) by $w_k u^{p-2} \zeta^2$, integrate over $2Q$ and integrate by parts. Here $0 \leq \zeta \leq 1$ is a smooth cut-off function equal to 1 on Q , vanishing outside $2Q$ and satisfying $r|\nabla \zeta| + r^2|\zeta_t| \leq C$ for some $C > 0$ independent of Q . This gives

$$\begin{aligned} & \int_{2Q} (w_k)_t w_k u^{p-2} \zeta^2 \, dX \, dt + \int_{2Q} a_{ij} (\partial_j w_k) \partial_i (w_k u^{p-2} \zeta^2) \, dX \, dt \\ &= - \int_{2Q} (\partial_k a_{ij}) w_j \partial_i (w_k u^{p-2} \zeta^2) \, dX \, dt - \int_{2Q} b_i w_i \partial_k (w_k u^{p-2} \zeta^2) \, dX \, dt. \end{aligned} \quad (4-18)$$

We rearrange and group similar terms together:

$$\begin{aligned} & \frac{1}{2} \int_{2Q} [(w_k u^{p/2-1} \zeta)_t]^2 \, dX \, dt - \frac{p-2}{2} \int_{2Q} w_k^2 u^{p-3} u_t \zeta^2 \, dX \, dt \\ &+ \int_{2Q} A(\nabla(w_k \zeta) u^{p/2-1}) \cdot (\nabla(w_k \zeta) u^{p/2-1}) \, dX \, dt \\ &+ (p-2) \int_{2Q} A(\nabla(w_k \zeta) u^{p/2-1}) \cdot ((\nabla u) w_k u^{p/2-2} \zeta) \, dX \, dt \\ &= \int_{2Q} |w_k|^2 u^{p-2} \zeta \zeta_t \, dX \, dt + \int_{2Q} |w_k|^2 u^{p-2} A \nabla \zeta \cdot \nabla \zeta \, dX \, dt - \int_{2Q} b_i w_i \partial_k (w_k \zeta) u^{p-2} \zeta \, dX \, dt \\ &- (p-2) \int_{2Q} b_i w_i ((\partial_k u) w_k u^{p/2-2} \zeta) u^{p/2-1} \zeta \, dX \, dt \\ &- \int_{2Q} b_i w_i w_k u^{p-2} \zeta \zeta_k \, dX \, dt - \int_{2Q} (\partial_k a_{ij}) w_j w_k u^{p-2} \zeta \zeta_i \, dX \, dt \\ &- \int_{2Q} (\partial_k a_{ij}) w_j (\partial_i w_k \zeta) u^{p-2} \zeta \, dX \, dt \\ &- (p-2) \int_{2Q} (\partial_k a_{ij}) w_j ((\partial_i u) w_k u^{p/2-2} \zeta) u^{p/2-1} \zeta \, dX \, dt. \end{aligned} \quad (4-19)$$

All the terms after the equal sign are “error” terms since they either contain a derivative of ζ , or coefficients ∇A or B . These will be handled using the Cauchy–Schwarz inequality and the estimates for $|\nabla A|, |B| \leq K/r$. The four main terms are on the left-hand side of (4-19). The term that needs further work is the second term, and we use the PDE (1-1) for u_t . This gives

$$\begin{aligned}
 & -\frac{p-2}{2} \int_{2Q} w_k^2 u^{p-3} u_t \zeta^2 \, dX \, dt \\
 & = -\frac{p-2}{2} \int_{2Q} w_k^2 u^{p-3} \operatorname{div}(A \nabla u) \zeta^2 \, dX \, dt - \frac{p-2}{2} \int_{2Q} w_k^2 u^{p-3} B \cdot W \zeta^2 \, dX \, dt. \tag{4-20}
 \end{aligned}$$

Again the second term will be an “error” term. For the first term, we observe the equality

$$u^{p-3} \operatorname{div}(A \nabla u) = \operatorname{div}(A(\nabla u) u^{p-3}) - (p-3)A((\nabla u) u^{p/2-2}) \cdot ((\nabla u) u^{p/2-2}).$$

It follows (by integrating by parts) that

$$\begin{aligned}
 & -\frac{p-2}{2} \int_{2Q} w_k^2 u^{p-3} \operatorname{div}(A \nabla u) \zeta^2 \, dX \, dt \\
 & = (p-2) \int_{2Q} A(\nabla(w_k \zeta) u^{p/2-1}) \cdot ((\nabla u) w_k u^{p/2-2} \zeta) \, dX \, dt \\
 & \quad + \frac{(2-p)(3-p)}{2} \int_{2Q} A((\nabla u) w_k u^{p/2-2} \zeta) \cdot ((\nabla u) w_k u^{p/2-2} \zeta) \, dX \, dt. \tag{4-21}
 \end{aligned}$$

We now group all main terms together; these are the first, second and fourth terms on the left-hand side of (4-19) and the terms of (4-21). This gives

$$\begin{aligned}
 \text{LHS of (4-19)} & = \frac{1}{2} \int_{2Q} [(w_k u^{p/2-1} \zeta)^2]_t \, dX \, dt \\
 & \quad + \int_{2Q} A(\nabla(w_k \zeta) u^{p/2-1}) \cdot (\nabla(w_k \zeta) u^{p/2-1}) \, dX \, dt \\
 & \quad + 2(p-2) \int_{2Q} A(\nabla(w_k \zeta) u^{p/2-1}) \cdot ((\nabla u) w_k u^{p/2-2} \zeta) \, dX \, dt \\
 & \quad + \frac{(2-p)(3-p)}{2} \int_{2Q} A((\nabla u) w_k u^{p/2-2} \zeta) \cdot ((\nabla u) w_k u^{p/2-2} \zeta) \, dX \, dt \\
 & = \frac{1}{2} \int_{2Q} [(w_k u^{p/2-1} \zeta)^2]_t \, dX \, dt \\
 & \quad + \left(1 - \frac{2(2-p)}{3-p}\right) \int_{2Q} A(\nabla(w_k \zeta) u^{p/2-1}) \cdot (\nabla(w_k \zeta) u^{p/2-1}) \, dX \, dt \\
 & \quad + \int_{2Q} A\left(\sqrt{\frac{2(2-p)}{3-p}} [\nabla(w_k \zeta) u^{p/2-1}] - \sqrt{\frac{(2-p)(3-p)}{2}} [(\nabla u) w_k u^{p/2-2} \zeta]\right) \\
 & \quad \cdot \left(\sqrt{\frac{2(2-p)}{3-p}} [\nabla(w_k \zeta) u^{p/2-1}] - \sqrt{\frac{(2-p)(3-p)}{2}} [(\nabla u) w_k u^{p/2-2} \zeta]\right) \, dX \, dt \\
 & \geq \frac{1}{2} \int_{2Q} [(w_k u^{p/2-1} \zeta)^2]_t \, dX \, dt + \frac{(p-1)\lambda}{3-p} \int_{2Q} |\nabla(w_k \zeta) u^{p/2-1}|^2 \, dX \, dt. \tag{4-22}
 \end{aligned}$$

Here we have first completed the square (using the symmetry of A), and then used the ellipticity of the matrix A . The important point is that for all $1 < p < 2$ the coefficient $(p - 1)\lambda/(3 - p)$ is positive.

We also we could have completed the square differently and, instead of (4-22), obtained the estimate

$$\text{LHS of (4-19)} \geq \frac{1}{2} \int_{2Q} [(w_k u^{p/2-1} \zeta)^2]_t \, dX \, dt + \frac{(p-1)(2-p)\lambda}{2} \int_{2Q} |(\nabla u) w_k u^{p/2-2} \zeta|^2 \, dX \, dt. \quad (4-23)$$

It follows that we could average (4-22) and (4-23) and have both

$$\int_{2Q} |\nabla(w_k \zeta) u^{p/2-1}|^2 \, dX \, dt \quad \text{and} \quad \int_{2Q} |(\nabla u) w_k u^{p/2-2} \zeta|^2 \, dX \, dt$$

in the estimate with small positive constants.

Now we briefly mention how all the error terms of (4-19), (4-20) and (4-22) can be handled. Some can be immediately estimated from above by

$$r^{-2} \int_{2Q} |W|^2 u^{p-2} \, dX \, dt,$$

where the scaling factor r^{-2} comes from the estimates on $\nabla \zeta$, ζ_t , $|\nabla A|$ and $|B|$. For other terms (for example the third term of fourth line of (4-19) or the term on the fifth line) we use Cauchy–Schwarz. One of the terms in the product will be

$$\left(r^{-2} \int_{2Q} |W|^2 u^{p-2} \, dX \, dt \right)^{1/2},$$

while the other term is one of

$$\left(\int_{2Q} |\nabla(w_k \zeta) u^{p/2-1}|^2 \, dX \, dt \right)^{1/2} \quad \text{or} \quad \left(\int_{2Q} |(\nabla u) w_k u^{p/2-2} \zeta|^2 \, dX \, dt \right)^{1/2}.$$

It follows using the ε -Cauchy–Schwarz inequality that we can hide these on the left-hand side of (4-19). Finally, we put everything together by summing over all k and recalling that $W = \nabla u$. This gives for some constant $\varepsilon = \varepsilon(p, \lambda, n) > 0$ with $\varepsilon \rightarrow 0$ as $p \rightarrow 1$,

$$\sup_{\tau} \int_{Q \cap \{t=\tau\}} |\nabla u|^2 u^{p-2} \, dX + \varepsilon \int_Q |\nabla^2 u|^2 u^{p-2} \, dX \, dt + \varepsilon \int_Q |\nabla u|^4 u^{p-4} \, dX \, dt \leq Cr^{-2} \int_{2Q} |\nabla u|^2 u^{p-2} \, dX \, dt. \quad (4-24)$$

In particular (4-16) holds. □

After using (4-16) in (4-15) we can conclude the following.

Lemma 4.5. *Let u be a nonnegative solution of (1-1) with matrix A satisfying the ellipticity hypothesis and the coefficients satisfying the bound $|\nabla A|, |B| \leq K/\delta$. Then given $a > 0$ there exists a constant $C = C(\Lambda, \lambda, a, K, p, n)$ such that*

$$A_{p,a}(u)(X, t) \leq CS_{p,2a}(u)(X, t). \quad (4-25)$$

From this we have the global estimate

$$\|A_{p,a}(u)\|_{L^p(\partial\Omega)}^p \leq C_2 \|S_{p,a}(u)\|_{L^p(\partial\Omega)}^p. \quad (4-26)$$

As far as the proof goes, the calculations above clearly work for solutions u with a uniform bound $u \geq \varepsilon > 0$. Hence considering $v_\varepsilon = u + \varepsilon$ and then taking the limit $\varepsilon \rightarrow 0+$, using Fatou’s lemma yields (4-25) for all nonnegative u , where we have used the convention that $|\nabla u|^2 u^{p-2} = 0$ whenever $u = 0$ and $\nabla u = 0$ with a similar convention for the second gradient in $A_{p,a}$.

5. Bounding the p -adapted square function by the nontangential maximum function

We slightly abuse notation and only work on a Carleson region $T(\Delta_r)$ in the upper half-space U even though we formulate the following lemmas on any admissible domain Ω . The equivalence of these formulations via the pullback map ρ is discussed in Section 2C and [Dindoš and Hwang 2018], and hence we omit the details. We start with a local bound of the p -adapted square function by the nontangential maximal function.

Lemma 5.1. *Let Ω be an admissible domain from Definition 2.10 with character (ℓ, η, N, d) . Let $1 < p < 2$ and u be a nonnegative solution of (1-1), with the Carleson conditions (1-7) and (1-8) on the coefficients A and B . Then there exists a constant $C = C(\lambda, \Lambda, N, C_0)$ such that for any solution u with boundary data f on any ball $\Delta_r \subset \partial\Omega$ with $r \leq \min\{d/4, d/(4C_0)\}$ we have*

$$\int_{T(\Delta_r)} |\nabla u|^2 |u|^{p-2} x_0 \, dx_0 \, dx \, dt \leq C(1 + \|\mu\|)(1 + \ell^2) \int_{\Delta_{2r}} (N^{2r}(u))^2 \, dx \, dt. \tag{5-1}$$

In addition, we have the following global result.

Lemma 5.2. *Let Ω be an admissible domain with smooth boundary $\partial\Omega$. Let $1 < p < 2$ and u be a weak nonnegative solution of (1-1) satisfying (2-34), (2-35), (2-37) and (2-38) with Dirichlet boundary data $f \in L^p(\partial\Omega)$. Then there exist positive constants C_1 and C_2 independent of u such that for small $r_0 > 0$ we have*

$$\begin{aligned} & \frac{C_1}{2} \int_0^{r_0/2} \int_{\partial\Omega} |\nabla u|^2 |u|^{p-2} x_0 \, dx \, dt \, dx_0 + \frac{2}{r_0} \int_0^{r_0} \int_{\partial\Omega} u^p(x_0, x, t) \, dx \, dt \, dx_0 \\ & \leq \int_{\partial\Omega} u^p(r_0, x, t) \, dx \, dt + \int_{\partial\Omega} u^p(0, x, t) \, dx \, dt \\ & \quad + C_2(\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}) \int_{\partial\Omega} (N^{2r}(u))^p \, dx \, dt. \end{aligned} \tag{5-2}$$

Proof of Lemmas 5.1 and 5.2. Let $Q_r(y, s)$ be a parabolic cube on the boundary with $r < d$ and let ζ be a smooth cut-off function independent of the x_0 -variable. As long as there is no ambiguity we suppress the argument of Q_r and extensively use the Einstein summation convention. Let ζ be supported in Q_{2r} , equal 1 in Q_r and satisfy the estimate $r|\nabla\zeta| + r^2|\zeta_t| \leq C$ for some constant C .

We start by estimating

$$\int_0^r \int_{Q_{2r}} |u|^{p-2} \frac{a_{ij}}{a_{00}} (\partial_i u)(\partial_j u) \zeta^2 x_0 \, dx \, dt \, dx_0, \tag{5-3}$$

where by ellipticity we have

$$\frac{\lambda}{\Lambda} \int_0^r \int_{Q_r} |\nabla u|^2 |u|^{p-2} x_0 \, dx \, dt \, dx_0 \leq \int_0^r \int_{Q_{2r}} |u|^{p-2} \frac{a_{ij}}{a_{00}} (\partial_i u)(\partial_j u) \zeta^2 x_0 \, dx \, dt \, dx_0.$$

Now we integrate by parts whilst noting that $\nu = (1, 0, 0, \dots, 0)$ since the domain is $\{x_0 > 0\}$:

$$\begin{aligned}
 & \int_0^r \int_{Q_{2r}} |u|^{p-2} \frac{a_{ij}}{a_{00}} (\partial_i u) (\partial_j u) \zeta^2 x_0 \, dx \, dt \, dx_0 \\
 &= \frac{1}{p} \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_j (|u(r, x, t)|^p) r \zeta^2 \, dx \, dt - \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_i (a_{ij} \partial_j u) \zeta^2 x_0 \, dx \, dt \, dx_0 \\
 &\quad - \int_0^r \int_{Q_{2r}} \partial_i \left(\frac{1}{a_{00}} \right) |u|^{p-2} u a_{ij} \partial_j u \zeta^2 x_0 \, dx \, dt \, dx_0 - 2 \int_0^r \int_{Q_{2r}} \frac{a_{ij}}{a_{00}} |u|^{p-2} u (\partial_j u) \zeta \partial_i \zeta x_0 \, dx \, dt \, dx_0 \\
 &\quad - \int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} |u|^{p-2} u (\partial_j u) \zeta^2 \, dx \, dt \, dx_0 - \int_0^r \int_{Q_{2r}} \frac{a_{ij}}{a_{00}} \partial_i (|u|^{p-2}) u (\partial_j u) \zeta^2 \, dx \, dt \, dx_0 \\
 &= I + II + III + IV + V + VI. \tag{5-4}
 \end{aligned}$$

Our strategy is to further estimate all these terms and then group similar terms together. First consider II ; we use that u is a solution to (1-1):

$$II = - \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u u_t \zeta^2 x_0 \, dx \, dt \, dx_0 + \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u b_i \partial_i u \zeta^2 x_0 \, dx \, dt \, dx_0 = II_1 + II_2.$$

Using the identity $2x_0 = \partial_0 x_0^2$ we integrate by parts in x_0 to obtain

$$\begin{aligned}
 II_1 &= -\frac{1}{2} \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u u_t \zeta^2 \partial_0 x_0^2 \, dx \, dt \, dx_0 \\
 &= -\frac{1}{2} \int_{Q_{2r}} \frac{1}{a_{00}} |u(r, x, t)|^{p-2} u(r, x, t) u_t(r, x, t) \zeta^2 r^2 \, dx \, dt + \frac{1}{2} \int_0^r \int_{Q_{2r}} \partial_0 \left(\frac{1}{a_{00}} \right) |u|^{p-2} u u_t \zeta^2 x_0^2 \, dx \, dt \, dx_0 \\
 &\quad + \frac{p-1}{2} \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} \partial_0 u u_t \zeta^2 x_0^2 \, dx \, dt \, dx_0 + \frac{1}{2} \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_0 \partial_t u \zeta^2 x_0^2 \, dx \, dt \, dx_0 \\
 &= II_{11} + II_{12} + II_{13} + II_{14}.
 \end{aligned}$$

Consider the boundary term II_{11} and we integrate by parts in t :

$$\begin{aligned}
 II_{11} &= -\frac{1}{4} \int_{Q_{2r}} \frac{1}{a_{00}} |u(r, x, t)|^{p-2} \partial_t (u^2(r, x, t)) \zeta^2 r^2 \, dx \, dt \\
 &= \frac{1}{4} \int_{Q_{2r}} \partial_t \left(\frac{1}{a_{00}} \right) |u(r, x, t)|^{p-2} u^2(r, x, t) \zeta^2 r^2 \, dx \, dt \\
 &\quad + \frac{1}{2} \int_{Q_{2r}} \frac{1}{a_{00}} |u(r, x, t)|^{p-2} u^2(r, x, t) \zeta \zeta_t r^2 \, dx \, dt \\
 &\quad + \frac{p-2}{4} \int_{Q_{2r}} \frac{1}{a_{00}} |u(r, x, t)|^{p-2} u(r, x, t) u_t(r, x, t) \zeta^2 r^2 \, dx \, dt \\
 &= II_{111} + II_{112} + II_{113}.
 \end{aligned}$$

Since $p < 2$, so $p - 2 < 0$, we can absorb II_{113} into II_{11} and save II_{12} to bound later on.

Considering II_{14} , we swap the order of differentiation on $\partial_0 \partial_t u$ and integrate by parts in t to show

$$\begin{aligned} II_{14} &= \frac{1}{2} \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_t \partial_0 u \zeta^2 x_0^2 \, dx \, dt \, dx_0 \\ &= -\frac{1}{2} \int_0^r \int_{Q_{2r}} \partial_t \left(\frac{1}{a_{00}} \right) |u|^{p-2} u \partial_0 u \zeta^2 x_0^2 \, dx \, dt \, dx_0 - \frac{p-1}{2} \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u_t \partial_0 u \zeta^2 x_0^2 \, dx \, dt \, dx_0 \\ &\quad - \int_0^r \int_{Q_{2r}} \frac{1}{a_{00}} |u|^{p-2} u \partial_0 u \zeta \zeta_t x_0^2 \, dx \, dt \, dx_0 \\ &= II_{141} + II_{142} + II_{143}. \end{aligned}$$

Observe that $II_{142} = -II_{13}$ so these terms cancel. We bound II_{141} by

$$\begin{aligned} II_{141} &= \frac{1}{2} \int_0^r \int_{Q_{2r}} \frac{\partial_t a_{00}}{a_{00}^2} |u|^{p-2} u \partial_0 u \zeta^2 x_0^2 \, dx \, dt \, dx_0 \\ &\lesssim \left(\int_0^r \int_{Q_{2r}} |A_t|^2 |u|^p x_0^3 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2}. \end{aligned}$$

Two parts of II_1 we have left to bound are II_{112} and II_{143} . Both of these integrals involve $\zeta \zeta_t$ and therefore if ζ is a partition of unity, when we sum over that partition these terms sum to 0.

The terms II_2 and III are simply dealt with by

$$\begin{aligned} II_2 &\lesssim \left(\int_0^r \int_{Q_{2r}} |B|^2 |u|^p x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2}, \\ III &\lesssim \left(\int_0^r \int_{Q_{2r}} |\nabla A|^2 |u|^p x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2}. \end{aligned}$$

The integral in the term IV contains the terms $\zeta \partial_i \zeta$ and as before if ζ is a partition of unity then after summing this term cancels out. Therefore the terms that we have yet to estimate are I , V , and VI .

We consider V in the two cases $j = 0$ and $j \neq 0$ separately. Since ζ is independent of x_0 by the fundamental theorem of calculus

$$\begin{aligned} V_{\{j=0\}} &= - \int_0^r \int_{Q_{2r}} |u|^{p-2} u (\partial_0 u) \zeta^2 \, dx \, dt \, dx_0 = -\frac{1}{p} \int_0^r \int_{Q_{2r}} \partial_0 (|u|^p \zeta^2) \, dx \, dt \, dx_0 \\ &= \frac{1}{p} \int_{Q_{2r}} |u(0, x, t)|^p \zeta^2 \, dx \, dt - \frac{1}{p} \int_{Q_{2r}} |u(r, x, t)|^p \zeta^2 \, dx \, dt. \end{aligned}$$

For the $j \neq 0$ case we use that $\partial_0 x_0 = 1$ and integrate this case by parts in x_0 :

$$\begin{aligned} V_{\{j \neq 0\}} &= -\frac{1}{p} \int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_j (|u|^p) \zeta^2 \, dx \, dt \, dx_0 \\ &= -\frac{1}{p} \int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_j (|u|^p) \zeta^2 \partial_0 x_0 \, dx \, dt \, dx_0 \\ &= -\frac{1}{p} \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_j (|u(r, x, t)|^p) \zeta^2 r \, dx \, dt + \frac{1}{p} \int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_j \partial_0 (|u|^p) \zeta^2 x_0 \, dx \, dt \, dx_0 \\ &\quad + \frac{1}{p} \int_0^r \int_{Q_{2r}} \partial_0 \left(\frac{a_{0j}}{a_{00}} \right) \partial_j (|u|^p) \zeta^2 x_0 \, dx \, dt \, dx_0 \\ &= V_1 + V_2 + V_3. \end{aligned}$$

Since $V_1 = -I_{\{j \neq 0\}}$, they cancel out. For V_2 we integrate by parts in x_j :

$$\begin{aligned} V_2 &= - \sum_{j \neq 0} \frac{1}{p} \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_0(|u(r, x, t)|^p) \zeta^2 r \, dx \, dt \\ &\quad - \frac{1}{p} \int_0^r \int_{Q_{2r}} \partial_j \left(\frac{a_{0j}}{a_{00}} \right) \partial_0(|u|^p) \zeta^2 x_0 \, dx \, dt \, dx_0 - \frac{2}{p} \int_0^r \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_0(|u|^p) \zeta \partial_j \zeta x_0 \, dx \, dt \, dx_0 \\ &= V_{21} + V_{22} + V_{23}. \end{aligned}$$

The terms V_{22} and V_3 are of the same type and can be estimated as *III* by

$$\begin{aligned} &\left| \int_0^r \int_{Q_{2r}} \nabla \left(\frac{a_{0j}}{a_{00}} \right) \nabla(|u|^p) \zeta^2 x_0 \, dx \, dt \, dx_0 \right| \\ &\quad \lesssim \int_0^r \int_{Q_{2r}} |u|^{p-1} |\nabla u| |\nabla A| \zeta^2 x_0 \, dx \, dt \, dx_0 \\ &\quad \lesssim \left(\int_0^r \int_{Q_{2r}} |\nabla A|^2 |u|^p \zeta^2 x_0 \, dx \, dt \, dx_0 \right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} \zeta^2 x_0 \, dx \, dt \, dx_0 \right)^{1/2}. \end{aligned}$$

The final term from (5-4) to estimate is *VI*:

$$VI = - \int_0^r \int_{Q_{2r}} \frac{a_{ij}}{a_{00}} \partial_i(|u|^{p-2}) u (\partial_j u) \zeta^2 \, dx \, dt \, dx_0 = (2-p) \int_0^r \int_{Q_{2r}} \frac{a_{ij}}{a_{00}} |u|^{p-2} (\partial_i u) (\partial_j u) \zeta^2 \, dx \, dt \, dx_0$$

and since $2 - p < 1$ we can hide *VI* in the left-hand side of (5-4).

We are now at the stage where we can group all the similar terms and estimate them. There are four different types of terms:

$$\begin{aligned} J_1 &= I_{\{j=0\}} + II_{111} + V_{\{j=0\}} + V_{21}, & J_2 &= II_{12}, \\ J_3 &= II_{141} + II_2 + III + \sum_{j \neq 0} V_{22} + \sum_{j \neq 0} V_3, & J_4 &= II_{112} + II_{143} + IV + \sum_{j \neq 0} V_{23}. \end{aligned}$$

We shall use the following standard result multiple times to deal with terms containing $|\nabla A|^2$, $|A_t|$ or $|B|$; a reference for this is [Stein 1993, p. 59]. Let μ be a Carleson measure and U the upper half-space. Then for any function u we have

$$\int_U |u|^p \, d\mu \leq \|\mu\|_C \|N(u)\|_{L^p(\mathbb{R}^n)}^p, \tag{5-5}$$

with a local version holding on Carleson boxes as well.

First we consider J_1 , which consists of boundary terms at $(0, x, t)$ and (r, x, t) :

$$\begin{aligned} J_1 &= \frac{1}{p} \int_{Q_{2r}} \partial_0(|u(r, x, t)|^p) \zeta^2 r \, dx \, dt - \frac{1}{4} \int_{Q_{2r}} \frac{\partial_t a_{00}}{a_{00}^2} |u(r, x, t)|^{p-2} u^2(r, x, t) \zeta^2 r^2 \, dx \, dt \\ &\quad + \frac{1}{p} \int_{Q_{2r}} |u(0, x, t)|^p \zeta^2 \, dx \, dt - \frac{1}{p} \int_{Q_{2r}} |u(r, x, t)|^p \zeta^2 \, dx \, dt \\ &\quad - \sum_{j \neq 0} \frac{1}{p} \int_{Q_{2r}} \frac{a_{0j}}{a_{00}} \partial_0(|u(r, x, t)|^p) \zeta^2 r \, dx \, dt. \end{aligned}$$

The second term in J_1 , originating from II_{111} , has the bound

$$\begin{aligned} II_{111} &= -\frac{1}{4} \int_{Q_{2r}} \frac{\partial_t a_{00}}{a_{00}^2} |u(r, x, t)|^{p-2} u^2(r, x, t) \zeta^2 r^2 \, dx \, dt \\ &\leq \frac{1}{4\lambda^2} \int_{Q_{2r}} |A_t| |u(r, x, t)|^p \zeta^2 r^2 \, dx \, dt \leq \frac{\|\mu_2\|_{C,2r}^{1/2}}{\lambda^2} \|N^r(u)\|_{L^p(Q_{2r})}^p. \end{aligned}$$

For the term J_2 , we have

$$\begin{aligned} J_2 &= \frac{1}{2} \int_0^r \int_{Q_{2r}} \partial_0 \left(\frac{1}{a_{00}} \right) |u|^{p-2} u u_t \zeta^2 x_0^2 \, dx \, dt \, dx_0 \\ &\leq \frac{1}{2\lambda^2} \left(\int_0^r \int_{Q_{2r}} |\nabla A|^2 |u|^p x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |u_t|^2 |u|^{p-2} x_0^3 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} \\ &\leq \frac{1}{\lambda^2} (\|\mu_2\|_{C,2r} \|N^r(u)\|_{L^p(Q_{2r})}^p)^{1/2} \left(\int_0^r \int_{Q_{2r}} |u_t|^2 |u|^{p-2} x_0^3 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2}. \end{aligned}$$

With a constant $C_3 = C_3(\lambda, \Lambda, n)$ we can bound J_3 by

$$\begin{aligned} J_3 &\leq C_3 \left(\int_0^r \int_{Q_{2r}} (x_0 |\nabla A|^2 + x_0 |B|^2 + x_0^3 |A_t|^2) |u|^p \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} \\ &\leq C_3 (\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r}) \|N^r(u)\|_{L^p(Q_{2r})}^p)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2}. \end{aligned}$$

Finally, J_4 consists of terms of the types $\zeta \partial_t \zeta$ and $\zeta \partial_i \zeta$. Later we take ζ to be a partition of unity and so when we sum up over the partition, all the terms in J_4 sum to 0.

Therefore after all these calculations

$$\begin{aligned} &\int_0^r \int_{Q_{2r}} |u|^{p-2} \frac{a_{ij}}{a_{00}} (\partial_i u)(\partial_j u) \zeta^2 x_0 \, dx \, dt \, dx_0 \\ &= J_1 + J_2 + J_3 + J_4 \\ &\leq \frac{n\Lambda}{\lambda} \int_{Q_{2r}} \partial_0 (|u(r, x, t)|^p) \zeta^2 r \, dx \, dt + \int_{Q_{2r}} |u(0, x, t)|^p \zeta^2 \, dx \, dt \\ &\quad - \int_{Q_{2r}} |u(r, x, t)|^p \zeta^2 \, dx \, dt + \frac{\|\mu_2\|_{C,2r}^{1/2}}{\lambda^2} \|N^r(u)\|_{L^p(Q_{2r})}^p \\ &\quad + \frac{1}{\lambda^2} (\|\mu_2\|_{C,2r} \|N^r(u)\|_{L^p(Q_{2r})}^p)^{1/2} \left(\int_0^r \int_{Q_{2r}} |u_t|^2 |u|^{p-2} x_0^3 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} \\ &\quad + C_3 (\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r}) \|N^r(u)\|_{L^p(Q_{2r})}^p)^{1/2} \left(\int_0^r \int_{Q_{2r}} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \, dx \, dt \, dx_0 \right)^{1/2} + J_4. \tag{5-6} \end{aligned}$$

By assuming that Ω is smooth as well as an admissible domain (see [Definition 2.10](#)) there exists a collar neighbourhood V of $\partial\Omega$ in \mathbb{R}^{n+1} such that $\Omega \cap V$ can be globally parametrised by $(0, r) \times \partial\Omega$ for some small $r > 0$; see [Remark 2.20](#) and [\[Dindoš and Hwang 2018\]](#) for details. Using [Definition 2.10](#), there is a collection of charts covering $\partial\Omega$ with bounded overlap, say by M . We consider a partition

of unity of these charts ζ_j , with ζ_j having the same definition, support and estimates as ζ before, and $\sum_j \zeta_j = 1$ everywhere. Therefore, when we sum (5-6) over this partition of unity the term on the left-hand side is bounded below by

$$\frac{1}{\Lambda} \int_0^r \int_{\partial\Omega} |u|^{p-2} (A \nabla u \cdot \nabla u)_{x_0} \, dx \, dt \, dx_0,$$

which is comparable to the truncated p -adapted square function $\|S_p^r(u)\|_{L^p(\partial\Omega)}^p$. Therefore, remembering that after summing $J_4 = 0$, for any $\varepsilon > 0$ we have

$$\begin{aligned} \frac{\lambda}{\Lambda} \|S_p^r(u)\|_{L^p(\partial\Omega)}^p &\sim \frac{\lambda}{\Lambda} \int_0^r \int_{\partial\Omega} |u|^{p-2} |\nabla u|^2 x_0 \, dx \, dt \, dx_0 \\ &\leq \frac{n\Lambda}{\lambda} \int_{\partial\Omega} \partial_0(|u(r, x, t)|^p) r \, dx \, dt + \int_{\partial\Omega} |u(0, x, t)|^p \, dx \, dt - \int_{\partial\Omega} |u(r, x, t)|^p \, dx \, dt \\ &\quad + \frac{M \|\mu_2\|_{C,2r}^{1/2}}{\lambda^2} \|N^r(u)\|_{L^p(\partial\Omega)}^p + \frac{\|\mu_2\|_{C,2r}}{4\varepsilon\lambda^2} \|N^r(u)\|_{L^p(\partial\Omega)}^p \\ &\quad + \varepsilon \int_0^r \int_{\partial\Omega} |u_t|^2 |u|^{p-2} x_0^3 \zeta^2 \, dx \, dt \, dx_0 + C_3 \frac{\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r}}{4\varepsilon} \|N^r(u)\|_{L^p(\partial\Omega)}^p \\ &\quad + \varepsilon \int_0^r \int_{\partial\Omega} |\nabla u|^2 |u|^{p-2} x_0 \zeta^2 \, dx \, dt \, dx_0. \end{aligned} \tag{5-7}$$

By applying Lemma 4.5 to the p -adapted area function in (5-7) we see that the p -adapted square function on the right-hand side of (5-7) is always multiplied by ε . By choosing ε small enough we can absorb this p -adapted square function into the left-hand side yielding

$$\begin{aligned} C_1 \|S_p^r(u)\|_{L^p(\partial\Omega)}^p &\leq \int_{\partial\Omega} \partial_0(|u(r, x, t)|^p) r \, dx \, dt + \int_{\partial\Omega} |u(0, x, t)|^p \, dx \, dt - \int_{\partial\Omega} |u(r, x, t)|^p \, dx \, dt \\ &\quad + C_2 (\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}) \|N^r(u)\|_{L^p(\partial\Omega)}^p. \end{aligned} \tag{5-8}$$

We integrate (5-8) in the r -variable, average over $[0, r_0]$ and use the identity $(\partial_0|u|^p)_{x_0} = \partial_0(|u|^p x_0) - |u|^p$ to give

$$\begin{aligned} C_1 \int_0^{r_0} \int_{\partial\Omega} \left(x_0 - \frac{x_0^2}{r_0}\right) |\nabla u|^2 |u|^{p-2} \, dx \, dt \, dx_0 &+ \frac{2}{r_0} \int_0^{r_0} \int_{\partial\Omega} |u(x_0, x, t)|^p \, dx \, dt \, dx_0 \\ &\leq \int_{\partial\Omega} |u(r_0, x, t)|^p \, dx \, dt + \int_{\partial\Omega} |u(0, x, t)|^p \, dx \, dt \\ &\quad + C_2 (\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}) \|N^r(u)\|_{L^p(\partial\Omega)}^p. \end{aligned} \tag{5-9}$$

Finally truncating the first integral on the left-hand side to $[0, r_0/2]$ gives

$$\begin{aligned} \frac{C_1}{2} \int_0^{r_0/2} \int_{\partial\Omega} |\nabla u|^2 |u|^{p-2} x_0 \, dx \, dt \, dx_0 &+ \frac{2}{r_0} \int_0^{r_0} \int_{\partial\Omega} |u(x_0, x, t)|^p \, dx \, dt \, dx_0 \\ &\leq \int_{\partial\Omega} |u(r_0, x, t)|^p \, dx \, dt + \int_{\partial\Omega} |u(0, x, t)|^p \, dx \, dt \\ &\quad + C_2 (\|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}) \|N^r(u)\|_{L^p(\partial\Omega)}^p. \end{aligned} \tag{5-10}$$

The local estimate for [Lemma 5.1](#) is obtained (exactly as in [\[Dindoš and Hwang 2018\]](#)) if we do not sum over all the coordinate patches but instead use the estimates derived for a single boundary cube Q_r in [\(5-6\)](#). □

We need to control the first integral on the right-hand side of [\(5-2\)](#) to achieve our goal of controlling the p -adapted square function. Thankfully this has already been done for us in the proof of [\[Dindoš and Hwang 2018, Corollary 5.3\]](#), which we encapsulate below.

Lemma 5.3. *Let Ω be as in [Lemma 5.2](#) and u be a nonnegative solution to [\(1-1\)](#). For a small $r_0 > 0$ depending on the geometry of the domain Ω , there exists a constant C such that for $\varepsilon = \|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}$*

$$\int_{\partial\Omega} u(r_0, x, t)^p \, dx \, dt \leq \frac{2}{r_0} \int_0^{r_0} \int_{\partial\Omega} u(x_0, x, t)^p \, dx \, dt \, dx_0 + C\varepsilon \|N^{r_0}(u)\|_{L^p(\partial\Omega)}^p.$$

Combining [Lemmas 5.2](#) and [5.3](#) gives us the desired result.

Corollary 5.4. *Let Ω be as in [Lemma 5.2](#) and u be a nonnegative solution to [\(1-1\)](#). For a small $r_0 > 0$ depending on the geometry of the domain Ω , there exist constants $C_1, C_2 > 0$ such that for $\varepsilon = \|\mu_1\|_{C,2r} + \|\mu_2\|_{C,2r} + \|\mu_2\|_{C,2r}^{1/2}$*

$$\begin{aligned} \|S_p^{r_0/2}(u)\|_{L^p(\partial\Omega)}^p &\sim \int_0^{r_0/2} \int_{\partial\Omega} |\nabla u|^2 |u|^{p-2} x_0 \, dx \, dt \, dx_0 \\ &\leq C_1 \int_{\partial\Omega} |u(0, x, t)|^p \, dx \, dt + C_2\varepsilon \|N^{r_0}(u)\|_{L^p(\partial\Omega)}^p. \end{aligned} \tag{5-11}$$

6. Bounding the nontangential maximum function by the p -adapted square function

Our goal in this section has been vastly simplified due to [\[Rivera-Noriega 2003\]](#) proving a local good- λ inequality. We use this to bound the nontangential maximum function by the p -adapted square function. We first bound the nontangential maximum function by the usual L^2 -based square function $S_2(u)$ but a simple argument from [\[Dindoš et al. 2007, \(3.41\)\]](#) shows that for $1 < p < 2$ and any $\varepsilon > 0$ we have

$$\|S_2^r(u)\|_{L^p(\partial\Omega)} \leq C_\varepsilon \|S_p^r(u)\|_{L^p(\partial\Omega)} + \varepsilon \|N^r(u)\|_{L^p(\partial\Omega)}, \tag{6-1}$$

with a local version of this statement holding as well.

The good- λ inequality from [\[Rivera-Noriega 2003, p. 508\]](#) is expressed in the following lemma.

Lemma 6.1. *Let v be a solution to [\(2-29\)](#) and $v(X, t) = 0$ for some point $(X, t) \in Q_r$. Let $E = \{(0, x, t) \in Q_r : S_{2,a}(v) \leq \lambda\}$ and $q > 2$. Then*

$$|\{(0, x, t) \in Q_r : N_a(v) > \lambda\}| \lesssim |\{(0, x, t) \in Q_r : S_{2,a}(v) > \lambda\}| + \frac{1}{\lambda^q} \int_E S_{2,a}(v)^q \, dx \, dt. \tag{6-2}$$

If $p \geq 2$ then the following lemma is immediate from [\[Dindoš and Hwang 2018, Lemma 6.1\]](#), which is an adaptation of [\[Rivera-Noriega 2003, Theorem 1.3 and Proposition 5.3\]](#).

Lemma 6.2. *Let v be a solution to (2-29) in U and the coefficients of (2-29) satisfy the Carleson estimates (2-34), (2-35), (2-37) and (2-38) on all parabolic balls of size $\leq r_0$. Then there exists a constant C such that for any $r \in (0, r_0/8)$*

$$\int_{Q_r} N_{a/12}(v)^p \, dx \, dt \leq C \left(\int_{Q_{2r}} A_{2,a}(v)^p \, dx \, dt + \int_{Q_{2r}} S_{2,a}(v)^p \, dx \, dt \right) + r^{n+1} |v(A_{\Delta_r})|^p, \tag{6-3}$$

where A_{Δ_r} is a corkscrew point of the boundary ball Δ_r . That is, a point $2r^2$ later in time than the centre of Δ_r and at a distance comparable to r from the boundary and r from the centre of the ball Δ_r .

Proof. We first assume that $v(X, t) = 0$ for some $(X, t) \in Q_r$ and then we have the good- λ inequality (6-2). The passage from this good- λ inequality to a local L^p estimate is standard in the spirit of [Fefferman and Stein 1972]. We remove the assumption $v(X, t) = 0$ for the cost of adding the $r^{n+1} |v(A_{\Delta_r})|^p$ term in the same way as [Rivera-Noriega 2003; Dindoš and Hwang 2018]. □

From this local estimate, we can obtain the following global L^p estimate by the same proof as the global L^2 estimate from [Dindoš and Hwang 2018, Theorem 6.3].

Theorem 6.3. *Let u be a solution to (1-1) and the coefficients of (1-1) satisfy the Carleson estimates (2-36) and (2-39) then*

$$\|N^r(u)\|_{L^p(\partial\Omega)} \lesssim \|S_2^r(u)\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\partial\Omega)} \tag{6-4}$$

and by (6-1)

$$\|N^r(u)\|_{L^p(\partial\Omega)} \lesssim \|S_p^r(u)\|_{L^p(\partial\Omega)} + \|u\|_{L^p(\partial\Omega)}. \tag{6-5}$$

7. Proof of Theorem 1.1

We only consider the case $1 < p < 2$ and use interpolation to obtain solvability for $p \geq 2$. First assume either the stronger Carleson condition of (2-39), or (1-7) and (1-8) hold. Therefore the Carleson conditions on the pullback coefficients (2-34), (2-35), (2-37) and (2-38) hold.

Without loss of generality, by Remark 2.20, we may assume that our domain is smooth. Consider $f^+ = \max\{0, f\}$ and $f^- = \max\{0, -f\}$, where $f \in C_0(\partial\Omega)$, and denote the corresponding solutions with these boundary data by u^+ and u^- respectively. Hence we may apply Corollary 5.4 separately to u^+ and u^- . By the maximum principle, these two solutions are nonnegative. It follows that for any such nonnegative u we have

$$\|S_p^r(u)\|_{L^p(\partial\Omega)}^p \leq C \|f\|_{L^p(\partial\Omega)}^p + C (\|\mu\|_C^{1/2} + \|\mu\|_C) \|N^{2r}(u)\|_{L^p(\partial\Omega)}^p$$

and Theorem 6.3 gives

$$\|N^r(u)\|_{L^p(\partial\Omega)}^p \leq C \|f\|_{L^p(\partial\Omega)}^p + C \|S_p^{2r}(u)\|_{L^p(\partial\Omega)}^p,$$

where $\|\mu\|_C$ is the Carleson norm of (1-7) on Carleson regions of size $\leq r_0$. As noted earlier, if, for example, Ω is of VMO type then the size of μ appearing in this estimate will only depend on the Carleson norm of coefficients on Ω , provided we only consider small Carleson regions. Hence we can choose r_0

small enough (depending on the domain Ω) such that the Carleson norm after the pullback is only twice the original Carleson norm of the coefficients over all balls of size $\leq r_0$.

Since we are assuming $\|\mu\|_C$ is small, clearly we also have $\|\mu\|_C \leq C\|\mu\|_C^{1/2}$. By rearranging these two inequalities and combining estimates for u^+ and u^- , we obtain, for $0 < r \leq r_0/8$,

$$\|N^r(u)\|_{L^p(\partial\Omega)}^p \leq C\|f\|_{L^p(\partial\Omega)}^p + C\|\mu\|_C^{1/2}\|N^{4r}(u)\|_{L^p(\partial\Omega)}^p.$$

By a simple geometric argument in [Dindoš and Hwang 2018] involving cones of different apertures, Lemmas 3.4 and 3.7 show there exists a constant M such that

$$\|N^{4r}(u)\|_{L^p(\partial\Omega)}^p \leq M\|N^r(u)\|_{L^p(\partial\Omega)}^p. \tag{7-1}$$

It follows that if $CM\|\mu\|_C^{1/2} < \frac{1}{2}$ by combining the last two inequalities we obtain

$$\|N^r(u)\|_{L^p(\partial\Omega)}^p \leq 2C\|f\|_{L^p(\partial\Omega)}^p,$$

which is the desired estimate (for the truncated version of nontangential maximum function). The result with the nontruncated version of the nontangential maximum function $N(u)$ follows as our domain is bounded in space and hence (7-1) can be iterated finitely many times until the nontangential cones have sufficient height to cover the whole domain.

Finally, we comment on how the Carleson condition (2-39) can be relaxed to the weaker condition (1-6). The idea is the same as [Dindoš and Hwang 2018, Theorem 3.1]. As shown there, if the operator \mathcal{L} satisfies the weaker condition (1-6), then it is possible (via mollification of coefficients) to find another operator \mathcal{L}_1 which is a small perturbation of the operator \mathcal{L} and \mathcal{L}_1 satisfies (2-39). The solvability of the L^p Dirichlet problem in the range $1 < p < 2$ for \mathcal{L}_1 follows by our previous arguments. However, as \mathcal{L} is a small perturbation of the operator \mathcal{L}_1 we have by the perturbation argument of [Sweezy 1998] L^p solvability of \mathcal{L} as well.

Finally, for larger values of p we use the maximum principle and interpolation to obtain solvability results in the full range $1 < p < \infty$. □

Appendix: proofs of results from Section 2

Proof of Theorem 2.3. We begin by proving the equivalence of (3) and (6) using ideas from [Strichartz 1980] and write $F = \mathbb{D}\phi$, where F is a tempered distribution. Let

$$\varphi^k = \chi_{\tilde{Q}_1(0,0)} - \chi_{\tilde{Q}_1(e_k)}.$$

Then for $1 \leq k \leq n - 1$

$$\begin{aligned} \widehat{\varphi}^k(\xi, \tau) &= \frac{2 \sin^2(\xi_k/2)}{\xi_k} \frac{1 - e^{-i\tau}}{i\tau} \prod_{j \neq k}^{n-1} \frac{1 - e^{-i\xi_j}}{i\xi_j}, \\ \widehat{\varphi}^n(\xi, \tau) &= \frac{2 \sin^2(\tau/2)}{\tau} \prod_{j=1}^{n-1} \frac{1 - e^{-i\xi_j}}{i\xi_j}, \end{aligned} \tag{A-1}$$

with $\widehat{\varphi}^k(\xi, \tau) \sim \xi_k$ for small ξ_k and $1 \leq k \leq n - 1$. We let

$$\widehat{\psi}^u = \frac{e^{i(\xi,0) \cdot u} - 1}{\|(\xi, \tau)\|}$$

and denote by $\psi_\rho^u(x, t)$ the usual parabolic dilation by ρ , that is,

$$\psi_\rho^u(x, t) = \rho^{-(n+1)} \psi^u(x/\rho, t/\rho^2).$$

It is worth noting that $(\varphi^k * \psi^u)_\rho = \varphi_\rho^k * \psi_\rho^u$. Therefore we may rewrite (6.a), by Remark 2.4, as

$$\sup_{Q_r} \sum_{k=1}^{n-1} \frac{1}{|Q_r|} \int_{Q_r} \int_{u \in \mathbb{S}^{n-1}} \int_0^r (\psi_\rho^u * \varphi_\rho^k * F)^2 \frac{d\rho}{\rho} du dx dt \sim B_{(6.a)}. \tag{A-2}$$

Similarly if we let

$$\widehat{\psi}_n^u = \frac{e^{i(0,\tau) \cdot u} - 1}{\|(\xi, \tau)\|} \tag{A-3}$$

then we may rewrite (6.b) as

$$\sup_{Q_r} \frac{1}{|Q_r|} \int_{Q_r} \int_{u \in \mathbb{S}^{n-1}} \int_0^r (\psi_{n,\rho}^u * F)^2 \frac{d\rho}{\rho} du dx dt \sim B_{(6.b)}. \tag{A-4}$$

The functions $\varphi^k * \psi^u$ and ψ_n^u all satisfy the following conditions for some $\varepsilon_i > 0$:

$$\begin{aligned} \int \psi dx dt &= 0, \\ |\psi(x, t)| &\lesssim \|(x, t)\|^{-n-1-\varepsilon_1} \quad \text{for } \|(x, t)\| \geq a > 0, \\ |\widehat{\psi}(\xi, \tau)| &\lesssim \|(\xi, \tau)\|^{\varepsilon_2} \quad \text{for } \|(\xi, \tau)\| \leq 1, \\ |\widehat{\psi}(\xi, \tau)| &\lesssim \|(\xi, \tau)\|^{-\varepsilon_3} \quad \text{for } \|(\xi, \tau)\| \geq 1. \end{aligned} \tag{A-5}$$

Therefore if $\mathbb{D}\phi = F \in \text{BMO}(\mathbb{R}^n)$ then $B_{(6.a)} \lesssim \|\mathbb{D}\phi\|_*^2$ and $B_{(6.b)} \lesssim \|\mathbb{D}\phi\|_*^2$ by [Strichartz 1980, Theorem 2.1]; this shows (3) implies (6).

For the converse, we proceed via an analogue of the proof of [Strichartz 1980, Theorem 2.6]. Consider

$$\widehat{\theta}(\xi, \tau) = \|(\xi, \tau)\| \widehat{\zeta}(\xi, \tau),$$

where $\zeta \in C_0^\infty(\mathbb{R})$. Let H_0^1 be the dense subclass of continuous H^1 functions g such that g and all its derivatives decay rapidly; see [Stein 1970, p. 225]. Via an analogue of [Fefferman and Stein 1972, Theorem 3; Strichartz 1980, Lemma 2.7] by assuming (6.a) and (6.b) if $g \in H_0^1(\mathbb{R}^n)$ then for each $1 \leq k \leq n - 1$

$$\left| \int_{\mathbb{S}^{n-1}} \int_0^\infty \iint_{\mathbb{R}^{n-1} \times \mathbb{R}} \psi_\rho^u * \varphi_\rho^k * F(x, t) \theta_\rho * g(x, t) dx dt \frac{d\rho}{\rho} du \right| \lesssim B_{(6.a)}^{1/2} \|g\|_{H^1}, \tag{A-6}$$

$$\left| \int_{\mathbb{S}^{n-1}} \int_0^\infty \iint_{\mathbb{R}^{n-1} \times \mathbb{R}} \psi_{n,\rho}^u * F(x, t) \theta_\rho * g(x, t) dx dt \frac{d\rho}{\rho} du \right| \lesssim B_{(6.b)}^{1/2} \|g\|_{H^1}. \tag{A-7}$$

For $1 \leq k \leq n - 1$ let

$$\begin{aligned}
 m_k(\xi, \tau) &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{\widehat{\psi}^u(-\rho\xi, -\rho^2\tau)\widehat{\varphi}^k(-\rho\xi, -\rho^2\tau)\|(\xi, \tau)\|\zeta(\rho\|(\xi, \tau)\|)}{\widehat{\psi}^u(-\rho\xi, -\rho^2\tau)\|(\xi, \tau)\|\zeta(\rho\|(\xi, \tau)\|)} d\rho du, \\
 m_n(\xi, \tau) &= \int_{\mathbb{S}^{n-1}} \int_0^\infty \frac{\widehat{\psi}_n^u(-\rho\xi, -\rho^2\tau)\|(\xi, \tau)\|\zeta(\rho\|(\xi, \tau)\|)}{\widehat{\psi}_n^u(-\rho\xi, -\rho^2\tau)\|(\xi, \tau)\|\zeta(\rho\|(\xi, \tau)\|)} d\rho du.
 \end{aligned}
 \tag{A-8}$$

All of these functions m_i are homogeneous of degree zero, smooth away from the origin and the associated Fourier multipliers M_k , for $1 \leq k \leq n$, are Calderón–Zygmund operators that preserve the class H_{00}^1 and are bounded on H^1 .

The nondegeneracy condition from [Calderón and Torchinsky 1975] on the family of functions $\{m_k\}_{k=1}^n$ holds — that is, the property that $\sum_k |m_k(r\xi, r^2\tau)|^2$ does not vanish identically in r for $(\xi, \tau) \neq (0, 0)$. Therefore by [Calderón and Torchinsky 1975; 1977] we can find smooth homogeneous functions $u_{k,j}(\xi, \tau)$ of degree zero and positive numbers r_j such that for all $(\xi, \tau) \neq (0, 0)$

$$\sum_{k=1}^n \sum_{j=1}^{j_0} m_{k,r_j}(\xi, \tau)u_{k,j}(\xi, \tau) = 1,
 \tag{A-9}$$

where m_{k,r_j} are as m_k but with $r_j\rho$ replacing ρ in the arguments of $\widehat{\psi}^u$, $\widehat{\varphi}^k$ and $\widehat{\psi}_2^u$ in (A-8) (but not ζ).

Let $M_{k,j}$ and $U_{k,j}$ be the Fourier multiplier operators associated to their respective multipliers m_{k,r_j} and $u_{k,j}$. Then $\sum \sum M_{k,j}U_{k,j}g = g$ for all $g \in H_{00}^1$. By [Fefferman and Stein 1972, Theorem 3; Strichartz 1980, Lemma 2.7] there exists $h_{k,j} \in \text{BMO}(\mathbb{R}^n)$ such that $\|h_{k,j}\|_*^2 \lesssim B_{(6.a)}$ or $B_{(6.b)}$, and $(h_{k,j}, g) = (F, M_{k,j}g)$ for all $g \in H_{00}^1$. If we replace g by $U_{j,k}g \in H_{00}^1$ in the previous identity and sum over j and k we obtain $(h, g) = (F, g)$ for all $g \in H_{00}^1$, where $h = \sum_{k,j} U_{k,j}^*h_{k,j}$; furthermore by the BMO condition on $h_{k,j}$, we have $\|h\|_*^2 \lesssim B_{(6.a)} + B_{(6.b)}$. The identity (A-9) does not need to hold at the origin; therefore $\widehat{h} - \widehat{F}$ may be supported at the origin and hence $F = h + p$, where p is a polynomial. Due to the assumption $\phi \in \text{Lip}(1, \frac{1}{2})$, clearly F must be a tempered distribution. Hence as in [Strichartz 1980] we may conclude $F = h \in \text{BMO}(\mathbb{R}^n)$. This implies equivalence of (3) and (6).

Similarly we may prove the equivalence of (4) and (5) to (3). The changes needed are outlined below.

We first look at (5) \iff (3). In this instance we replace the convolutions $\varphi^k * \psi^u$ by

$$\widehat{\psi}_1^u(\xi, \tau) = \frac{e^{i(\xi,0)\cdot u} - 2 - e^{-i(\xi,0)\cdot u}}{\|(\xi, \tau)\|},$$

which corresponds to (5.a), and we keep the convolution ψ_n^u as it is in (A-3). The same proof then goes through to give that (5) holds if and only if (3) holds with equivalent norms, as in (2-13).

We now consider (4) \iff (3). This case is stated in [Rivera-Noriega 2003, Proposition 3.2]. Again the proof proceeds as above with one convolution

$$\widehat{\psi}^u(\xi, \tau) = \frac{e^{i(\xi,\tau)\cdot u} - 2 - e^{-i(\xi,\tau)\cdot u}}{\|(\xi, \tau)\|}. \quad \square$$

Proof of Theorem 2.8. Without loss of generality, we only consider the case $\eta < 1$. When $\eta \geq 1$ the existence of an extension with $\|\mathbb{D}\Phi\|_* \lesssim \eta + \ell$ requires a much simpler argument.

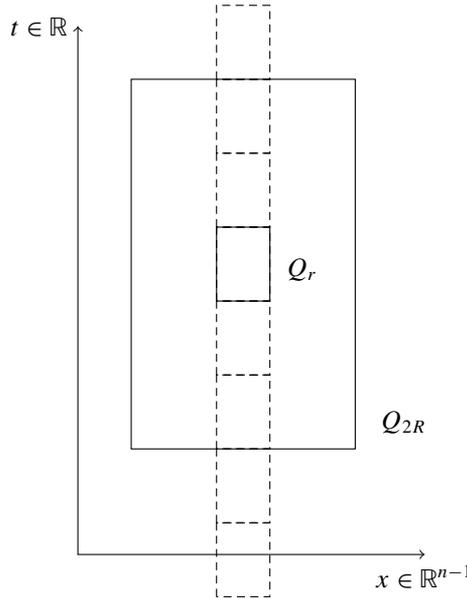


Figure 1. The reflection and tiling of the cube $Q_r \subset Q_{2R}$ defined in (A-10).

By (2-20) there exists $f \in C_\delta$ such that $\|\nabla\phi - f\|_{*, Q_{8d}} \leq 2\eta$ and a scale $0 < r_0 = r_0(\delta) \leq d$ such that

$$\|f\|_{*, Q_{8d}, r_0} \leq 2\eta.$$

Let $d' = \eta \min(r_0, r_1)/2$ and consider some $r \leq d'$ and $Q_r \subset Q_{4d}$. Find a natural number k such that $R = 2^k r$ and $R\eta/2 < r \leq R\eta$. By our choice of d' the cube Q_{2R} , which is an enlargement of Q_r by a factor 2^{k+1} , is still contained in the original cube Q_{8d} .

It follows that

$$\begin{aligned} &\|\nabla\phi\|_{*, Q_{2R}} \lesssim \eta, \\ &\sup_{\substack{Q_s = I_s \times I_s \\ Q_s \subset Q_{2R}}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi(x, t) - \phi(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \leq \eta^2. \end{aligned}$$

Without loss of generality, we may now assume that the cube Q_{2R} is centred at the origin $(0, 0)$ and that $\phi(0, 0) = 0$, since the BMO norm is invariant under translation and ignores constants. We first define $\tilde{\phi}$ as an extension in time via reflection and tiling of the cube Q_r :

$$\tilde{\phi}(x, t) = \begin{cases} \phi(x, t), & t \in [-r^2, r^2] + 4kr^2, \\ \phi(x, 2r^2 - t), & t \in [r^2, 3r^2] + 4kr^2, k \in \mathbb{Z}. \end{cases} \tag{A-10}$$

See Figure 1 for an illustration of this. Clearly $\tilde{\phi}$ coincides with ϕ on Q_r .

It follows that $\tilde{\phi}$ is a function $\tilde{\phi} : \{|x|_\infty < 2R\} \times \mathbb{R} \rightarrow \mathbb{R}$ and $(\nabla\tilde{\phi})_{Q_r} = (\nabla\phi)_{Q_r}$. Consider a cut-off function ρ such that

$$\rho(x) = \begin{cases} 1 & \text{if } |x|_\infty < r, \\ 0 & \text{if } |x|_\infty > 2R, \end{cases}$$

and $|\nabla\rho| \lesssim 1/R \lesssim \eta/r$. Finally define

$$\Phi = \tilde{\phi}\rho + (1 - \rho)(x \cdot (\nabla\tilde{\phi})_{Q_r}). \tag{A-11}$$

Clearly Φ is well-defined on $\mathbb{R}^{n-1} \times \mathbb{R}$ as $\rho = 0$ outside the support of $\tilde{\phi}$. We claim that Φ satisfies (i)–(iv) of [Theorem 2.8](#), which we establish in a sequence of lemmas below. Observe also that from our definition of Φ we have

$$\nabla\Phi = (\nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r})\rho + \nabla\rho(\tilde{\phi} - x \cdot (\nabla\tilde{\phi})_{Q_r}) + (\nabla\tilde{\phi})_{Q_r}, \tag{A-12}$$

completing the proof. □

We start with a couple of lemmas that allow us to reduce our claim to the dyadic case; this is to make the geometry easier to handle.

Lemma A.1 ([\[Jones 1980, Lemma 2.3\]](#), see also [\[Strichartz 1980, Theorem 2.8\]](#)). *Let f be defined on \mathbb{R}^n and*

$$\sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| \leq c(\eta), \tag{A-13}$$

where the supremum is taken over all dyadic cubes $Q \subset \mathbb{R}^n$. Further, assume that

$$\sup_{Q_1, Q_2} |f_{Q_1} - f_{Q_2}| \leq c(\eta), \tag{A-14}$$

where the supremum is taken over all dyadic cubes Q_1, Q_2 of equal edge length with a touching edge. Then

$$\|f\|_* \lesssim c(\eta).$$

Below $l(Q_s) = s$ denotes the radius of a parabolic cube.

Lemma A.2 [\[Jones 1980, Lemma 2.1 and pp. 44-45\]](#). *Let $f \in \text{BMO}(Q)$ and $Q_0 \subset Q_1 \subset Q$. Then*

$$|f_{Q_0} - f_{Q_1}| \lesssim \log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right) \|f\|_{*,Q}. \tag{A-15}$$

Furthermore, the same proof in [\[Jones 1980\]](#) gives the following slightly stronger result:

$$\frac{1}{|Q_0|} \int_{Q_0} |f - f_{Q_1}| \lesssim \log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right) \|f\|_{*,Q}. \tag{A-16}$$

If $Q_0, Q_1 \subset Q$ and $l(Q_0) \leq l(Q_1)$ but they are not necessarily nested then

$$|f_{Q_0} - f_{Q_1}| \lesssim \left(\log\left(2 + \frac{l(Q_1)}{l(Q_0)}\right) + \log\left[2 + \frac{\text{dist}(Q_0, Q_1)}{l(Q_1)}\right] \right) \|f\|_{*,Q}. \tag{A-17}$$

If the cubes Q_0, Q_1 and Q are dyadic then we may replace BMO by dyadic BMO.

There is a typo at the top of [\[Jones 1980, p. 45\]](#). It should read $l(Q_k) \leq l(Q_j)$ (it currently reads the converse).

Claim A.3. Let $\tilde{\phi}$ be defined as in (A-10), $\|\nabla\phi\|_{*,Q_{2R}} \lesssim \eta$, and let Q be dyadic with $r \leq l(Q) \leq 2R$. Then

$$\frac{1}{|Q|} \int_Q |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}. \tag{A-18}$$

Proof of claim. Let $N \in \mathbb{N}$ be such that $l(Q) = 2^N l(Q_r)$. Let $\{Q^i\}$ be the $2^{N(n-1)}$ dyadic cubes that are translations of Q_r and partition $Q \cap \{|t| \leq r^2\}$. Then by Lemma A.2

$$\begin{aligned} \frac{1}{|Q|} \int_Q |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| &= \sum_i \frac{2^{2N}|Q^i|}{|Q|} \frac{1}{|Q^i|} \int_{Q^i} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \\ &\leq \sum_i \frac{2^{2N}|Q^i|}{|Q|} \left(\frac{1}{|Q^i|} \int_{Q^i} |\nabla\phi - \nabla\phi_{Q^i}| + |\nabla\phi_{Q^i} - \nabla\phi_{Q_r}| \right) \\ &\lesssim (\eta + \eta \log(2 + R/r)) \lesssim \eta + \eta \log(1 + 1/\eta) \lesssim_\varepsilon \eta^{1-\varepsilon}. \quad \square \end{aligned}$$

Lemma A.4 [Stegenga 1976]. Let $g, h \in L^1_{\text{loc}}$. Then

$$\frac{1}{|Q|} \int_Q |gh - (gh)_Q| \leq \frac{2}{|Q|} \int_Q |g(h - h_Q)| + \frac{|h_Q|}{|Q|} \int_Q |g - g_Q|. \tag{A-19}$$

Proof. This small reduction is from [Stegenga 1976, p. 582]. First observe

$$gh - (gh)_Q = g(h - h_Q) + h_Q(g - g_Q) + g_Q h_Q - (gh)_Q$$

and

$$|g_Q h_Q - (gh)_Q| = \left| \frac{1}{|Q|} \int_Q gh_Q - \frac{1}{|Q|} \int_Q gh \right| \leq \frac{1}{|Q|} \int_Q |g(h - h_Q)|.$$

Hence

$$\left| \frac{1}{|Q|} \int_Q |gh - (gh)_Q| - \frac{|h_Q|}{|Q|} \int_Q |g - g_Q| \right| \leq 2 \frac{1}{|Q|} \int_Q |g(h - h_Q)|, \tag{A-20}$$

completing the proof. □

We can now prove (iii) of Theorem 2.8.

Lemma A.5. Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined as in (A-11) with $\|\nabla\phi\|_{*,Q_{2R}} \lesssim \eta$. Then $\nabla\Phi \in \text{BMO}(\mathbb{R}^n)$ and for all $0 < \varepsilon < 1$

$$\|\nabla\Phi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta\ell. \tag{A-21}$$

Proof. Recall $\nabla\Phi = (\nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r})\rho + \nabla\rho(\tilde{\phi} - x \cdot (\nabla\tilde{\phi})_{Q_r}) + (\nabla\tilde{\phi})_{Q_r}$; we can ignore the constant term as the BMO norm doesn't see it. Let $\psi = \nabla\tilde{\phi} - (\nabla\tilde{\phi})_{Q_r}$ and $\theta = \tilde{\phi} - x \cdot (\nabla\tilde{\phi})_{Q_r}$. We want to bound $\|\rho\psi\|_*$ and $\|\nabla\rho\theta\|_*$. We first tackle the term $\|\rho\psi\|_*$.

Step 1: (A-14) holds; that is, $\sup_{Q_1, Q_2} |(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \leq c(\eta)$ for Q_1, Q_2 dyadic cubes of equal side length and with a touching edge.

Since $\tilde{\phi}$ is the extension in the time direction by reflection and tiling (see (A-10)), and Q_1, Q_2 and Q_r are all dyadic cubes, we may assume that if $l(Q_1) \leq r$ then $Q_1, Q_2 \subset \{|t| < r^2\}$, and if $l(Q_1) > r$ then $\{|t| < r^2\} \subset Q_1$.

If $Q_1, Q_2 \subset Q_{2R}$ then $|(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \lesssim \|\rho\psi\|_{*, \text{ dyadic}, Q_{2R}}$. Therefore, if we show (A-13) for $f = \rho\psi$ then by Lemmas A.2 and A.4 clearly

$$|(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \lesssim \|\rho\psi\|_{*, \text{ dyadic}, Q_{2R}} \leq \|\psi\|_{*, \text{ dyadic}, Q_{2R}} \leq \|\nabla\tilde{\phi}\|_{*, \text{ dyadic}, Q_{2R}} \lesssim \eta.$$

Now look at the other cases: $Q_1 \subset Q_{2R}$ and $Q_2 \cap Q_{2R} = \emptyset$, or $Q_{2R} \subset Q_1$ and $Q_2 \cap Q_{2R} = \emptyset$. In both cases, we wish to control $|(\rho\psi)_{Q_1}|$.

Step 1.a: Case $Q_1 \subset Q_{2R}$, $Q_2 \cap Q_{2R} = \emptyset$ and $l(Q_1) \lesssim R\eta/\ell$.

Q_1 is small here and touches the boundary of Q_{2R} . This means that $\|\rho\|_{L^\infty(Q_1)} \lesssim l(Q_1)/R$ since ρ is 0 outside Q_{2R} . Therefore we apply the trivial bound

$$|(\rho\psi)_{Q_1}| \leq \|\rho\|_{L^\infty(Q_1)} \|\psi\|_{L^\infty(Q_1)} \lesssim \frac{l(Q_1)}{R} \ell \lesssim \eta.$$

Step 1.b: Case $Q_1 \subset Q_{2R}$, $Q_2 \cap Q_{2R} = \emptyset$ and $R\eta/\ell \lesssim l(Q_1) \leq 2R$.

Since $Q_1 \subset Q_{2R}$ we have $R\eta/\ell \lesssim l(Q_1) \leq 2R$. Q_1 is dyadic so there exists $N \in \mathbb{Z}$ such that $l(Q_1) = 2^N l(Q_r)$.

Step 1.b.i: $N \leq 0$.

This means that $l(Q_1) \leq l(Q_r)$ and so by the reflection and tiling in time, (A-10), we may assume $Q_1 \subset \{|t| \leq r^2\}$ and by Lemma A.2

$$\begin{aligned} |(\rho\psi)_{Q_1}| &\leq |\psi|_{Q_1} = \frac{1}{|Q_1|} \int_{Q_1} |\nabla\phi - \nabla\phi_{Q_r}| \leq \frac{1}{|Q_1|} \int_{Q_1} |\nabla\phi - \nabla\phi_{Q_1}| + |\nabla\phi_{Q_1} - \nabla\phi_{Q_r}| \\ &\lesssim \eta + \eta \log(1 + \ell) + \eta \log(1 + 1/\eta) \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta \log(1 + \ell). \end{aligned}$$

Step 1.b.ii: $N > 0$.

By Claim A.3 we obtain

$$|(\rho\psi)_{Q_1}| \leq |\psi|_{Q_1} = \frac{1}{|Q_1|} \int_{Q_1} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 1.c: Case $Q_{2R} \subset Q_1$, $Q_2 \cap Q_{2R} = \emptyset$ so $l(Q_1) \geq 2R$.

Let N satisfy $l(Q_1) = 2^N l(Q_{2R})$, the number of dyadic generations separating Q_1 and Q_{2R} . Then Q_1 overlaps Q_{2R} (and its dyadic translates in time) exactly 2^{2N} times. Therefore by Claim A.3,

$$|(\rho\psi)_{Q_1}| \leq |\psi|_{Q_1} \leq \frac{2^{2N}}{|Q_1|} \int_{Q_{2R}} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \leq \frac{2^{2N}}{2^{N(n+1)} |Q_{2R}|} \int_{Q_{2R}} |\nabla\tilde{\phi} - \nabla\tilde{\phi}_{Q_r}| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Hence, modulo the unproved statement $\|\rho\psi\|_{*, \text{ dyadic}, Q_{2R}} \lesssim \eta$ we have shown

$$|(\rho\psi)_{Q_1} - (\rho\psi)_{Q_2}| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta \log(1 + \ell).$$

Step 2: (A-13) holds; that is, $\|\rho\psi\|_{*, \text{ dyadic}} \lesssim c(\eta)$.

To apply Lemma A.4 we need to control two terms,

$$\sup_{Q \text{ dyadic}} \|\rho\|_{L^\infty(Q)} \frac{1}{|Q|} \int_Q |\psi - \psi_Q|$$

and

$$\sup_{Q \text{ dyadic}} \frac{|\psi_Q|}{|Q|} \int_Q |\rho - \rho_Q|.$$

Step 2.a: Estimating

$$\sup_{Q \text{ dyadic}} \|\rho\|_{L^\infty(Q)} \frac{1}{|Q|} \int_Q |\psi - \psi_Q|.$$

In all the following cases we bound $\|\rho\|_{L^\infty(Q)} \leq 1$.

Step 2.a.i: Case $l(Q) \leq r$.

As before, by the reflection and tiling in time, we may assume $Q \subset \{|t| \leq r^2\}$ and so $\nabla \tilde{\phi} = \nabla \phi$ on Q .

Hence

$$\frac{1}{|Q|} \int_Q |\psi - \psi_Q| = \frac{1}{|Q|} \int_Q |\nabla \tilde{\phi} - (\nabla \tilde{\phi})_Q| = \frac{1}{|Q|} \int_Q |\nabla \phi - (\nabla \phi)_Q| \lesssim \eta.$$

Step 2.a.ii: Case $r < l(Q) \leq 2R$.

Applying [Claim A.3](#) gives

$$\frac{1}{|Q|} \int_Q |\psi - \psi_Q| \leq |\psi|_Q \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.a.iii: Case $2R < l(Q)$.

From [Step 1.c](#) it follows that

$$\frac{1}{|Q|} \int_Q |\psi - \psi_Q| \leq |\psi|_Q \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.b: Estimating

$$\sup_{Q \text{ dyadic}} \frac{|\psi_Q|}{|Q|} \int_Q |\rho - \rho_Q|.$$

We have the following three cases to consider.

Step 2.b.i: Case $Q \subset Q_{2R}$, $l(Q) \leq r$ and $Q \subset \{|t| \leq r^2\}$.

Because the cube Q might not be touching the boundary we can't follow [Section 7](#) and bound

$$\frac{1}{|Q|} \int_Q |\rho - \rho_Q|$$

by $\|\rho\|_{L^\infty(Q)}$, which here is likely be 1. However, we can use the mean value theorem and get a better bound. By the intermediate value theorem there exists $(z, \tau) \in Q$ such that $\rho(z) = \rho_Q$ and using that ρ is independent of time and $|\nabla \rho| \lesssim 1/R$ we have

$$|\rho(x) - \rho_Q| = |\rho(x) - \rho(z)| \leq |\nabla \rho| l(Q) \lesssim \frac{l(Q)}{R} \leq \frac{l(Q)}{r}.$$

Then applying [Lemma A.2](#) gives

$$\begin{aligned} \frac{|\psi_Q|}{|Q|} \int_Q |\rho - \rho_Q| &\lesssim \frac{l(Q)}{r} \left| \frac{1}{|Q|} \int_Q \nabla \tilde{\phi} - \nabla \tilde{\phi}_Q \right| \leq \frac{l(Q)}{r} \frac{1}{|Q|} \int_Q |\nabla \phi - \nabla \phi_Q| \\ &\lesssim \frac{l(Q)}{r} \log\left(2 + \frac{r}{l(Q)}\right) \eta \lesssim \eta. \end{aligned}$$

Step 2.b.ii: Case $Q \subset Q_{2R}$ and $r < l(Q) \leq 2R$.

This case is a straightforward application of [Claim A.3](#):

$$\frac{|\psi_Q|}{|Q|} \int_Q |\rho - \rho_Q| \leq |\psi_Q| \lesssim_\varepsilon \eta^{1-\varepsilon}.$$

Step 2.b.iii: Case $Q_{2R} \subset Q$ so $l(Q) > 2R$.

This follows similarly to [Step 1.c](#); let N be defined as there and

$$\frac{|\psi_Q|}{|Q|} \int_Q |\rho - \rho_Q| \leq \frac{1}{|Q|} \left| \int_Q \nabla \phi - \nabla \phi_{Q_{2R}} \right| \leq \frac{2^{2N}}{2^{N(n+1)}} \|\nabla \phi\|_{*, Q_{2R}} \leq \eta.$$

Therefore by [Lemma A.1](#), $\|\rho\psi\|_* \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta \log(1 + \ell)$.

It remains to tackle the harder piece $\nabla \rho \theta = \nabla \rho(\tilde{\phi} - x \cdot \nabla \tilde{\phi}_{Q_r})$. Recall that $\text{supp}(\nabla \rho) = \{x \leq |x|_\infty \leq 2R\}$.

Step 3: [\(A-14\)](#) holds; that is, $\sup_{Q_1, Q_2} |(\nabla \rho \theta)_{Q_1} - (\nabla \rho \theta)_{Q_2}| \leq c(\eta)$, where Q_1, Q_2 are dyadic with a touching edge and $l(Q_1) = l(Q_2)$.

There are two different cases to consider:

- (1) $Q_1 \cap \text{supp}(\nabla \rho) \neq \emptyset$ and $Q_2 \cap \text{supp}(\nabla \rho) \neq \emptyset$.
- (2) $Q_1 \cap \text{supp}(\nabla \rho) \neq \emptyset$ and $Q_2 \cap \text{supp}(\nabla \rho) = \emptyset$.

Again (1) is controlled by $\|\nabla \rho \theta\|_{*, \text{dyadic}, Q_{2R}}$ by [Lemma A.2](#) so we only have to deal with (2) and bound $\sup_{Q_1 \text{ dyadic}} |(\nabla \rho \theta)_{Q_1}|$.

Step 3.a: Case $Q_1 \subset Q_{2R}$ and $l(Q_1) \lesssim R\eta/\ell$.

In this case Q_1 touches the boundary of the support of $\nabla \rho$ so we have the estimate $\|\nabla \rho\|_{L^\infty(Q_1)} \lesssim l(Q_1)/R^2$ since $|\nabla^2 \rho| \lesssim 1/R^2$. Also $\phi(0, 0) = 0$ and $\phi \in \text{Lip}(1, \frac{1}{2})$ so

$$\|\tilde{\phi}(x, t)\|_{L^\infty(Q_1)} \leq \|\phi(x, t)\|_{L^\infty(Q_{2R})} \lesssim \ell R.$$

Finally $\|x \cdot \nabla \tilde{\phi}_{Q_r}\|_{L^\infty(Q_{2R})} \lesssim \ell R$. Therefore

$$|(\nabla \rho \theta)_{Q_1}| \leq \|\nabla \rho\|_{L^\infty(Q_1)} |\theta|_{Q_1} \lesssim \frac{l(Q_1)}{R^2} \frac{1}{|Q_1|} \int_{Q_1} |\tilde{\phi}(x, t) - x \cdot \nabla \tilde{\phi}_{Q_r}| \, dx \, dt \lesssim \frac{l(Q_1)}{R^2} \ell R \lesssim \eta.$$

Step 3.b: Case $Q_1 \subset Q_{2R}$ and $R\eta/\ell \lesssim l(Q_1) \leq 2R$.

By the fundamental theorem of calculus, we may write

$$\tilde{\phi}(x, t) - \tilde{\phi}\left(r \frac{x}{|x|}, t\right) = x \cdot \int_{r/|x|}^1 \nabla \tilde{\phi}(\lambda x, t) \, d\lambda.$$

Therefore, we have

$$\begin{aligned} |(\nabla \rho \theta)_{Q_1}| &\leq |\nabla \rho| |\theta|_{Q_1} \\ &= |\nabla \rho| \left| \tilde{\phi}\left(r \frac{x}{|x|}, t\right) + x \cdot \int_{r/|x|}^1 (\nabla \tilde{\phi}(\lambda x, t) - \nabla \tilde{\phi}_{Q_r}) \, d\lambda + x \cdot \frac{r}{|x|} \nabla \tilde{\phi}_{Q_r} \right|_{Q_1} \\ &\lesssim \frac{1}{R} \left\| \tilde{\phi}\left(r \frac{x}{|x|}, t\right) \right\|_{L^\infty(Q_1)} + \frac{R}{R} \frac{1}{|Q_1|} \int_{Q_1} \left(\int_{r/|x|}^1 |\nabla \tilde{\phi}(\lambda x, t) - \nabla \tilde{\phi}_{Q_r}| \, d\lambda \right) \, dx \, dt + \frac{\eta R \ell}{R}. \end{aligned}$$

Since $\tilde{\phi}$ defined by (A-10) is tiled and reflected in time on cubes of scale r , and $(rx/|x|, 0) \in Q_r$ we control the first term above by

$$\frac{1}{R} \left\| \tilde{\phi} \left(r \frac{x}{|x|}, t \right) - 0 \right\|_{L^\infty(Q_1)} \leq \frac{1}{R} \|\phi - \phi(0, 0)\|_{L^\infty(Q_r)} \lesssim \frac{\ell r}{R} \lesssim \ell \eta.$$

Recall that $r \sim \eta R$, $R\eta/\ell \lesssim l(Q_1) \leq 2R$ and $r \leq |x|_\infty \leq 2R$ so $\eta/2 \leq \lambda \leq 1$. We apply Fubini to the second term:

$$\frac{1}{|Q_1|} \int_{Q_1} \left(\int_{r/|x|}^1 |\nabla \tilde{\phi}(\lambda x, t) - \nabla \tilde{\phi}_{Q_r}| d\lambda \right) dx dt \leq \frac{1}{|Q_1|} \int_{\eta/2}^1 \int_{Q_1} |\nabla \tilde{\phi}(\lambda x, t) - \nabla \tilde{\phi}_{Q_r}| dx dt d\lambda.$$

Let \tilde{Q}_1 be the set formed by Q_1 under the transformation $(x, t) \mapsto (\lambda x, t)$. We may further cover \tilde{Q}_1 by $\sim \lambda^{-2}$ translations of λQ_1 with $|\lambda Q_1|/|\tilde{Q}_1| \lesssim \lambda^2$. Therefore a similar proof to Claim A.3, using Lemma A.2, gives

$$\begin{aligned} \frac{1}{|Q_1|} \int_{Q_1} |\nabla \tilde{\phi}(\lambda x, t) - \nabla \tilde{\phi}_{Q_r}| dx dt &= \frac{1}{|\tilde{Q}_1|} \int_{\tilde{Q}_1} |\nabla \tilde{\phi} - \nabla \tilde{\phi}_{Q_r}| \\ &\lesssim \lambda^{-2} \frac{\lambda^2}{|s Q_1|} \int_{s Q_1} |\nabla \tilde{\phi} - \nabla \tilde{\phi}_{Q_r}| \lesssim \eta \log \left(2 + \frac{r}{sl(Q_1)} \right) \lesssim \eta \log \left(1 + \frac{\ell}{\eta^2} \right) \\ &\lesssim_\varepsilon \eta^{1-\varepsilon} + \eta \log(1 + \ell) \end{aligned}$$

and hence after harmlessly integrating in λ we can control the second term by

$$\int_{\eta/2}^1 \eta \log \left(1 + \frac{\ell}{\eta^2} \right) d\lambda \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta \log(1 + \ell).$$

Step 3.c: Case $l(Q_1) \geq 2R$.

As before in Step 1.c, $|(\nabla \rho \theta)_{Q_1}| \leq |(\nabla \rho \theta)_{Q_{2R}}|$, which can be further controlled by cubes that tile $\text{supp}(\nabla \rho)$. Therefore, this case is bounded as in Section 7.

Step 4: (A-13) holds; that is, $\|\nabla \rho \theta\|_{*, \text{dyadic}} \lesssim c(\eta)$.

Here we have three cases to consider:

- (1) $Q \subset Q_{2R}$.
- (2) $Q \subset \mathbb{R}^n \setminus \text{supp}(\nabla \rho)$.
- (3) $Q_{2R} \subset Q$.

Case (2) is obvious. Case (3) reduces down to (1) by Step 1.c, the reflection and tiling of $\tilde{\phi}$, and $\text{supp}(\nabla \rho)$.

Case (1): Using Lemma A.4 this reduces down to showing that

- (a) $\frac{|\theta_Q|}{|Q|} \int_Q |\nabla \rho - (\nabla \rho)_Q| \lesssim c(\eta)$,
- (b) $\frac{1}{|Q|} \int_Q |\nabla \rho(\theta - \theta_Q)| \lesssim c(\eta)$

for Q dyadic and $Q \subset Q_{2R}$.

Step 4.a: (a) holds for Q dyadic and $Q \subset Q_{2R}$.

Step 4.a.i: Case $Q \subset Q_{2R}$ and $l(Q) \lesssim R\eta/\ell$.

By the naive bounds in [Step 3.a](#), $|\theta|_Q \lesssim \ell R$. If we use the mean value theorem for $\nabla \rho$ similar to [Section 7](#) then

$$\frac{1}{|Q|} \int_Q |\nabla \rho - (\nabla \rho)_Q| \lesssim |\nabla^2 \rho| l(Q) \lesssim \frac{l(Q)}{R^2}.$$

Therefore

$$\frac{|\theta_Q|}{|Q|} \int_Q |\nabla \rho - (\nabla \rho)_Q| \lesssim \ell R \frac{l(Q)}{R^2} \lesssim \eta.$$

Step 4.a.ii: Case $Q \subset Q_{2R}$ and $R\eta/\ell \lesssim l(Q) \leq 2R$.

Here we apply the same technique as [Section 7](#):

$$\frac{|\theta_Q|}{|Q|} \int_Q |\nabla \rho - (\nabla \rho)_Q| \leq |\theta|_Q |\nabla \rho| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta \log(1 + \ell).$$

Step 4.b: (b) holds for Q dyadic and $Q \subset Q_{2R}$.

$$\frac{1}{|Q|} \int_Q |\nabla \rho(\theta - \theta_Q)| \lesssim \frac{1}{R} \frac{1}{|Q|} \int_Q |\theta - \theta_Q|.$$

We split this into the now-usual cases.

Step 4.b.i: Case $l(Q) \lesssim R\eta/\ell$.

By the intermediate and mean value theorems $|\tilde{\phi} - \tilde{\phi}_Q| \lesssim l(Q)\ell$ and $|x - x_Q| \lesssim l(Q)$ so

$$\frac{1}{R} \frac{1}{|Q|} \int_Q |\theta - \theta_Q| = \frac{1}{R} \frac{1}{|Q|} \int_Q |\tilde{\phi} - \tilde{\phi}_Q - x \cdot \nabla \tilde{\phi}_Q + (x \cdot \nabla \tilde{\phi}_Q)_Q| \lesssim \frac{1}{R} l(Q)\ell \lesssim \eta.$$

Step 4.b.ii: Case $R\eta/\ell \lesssim l(Q) < 2R$.

$$\frac{1}{R} \frac{1}{|Q|} \int_Q |\theta - \theta_Q| \lesssim \frac{1}{R} |\theta|_Q,$$

and then applying the result from [Section 7](#) gives

$$\frac{1}{|Q|} \int_Q |\nabla \rho(\theta - \theta_Q)| \lesssim_\varepsilon \eta^{1-\varepsilon} + \eta \log(1 + \ell).$$

Therefore by [Lemma A.1](#) we have shown $\nabla \Phi \in \text{BMO}(\mathbb{R}^n)$ and the bound [\(A-21\)](#) holds. □

To finish proving [Theorem 2.8](#) we need to establish (iv).

Lemma A.6. *Let $\Phi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined in [\(A-11\)](#) with*

$$\sup_{\substack{Q_s = J_s \times I_s \\ Q_s \subset Q_{8d}, s \leq r_1}} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\phi(x, t) - \phi(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \leq \eta^2 \tag{A-22}$$

then Φ satisfies

$$\sup_{Q_s = J_s \times I_s} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\Phi(x, t) - \Phi(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \lesssim \eta^2. \tag{A-23}$$

Proof. Trivially since Φ is defined globally

$$\sup_{Q_s=J_s \times I_s} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\Phi(x, t) - \Phi(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \leq \sup_{Q_s=J_s \times I_s} \frac{1}{|Q_s|} \int_{Q_s} \int_{I_s} \frac{|\tilde{\phi}(x, t) - \tilde{\phi}(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx,$$

where we interpret the value of $\tilde{\phi}$ where it is undefined as 0, i.e., $\tilde{\phi}(x, t) = 0$ when $(x, t) \notin \text{supp}(\tilde{\phi})$. It remains to establish

$$\sup_{I_s} \frac{1}{|I_s|} \int_{I_s} \int_{I_s} \frac{|\tilde{\phi}(x, t) - \tilde{\phi}(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \lesssim \sup_{I_s \subset I_r} \frac{1}{|I_s|} \int_{I_s} \int_{I_s} \frac{|\phi(x, t) - \phi(x, \tau)|^2}{|t - \tau|^2} d\tau dt dx \quad (\text{A-24})$$

pointwise in x , where $Q_r = J_r \times I_r$ and is used to define Φ in (A-11). To simplify our notation, we drop the dependence on the spatial variables in $\tilde{\phi}$ and ϕ . We also set $A := I_s$. Recall from (A-10) that

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in [-r^2, r^2] + 4kr^2, \\ \phi(2r^2 - t), & t \in [r^2, 3r^2] + 4kr^2, \end{cases}$$

for $k \in \mathbb{Z}$. Let $I_k = [-r^2, r^2] + 4kr^2$ and $J_k = [r^2, 3r^2] + 4kr^2$ be intervals in time for $k \in \mathbb{Z}$. We partition A into disjoint pieces $A = \bigcup_i I_i \bigcup_j J_j \cup A_1 \cup A_2$, where A_1 and A_2 are pieces that don't contain either I_i or J_j .

If $A = A_1 \cup A_2$, we may as well assume (by translation and reflection) that $A_1 = [a, r^2]$, $A_2 = [r^2, b]$. Let τ' , b' and A'_2 be the images of τ , b and A_2 respectively under the map $\tau \mapsto 2r^2 - \tau$. Without loss of generality we only consider the case $|A_1| > |A_2|$. Since $|t - \tau| = |t - r^2| + |\tau' - r^2| \geq |t - \tau'|$ we have for $t \in A_1$, $\tau \in A_2$

$$\begin{aligned} \int_{A_1} \int_{A_2} \frac{|\tilde{\phi}(t) - \tilde{\phi}(\tau)|^2}{|t - \tau|^2} d\tau dt &= \int_a^{r^2} \int_{b'}^{r^2} \frac{|\phi(t) - \phi(\tau')|^2}{|t - (2t^2 - \tau')|^2} d\tau' dt \\ &\leq \int_a^{r^2} \int_{b'}^{r^2} \frac{|\phi(t) - \phi(\tau')|^2}{|t - \tau'|^2} d\tau' dt \leq \int_{A_1} \int_{A_1} \frac{|\phi(t) - \phi(\tau')|^2}{|t - \tau'|^2} d\tau' dt. \end{aligned}$$

Therefore

$$\frac{1}{|A|} \int_A \int_A \frac{|\tilde{\phi}(t) - \tilde{\phi}(\tau)|^2}{|t - \tau|^2} d\tau dt = \frac{1}{|A|} \left(\int_{A_1} \int_{A_1} + 2 \int_{A_1} \int_{A_2} + \int_{A_2} \int_{A_2} \right) \frac{|\phi(t) - \phi(\tau')|^2}{|t - \tau'|^2} d\tau' dt \lesssim \eta^2.$$

In the general case when $A = \bigcup_{i \in \mathcal{I}} I_i \bigcup_{j \in \mathcal{J}} J_j \cup A_1 \cup A_2$ we write the double integral over A in terms of integrals

$$\sum_{i, k \in \mathcal{I}} \int_{I_i} \int_{I_k} \frac{|\tilde{\phi}(t) - \tilde{\phi}(\tau)|^2}{|t - \tau|^2} d\tau dt, \quad \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \int_{I_i} \int_{J_j} \frac{|\tilde{\phi}(t) - \tilde{\phi}(\tau)|^2}{|t - \tau|^2} d\tau dt$$

and integrals that involve sets A_1 or A_2 or both (those are handled similarly to the earlier calculation).

Dealing with the first case, if $i \neq k$, $t \in I_i$ and $\tau \in I_k$ then $|t - \tau| \sim r^2|i - k|$; if $i = k$ then $|t - \tau| = |t' - \tau'|$. Therefore

$$\begin{aligned} \sum_{i,k \in \mathcal{I}} \int_{I_i} \int_{I_k} \frac{|\tilde{\phi}(t) - \tilde{\phi}(\tau)|^2}{|t - \tau|^2} d\tau dt &\sim \sum_{i \in \mathcal{I}} \int_{I_0} \int_{I_0} \frac{|\phi(t) - \phi(\tau)|^2}{|t - \tau|^2} d\tau dt + \sum_{\substack{i,k \in \mathcal{I} \\ i \neq k}} \frac{1}{r^4|i - k|^2} \int_{I_0} \int_{I_0} |\phi(t) - \phi(\tau)|^2 d\tau dt \\ &\leq \sum_{i \in \mathcal{I}} \int_{I_0} \int_{I_0} \frac{|\phi(t) - \phi(\tau)|^2}{|t - \tau|^2} d\tau dt + \sum_{\substack{i,k \in \mathcal{I} \\ i \neq k}} \frac{1}{|i - k|^2} \int_{I_0} \int_{I_0} \frac{|\phi(t) - \phi(\tau)|^2}{|t - \tau|^2} d\tau dt \\ &\lesssim |\mathcal{I}| \int_{I_0} \int_{I_0} \frac{|\phi(t) - \phi(\tau)|^2}{|t - \tau|^2} d\tau dt. \end{aligned}$$

In the second case

$$\begin{aligned} \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \int_{I_i} \int_{J_j} \frac{|\tilde{\phi}(t) - \tilde{\phi}(\tau)|^2}{|t - \tau|^2} d\tau dt &\lesssim \sum_{\substack{i \in \mathcal{I}, j \in \mathcal{J} \\ |i - j| \leq 1}} \int_{I_0} \int_{I_0} \frac{|\phi(t) - \phi(\tau)|^2}{|t - \tau|^2} + \sum_{\substack{i \in \mathcal{I}, j \in \mathcal{J} \\ |i - j| \geq 2}} \frac{1}{r^4(|i - j| - 1)^2} \int_{I_0} \int_{I_0} |\phi(t) - \phi(\tau)|^2 d\tau dt \\ &\lesssim (|\mathcal{I}| + |\mathcal{J}|) \int_{I_0} \int_{I_0} \frac{|\phi(t) - \phi(\tau)|^2}{|t - \tau|^2} d\tau dt. \end{aligned}$$

Since $|A| \sim (|\mathcal{I}| + |\mathcal{J}|)|I_0|$ and I_0 is one of the time intervals considered in the supremum of (A-24),

$$\frac{1}{|A|} \int_A \int_A \frac{|\phi(t) - \phi(\tau)|^2}{|t - \tau|^2} d\tau dt \sim \frac{1}{|I_0|} \int_{I_0} \int_{I_0} \frac{|\phi(t) - \phi(\tau)|^2}{|t - \tau|^2} d\tau dt \lesssim \eta^2. \quad \square$$

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