ANALYSIS & PDE

Volume 13 No. 5 2020

PIERRE-DAMIEN THIZY

WHEN DOES A PERTURBED MOSER–TRUDINGER INEQUALITY ADMIT AN EXTREMAL?

msp

ANALYSIS AND PDE
Vol. 13, No. 5, 2020
https://doi.org/10.2140/apde.2020.13.1371

WHEN DOES A PERTURBED MOSER–TRUDINGER INEQUALITY ADMIT AN EXTREMAL?

PIERRE-DAMIEN THIZY

We are interested in several questions raised mainly by Mancini and Martinazzi (2017) (see also work of McLeod and Peletier (1989) and Pruss (1996)). We consider the perturbed Moser–Trudinger inequality $I_\alpha^g(\Omega)$ at the critical level $\alpha = 4\pi$, where g, satisfying $g(t) \to 0$ as $t \to +\infty$, can be seen as a perturbation with respect to the original case $g \equiv 0$. Under some additional assumptions, ensuring basically that g does not oscillate too fast as $t \to +\infty$, we identify a new condition on g for this inequality to have an extremal. This condition covers the case $g \equiv 0$ studied by Carleson and Chang (1986), Struwe (1988), and Flucher (1992). We prove also that this condition is sharp in the sense that, if it is not satisfied, $I_{4\pi}^g(\Omega)$ may have no extremal.

1. Introduction

Let Ω be a smooth, bounded domain of $\mathbb{R}^2$ and let $H_0^1 = H_0^1(\Omega)$ be the standard Sobolev space, obtained as the completion of the set of smooth functions with compact support in Ω, with respect to the norm $\|\cdot\|_{H_0^1}$ given by

$$\|u\|_{H_0^1}^2 = \int_\Omega |\nabla u(x)|^2 \, dx.$$

Throughout the paper, Ω is assumed to be connected. Let g be such that

$$g \in C^1(\mathbb{R}), \qquad \lim_{s \to +\infty} g(s) = 0, \qquad g(t) > -1 \quad \text{and} \quad g(t) = g(-t) \quad \text{for all } t \tag{1-1}$$

(see also Remark 1.6). Then, we have

$$C_{g,\alpha}(\Omega) := \sup_{u \in H_0^1 : \|u\|_{H_0^1}^2 \le \alpha} \int_\Omega (1 + g(u)) \exp(u^2) \, dx \tag{$I_\alpha^g(\Omega)$}$$

is finite for $0 < \alpha \le 4\pi$ and equals $+\infty$ for $\alpha > 4\pi$. This result was first obtained in [Moser 1971] in the unperturbed case $g \equiv 0$. Still by that work, we easily extend the $g \equiv 0$ case to the case of g as in (1-1). Finally, [Moser 1971] gives also the existence of an extremal for $(I_\alpha^g(\Omega))$ if $0 < \alpha < 4\pi$ (see Lemma 3.1). If now $\alpha = 4\pi$, getting the existence of an extremal is more challenging; however, Carleson and Chang [1986], Struwe [1988] and Flucher [1992] were also able to prove that $(I_{4\pi}^0(\Omega))$ admits an extremal in the unperturbed case $g \equiv 0$. Yet, surprisingly, McLeod and Peletier [1989] conjectured that there should exist a g as in (1-1) such that $(I_{4\pi}^g(\Omega))$ does not admit any extremal function. Through a nice but very

MSC2010: 35B33, 35B44, 35J15, 35J61.
Keywords: Moser–Trudinger inequality, blow-up analysis, elliptic equations, extremal function.

implicit procedure, Pruss [1996] was able prove that such a g does exist. Observe that, since $g(u) \to 0$ as $u \to +\infty$ in (1-1), $(1+g(u))\exp(u^2)$ in $(I_\alpha^g(\Omega))$ seems like a very mild perturbation of $\exp(u^2)$ as $u \to +\infty$ and then, this naturally raises the following question:

Question 1.1. To what extent does the existence of an extremal for the critical Moser–Trudinger inequality $(I_{4\pi}^0(\Omega))$ really depend on asymptotic properties of the function $t \mapsto \exp(t^2)$ as $t \to +\infty$?

To investigate this question, we may rephrase it as follows: for what g satisfying (1-1) does $(I_{4\pi}^g(\Omega))$ admit an extremal? This is Open Problem 2 in [Mancini and Martinazzi 2017], stated in this paper for $\Omega = \mathbb{D}^2$, the unit disk of $\mathbb{R}^2$. In order to state our main general result, we introduce now some notation. For a first reading, one can go directly to Corollary 1.3, which aims to give a less general but more readable statement. We let $H : (0, +\infty) \to \mathbb{R}$ be given by

$$H(t) = 1 + g(t) + \frac{g'(t)}{2t}, \tag{1-2}$$

so that we have

$$[(1+g(t))\exp(t^2)]' = 2tH(t)\exp(t^2). \tag{1-3}$$

We set $tH(t) = 0$ for $t = 0$, so that $t \mapsto tH(t)$ is continuous at 0 by (1-1). This function H comes into play, since the Euler–Lagrange associated to $(I_\alpha^g(\Omega))$ reads as

$$\begin{cases} \Delta u = \lambda u H(u)\exp(u^2) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1-4}$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier and $\Delta = -\partial_{xx} - \partial_{yy}$ (see also Lemma 3.1 below). Now, we make some further assumptions on the behavior of g at $+\infty$ and at 0. First, we assume that there exist $\delta_0 \in (0, 1)$ and a sequence of real numbers $A = (A(\gamma))_\gamma$ such that:

(1-5a) $H(\gamma - t/\gamma) = H(\gamma)(1 + A(\gamma)t + o(|A(\gamma)| + \gamma^{-4}))$ in $C^0_{\mathrm{loc}}(\mathbb{R}_t)$ as $\gamma \to +\infty$.

(1-5b) There exists $C > 0$ such that $|H(\gamma - t/\gamma) - H(\gamma)| \le C|H(\gamma)|(|A(\gamma)| + \gamma^{-4})\exp(\delta_0 t)$ for all $\gamma \gg 1$ and all $0 \le t \le \gamma^2$.

(1-5c) $\lim_{\gamma\to+\infty} A(\gamma) = 0$.

In (1-5a) and (1-6a), γ is a parameter and the $C^0_{\mathrm{loc}}([0, +\infty))$ convergence is in the t-variable. We also assume that there exist $\delta_0' \in (0, 1)$, $\kappa \ge 0$, $\tilde{\varepsilon}_0 \in \{-1, +1\}$, F given by $F(t) := \tilde{\varepsilon}_0 t^\kappa$, and a sequence $B = (B(\gamma))_\gamma$ of positive real numbers such that:

(1-6a) $(t/\gamma)H(t/\gamma) = B(\gamma)F(t) + o(|B(\gamma)| + \gamma^{-1})$ in $C^0_{\mathrm{loc}}((0, +\infty)_t)$ as $\gamma \to +\infty$.

(1-6b) There exists $C > 0$ such that $|(t/\gamma)H(t/\gamma)| \le C(|B(\gamma)| + \gamma^{-1})\exp(\delta_0' t)$ for all $\gamma \gg 1$ and all $0 \le t \le \gamma^2$.

Observe that we may have $B(\gamma) = o(\gamma^{-1})$ as $\gamma \to +\infty$, in which case the precise formula for F is not really significant. Since $t \mapsto (1+g(t))\exp(t^2)$ is an even C^1 function, we have

$$\lim_{\gamma\to+\infty} B(\gamma) = 0, \tag{1-7}$$

in view of (1-3) and (1-6). Following rather standard notation, we may split the Green's function G of Δ, with zero Dirichlet boundary conditions in Ω, according to

$$G_x(y) = \frac{1}{4\pi}\left(\log\frac{1}{|x-y|^2} + \mathcal{H}_x(y)\right) \tag{1-8}$$

for all $x \neq y$ in Ω, where $\mathcal{H}_x$ is harmonic in Ω and coincides with $-\log 1/|x-\cdot|^2$ in $\partial\Omega$. Then the Robin function $x \mapsto \mathcal{H}_x(x)$ is smooth in Ω, and goes to $-\infty$ as $x \to \partial\Omega$, so that we may set

$$\begin{aligned} M &= \max_{x\in\Omega} \mathcal{H}_x(x), \\ K_\Omega &= \{y \in \Omega : \mathcal{H}_y(y) = M\}, \\ S &= \max_{z\in K_\Omega} \int_\Omega G_z(y) F(4\pi G_z(y))\, dy, \end{aligned} \tag{1-9}$$

where F is as in (1-6). For $N \geq 1$, we let g_N be given by

$$(1+g_N(t))\exp(t^2) = (1+g(t))(1+t^2) + (1+g(t))\left(\sum_{k=N+1}^{+\infty}\frac{t^{2k}}{k!}\right), \tag{1-10}$$

so that $g_N \leq g$, $g_N(0) = g(0)$ for all $N \geq 1$, while $g = g_N$ for $N = 1$. We also set

$$\Lambda_g(\Omega) := \max_{u\in H_0^1 : \|u\|^2_{H_0^1} \leq 4\pi} \int_\Omega \big((1+g(u))(1+u^2) - (1+g(0))\big)\, dx. \tag{1-11}$$

We are now in position to state our main result, giving a new, very general and basically sharp picture about the existence of an extremal for the perturbed Moser–Trudinger inequality $(I^g_{4\pi}(\Omega))$.

Theorem 1.2 (existence and nonexistence of an extremal). *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that (1-1) and (1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Assume that*

$$l = \lim_{\gamma\to+\infty} \frac{\gamma^{-4} + \frac{1}{2}A(\gamma) + 4\gamma^{-3}\exp(-1-M)B(\gamma)S}{\gamma^{-4} + |A(\gamma)| + \gamma^{-3}|B(\gamma)|} \tag{1-12}$$

exists, where M and S are given by (1-9). Then:

(1) *If $l > 0$ or $\Lambda_g(\Omega) \geq \pi\exp(1+M)$, then $(I^g_{4\pi}(\Omega))$ admits an extremal, where $\Lambda_g(\Omega)$ is as in (1-11).*

(2) *If $l < 0$ and $\Lambda_g(\Omega) < \pi\exp(1+M)$, there exists $N_0 \geq 1$ such that $(I^{g_N}_{4\pi}(\Omega))$ admits no extremal for all $N \geq N_0$, where g_N is given by (1-10).*

Observe that, for all given $N \geq 1$, g_N satisfies (1-1) and (1-5)–(1-6), with the same A, B and F as the original g, in view of $H(\gamma) \to 1$ as $\gamma \to +\infty$; see (3-3). Moreover it is clear that $\Lambda_{g_N}(\Omega) \leq \Lambda_g(\Omega)$. Then, this second assertion in Theorem 1.2 proves that the assumptions on g in the first assertion are basically sharp to get the existence of an extremal for $(I^g_{4\pi}(\Omega))$. As a remark, Pruss [1996] concludes that the existence of an extremal for the critical Moser–Trudinger inequality is in some sense accidental and relies on nonasymptotic properties of $\exp(u^2)$. Theorem 1.2 clarifies this tricky situation: the existence or nonexistence of an extremal for $(I^g_{4\pi}(\Omega))$ may really depend on a balance of the asymptotic properties of g

both at infinity (given by $A(\gamma)$) and at zero (given by $B(\gamma)$). Yet, it may also depend on the nonasymptotic quantity $\Lambda_g(\Omega)$ (see Corollary 1.4). Observe that $\Lambda_0(\Omega) = 4\pi/\lambda_1(\Omega)$ in the unperturbed case $g \equiv 0$, where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Δ in Ω.

From now on, we illustrate Theorem 1.2 by two corollaries dealing with less general but more explicit situations. Let $c, c' \in \mathbb{R}$, $(a, b), (a', b') \in \mathcal{E}$, where

$$\mathcal{E} = \{(a, b) \in [0, +\infty) \times \mathbb{R} : b > 0 \text{ if } a = 0\}. \tag{1-13}$$

Let $R' > 0$ be a large positive constant. If one picks g such that

$$g(t) = \begin{cases} g_0(t) := g(0) + ct^{a+1}[\log(1/t)]^{-b} & \text{in } (0, 1/R'], \\ g_\infty(t) := c't^{-a'}[\log t]^{-b'} & \text{in } [R', +\infty), \end{cases} \tag{1-14}$$

l in (1-12) of Theorem 1.2 can be made more explicit. Indeed, we can then set

$$\begin{aligned} B(\gamma) &= \frac{1+g(0)}{\gamma} + \frac{c(a+1)}{2\gamma^a(\log\gamma)^b}, \\ F(t) &= \begin{cases} t^{\min(a,1)} & \text{if } c \neq 0, \\ t & \text{otherwise}, \end{cases} \\ A(\gamma) &= c' \times \begin{cases} a'\gamma^{-(a'+2)}(\log\gamma)^{-b'} & \text{if } a' > 0, \\ b'\gamma^{-2}(\log\gamma)^{-(b'+1)} & \text{if } a' = 0 \end{cases} \end{aligned} \tag{1-15}$$

(see also Lemma 3.3). Theorem 1.2 is even more explicit in the particular case $\Omega = \mathbb{D}^2$. Indeed, in this case we have that $K_{\mathbb{D}^2} = \{0\}$ in (1-9) and

$$G_0(x) = \frac{1}{2\pi}\log\frac{1}{|x|}.$$

Still on the unit disk $\mathbb{D}^2$, it is known that

$$\Lambda_0(\mathbb{D}^2) = \frac{4\pi}{\lambda_1(\mathbb{D}^2)} < \pi e \tag{1-16}$$

($\lambda_1(\mathbb{D}^2) \simeq 5.78$). Property (1-16) shows in particular that the second assertion $\Lambda_0(\mathbb{D}^2) \geq \pi e$ of Theorem 1.2(1) is not satisfied. In some sense, this is an additional motivation for the nice approach of [Carleson and Chang 1986], proving the existence of an extremal for $(I^0_{4\pi}(\mathbb{D}^2))$ via asymptotic analysis. As an illustration and a very particular case of Theorem 1.2, we get the following corollary.

Corollary 1.3 (case $\Omega = \mathbb{D}^2$). *Assume that or $\Omega = \mathbb{D}^2$. Let $c' \neq 0$ and $(a', b') \in \mathcal{E}$ be given, where $\mathcal{E}$ is as in (1-13). Let g_∞ be as in (1-14):*

(1) *If we assume $a' > 2$ or $c' > 0$, then for any even function $g \in C^1(\mathbb{R})$ such that $g > -1$,*

$$(g - g(0))^{(i)}(t) = o(t^{2-i}) \tag{1-17}$$

as $t \to 0$ and

$$g^{(i)}(t) = g^{(i)}_\infty(t)(1 + o(1)) \tag{1-18}$$

as $t \to +\infty$ for all $i \in \{0, 1\}$, the inequality $(I^g_{4\pi}(\mathbb{D}^2))$ admits an extremal.

(2) *If we assume* $a' < 2$ *and* $c' < 0$, *there exists an even function* $g \in C^1(\mathbb{R})$ *such that* $g > -1$ *and* (1-17) *and* (1-18) *hold true*, *while* $(I^g_{4\pi}(\mathbb{D}^2))$ *admits no extremal.*

Our main concern in Corollary 1.3 is to write a readable statement. In this result, the existence of an extremal in the unperturbed case $g \equiv 0$ is recovered for quickly decaying g's, namely if $a' > 2$; see [Mancini and Martinazzi 2017]. But a threshold phenomenon appears (only if $c' < 0$) and there are no more extremals for less decaying g's, namely for $a' < 2$. Note that Theorem 1.2 also allows us to point out the existence of a threshold $c' < 0$ in the border case $a' = 2,\ b' = 0$ (see Remark 1.5). Indeed, proving Corollary 1.3 basically reduces to giving an explicit formula for l in (1-12), which only depends on Ω and on the asymptotics of g at $+\infty$ and at 0. On the contrary, we do not care about the precise asymptotics of g in the following corollary, thus illustrating the role of $\Lambda_g(\Omega)$ in Theorem 1.2.

Corollary 1.4 (extremal for $\Lambda_g(\Omega)$ large). *Let* Ω *be a smooth bounded domain of* $\mathbb{R}^2$. *Let* $\lambda_1(\Omega) > 0$ *be the first Dirichlet eigenvalue of* Δ *in* Ω *and* M *be given as in* (1-9). *Let* $\bar{A}$ *be such that* $4(1+\bar{A}) > \lambda_1(\Omega)\exp(1+M)$ *and let* $C > \bar{A}$ *be given. Then there exists* $R \gg 1$ *such that* $(I^g_{4\pi}(\Omega))$ *admits an extremal for all* g *satisfying* (1-1) *and*

$$g(0) = \bar{A}, \qquad g \geq g(0) \quad \text{in } [1/R, R] \qquad \text{and} \qquad |g| \leq C \quad \text{in } \mathbb{R}. \tag{1-19}$$

We give now an overview of the proof of Theorem 1.2, since it is a bit intricate. First, we comment on part (1). For all $0 < \varepsilon \ll 1$ small, we start by picking an extremal function u_ε for $(I^g_{4\pi(1-\varepsilon)}(\mathbb{D}^2))$. Under the assumptions of part (1), we only need to rule out the case where (2-1) holds true, as described in the proof of Theorem 1.2(1) in Section 2. Then we assume by contradiction that (2-1) holds true. By Lemma 3.4, Case 2, we get expansions of the u_ε's, and then expansions both of the Moser–Trudinger functional (see (2-4)) and of the Dirichlet energy (see (2-5)). These results are gathered in Proposition 2.1 below, whose proof (see Section 4) amounts to showing that not only M but also S in (1-9) may have to be attained at a blow-up point of our sequence of *maximizers* $(u_\varepsilon)_\varepsilon$ (see Lemma 4.1). Observe that this two-fold maximization property is necessary to get a sharp picture in Theorem 1.2. Moreover, this is not seen when restricting to the case $\Omega = \mathbb{D}^2$, where K_Ω in (1-9) contains only the single point 0, so that expanding the Dirichlet energy of a blowing-up sequence of *critical points* $(u_\varepsilon)_\varepsilon$ is sufficient; see [Mancini and Martinazzi 2017]. Theorem 1.2(1) is eventually obtained by getting a contradiction with (2-1): either by comparing (2-4) with our assumption $\Lambda_g(\Omega) \geq \pi\exp(1+M)$, or by comparing $\|u_\varepsilon\|^2_{H^1_0} = 4\pi(1-\varepsilon)$ (see (3-8) in Lemma 3.4) and (2-5) with our assumption $l > 0$.

Now we comment on part (2). Making our assumptions of part (2) and assuming also by contradiction that there exists an extremal function u_ε for $(I^{g_{N_\varepsilon}}_{4\pi}(\Omega))$ such that $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$, we get from Lemma 3.4, Case 1 that our assumption $\Lambda_g(\Omega) < \pi\exp(1+M)$ automatically implies (2-1) (see Step 2), so that we may get expansions of the u_ε's and then (2-10). This gives a contradiction by comparing

$$\|u_\varepsilon\|^2_{H^1_0} = 4\pi$$

and (2-10) with our assumption $l < 0$, as developed in the proof of Theorem 1.2(2) in Section 2. These key ingredients are gathered in Proposition 2.3. In comparison with the expansions of part (1), the key observation is that the delicate N_ε-dependence generates additional terms which may only *reduce* the Dirichlet energy, as explained in the proof of Proposition 2.3 of Section 4.

Overall, the proof of Lemma 3.4, Case 1 is the most delicate part: we need to use first that the u_ε's are *maximizers* to check that we are in a Moser–Trudinger critical regime (see Step 2 and Remark 3.5) and that the pointwise and global gradient estimate (3-52) is true. In both Cases 1 and 2, resuming the approach of [Druet and Thizy 2017], this last point is the key ingredient to be in position to use the radial model B_ε studied in the Appendix. To conclude, the case of a general domain Ω addressed by Theorem 1.2 requires sharp estimates, not only at small scales close to a blow-up point x_ε, as performed in the radial case by [Mancini and Martinazzi 2017], but also in the whole Ω (see (3-99) or (4-1)). This allows in particular to get a useful accurate expansion of the Lagrange multiplier λ_ε in (4-12), when proving Proposition 2.1. As a remark, in the process of the proof below (see Remark 2.2), we answer the very interesting Open Problem 6 of [Mancini and Martinazzi 2017].

Remark 1.5 (links between Theorem 1.2 and [Carleson and Chang 1986; Flucher 1992; Mancini and Martinazzi 2017; Struwe 1988]). For $\Omega = \mathbb{D}^2$, part (1) of Theorem 1.2 implies in general [Mancini and Martinazzi 2017, Corollary 3], which gives itself the existence of an extremal function for $(I^0_{4\pi}(\Omega))$ pioneered by [Carleson and Chang 1986] in the original case $g \equiv 0$. Even if both [Mancini and Martinazzi 2017, Corollary 3] and Theorem 1.2 are much more general, we restrict there for simplicity to g's satisfying (1-17) and coinciding with g_∞ for all $t \gg 1$; see (1-14). Then [Mancini and Martinazzi 2017, Corollary 3] covers the fast decaying case $a' > 2$ (or $c' = 0$) on the disk. By (1-15), thanks to the explicit formulas above (1-16) for $\Omega = \mathbb{D}^2$ and since $\int_{\mathbb{D}^2} (\log|x|)^2\,dx = \frac{\pi}{2}$, it is easy to check that we have in this latter case that $l > 0$ in (1-12), since we have

$$\gamma^{-4} + \tfrac{1}{2}A(\gamma) + 4\gamma^{-3}\exp(-1-M)B(\gamma)S = \gamma^{-4}\Big(1 + \frac{2}{e}(1+g(0))\Big) + o(\gamma^{-4})$$

as $\gamma \to +\infty$. Pushing further their asymptotic analysis, Mancini and Martinazzi [2017] cover also the case $a' = 2$ and then suspect (see Theorem 4–Open Problem 2 in that work) that there could be no extremal function for $(I^g_{4\pi}(\mathbb{D}^2))$, if, in addition, c' is a sufficiently large negative constant. Corollary 1.4 claims that there can actually be an extremal for such a g, whatever c' is, and even independently from the precise behavior of g close to 0 or $+\infty$. However, part (2) of Theorem 1.2 gives with (1-15) the following picture in this threshold case $a' = 2$:

$$\begin{aligned} &\text{if } c' > -\Big(1 + \frac{2}{e}(1+g(0))\Big) \text{ or } \Lambda_g(\mathbb{D}^2) \ge \pi e, \text{ there is an extremal for } (I^g_{4\pi}(\mathbb{D}^2)),\\ &\text{if } c' < -\Big(1 + \frac{2}{e}(1+g(0))\Big),\ \Lambda_g(\mathbb{D}^2) < \pi e, \text{ and } N \gg 1, \text{ there is no extremal for } (I^{g_N}_{4\pi}(\mathbb{D}^2)). \end{aligned}$$

Observe that there are many ways of building such g's satisfying $\Lambda_g(\mathbb{D}^2) < \pi e$: one is given in the proof of Corollary 1.3 in Section 2 (see also (1-16)). As observed just below Theorem 1.2, this gives a basically sharp picture about how far we can get the existence of an extremal function for $(I^g_{4\pi}(\Omega))$, *relying only on the asymptotic properties of g* (see Question 1.1). Theorem 1.2 gives a similar picture on any domain Ω, and then gives back (for $c' = 0$) the results of [Flucher 1992; Struwe 1988]. Stronger perturbations, for instance $a' < 2$ or even $a' = 0$ and $b' > 0$, are also covered by Theorem 1.2.

We conclude this introductory section by the following remark about the relevance of the assumption (1-1) on g introduced in [Mancini and Martinazzi 2017]. We also mention the nice and early result of [de Figueiredo and Ruf 1995].

Remark 1.6 (about assumption (1-1)). Indeed, assume that g is a C^1, even function such that $1+g > 0$ in $\mathbb{R}$. Assume also that $\bar{g} = \lim_{t\to+\infty} g(t) \in [-1, +\infty]$ exists. Firstly, if $\bar{g} = +\infty$, it is easy to check with the test functions of Step 1 that $C_{g,4\pi}(\Omega) = +\infty$. Secondly, if $\bar{g} = -1$, it follows from standard integration theory (see for instance [Mancini and Martinazzi 2017, Lemma 7]) and from Moser's result [1971] that there exists an extremal function for $(I^g_{4\pi}(\Omega))$. Thus, up to replacing $1+g$ by $(1+g)/(1+\bar{g})$, we have that (1-1) holds true in the remaining more sensitive case $\bar{g} \in (-1, +\infty)$. To end this remark, we mention that [de Figueiredo and Ruf 1995] already studied (1-4) in $\mathbb{D}^2$, permitting one to recover the existence of an extremal in some subcases where $\bar{g} = -1$. First, assuming that H given by (1-3) is positive in $(0, +\infty)$, it is clear that a nonnegative extremal for $(I^g_{4\pi}(\Omega))$ turns out to be a positive solution of (1-4) (for some $\lambda > 0$). Now following [de Figueiredo and Ruf 1995], assume also that $\Omega = \mathbb{D}^2$, that $t \mapsto tH(t)$ is C^2 and that, given $a > 0$, there exist $K, C, \sigma > 0$ such that $tH(t) = Kt^{-a}$ for all $t \gg 1$ and such that $H(t) \le CKt^{\sigma}$ for all $t > 0$ close to 0. Then, [de Figueiredo and Ruf 1995, Theorem 1.1] allows us to claim that there exists no positive solution of (1-4) for all $0 < \lambda \ll 1$ small enough if $a \ge 1$, while there exists a family of positive solutions of (1-4) blowing-up as $\lambda \to 0$ if $a < 1$. From by now standard arguments, this first property directly gives back the existence of an extremal in the subcase $a \ge 1$. However, observe that $\bar{g} = -1$ for all $a > 0$, since $1 + g(t) \sim 2Ke^{-t^2} \int_1^t s^{-a} e^{s^2}\, ds = O(t^{a+1}) \to 0$ as $t \to +\infty$, so that an extremal also exists in the subcase $a \in (0, 1)$. Actually we assert that a more precise analysis in the spirit of [Mancini and Martinazzi 2017] allows us to exclude that the aforementioned blow-up solutions of (1-4) are maximizers and to recover the existence of an extremal also in the subcase $a \in (0, 1)$ through this approach using the Euler–Lagrange equation.

2. Proof of the main results

We begin by proving Corollary 1.3, assuming that Theorem 1.2 holds true.

Proof of Corollary 1.3. The first part of Corollary 1.3 is a direct consequence of the first part of Theorem 1.2: plugging the formulas of (1-15) in (1-12), we get that $l > 0$ for g as in case (1) of Corollary 1.3. In order to prove the second part of Corollary 1.3, we apply the second part of Theorem 1.2. Let χ be a smooth nonnegative function in $\mathbb{R}$ such that $\chi(t) = 0$ for all $t \le \frac{1}{2}$ and $\chi(t) = 1$ for all $t \ge 1$. By the Sobolev inequality and standard integration theory, we can check that $g_R := g_\infty \times \chi(\cdot/R)$ satisfies $\Lambda_{g_R}(\mathbb{D}^2) \to \Lambda_0(\mathbb{D}^2)$ as $R \to +\infty$. Then, by (1-15), (1-16), assuming $a' < 2$, $c' < 0$, the second part of Theorem 1.2 applies, starting from $g = g_R$ for $R \gg 1$ fixed sufficiently large. Observe that, for all given $N \gg 1$, $(g_R)_N$ (given by (1-10) for $g = g_R$) satisfies (1-17)–(1-18). □

Proof of Corollary 1.4. Let Ω, $\bar{A}$, $\lambda_1(\Omega)$, C be as in the statement of the corollary. It is sufficient to prove that there exists $R \gg 1$ such that for all g satisfying (1-1) and (1-19), we have $\Lambda_g(\Omega) \ge \pi \exp(1+M)$, where $\Lambda_g(\Omega)$ is as in (1-11). Let $v > 0$ in Ω be the first eigenvalue of Δ normalized according to $\|v\|^2_{H^1_0} = 4\pi$. For all g satisfying (1-19), we have

$$\begin{aligned}\Lambda_g(\Omega) &\geq \int_\Omega ((1+g(0))v^2 + (g(v)-g(0))(1+v^2))\,dx \\ &\geq (1+\bar{A})\frac{4\pi}{\lambda_1(\Omega)} + \int_{\{v\notin[1/R,R]\}} (g(v)-g(0))(1+v^2)\,dx,\end{aligned}$$

and, since we have

$$\left|\int_{\{v\notin[1/R,R]\}} (g(v)-g(0))(1+v^2)\,dx\right| \leq (|\bar{A}|+C)(1+\|v\|^2_{L^\infty})|\{v \notin [1/R,R]\}| \to 0$$

as $R \to +\infty$, we get the result using that $4(1+\bar{A}) > \lambda_1(\Omega)\exp(1+M)$. □

The following result is the core of the argument to get the existence of an extremal in Theorem 1.2(1). Its proof is postponed until Section 4. It uses the tools developed in [Druet and Thizy 2017] that allow us to push the asymptotic analysis of a concentrating sequence of extremals $(u_\varepsilon)_\varepsilon$ further than in previous works. In the process of the proof of Proposition 2.1 (see Lemma 4.1), we show first that a concentration point $\bar{x}$ of such u_ε's realizes M in (1-9). But in the case where $|B(\gamma)|$ matters in (1-12) or, in other words, where $\gamma^3|A(\gamma)| + \gamma^{-1} \lesssim |B(\gamma)|$ as $\gamma \to +\infty$, we also show that S in (1-9) has to be attained at $\bar{x}$.

Proposition 2.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that (1-1) and (1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I^g_{4\pi(1-\varepsilon)}(\Omega))$ for all $0 < \varepsilon \ll 1$. Assume that*

$$u_\varepsilon \rightharpoonup 0 \quad \text{in } H^1_0 \tag{2-1}$$

as $\varepsilon \to 0$. Then, $\|u_\varepsilon\|^2_{H^1_0} = 4\pi(1-\varepsilon)$, there exists a sequence $(\lambda_\varepsilon)_\varepsilon$ of real numbers such that u_ε solves in H^1_0

$$\begin{cases}\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon H(u_\varepsilon)\exp(u_\varepsilon^2), & u_\varepsilon > 0 \text{ in } \Omega,\\ u_\varepsilon = 0 \text{ on } \partial\Omega,\end{cases} \tag{2-2}$$

$u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ $(0<\theta<1)$ and we have

$$\gamma_\varepsilon := \max_{y\in\Omega} u_\varepsilon \to +\infty. \tag{2-3}$$

Moreover, we have

$$\lim_{\varepsilon\to 0}\int_\Omega (1+g(u_\varepsilon))\exp(u_\varepsilon^2)\,dx = |\Omega|(1+g(0)) + \pi\exp(1+M) \tag{2-4}$$

and

$$\|u_\varepsilon\|^2_{H^1_0} = 4\pi\big(1 + I(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)|)\big) \tag{2-5}$$

as $\varepsilon \to 0$, where

$$I(\gamma_\varepsilon) := \gamma_\varepsilon^{-4} + \tfrac{1}{2}A(\gamma_\varepsilon) + 4\gamma_\varepsilon^{-3}\exp(-1-M)B(\gamma_\varepsilon)S, \tag{2-6}$$

where $|\Omega|$ stands for the volume of the domain Ω and where M and S are as in (1-9).

Remark 2.2. Let g, H be such that (1-1), (1-2), (1-5)–(1-7) hold true. Let u_ε be a maximizer for $(I^g_{4\pi(1-\varepsilon)})$ such that (2-1) holds true, as in Proposition 2.1. Then, for such a sequence $(u_\varepsilon)_\varepsilon$ satisfying in particular (2-2) and (2-3), we get in the process of the proof (see (3-16) below) that the term $I(\gamma_\varepsilon)$ in (2-5) is necessarily smaller than $o(\gamma_\varepsilon^{-2})$ as $\varepsilon \to 0$. Moreover this threshold $o(\gamma_\varepsilon^{-2})$ is sharp, in the sense that this term may be for instance of size $\gamma_\varepsilon^{-(2+a')}$ for all given $a' \in (0, 2]$. This can be seen by picking an appropriate g such that $I^g_{4\pi}(\Omega)$ has no extremal, as in Corollary 1.3, and by using Proposition 2.1. Observe that, for such a g, assumption (2-1) is indeed automatically true. This gives an answer to Open Problem 6 in [Mancini and Martinazzi 2017].

Proof of Theorem 1.2(1): existence of an extremal for $(I^g_{4\pi}(\Omega))$. We first prove the existence of an extremal stated in part (1) of Theorem 1.2. Let g be such that (1-1) and (1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Assume either that $l > 0$ in (1-12) or that $\Lambda_g(\Omega) \ge \pi \exp(1 + M)$. Using Lemma 3.1, let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I^g_{4\pi(1-\varepsilon)}(\Omega))$ for all $0 < \varepsilon \ll 1$. Then, up to a subsequence, $(u_\varepsilon)_\varepsilon$ converges a.e. and weakly in H^1_0 to some u_0. Independently, we check that

$$\lim_{\varepsilon\to 0} C_{g,4\pi(1-\varepsilon)}(\Omega) = C_{g,4\pi}(\Omega), \tag{2-7}$$

where $C_{g,\alpha}(\Omega)$ is as in $(I^g_\alpha(\Omega))$. Indeed, if one assumes by contradiction that the $C_{g,4\pi(1-\varepsilon)}(\Omega)$'s increase to some $\bar{l} < C_{g,4\pi}(\Omega)$ as $\varepsilon \to 0$, then we may choose some nonnegative u such that $\|u\|^2_{H^1_0} \le 4\pi$ and

$$\int_\Omega (1 + g(u)) \exp(u^2)\, dx > \bar{l}.$$

But, picking $v_\varepsilon = u\sqrt{1-\varepsilon}$, we have $\|v_\varepsilon\|^2_{H^1_0} \le 4\pi(1-\varepsilon)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega (1 + g(v_\varepsilon)) \exp(v_\varepsilon^2)\, dx = \int_\Omega (1 + g(u)) \exp(u^2)\, dx$$

by the dominated convergence theorem, using (1-1), $v_\varepsilon^2 \le u^2$ and $\exp(u^2) \in L^1(\Omega)$. But this contradicts the definition of $\bar{l}$ and concludes the proof of (2-7). Now, by (2-7) and since $\|u_0\|^2_{H^1_0} \le 4\pi$, in order to get that u_0 is the extremal for $(I^g_{4\pi}(\Omega))$ we look for, it is sufficient to prove that

$$\lim_{\varepsilon\to 0} \int_\Omega (1 + g(u_\varepsilon)) \exp(u_\varepsilon^2)\, dx = \int_\Omega (1 + g(u_0)) \exp(u_0^2)\, dx. \tag{2-8}$$

If $u_0 = 0$, then Proposition 2.1 gives a contradiction: either by (2-4) and (2-7) if $\Lambda_g(\Omega) \ge \pi \exp(1 + M)$, since it is clear that

$$C_{g,4\pi}(\Omega) > \Lambda_g(\Omega) + (1 + g(0))|\Omega|,$$

or by (2-5) and (2-6) if $l > 0$, since $\|u_\varepsilon\|_{H^1_0} \le 4\pi$. Thus, we necessarily have that $u_0 \ne 0$. Then, noting that

$$\|u_\varepsilon - u_0\|^2_{H^1_0} \le 4\pi - \|u_0\|^2_{H^1_0} + o(1),$$

the standard Moser–Trudinger inequality $(I^0_{4\pi}(\Omega))$ and Vitali's theorem give that (2-8) still holds true, and part (1) of Theorem 1.2 is proved in any case. □

The following proposition is the core of the argument to get the nonexistence of an extremal in Theorem 1.2(2). Its proof is postponed until Section 4.

Proposition 2.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that (1-1) and (1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Assume that $\Lambda_g(\Omega) < \pi \exp(1+M)$, where M is as in (1-9) and $\Lambda_g(\Omega)$ as in (1-11). Assume that there exists a sequence of positive integers $(N_\varepsilon)_\varepsilon$ such that*

$$\lim_{\varepsilon\to 0} N_\varepsilon = +\infty \tag{2-9}$$

and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits a nonnegative extremal u_ε for all $\varepsilon > 0$, where g_{N_ε} is as in (1-10). Then we have (2-1) and $\|u_\varepsilon\|^2_{H_0^1} = 4\pi$ for all $0 < \varepsilon \ll 1$. Moreover, we have $u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ $(0 < \theta < 1)$, (2-3) and

$$\|u_\varepsilon\|^2_{H_0^1} \le 4\pi\big(1 + I(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)|)\big) \tag{2-10}$$

as $\varepsilon \to 0$, where $I(\gamma_\varepsilon)$ is given by (2-6).

Proof of Theorem 1.2(2): nonexistence of an extremal for $(I_{4\pi}^{g_N}(\Omega))$, $N \ge N_0$. Let g be such that (1-1) and 1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Assume $l < 0$ and $\Lambda_g(\Omega) < \pi \exp(1+M)$, where l is as in (1-12), Λ_g is as in (1-11) and M is as in (1-9). In order to prove part (2) of Theorem 1.2, we assume by contradiction that there exists a sequence $(N_\varepsilon)_\varepsilon$ of positive integers satisfying (2-9) and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits an extremal for g_{N_ε} as in (1-10). We let $(u_\varepsilon)_\varepsilon$ be a sequence of nonnegative functions such that u_ε is a maximizer for $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ for all $\varepsilon > 0$. But this is not possible by Proposition 2.3, since $\|u_\varepsilon\|^2_{H_0^1} = 4\pi$ contradicts (2-10), since we also assume now $l < 0$. □

3. Blow-up analysis in the strongly perturbed Moser–Trudinger regime

We now aim to prove the main blow-up analysis results that we need to get both Propositions 2.1 and 2.3. The following preliminary lemma deals with the existence of an extremal for the perturbed Moser–Trudinger inequality $(I_\alpha^g(\Omega))$ in the subcritical case $0 < \alpha < 4\pi$. Its proof relies on integration theory combined with $(I_{4\pi}^0(\Omega))$ and on standard variational techniques. It is omitted here and the interested reader may find more details in the proof of Proposition 6 of [Mancini and Martinazzi 2017].

Lemma 3.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that (1-1) holds true. Then, $(I_\alpha^g(\Omega))$ admits a nonnegative extremal u_α for all $0 < \alpha < 4\pi$. Moreover, we have that*

(1) *either $\|u_\alpha\|^2_{H_0^1} < \alpha$ and $u_\alpha H(u_\alpha) = 0$ a.e., or*

(2) *$\|u_\alpha\|^2_{H_0^1} = \alpha$ and there exists $\lambda \in \mathbb{R}$ such that u_α solves in H_0^1 the Euler–Lagrange equation (1-4).*

Remark 3.2. The first alternative in Lemma 3.1 may occur in general, but does not if $t \mapsto (1+g(t))\exp(t^2)$ increases in $(0, +\infty)$.

The following lemma investigates more precisely the behavior of g and H when we assume (1-1) together with (1-5)–(1-6).

Lemma 3.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that (1-1), (1-5) and (1-6) hold true for H as in (1-2), and let A, B and δ_0, δ_0', F, κ be thus given. Then*:

(3-1a) *We have*

$$\left(1+g\left(\frac{t}{\gamma}\right)\right)\exp\left(\frac{t^2}{\gamma^2}\right)=(1+g(0))+\frac{2B(\gamma)F(t)t}{\gamma(\kappa+1)}+o\left(\frac{|B(\gamma)|}{\gamma}+\frac{1}{\gamma^2}\right)$$

in $C^0_{\mathrm{loc}}((0,+\infty)_t)$ as $\gamma\to+\infty$.

(3-1b) *There exists $C>0$ such that*

$$\left|\left(1+g\left(\frac{t}{\gamma}\right)\right)\exp\left(\frac{t^2}{\gamma^2}\right)-(1+g(0))\right|\le C\left(\frac{|B(\gamma)|}{\gamma}+\frac{1}{\gamma^2}\right)t\exp(\delta_0' t)$$

for all $\gamma\gg 1$ and all $0\le t\le 2\gamma$.

(3-1c) $\|g\|_{L^\infty(\mathbb{R})}<+\infty$.

Additionally:

(3-2a) *We have*

$$1+g\left(\gamma-\frac{t}{\gamma}\right)=H(\gamma)\big(1+A(\gamma)\big(t+\tfrac{1}{2}\big)+o(|A(\gamma)|+\gamma^{-4})\big)$$

in $C^0_{\mathrm{loc}}(\mathbb{R}_t)$ as $\gamma\to+\infty$.

(3-2b) *There exists $C>0$ such that*

$$\left|1+g\left(\gamma-\frac{t}{\gamma}\right)-H(\gamma)\right|\le C|H(\gamma)|(|A(\gamma)|+\gamma^{-4})\exp(\delta_0 t)$$

for all $\gamma\gg 1$ and all $0\le t\le 2\gamma$.

In particular, we have

$$H(\gamma)\to 1\quad \text{as } \gamma\to+\infty. \tag{3-3}$$

Proof of Lemma 3.3. We first prove (3-3). Using (1-3), we write

$$(1+g(r))\exp(r^2)-(1+g(0))=2\int_0^r sH(s)\exp(s^2)\,ds \tag{3-4}$$

for all $r\ge 0$. Then, as $\gamma\to+\infty$, setting $r=\gamma$, we can write

$$\begin{aligned}1+g(\gamma)&=\exp(-\gamma^2)(1+g(0))+2\int_0^{\gamma^2}\left(1-\frac{u}{\gamma^2}\right)H\left(\gamma-\frac{u}{\gamma}\right)\exp\left(-2u+\frac{u^2}{\gamma^2}\right)du\\&=O(\exp(-\gamma^2))+2H(\gamma)\int_0^{\gamma^2}\left(1-\frac{u}{\gamma^2}\right)\exp\left(-2u+\frac{u^2}{\gamma^2}\right)du\\&\qquad+O\left(|H(\gamma)|(|A(\gamma)|+\gamma^{-4})\int_0^{\gamma^2}\exp(-(1-\delta_0)u)\exp\left(-u\left(1-\frac{u}{\gamma^2}\right)\right)du\right)\\&=O(\exp(-\gamma^2))+H(\gamma)(1-\exp(-\gamma^2))+o(H(\gamma)),\end{aligned}$$

using (1-5). This proves (3-3) since g satisfies (1-1). Observe that (3-1a) and (3-1b) follow from (1-6) and (3-4) with $r = t/\gamma$, while (3-1c) is a straightforward consequence of (1-1). We prove now (3-2b). As $\gamma \to +\infty$, we write for all $0 \le t \le \gamma$

$$\begin{aligned}
&\left(1+g\left(\gamma-\frac{t}{\gamma}\right)\right)\exp\left(\left(\gamma-\frac{t}{\gamma}\right)^2\right)-(1+g(\gamma-1))\exp((\gamma-1)^2)\\
&=2\int_{\gamma-1}^{\gamma-t/\gamma} rH(r)\exp(r^2)\,dr\\
&=2\int_t^{\gamma}\left(1-\frac{u}{\gamma^2}\right)H\left(\gamma-\frac{u}{\gamma}\right)\exp\left(\gamma^2-2u+\frac{u^2}{\gamma^2}\right)du\\
&=H(\gamma)\left(\exp\left(\left(\gamma-\frac{t}{\gamma}\right)^2\right)-\exp((\gamma-1)^2)\right)+O\left(|H(\gamma)|(|A(\gamma)|+\gamma^{-4})\int_t^{\gamma}\exp(\gamma^2-(2-\delta_0)u)\,du\right), \qquad (3\text{-}5)
\end{aligned}$$

using (1-5b). Multiplying the above identity by $\exp(-(\gamma-(t/\gamma))^2)$, using $t \le \gamma$, (1-1) and (3-3), (3-2b) easily follows. Using now (1-5a) in the second-to-last line of (3-5), we also get (3-2a). ☐

In the sequel, for all integers $N \ge 1$, we let φ_N be given by (see also (3-36) below)

$$\varphi_N(t) = \sum_{k=N+1}^{+\infty} \frac{t^k}{k!}. \qquad (3\text{-}6)$$

The main results of this section are stated in the following lemma.

Lemma 3.4. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that (1-1) and (1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Let $(\alpha_\varepsilon)_\varepsilon$ be a sequence of numbers in $(0, 4\pi]$. Let $(N_\varepsilon)_\varepsilon$ be a sequence of positive integers. Assume*

$$\lim_{\varepsilon\to 0}\alpha_\varepsilon = 4\pi \quad \text{and} \quad u_\varepsilon \ge 0 \text{ is an extremal for } (I^{g_{N_\varepsilon}}_{\alpha_\varepsilon}(\Omega)) \qquad (3\text{-}7)$$

for all $0 < \varepsilon \ll 1$, where g_{N_ε} is as in (1-10). Assume in addition that we are in one of the following two cases:

Case 1: $\lim_{\varepsilon\to 0} N_\varepsilon = +\infty$, $\alpha_\varepsilon = 4\pi$ for all ε, and $\Lambda_g(\Omega) < \pi\exp(1+M)$, where $\Lambda_g(\Omega)$ is as in (1-11) and M is as in (1-9).

Case 2: $N_\varepsilon = 1$ for all ε and (2-1) holds true.

Then, up to a subsequence,

$$\|u_\varepsilon\|^2_{H^1_0} = \alpha_\varepsilon, \qquad (3\text{-}8)$$

and $u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ $(0 < \theta < 1)$ solves

$$\begin{cases} \Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon H_{N_\varepsilon}(u_\varepsilon)\exp(u_\varepsilon^2), & u_\varepsilon > 0 \text{ in } \Omega,\\ u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \qquad (3\text{-}9)$$

where $H_N(t) = 1 + g_N(t) + g'_N(t)/(2t)$. Moreover, by (2-4) we have

$$\lambda_\varepsilon = \frac{4+o(1)}{\gamma_\varepsilon^2\exp(1+M)}, \qquad (3\text{-}10)$$

$$A(\gamma_\varepsilon) - 2\xi_\varepsilon = o(\tilde{\zeta}_\varepsilon), \tag{3-11}$$

$$x_\varepsilon \to \bar{x} \quad (\bar{x} \in K_\Omega) \tag{3-12}$$

as $\varepsilon \to 0$, *where* x_ε, γ_ε *satisfy*

$$u_\varepsilon(x_\varepsilon) = \max_\Omega u_\varepsilon = \gamma_\varepsilon \to +\infty \tag{3-13}$$

as $\varepsilon \to 0$, *where* ξ_ε *is given by*

$$\xi_\varepsilon = \frac{\gamma_\varepsilon^{2(N_\varepsilon-1)}}{\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2)(N_\varepsilon-1)!}, \tag{3-14}$$

and where $\tilde{\zeta}_\varepsilon$ *is given by*

$$\tilde{\zeta}_\varepsilon = \max\left(\frac{1}{\gamma_\varepsilon^2}, |A(\gamma_\varepsilon)|, \xi_\varepsilon\right). \tag{3-15}$$

Finally, (3-97)–(3-99) *below hold true for* μ_ε *as in* (3-40) *and* t_ε *as in* (3-41).

Observe that $N_\varepsilon = 1$ in Case 2 reduces to say that $g_{N_\varepsilon} = g$. From (3-30) obtained in the process of the proof below, we get that $\xi_\varepsilon = o(1/\gamma_\varepsilon^2)$ in Case 2, so that (3-11) is then equivalent to

$$A(\gamma_\varepsilon) = o\left(\frac{1}{\gamma_\varepsilon^2}\right), \tag{3-16}$$

as discussed in Remark 2.2.

Proof of Lemma 3.4. We start by several basic steps. First, a test function computation gives the following result.

Step 1. *For all* g *such that* (1-1) *holds true*, *we have*

$$C_{g,4\pi}(\Omega) \geq |\Omega|(1+g(0)) + \pi \exp(1+M),$$

where $C_{g,4\pi}(\Omega)$ *is as in* $(I_\alpha^g(\Omega))$ $(\alpha = 4\pi)$ *and where* M *is as in* (1-9).

Proof of Step 1. In order to get Step 1, it is sufficient to prove that there exist functions $f_\varepsilon \in H_0^1$ such that $\|f_\varepsilon\|_{H_0^1}^2 = 4\pi$ and such that

$$\int_\Omega (1+g(f_\varepsilon)) \exp(f_\varepsilon^2)\, dy \geq |\Omega|(1+g(0)) + \pi \exp(1+M) + o(1) \tag{3-17}$$

as $\varepsilon \to 0$. In order to reuse these computations later, we fix any sequence $(z_\varepsilon)_\varepsilon$ of points in Ω such that

$$\frac{\varepsilon^2}{d(z_\varepsilon, \partial\Omega)^2} = o\left(\left(\log \frac{1}{\varepsilon}\right)^{-1}\right). \tag{3-18}$$

For $0 < \varepsilon < 1$, we let v_ε be given by

$$v_\varepsilon(y) = \log \frac{1}{\varepsilon^2 + |y - z_\varepsilon|^2} + \mathcal{H}_{z_\varepsilon,\varepsilon},$$

where $\mathcal{H}_{z_\varepsilon,\varepsilon}$ is harmonic in Ω and such that v_ε is zero on $\partial\Omega$. Then, by the maximum principle and (1-8), we have

$$\mathcal{H}_{z_\varepsilon,\varepsilon}(y) = \mathcal{H}_{z_\varepsilon}(y) + O\left(\frac{\varepsilon^2}{d(z_\varepsilon, \partial\Omega)^2}\right) \quad \text{for all } y \in \Omega, \tag{3-19}$$

where $\mathcal{H}_{z_\varepsilon}$ is as in (1-8). Then, integrating by parts, we compute

$$\begin{aligned}\|v_\varepsilon\|^2_{H^1_0} &= \int_\Omega v_\varepsilon \Delta v_\varepsilon \, dy \\ &= \int_\Omega \frac{4}{\varepsilon^2(1+|z_\varepsilon - y|^2/\varepsilon^2)^2}\left(\log\frac{1}{\varepsilon^2} + \log\frac{1}{1+|y-z_\varepsilon|^2/\varepsilon^2} + \mathcal{H}_{z_\varepsilon,\varepsilon}(y)\right) dy \\ &= 4\pi\left(\log\frac{1}{\varepsilon^2} + o(1)\right) - 4\pi(1+o(1)) + 4\pi(\mathcal{H}_{z_\varepsilon}(z_\varepsilon) + o(1)) \\ &= 4\pi\left(\log\frac{1}{\varepsilon^2} - 1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon)\right) + o(1),\end{aligned} \tag{3-20}$$

where the change of variable $z = (y - z_\varepsilon)/\varepsilon$, (3-18), (3-19) and

$$\mathcal{H}_{z_\varepsilon}(z_\varepsilon + \varepsilon z) = \mathcal{H}_{z_\varepsilon}(z_\varepsilon) + O\left(\frac{\varepsilon|z|}{d(z_\varepsilon, \partial\Omega)}\right), \tag{3-21}$$

(see for instance Appendix B in [Druet and Thizy 2017]) are used. From now on $\liminf_{\varepsilon\to 0} d(z_\varepsilon, \partial\Omega) > 0$ is assumed. Let f_ε be given by $4\pi v_\varepsilon^2 = f_\varepsilon^2 \|v_\varepsilon\|^2_{H^1_0}$. We can write

$$f_\varepsilon(y)^2 = \left(\left(\log\frac{1}{|z_\varepsilon - y|^2 + \varepsilon^2}\right)^2 + 2\mathcal{H}_{z_\varepsilon,\varepsilon}(y)\log\frac{1}{|z_\varepsilon - y|^2+\varepsilon^2} + \mathcal{H}_{z_\varepsilon,\varepsilon}(y)^2\right) \times \left(\log\frac{1}{\varepsilon^2}\left(1 + \frac{\mathcal{H}_{z_\varepsilon}(z_\varepsilon) - 1}{\log 1/\varepsilon^2} + o\left(\frac{1}{\log 1/\varepsilon}\right)\right)\right)^{-1},$$

using (3-20). Then, writing

$$\log\frac{1}{|z_\varepsilon - y|^2 + \varepsilon^2} = \log\frac{1}{\varepsilon^2} + \log\frac{1}{1+|z_\varepsilon - y|^2/\varepsilon^2},$$

we get

$$\begin{aligned}&\int_{B_{z_\varepsilon}(\check{r}_\varepsilon)\cap\Omega} (1+g(f_\varepsilon))\exp(f_\varepsilon^2)\, dy \\ &\quad = \int_{B_{z_\varepsilon}(\check{r}_\varepsilon)\cap\Omega} (1+o(1))\frac{\exp(-2\check{t}_\varepsilon(y) + 2\mathcal{H}_{z_\varepsilon,\varepsilon}(y) - \mathcal{H}_{z_\varepsilon}(z_\varepsilon) + 1)}{\varepsilon^2} \\ &\qquad\qquad \times \exp\left(\frac{\check{t}_\varepsilon^2}{\log 1/\varepsilon^2} + O\left(\frac{1+\check{t}_\varepsilon}{\log 1/\varepsilon^2} + \frac{1+\check{t}_\varepsilon^2}{(\log 1/\varepsilon^2)^2}\right)\right) dy \\ &\quad = \pi\exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon) + 1)(1+o(1))\end{aligned} \tag{3-22}$$

as $\varepsilon \to 0$, using (1-1), (3-19) and (3-21), where $\check{t}_\varepsilon(y) = \log(1 + |z_\varepsilon - y|^2/\varepsilon^2)$ and where $\check{r}_\varepsilon$ is given by

$$\log\left(1 + \frac{\check{r}_\varepsilon^2}{\varepsilon^2}\right) = \frac{1}{2}\log\frac{1}{\varepsilon^2}.$$

Now, we can check that

$$f_\varepsilon(y)^2 \le \left(\log\frac{1}{\varepsilon^2}+O(1)\right)^{-1}\left(\log\frac{1}{|z_\varepsilon-y|^2}+O(1)\right)^2$$
$$\le \left(\log\frac{1}{|z_\varepsilon-y|^2}+O(1)\right)\left(\tfrac{1}{2}+o(1)\right) \quad \text{for all } y\in\Omega\backslash B_{z_\varepsilon}(\check r_\varepsilon),$$

using (1-8), (3-19) and our definition of $\check r_\varepsilon$, so that we also get

$$\int_{\Omega\backslash B_{z_\varepsilon}(\check r_\varepsilon)}(1+g(f_\varepsilon))\exp(f_\varepsilon^2)\,dy \to (1+g(0))|\Omega| \tag{3-23}$$

as $\varepsilon\to 0$, by the dominated convergence theorem, using (1-1). Property (3-17) and then Step 1 follow from (3-22) and (3-23), choosing $z_\varepsilon \in K_\Omega$ as in (1-9). □

From now on, we make the assumptions of Lemma 3.4. In particular, we assume that either Case 1 or Case 2 holds true. Given an integer $N\ge 1$, observe that Step 1 applies to g_N, since g_N satisfies (1-1), if g does. Then, using $\alpha_\varepsilon = 4\pi$ in Case 1, or (2-7) and $g_{N_\varepsilon}=g$ in Case 2, we get

$$|\Omega|(1+g(0))+\pi\exp(1+M) \le \begin{cases} C_{g_{N_\varepsilon},4\pi} & \text{in Case 1,}\\ C_{g,\alpha_\varepsilon}+o(1) & \text{in Case 2}\end{cases} \tag{3-24}$$

as $\varepsilon\to 0^+$, where $C_{g,\alpha}(\Omega)$ is as in formula $(I_\alpha^g(\Omega))$ and where M is as in (1-9). Let us rewrite now (3-9) in a more convenient way. Let Ψ_N be given by

$$\Psi_N(t) = (1+g_N(t))\exp(t^2). \tag{3-25}$$

Observe in particular that

$$(1+g(t))(1+t^2)\le \Psi_N(t)\le (1+g(t))\exp(t^2)$$

for all t and all N, by (1-1). Using (1-2), (1-3) and (1-10), we may rewrite (3-9) as

$$\begin{cases}\Delta u_\varepsilon = \tfrac{1}{2}\lambda_\varepsilon \Psi'_{N_\varepsilon}(u_\varepsilon), & u_\varepsilon>0 \text{ in } \Omega,\\ u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{3-26}$$

with

$$\Psi'_N(t) = 2tH(t)(1+t^2+\varphi_N(t^2)) + 2t(1+g(t))\left(\frac{t^{2N}}{N!}-t^2\right)$$
$$= 2tH(t)\varphi_N(t^2)+2t\left(1+\frac{t^{2N}}{N!}\right)(1+g(t))+g'(t)(1+t^2). \tag{3-27}$$

Indeed, in (3-9), it turns out that

$$H_N(t) = \frac{\Psi'_N(t)\exp(-t^2)}{2t}. \tag{3-28}$$

Observe that by (1-1) and (3-3), using the first line of (3-27), we clearly have that there exists $C>0$ such that

$$|\Psi'_{N_\varepsilon}(t)|\le Ct\exp(t^2) \tag{3-29}$$

for all $t\ge 0$ and all ε. In Case 2, (2-1) is assumed to be true.

Step 2. *Assume that we are in Case 1. Then (2-1) holds true. Moreover, if $\gamma_\varepsilon := \operatorname{ess\,sup} u_\varepsilon < +\infty$ for all ε, we have*

$$\liminf_{\varepsilon\to 0} \underbrace{\frac{\varphi_{N_\varepsilon}(\gamma_\varepsilon^2)}{\exp(\gamma_\varepsilon^2)}}_{:=\delta_\varepsilon\in(0,1)} > 0, \tag{3-30}$$

and, in other words,

$$\liminf_{\varepsilon\to 0} \frac{\gamma_\varepsilon^2 - N_\varepsilon}{\sqrt{N_\varepsilon}} > -\infty, \tag{3-31}$$

where φ_N is as in (3-6).

Proof of Step 2. By (3-7) and (3-24), we get

$$\int_\Omega \Psi_{N_\varepsilon}(u_\varepsilon)\, dy \geq (1+g(0))|\Omega| + \pi \exp(1+M). \tag{3-32}$$

Writing now

$$\Psi_N(t) = (1+g(0)) + \big((1+g(t))(1+t^2) - (1+g(0))\big) + (1+g(t))\varphi_N(t^2)$$

and using (1-1) we also get

$$\int_\Omega \Psi_{N_\varepsilon}(u_\varepsilon)\, dy \leq (1+g(0))|\Omega| + \Lambda_g(\Omega) + \int_\Omega (1+g(u_\varepsilon))\varphi_{N_\varepsilon}(u_\varepsilon^2)\, dy, \tag{3-33}$$

where Λ_g is as in (1-11). Then by (1-1) and Case 1, we get from (3-32) and (3-33) that

$$\liminf_{\varepsilon\to 0} \int_\Omega \varphi_{N_\varepsilon}(u_\varepsilon^2)\, dy > 0. \tag{3-34}$$

Up to a subsequence, $u_\varepsilon \rightharpoonup u_0$ in H_0^1 for some $u_0 \in H_0^1$ such that $\|u_0\|_{H_0^1}^2 \leq 4\pi$. Let $0 < \beta \ll 1$ be given. First we have

$$u_\varepsilon^2 \leq (1+\beta)(u_\varepsilon - u_0)^2 + \Big(1 + \frac{1}{\beta}\Big)u_0^2.$$

Independently, by the Moser–Trudinger inequality, we have

$$u \in H_0^1 \quad \Longrightarrow \quad \text{for all } p \in [1,+\infty),\ \ \exp(u^2) \in L^p. \tag{3-35}$$

Therefore, if $u_0 \not\equiv 0$ and $\lim_{\varepsilon\to 0} \|u_\varepsilon - u_0\|_{H_0^1}^2 < 4\pi$, then there exists $p_0 > 1$ such that $(\exp(u_\varepsilon^2))_\varepsilon$ is bounded in L^{p_0}, by Moser's and Hölder's inequalities. Then, by Vitali's theorem, since $\varphi_{N_\varepsilon} \leq \exp$ in $[0,+\infty)$ and since $N_\varepsilon \to +\infty$ in Case 1, we get

$$u_0 \not\equiv 0 \quad \Longrightarrow \quad \int_\Omega \varphi_{N_\varepsilon}(u_\varepsilon^2)\, dy = o(1)$$

as $\varepsilon \to 0$, which proves (2-1), in view of (3-34). Noting that the function $t \mapsto \varphi_N(t)\exp(-t)$ increases in $[0,+\infty)$, we can write

$$\int_\Omega \varphi_{N_\varepsilon}(u_\varepsilon^2)\, dy \leq \frac{\varphi_{N_\varepsilon}(\gamma_\varepsilon^2)}{\exp(\gamma_\varepsilon^2)} \int_\Omega \exp(u_\varepsilon^2)\, dy$$

and conclude that (3-30) holds true by (3-34) and Moser's inequality. Observe that

$$\varphi_N(\Gamma) = \exp(\Gamma)\int_0^\Gamma \exp(-s)\frac{s^N}{N!}\,ds. \tag{3-36}$$

Setting $\Gamma = \gamma_\varepsilon^2$, $N = N_\varepsilon$ and $s = N_\varepsilon + u\sqrt{N_\varepsilon}$, we get (3-31) from (3-30), using Stirling's formula and

$$\left(1+\frac{u}{\sqrt{N}}\right)^N e^{-u\sqrt{N}} \le e^{-u^2/2}$$

for $-\sqrt{N} < u < 0$. □

The next steps applies in both Case 1 and Case 2.

Step 3. *We have that (3-8), (3-9) hold true, and that u_ε is in $C^{1,\theta}(\bar{\Omega})$.*

Proof of Step 3. Assume by contradiction that (3-8) does not hold true, or in other words that $\|u_\varepsilon\|^2_{H_0^1} < \alpha_\varepsilon$ for all $\varepsilon \ll 1$, up to a subsequence; then it follows from the fact that u_ε is an (unconstrained) critical point of our functional that $\Psi'_\varepsilon(u_\varepsilon) = 0$ a.e. in Ω. The key property is now that the Lebesgue measure of $\{t_0 < u_\varepsilon \le t_1\}$ is positive for all $0 \le t_0 < t_1 \le \gamma_\varepsilon$, as it follows by $\int_\Omega |\nabla T u_\varepsilon|^2 > 0$, where $Tu_\varepsilon \in H_0^1$ is the truncation of $u_\varepsilon - t_0$ as 0 when $u_\varepsilon \le t_0$ and as $t_1 - t_0$ when $u_\varepsilon > t_1$; this shows that $\Psi'_{N_\varepsilon} = 0$ in $(0, \gamma_\varepsilon)$ and then

$$(1+g(t)) = \frac{1+g(0)}{1+t^2+\varphi_{N_\varepsilon}(t^2)} \tag{3-37}$$

for all $t \in [0, \gamma_\varepsilon)$. If $\gamma_\varepsilon = +\infty$, a contradiction arises; then $\gamma_\varepsilon < +\infty$ and one can use Step 2 to show that $\gamma_\varepsilon \to +\infty$, still reaching a contradiction. Then (3-8) is proved, so that (3-9) holds true in H_0^1. Thus for all given ε, u_ε is uniformly bounded and then in $C^{1,\theta}$ by (3-9) and elliptic theory. We also use there that g appearing in the formula (3-27) of Ψ'_N is assumed to be C^1 in (1-1). □

The previous steps give in particular that (3-13) makes sense and holds true.

Step 4. *It holds that $\lambda_\varepsilon > 0$ for all $0 < \varepsilon \ll 1$. Moreover*

$$\lambda_\varepsilon \to 0 \tag{3-38}$$

as $\varepsilon \to 0$, where λ_ε is as in (3-9).

Proof of Step 4. By (2-1), we have $u_\varepsilon \to 0$ a.e. and in L^p for all $p < +\infty$. Since $\int_{u_\varepsilon \le M_0} \Psi_{N_\varepsilon}(u_\varepsilon)\,dx \to (1+g(0))|\Omega|$, by (3-24) one has

$$\liminf_{\varepsilon\to 0} \int_{u_\varepsilon > M_0} \Psi_{N_\varepsilon}(u_\varepsilon)\,dx \ge \pi \exp(1+M)$$

for all given $M_0 > 0$; one can now use (3-27) with (1-1), (3-3) and some standard integration argument to get

$$\liminf_{\varepsilon\to 0} \int_\Omega [\Psi'_{N_\varepsilon}(u_\varepsilon) + 2(1+g(u_\varepsilon))u_\varepsilon^3]u_\varepsilon\,dx = +\infty. \tag{3-39}$$

Then, multiplying (3-26) by u_ε and integrating by parts, we get $\lambda_\varepsilon > 0$ and

$$4\pi + o(1) = \int_\Omega |\nabla u_\varepsilon|^2\,dx \gg \lambda_\varepsilon,$$

which proves (3-38). □

Then, using (3-3), we may let $\mu_\varepsilon > 0$ be given by

$$\lambda_\varepsilon H(\gamma_\varepsilon)\mu_\varepsilon^2\gamma_\varepsilon^2\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2) = 4, \tag{3-40}$$

where φ_N is as in (3-6). Before starting the core of the proof, we would like to make a parenthetical remark.

Remark 3.5. Case 1 is particularly delicate to handle, since the nonlinearities $(\Psi'_{N_\varepsilon})_\varepsilon$ are not of *uniform critical growth*, even in the very general framework of [Druet 2006, Definition 1]. A more intuitive way to see this is the following: if $(\tilde{\gamma}_\varepsilon)_\varepsilon$ is a sequence of positive real numbers such that $\tilde{\gamma}_\varepsilon \to +\infty$, but not too fast, in the sense that $\tilde{\gamma}_\varepsilon^2 \ll N_\varepsilon$, then it can be checked with (1-1) and (3-3) that

$$\tfrac{1}{2}\lambda_\varepsilon \Psi'_{N_\varepsilon}(\tilde{\gamma}_\varepsilon) = \tilde{\lambda}_\varepsilon(1+o(1))\tilde{\gamma}_\varepsilon^{2N_\varepsilon+1}$$

as $\varepsilon \to 0$, where $\tilde{\lambda}_\varepsilon = \lambda_\varepsilon/(N_\varepsilon!)$. Then, in the regime $0 \le u_\varepsilon \le \tilde{\gamma}_\varepsilon$, at least formally, (3-26) looks at first order like the Lane–Emden problem, namely

$$\begin{cases} \Delta u_\varepsilon = \tilde{\lambda}_\varepsilon u_\varepsilon^{2N_\varepsilon+1}, \quad u_\varepsilon > 0 \text{ in } \Omega, \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \\ N_\varepsilon \to +\infty, \end{cases}$$

for which very interesting, but very different concentration phenomena were pointed out; see for instance [Adimurthi and Grossi 2004; De Marchis et al. 2016; 2017; Esposito et al. 2006; Ren and Wei 1994; 1996]. A real difficulty in concluding the subsequent proofs is to extend the analysis developed in [Adimurthi and Druet 2004; Druet 2006; Druet and Thizy 2017] for the Moser–Trudinger "purely critical" regime, in order to deal also with such other intermediate regimes. As a last remark, a much simpler version of the techniques developed here permits us also to answer some open questions about the Lane–Emden problem, as performed in [Thizy 2019].

We let t_ε be given by

$$t_\varepsilon(x) = \log\left(1 + \frac{|x-x_\varepsilon|^2}{\mu_\varepsilon^2}\right). \tag{3-41}$$

Here and in the sequel, for a radially symmetric function f around of x_ε (resp. around 0), we will often write $f(r)$ instead of $f(x)$ for $|x-x_\varepsilon| = r$ (resp. $|x| = r$).

Step 5. *We have*

$$\gamma_\varepsilon(\gamma_\varepsilon - u_\varepsilon(x_\varepsilon + \mu_\varepsilon\,\cdot)) \to T_0 := \log(1+|\cdot|^2) \quad \textit{in } C^{1,\theta}_{\mathrm{loc}}(\mathbb{R}^2), \tag{3-42}$$

where $\gamma_\varepsilon, x_\varepsilon$ are as in (3-13) and μ_ε is as in (3-40). Moreover, we have

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon\gamma_\varepsilon^2 > 0. \tag{3-43}$$

At this stage, by taking the log of (3-40), by estimating λ_ε with (3-38) and (3-43) we get from (3-3) and (3-30) that

$$\log\frac{1}{\mu_\varepsilon^2} = \gamma_\varepsilon^2(1+o(1)) \tag{3-44}$$

as $\varepsilon \to 0$. Observe in particular that (3-44) holds true in Case 1.

Proof of Step 5. We first sketch the proof of (3-42). In Case 2, (3-42) follows closely Step 1 of the proof of [Druet 2006, Proposition 1]. Thus, we focus now on the proof of (3-42) in Case 1. Observe that

$$\sup_{t\in\mathbb{R}} \frac{t^{2N}}{N!}\exp(-t^2) = \frac{N^N}{N!}\exp(-N) \underset{N\to+\infty}{=} \frac{1+o(1)}{\sqrt{2\pi N}} \tag{3-45}$$

by Stirling's formula. Then, by (1-1), (3-3), (3-13), (3-27) and (3-30), we have

$$\begin{aligned}\tfrac{1}{2}\Psi'_{N_\varepsilon}(u_\varepsilon) &= u_\varepsilon H(u_\varepsilon)\varphi_{N_\varepsilon}(u_\varepsilon^2) + u_\varepsilon(1+g(u_\varepsilon))\frac{u_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} + O(\gamma_\varepsilon^3)\\ &\le (1+o(1))\gamma_\varepsilon\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2).\end{aligned} \tag{3-46}$$

Observe that, by (3-13) and elliptic theory, we must have $\sup_\Omega \lambda_\varepsilon \Psi'_{N_\varepsilon}(u_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Then, (3-46) implies $\lambda_\varepsilon\gamma_\varepsilon\varphi_{N_\varepsilon-1}(\gamma_\varepsilon^2) \to +\infty$ and then $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$, by (3-40). Let τ_ε be given in $(\Omega - x_\varepsilon)/\mu_\varepsilon$ by

$$u_\varepsilon(x_\varepsilon + \mu_\varepsilon\,\cdot\,) = \gamma_\varepsilon - \frac{\tau_\varepsilon}{\gamma_\varepsilon}.$$

Then, since $\Delta\tau_\varepsilon = -\mu_\varepsilon^2\gamma_\varepsilon(\Delta u_\varepsilon)(x_\varepsilon + \mu_\varepsilon\,\cdot\,)$, we get from (3-26), (3-40) and (3-46) that there exists $C > 0$ such that $|\Delta\tau_\varepsilon| \le C$, while $\tau_\varepsilon \ge 0$, $\tau_\varepsilon(0) = 0$. As in [Druet 2006, p. 231], we have $\mu_\varepsilon = o(d(x_\varepsilon, \partial\Omega))$. Then, by standard elliptic theory, there exists τ_0 such that

$$\tau_\varepsilon \to \tau_0 \quad \text{in } C^{1,\theta}_{\text{loc}}(\mathbb{R}^2) \tag{3-47}$$

as $\varepsilon \to 0$. Note that for all Γ, $T > 0$ and all N, we have

$$\varphi_N(T) = \varphi_N(\Gamma)\exp(-(\Gamma - T)) - \exp(T)\int_T^\Gamma \exp(-s)\frac{s^N}{N!}\,ds. \tag{3-48}$$

Writing the previous identity for $N = N_\varepsilon - 1$, $\Gamma = \gamma_\varepsilon^2$ and $T = u_\varepsilon^2 = \gamma_\varepsilon^2 - 2\tau_\varepsilon + \tau_\varepsilon^2/\gamma_\varepsilon^2$, noting from (3-45) and (3-47) that

$$\int_{u_\varepsilon^2}^{\gamma_\varepsilon^2} \exp(-s)\frac{s^{N_\varepsilon-1}}{(N_\varepsilon-1)!}\,ds = O\left(\frac{1}{\sqrt{N_\varepsilon}}\right)$$

in $\mathbb{R}^2_{\text{loc}}$ and resuming the arguments to get (3-46), we get

$$\Delta(-\tau_0) = 4\exp(-2\tau_0)$$

using also (3-26), (3-30) and (3-40). Now, choosing $R \gg 1$ such that $|g(t)| < 1$ and $H(t) > 0$ for all $t \ge R$, we easily see that there exists $C_R > 0$ such that

$$u_\varepsilon[\Psi'_{N_\varepsilon}(u_\varepsilon)]^- \le C_R|u_\varepsilon| + 4u_\varepsilon^4, \tag{3-49}$$

by (1-1), (3-3) and (3-27), where $t^- = -\min(t, 0)$. Then, we have

$$\frac{\lambda_\varepsilon}{2}\int_\Omega u_\varepsilon[\Psi'_{N_\varepsilon}(u_\varepsilon)]^+\,dy = 4\pi + o(1),$$

by (3-8), (3-26), (3-38) and (3-49), where $t^+ = \max(t, 0)$. For all $A \gg 1$, we get

$$4 \int_{B_0(A)} \exp(-2\tau_0)\, dy \le \liminf_{\varepsilon \to 0} \frac{\lambda_\varepsilon}{2} \int_\Omega u_\varepsilon [\Psi'_{N_\varepsilon}(u_\varepsilon)]^+ \, dy,$$

by (3-47) and, since A is arbitrary, we get then that $\int_{\mathbb{R}^2} \exp(-2\tau_0)\, dy < +\infty$. Thus, by the classification result of [Chen and Li 1991], since $\tau_0 \ge 0$ and $\tau_0(0) = 0$, we get $\tau_0(y) = \log(1 + |y|^2)$. Thus (3-42) is proved by (3-47). Similarly, we may also choose some A_ε's such that $A_\varepsilon \to +\infty$ and such that

$$\frac{\lambda_\varepsilon}{2} \int_{B_{x_\varepsilon}(A_\varepsilon \mu_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon)\, dy = \frac{2\pi + o(1)}{\gamma_\varepsilon^2}.$$

We use (3-45) to write

$$\frac{\varphi_{N_\varepsilon}(\gamma_\varepsilon^2)}{\varphi_{N_\varepsilon - 1}(\gamma_\varepsilon^2)} = 1 - \frac{\gamma_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\, \varphi_{N_\varepsilon - 1}(\gamma_\varepsilon^2)} = 1 + o(1)$$

as $\varepsilon \to 0$. Thus, since $0 < \Psi_{N_\varepsilon}(t) \le (1 + g(t)) \exp(t^2)$ for all $t \ge 0$, and since $C_{g,4\pi}(\Omega) < +\infty$, we get (3-43) from (1-1). □

By Step 5 and estimates in its proof, since we assume $\|u_\varepsilon\|_{H_0^1}^2 \le 4\pi$, we get that

$$\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \int_{\Omega \setminus B_{x_\varepsilon}(R\mu_\varepsilon)} (\Delta u_\varepsilon(y))^+ u_\varepsilon \, dy = 0. \tag{3-50}$$

We let Ω_ε be given by

$$\Omega_\varepsilon = \begin{cases} \{y \in \Omega : \varphi_{N_\varepsilon - 1}(u_\varepsilon(y)^2) \ge u_\varepsilon(y)^2 + 1\} & \text{in Case 1,} \\ \Omega & \text{in Case 2.} \end{cases}$$

Now, despite the difficulty pointed out in Remark 3.5, we are able to get the following weak, but global pointwise estimates.

Step 6. *There exists $C > 0$ such that*

$$|\cdot - x_\varepsilon|^2 |\Delta u_\varepsilon| u_\varepsilon \le C \quad \text{in } \Omega_\varepsilon \tag{3-51}$$

and such that

$$|\cdot - x_\varepsilon| |\nabla u_\varepsilon| u_\varepsilon \le C \quad \text{in } \Omega_\varepsilon \tag{3-52}$$

for all ε.

In Case 2, it is not so difficult to adapt the arguments of [Druet 2006, §3,4] to get Step 6. Thus, in the proof of Step 6 just below, we assume that we are in Case 1. Then observe that $\Omega_\varepsilon \ne \varnothing$ by Step 2. Given $\eta_0 \in (0, 1)$, writing

$$\varphi_{N_\varepsilon - 1}(tN_\varepsilon) = \frac{t^{N_\varepsilon} N_\varepsilon^{N_\varepsilon}}{N_\varepsilon!} \left(\sum_{k=0}^{+\infty} t^k + o(1) \right) = \frac{(et)^{N_\varepsilon}}{\sqrt{2\pi N_\varepsilon}} \left(\frac{1}{1-t} + o(1) \right),$$

by Stirling's formula, where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly in $|t| \le \eta_0$, the unique positive solution Γ_ε of $\varphi_{N_\varepsilon - 1}(\Gamma_\varepsilon) = \Gamma_\varepsilon + 1$ satisfies $\Gamma_\varepsilon = (1 + o(1))(N_\varepsilon / e)$. Then, since $\varphi_{N_\varepsilon - 1}/(1 + \cdot\,)$ increases in $(0, +\infty)$, we clearly get

$$(1+o(1))\frac{N_\varepsilon}{e} \le \min_{\Omega_\varepsilon} u_\varepsilon^2. \tag{3-53}$$

Observe also that (3-53) almost characterizes Ω_ε in the following sense: given $\delta > 0$, for all $\varepsilon \ll 1$ so that $(1+\delta)(N_\varepsilon/e) \ge \Gamma_\varepsilon$, one has that $u_\varepsilon(y)^2 \ge (1+\delta)(N_\varepsilon/e)$ implies $y \in \Omega_\varepsilon$.

Proof of Step 6, formula (3-51). As previously mentioned, we still assume that we are in Case 1. Thus, in particular, we assume that $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Assume now by contradiction that

$$\max_{y\in\Omega_\varepsilon} |y-x_\varepsilon|^2|\Delta u_\varepsilon(y)|u_\varepsilon(y) = |y_\varepsilon - x_\varepsilon|^2|\Delta u_\varepsilon(y_\varepsilon)|u_\varepsilon(y_\varepsilon) \to +\infty \tag{3-54}$$

as $\varepsilon \to 0$, for some y_ε's such that $y_\varepsilon \in \Omega_\varepsilon$. First for any sequence $(\check{z}_\varepsilon)_\varepsilon$ such that $\check{z}_\varepsilon \in \Omega_\varepsilon$, we have $\Delta u_\varepsilon(\check{z}_\varepsilon) > 0$, $g'(u_\varepsilon(\check{z}_\varepsilon)) = o(u_\varepsilon(\check{z}_\varepsilon))$ and

$$\Psi'_{N_\varepsilon}(u_\varepsilon(\check{z}_\varepsilon)) = (1+o(1))2u_\varepsilon(\check{z}_\varepsilon)\varphi_{N_\varepsilon-1}(u_\varepsilon(\check{z}_\varepsilon)^2) \tag{3-55}$$

as $\varepsilon \to 0$, using (1-1), (3-3), (3-27) and (3-53). Additionally, we have

$$u_\varepsilon(y_\varepsilon) \to +\infty \tag{3-56}$$

as $\varepsilon \to 0$. Let $\nu_\varepsilon > 0$ be given by

$$\nu_\varepsilon^2|\Delta u_\varepsilon(y_\varepsilon)|u_\varepsilon(y_\varepsilon) = 1.$$

Then, using also (3-54), we have

$$\lim_{\varepsilon\to 0}\frac{|y_\varepsilon - x_\varepsilon|}{\nu_\varepsilon} = +\infty, \tag{3-57}$$

and, in view of Step 5,

$$\lim_{\varepsilon\to 0}\frac{|y_\varepsilon - x_\varepsilon|}{\mu_\varepsilon} = +\infty. \tag{3-58}$$

For $R > 0$, we set $\Omega_{R,\varepsilon} = B_{y_\varepsilon}(R\nu_\varepsilon) \cap \Omega$ and $\widetilde{\Omega}_{R,\varepsilon} = (\Omega_{R,\varepsilon} - y_\varepsilon)/\nu_\varepsilon$. Up to harmless rotations and since Ω is smooth, we may assume that there exists $B \in [0, +\infty]$ such that $\widetilde{\Omega}_{R,0} \to (-\infty, B) \times \mathbb{R}$ as $R \to +\infty$, where $\widetilde{\Omega}_{R,\varepsilon} \to \widetilde{\Omega}_{R,0}$ as $\varepsilon \to 0$. In this proof, for $z \in \widetilde{\Omega}_{R,\varepsilon}$, we write $z_\varepsilon = y_\varepsilon + \nu_\varepsilon z \in \Omega_{R,\varepsilon}$. Let $\tilde{u}_\varepsilon$ be given by

$$\tilde{u}_\varepsilon(z) = u_\varepsilon(y_\varepsilon)(u_\varepsilon(z_\varepsilon) - u_\varepsilon(y_\varepsilon)), \tag{3-59}$$

so that we get

$$(\Delta\tilde{u}_\varepsilon)(z) = \frac{(\Delta u_\varepsilon)(z_\varepsilon)}{(\Delta u_\varepsilon)(y_\varepsilon)} = \frac{\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))}{\Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon))}. \tag{3-60}$$

First, we prove that for all $R > 0$ there exists $C_R > 0$ such that

$$|\Delta\tilde{u}_\varepsilon| \le C_R \quad \text{in } \widetilde{\Omega}_{R,\varepsilon} \tag{3-61}$$

for all $0 < \varepsilon \ll 1$. Otherwise, by (3-60), assume by contradiction that there exists $z_\varepsilon \in \Omega_{R,\varepsilon}$ such that

$$|\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| \gg \Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon)) \tag{3-62}$$

as $\varepsilon \to 0$. If, still by contradiction, $z_\varepsilon \notin \Omega_\varepsilon$, we have $u_\varepsilon(z_\varepsilon) < u_\varepsilon(y_\varepsilon)$ and

$$\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) < \varphi_{N_\varepsilon-1}(u_\varepsilon(y_\varepsilon)^2),$$

by the definition of Ω_ε and since $\varphi_N/(1+\cdot)$ increases in $[0,+\infty)$, and then

$$|\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| \lesssim u_\varepsilon(z_\varepsilon)(1+u_\varepsilon(z_\varepsilon)^2+\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2)) \lesssim \Psi'_{N_\varepsilon}(u_\varepsilon(y_\varepsilon)),$$

using (1-1), (3-3), (3-27), (3-55) and $y_\varepsilon \in \Omega_\varepsilon$ again. This contradicts (3-62) and then it must be the case that $z_\varepsilon \in \Omega_\varepsilon$. Thus, since y_ε is a maximizer on Ω_ε in (3-54), we get from (3-57) and (3-62) that $u_\varepsilon(z_\varepsilon) \ll u_\varepsilon(y_\varepsilon)$. But this is not possible by (3-55) and (3-62), which proves (3-61). Now we prove that, for all $R>0$,

$$\limsup_{\varepsilon\to 0} \sup_{z\in\widetilde{\Omega}_{R,\varepsilon}} \tilde{u}_\varepsilon(z) \le 0. \tag{3-63}$$

Until the end of this proof, we set $\tilde{\gamma}_\varepsilon := u_\varepsilon(y_\varepsilon)$. If (3-63) does not hold true, since $\tilde{u}_\varepsilon(0)=0$ and by continuity, we may assume that there exist $z_\varepsilon \in \Omega_{R,\varepsilon}$ such that

$$\beta_\varepsilon := [\tilde{\gamma}_\varepsilon(u_\varepsilon(z_\varepsilon)-\tilde{\gamma}_\varepsilon)] \to \beta_0 \in (0,+\infty) \tag{3-64}$$

as $\varepsilon \to 0$. Since $u_\varepsilon(z_\varepsilon) > u_\varepsilon(y_\varepsilon)$ for $0<\varepsilon\ll 1$ by (3-64), we have $z_\varepsilon \in \Omega_\varepsilon$. Moreover, since y_ε is maximizing in (3-54), we then get from (3-55), (3-56) and (3-57) that

$$\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) \le (1+o(1))\,\varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2).$$

Independently, since φ_N is convex, we get

$$\begin{aligned}\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2) &\ge \varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2)+\varphi'_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2)(u_\varepsilon(z_\varepsilon)^2-\tilde{\gamma}_\varepsilon^2)\\ &\ge (1+2\beta_0(1+o(1)))\varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2),\end{aligned} \tag{3-65}$$

using (3-64) and $\varphi'_N(t) \ge \varphi_N(t)$ for $t\ge 0$. But (3-64)–(3-65) cannot hold true simultaneously, which proves (3-63). As in [Druet 2006, p. 231], $\tilde{u}_\varepsilon(0)=0$, $u_\varepsilon = 0$ on $\partial\Omega$, (3-61) and (3-63) imply

$$\lim_{\varepsilon\to 0} \frac{d(y_\varepsilon,\partial\Omega)}{\nu_\varepsilon} = +\infty. \tag{3-66}$$

Moreover, by standard elliptic theory, $\tilde{u}_\varepsilon(0)=0$, (3-61), (3-63) and (3-66) give

$$\tilde{u}_\varepsilon \to u_0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2) \tag{3-67}$$

as $\varepsilon\to 0$, for some $u_0 \in C^1(\mathbb{R}^2)$. Given $R>0$, we prove now

$$\liminf_{\varepsilon\to 0} \inf_{z\in\widetilde{\Omega}_{R,\varepsilon}} (\Delta\tilde{u}_\varepsilon)(z) > 0. \tag{3-68}$$

Using (3-27), (3-56) and (3-67), we have

$$\Psi'_{N_\varepsilon}(u_\varepsilon) = 2\tilde{\gamma}_\varepsilon\varphi_{N_\varepsilon-1}(u_\varepsilon^2)(1+o(1))+o(\tilde{\gamma}_\varepsilon^3),$$

uniformly in $\Omega_{R,\varepsilon}$. Then, coming back to (3-60), using (3-55) and $y_\varepsilon \in \Omega_\varepsilon$, we get

$$(\Delta\tilde{u}_\varepsilon)(z) = (1+o(1))\frac{\varphi_{N_\varepsilon-1}(u_\varepsilon(z_\varepsilon)^2)}{\varphi_{N_\varepsilon-1}(\tilde{\gamma}_\varepsilon^2)} + o(1),$$

uniformly in $z \in \widetilde{\Omega}_{R,\varepsilon}$. Now, we write (3-48) with $\Gamma = \tilde{\gamma}_\varepsilon^2$ and $T = u_\varepsilon^2$, where u_ε stands for $u_\varepsilon(z_\varepsilon)$ here and below. Then, in order to conclude the proof of (3-68), using also (3-36), it is sufficient to check that there exists $\eta_R < 1$ such that

$$I_\varepsilon := \frac{\exp(u_\varepsilon^2)}{\varphi_{\widetilde{N}_\varepsilon}(\tilde{\gamma}_\varepsilon^2)\exp(-(\tilde{\gamma}_\varepsilon^2 - u_\varepsilon^2))}\int_{u_\varepsilon^2}^{\tilde{\gamma}_\varepsilon^2}\exp(-s)\frac{s^{\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!}\,ds = \frac{\int_{u_\varepsilon^2}^{\tilde{\gamma}_\varepsilon^2}\exp(-s)(s^{\widetilde{N}_\varepsilon}/\widetilde{N}_\varepsilon!)\,ds}{\int_0^{\tilde{\gamma}_\varepsilon^2}\exp(-s)(s^{\widetilde{N}_\varepsilon}/\widetilde{N}_\varepsilon!)\,ds} \le \eta_R \tag{3-69}$$

for all $0 < \varepsilon \ll 1$, uniformly in $\Omega_{R,\varepsilon}$, where $\widetilde{N}_\varepsilon = N_\varepsilon - 1$. If $u_\varepsilon \ge \tilde{\gamma}_\varepsilon$, the last inequality in (3-69) is obvious. If now $u_\varepsilon < \tilde{\gamma}_\varepsilon$, we write

$$I_\varepsilon = \frac{\int_{u_\varepsilon^2-\tilde{\gamma}_\varepsilon^2}^0 \exp(-t)(1+t/\tilde{\gamma}_\varepsilon^2)^{\widetilde{N}_\varepsilon}\,dt}{\int_{-\tilde{\gamma}_\varepsilon^2}^0 \exp(-t)(1+t/\tilde{\gamma}_\varepsilon^2)^{\widetilde{N}_\varepsilon}\,dt} \le \frac{\int_{u_\varepsilon^2-\tilde{\gamma}_\varepsilon^2}^0 \exp(t(\widetilde{N}_\varepsilon/\tilde{\gamma}_\varepsilon^2 - 1) + O(\widetilde{N}_\varepsilon t^2/\tilde{\gamma}_\varepsilon^4))\,dt}{\int_{2(u_\varepsilon^2-\tilde{\gamma}_\varepsilon^2)}^0 \exp(t(\widetilde{N}_\varepsilon/\tilde{\gamma}_\varepsilon^2 - 1) + O(\widetilde{N}_\varepsilon t^2/\tilde{\gamma}_\varepsilon^4))\,dt} \le \eta_R$$

using (3-67), where I_ε is as in (3-69). We get the last inequality using (3-53) and $y_\varepsilon \in \Omega_\varepsilon$: (3-69) and then (3-68) are proved in any case. Let $R > 0$ be given. By (3-57), (3-58) and (3-68), we clearly get

$$\int_{\Omega\setminus B_{x_\varepsilon}(R\mu_\varepsilon)}(\Delta u_\varepsilon(y))^+ u_\varepsilon\,dy \ge \int_{B_{y_\varepsilon}(\nu_\varepsilon)}\Delta u_\varepsilon(y)\,u_\varepsilon(y)\,dy$$

for all ε small enough. Using now (3-56) and (3-67), we write that $u_\varepsilon = \tilde{\gamma}_\varepsilon(1+o(1))$ uniformly in $B_{y_\varepsilon}(\nu_\varepsilon)$, so that

$$\liminf_{\varepsilon\to 0}\int_{B_{y_\varepsilon}(\nu_\varepsilon)}\Delta u_\varepsilon(y)\,u_\varepsilon(y)\,dy = \liminf_{\varepsilon\to 0}\int_{B_0(1)}\Delta\tilde{u}_\varepsilon(z)(1+o(1))\,dz > 0,$$

by (3-68). Since this last term is independent of $R > 0$, this contradicts (3-50), which concludes the proof of (3-51). □

Proof of Step 6, formula (3-52). Remember that we assume that Case 1 holds true. Assume then by contradiction that there exists $(y_\varepsilon)_\varepsilon$ such that $y_\varepsilon \in \Omega_\varepsilon$ and

$$\max_{y\in\Omega_\varepsilon}|y-x_\varepsilon||\nabla u_\varepsilon(y)|u_\varepsilon(y) = |y_\varepsilon - x_\varepsilon||\nabla u_\varepsilon(y_\varepsilon)|u_\varepsilon(y_\varepsilon) := C_\varepsilon \to +\infty \tag{3-70}$$

as $\varepsilon \to 0$. Then, by (3-53), (3-56) holds true. Let $\nu_\varepsilon > 0$ be given by

$$\nu_\varepsilon = \min(|x_\varepsilon - y_\varepsilon|, d(y_\varepsilon, \partial\Omega)). \tag{3-71}$$

For all $R > 1$ and all ε, we let $\Omega_{R,\varepsilon}$ and $\widetilde{\Omega}_{R,\varepsilon}$ be given by the formulas above (3-59). Let w_ε be given by

$$w_\varepsilon(z) = u_\varepsilon(y_\varepsilon + \nu_\varepsilon z).$$

Since $\|u_\varepsilon\|_{H_0^1}^2 \le 4\pi$, we get from Moser's inequality that $\int_\Omega \exp(u_\varepsilon^2)\,dy = O(1)$ and then that, for all given $p \ge 1$,

$$\|\nu_\varepsilon^{2/p} w_\varepsilon\|_{L^p(\widetilde{\Omega}_{R,\varepsilon})} = O(1) \tag{3-72}$$

for all ε. Set $\tilde{x}_\varepsilon = (x_\varepsilon - y_\varepsilon)/\nu_\varepsilon$. Now, for any given $R > 1$ and any sequence $(z_\varepsilon)_\varepsilon$ such that $z_\varepsilon \in \Omega_{R,\varepsilon}\backslash\{x_\varepsilon\}$ (i.e., $\tilde{z}_\varepsilon := (z_\varepsilon - y_\varepsilon)/\nu_\varepsilon \in \widetilde{\Omega}_{R,\varepsilon}\backslash\{\tilde{x}_\varepsilon\}$), we get

$$|\Delta w_\varepsilon(\tilde{z}_\varepsilon)| = \nu_\varepsilon^2 |\Delta u_\varepsilon(z_\varepsilon)| \lesssim \begin{cases} 1/(u_\varepsilon(z_\varepsilon)|\tilde{z}_\varepsilon - \tilde{x}_\varepsilon|^2) & \text{if } z_\varepsilon \in \Omega_\varepsilon, \\ \lambda_\varepsilon \nu_\varepsilon^2 |\Psi'_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| = O(\lambda_\varepsilon \nu_\varepsilon^2 (1 + u_\varepsilon(z_\varepsilon)^3)) & \text{if } z_\varepsilon \notin \Omega_\varepsilon, \end{cases}$$

using (3-51) for the first line, and (3-27) for the second one. Then, using either (3-53) or (3-38) with (3-72), we get

$$\|\Delta w_\varepsilon\|_{L^p(\widetilde{\Omega}_{R,\varepsilon}\backslash B_{\tilde{x}_\varepsilon}(1/R))} \to 0 \tag{3-73}$$

as $\varepsilon \to 0$. Independently, since $\|u_\varepsilon\|_{H_0^1} = O(1)$, we easy get

$$\int_{\widetilde{\Omega}_{R,\varepsilon}} |\nabla w_\varepsilon|^2 \, dz = O(1). \tag{3-74}$$

Observe that $|\tilde{x}_\varepsilon| \geq 1$. Now, we claim that up to a subsequence,

$$\nu_\varepsilon \to 0 \quad \text{and} \quad \frac{d(y_\varepsilon, \partial\Omega)}{|x_\varepsilon - y_\varepsilon|} \to +\infty \tag{3-75}$$

as $\varepsilon \to 0$. In particular, by (3-71), this implies $\nu_\varepsilon = |x_\varepsilon - y_\varepsilon|$. Now we prove (3-75). Indeed, if we assume by contradiction that (3-75) does not hold, for all $R \gg 1$ sufficiently large, we get that the $(w_\varepsilon/u_\varepsilon(y_\varepsilon))$'s converge locally out of $B_{\tilde{x}_\varepsilon}\left(\frac{1}{2}\right)$ to some C^1 function which is 1 at 0 and 0 on the nonempty and smooth boundary of $\lim_{R\to+\infty} \lim_{\varepsilon\to 0} \widetilde{\Omega}_{R,\varepsilon}$ (maybe after a harmless rotation). We use here the Harnack inequality and elliptic theory with (3-56), (3-73) (with $p > 2$) and (3-74), since $u_\varepsilon = 0$ in $\partial\Omega$. This clearly contradicts (3-74) and (3-75) is proved. Up to a subsequence, we may now assume

$$\tilde{x}_\varepsilon \to \tilde{x}, \quad |\tilde{x}| = 1, \tag{3-76}$$

as $\varepsilon \to 0$. By (3-56), (3-73), (3-74), and similar arguments including again Harnack's principle, we get

$$\frac{w_\varepsilon}{u_\varepsilon(y_\varepsilon)} \to 1 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^2\backslash\{\tilde{x}\}), \tag{3-77}$$

using also (3-75). By (3-72) and (3-77), we get that for all $p \geq 1$

$$\nu_\varepsilon^{2/p} u_\varepsilon(y_\varepsilon) = O(1) \tag{3-78}$$

as $\varepsilon \to 0$. Let now $\tilde{w}_\varepsilon$ be given by

$$\tilde{w}_\varepsilon = \frac{w_\varepsilon - w_\varepsilon(0)}{\nu_\varepsilon |\nabla u_\varepsilon(y_\varepsilon)|},$$

so that $|\nabla \tilde{w}_\varepsilon(0)| = 1$. For any given $R > 1$ and any sequence $(z_\varepsilon)_\varepsilon$ such that $\tilde{z}_\varepsilon := (z_\varepsilon - y_\varepsilon)/\nu_\varepsilon \in \widetilde{\Omega}_{R,\varepsilon}\backslash B_{\tilde{x}}(1/R)$, we get

$$|\Delta \tilde{w}_\varepsilon(\tilde{z}_\varepsilon)| = \frac{u_\varepsilon(y_\varepsilon)}{C_\varepsilon} |\Delta w_\varepsilon(\tilde{z}_\varepsilon)| \lesssim \begin{cases} 1/(C_\varepsilon |\tilde{z}_\varepsilon - \tilde{x}_\varepsilon|^2) & \text{if } z_\varepsilon \in \Omega_\varepsilon, \\ (\lambda_\varepsilon / C_\varepsilon) \nu_\varepsilon^2 u_\varepsilon(y_\varepsilon)^4 & \text{if } z_\varepsilon \notin \Omega_\varepsilon \end{cases}$$

for all ε, using (3-51), (3-70) and (3-77). Then, since $\lambda_\varepsilon = o(1)$, we get from (3-70), (3-75) and (3-78) (with $p \geq 4$) that

$$\Delta \tilde{w}_\varepsilon \to 0 \quad \text{in } L^\infty_{\text{loc}}(\mathbb{R}^2 \backslash \{\tilde{x}\}) \tag{3-79}$$

as $\varepsilon \to 0$. By (3-70), (3-76) and (3-77), given $R > 1$ and $\tilde{z}_\varepsilon \in \widetilde{\Omega}_{R,\varepsilon} \backslash B_{\tilde{x}}(1/R)$, we get

$$|\nabla \tilde{w}_\varepsilon(\tilde{z}_\varepsilon)| = \frac{|\nabla u_\varepsilon(z_\varepsilon)|}{|\nabla u_\varepsilon(y_\varepsilon)|} \leq \frac{u_\varepsilon(y_\varepsilon)}{u_\varepsilon(z_\varepsilon)} \frac{1}{|\tilde{x}_\varepsilon - \tilde{z}_\varepsilon|} \leq \frac{1+o(1)}{|\tilde{x}_\varepsilon - \tilde{z}_\varepsilon|} \tag{3-80}$$

for all $0 < \varepsilon \ll 1$. Then, by (3-79), (3-80) and since $\tilde{w}_\varepsilon(0) = 0$, there exists a harmonic function $\mathcal{H}$ in $\mathbb{R}^2 \backslash \{\tilde{x}\}$ such that $\lim_{\varepsilon \to 0} \tilde{w}_\varepsilon = \mathcal{H}$ in $C^1_{\text{loc}}(\mathbb{R}^2 \backslash \{\tilde{x}\})$. Now, for all given $\beta > 0$, integrating by parts, we get

$$\int_{\partial B_{x_\varepsilon}(\beta \nu_\varepsilon)} u_\varepsilon \partial_\nu u_\varepsilon \, d\sigma = C_\varepsilon \left(\int_{\partial B_{\tilde{x}}(\beta)} \partial_\nu \mathcal{H} d\sigma + o(1) \right) \leq \int_\Omega |\nabla u_\varepsilon|^2 \, dy + \int_\Omega u_\varepsilon (\Delta u_\varepsilon)^+ \, dy = O(1),$$

using (3-70) and (3-77), as $\varepsilon \to 0$. Since $C_\varepsilon \to +\infty$, this implies $\int_{\partial B_{\tilde{x}}(\beta)} \partial_\nu \mathcal{H} \, d\sigma = 0$. Then, also by (3-80), β being arbitrary, $\mathcal{H}$ is bounded around $\tilde{x}$ and then the singularity at $\tilde{x}$ is removable. By the Liouville theorem, $\mathcal{H}$ is constant in $\mathbb{R}^2$, which is not possible since $|\nabla \tilde{w}_\varepsilon(0)| = |\nabla \mathcal{H}(0)| = 1$. This concludes the proof of (3-52). □

Remark 3.6. We do not assume that the continuous function Ψ'_{N_ε} is positive and increasing in $[0, +\infty)$. Then, standard moving plane techniques [Adimurthi and Druet 2004; Gidas et al. 1979; Han 1991; de Figueiredo et al. 1982] do not apply. We use in the proof below the variational characterization (3-7) of the u_ε's to get that $\bar{x} \in K_\Omega$, K_Ω as in (1-9), and that, in particular, $\bar{x} \notin \partial\Omega$ in (3-12).

Let B_ε be the radial solution around x_ε of

$$\begin{cases} \Delta B_\varepsilon = \frac{1}{2} \lambda_\varepsilon \Psi'_{N_\varepsilon}(B_\varepsilon), \\ B_\varepsilon(x_\varepsilon) = \gamma_\varepsilon, \end{cases} \tag{3-81}$$

where γ_ε is still given by (3-13). Let $\bar{u}_\varepsilon$ be given by

$$\bar{u}_\varepsilon(z) = \frac{1}{2\pi |x_\varepsilon - z|} \int_{\partial B_{x_\varepsilon}(|x_\varepsilon - z|)} u_\varepsilon \, d\sigma \tag{3-82}$$

for all $z \neq x_\varepsilon$ and $\bar{u}_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$. Let $\varepsilon_0 \in (\sqrt{1/e}, 1)$ be given. Let $\rho_\varepsilon > 0$ be given by

$$t_\varepsilon(\rho_\varepsilon) = (1 - \varepsilon_0)\gamma_\varepsilon^2. \tag{3-83}$$

By (3-44), we have

$$\rho_\varepsilon^2 = \exp(-(\varepsilon_0 + o(1))\gamma_\varepsilon^2). \tag{3-84}$$

Let r_ε be given by

$$r_\varepsilon = \sup\{r \in (0, \rho_\varepsilon] : |\bar{u}_\varepsilon - B_\varepsilon| \leq 1/\gamma_\varepsilon \text{ in } B_{x_\varepsilon}(r)\}. \tag{3-85}$$

Observe that $r_\varepsilon \gg \mu_\varepsilon$ by Step 5 and the Appendix. Then, we state the following key result.

Step 7. *We have*

$$\bar{u}_\varepsilon(r_\varepsilon) = B_\varepsilon(r_\varepsilon) + o\left(\frac{1}{\gamma_\varepsilon}\right) \tag{3-86}$$

and then $r_\varepsilon = \rho_\varepsilon$ for all $0 < \varepsilon \ll 1$. Moreover, there exists $C > 0$ such that

$$|\nabla(B_\varepsilon - u_\varepsilon)| \le \frac{C}{\rho_\varepsilon \gamma_\varepsilon} \quad \text{in } B_{x_\varepsilon}(\rho_\varepsilon) \tag{3-87}$$

for all $0 < \varepsilon \ll 1$, where $(x_\varepsilon)_\varepsilon$ is as in (3-13), B_ε is as in (3-81), $\bar{u}_\varepsilon$ is as in (3-82), ρ_ε is as in (3-83) and r_ε is as in (3-85).

Since $B_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$, (3-87) obviously implies

$$|B_\varepsilon - u_\varepsilon| \le C \frac{|\cdot - x_\varepsilon|}{\rho_\varepsilon \gamma_\varepsilon} \quad \text{in } B_{x_\varepsilon}(\rho_\varepsilon) \tag{3-88}$$

for all $0 < \varepsilon \ll 1$. Then, combined with the Appendix, Step 7 provides pointwise estimates of the u_ε's in $B_{x_\varepsilon}(\rho_\varepsilon)$.

Proof of Step 7. The proof of Step 7 follows the lines of [Druet and Thizy 2017, Section 3]. We only recall here the argument in the more delicate Case 1. Let v_ε be given by

$$u_\varepsilon = B_\varepsilon + v_\varepsilon. \tag{3-89}$$

By the Appendix, we have that B_ε is well-defined, radially decreasing in $B_{x_\varepsilon}(\rho_\varepsilon)$, and

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + o\left(\frac{t_\varepsilon}{\gamma_\varepsilon}\right) \tag{3-90}$$

uniformly in $B_{x_\varepsilon}(\rho_\varepsilon)$ as $\varepsilon \to 0$, where t_ε is given by (3-41). Then, we get first from (3-83) and (3-90) the lower bound

$$\min_{B_{x_\varepsilon}(r_\varepsilon)} B_\varepsilon \ge \gamma_\varepsilon(\varepsilon_0 + o(1)).$$

Let us introduce now an intermediate radius $\tilde{r}_\varepsilon$ given by

$$\tilde{r}_\varepsilon = \sup\left\{r \in (0, r_\varepsilon] : \tfrac{1}{2}\varepsilon_0 \gamma_\varepsilon |x_\varepsilon - \cdot||\nabla u_\varepsilon| \le C \text{ in } B_{x_\varepsilon}(r)\right\}$$

for C as in (3-52). We prove now that $\tilde{r}_\varepsilon = r_\varepsilon$ for all $\varepsilon \ll 1$. Indeed, by Wirtinger's inequality on $\partial B_0(r)$, $0 < r \le \tilde{r}_\varepsilon$, we have

$$|\bar{u}_\varepsilon - u_\varepsilon| \le \frac{2C}{\varepsilon_0 \gamma_\varepsilon} \pi,$$

so that, by (3-85),

$$|v_\varepsilon| = |B_\varepsilon - u_\varepsilon| \le \left(\frac{2\pi C}{\varepsilon_0} + 1\right) \gamma_\varepsilon^{-1}$$

in $B_{x_\varepsilon}(\tilde{r}_\varepsilon)$. Then, we get a lower bound on u_ε as well, namely

$$\min_{B_{x_\varepsilon}(\tilde{r}_\varepsilon)} u_\varepsilon \ge \gamma_\varepsilon(\varepsilon_0 + o(1)), \tag{3-91}$$

so that, by (3-52), the condition in the definition of $\tilde{r}_\varepsilon$ never saturates: $\tilde{r}_\varepsilon = r_\varepsilon$ for all $\varepsilon \ll 1$. Observe for this that (3-91) combined with (3-53) (see also the paragraph below (3-53)) and with our assumption

$e\varepsilon_0^2 > 1$ implies $B_{x_\varepsilon}(\tilde{r}_\varepsilon) \subset \Omega_\varepsilon$. Observe in particular that (3-31) provides $\gamma_\varepsilon^2 \geq N_\varepsilon(1+o(1))$. Summarizing what we have just obtained in $B_{x_\varepsilon}(r_\varepsilon)$, we may write

$$\| |x_\varepsilon - \cdot| |\nabla u_\varepsilon| \|_{L^\infty(B_{x_\varepsilon}(r_\varepsilon))} = O\left(\frac{1}{\gamma_\varepsilon}\right),$$

and

$$\|v_\varepsilon\|_{L^\infty(B_{x_\varepsilon}(r_\varepsilon))} = O\left(\frac{1}{\gamma_\varepsilon}\right). \tag{3-92}$$

We also have

$$B_\varepsilon \leq \gamma_\varepsilon \tag{3-93}$$

in $B_{x_\varepsilon}(r_\varepsilon)$. By combining (3-26) and (3-81), (3-92) allows us to linearize (3-81) to control v_ε. More precisely, (1-5) and Lemma 3.3 permit us to compute the variations of Ψ'_{N_ε} in (3-27), even if g is only C^1 in (1-1), so that Ψ'_{N_ε} is only continuous. Namely, we get from (1-5a) and (1-5c) and from Lemma 3.3 (for $\gamma = B_\varepsilon$) that

$$|\Delta v_\varepsilon| = |\Delta(u_\varepsilon - B_\varepsilon)| \leq C' \lambda_\varepsilon \gamma_\varepsilon^2 \varphi_{N_\varepsilon - 2}(B_\varepsilon^2)\left[|v_\varepsilon| + o\left(\frac{1}{\gamma_\varepsilon}\right)\right] \quad \text{in } B_{x_\varepsilon}(r_\varepsilon)$$

for all ε, using (3-48), (3-91)–(3-93) and some computations. Then, (3-90) gives

$$|\Delta v_\varepsilon| \leq C'' \frac{\exp(-2t_\varepsilon(1+o(1)) + t_\varepsilon^2/\gamma_\varepsilon^2)}{\mu_\varepsilon^2}\left[|v_\varepsilon| + o\left(\frac{1}{\gamma_\varepsilon}\right)\right] \quad \text{in } B_{x_\varepsilon}(r_\varepsilon) \tag{3-94}$$

using (3-30), (3-40) and (3-45). Starting now from (3-92)–(3-94), we can compute and argue as in [Druet and Thizy 2017, Section 3] in order to get (3-86)–(3-87). □

Conclusion of the proof of Lemma 3.4. Let $\varepsilon'_0 \in (\varepsilon_0, 1)$ be fixed and let $\rho'_\varepsilon > 0$ be given by

$$t_\varepsilon(\rho'_\varepsilon) = (1 - \varepsilon'_0)\gamma_\varepsilon^2, \tag{3-95}$$

so that, by (3-44),

$$(\rho'_\varepsilon)^2 = \exp(-\varepsilon'_0(1+o(1))\gamma_\varepsilon^2). \tag{3-96}$$

In order to conclude the proof of Lemma 3.4, by Steps 1–7, it remains to prove (2-4), (3-10)–(3-12), that

$$\left| u_\varepsilon(y) - \frac{4\pi G_{x_\varepsilon}(y)}{\gamma_\varepsilon} \right| = o\left(\frac{G_{x_\varepsilon}(y)}{\gamma_\varepsilon}\right) \tag{3-97}$$

uniformly in $B_{x_\varepsilon}(\rho'_\varepsilon)^c$, that

$$u_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)\frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + o\left(\bar{t}_\varepsilon \frac{\zeta_\varepsilon}{\gamma_\varepsilon}\right) \tag{3-98}$$

uniformly in $B_{x_\varepsilon}(\rho'_\varepsilon)$, where the $S_{i,\varepsilon}$'s are as in (A-5), and that

$$\begin{aligned} u_\varepsilon(y) = G_{x_\varepsilon}(y)\left(\frac{4\pi}{\gamma_\varepsilon} + \sum_{i=0}^{1} \frac{A_i}{\gamma_\varepsilon^{3+2i}} + \frac{A_2(A(\gamma_\varepsilon) - 2\xi_\varepsilon)}{\gamma_\varepsilon}\right) \\ + \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^2 \exp(1 + \mathcal{H}_{x_\varepsilon}(x_\varepsilon))} \int_\Omega G_y(x) F(4\pi G_{x_\varepsilon}(x))\,dx + o\left(\frac{\zeta_\varepsilon}{\gamma_\varepsilon} G_{x_\varepsilon}(y) + \frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^2}\right), \end{aligned} \tag{3-99}$$

uniformly in $B_{x_\varepsilon}(\rho'_\varepsilon)^c$, as $\varepsilon \to 0$, where F and $B(\gamma_\varepsilon)$ are given in (1-6), where the A_i's are as in (A-3), and where ζ_ε is given in (A-8).

(1) In this first point, we aim to get pointwise estimates of the u_ε's out of $B_{x_\varepsilon}(\rho'_\varepsilon)$. Let G be the Green's function in (1-8). It is known that (see for instance [Druet and Thizy 2017, Appendix B]) there exists $C > 0$ such that

$$|\nabla_y G_x(y)| \le \frac{C}{|x-y|} \quad \text{and} \quad 0 < G_x(y) \le \frac{1}{2\pi} \log \frac{C}{|x-y|} \tag{3-100}$$

for all $x, y \in \Omega$, $x \ne y$. By (3-87) and since $\|u_\varepsilon\|^2_{H^1_0} \le 4\pi$, it is possible to prove (see for instance the proof of [Druet and Thizy 2017, Claim 4.6]) that, given $p < 1/\varepsilon'_0$,

$$\|\exp(u_\varepsilon^2)\|_{L^p(B_{x_\varepsilon}(\rho'_\varepsilon/2)^c)} = O(1) \tag{3-101}$$

for all ε, where $B_{x_\varepsilon}(\rho'_\varepsilon/2)^c = \Omega \backslash B_{x_\varepsilon}(\rho'_\varepsilon/2)$. In the sequel, $p' > 1$ is chosen such that

$$\frac{1}{p} + \frac{1}{p'} < 1.$$

Let now $(z_\varepsilon)_\varepsilon$ be any sequence of points in $B_{x_\varepsilon}(\rho'_\varepsilon)^c$. By the Green's representation formula and (3-26), we can write

$$u_\varepsilon(z_\varepsilon) = \frac{\lambda_\varepsilon}{2} \int_\Omega G_{z_\varepsilon}(y) \Psi'_{N_\varepsilon}(u_\varepsilon(y))\, dy.$$

By (3-100), we have that there exists $C > 0$ such that

$$|G_{z_\varepsilon}(x_\varepsilon) - G_{z_\varepsilon}| \le C \frac{|x_\varepsilon - \cdot|}{\rho'_\varepsilon} \tag{3-102}$$

in $B_{x_\varepsilon}(\rho'_\varepsilon/2)$ for all ε. Set $\bar{t}_\varepsilon = 1 + t_\varepsilon$. By (3-44) and (3-84), we have

$$\frac{|\cdot - x_\varepsilon|}{\gamma_\varepsilon \rho_\varepsilon} = o\left(\frac{\bar{t}_\varepsilon}{\gamma_\varepsilon^5}\right) \quad \text{in } \tilde{\Omega}_\varepsilon := \{y : t_\varepsilon(y) \le \gamma_\varepsilon\}$$

as $\varepsilon \to 0$, and then, by (3-88), (A-9) holds true for v_ε as in (3-89). Independently, using (3-29), (3-40), (3-88) and (A-3) with (A-7), we clearly get that there exists $C > 0$ such that

$$\lambda_\varepsilon |\Psi'_{N_\varepsilon}(u_\varepsilon)| \le C \frac{\exp(-2t_\varepsilon + t_\varepsilon^2/\gamma_\varepsilon^2)}{\mu_\varepsilon^2 \gamma_\varepsilon} \quad \text{in } B_{x_\varepsilon}\left(\tfrac{1}{2}\rho'_\varepsilon\right) \tag{3-103}$$

for all ε. Then, we get

$$\begin{aligned} u_\varepsilon(z_\varepsilon) &= G_{z_\varepsilon}(x_\varepsilon) \int_{B_{x_\varepsilon}(\rho'_\varepsilon/2)} \frac{\lambda_\varepsilon \Psi'_{N_\varepsilon}(u_\varepsilon)}{2}\, dy + O\left(\int_{B_{x_\varepsilon}(\rho'_\varepsilon/2)} \frac{\exp(-2t_\varepsilon + t_\varepsilon^2/\gamma_\varepsilon^2)|\cdot - x_\varepsilon|}{\mu_\varepsilon^2 \gamma_\varepsilon \rho'_\varepsilon}\, dy\right) + O(\lambda_\varepsilon \|u_\varepsilon\|_{L^{p'}}) \\ &= G_{z_\varepsilon}(x_\varepsilon) \frac{4\pi}{\gamma_\varepsilon}\left(1 + \frac{1}{\gamma_\varepsilon^2} + \frac{A(\gamma_\varepsilon) - 2\xi_\varepsilon}{2} + o(\tilde{\zeta}_\varepsilon)\right) + o\left(\frac{1}{\gamma_\varepsilon}\right) + o(\|u_\varepsilon\|_{L^{p'}}), \end{aligned} \tag{3-104}$$

where $\tilde{\zeta}_\varepsilon$ is given by (3-15). We start by focusing on the first equality of (3-104): (3-102) and (3-103) are used to get the first two terms. The last term is obtained from (3-29), (3-100), (3-101) and Hölder's inequality. We focus now on the second equality of (3-104), resuming the previous one term by term:

The first term is easily computed by integrating (A-9) in $\widetilde{\Omega}_\varepsilon$ and by plugging the values of the A_i's from (A-2)–(A-4) on the one hand, and by estimating roughly in $B_{x_\varepsilon}(\rho'_\varepsilon)\backslash\widetilde{\Omega}_\varepsilon$ with (3-103) on the other hand. The last term obviously follows from $\lambda_\varepsilon = o(1)$. As for the $o(1/\gamma_\varepsilon)$, we get first $O(\mu_\varepsilon/(\rho'_\varepsilon\gamma_\varepsilon))$ using $\varepsilon_0 > \frac{1}{2}$, which clearly concludes by (3-95). Using first that $u_\varepsilon \le \gamma_\varepsilon$ and (3-84) in $B_{x_\varepsilon}(\rho_\varepsilon)$, and then (3-104) with (3-100) in $\Omega\backslash B_{x_\varepsilon}(\rho_\varepsilon)$, we get

$$\|u_\varepsilon\|_{L^{p'}} = o\left(\frac{1}{\gamma_\varepsilon} + \|u_\varepsilon\|_{L^{p'}}\right) + O\left(\frac{1}{\gamma_\varepsilon}\right).$$

This implies with (3-104)

$$u_\varepsilon(z_\varepsilon) = \frac{4\pi G_{z_\varepsilon}(x_\varepsilon)}{\gamma_\varepsilon}\left(1 + \frac{1}{\gamma_\varepsilon^2} + \frac{A(\gamma_\varepsilon) - 2\xi_\varepsilon}{2} + o(\tilde{\xi}_\varepsilon)\right) + o\left(\frac{1}{\gamma_\varepsilon}\right). \tag{3-105}$$

(2) In this second point, we prove

$$\lambda_\varepsilon \le \frac{4 + o(1)}{\gamma_\varepsilon^2 \exp(1 + M)} \tag{3-106}$$

as $\varepsilon \to 0$, for M as in (1-9). Observe that (3-105) implies

$$u_\varepsilon = (1 + o(1))\frac{4\pi G_{x_\varepsilon} + o(1)}{\gamma_\varepsilon}$$

in $\Omega\backslash B_{x_\varepsilon}(\rho_\varepsilon)$. By (1-1) and (3-100), our definition of ρ_ε and the dominated convergence theorem, this implies

$$\lim_{\varepsilon\to 0}\int_{\Omega\backslash B_{x_\varepsilon}(\rho_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon)\,dy = |\Omega|(1 + g(0)). \tag{3-107}$$

Independently, (A-7) and (3-88) give

$$u_\varepsilon = \gamma_\varepsilon - \frac{(1 + o(1))t_\varepsilon}{\gamma_\varepsilon} \tag{3-108}$$

in $B_{x_\varepsilon}(\rho_\varepsilon)$, since $\mu_\varepsilon \ll \rho_\varepsilon$. Then, using (3-30), (3-45), $\varepsilon_0^2 > 1/e$ and resuming the arguments to get (3-55), we have

$$\Psi_{N_\varepsilon}(u_\varepsilon) = (1 + o(1))\varphi_{N_\varepsilon - 1}(u_\varepsilon^2) \quad \text{and} \quad \Psi'_{N_\varepsilon}(u_\varepsilon) = 2(1 + o(1))\, u_\varepsilon\varphi_{N_\varepsilon - 1}(u_\varepsilon^2) \tag{3-109}$$

in $B_{x_\varepsilon}(\rho_\varepsilon)$. Independently, observe that, for all $\Gamma, \delta > 0$,

$$\varphi_N(\Gamma) = \delta\exp(\Gamma) \quad \Longrightarrow \quad \text{for all } T \in [0, \Gamma],\ \ \varphi_N(T) \le \delta\exp(T), \tag{3-110}$$

since $\varphi'_N \ge \varphi_N$ in $[0, +\infty]$. Then we get

$$\int_{B_{x_\varepsilon}(\rho_\varepsilon)} \Psi_{N_\varepsilon}(u_\varepsilon)\,dy = \frac{4\pi(1 + o(1))}{\gamma_\varepsilon^2\lambda_\varepsilon} \tag{3-111}$$

as $\varepsilon \to 0$, by (3-30), (3-40), (3-108), (3-109), with (3-48) for $|y - x_\varepsilon| \lesssim \mu_\varepsilon$, or with (3-110) and the dominated convergence theorem for $|y - x_\varepsilon| \gg \mu_\varepsilon$. Then, because of (3-7), we get that (3-106) holds true, by combining (3-107), (3-111) with (3-24).

(3) In this point, we conclude the proof of (3-10), and prove (2-4) and (3-12). For $R > 1$, let $\chi_{\varepsilon,R}$ be given in $\Omega_{\varepsilon,R} := \Omega\backslash B_{x_\varepsilon}(R\mu_\varepsilon)$ by

$$\chi_{\varepsilon,R} = 4\pi\Lambda_{\varepsilon,R} G_{x_\varepsilon}$$

for $\Lambda_{\varepsilon,R} > 0$ to be chosen later such that

$$\chi_{\varepsilon,R} \le u_\varepsilon \quad \text{on } \partial B_{x_\varepsilon}(R\mu_\varepsilon). \tag{3-112}$$

Integrating by parts, we can write

$$\begin{aligned}\int_{\Omega_{\varepsilon,R}} |\nabla u_\varepsilon|^2\,dy &= \int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2\,dy - 2\int_{\partial B_{x_\varepsilon}(R\mu_\varepsilon)} (\partial_\nu \chi_{\varepsilon,R})(u_\varepsilon - \chi_{\varepsilon,R})\,d\sigma + \int_{\Omega_{\varepsilon,R}} |\nabla(u_\varepsilon - \chi_{\varepsilon,R})|^2\,dy \\ &\ge \int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2\,dy,\end{aligned} \tag{3-113}$$

where ν is the unit outward normal to the boundary of $B_{x_\varepsilon}(R\mu_\varepsilon)$, using (3-112). Indeed, by [Druet and Thizy 2017, Appendix B] for instance, since $d(x_\varepsilon, \partial\Omega) \gg \mu_\varepsilon$ by Step 5, we have

$$\partial_\nu G_{x_\varepsilon} = -\frac{1}{2\pi R\mu_\varepsilon} + O\left(\frac{1}{d(x_\varepsilon, \partial\Omega)}\right) \quad \text{on } \partial B_{x_\varepsilon}(R\mu_\varepsilon). \tag{3-114}$$

Now, by (3-3), (3-40), (3-42), (3-45), (3-84), in order to have (3-112), we can choose $\Lambda_{\varepsilon,R}$ such that

$$\Lambda_{\varepsilon,R} = \frac{1}{\gamma_\varepsilon}\left(1 - \frac{\log(1+R^2) + o(1)}{\gamma_\varepsilon^2}\right)\left(1 + \frac{\log(\delta_\varepsilon\lambda_\varepsilon\gamma_\varepsilon^2/(4R^2)) + \mathcal{H}_{x_\varepsilon}(x_\varepsilon) + o(1)}{\gamma_\varepsilon^2}\right)^{-1}, \tag{3-115}$$

with $\delta_\varepsilon \in (0, 1]$ as in (3-30). In (3-115), we use

$$|\mathcal{H}_{x_\varepsilon} - \mathcal{H}_{x_\varepsilon}(x_\varepsilon)| = O\left(\frac{\mu_\varepsilon}{d(x_\varepsilon, \partial\Omega)}\right) = o(1)$$

uniformly in $\partial B_{x_\varepsilon}(R\mu_\varepsilon)$, using Step 5 and computing as in (3-21). Now, by (1-8), (3-44), (3-84), and (3-114), we compute and get first that

$$\begin{aligned}\int_{\Omega_{\varepsilon,R}} |\nabla \chi_{\varepsilon,R}|^2\,dy &\ge -\int_{\partial B_{x_\varepsilon}(R\mu_\varepsilon)} (\partial_\nu \chi_{\varepsilon,R})\chi_{\varepsilon,R}\,d\sigma \\ &\ge 4\pi\left(1 - \frac{2\log(1+R^2) + o(1)}{\gamma_\varepsilon^2}\right)\left(1 + \frac{\log(\delta_\varepsilon\lambda_\varepsilon\gamma_\varepsilon^2/(4R^2)) + \mathcal{H}_{x_\varepsilon}(x_\varepsilon) + o(1)}{\gamma_\varepsilon^2}\right)^{-1},\end{aligned}$$

using also (3-115). Independently, we compute and get also that

$$\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} |\nabla u_\varepsilon|^2\,dy = \frac{4\pi}{\gamma_\varepsilon^2}\left(\log(1+R^2) - \frac{R^2}{1+R^2} + o(1)\right),$$

by (3-42). Thus, since $\|u_\varepsilon\|_{H_0^1}^2 \le 4\pi$ and by (3-7) and (3-113), we eventually get

$$\frac{\log\delta_\varepsilon\lambda_\varepsilon + \mathcal{H}_{x_\varepsilon}(x_\varepsilon)}{\gamma_\varepsilon^2} \ge o(1).$$

Moreover, using the definition (1-9) of M, (3-106), $\delta_\varepsilon \le 1$ and that $R > 0$ may be arbitrarily large, we get

$$\delta_\varepsilon \to 1, \tag{3-116}$$

and that (3-10) and (3-12) hold true. As a remark, in Case 2 where $N_\varepsilon = 1$, (3-116) is a direct consequence of the definition (3-30) of δ_ε. Then, (2-4) follows from (3-10), (3-107) and (3-111).

(4) Now we prove (3-11). Since $\varepsilon_0' > \varepsilon_0$, we get from (3-84), (3-88), (3-96) and (A-7) that

$$u_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} - \frac{t_\varepsilon}{\gamma_\varepsilon^3} - (A(\gamma_\varepsilon) - 2\xi_\varepsilon)\frac{t_\varepsilon}{2\gamma_\varepsilon} + o\left(\frac{t_\varepsilon \tilde{\zeta}_\varepsilon}{\gamma_\varepsilon}\right) \tag{3-117}$$

uniformly in $\{y \in B_{x_\varepsilon}(\rho_\varepsilon') : t_\varepsilon \geq \gamma_\varepsilon/4\}$, using also (A-3). Then, noting that the averages of (3-105) and (3-117) have to match on $\partial B_{x_\varepsilon}(\rho_\varepsilon')$, we compute and get

$$\lambda_\varepsilon = \frac{4}{\gamma_\varepsilon^2 \exp\left(1 + M + \frac{1}{2}\gamma_\varepsilon^2(A(\gamma_\varepsilon) - 2\xi_\varepsilon) + o(\tilde{\zeta}_\varepsilon \gamma_\varepsilon^2)\right)}, \tag{3-118}$$

by (3-12), (3-116) and (3-40) with (3-3) and (3-45). Observe in particular that

$$1 \lesssim \gamma_\varepsilon^{-2} G_{x_\varepsilon} \lesssim 1, \quad 1 \lesssim \gamma_\varepsilon^{-2} t_\varepsilon \lesssim 1$$

on $\partial B_{x_\varepsilon}(\rho_\varepsilon')$, by (3-95) and (3-96) with (1-8) and (3-12). By (3-10) and (3-118), (3-11) is proved.

(5) Here, we conclude the proof of Lemma 3.4. As an immediate consequence of (3-105), we get that (3-97) holds true. Pushing now one step further the above computations with very similar arguments, we get that (3-98) holds true as well. At last, using in particular (3-10) with (1-6) to improve the estimates in point (1) of this proof, we get (3-99). □

Lemma 3.4 is proved. □

4. Proof of Proposition 2.1

Proof of Proposition 2.1. We make the assumptions of Lemma 3.4 in Case 2 with $\alpha_\varepsilon = 4\pi(1-\varepsilon)$. In particular, we assume that u_ε is a maximizer for $(I^g_{4\pi(1-\varepsilon)}(\Omega))$, for all $0 < \varepsilon \ll 1$, and that (2-1) holds true. Then, Lemma 3.4 in Case 2 will be currently applied in the sequel. In particular, we may let $\lambda_\varepsilon, \gamma_\varepsilon, x_\varepsilon, \mu_\varepsilon$ be thus given and it only remains to prove (2-5)–(2-6) to get Proposition 2.1.

Let $z \in \Omega$ be given. In view of (3-99), for $\gamma, \mu > 0$, we let now $U_{\mu,\gamma,z}$ be given by

$$\begin{aligned} U_{\mu,\gamma,z}(x) = \frac{1}{\gamma}\left(-\log\left(1 + \frac{|x-z|^2}{\mu^2}\right) + \log\frac{1}{\mu^2} + \mathcal{H}_{-1,\mu,z}(x)\right) \\ + \sum_{i=0}^{1} \frac{1}{\gamma^{3+2i}}\left(S_i\left(\frac{x-z}{\mu}\right) + \frac{A_i}{4\pi}\left(\log\frac{1}{\mu^2} + \mathcal{H}_{i,\mu,z}(x)\right) - B_i\right) \\ + \frac{A(\gamma)}{\gamma}\left(S_2\left(\frac{x-z}{\mu}\right) + \frac{A_2}{4\pi}\left(\log\frac{1}{\mu^2} + \mathcal{H}_{2,\mu,z}(x)\right) - B_2\right) \\ + \frac{4B(\gamma)}{\gamma^2\exp(1 + \mathcal{H}_z(z))}\int_\Omega G_x(y) F(4\pi G_z(y))\, dy, \end{aligned} \tag{4-1}$$

where the S_i are given by (A-2), where the A_i, B_i are as in (A-3), where $\mathcal{H}$ is as in (1-8), where the $\mathcal{H}_{j,\mu,z}$ are harmonic in Ω and given by

$$\mathcal{H}_{-1,\mu,z} = -\log\left(\frac{1}{\mu^2+|z-\cdot|^2}\right) \quad \text{or} \quad \mathcal{H}_{j,\mu,z} = -\frac{4\pi}{A_j}\left(S_j\left(\frac{\cdot - z}{\mu}\right) - B_j\right) + \log \mu^2$$

on $\partial\Omega$, for $j \in \{0, 1, 2\}$. By the maximum principle and (A-3), we have that $\mathcal{H}_{j,\mu,z}(x) \to \mathcal{H}_z(x)$ and $|\partial_\mu \mathcal{H}_{j,\mu,z}(x)| \le C\mu$ uniformly in $x \in \Omega$ as $\mu \to 0$, for all j. Then, setting $f_\gamma(\mu) = \gamma^{-1} U_{\mu,\gamma,z}(z) - 1$, using that $S_i(0) = 0$ and (4-1), it may be easily checked that $f_\gamma(\mu) = -\gamma^{-2}\log\mu^2(1+o(1)) - 1$, C^1-uniformly in $\mu \in (0, \mu(\gamma))$ as $\gamma \to +\infty$, where $\mu(\gamma)$ is given by $-\log\mu(\gamma)^2 = \frac{1}{2}\gamma^2$. In particular, there exists $\tilde{\gamma} \gg 1$ such that $\lim_{\mu\to 0} f_\gamma(\mu) = +\infty$, $f_\gamma(\mu(\gamma)) < 0$ and $f'_\gamma < 0$ in $(0, \mu(\gamma))$, so that there exists a unique $\tilde{\mu}(\gamma, z) \in (0, \mu(\gamma))$ such that $f_\gamma(\tilde{\mu}(\gamma, z)) = 0$ for all $\gamma \ge \tilde{\gamma}$. Fixing K a compact subset of Ω, it is clear that $\tilde{\gamma}$ can be chosen independent of $z \in K$; in particular, we may let $\tilde{\mu}_\varepsilon := \tilde{\mu}(\gamma_\varepsilon, z)$ be the unique $\mu \in (0, \mu(\gamma_\varepsilon))$ given by

$$U_{\mu,\gamma_\varepsilon,z}(z) = \gamma_\varepsilon \tag{4-2}$$

for all ε small. We write from now on $\widetilde{\mathcal{H}}_{j,\varepsilon,z} := \mathcal{H}_{j,\tilde{\mu}_\varepsilon,z}$ and $U_{\varepsilon,z} := U_{\tilde{\mu}_\varepsilon,\gamma_\varepsilon,z}$. The following result concludes the proof of Proposition 2.1.

Lemma 4.1. *We have*

$$S = \int_\Omega G_{\bar{x}}(y) F(4\pi G_{\bar{x}}(y))\, dy \quad \text{if } \frac{\gamma_\varepsilon^{-3} B(\gamma_\varepsilon)}{\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)|} \not\to 0 \tag{4-3}$$

as $\varepsilon \to 0$, *where* S *is as in* (1-9) *and* $\bar{x}$ *as in* (3-12). *Moreover*, (2-5) *holds true in any case.*

Proof of Lemma 4.1. Let K be a compact subset of Ω and $(z_\varepsilon)_\varepsilon$ be a given sequence of points of K. For simplicity, we let in the proof below $\check{\zeta}_\varepsilon$ be given by

$$\check{\zeta}_\varepsilon = \max\left(\frac{1}{\gamma_\varepsilon^4}, |A(\gamma_\varepsilon)|, \frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^3}\right). \tag{4-4}$$

(1) We first derive the following more explicit expression of the $\tilde{\mu}_\varepsilon$ from (4-2):

$$\begin{aligned}\frac{4}{\tilde{\mu}_\varepsilon^2 \exp(\gamma_\varepsilon^2)\gamma_\varepsilon^2} = \frac{4}{\gamma_\varepsilon^2 \exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))} &(1 + O(\check{\zeta}_\varepsilon + \gamma_\varepsilon^4 |A(\gamma_\varepsilon)|^2)) \\ &\times \left(1 - \frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2} - \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon \exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))} \int_\Omega G_{z_\varepsilon}(y) F(4\pi G_{z_\varepsilon}(y))\, dy\right)\end{aligned} \tag{4-5}$$

as $\varepsilon \to 0$. By the maximum principle and (A-3), we get that there exists $C_K > 0$ such that $|\widetilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon}| \le C_K$ in Ω, so that, by elliptic theory, the $\widetilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon}$'s are also bounded in $C^1_{\mathrm{loc}}(\Omega)$ for all ε and j. We get from (4-2) that $|\log 1/\tilde{\mu}_\varepsilon^2 - \gamma_\varepsilon^2| \le C'_K$, and then that

$$|\widetilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon} - \mathcal{H}_{z_\varepsilon}| \le C''_K \gamma_\varepsilon^8 \exp(-\gamma_\varepsilon^2) \quad \text{in } \Omega, \tag{4-6}$$

for all $0<\varepsilon\ll 1$ and $j\in\{-1,\dots,2\}$, by the maximum principle, (1-8) and (A-3). Rewriting then (4-2) as

$$\gamma_\varepsilon^2=\log\frac{1}{\tilde\mu_\varepsilon^2}\left(1+\frac{A_0}{4\pi\gamma_\varepsilon^2}+\frac{A_1}{4\pi\gamma_\varepsilon^4}+\frac{A(\gamma_\varepsilon)A_2}{4\pi}\right)+\mathcal{H}_{z_\varepsilon}(z_\varepsilon)\left(1+\frac{A_0}{4\pi\gamma_\varepsilon^2}\right)\\ -\frac{B_0}{\gamma_\varepsilon^2}+\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))}\int_\Omega G_{z_\varepsilon}(y)F(4\pi G_{z_\varepsilon}(y))\,dy+O(\gamma_\varepsilon^{-4}+|A(\gamma_\varepsilon)|),$$

we easily get (4-5), using (3-16) and (A-3) with $A_1/(4\pi)-A_0^2/(16\pi^2)-B_0=0$.

(2) We prove now that

$$\int_\Omega|\nabla U_{\varepsilon,z_\varepsilon}|^2\,dx=4\pi(1+I_{z_\varepsilon}(\gamma_\varepsilon)+o(\check\zeta_\varepsilon)) \tag{4-7}$$

as $\varepsilon\to 0$, where $I_{z_\varepsilon}(\gamma_\varepsilon)$ is given by

$$I_{z_\varepsilon}(\gamma_\varepsilon)=\gamma_\varepsilon^{-4}+\frac{A(\gamma_\varepsilon)}{2}+\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^3\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))}\int_\Omega G_{z_\varepsilon}(y)F(4\pi G_{z_\varepsilon}(y))\,dy \tag{4-8}$$

and where $U_{\varepsilon,z_\varepsilon}$ is given by (4-1)–(4-2). By (1-6) and elliptic theory,

$$\left(x\mapsto\int_\Omega G_x(y)F(4\pi G_{z_\varepsilon}(y))\,dy\right)_\varepsilon \text{ is a bounded sequence in } C^1(\overline\Omega). \tag{4-9}$$

By the construction of the $\widetilde{\mathcal{H}}_{j,\varepsilon,z_\varepsilon}$, we can write

$$\begin{aligned}\int_\Omega&|\nabla U_{\varepsilon,z_\varepsilon}(y)|^2\,dy\\ &=\int_\Omega \Delta U_{\varepsilon,z_\varepsilon}(y)\,U_{\varepsilon,z_\varepsilon}(y)\,dy,\\ &=\int_{\{y:\tilde t_\varepsilon(y)\le\gamma_\varepsilon\}}\left(\frac{\Delta(-\tilde t_\varepsilon)}{\gamma_\varepsilon}+\frac{\Delta\widetilde S_{0,\varepsilon}}{\gamma_\varepsilon^3}+\frac{\Delta\widetilde S_{1,\varepsilon}}{\gamma_\varepsilon^5}+\frac{A(\gamma_\varepsilon)\Delta\widetilde S_{2,\varepsilon}}{\gamma_\varepsilon}\right)\\ &\qquad\times\left(\gamma_\varepsilon-\frac{\tilde t_\varepsilon}{\gamma_\varepsilon}+\frac{\widetilde S_{0,\varepsilon}}{\gamma_\varepsilon^3}+O\left(\left(\frac{|A(\gamma_\varepsilon)|}{\gamma_\varepsilon}+\frac{1}{\gamma_\varepsilon^5}\right)(1+\tilde t_\varepsilon)+\frac{|y-z_\varepsilon|}{\gamma_\varepsilon}\right)\right)dy+o(\gamma_\varepsilon^{-4})\\ &\quad+\int_{\{y:\tilde t_\varepsilon(y)\ge\gamma_\varepsilon(\gamma_\varepsilon-1)\}}\left(O(\tilde\mu_\varepsilon^2\gamma_\varepsilon^4)+\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon^2\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))}F(4\pi G_{z_\varepsilon}(y))\right)\\ &\qquad\times\left(\frac{4\pi G_{z_\varepsilon}(y)}{\gamma_\varepsilon}+O\left(\frac{G_{z_\varepsilon}(y)}{\gamma_\varepsilon^3}+\frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^2}\right)\right)dy,\end{aligned} \tag{4-10}$$

where $\tilde t_\varepsilon(y)=\log(1+|y-z_\varepsilon|^2/\tilde\mu_\varepsilon^2)$ and $\widetilde S_{i,\varepsilon}=S_i(|y-z_\varepsilon|/\tilde\mu_\varepsilon)$. We use also here (1-8) with (3-16), and the estimates of point (1) of this proof, including (4-5)–(4-6). The integral on $\{\tilde t_\varepsilon\in(\gamma_\varepsilon,\gamma_\varepsilon(\gamma_\varepsilon-1))\}$ gives an $o(\gamma_\varepsilon^{-4})$-term. Estimate (4-7) follows from (4-10), the Appendix and some computations that we do not develop here again; see also [Mancini and Martinazzi 2017, §5].

(3) We prove now that

$$\int_\Omega|\nabla u_\varepsilon|^2\,dx=4\pi(1+I_{x_\varepsilon}(\gamma_\varepsilon)+o(\check\zeta_\varepsilon)) \tag{4-11}$$

as $\varepsilon \to 0$, where $I_{x_\varepsilon}(\gamma_\varepsilon)$ is given by (4-8), for $(x_\varepsilon)_\varepsilon$ as in (3-13). Now, we can push one step further the argument involving (3-118), writing now that both formulas (3-98) and (3-99) must also coincide on $\partial B_{x_\varepsilon}(\rho'_\varepsilon)$, where $\rho'_\varepsilon > 0$ is as in (3-95). We compute and then get for μ_ε in (3-40) the analogue of (4-5) for $\tilde{\mu}_\varepsilon$

$$\begin{aligned}\lambda_\varepsilon H(\gamma_\varepsilon) &= \frac{4}{\mu_\varepsilon^2 \exp(\gamma_\varepsilon^2)\gamma_\varepsilon^2}\left(1+o\left(\frac{1}{\gamma_\varepsilon^4}\right)\right)\\ &= \frac{4}{\gamma_\varepsilon^2 \exp(1+\mathcal{H}_{x_\varepsilon}(x_\varepsilon))}(1+o(\gamma_\varepsilon^2\check{\zeta}_\varepsilon))\\ &\qquad \times\left(1-\frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2}-\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon \exp(1+\mathcal{H}_{x_\varepsilon}(x_\varepsilon))}\int_\Omega G_{x_\varepsilon}(y)F(4\pi G_{x_\varepsilon}(y))\,dy\right), \end{aligned} \tag{4-12}$$

using (1-8), (3-16), (A-3)–(A-7). Independently, integrating by parts, resuming some computations in the Appendix and using (2-2), (3-12), (3-44), point (1), and (3-97)–(3-99) (see also (3-89) and (A-9)), we get

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx = \int_\Omega u_\varepsilon(\lambda_\varepsilon H(u_\varepsilon)u_\varepsilon \exp(u_\varepsilon^2))\,dx, = \int_\Omega U_{\varepsilon,x_\varepsilon}\Delta U_{\varepsilon,x_\varepsilon}\,dx + o(\check{\zeta}_\varepsilon). \tag{4-13}$$

In order to get the second equality and to apply the dominated convergence theorem, it may be useful to split Ω as

$$\Omega = \{y : t_\varepsilon(y) \le \gamma_\varepsilon\} \cup \left\{y : t_\varepsilon(y) > \gamma_\varepsilon \text{ and } \log\frac{1}{|x_\varepsilon - y|^2} \ge \frac{1-\delta'_0}{2}\gamma_\varepsilon^2\right\} \cup \left\{y : \log\frac{1}{|x_\varepsilon - y|^2} < \frac{1-\delta'_0}{2}\gamma_\varepsilon^2\right\},$$

where δ'_0 is as in (1-6), and to use the first line of (4-12) with (1-5) (resp. with (3-29)) in the first region (resp. in the second region), or (1-6)–(1-7) in the last region. Observe that the argument here is to show that $U_{\varepsilon,x_\varepsilon}$ (resp. $\Delta U_{\varepsilon,x_\varepsilon}$) is in some sense the main part of the expansion of u_ε (resp. Δu_ε). Thus we get (4-11) from (4-7) and (4-13).

(4) We prove now that, for any fixed sequence $(\eta_\varepsilon)_\varepsilon$ of real numbers such that $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$, we have

$$\begin{aligned}\int_\Omega (1+g(V_{\varepsilon,z_\varepsilon}))\exp(V_{\varepsilon,z_\varepsilon}^2)\,dy &= |\Omega|(1+g(0))+\pi\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))(1-\eta_\varepsilon\gamma_\varepsilon^2)\\ &\qquad \times H(\gamma_\varepsilon)\left(1+\gamma_\varepsilon^2 I_{z_\varepsilon}(\gamma_\varepsilon)+\frac{1}{\gamma_\varepsilon^2}+o(\gamma_\varepsilon^2(\check{\zeta}_\varepsilon+|\eta_\varepsilon|))\right)\\ &\qquad \times\left(1+\frac{8B(\gamma_\varepsilon)}{\gamma_\varepsilon(\kappa+1)\exp(1+\mathcal{H}_{z_\varepsilon}(z_\varepsilon))}\int_\Omega G_{z_\varepsilon}(y)F(4\pi G_{z_\varepsilon}(y))\,dy\right),\end{aligned} \tag{4-14}$$

where κ is as in (1-6) and where $V_{\varepsilon,z_\varepsilon} \ge 0$ is given by

$$V_{\varepsilon,z_\varepsilon}^2 = (1-\eta_\varepsilon)U_{\varepsilon,z_\varepsilon}^2, \tag{4-15}$$

where $U_{\varepsilon,z_\varepsilon}$ is given in (4-1). Computations in the spirit of the proof of (4-13) give

$$\int_\Omega (1+g(U_{\varepsilon,x_\varepsilon}))\exp(U_{\varepsilon,x_\varepsilon}^2)\,dy = \int_\Omega (1+g(u_\varepsilon))\exp(u_\varepsilon^2)\,dy + o(\gamma_\varepsilon^2\check{\zeta}_\varepsilon), \tag{4-16}$$

not only by combining (1-1), (1-5)–(1-6), Lemma 3.3, (3-12), (3-97)–(3-99) and the Appendix, and by splitting Ω as in (4-10), but also by using (4-5) and (4-12). In particular, once (4-14) is proved, choosing $\eta_\varepsilon = 0$ and $z_\varepsilon = x_\varepsilon$, we get from (4-16) that

$$\begin{aligned}\int_\Omega (1+g(u_\varepsilon))\exp(u_\varepsilon^2)\,dy &= |\Omega|(1+g(0))+\pi\exp(1+\mathcal{H}_{x_\varepsilon}(x_\varepsilon))H(\gamma_\varepsilon)\\ &\quad\times\left(1+\gamma_\varepsilon^2 I_{x_\varepsilon}(\gamma_\varepsilon)+\frac{1}{\gamma_\varepsilon^2}+o(\gamma_\varepsilon^2\check{\zeta}_\varepsilon)\right)\\ &\quad\times\left(1+\frac{8B(\gamma_\varepsilon)}{\gamma_\varepsilon(\kappa+1)\exp(1+\mathcal{H}_{x_\varepsilon}(x_\varepsilon))}\int_\Omega G_{x_\varepsilon}(y)F(4\pi G_{x_\varepsilon}(y))\,dy\right).\end{aligned} \tag{4-17}$$

It remains to prove (4-14). We compute and get

$$U_{\varepsilon,z_\varepsilon}(y)^2 = \gamma_\varepsilon^2 - 2\tilde{t}_\varepsilon + \frac{\tilde{t}_\varepsilon^2}{\gamma_\varepsilon^2} + \frac{2\widetilde{S}_{0,\varepsilon}}{\gamma_\varepsilon^2} + O\big((|A(\gamma_\varepsilon)|+\gamma_\varepsilon^{-4})(1+\tilde{t}_\varepsilon(y)^2)+|y-z_\varepsilon|\big) \tag{4-18}$$

for all y such that $\tilde{t}_\varepsilon(y)\le\gamma_\varepsilon$, using (1-7), (4-1)–(4-2), (4-5), (4-9) and (A-3). Then we get

$$\begin{aligned}&\int_{\{\tilde{t}_\varepsilon\le\gamma_\varepsilon\}}(1+g(V_{\varepsilon,z_\varepsilon}))\exp(V_{\varepsilon,z_\varepsilon}^2)\,dy\\ &\quad=\int_{\{\tilde{t}_\varepsilon\le\gamma_\varepsilon\}}H(\gamma_\varepsilon)(1+O(|A(\gamma_\varepsilon)|\exp(\delta_0\tilde{t}_\varepsilon)))\exp(\gamma_\varepsilon^2)\exp(-2\tilde{t}_\varepsilon)\exp(-\eta_\varepsilon\gamma_\varepsilon^2)\\ &\qquad\qquad\times\exp\left(\frac{\tilde{t}_\varepsilon^2+2\widetilde{S}_{0,\varepsilon}}{\gamma_\varepsilon^2}\right)\exp\big(O((|\eta_\varepsilon|+|A(\gamma_\varepsilon)|+\gamma_\varepsilon^{-4})(1+\tilde{t}_\varepsilon^2))+|y-z_\varepsilon|\big)\,dy,\end{aligned}$$

using (3-2) and (4-15) with (4-18). Then combining $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$, (3-16), (4-5), computing explicitly $\int_{\mathbb{R}^2}\exp(-2T_0)S_0\,dy = 0$ and $\int_{\mathbb{R}^2}\exp(-2T_0)T_0^2\,dy = 2\pi$ for T_0 as in (3-42), we get

$$\begin{aligned}&\int_{\{\tilde{t}_\varepsilon\le\gamma_\varepsilon\}}(1+g(V_{\varepsilon,z_\varepsilon}))\exp(V_{\varepsilon,z_\varepsilon}^2)\,dy\\ &=\frac{(1-\eta_\varepsilon\gamma_\varepsilon^2)H(\gamma_\varepsilon)\exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon)+1)}{4}\big(1+o(\gamma_\varepsilon^2(|A(\gamma_\varepsilon)|+|\eta_\varepsilon|)+\gamma_\varepsilon^{-2})\big)\\ &\quad\times\left(1+\frac{\gamma_\varepsilon^2A(\gamma_\varepsilon)}{2}+\frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon\exp(\mathcal{H}_{z_\varepsilon}(z_\varepsilon)+1)}\int_\Omega G_{z_\varepsilon}(x)F(4\pi G_{z_\varepsilon}(x))\,dx+o\left(\frac{B(\gamma_\varepsilon)}{\gamma_\varepsilon}\right)\right)4\pi\left(1+\frac{2}{\gamma_\varepsilon^2}\right).\end{aligned} \tag{4-19}$$

Independently, we get from (1-6), (3-1) (parts (a) and (b) in $\{y, 4\pi G_{z_\varepsilon}(y)\le\frac{1}{2}\gamma_\varepsilon\}$, or part (c) otherwise), (4-1), (4-5) and the dominated convergence theorem that

$$\begin{aligned}&\int_{\{\tilde{t}_\varepsilon\ge\gamma_\varepsilon\}}(1+g(V_{\varepsilon,z_\varepsilon}))\exp(V_{\varepsilon,z_\varepsilon}^2)\,dy\\ &\qquad=|\Omega|(1+g(0))+\frac{8\pi B(\gamma_\varepsilon)}{\gamma_\varepsilon(\kappa+1)}\int_\Omega G_{z_\varepsilon}(y)F(4\pi G_{z_\varepsilon}(y))\,dy+o\left(\frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon}+\frac{1}{\gamma_\varepsilon^2}\right).\end{aligned} \tag{4-20}$$

Combining (4-19) and (4-20), we conclude that (4-14) holds true, using (3-3) and (4-5).

(5) We are now in position to conclude the proof of Lemma 4.1. Let $\bar{x}_0$ be a point in the compact set $K_\Omega \Subset \Omega$ where S is attained in the third equation of (1-9). Let η_ε be given by

$$(1-\eta_\varepsilon) = \frac{4\pi(1-\varepsilon)}{\|U_{\varepsilon,\bar{x}_0}\|_{H_0^1}^2}. \tag{4-21}$$

First, we can check that

$$\eta_\varepsilon = I_{\bar{x}_0}(\gamma_\varepsilon) - I_{x_\varepsilon}(\gamma_\varepsilon) + o(\check{\zeta}_\varepsilon), \tag{4-22}$$

so that the condition $\eta_\varepsilon = o(\gamma_\varepsilon^{-2})$ above (4-14) is satisfied, using (1-7), (3-7), (3-16), (4-7) and (4-11). Additionally, we have $\|V_{\varepsilon,\bar{x}_0}\|_{H_0^1}^2 = 4\pi(1-\varepsilon)$, by our choice (4-21) of η_ε, and then, by (3-7),

$$\int_\Omega (1+g(u_\varepsilon))\exp(u_\varepsilon^2)\,dy \geq \int_\Omega (1+g(V_{\varepsilon,\bar{x}_0}))\exp(V_{\varepsilon,\bar{x}_0}^2)\,dy;$$

this implies, in view of (4-14), (4-17), (4-22) and of our choice of $\bar{x}_0$, that (4-3) is true and then, by (4-11) again, that (2-5)–(2-6) are true as well. This concludes the proof of Lemma 4.1. □

Proposition 2.1 is proved. □

Proof of Proposition 2.3. Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Let g be such that (1-1) and (1-5)–(1-6) hold true for H as in (1-2), and let A, B and F be thus given. Assume that $\Lambda_g(\Omega) < \pi\exp(1+M)$, where M is as in (1-9) and $\Lambda_g(\Omega)$ as in (1-11). Assume that there exists a sequence of positive integers $(N_\varepsilon)_\varepsilon$ such that (2-9) holds true and such that $(I_{4\pi}^{g_{N_\varepsilon}}(\Omega))$ admits a nonnegative extremal u_ε for all $\varepsilon > 0$, where g_{N_ε} is as in (1-10). Then, by Lemma 3.4 in Case 1, we have (2-1) and that (3-8) holds true for $\alpha_\varepsilon = 4\pi$ for all $0 < \varepsilon \ll 1$. Moreover, we have $u_\varepsilon \in C^{1,\theta}(\bar{\Omega})$ $(0<\theta<1)$ and (2-3) by (3-13). In order to conclude the proof of Proposition 2.3, it remains to prove (2-10). Still by Lemma 3.4 in Case 1, (3-97)–(3-99) and (A-9) (v_ε as in (3-89)) hold true. Concerning (3-97)–(3-99) and (A-9), observe that, contrary to Case 2, the term ξ_ε cannot be neglected in Case 1, which we are facing here. Indeed, using also now (3-30), (3-40), (3-110) and (A-9), we can resume computations of (4-10), (4-13) and the Appendix (now with (3-11)) to get

$$\|u_\varepsilon\|_{H_0^1}^2 = 4\pi\big(1 + \check{I}(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)| + \xi_\varepsilon)\big)$$

as $\varepsilon \to 0$, where

$$\check{I}(\gamma_\varepsilon) := \gamma_\varepsilon^{-4} + \tfrac{1}{2}(A(\gamma_\varepsilon) - 2\xi_\varepsilon) + 4\gamma_\varepsilon^{-3}\exp(-1-M)B(\gamma_\varepsilon)S,$$

so that (2-10) holds true, which concludes the proof. □

Appendix: Radial analysis

Let $(x_\varepsilon)_\varepsilon$ be a sequence of points in $\mathbb{R}^2$ and $(\gamma_\varepsilon)_\varepsilon$ be a sequence of positive real numbers such that $\gamma_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Let g be such that (1-1) and (1-5) holds true for H as in (1-2), and let A be thus given. Let $(N_\varepsilon)_\varepsilon$ be a sequence of integers. We assume that we are in one of the following two cases:

Case 1′: $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$, and (3-30)—(3-31) hold true.

Case 2′: $N_\varepsilon = 1$ for all ε.

Let B_ε be the radial solution around x_ε in $\mathbb{R}^2$ of (3-81), for Ψ_N as in (3-25), where $(\lambda_\varepsilon)_\varepsilon$ is any given sequence of positive real numbers. Let T_0 be given in $\mathbb{R}^2$ by

$$T_0(x) = \log(1+|x|^2). \tag{A-1}$$

Let $S_i,\ i=0,1,2$, be the radially symmetric solutions around 0 in $\mathbb{R}^2$ of

$$\begin{aligned}
&\Delta S_0 - 8\exp(-2T_0)S_0 = 4\exp(-2T_0)(T_0^2 - T_0),\\
&\Delta S_1 - 8\exp(-2T_0)S_1 = 4\exp(-2T_0)\big(S_0 + 2S_0^2 - 4T_0S_0 + 2S_0T_0^2 - T_0^3 + \tfrac{1}{2}T_0^4\big),\\
&\Delta S_2 - 8\exp(-2T_0)S_2 = 4\exp(-2T_0)T_0
\end{aligned} \tag{A-2}$$

such that $S_i(0)=0$. In the sequel, we will use the C^1 expansions of the S_i's given by

$$\begin{aligned}
S_0(r) &= \frac{A_0}{4\pi}\log\frac{1}{r^2} + B_0 + O(\log(r)^2r^{-2}), \quad \text{where } A_0 = 4\pi,\ B_0 = \frac{\pi^2}{6}+2,\\
S_1(r) &= \frac{A_1}{4\pi}\log\frac{1}{r^2} + B_1 + O(\log(r)^4r^{-2}), \quad \text{where } A_1 = 4\pi\Big(3+\frac{\pi^2}{6}\Big),\ B_1\in\mathbb{R},\\
S_2(r) &= \frac{A_2}{4\pi}\log\frac{1}{r^2} + B_2 + O(\log(r)r^{-2}), \quad \text{where } A_2 = 2\pi,\ B_2\in\mathbb{R},
\end{aligned} \tag{A-3}$$

as $r=|x|\to+\infty$. Note that in particular

$$A_i = \int_{\mathbb{R}^2}\Delta S_i\,dx. \tag{A-4}$$

The explicit formula for S_0

$$S_0(r) = -T_0(r) + \frac{2r^2}{1+r^2} - \tfrac{1}{2}T_0(r)^2 + \frac{1-r^2}{1+r^2}\int_1^{1+r^2}\frac{\log t}{1-t}\,dt,$$

and the expansions in (A-3) are derived in [Malchiodi and Martinazzi 2014; Mancini and Martinazzi 2017]. Let $\varepsilon_0\in(\sqrt{1/e},1)$ be given. Let μ_ε be given by (3-40) and t_ε by (3-41). Let $\rho_\varepsilon>0$ be given by (3-83) and satisfying (3-84). Let $S_{i,\varepsilon}$ be then given by

$$S_{i,\varepsilon}(x) = S_i\left(\frac{|x-x_\varepsilon|}{\mu_\varepsilon}\right) \tag{A-5}$$

for $i=0,1,2$. Let $\xi_\varepsilon>0$ be given by (3-14). In Case 1′ where $N_\varepsilon\to+\infty$ as $\varepsilon\to0$, we get $\xi_\varepsilon = O(N_\varepsilon^{-1/2})$ by (3-30) and (3-45). Then, in any case, we clearly have

$$\xi_\varepsilon\to0 \tag{A-6}$$

as $\varepsilon\to0$. Then we are in position to state the main result of this section.

Proposition A.1. *We have*

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon)-2\xi_\varepsilon)\frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + o\left(t_\varepsilon\left(\frac{1}{\gamma_\varepsilon^5} + \frac{|A(\gamma_\varepsilon)|+\xi_\varepsilon}{\gamma_\varepsilon}\right)\right) \tag{A-7}$$

uniformly in $[0,\rho_\varepsilon]$ *as* $\varepsilon\to0$.

In particular, using also (1-1) and (3-3), it can be checked that B_ε is positive and radially decreasing in $[0, \rho_\varepsilon]$. Observe also that $\xi_\varepsilon \ll \gamma_\varepsilon^{-4}$ can be seen as a remainder term in Case 2′. Let $\zeta_\varepsilon > 0$ be given by

$$\zeta_\varepsilon = \max\left(\frac{1}{\gamma_\varepsilon^4}, |A(\gamma_\varepsilon)|, \xi_\varepsilon\right). \tag{A-8}$$

Set $\bar{t}_\varepsilon = 1 + t_\varepsilon$. Resuming the computations below, we get as a byproduct of Proposition A.1 that $v_\varepsilon = o(\bar{t}_\varepsilon \gamma_\varepsilon^{-5})$ implies

$$\frac{\lambda_\varepsilon \Psi_\varepsilon'(B_\varepsilon + v_\varepsilon)}{2} = \frac{4\ \exp(-2t_\varepsilon)}{\mu_\varepsilon^2 \gamma_\varepsilon}\left[1 + \frac{(\Delta S_0)((\cdot - x_\varepsilon)/\mu_\varepsilon)}{\gamma_\varepsilon^2} + \frac{(\Delta S_1)((\cdot - x_\varepsilon)/\mu_\varepsilon)}{\gamma_\varepsilon^4} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)(\Delta S_2)\left(\frac{\cdot - x_\varepsilon}{\mu_\varepsilon}\right) + o(\zeta_\varepsilon \exp(\tilde{\delta}_0 t_\varepsilon))\right] \tag{A-9}$$

uniformly in $\{y : t_\varepsilon(y) \le \gamma_\varepsilon\}$, for some given $\tilde{\delta}_0 \in (\delta_0, 1)$, for δ_0 as in (1-5).

Proof of Proposition A.1. Since both arguments are very similar to prove for Case 1′ and Case 2′, for the sake of readability, we only write the proof of Proposition A.1 in the more delicate Case 1′. Then, assume that we are in Case 1′. We let τ_ε be given by

$$B_\varepsilon = \gamma_\varepsilon - \frac{\tau_\varepsilon}{\gamma_\varepsilon}. \tag{A-10}$$

Let $\bar{w}_\varepsilon$ be given by

$$B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^3} + \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^5} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)\frac{S_{2,\varepsilon}}{\gamma_\varepsilon} + \frac{\zeta_\varepsilon \bar{w}_\varepsilon}{\gamma_\varepsilon}. \tag{A-11}$$

Let $\bar{\delta} > 0$ be fixed and let $\bar{r}_\varepsilon \ge 0$ be given by

$$\bar{r}_\varepsilon = \sup\{r > 0 : |\bar{w}_\varepsilon| \le \bar{\delta} t_\varepsilon \text{ in } [0, r]\}. \tag{A-12}$$

Now, since $\bar{\delta} > 0$ may be arbitrarily small, in order to get Proposition A.1, it is sufficient to prove that $\bar{r}_\varepsilon = \rho_\varepsilon$ for all $0 < \varepsilon \ll 1$. Using (A-12), we perform computations in $[0, \bar{r}_\varepsilon]$ and the subsequent $o(1)$ are uniformly small in this set as $\varepsilon \to 0$. First, by (1-5), (A-3), (A-6) and (A-12), we have

$$\tau_\varepsilon = t_\varepsilon(1 + o(1)). \tag{A-13}$$

Observe that, as soon as we have $\Delta B_\varepsilon > 0$ in $[0, \bar{r}_\varepsilon]$, the solution B_ε is radially decreasing and (3-93) holds true in $[0, \bar{r}_\varepsilon]$. Let L_ε^H and L_ε^g be given by

$$H(B_\varepsilon) = H(\gamma_\varepsilon)(1 + L_\varepsilon^H), \quad \text{and then} \quad (1 + g(B_\varepsilon)) = H(\gamma_\varepsilon)(1 + L_\varepsilon^H + L_\varepsilon^g). \tag{A-14}$$

In view of (A-10) and (A-13), estimates of L_ε^H, L_ε^g are given by (1-5) and (3-2), respectively. We are now in position to expand the right-hand side of (3-81). From now on, it is convenient to write

$$\widetilde{N}_\varepsilon = N_\varepsilon - 1. \tag{A-15}$$

Going back to (3-27), we have

$$\frac{\Psi'_{N_\varepsilon}(B_\varepsilon)}{2} = B_\varepsilon H(\gamma_\varepsilon)\left[(1+L_\varepsilon^H)(1+\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)) + L_\varepsilon^g\left(\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} - B_\varepsilon^2\right)\right]. \tag{A-16}$$

By (3-83), (A-10) and (A-13) and since $\bar{r}_\varepsilon \le \rho_\varepsilon$, we have

$$\min_{[0,\bar{r}_\varepsilon]} B_\varepsilon \ge (\varepsilon_0 + o(1))\gamma_\varepsilon \to +\infty \tag{A-17}$$

as $\varepsilon \to 0$. Thus, by Stirling's formula, we get

$$\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} \ge \exp\left(N_\varepsilon\left(\log\frac{\gamma_\varepsilon^2}{N_\varepsilon} + (\log\varepsilon_0^2 + 1) + o(1)\right)\right)$$

and then, for all given integers $k \ge 0$,

$$B_\varepsilon^k = o(1)\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} \tag{A-18}$$

in $[0,\bar{r}_\varepsilon]$ as $\varepsilon \to 0$, using $\varepsilon_0^2 > 1/e$ with (3-31). Similarly, for all given integers $k \ge 0$, we have

$$\frac{B_\varepsilon^k}{\varphi_{N_\varepsilon}(B_\varepsilon^2)} = o(1) \tag{A-19}$$

in $[0,\bar{r}_\varepsilon]$ as $\varepsilon \to 0$. Then, by (3-40), (A-10), (A-19) and (A-18), we may rewrite (A-16) as

$$\frac{\lambda_\varepsilon \Psi'_{N_\varepsilon}(B_\varepsilon)}{2} = \frac{4}{\mu_\varepsilon^2\gamma_\varepsilon}\left(1 - \frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right)\left[O(\exp(-\gamma_\varepsilon^2)) + \frac{\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}{\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)}\left(1 + L_\varepsilon^H + O\left(\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}L_\varepsilon^g\right)\right)\right] \tag{A-20}$$

in $[0,\bar{r}_\varepsilon]$, as $\varepsilon \to 0$. Indeed, by (A-17), we have

$$L_\varepsilon^H = o(1) \quad \text{and} \quad L_\varepsilon^g = o(1) \tag{A-21}$$

in $[0,\bar{r}_\varepsilon]$ as $\varepsilon \to 0$, using (1-1), (3-3) and (A-14). In (A-20), the term $O(\exp(-\gamma_\varepsilon^2))$ is equal to $(1+L_\varepsilon^H)/\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)$ and we thus get this control by (3-30) and (A-21). In the following lines, we expand the terms of (A-20). By (3-48) with $\Gamma = \gamma_\varepsilon^2$ and $T = B_\varepsilon^2$, we get

$$\frac{\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}{\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)} = \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) - F_\varepsilon,$$

where F_ε satisfies in $[0,\bar{r}_\varepsilon]$

$$\begin{aligned} F_\varepsilon &= \frac{B_\varepsilon^{2\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)}\int_0^{\gamma_\varepsilon^2 - B_\varepsilon^2} \exp(-u)\left(1 + \frac{u}{B_\varepsilon^2}\right)^{\widetilde{N}_\varepsilon} du \\ &= \frac{\exp(B_\varepsilon^2)}{\varphi_{\widetilde{N}_\varepsilon}(\gamma_\varepsilon^2)}\int_{B_\varepsilon^2}^{\gamma_\varepsilon^2} \exp(-s)\frac{s^{\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!}\,ds = \xi_\varepsilon \exp(B_\varepsilon^2 - \gamma_\varepsilon^2)\int_{B_\varepsilon^2 - \gamma_\varepsilon^2}^{0} \exp(-y)\left(1 + \frac{y}{\gamma_\varepsilon^2}\right)^{\widetilde{N}_\varepsilon} dy. \end{aligned} \tag{A-22}$$

By (A-10) and (A-11), we may write

$$\tau_\varepsilon = t_\varepsilon - \frac{S_{0,\varepsilon}}{\gamma_\varepsilon^2} - \frac{S_{1,\varepsilon}}{\gamma_\varepsilon^4} - (A(\gamma_\varepsilon) - 2\xi_\varepsilon)S_{2,\varepsilon} - \zeta_\varepsilon\bar{w}_\varepsilon.$$

Then, keeping in mind (A-3), (A-6), (A-12), (A-13) and $t_\varepsilon \le \gamma_\varepsilon^2$, we may compute

$$\exp(B_\varepsilon^2-\gamma_\varepsilon^2)=\exp\left(-2\tau_\varepsilon+\frac{\tau_\varepsilon^2}{\gamma_\varepsilon^2}\right)=\exp\left[-2\tau_\varepsilon+\frac{1}{\gamma_\varepsilon^2}\left(t_\varepsilon^2-\frac{2t_\varepsilon S_{0,\varepsilon}}{\gamma_\varepsilon^2}+O(\zeta_\varepsilon\bar{t}_\varepsilon^2)\right)\right] \tag{A-23}$$

in $[0,\bar{r}_\varepsilon]$ as $\varepsilon\to 0$. Observe that

$$\left|\exp(y)-\sum_{j=0}^{N}\frac{y^j}{j!}\right|\le\frac{|y|^{N+1}}{(N+1)!}\exp(|y|)$$

for all $y\in\mathbb{R}$ and all integers $N\ge 0$. Then we draw from (A-23) that

$$\begin{aligned}
&\left(1-\frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right)\exp(B_\varepsilon^2-\gamma_\varepsilon^2)\\
&\quad=\exp(-2t_\varepsilon)\left[1+\frac{1}{\gamma_\varepsilon^2}(2S_{0,\varepsilon}+t_\varepsilon^2-t_\varepsilon)+\frac{1}{\gamma_\varepsilon^4}\big(2S_{1,\varepsilon}+2S_{0,\varepsilon}^2+\tfrac{1}{2}t_\varepsilon^4+2S_{0,\varepsilon}t_\varepsilon^2-4S_{0,\varepsilon}t_\varepsilon-t_\varepsilon^3+S_{0,\varepsilon}\big)\right.\\
&\qquad\left.+2(A(\gamma_\varepsilon)-2\xi_\varepsilon)S_{2,\varepsilon}+2\zeta_\varepsilon\bar{w}_\varepsilon+O\left(\left(\frac{\bar{t}_\varepsilon^6}{\gamma_\varepsilon^6}+\frac{\zeta_\varepsilon\bar{t}_\varepsilon^3}{\gamma_\varepsilon^2}+\zeta_\varepsilon^2\bar{t}_\varepsilon^3\right)\exp\left(o(t_\varepsilon)+\frac{t_\varepsilon^2}{\gamma_\varepsilon^2}\right)\right)\right]
\end{aligned} \tag{A-24}$$

in $[0,\bar{r}_\varepsilon]$ as $\varepsilon\to 0$. Independently, by (3-30), (3-45), (A-10), (A-12), (A-13) and since $B_\varepsilon(x_\varepsilon)=\gamma_\varepsilon$, for all given $R>0$, we have

$$\left\|\frac{B_\varepsilon^{2\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}+\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}\right\|_{L^\infty([0,\min(R\mu_\varepsilon,\bar{r}_\varepsilon)]}=O\left(\frac{1}{\sqrt{N_\varepsilon}}\right)\quad\text{and}\quad\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{\widetilde{N}_\varepsilon}(B_\varepsilon^2)}\le 1 \tag{A-25}$$

in $[0,\bar{r}_\varepsilon]$, the second inequality being obvious by (3-6) and (A-15). In the sequel, by (3-31), we may assume that

$$\beta_\varepsilon:=\frac{\widetilde{N}_\varepsilon}{\gamma_\varepsilon^2}\quad\text{satisfies}\quad\lim_{\varepsilon\to 0}\beta_\varepsilon=\beta_0\in[0,1], \tag{A-26}$$

up to a subsequence. Now, we give estimates for F_ε given in (A-22). Up to a subsequence, we can split our results according to the following two cases:

$$\lim_{\varepsilon\to 0}\frac{\gamma_\varepsilon^2-\widetilde{N}_\varepsilon}{\sqrt{\widetilde{N}_\varepsilon}}=+\infty, \tag{A-27a}$$

$$\frac{\gamma_\varepsilon^2-\widetilde{N}_\varepsilon}{\sqrt{\widetilde{N}_\varepsilon}}=O(1). \tag{A-27b}$$

Observe that, since we assume (3-31), all the possible situations are considered in (A-27). Let $(r_\varepsilon)_\varepsilon$ be any sequence such that

$$r_\varepsilon\in[0,\bar{r}_\varepsilon] \tag{A-28}$$

for all ε. We prove that, in the case of (A-27a),

$$F_\varepsilon(r_\varepsilon)=\begin{cases}O\big(\xi_\varepsilon\gamma_\varepsilon\exp\big(-2t_\varepsilon(r_\varepsilon)(\beta_0+o(1))\big)\big) & \text{if } B_\varepsilon(r_\varepsilon)^2\ge\widetilde{N}_\varepsilon+\sqrt{\widetilde{N}_\varepsilon},\\ O\big(\exp\big(-(1+\varepsilon_0+o(1))t_\varepsilon(r_\varepsilon)\big)\big) & \text{if } B_\varepsilon(r_\varepsilon)^2<\widetilde{N}_\varepsilon+\sqrt{\widetilde{N}_\varepsilon},\end{cases} \tag{A-29}$$

while we get in the case of (A-27b)

$$F_\varepsilon(r_\varepsilon) = \begin{cases} 2t_\varepsilon(r_\varepsilon)\xi_\varepsilon \exp\big(-2t_\varepsilon(r_\varepsilon)(1+o(1))\big) & \text{if } t_\varepsilon(r_\varepsilon) = o(\gamma_\varepsilon), \\ O\big(t_\varepsilon(r_\varepsilon)\xi_\varepsilon \exp\big(-(1+\varepsilon_0+o(1))t_\varepsilon(r_\varepsilon)\big)\big) & \text{if } \gamma_\varepsilon = O(t_\varepsilon(r_\varepsilon)). \end{cases} \tag{A-30}$$

Now we prove (A-29). We start with the first estimate of (A-29). Then, we assume that $B_\varepsilon(r_\varepsilon)^2 \geq \widetilde{N}_\varepsilon + \sqrt{\widetilde{N}_\varepsilon}$, and thus in particular that

$$1 - \frac{\widetilde{N}_\varepsilon}{B_\varepsilon(r_\varepsilon)^2} \geq \frac{1+o(1)}{\sqrt{\widetilde{N}_\varepsilon}}. \tag{A-31}$$

Writing now F_ε according to the first formula of (A-22), using (3-93), (A-17) and

$$\log(1+t) \leq t \quad \text{for all } t > -1, \tag{A-32}$$

we get first that

$$F_\varepsilon(r_\varepsilon) \leq \xi_\varepsilon \exp(-2\tau_\varepsilon(r_\varepsilon)\beta_\varepsilon) \int_0^{\gamma_\varepsilon^2 - B_\varepsilon^2} \exp\left(-y\left(1 - \frac{\widetilde{N}_\varepsilon}{B_\varepsilon(r_\varepsilon)^2}\right)\right) dy,$$

and conclude the proof of the first estimate of (A-29), by (3-31), (A-13) and (A-31). In order to prove the second estimate of (A-29), it is sufficient to write F_ε according to the second formula of (A-22), to check that

$$\int_{\mathbb{R}} \exp(-s) \frac{s^{\widetilde{N}_\varepsilon}}{\widetilde{N}_\varepsilon!} \, ds = 1,$$

that $r_\varepsilon \leq \bar{r}_\varepsilon \leq \rho_\varepsilon$ implies

$$t_\varepsilon(\bar{r}_\varepsilon) \leq (1-\varepsilon_0)\gamma_\varepsilon^2, \tag{A-33}$$

and to use (A-10), (A-13) and (3-30). Now we turn to the proof of (A-30). Then, we assume (A-27b) holds true and in particular

$$1 - \beta_\varepsilon = O\left(\frac{1}{\gamma_\varepsilon}\right) \quad \text{in (A-27b)}. \tag{A-34}$$

Writing F_ε according to the third estimate of (A-22), we get

$$F_\varepsilon = \xi_\varepsilon \exp\left(-\tau_\varepsilon\left(2 - \frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right)\right)(\gamma_\varepsilon^2 - B_\varepsilon^2) \int_0^1 \exp\left((\gamma_\varepsilon^2 - B_\varepsilon^2)y + \widetilde{N}_\varepsilon \log\left(1 - \frac{(\gamma_\varepsilon^2 - B_\varepsilon^2)y}{\gamma_\varepsilon^2}\right)\right) dy \tag{A-35}$$

at r_ε. Expanding the log, we easily get the first estimate of (A-30) from (A-13), (A-34), (A-35) and the assumption $t_\varepsilon(r_\varepsilon) = o(\gamma_\varepsilon)$. The second estimate of (A-30) can also be obtained from (A-35) by (A-13), (A-32), (A-33) and (A-34). This concludes the proof of (A-30). Now, we prove that, in the case of (A-27a), we have

$$\int_0^{\bar{r}_\varepsilon} F_\varepsilon(r) r \, dr = o\left(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4}\right). \tag{A-36}$$

Since $r_\varepsilon \le \rho_\varepsilon$, we get from (3-14), (3-30), (3-31), (A-29) and by Stirling's formula that

$$\int_{\{r\in[0,\bar r_\varepsilon],\,B_\varepsilon(r)^2\ge \widetilde N_\varepsilon+\sqrt{\widetilde N_\varepsilon}\}} F_\varepsilon(r) r\,dr \lesssim \exp(\gamma_\varepsilon^2[f(\beta_\varepsilon)+O((\log\gamma_\varepsilon)/\gamma_\varepsilon^2)]) \begin{cases} \mu_\varepsilon^2 & \text{if } \beta_0>\frac12,\\ \mu_\varepsilon^2\exp\big(\gamma_\varepsilon^2(1-\varepsilon_0)(1-2\beta_0+o(1))\big) & \text{if } \beta_0\le\frac12,\end{cases} \tag{A-37}$$

where f is the continuous function in $[0, 1]$ given for $\beta\in(0, 1]$ by

$$f(\beta)=\beta\log\frac{1}{\beta}+\beta-1.$$

Independently, since $\bar r_\varepsilon\le\rho_\varepsilon$, if

$$r_\varepsilon\in J_\varepsilon:=\big\{r\in[0,\bar r_\varepsilon]: B_\varepsilon(r)^2<\widetilde N_\varepsilon+\sqrt{\widetilde N_\varepsilon}\big\},$$

then $J_\varepsilon\ne\varnothing$ and $\gamma_\varepsilon^2\lesssim \widetilde N_\varepsilon$, by (A-10), (A-13) and (A-33). Thus we have

$$\gamma_\varepsilon\lesssim\sqrt{\widetilde N_\varepsilon}\ll t_\varepsilon(r_\varepsilon),$$

using that we are in the case (A-27a) for the last estimate. Then, we get from (A-29) that

$$\int_{J_\varepsilon} F_\varepsilon(r) r\,dr\lesssim\int_{\{r\le\rho_\varepsilon,\,t_\varepsilon\ge\gamma_\varepsilon\}}\exp\big(-(1+\varepsilon_0+o(1))t_\varepsilon(r)\big) r\,dr=o\bigg(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4}\bigg). \tag{A-38}$$

Observe that f and $\beta\mapsto f(\beta)+\frac12(1-2\beta)$ are negative in $[0, 1)$ and $\big[0,\frac12\big]$ respectively. Moreover, because of (A-27a) and by (3-31), we can check that

$$\beta_\varepsilon=\frac{\widetilde N_\varepsilon}{\gamma_\varepsilon^2}\le\frac{1}{1+1/\sqrt{\widetilde N_\varepsilon}}\le 1-\frac{1+o(1)}{\sqrt{\widetilde N_\varepsilon}}\le 1-\frac{1+o(1)}{\gamma_\varepsilon},$$

since $\gamma_\varepsilon^2\ge\widetilde N_\varepsilon+\sqrt{\widetilde N_\varepsilon}$, and then that

$$0<-f(\beta_\varepsilon)\lesssim\frac{1}{\gamma_\varepsilon}. \tag{A-39}$$

Thus, we get (A-36) from the first estimate of (A-37) with (A-39), from the second estimate of (A-37) with $1-\varepsilon_0<1-\sqrt{1/e}<\frac12$ and from (A-38). Computing as in (A-37), we get also that

$$\xi_\varepsilon=o\bigg(\frac{1}{\gamma_\varepsilon^4}\bigg) \tag{A-40}$$

in (A-27a) (see (A-39)). By (A-13) and the second part of (A-25), using that $\bar r_\varepsilon\le\rho_\varepsilon$, we may rewrite (A-20) as

$$\begin{aligned}\frac{\lambda_\varepsilon\Psi'_{N_\varepsilon}(B_\varepsilon)}{2} &=\frac{4}{\mu_\varepsilon^2\gamma_\varepsilon}\bigg[\bigg(1-\frac{\tau_\varepsilon}{\gamma_\varepsilon^2}+L_\varepsilon^H\bigg)\exp(B_\varepsilon^2-\gamma_\varepsilon^2)-F_\varepsilon+O\bigg(\frac{t_\varepsilon}{\gamma_\varepsilon^2}|F_\varepsilon|+\exp(-\gamma_\varepsilon^2)\bigg)\\ &\quad+O\bigg(\bigg(\frac{t_\varepsilon}{\gamma_\varepsilon^2}\exp(B_\varepsilon^2-\gamma_\varepsilon^2)+|F_\varepsilon|\bigg)(|L_\varepsilon^H|+|L_\varepsilon^g|)\bigg)+O\bigg(|L_\varepsilon^g|\exp(B_\varepsilon^2-\gamma_\varepsilon^2)\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!\,\varphi_{\widetilde N_\varepsilon}(B_\varepsilon^2)}\bigg)\bigg].\end{aligned} \tag{A-41}$$

By (3-84), we clearly have

$$\int_0^{\rho_\varepsilon} \exp(-\gamma_\varepsilon^2) r\, dr = o\left(\frac{\mu_\varepsilon^2}{\gamma_\varepsilon^4}\right). \tag{A-42}$$

Integrating by parts, observe that $\bar{w}_\varepsilon$ given by (A-11) satisfies

$$\bar{w}_\varepsilon(0) = 0 \quad \text{and} \quad -r_\varepsilon \bar{w}'_\varepsilon(r_\varepsilon) = \int_0^{r_\varepsilon} (\Delta \bar{w}_\varepsilon) r\, dr, \tag{A-43}$$

where, still using radial notation,

$$\bar{w}'_\varepsilon(r) = \frac{d\bar{w}_\varepsilon}{dr}(r).$$

From now on, we estimate $\bar{w}_\varepsilon$ in $[0, \bar{r}_\varepsilon]$ with (A-43). By (1-5) and (A-14), we may expand L_ε^H in (A-41). Now, since (3-81) holds true, by taking the laplacian of B_ε, we get from (A-11) and (A-41) an estimate of $\Delta \bar{w}_\varepsilon$ and then of the right-hand side of (A-43) for r_ε still as in (A-28). The key observation is that the precise form of the ODEs in (A-2) generates a cancellation, when plugging (A-24) in (A-41). The lower-order terms when taking the laplacian of (A-11) are estimated thanks to (A-3). We are left with estimating the lower-order terms in (A-41), in both cases of (A-27). Assume first that we are in the case of (A-27a). Estimating these lower-order terms amounts to gathering the appropriate previous estimates (see (A-21), (A-25), (A-29), (A-36), (A-40), (A-42)). This gives after some slightly long, but elementary computations that

$$\int_0^{r_\varepsilon} |(\Delta \bar{w}_\varepsilon)| r\, dr = O\left(\|\bar{w}'_\varepsilon\|_{L^\infty([0,r_\varepsilon])} \int_0^{r_\varepsilon/\mu_\varepsilon} \frac{\mu_\varepsilon r^2\, dr}{(1+r^2)^{1+\varepsilon_0+o(1)}}\right) + o\left(\int_0^{r_\varepsilon/\mu_\varepsilon} \frac{r\, dr}{(1+r^2)^{1+\varepsilon_0+o(1)}}\right). \tag{A-44}$$

We also use (1-5) and (3-2) to estimate L_ε^H and L_ε^g. The first term in the right-hand side of (A-44) uses that, for all $r \in [0, r_\varepsilon]$,

$$|\bar{w}_\varepsilon(r)| \le r \|\bar{w}'_\varepsilon\|_{L^\infty([0,r_\varepsilon])}.$$

Observe now that (A-44) still holds true in the case of (A-27b), replacing (A-29), (A-36) and (A-40) by (A-30) in the above argument. Since $\varepsilon_0 > \frac{1}{2}$, we clearly get from (A-43) and (A-44) that, in both cases of (A-27),

$$r_\varepsilon |\bar{w}'_\varepsilon(r_\varepsilon)| = O\left(\|\bar{w}'_\varepsilon\|_{L^\infty([0,r_\varepsilon])} \frac{\mu_\varepsilon (r_\varepsilon/\mu_\varepsilon)^3}{1+(r_\varepsilon/\mu_\varepsilon)^3}\right) + o\left(\frac{(r_\varepsilon/\mu_\varepsilon)^2}{1+(r_\varepsilon/\mu_\varepsilon)^2}\right). \tag{A-45}$$

Now we prove that

$$\mu_\varepsilon \|\bar{w}'_\varepsilon\|_{L^\infty([0,\bar{r}_\varepsilon])} = o(1). \tag{A-46}$$

If (A-46) does not hold true, then, by (A-45), there exists $s_\varepsilon \in [0, \bar{r}_\varepsilon]$ such that $s_\varepsilon = O(\mu_\varepsilon)$, $\mu_\varepsilon = O(s_\varepsilon)$,

$$|\bar{w}'_\varepsilon(s_\varepsilon)| = \|\bar{w}'_\varepsilon\|_{L^\infty([0,\bar{r}_\varepsilon])} \quad \text{and} \quad \limsup_{\varepsilon\to 0} \mu_\varepsilon |\bar{w}'_\varepsilon(s_\varepsilon)| > 0. \tag{A-47}$$

In particular, up to a subsequence, we may assume that there exists $\alpha_0 \in (0, +\infty]$ such that $\bar{r}_\varepsilon/\mu_\varepsilon \to \alpha_0$ as $\varepsilon \to 0$. Let $\tilde{w}_\varepsilon$ be given by

$$\tilde{w}_\varepsilon(y) = \bar{w}_\varepsilon(\mu_\varepsilon y)/(\mu_\varepsilon \|\bar{w}'_\varepsilon\|_{L^\infty([0,\bar{r}_\varepsilon])}).$$

By (A-45) and (A-47), we get that $(\|(1+\cdot)\tilde{w}_\varepsilon'\|_{L^\infty([0,\bar{r}_\varepsilon/\mu_\varepsilon])})_\varepsilon$ is a bounded sequence. Then, computing as in (A-44) and by radial elliptic theory with (3-81), we get $\tilde{w}_\varepsilon \to \tilde{w}_0$ in $C^2([0,\alpha_0])$ if $\alpha_0 < +\infty$ or in $C^2_{\text{loc}}([0,\alpha_0))$ if $\alpha_0 = +\infty$, where $\tilde{w}_0$ solves

$$\begin{cases} \Delta \tilde{w}_0 = 8\exp(-2T_0)\tilde{w}_0 \ \text{ in } B_0(\alpha_0), \\ \tilde{w}_0(0) = 0, \\ \tilde{w}_0 \text{ is radial around } 0 \in \mathbb{R}^2, \end{cases}$$

still making usual radial identifications, and where T_0 is given in (A-1). By the standard theory of radial elliptic equations, this implies $\tilde{w}_0 \equiv 0$, which contradicts (A-47) and proves (A-46). Then, since $\bar{w}_\varepsilon(0) = 0$ and by the fundamental theorem of calculus, we get from (A-45) with (A-46) that $\bar{r}_\varepsilon = \rho_\varepsilon$ in (A-12). By the discussion just above (A-13), this concludes the proof of Proposition A.1. □

Acknowledgements

The author warmly thanks the referees for several substantial remarks, and valuable suggestions and comments.

References

[Adimurthi and Druet 2004] Adimurthi and O. Druet, "Blow-up analysis in dimension 2 and a sharp form of Trudinger–Moser inequality", *Comm. Partial Differential Equations* **29**:1-2 (2004), 295–322. MR Zbl

[Adimurthi and Grossi 2004] Adimurthi and M. Grossi, "Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity", *Proc. Amer. Math. Soc.* **132**:4 (2004), 1013–1019. MR Zbl

[Carleson and Chang 1986] L. Carleson and S.-Y. A. Chang, "On the existence of an extremal function for an inequality of J. Moser", *Bull. Sci. Math.* (2) **110**:2 (1986), 113–127. MR Zbl

[Chen and Li 1991] W. X. Chen and C. Li, "Classification of solutions of some nonlinear elliptic equations", *Duke Math. J.* **63**:3 (1991), 615–622. MR Zbl

[De Marchis et al. 2016] F. De Marchis, I. Ianni, and F. Pacella, "Asymptotic analysis for the Lane–Emden problem in dimension two", preprint, 2016. arXiv

[De Marchis et al. 2017] F. De Marchis, I. Ianni, and F. Pacella, "Asymptotic profile of positive solutions of Lane–Emden problems in dimension two", *J. Fixed Point Theory Appl.* **19**:1 (2017), 889–916. MR Zbl

[Druet 2006] O. Druet, "Multibumps analysis in dimension 2: quantification of blow-up levels", *Duke Math. J.* **132**:2 (2006), 217–269. MR Zbl

[Druet and Thizy 2017] O. Druet and P.-D. Thizy, "Multi-bumps analysis for Trudinger–Moser nonlinearities, I: Quantification and location of concentration points", preprint, 2017. To appear in *J. Eur. Math. Soc.* arXiv

[Esposito et al. 2006] P. Esposito, M. Musso, and A. Pistoia, "Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent", *J. Differential Equations* **227**:1 (2006), 29–68. MR Zbl

[de Figueiredo and Ruf 1995] D. G. de Figueiredo and B. Ruf, "Existence and non-existence of radial solutions for elliptic equations with critical exponent in $\mathbb{R}^2$", *Comm. Pure Appl. Math.* **48**:6 (1995), 639–655. MR Zbl

[de Figueiredo et al. 1982] D. G. de Figueiredo, P.-L. Lions, and R. D. Nussbaum, "A priori estimates and existence of positive solutions of semilinear elliptic equations", *J. Math. Pures Appl.* (9) **61**:1 (1982), 41–63. MR Zbl

[Flucher 1992] M. Flucher, "Extremal functions for the Trudinger–Moser inequality in 2 dimensions", *Comment. Math. Helv.* **67**:3 (1992), 471–497. MR Zbl

[Gidas et al. 1979] B. Gidas, W. M. Ni, and L. Nirenberg, "Symmetry and related properties via the maximum principle", *Comm. Math. Phys.* **68**:3 (1979), 209–243. MR Zbl

[Han 1991] Z.-C. Han, "Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8**:2 (1991), 159–174. MR Zbl

[Malchiodi and Martinazzi 2014] A. Malchiodi and L. Martinazzi, "Critical points of the Moser–Trudinger functional on a disk", *J. Eur. Math. Soc.* **16**:5 (2014), 893–908. MR Zbl

[Mancini and Martinazzi 2017] G. Mancini and L. Martinazzi, "The Moser–Trudinger inequality and its extremals on a disk via energy estimates", *Calc. Var. Partial Differential Equations* **56**:4 (2017), art. id. 94. MR Zbl

[McLeod and Peletier 1989] J. B. McLeod and L. A. Peletier, "Observations on Moser's inequality", *Arch. Ration. Mech. Anal.* **106**:3 (1989), 261–285. MR Zbl

[Moser 1971] J. Moser, "A sharp form of an inequality by N. Trudinger", *Indiana Univ. Math. J.* **20** (1971), 1077–1092. MR Zbl

[Pruss 1996] A. R. Pruss, "Nonexistence of maxima for perturbations of some inequalities with critical growth", *Canad. Math. Bull.* **39**:2 (1996), 227–237. MR Zbl

[Ren and Wei 1994] X. Ren and J. Wei, "On a two-dimensional elliptic problem with large exponent in nonlinearity", *Trans. Amer. Math. Soc.* **343**:2 (1994), 749–763. MR Zbl

[Ren and Wei 1996] X. Ren and J. Wei, "Single-point condensation and least-energy solutions", *Proc. Amer. Math. Soc.* **124**:1 (1996), 111–120. MR Zbl

[Struwe 1988] M. Struwe, "Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **5**:5 (1988), 425–464. MR Zbl

[Thizy 2019] P.-D. Thizy, "Sharp quantization for Lane–Emden problems in dimension two", *Pacific J. Math.* **300**:2 (2019), 491–497. MR Zbl

Received 6 Feb 2018. Revised 13 Mar 2019. Accepted 12 May 2019.

PIERRE-DAMIEN THIZY: pierre-damien.thizy@univ-lyon1.fr
Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, Villeurbanne, France

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2020 is US $340/year for the electronic version, and $550/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 13 No. 5 2020