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SHARP VARIATION-NORM ESTIMATES FOR OSCILLATORY INTEGRALS RELATED TO CARLESON'S THEOREM

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We prove variation-norm estimates for certain oscillatory integrals related to Carleson's theorem. Bounds for the corresponding maximal operators were first proven by Stein and Wainger. Our estimates are sharp in the range of exponents, up to endpoints. Such variation-norm estimates have applications to discrete analogues and ergodic theory. The proof relies on square function estimates for Schrödinger-like equations due to Lee, Rogers and Seeger. In dimension 1, our proof additionally relies on a local smoothing estimate. Though the known endpoint local smoothing estimate by Rogers and Seeger is more than sufficient for our purpose, we also give a proof of certain local smoothing estimates using Bourgain–Guth iteration and the Bourgain–Demeter ℓ^2 decoupling theorem. This may be of independent interest, because it improves the previously known range of exponents for spatial dimensions $n \geq 4$.

1. Introduction

Let $n \geq 1$ and $\alpha > 1$ be fixed. Given a Calderón–Zygmund kernel $K : \mathbb{R}^n \to \mathbb{R}$ we define a modulated singular integral by

$$\mathcal{H}^{(u)} f(x) := \int_{\mathbb{R}^n} f(x-t) e^{iu|t|^\alpha} K(t)\, dt, \quad u \in \mathbb{R}. \tag{1-1}$$

The maximal operator

$$\sup_{u \in \mathbb{R}} |\mathcal{H}^{(u)} f| \tag{1-2}$$

was introduced in [Stein and Wainger 2001] as a generalization of the Carleson operator studied in [Carleson 1966; Fefferman 1973; Lacey and Thiele 2000]. In this paper, we study variation-norm estimates for the family $\{\mathcal{H}^{(u)} f\}_{u \in \mathbb{R}}$. Apart from the intrinsic interest in such bounds, another strong motivation is given by the connection to certain discrete analogues of (1-2) that are the subject of recent works [Krause and Lacey 2017; Krause 2018] (see Section 1A below).

If $\mathcal{J}$ is a subset of $\mathbb{R}$ and $\{a_u : u \in \mathcal{J}\}$ is a family of complex numbers indexed by $\mathcal{J}$, then for any $1 \leq r < \infty$ the r-variational norm of $\{a_u\}_{u \in \mathcal{J}}$ is defined to be

$$V^r\{a_u : u \in \mathcal{J}\} := \sup_{J \in \mathbb{N}} \sup_{\substack{u_0, u_1, \ldots, u_J \in \mathcal{J} \\ u_0 < u_1 < \cdots < u_J}} \left(\sum_{j=1}^{J} |a_{u_j} - a_{u_{j-1}}|^r \right)^{1/r}.$$

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Closely related to it is the jump function of the family $\{a_u\}_{u\in\mathcal{J}}$: for $\lambda > 0$, the λ-jump function of $\{a_u\}_{u\in\mathcal{J}}$, namely $N_\lambda\{a_u : u \in \mathcal{J}\}$, is defined to be the supremum of all positive integers N for which there exists a strictly increasing sequence $s_1 < t_1 < s_2 < t_2 < \cdots < s_N < t_N$, all of which are in $\mathcal{J}$, such that

$$|a_{t_j} - a_{s_j}| > \lambda$$

for all $j = 1, \ldots, N$. For $r \in (1, \infty)$ and $p \in (1, \infty)$, we will study the L^p mapping properties of the maps

$$f \longmapsto V^r\{\mathcal{H}^{(u)} f : u \in \mathbb{R}\},$$
$$f \longmapsto \lambda[N_\lambda\{\mathcal{H}^{(u)} f : u \in \mathbb{R}\}]^{1/r}, \quad \lambda > 0.$$

Henceforth f will always be a Schwartz function on $\mathbb{R}^n$; the goal is to establish a priori bounds for all such f. If in dimension $n = 1$ we take $\alpha = 1$ and replace $|t|$ by t, then this corresponds to the variation-norm Carleson operator, which has been studied in [Oberlin, Seeger, Tao, Thiele, and Wright 2012; Uraltsev 2016]. We refer the reader to [Bourgain 1989; Pisier and Xu 1988; Campbell, Jones, Reinhold, and Wierdl 2000; 2003; Jones, Seeger, and Wright 2008] for earlier results concerning jump function and variation-norm inequalities for other operators arising in harmonic analysis.

Let us assume that K is a homogeneous Calderón–Zygmund kernel, in the sense that

$$K(x) = \text{p.v.}\, \frac{\Omega(x)}{|x|^n}$$

for some function Ω that is smooth on $\mathbb{R}^n \setminus \{0\}$, homogeneous of degree 0. The assumption that K is homogeneous is not strictly necessary. It is there to help simplify the presentation of the proof of the theorem. We also assume that $\int_{\mathbb{S}^{n-1}} \Omega(x)\, d\sigma(x) = 0$, where σ denotes the surface measure on $\mathbb{S}^{n-1}$.

Theorem 1.1. *Let $n \geq 1$, $\alpha \in (1, \infty)$ and define $\mathcal{H}^{(u)}$ as in* (1-1). *If $r \in (2, \infty)$, $p \in (1, \infty)$ and $r > p'/n$, then we have*

$$\|V^r\{\mathcal{H}^{(u)} f : u \in \mathbb{R}\}\|_p \leq C\|f\|_p. \tag{1-3}$$

In addition, if $n \geq 2$ and $p \in (2n/(2n-1), \infty)$, then

$$\left\|\lambda\sqrt{N_\lambda\{\mathcal{H}^{(u)} f : u \in \mathbb{R}\}}\right\|_p \leq C\|f\|_p.$$

Here the constant C is allowed to depend on n, α, p and r.

Moreover, up to endpoints, we show that this is the best we can expect:

Theorem 1.2. *The estimate* (1-3) *fails if $r < p'/n$.*

Thus, the range of exponents for which estimate (1-3) holds is given by the quadrilateral in Figure 1 below (up to endpoints).

It is natural to ask what happens when α is less than 1. Our methods do not seem to be able to handle this case. But if $n = 1$, an easy adaptation of our methods allows us to obtain a positive result where the phase function $|t|^\alpha$ in (1-1) is replaced by $\text{sgn}(t)|t|^\alpha$. In particular, if α is an odd positive integer, we may replace $|t|^\alpha$ in (1-1) by t^α and still obtain a positive result.

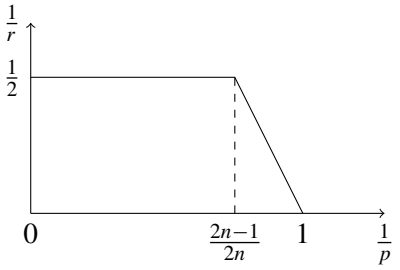

Figure 1. The range of exponents for which estimate (1-3) holds.

The inequality (1-3) can be understood as an extension of the well-known result from [Stein and Wainger 2001] (also see [Guo 2016] for the case when α is not an integer):

$$\big\| \sup_{u\in\mathbb{R}} |\mathcal{H}^{(u)} f| \big\|_p \lesssim \|f\|_p \quad \text{for every } p > 1. \tag{1-4}$$

1A. ***Connection with discrete analogues.*** Further motivation stems from the study of a discrete analogue of the maximal operator (1-2). Fix an integer $d \geq 2$ and let $u \in \mathbb{R}$. Consider the following operator $\mathcal{H}^{(u)}_{\mathbb{Z}}$ acting on functions $f : \mathbb{Z} \to \mathbb{C}$:

$$\mathcal{H}^{(u)}_{\mathbb{Z}} f(x) = \sum_{t\in\mathbb{Z}\setminus\{0\}} f(x-t) e^{iut^d} \frac{1}{t}, \quad x \in \mathbb{Z}.$$

This is a discrete analogue of our operator $\mathcal{H}^{(u)}$ for $n = 1$ and $\alpha = d$. Bounding the associated maximal operator $f \mapsto \sup_{u\in\mathbb{R}} |\mathcal{H}^{(u)}_{\mathbb{Z}} f|$ on $\ell^p(\mathbb{Z})$ is significantly more difficult than bounding Stein and Wainger's maximal operator and until recently, no such bounds were known. For the recent progress on this problem and further discussion of discrete analogues, we refer to [Krause 2018; Krause and Lacey 2017]. A careful analysis of the multiplier of $\mathcal{H}^{(u)}_{\mathbb{Z}}$, which is much in the spirit of the Hardy–Littlewood circle method, reveals a natural splitting of the problem into a number-theoretic and an analytic component. In the case $p = 2$, the core estimate for the analytic component is a variant of Bourgain's classical maximal multifrequency lemma [1989, Lemma 4.1]. The precise statements can be found in [Krause and Lacey 2017, Section 3; Krause 2018, Sections 5 and 10.2]; see, in particular, Theorem 3.5 of [Krause and Lacey 2017]. Using a small refinement of our Theorem 1.1 (see Theorem B.3 below), together with the argument from [Bourgain 1989], one can obtain an alternative simple proof of (a small extension of) Theorem 3.5 of [Krause and Lacey 2017]; we include some details in Appendix B.

Discrete analogues are intimately related to ergodic theorems and this connection provides a further application of our variation-norm estimates. Krause [2018, Theorem 1.2] made use of a variant of the estimate (1-3) in his recent work on a pointwise ergodic theorem.

1B. ***Outline of the proof.*** We now briefly describe an outline of the proof of Theorem 1.1. To control the left-hand side of the estimate (1-3), we split the contribution into two parts: *long variations* and *short variations*. For each $j \in \mathbb{Z}$, define the short variation on the u-interval $[2^{j\alpha}, 2^{(j+1)\alpha}]$ by

$$V^r_j \mathcal{H} f(x) := V^r\{\mathcal{H}^{(u)} f(x) : u \in [2^{j\alpha}, 2^{(j+1)\alpha}]\}.$$

Also define

$$S_r(\mathcal{H}f)(x) := \left(\sum_{j\in\mathbb{Z}} |V_j^r \mathcal{H}f(x)|^r\right)^{1/r},$$

$$N_\lambda^{\mathrm{dyad}}(\mathcal{H}f)(x) := N_\lambda\{\mathcal{H}^{(2^{j\alpha})}f(x) : j \in \mathbb{Z}\}.$$

We will use the following lemma (see, for example, [Jones, Seeger, and Wright 2008]):

Lemma 1.3. *For $r \in [2,\infty)$ we have*

$$\lambda[N_\lambda\{\mathcal{H}^{(u)}f : u > 0\}]^{1/r} \lesssim S_r(\mathcal{H}f) + \lambda[N_{\lambda/3}^{\mathrm{dyad}}(\mathcal{H}f)]^{1/r}$$

uniformly in $\lambda > 0$.

(Hereafter, $A \lesssim B$ means $A \le C\,B$ for some absolute constant C.)

By this lemma, and by Bourgain's argument [1989] of passing from jump norms to variation-norms (see also [Jones, Seeger, and Wright 2008, Section 2]), to prove Theorem 1.1 it suffices to prove the following two propositions.

Proposition 1.4. *For every $p \in (1,\infty)$ and $r \in [2,\infty)$ we have*

$$\|\lambda[N_{\lambda/3}^{\mathrm{dyad}}(\mathcal{H}f)]^{1/r}\|_p \lesssim \|f\|_p$$

uniformly in $\lambda > 0$.

Proposition 1.5. *Let $n \ge 1$ and $p \in (1,\infty)$, $r \in (2,\infty)$ with $r > p'/n$. Then we have*

$$\|S_r(\mathcal{H}f)\|_p \lesssim \|f\|_p.$$

If $n \ge 2$, then the inequality also holds for $r = 2$.

The proof of Proposition 1.4 depends on a jump function inequality of [Jones, Seeger, and Wright 2008] that is based on a Lépingle inequality for martingales.

By interpolation with the inequality (1-4) of [Stein and Wainger 2001], it suffices to consider the case $p \in (2n/(2n-1),\infty)$ to prove Proposition 1.5. The proof of Proposition 1.5 then depends on a square function estimate for Schrödinger-like equations, which is due to [Lee, Rogers, and Seeger 2012]. In one dimension, we additionally need a local smoothing estimate for these equations. The following local smoothing result is more than sufficient for our needs: indeed we will only need the following estimate for $n = 1$ and some $p < \infty$. We are including the full theorem here only because it may be of independent interest.

Theorem 1.6. *Let $\gamma > 1$ be a real number and let I be a compact time interval. For any dimension $n \ge 1$ and exponent $p < \infty$ satisfying*

$$\begin{cases} p > \dfrac{2(4n+7)}{4n+1} & \text{if } n \equiv -1 \pmod 3, \\ p > \dfrac{2n+3}{n} & \text{if } n \equiv 0 \pmod 3, \\ p > \dfrac{4(n+2)}{2n+1} & \text{if } n \equiv 1 \pmod 3, \end{cases} \tag{1-5}$$

we have

$$\left(\int_{\mathbb{R}^n\times I}\left|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)e^{it|\xi|^\gamma}\,d\xi\right|^p dx\,dt\right)^{1/p} \lesssim_\epsilon \|f\|_{W^{\beta+\epsilon,p}(\mathbb{R}^n)} \tag{1-6}$$

whenever $\epsilon>0$ *and*

$$\frac{\beta}{\gamma}=n\Big(\frac{1}{2}-\frac{1}{p}\Big)-\frac{1}{p}.$$

Here we write $W^{s,p}(\mathbb{R}^n)=(I-\Delta)^{-s/2}L^p(\mathbb{R}^n)$ *to denote the standard Bessel potential space.*

Let us take a moment to compare Theorem 1.6 with results in the existing literature. Rogers [2008] considered the case $\gamma=2$, namely a local smoothing estimate for the Schrödinger propagator $e^{it\Delta}$. He proved that (1-6) holds whenever $\gamma=2$, $p\in(2+4/(n+1),\infty)$ and $\epsilon>0$ (in the rest of this section β will always be as specified in Theorem 1.6). This was improved subsequently by Rogers and Seeger [2010], who obtained the endpoint case $\epsilon=0$ for all $\gamma>1$: they established that (1-6) holds with $\epsilon=0$ for all $p\in(2+4/(n+1),\infty)$ and all $\gamma>1$. In particular, this implies Theorem 1.6 for $n=1,2,3$. Theorem 1.6 gives a larger range of p in dimensions $n\geq 4$, albeit with an ϵ-loss in smoothness. We also note that in the case $\gamma=2$ (i.e., for the Schrödinger propagator), Lee, Rogers and Seeger [2013] obtained an improvement of the aforementioned result of [Rogers and Seeger 2010]; in particular, in Proposition 5.2 of [Lee, Rogers, and Seeger 2013], they proved that if the dual Fourier restriction conjecture holds at an exponent q_0, in the sense that

$$\|Ef\|_{L^{q_0}(\mathbb{R}^{n+1})}\lesssim\|f\|_{L^{q_0}([0,1]^n)}$$

for some exponent $\gamma_0<2(n+3)/(n+1)$, where E is the Fourier extension operator for the paraboloid in $\mathbb{R}^{n+1}$ given by

$$Ef(x,t)=\int_{[0,1]^n} f(\xi)e^{i(x\cdot\xi+t|\xi|^2)}\,d\xi,\quad (x,t)\in\mathbb{R}^n\times\mathbb{R}, \tag{1-7}$$

then (1-6) holds for $\gamma=2$ with $\epsilon=0$ whenever $p\in(q_*,\infty)$, where q_* is defined by

$$q_*:=\frac{2(n+3)}{n+1}(1-\gamma(n,q_0)),\quad \text{with } \gamma(n,q_0):=\frac{1/q_0-(n+1)/(2(n+3))}{n((n+1)/2-(n+2)/q_0)}.$$

A direct computation shows that

$$q_*=2+\frac{4}{n}-\frac{2}{n^2-n(4-q_0)/(q_0-2)}.$$

As a result, even if one can establish (1-7) in all dimensions n with $q_0=q_0(n)$ that decays like $q_0(n)=2+(2+\lambda)/n+O(1/n^2)$ for some $\lambda>0$ (the Fourier restriction conjecture shows that the best one can hope for is $\lambda=0$), using the above result of Lee, Rogers and Seeger, one can only establish the local smoothing estimate (1-6) for $p\in(q_*(n),\infty)$, where

$$q_*(n)=2+\frac{4}{n}+O\Big(\frac{1}{n^2}\Big).$$

On the contrary, if $p_*(n)$ is the Bourgain–Guth exponent given by the right-hand sides of (1-5), we see that

$$p_*(n) = 2 + \frac{3}{n} + O\left(\frac{1}{n^2}\right),$$

so our range of the exponent p is larger than that of Lee, Rogers and Seeger in high dimensions n, even for the Schrödinger equation case.

Contrary to [Rogers and Seeger 2010], which relied on bilinear restriction estimates, our proof of Theorem 1.6 relies on the Bourgain–Guth argument [2011] (see also the presentation in [Bourgain and Demeter 2017]), and the Bourgain–Demeter decoupling inequality [2015]; see [Wolff 2000; Łaba and Wolff 2002] for some earlier foundational work on decoupling inequalities, and their applications to local smoothing estimates. The multilinear estimates developed in [Guth 2018] might be useful in establishing (1-6) for a larger range of exponents, but we did not pursue this here.

Organization of the paper. In Section 2 we state two preliminary results, namely a consequence of the classical Lépingle inequality, and a consequence of the Plancherel–Pólya inequality. In Section 3 we control long jumps; that is, we will prove Proposition 1.4. The treatment for short jumps (that is, the proof of Proposition 1.5) will be split into two parts. In Section 4 we prove Proposition 1.5 in two special cases: $n \geq 2,\ p > 2(n+2)/n$, and $n = 1,\ p > 2$. These are the main cases to be considered. In Section 5 we indicate the modifications necessary to prove the remaining case of Proposition 1.5: namely, $n \geq 2$ and $2n/(2n-1) < p \leq 2(n+2)/n$. The proof of Theorem 1.2 is in Section 6. In Section 7 we provide the proof of a vector-valued generalization of a multiplier theorem of [Seeger 1988], which we used in the proof of the short jump estimates in Section 4. In Appendix A we prove the local smoothing estimates in Theorem 1.6. In Appendix B we refine our Theorem 1.1 by obtaining a good bound on the growth of the constant C in (1-3) as $p = r \to 2^+$ (see Theorem B.3), and use it to provide an alternative simple proof of a maximal multifrequency estimate of Krause and Lacey [2017, Theorem 3.5].

2. Prerequisites

2A. ***A jump function inequality of Jones, Seeger and Wright.*** We recall a jump function inequality for convolutions with dyadic dilations of a fixed measure from [Jones, Seeger, and Wright 2008, Theorem 1.1]. It is a consequence of the more classical Lépingle inequality for martingales.

Proposition 2.1 [Jones, Seeger, and Wright 2008]. *Let σ be a compactly supported finite nonnegative Borel measure on $\mathbb{R}^n$ whose Fourier transform satisfies*

$$|\hat{\sigma}(\xi)| \leq C|\xi|^{-a}$$

for some $a > 0$. For $k \in \mathbb{Z}$, define σ_k by

$$\int_{\mathbb{R}^n} f(x)\, d\sigma_k(x) = \int_{\mathbb{R}^n} f(2^{-k}x)\, d\sigma(x).$$

Then

$$\big\| \lambda \sqrt{N_\lambda\{f * \sigma_k : k \in \mathbb{Z}\}} \big\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$, uniformly in $\lambda > 0$.

We will apply this proposition as follows. Let S be a nonnegative smooth function with compact support in $[-1, 1]^n$ and $\int_{\mathbb{R}^n} S(x)\,dx = 1$. For $k \in \mathbb{Z}$ and any Schwartz function f on $\mathbb{R}^n$, let

$$S_k f(x) = f * S_k(x),$$

where $S_k(x) = 2^{kn} S(2^k x)$. If σ is the measure on $\mathbb{R}^n$ given by

$$\int_{\mathbb{R}^n} f(x)\,d\sigma(x) = \int_{\mathbb{R}^n} f(x) S(x)\,dx,$$

then $\sigma_k(x)$ coincides with $S_k(x)dx$, and hence $f * \sigma_k = S_k f$ for all $k \in \mathbb{Z}$. Proposition 2.1 then gives

$$\big\|\lambda\sqrt{N_\lambda\{S_k f : k \in \mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{2-1}$$

for all $1 < p < \infty$, uniformly in $\lambda > 0$. Note that $\widehat{S}(0) = 1$ and $\widehat{S}(\xi)$ decreases rapidly to zero as $|\xi| \to \infty$. So later it helps to think of $\widehat{S}(\xi)$ as localized to $|\xi| \lesssim 1$, and interpret $S_k f$ as a localization of f to frequency $\lesssim 2^k$.

Next, let $\{c_\ell\}_{\ell=0}^\infty$ be a complex sequence with $|c_\ell| = O(2^{-\alpha\ell})$ for some $\alpha > 0$. Let $\widetilde{S}_k$ be the operator defined by

$$\widetilde{S}_k f := \sum_{\ell=0}^\infty c_\ell S_{k-\ell} f. \tag{2-2}$$

We will use (2-1) to prove that

$$\big\|\lambda\sqrt{N_\lambda\{\widetilde{S}_k f : k \in \mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{2-3}$$

for all $1 < p < \infty$, uniformly in $\lambda > 0$. Recall the definition of the jump norm $N_\lambda\{\widetilde{S}_k f(x) : k \in \mathbb{Z}\}$: it is the supremum of all positive integers N for which there exists a strictly increasing sequence $s_1 < t_1 < s_2 < t_2 < \cdots < s_N < t_N$, all of which are in $\mathbb{Z}$, such that

$$|\widetilde{S}_{t_j} f(x) - \widetilde{S}_{s_j} f(x)| > \lambda \tag{2-4}$$

for all $j = 1, \ldots, N$. But if $s_1 < t_1 < s_2 < t_2 < \cdots < s_N < t_N$ is as such, then for all $j = 1, \ldots, N$ we have

$$|S_{t_j-\ell} f(x) - S_{s_j-\ell} f(x)| \gtrsim 2^{\ell\alpha/2}\lambda$$

for at least one $\ell \ge 0$. Hence,

$$N_\lambda\{\widetilde{S}_k f(x) : k \in \mathbb{Z}\} \lesssim \sum_{\ell=0}^\infty N_{2^{\ell\alpha/2}\lambda}\{S_k f(x) : k \in \mathbb{Z}\},$$

which implies

$$\sqrt{N_\lambda\{\widetilde{S}_k f : k \in \mathbb{Z}\}} \lesssim \sum_{\ell=0}^\infty \sqrt{N_{2^{\ell\alpha/2}\lambda}\{S_k f : k \in \mathbb{Z}\}}.$$

This further implies

$$\big\|\lambda\sqrt{N_\lambda\{\widetilde{S}_k f : k \in \mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{\ell=0}^\infty 2^{-\ell\alpha/2} \big\|2^{\ell\alpha/2}\lambda\sqrt{N_{2^{\ell\alpha/2}\lambda}\{S_k f : k \in \mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_p.$$

This finishes the proof of the estimate (2-3).

2B. ***An inequality of Plancherel and Pólya.*** Next, let $F(u)$ be an L^2 function on $\mathbb{R}$ whose Fourier transform $\widehat{F}(\xi)$ is supported on the set $|\xi| \le 1$. Such an F is sometimes said to be in a Paley–Wiener space. An inequality of Plancherel and Pólya [1936; 1937] says that for any such F and any $r \in [1, \infty)$, we have

$$\sum_{j\in\mathbb{Z}} |F(j)|^r \le C_r \int_{\mathbb{R}} |F(u)|^r \, du, \tag{2-5}$$

where C_r is a constant independent of F. This holds because if $\widehat{F}$ is supported on $|\xi| \le 1$, then, by the uncertainty principle, F is essentially constant on every interval of length 1 (see also [Young 1980] for an alternative proof based on complex analysis).

From (2-5) we can deduce the following variation-norm estimate (see also page 6729 of [Jones, Seeger, and Wright 2008]):

Proposition 2.2. *Let $F(u)$ be a function on $\mathbb{R}$ whose Fourier transform $\widehat{F}(\xi)$ is supported on the set $\{|\xi| \le \lambda\}$. Then for every $1 \le q \le r < \infty$, we have*

$$V^r\{F(u) : u \in \mathbb{R}\} \le A_{q,r}\lambda^{1/q}\|F\|_{L^q}, \tag{2-6}$$

with a constant $A_{q,r}$ depending only on q and r.

Proof. By rescaling we may assume that $\lambda = 1$. Now let $k \in \mathbb{N}$ and $u_1 < \cdots < u_k$ be a strictly increasing sequence in $\mathbb{R}$. We let $\kappa(0) = 1$, $n_1 = \lfloor u_{\kappa(0)} \rfloor$ and let $\kappa(1)$ be the largest integer in $\{1, \dots, k\}$ such that $u_{\kappa(1)} < n_1 + 1$. If $\kappa(1) < k$, we let $n_2 = \lfloor u_{\kappa(1)+1} \rfloor$ and let $\kappa(2)$ be the largest integer in $\{1, \dots, k\}$ such that $u_{\kappa(2)} < n_2 + 1$. Clearly this process will terminate in finitely many, say m, steps. In this way we collect the points $u_1, \dots, u_k$ into intervals $[n_1, n_1 + 1]$, $[n_2, n_2 + 1]$, $\dots$, $[n_m, n_m + 1]$ of length at most 1. Now for $s = 1, \dots, m-1$, by the triangle inequality, we have

$$|F(u_{\kappa(s)}) - F(u_{\kappa(s)+1})|^r \lesssim |F(u_{\kappa(s)}) - F(n_s+1)|^r + |F(n_s+1)|^r + |F(n_{s+1})|^r + |F(n_{s+1}) - F(u_{\kappa(s)+1})|^r.$$

This shows

$$\begin{aligned}
&\sum_{i=1}^{k-1} |F(u_i) - F(u_{i+1})|^r \\
&\quad\lesssim \sum_{s=1}^{m} (|F(n_s)|^r + |F(n_s+1)|^r) + \sum_{s=1}^{m} \Bigg(|F(n_s) - F(u_{\kappa(s-1)})|^r \\
&\qquad\qquad + \sum_{\kappa(s-1)\le i < \kappa(s)} |F(u_i) - F(u_{i+1})|^r + |F(u_{\kappa(s)}) - F(n_s+1)|^r \Bigg).
\end{aligned}$$

(Indeed, for $s = 1$, we do not need the terms $|F(n_s)|^r$ and $|F(n_s) - F(u_{\kappa(s-1)})|^r$ on the right-hand side; similarly for $s = m$, we do not need the terms $|F(n_s + 1)|^r$ and $|F(u_{\kappa(s)}) - F(n_s + 1)|^r$. But there is no harm putting them in, which makes the expression on the right-hand side more symmetric.) By the mean-value theorem, for $s = 1, \dots, m$, we have

$$\begin{aligned}
&|F(n_s) - F(u_{\kappa(s-1)})|^r + \sum_{\kappa(s-1)\le i < \kappa(s)} |F(u_i) - F(u_{i+1})|^r + |F(u_{\kappa(s)}) - F(n_s + 1)|^r \\
&\qquad\qquad \le \|F'\|_{L^\infty}^r \Bigg(|n_s - u_{\kappa(s-1)}|^r + \sum_{\kappa(s-1)\le i < \kappa(s)} |u_i - u_{i+1}|^r + |u_{\kappa(s)} - (n_s + 1)|^r \Bigg),
\end{aligned}$$

and the quantity inside the parentheses in the last line is ≤ 1 since we have the elementary inequality

$$t_1^r + \cdots + t_\sigma^r \le (t_1 + \cdots + t_\sigma)^r$$

whenever $t_1, \ldots, t_\sigma \ge 0$ and $1 \le r < \infty$. Now since $\widehat{F}$ is supported on $|\xi| \le 1$, Bernstein's inequality implies

$$\|F'\|_{L^\infty} \lesssim_r \|F\|_{L^r}$$

whenever $1 \le r < \infty$. Altogether, we see that

$$\begin{aligned}\sum_{i=1}^{k-1} |F(u_i) - F(u_{i+1})|^r &\lesssim_r \|F\|_{L^r}^r + \sum_{s=1}^{m} (|F(n_s)|^r + |F(n_s+1)|^r) \\ &\lesssim_r \|F\|_{L^r}^r + \sum_{j\in\mathbb{Z}} |F(j)|^r \lesssim_r \|F\|_{L^r}^r\end{aligned}$$

whenever $1 \le r < \infty$, the last inequality following from (2-5). Since $\widehat{F}$ is supported on $\{|\xi| \le 1\}$ and $1 \le q \le r$, Bernstein's inequality again implies $\|F\|_{L^r} \lesssim_{q,r} \|F\|_{L^q}$. This completes the proof of (2-6). □

3. Long jump estimates

Our goal in this section is to prove Proposition 1.4. Indeed, we will prove something slightly stronger, including the case $0 < \alpha < 1$.

Proposition 3.1. *Fix $\alpha > 0$, $\alpha \ne 1$. For $1 < p < \infty$, we have*

$$\|\lambda \sqrt{N_\lambda\{\mathcal{H}^{(2^{k\alpha})} f : k \in \mathbb{Z}\}}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{3-1}$$

uniformly in $\lambda > 0$. Here $\mathcal{H}^{(2^{k\alpha})}$ is defined as in (1-1).

First we decompose $\mathcal{H}^{(2^{k\alpha})}$ into

$$\begin{aligned}\mathcal{H}^{(2^{k\alpha})} f(x) &= \int_{|t|\le 2^{-k}} f(x-t) e^{i2^{k\alpha}|t|^\alpha} K(t)\, dt + \int_{|t|>2^{-k}} f(x-t) e^{i2^{k\alpha}|t|^\alpha} K(t)\, dt \\ &=: \mathcal{H}_{k,-\infty} f(x) + \mathcal{H}_{k,\infty} f(x).\end{aligned}$$

In the term $\mathcal{H}_{k,-\infty} f$, we are integrating over small t, and the exponential $e^{i2^{k\alpha}|t|^\alpha}$ is approximately 1. This motivates us to further decompose $\mathcal{H}_{k,-\infty} f$ as

$$\begin{aligned}\mathcal{H}_{k,-\infty} f(x) &= \int_{|t|\le 2^{-k}} f(x-t) K(t)\, dt + \int_{|t|\le 2^{-k}} f(x-t)(e^{i2^{k\alpha}|t|^\alpha} - 1) K(t)\, dt \\ &=: \widetilde{\mathcal{H}}_{k,0} f(x) + \mathcal{H}_{k,0} f(x).\end{aligned} \tag{3-2}$$

For the other term, we take the decomposition

$$\mathcal{H}_{k,\infty} f(x) = \sum_{\ell=1}^{\infty} \mathcal{H}_{k,\ell} f(x) := \sum_{\ell=1}^{\infty} \int_{2^{-k+\ell-1}<|t|\le 2^{-k+\ell}} f(x-t) e^{i2^{k\alpha}|t|^\alpha} K(t)\, dt.$$

The former term in (3-2) is a truncated singular integration. We have:

Lemma 3.2 [Campbell, Jones, Reinhold, and Wierdl 2003, Theorem A].

$$\big\|\lambda\sqrt{N_\lambda\{\widetilde{\mathcal{H}}_{k,0}f : k\in\mathbb{Z}\}}\big\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$.

Hence it remains to estimate the jump norms of $\mathcal{H}_{k,0}f(x) + \sum_{\ell=1}^{\infty}\mathcal{H}_{k,\ell}f(x) = \sum_{\ell=0}^{\infty}\mathcal{H}_{k,\ell}f(x)$. To do so, we carry out a Littlewood–Paley decomposition. For each $\ell \geq 0$, apply

$$\mathcal{H}_{k,\ell}f = \mathcal{H}_{k,\ell}S_{k-\ell}f + \mathcal{H}_{k,\ell}(f - S_{k-\ell}f).$$

(see Section 2 for the precise definition of $S_k f$). Notice that $S_{k-\ell}f$ is approximately constant at the physical scale $2^{-k+\ell}$. Thus, $\mathcal{H}_{k,\ell}S_{k-\ell}f$ is almost just a multiple of $S_{k-\ell}f$. This motivates us to further take the decomposition

$$\mathcal{H}_{k,\ell}S_{k-\ell}f = c_\ell S_{k-\ell}f + (\mathcal{H}_{k,\ell}S_{k-\ell}f - c_\ell S_{k-\ell}f),$$

where

$$c_0 := \int_{|t|\leq 1}(e^{i|t|^\alpha} - 1)K(t)\,dt \quad \text{and} \quad c_\ell := \int_{1/2<|t|\leq 1} e^{i2^{\ell\alpha}|t|^\alpha}K(t)\,dt \quad \text{for } \ell \geq 1 \tag{3-3}$$

are constants. Here we choose the constants c_0 and c_ℓ as such because K is assumed to be homogeneous. Hence

$$\sum_{\ell=0}^{\infty}\mathcal{H}_{k,\ell}f(x) = \sum_{\ell=0}^{\infty}c_\ell S_{k-\ell}f + \sum_{\ell=0}^{\infty}(\mathcal{H}_{k,\ell}S_{k-\ell}f - c_\ell S_{k-\ell}f) + \sum_{\ell=0}^{\infty}\mathcal{H}_{k,\ell}(f - S_{k-\ell}f).$$

Since a simple integration-by-parts argument shows that $|c_\ell| = O(2^{-\alpha\ell})$, the contribution from the first term to the desired jump norm can be controlled using (2-3). To handle the latter two terms we use a square function. It suffices to show that

$$\sum_{\ell=0}^{\infty}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}|\mathcal{H}_{k,\ell}S_{k-\ell}f - c_\ell S_{k-\ell}f|^2\bigg)^{1/2}\bigg\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \tag{3-4}$$

$$\sum_{\ell=0}^{\infty}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}|\mathcal{H}_{k,\ell}(f - S_{k-\ell}f)|^2\bigg)^{1/2}\bigg\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \tag{3-5}$$

since the square functions dominate the desired jump norms pointwisely. To establish these estimates we apply a finer frequency decomposition. Let

$$\Delta(x) := 2^n S(2x) - S(x) \quad \text{and} \quad \Delta_k(x) := 2^{kn}\Delta(2^k x)$$

and write $\Delta_k f := f * \Delta_k$ so that

$$S_{k-\ell}f = \sum_{j=1}^{\infty}\Delta_{k-\ell-j}f \quad \text{and} \quad f - S_{k-\ell}f = \sum_{j=0}^{\infty}\Delta_{k-\ell+j}f.$$

By the triangle inequality, to prove (3-4) and (3-5), it suffices to prove the existence of some constant $\gamma > 0$ such that

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\mathcal{H}_{k,\ell}\Delta_{k-\ell-j} f - c_\ell \Delta_{k-\ell-j} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\gamma(j+\ell)} \|f\|_{L^p(\mathbb{R}^n)}, \tag{3-6}$$

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\mathcal{H}_{k,\ell}\Delta_{k-\ell+j} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\gamma(j+\ell)} \|f\|_{L^p(\mathbb{R}^n)} \tag{3-7}$$

for every $j, \ell \geq 0$ and every $1 < p < \infty$. Throughout the paper, we use γ to denote a positive real number that might vary from line to line, if not otherwise stated.

Now each of the estimates (3-6) and (3-7) holds for $1 < p < \infty$ without the small factors on the right, since $|\mathcal{H}_{k,\ell} f| \lesssim Mf$ where M is the Hardy–Littlewood maximal operator on $\mathbb{R}^n$, allowing us to invoke the Fefferman–Stein vector-valued inequality for the maximal function [Stein 1993, Chapter II.1]. Hence by real interpolation, it suffices to prove the case $p = 2$. To do so, fix $\alpha > 0$, $\alpha \neq 1$ and $\ell \in \mathbb{N}$. Let $m_\ell(\xi)$ be the multiplier defined by

$$m_0(\xi) := \int_{|t|\leq 1} (e^{i|t|^\alpha} - 1) e^{-it\cdot\xi} K(t)\, dt,$$
$$m_\ell(\xi) := \int_{1/2<|t|\leq 1} e^{i|2^\ell t|^\alpha} e^{-it\cdot\xi} K(t)\, dt \quad \text{for } \ell \geq 1.$$

Let $\tilde{m}_\ell(\xi)$ be the multiplier defined by

$$\tilde{m}_0(\xi) := \int_{|t|\leq 1} (e^{i|t|^\alpha} - 1)(e^{-it\cdot\xi} - 1) K(t)\, dt,$$
$$\tilde{m}_\ell(\xi) := \int_{1/2<|t|\leq 1} e^{i|2^\ell t|^\alpha} (e^{-it\cdot\xi} - 1) K(t)\, dt \quad \text{for } \ell \geq 1.$$

Since K is assumed to be homogeneous, for $\ell \geq 0$ the multiplier for $\mathcal{H}_{k,\ell}$ is $m_\ell(2^{-k+\ell}\xi)$. It follows that for $\ell \geq 0$ the multiplier for $\mathcal{H}_{k,\ell} - c_\ell$ is $\tilde{m}(2^{-k+\ell}\xi)$. Then (3-6) and (3-7) with $p = 2$ follow from the pointwise estimates for multipliers

$$\left(\sum_{k\in\mathbb{Z}} |\hat{\Delta}(2^{-k+\ell+j}\xi)\tilde{m}_\ell(2^{-k+\ell}\xi)|^2 \right)^{1/2} + \left(\sum_{k\in\mathbb{Z}} |\hat{\Delta}(2^{-k+\ell-j}\xi) m_\ell(2^{-k+\ell}\xi)|^2 \right)^{1/2} \lesssim 2^{-\gamma(\ell+j)}. \tag{3-8}$$

We need the following lemma, which is a consequence of the van der Corput lemma (details omitted):

Lemma 3.3. *We have*

$$|m_\ell(\xi)| \lesssim \min\{2^{-\gamma\ell}, 2^{\alpha\ell}|\xi|^{-\gamma}\} \quad \textit{for all } \xi \in \mathbb{R}. \tag{3-9}$$

In particular,

$$|m_\ell(\xi)| \lesssim (2^{-\gamma\ell} \cdot 2^{\alpha\ell}|\xi|^{-\gamma})^{1/2} \quad \textit{for all } \xi \in \mathbb{R}. \tag{3-10}$$

We also have

$$|\tilde{m}_\ell(\xi)| \lesssim \begin{cases} \min\{2^{-\gamma\ell}, |\xi|\} \lesssim 2^{-\gamma\ell/2}|\xi|^{1/2} & \textit{for } |\xi| \leq 1, \\ 1 & \textit{for } |\xi| \geq 1. \end{cases}$$

We are ready to prove (3-8). The estimate is invariant upon replacing ξ by 2ξ; hence we only need to prove it when $|\xi| \simeq 1$. First consider the first term on the left-hand side of (3-8). When $k \le 0$, we bound $|\tilde{m}_\ell(2^{-k+\ell}\xi)| \lesssim 1$ and $|\hat{\Delta}(2^{-k+\ell+j}\xi)| \lesssim 2^{-10(-k+\ell+j)}$. Summing over $k \le 0$, we obtain $2^{-10(\ell+j)}$.

When $k \ge 0$, we bound $|\tilde{m}_\ell(2^{-k+\ell}\xi)| \lesssim 2^{-\gamma\ell/2}2^{-k/2+\ell/2}$ and

$$|\hat{\Delta}(2^{-k+\ell+j}\xi)| \lesssim \begin{cases} 2^{-10(-k+\ell+j)} & \text{if } 0 \le k \le \ell + j, \\ 2^{-k+\ell+j} & \text{if } k \ge \ell + j. \end{cases}$$

Summing over $k \ge 0$, we obtain $2^{-\gamma(\ell+j)}$ for some $\gamma > 0$. This finishes the proof of the first half of (3-8).

Next we turn to the second term on the left-hand side of (3-8). What we need to prove can also be written as

$$\left(\sum_{k\in\mathbb{Z}} |\hat{\Delta}(2^k\xi)m_\ell(2^{k+j}\xi)|^2\right)^{1/2} \lesssim 2^{-\gamma(\ell+j)} \quad \text{for } |\xi| \simeq 1. \tag{3-11}$$

We work on two different cases. Let $C_\alpha > 0$ be a sufficiently large constant. Assume that we are in the case $j \ge C_\alpha \ell$. We bound the left-hand side of (3-11) by

$$\sum_{k\ge 0} 2^{-10k}2^{\alpha\ell}2^{-\gamma k-\gamma j} + \sum_{k<0} 2^k(2^{\alpha\ell}\cdot 2^{-\gamma\ell}2^{-\gamma k-\gamma j})^{1/2} \lesssim 2^{-\gamma(\ell+j)}.$$

Here for the case $k \ge 0$ we applied (3-9), and for the case $k < 0$ we applied (3-10).

Finally, we assume that $0 \le j \le C_\alpha \ell$. We bound the left-hand side of (3-11) by

$$\sum_{k\ge 0} 2^{-10k}2^{-\gamma\ell} + \sum_{k<0} 2^k 2^{-\gamma\ell} \lesssim 2^{-\gamma(\ell+j)}.$$

Here in both cases $k \ge 0$ and $k < 0$ we applied (3-9).

4. Short jump estimates for large p

We are now going to start the proof of Proposition 1.5. Recall that by interpolation, we only need to establish Proposition 1.5 when $p \in (2n/(2n-1), \infty)$ and $r \in (2, \infty)$ (see discussion following Proposition 1.5). In this section we will do so for all sufficiently large values of p. More precisely, let $\alpha > 1$, let $\mathcal{H}^{(u)}$ be as in (1-1), and let $V_j^r \mathcal{H} f(x) = V^r\{\mathcal{H}^{(u)} f(x) : u \in [2^{j\alpha}, 2^{(j+1)\alpha}]\}$. We prove

$$\left\| \left(\sum_{j\in\mathbb{Z}} |V_j^r(\mathcal{H}f)|^r \right)^{1/r} \right\|_p \lesssim \|f\|_p \tag{4-1}$$

whenever

$$p \in (2, \infty), \quad n = 1, \quad r \in (2, \infty) \tag{4-2}$$

or

$$p \in \left(2 + \frac{4}{n}, \infty\right), \quad n \ge 2, \quad r \in [2, \infty). \tag{4-3}$$

This proves Proposition 1.5 when $n = 1$. In the next section, we extend (4-1) to all $p \in (2n/(2n-1), \infty)$ when $n \ge 2$, $r \in [2, \infty)$. That would complete the proof of Proposition 1.5 when $n \ge 2$.

4A. ***Main tool*: *a square function estimate for the semigroup $e^{it(-\Delta)^{\lambda/2}}$.*** The main input to our proof of (4-1) under condition (4-2) or (4-3) is a square function estimate, due to [Lee, Rogers, and Seeger 2012]:

Proposition 4.1 [Lee, Rogers, and Seeger 2012]. (1) *Let $n = 1$, $p \in [2, \infty)$ and $\lambda > 1$. Then for any compact time interval I,*

$$\left\| \left(\int_I \left| \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) e^{it|\xi|^\lambda} \, d\xi \right|^2 dt \right)^{1/2} \right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}.$$

(2) *Let $n \geq 2$, $p \in (2(n+2)/n, \infty)$ and $\lambda > 1$. Then for any compact time interval I,*

$$\left\| \left(\int_I \left| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) e^{it|\xi|^\lambda} \, d\xi \right|^2 dt \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{\beta,p}(\mathbb{R}^n)},$$

with

$$\frac{\beta}{\lambda} = n\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{2}.$$

We will apply the above estimates with $\lambda = \alpha' := \alpha/(\alpha - 1)$ (remember $\alpha > 1$). Recall that we are interested in the variation of $\mathcal{H}^{(u)} f(x)$, where u is restricted to the range $[2^{j\alpha}, 2^{(j+1)\alpha}]$ for some $j \in \mathbb{Z}$. To estimate this, we decompose the kernel $e^{iu|t|^\alpha} K(t)$ into a part where oscillation plays no role and a part where the oscillation becomes important. More precisely, for $\ell \in \mathbb{Z}$, let

$$\mathcal{H}^{(u)}_\ell f(x) := \int_{\mathbb{R}^n} f(x-t) e^{iu|t|^\alpha} \varphi_\ell(t) K(t) \, dt, \tag{4-4}$$

where $\varphi_\ell(t) = \varphi_0(2^{-\ell}t)$ and φ_0 is radial, smooth and compactly supported on an annulus $\{|t| \simeq 1\}$ so that for $t \neq 0$ we have $\sum_{\ell\in\mathbb{Z}} \varphi_\ell(t) = 1$. When $u \simeq 2^{j\alpha}$, $|t| \simeq 2^{\ell-j}$, the phase $e^{iu|t|^\alpha}$ in (4-4) is approximately 1 precisely when $\ell < 0$. Thus, it makes sense to take the decomposition

$$\mathcal{H}^{(u)} f(x) = \sum_{\ell\in\mathbb{Z}} \mathcal{H}^{(u)}_{\ell-j} f(x) \tag{4-5}$$

and expect that the terms $\ell < 0$ in the above sum are essentially nonoscillatory.

It suffices to show that

$$\sum_{\ell\in\mathbb{Z}} \left\| \left(\sum_{j\in\mathbb{Z}} |V_j^r \mathcal{H}^{(u)}_{\ell-j} f|^r \right)^{1/r} \right\|_p \lesssim \|f\|_p. \tag{4-6}$$

To do so, we introduce a Littlewood–Paley decomposition in the x-variable. Let P_k be a multiplier operator defined by $\widehat{P_k f}(\xi) = \psi(2^{-k}\xi)\hat{f}(\xi)$, where ψ is a smooth function with compact support on the annulus $\frac{1}{2} \leq |\xi| \leq 2$ so that for $\xi \neq 0$, we have $\sum_{k\in\mathbb{Z}} \psi(2^{-k}\xi) = 1$. We further take the decomposition

$$\mathcal{H}^{(u)}_{\ell-j} f(x) = \sum_{k\in\mathbb{Z}} \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f(x). \tag{4-7}$$

We will estimate

$$\| \| V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f \|_{\ell_j^r} \|_p \tag{4-8}$$

for each $k, \ell \in \mathbb{Z}$, and sum the estimates at the end. (Hereafter, for compactness of notation, we write ℓ_j^r for the ℓ^r norm over all $j \in \mathbb{Z}$.)

4B. ***Estimates for $\ell \le -k/(2(\alpha+1))$: bounding the V_j^r norm by the $\dot{W}^{1,1}$ norm.*** First there are two simple estimates for (4-8). One way to estimate (4-8) is to bound the V_j^r norm by the V_j^1 norm, which in turn is bounded by the $\dot{W}^{1,1}$ norm on the u interval $[2^{j\alpha}, 2^{(j+1)\alpha}]$. We get

$$V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f(x) \lesssim \int_{|u|\simeq 2^{j\alpha}} 2^{(\ell-j)\alpha} \int_{|t|\simeq 2^{\ell-j}} |P_{j+k} f(x-t)|\,|K(t)|\,dt\,du \lesssim 2^{\ell\alpha} M P_{j+k} f(x),$$

where M is the Hardy–Littlewood maximal function, so by the Fefferman–Stein inequality and the Littlewood–Paley inequality, we have

$$\| \| V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f(x) \|_{\ell_j^r} \|_{L_x^p} \lesssim 2^{\ell\alpha} \| f \|_{L^p}, \quad 1 < p < \infty. \tag{4-9}$$

For the second simple estimate, recall that $\int_{|t|=R} K(t)\,d\sigma(t) = 0$ for all $R \in (0,\infty)$. Since φ was chosen to be radial, we have

$$\int_{\mathbb{R}^n} e^{iu|t|^\alpha} \varphi_{\ell-j}(t) K(t)\,dt = 0.$$

Thus, in computing $V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f(x)$, we could have instead computed the V_j^r norm of

$$\mathcal{H}^{(u)}_{\ell-j} P_{j+k} f(x) - P_{j+k} f(x) \int_{\mathbb{R}^n} e^{iu|t|^\alpha} \varphi_{\ell-j}(t) K(t)\,dt.$$

This expression is equal to

$$\int_{\mathbb{R}^n} [P_{j+k} f(x-t) - P_{j+k} f(x)] e^{iu|t|^\alpha} \varphi_{\ell-j}(t) K(t)\,dt.$$

The variational norm of this expression is controlled by its $\dot{W}^{1,1}$ norm in the u interval $[2^{j\alpha}, 2^{(j+1)\alpha}]$, which in turn is controlled by

$$2^{j+k} 2^{\ell-j} 2^{\ell\alpha} M \widetilde{P}_{j+k} f(x),$$

where $\widetilde{P}_{j+k}$ is a variant of the Littlewood–Paley projection P_{j+k}, so arguing as before, we see that

$$\| \| V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f(x) \|_{\ell_j^r} \|_{L_x^p} \lesssim 2^{\ell+k} 2^{\ell\alpha} \| f \|_{L^p}, \quad 1 < p < \infty. \tag{4-10}$$

We can sum (4-10) over all pairs (k,ℓ) with $\ell \le -k/(2(\alpha+1))$ and $k \le 0$. We can also sum (4-9) over all (k,ℓ) with $\ell \le -k/(2(\alpha+1))$ and $k \ge 0$. Thus, it remains to bound (4-8) when

$$\ell > -\frac{k}{2(\alpha+1)} \tag{4-11}$$

and sum over all such pairs of (k,ℓ).

4C. ***Estimates for $\ell > -k/(2(\alpha+1))$: division into three cases.*** First we look at $\mathcal{H}^{(u)}_{\ell-j} P_{j+k} f(x)$ in terms of its multiplier:

$$\mathcal{H}^{(u)}_{\ell-j} P_{j+k} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) \bigg(\psi(2^{-j-k}\xi) \int_{\mathbb{R}^n} e^{-it\cdot\xi} e^{iu|t|^\alpha} \varphi_{\ell-j}(t) K(t)\,dt \bigg) e^{ix\cdot\xi}\,dx.$$

The multiplier is an oscillatory integral in t with phase $\phi(t) = -t\cdot\xi + u|t|^\alpha$, which (assuming $|u| \simeq 2^{\alpha j}$ and $|\xi| \simeq 2^{j+k}$) has a critical point in the annulus $\{|t| \simeq 2^{\ell-j}\}$ if and only if $2^{k+\ell} \simeq 2^{\ell\alpha}$, that is, if and only if

$k=\ell(\alpha-1)+O(1)$. In that case, using stationary phase (see, for example, [Stein 1993, Chapter VIII.5.7] or [Sogge 1993, Theorem 1.2.1]), the multiplier can be written as

$$\psi(2^{-j-k}\xi)\big(e^{ic_\alpha(2^{-j\alpha}u)^{-1/(\alpha-1)}(2^{-j}|\xi|)^{\alpha'}}a(2^\ell 2^{-j}\xi,2^{\ell\alpha}2^{-j\alpha}u)+e(2^\ell 2^{-j}\xi,2^{\ell\alpha}2^{-j\alpha}u)\big),\tag{4-12}$$

where $\alpha'=\alpha/(\alpha-1)$, $c_\alpha=(\alpha-1)/\alpha^{\alpha'}$, $a\in S^{-n/2}(\mathbb{R}^{n+1})$ and $e\in S^{-\infty}(\mathbb{R}^{n+1})$. If there are no critical points in the annulus $\{|t|\simeq 2^{\ell-j}\}$, then the multiplier is simply

$$\psi(2^{-j-k}\xi)e(2^\ell 2^{-j}\xi,2^{\ell\alpha}2^{-j\alpha}u).\tag{4-13}$$

(In the above, by $a\in S^{-n/2}(\mathbb{R}^{n+1})$ we mean

$$|\partial_\xi^{\alpha'}\partial_u^{\alpha''}a(\xi,u)|\lesssim_\alpha(1+|\xi|+|u|)^{-n/2-|\alpha|}$$

for every multiindex $\alpha=(\alpha',\alpha'')\in\mathbb{Z}^{n+1}_{\geq 0}$, and by $e\in S^{-\infty}(\mathbb{R}^{n+1})$ we mean

$$|\partial_\xi^{\alpha'}\partial_u^{\alpha''}e(\xi,u)|\lesssim_{N,\alpha}(1+|\xi|+|u|)^{-N-|\alpha|}$$

for any positive integers N and any multiindex α.)

The above motivates us to consider three cases separately (under our earlier standing assumption (4-11)):

Case 1: $\ell\geq 0$, $k=\ell(\alpha-1)+O(1)$.

Case 2: $k>\ell(\alpha-1)+C$ for some $C>0$.

Case 3: $k<\ell(\alpha-1)-C$ for some $C>0$.

4D. ***Estimates in Case 1.*** Now we consider Case 1. Our goal is to bound (4-8) given k and ℓ as in Case 1. We proceed in a few steps.

4D1. *Application of Plancherel–Pólya.* First we will essentially show that if $r\in[2,\infty)$, then

$$\|\|V_j^r\mathcal{H}^{(u)}_{\ell-j}P_{j+k}f\|_{\ell_j^r}\|_{L_x^p}\lesssim 2^{\ell\alpha/q}\|\|\|\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}f(x)\|_{L_u^q}\|_{\ell_j^q}\|_{L_x^p}\tag{4-14}$$

for any $q\in[2,r]$ and any $p\in[1,\infty]$; here $\chi(u)$ is a smooth function with compact support on $\left[\frac12,2^{\alpha+1}\right]$ that is identically equal to 1 on $[1,2^\alpha]$. Indeed, when $n\geq 2$ (and p,r are as in (4-3)), we will only need (4-14) for $q=2$. But for $n=1$ (and p,r as in (4-2)), we will need (4-14) for both $q=2$ and $q=r$. We will see that this is the case after we prove (4-14).

To prove (4-14), let us temporarily write $g=P_{j+k}f$. As a function of u, $\mathcal{H}^{(u)}_{\ell-j}g$ has frequency morally supported on the annulus of size $\simeq 2^{(\ell-j)\alpha}$ centered at the origin. Thus, we introduce Littlewood–Paley projections in the u-variable (denoted by $P^{(2)}$ so that $P^{(2)}_{(\ell-j)\alpha}$ is projection onto frequency $\simeq 2^{(\ell-j)\alpha}$) and estimate

$$\begin{aligned}|V_j^r\mathcal{H}^{(u)}_{\ell-j}g(x)|&\\ &\leq|V^r(P^{(2)}_{\leq(\ell-j)\alpha}[\chi(2^{-j\alpha}u)\mathcal{H}^{(u)}_{\ell-j}g(x)])|+\sum_{k=1}^\infty|V^r(P^{(2)}_{(\ell-j+k)\alpha}[\chi(2^{-j\alpha}u)\mathcal{H}^{(u)}_{\ell-j}g(x)])|.\end{aligned}\tag{4-15}$$

(Here $P^{(2)}_{\leq(\ell-j)\alpha}:=\sum_{k\leq\ell-j}P^{(2)}_{k\alpha}$.)

The first term on the right-hand side of (4-15) is the main term and can be estimated using Proposition 2.2. In particular, it is bounded by

$$2^{(\ell-j)\alpha/q}\|\chi(2^{-j\alpha}u)\mathcal{H}^{(u)}_{\ell-j}g(x)\|_{L^q_u}$$

(recall $q \in [2, r]$). By changing variable in u, this is just

$$2^{\ell\alpha/q}\|\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}g(x)\|_{L^q_u}.$$

Hence the contribution of the first term of (4-15) to the left-hand side of (4-14) is bounded by

$$2^{\ell\alpha/q}\|\|\|\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}f(x)\|_{L^q_u}\|_{\ell^r_j}\|_{L^p_x}.$$

Since $r \geq q$, we have ℓ^r norm bounded by ℓ^q norm; hence the above is bounded by the right-hand side of (4-14).

On the other hand, for the second term on the right-hand side of (4-15), since $k > -C$, one can integrate by parts in u, using the fact that the multiplier for $P^{(2)}_{(\ell-j+k)\alpha}$ vanishes to infinite order at 0, and obtain

$$|P^{(2)}_{(\ell-j+k)\alpha}[\chi(2^{-j\alpha}u)\mathcal{H}^{(u)}_{\ell-j}g(x)]| \lesssim_N 2^{-k\alpha N}\widetilde{P}^{(2)}_{(\ell-j+k)\alpha}[\chi(2^{-j\alpha}u)\widetilde{\mathcal{H}}^{(u)}_{\ell-j}g(x)] \tag{4-16}$$

for any positive integer N, where $\widetilde{P}^{(2)}$ is a Littlewood–Paley projection similar to $P^{(2)}$, and $\widetilde{\mathcal{H}}^{(u)}_{\ell-j}$ is the same as $\mathcal{H}^{(u)}_{\ell-j}$ defined in (4-4), except that the cutoff φ is replaced by a smooth multiple $\tilde{\varphi}$ of φ. Hence by repeating the above argument, and summing over k using the additional convergence factors $2^{-k\alpha N}$ that we gained in (4-16), the contribution of the second term of (4-15) to the left-hand side of (4-14) is bounded by

$$2^{\ell\alpha/q}\|\|\|\chi(u)\widetilde{\mathcal{H}}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}f(x)\|_{L^q_u}\|_{\ell^q_j}\|_{L^p_x}. \tag{4-17}$$

Since $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ satisfy the same estimates, we will not distinguish the two, and declare that we can also bound (4-17) once we can bound the right-hand side of (4-14).

4D2. *Application of the square function estimate.* Now fix k, ℓ as in Case 1. In other words, fix $k, \ell \geq 0$ with $k = \ell(\alpha-1) + O(1)$. We will try to bound the right-hand side of (4-14) when $q = 2$. The multiplier for $\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}f$ is given by (4-12) with u replaced by $2^{j\alpha}u$. For $u \in \mathbb{R}$, let $\tilde{m}_u(\xi)$ be the multiplier

$$\tilde{m}_u(\xi) = \chi(u)\psi(2^{-k}\xi)(e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}}a(2^\ell\xi, 2^{\ell\alpha}u) + e(2^\ell\xi, 2^{\ell\alpha}u)), \tag{4-18}$$

where $a \in S^{-n/2}(\mathbb{R}^{n+1})$ and $e \in S^{-\infty}(\mathbb{R}^{n+1})$ are as in (4-12). Then the multiplier of the operator $\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j}P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. Now expand $\chi(u)a(2^\ell\xi, 2^{\ell\alpha}u)$ in Fourier series in u: let c be a small enough constant depending on α so that the support of $\chi(u)$ is contained in $[0, c^{-1}]$. Using the smoothness in the variable u, we get

$$\chi(u)a(2^\ell\xi, 2^{\ell\alpha}u) = \sum_{\kappa\in c\mathbb{Z}}(1+|\kappa|)^{-2}a_\kappa(2^\ell\xi)e^{i\kappa u}$$

for $u \in [0, c^{-1}]$, where $a_\kappa \in S^{-n/2}(\mathbb{R}^n)$ uniformly for every $\kappa \in c\mathbb{Z}$. Similarly, expand $\chi(u)e(2^\ell\xi, 2^{\ell\alpha}u)$ in Fourier series in u:

$$\chi(u)e(2^\ell\xi, 2^{\ell\alpha}u) = \sum_{\kappa\in c\mathbb{Z}}(1+|\kappa|)^{-2}e_\kappa(2^\ell\xi)e^{i\kappa u}$$

for $u \in [0, c^{-1}]$, where $e_\kappa \in S^{-\infty}(\mathbb{R}^n)$ uniformly for every $\kappa \in c\mathbb{Z}$. This shows

$$\tilde{m}_u(\xi) = \sum_{\kappa \in c\mathbb{Z}} (1+|\kappa|)^{-2} e^{i\kappa u} \psi(2^{-k}\xi)(a_\kappa(2^\ell \xi) e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}} + e_\kappa(2^\ell \xi))$$

for $u \in [0, c^{-1}]$. Temporarily let g be the function such that $\hat{g}(\xi) = \hat{f}(\xi)\psi(2^{-k}\xi)a_\kappa(2^\ell\xi)$; note that when $k \geq 0$, we have $\|g\|_{L^p_\beta(\mathbb{R}^n)} \lesssim 2^{k\beta} 2^{-(k+\ell)n/2} \|f\|_{L^p(\mathbb{R}^n)}$ by the Hörmander–Mikhlin multiplier theorem, with an implicit constant independent of κ. This is further bounded by $2^{\ell(\alpha-1)\beta} 2^{-\ell\alpha n/2}\|f\|_{L^p(\mathbb{R}^n)}$ since we are in Case 1, where $k = \ell(\alpha-1) + O(1)$. We apply Proposition 4.1 with g in place of f and obtain

$$\begin{aligned} \left\| \left\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\psi(2^{-k}\xi) a_\kappa(2^\ell\xi) e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}}\, d\xi \right\|_{L^2_u[0,c^{-1}]} \right\|_{L^p(\mathbb{R}^n)} & \\ \lesssim \begin{cases} 2^{-\ell\alpha/2}\|f\|_{L^p(\mathbb{R})} & \text{if } p \in [2,\infty) \text{ and } n = 1, \\ 2^{\ell\alpha[n(1/2-1/p)-1/2]} 2^{-\ell\alpha n/2}\|f\|_{L^p(\mathbb{R}^n)} & \text{if } p \in \left(2+\frac{4}{n}, \infty\right) \text{ and } n \geq 2. \end{cases} & \end{aligned} \tag{4-19}$$

We get a better decay if $a_\kappa(2^\ell\xi)e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}}$ above is replaced by $e_\kappa(2^\ell\xi)$. Summing over κ, and simplifying the exponent in the case $n \geq 2$, we get

$$\left\| \left\| \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi)\tilde{m}_u(\xi)\, d\xi \right\|_{L^2_u} \right\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} 2^{-\ell\alpha/2}\|f\|_{L^p(\mathbb{R})} & \text{if } p \in [2,\infty) \text{ and } n = 1, \\ 2^{-\ell\alpha/2} 2^{-\ell\alpha n/p}\|f\|_{L^p(\mathbb{R}^n)} & \text{if } p \in \left(2+\frac{4}{n}, \infty\right) \text{ and } n \geq 2. \end{cases}$$

But recall that the multiplier of the operator $\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. By scale invariance, we have

$$\left\| \left\| \chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f \right\|_{L^2_u} \right\|_{L^p(\mathbb{R}^n)} \lesssim \begin{cases} 2^{-\ell\alpha/2}\|f\|_{L^p(\mathbb{R})} & \text{if } p \in [2,\infty) \text{ and } n = 1, \\ 2^{-\ell\alpha/2} 2^{-\ell\alpha n/p}\|f\|_{L^p(\mathbb{R}^n)} & \text{if } p \in \left(2+\frac{4}{n}, \infty\right) \text{ and } n \geq 2 \end{cases} \tag{4-20}$$

for all $j \in \mathbb{Z}$, where the implicit constants are independent of j. (The Fourier series expansions used to remove the dependence on u are very reminiscent of the method used to prove L^2 boundedness of multipliers in S^0; see, for example, [Stein 1993, Chapter VI.2].)

Recall that our goal now is to bound the right-hand side of (4-14) when $q = 2$. Hence we need a vector-valued version of (4-20), where we will have an additional ℓ^2 norm over $j \in \mathbb{Z}$ inside the L^p norm on the left-hand side of (4-20). To do so, we need Proposition 4.2.

4D3. *Application of Seeger's theorem for multipliers with localized bounds.* First we state a vector-valued variant of a theorem of Seeger, about multipliers with localized bounds:

Proposition 4.2 [Jones, Seeger, and Wright 2008; Seeger 1988]. *Let $I \subset \mathbb{R}$ be a compact interval. Let $\{\tilde{m}_u(\xi) : u \in I\}$ be a family of Fourier multipliers on $\mathbb{R}^n$, each of which is compactly supported on $\left\{\xi : \frac{1}{2} \leq |\xi| \leq 2\right\}$ and satisfies*

$$\sup_{u\in I} |\partial_\xi^\tau \tilde{m}_u(\xi)| \leq B \quad \textit{for each } 0 \leq |\tau| \leq n+1$$

for some constant B. For $u \in I$ and $j \in \mathbb{Z}$, denote by $T_{u,j}$ the multiplier operator with multiplier $\tilde{m}_u(2^{-j}\xi)$. Fix some $p \in [2,\infty)$. Assume that there exists some constant A such that

$$\sup_{j\in\mathbb{Z}} \| \|T_{u,j} f\|_{L^2(I)} \|_{L^s(\mathbb{R}^n)} \leq A\|f\|_{L^s(\mathbb{R}^n)} \tag{4-21}$$

for both $s = p$ and $s = 2$. Then

$$\| \| \| T_{u,j} f \|_{L^2(I)} \|_{\ell^2(\mathbb{Z})} \|_{L^p(\mathbb{R}^n)} \lesssim A \left| \log\left(2 + \frac{B}{A}\right) \right|^{1/2-1/p} \| f \|_{L^p(\mathbb{R}^n)}.$$

This proposition was stated without proof on page 6737 of [Jones, Seeger, and Wright 2008]. It is a vector-valued analogue of Theorem 1 of [Seeger 1988], and we provide a proof of this proposition in Section 7 for the convenience of the reader.

Recall that our goal is to bound the right-hand side of (4-14) when $q = 2$. Also recall that if $\tilde{m}_u(\xi)$ is defined as in (4-18), and $T_{u,j}$ is the multiplier operator with multiplier $\tilde{m}_u(2^{-j}\xi)$ as in Proposition 4.2, then $T_{u,j} f$ is precisely $\chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f$. Thus, if we could apply Proposition 4.2, we would obtain a bound about the right-hand side of (4-14) when $q = 2$. To do so we verify the hypothesis of Proposition 4.2. From the explicit expression (4-18), we have

$$\sup_{u \in I} |\partial_\xi^\tau \tilde{m}_u(\xi)| \lesssim 2^{\ell N}$$

for some large positive integer N if $|\tau| \le n+1$. The hypothesis (4-21) for $s = p$ is given by (4-20), where A can be chosen to be relatively small if ℓ is large. On the other hand, by considering the L^∞ norm of the multipliers, we also get

$$\| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f \|_{L^2_u} \|_{L^2(\mathbb{R}^n)} \lesssim 2^{-\ell\alpha n/2} \| f \|_{L^2(\mathbb{R}^n)} \quad \text{for all } n \ge 1, \tag{4-22}$$

which gives us the hypothesis (4-21) for $s = 2$, where A can be chosen to be relatively small if ℓ is large.

More precisely, suppose first $n \ge 2$ and $p \in (2 + 4/n, \infty)$. Then we invoke (4-20) and (4-22). Since $2^{-\ell\alpha n/2} \le 2^{-\ell\alpha/2} 2^{-\ell\alpha n/p}$, we may apply Proposition 4.2 with $A = 2^{-\ell\alpha/2} 2^{-\ell\alpha n/p}$ and $B = 2^{\ell N}$ for some large positive integer N depending only on α. Thus, if $n \ge 2$ and $p \in (2 + 4/n, \infty)$, then we get

$$\| \| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x) \|_{L^2_u} \|_{\ell^2_j} \|_{L^p_x} \lesssim_\epsilon 2^{-\ell\alpha/2} 2^{-\ell\alpha n/p} 2^{\ell\epsilon} \| f \|_{L^p(\mathbb{R}^n)}$$

for any $\epsilon > 0$. Taking $q = 2$ in (4-14), this shows that

$$\| \| V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f \|_{\ell^r_j} \|_p \lesssim_\epsilon 2^{-\ell\alpha n/p} 2^{\ell\epsilon} \| f \|_{L^p(\mathbb{R}^n)} \quad \text{if } n \ge 2,\ p \in \left(2 + \frac{4}{n}, \infty\right) \text{ and } r \in [2, \infty).$$

Note that the power of 2 here is negative. So this estimate can be summed over all $\ell \ge 0$, and this gives the desired bound for (4-8) when $n \ge 2$, $p \in (2 + 4/n, \infty)$ and $r \in [2, \infty)$ for k, ℓ as in Case 1.

On the other hand, if $n = 1$ and $p \in [2, \infty)$, then in light of (4-20) and (4-22), we may apply Proposition 4.2 with $A = 2^{-\ell\alpha/2}$ and $B = 2^{\ell N}$ for some large positive integer N depending on α. We obtain

$$\| \| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x) \|_{L^2_u} \|_{\ell^2_j} \|_{L^p_x} \lesssim_\epsilon 2^{-\ell\alpha/2} 2^{\ell\epsilon} \| f \|_{L^p(\mathbb{R})} \quad \text{if } n = 1 \text{ and } p \in [2, \infty)$$

for any $\epsilon > 0$. Taking $q = 2$ in (4-14), this shows that

$$\| \| V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f \|_{\ell^r_j} \|_p \lesssim_\epsilon 2^{\ell\epsilon} \| f \|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in [2, \infty) \text{ and } r \in [2, \infty). \tag{4-23}$$

This is not good enough to be summed over all $\ell \ge 0$, so we need to gain a slightly better decay in ℓ. This is achieved via the local smoothing estimate in Theorem 1.6.

4D4. *Application of a local smoothing estimate in dimension $n = 1$.* The goal of this subsection is to prove that

$$\|\|V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f\|_{\ell_j^r}\|_p \lesssim 2^{-\ell\alpha/p}\|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p = r \in (4,\infty). \tag{4-24}$$

Assume for the moment that this has been established. Interpolating (4-24) against (4-23) using complex interpolation of vector-valued L^p spaces (see [Bergh and Löfström 1976, Theorem 5.1.2]), we get

$$\|\|V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f\|_{\ell_j^r}\|_p \lesssim 2^{-\gamma\ell}\|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in (2,\infty),\ r \in (2,\infty), \tag{4-25}$$

where $\gamma = \gamma(p, r)$ is a positive constant. This can be summed over all $\ell > 0$, and this gives the desired bound for (4-8) when $n = 1$, $p \in (2,\infty)$ and $r \in (2,\infty)$ for k, ℓ as in Case 1.

To prove (4-24) we use the local smoothing estimate in Theorem 1.6. Suppose $n = 1$, $p = r \in (4,\infty)$. We use (4-14) with $q = r = p$. Thus, the left-hand side of (4-24) is bounded up to a constant by

$$2^{\ell\alpha/p}\|\|\|\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x)\|_{L_u^p}\|_{\ell_j^p}\|_{L_x^p}. \tag{4-26}$$

Consider first

$$\begin{aligned}\Big\|\Big\|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat f(\xi)\tilde m_u(\xi)\,d\xi\Big\|_{L_u^p}\Big\|_{L_x^p}&\\ = \Big\|\Big\|\chi(u)\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat f(\xi)\psi(2^{-k}\xi)(e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}}a(2^\ell\xi,2^{\ell\alpha}u)+e(2^\ell\xi,2^{\ell\alpha}u))\,d\xi\Big\|_{L_u^p}\Big\|_{L_x^p}&.\end{aligned}$$

We first use Fubini's theorem to interchange the integrals in u and x, and use the Hörmander–Mikhlin multiplier theorem (for each fixed u) to get rid of the multiplier $a(2^\ell\xi, 2^{\ell\alpha}u)$. Since $k = \ell(\alpha-1)+O(1)$, this gives

$$\begin{aligned}\Big\|\Big\|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat f(\xi)\tilde m_u(\xi)\,d\xi\Big\|_{L_u^p}\Big\|_{L_x^p}&\\ \lesssim 2^{-\ell\alpha/2}\Big\|\Big\|\chi(u)\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat f(\xi)\psi(2^{-k}\xi)e^{ic_\alpha u^{-1/(\alpha-1)}|\xi|^{\alpha'}}\,d\xi\Big\|_{L_u^p}\Big\|_{L_x^p}&+2^{-\ell N}\|f\|_{L^p(\mathbb{R})}\end{aligned}$$

for any positive integer N. Thus, Theorem 1.6 applies, and when $k \ge 0$ we have

$$\Big\|\Big\|\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat f(\xi)\tilde m_u(\xi)\,d\xi\Big\|_{L_u^p}\Big\|_{L_x^p} \lesssim 2^{-\ell\alpha/2}2^{k\alpha'[(1/2-1/p)-1/p]}\|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in (4,\infty).$$

But recall that the multiplier of the operator $\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k}$ is precisely $\tilde m_u(2^{-j}\xi)$. By scale invariance, and remembering that $k = \ell(\alpha-1)+O(1)$, we have

$$\|\|\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x)\|_{L_u^p}\|_{L_x^p} \lesssim 2^{-2\ell\alpha/p}\|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in (4,\infty).$$

Replacing f by $\widetilde{P}_{j+k} f$, taking the ℓ_j^p norm on both sides, and using the Littlewood–Paley inequality (remember $p \ge 2$), we get

$$\|\|\|\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x)\|_{L_u^p}\|_{\ell_j^p}\|_{L_x^p} \lesssim 2^{-2\ell\alpha/p}\|f\|_{L^p(\mathbb{R})} \quad \text{if } n = 1,\ p \in (4,\infty).$$

Thus, (4-26) is $\lesssim 2^{-\ell\alpha/p}\|f\|_{L^p(\mathbb{R})}$. This establishes (4-24), and our treatment for Case 1 is now complete.

4E. ***Estimates in Cases 2 and 3: further gains over Case 1.*** Next we estimate (4-8) for k, ℓ as in Case 2. Fix k, ℓ such that $k > \ell(\alpha-1)+C$ for some positive constant C. If C is large enough, then the multiplier for $\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f$ is given by (4-13), since the phase function of the oscillatory integral defining the multiplier has no critical point in that case. For $u \in \mathbb{R}$, let $\tilde{m}_u(\xi)$ be the multiplier

$$\tilde{m}_u(\xi) = \chi(u)\psi(2^{-k}\xi)e(2^{\ell}\xi, 2^{\ell\alpha}u),$$

where $e \in S^{-\infty}(\mathbb{R}^{n+1})$ is as in (4-13). Then the multiplier of the operator $\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. For every $N \in \mathbb{N}$ we can write

$$\tilde{m}_u(\xi) = 2^{-(k+\ell)N}\chi(u)\psi(2^{-k}\xi)\tilde{e}_N(2^{\ell}\xi, 2^{\ell\alpha}u)$$

for some symbol $\tilde{e}_N \in S^{-\infty}(\mathbb{R}^{n+1})$. Thus, applying Proposition 4.2 as in the proof of (4-23), we get

$$\| \| V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f \|_{\ell_j^r} \|_p \lesssim_N 2^{-(k+\ell)N} \| f \|_{L^p(\mathbb{R}^n)}$$

whenever one of the following two conditions is fulfilled: $n = 1$, $p \in [2, \infty)$ and $r \in [2, \infty)$, or $n \geq 2$, $p \in (2 + 4/n, \infty)$, and $r \in [2, \infty)$. The right-hand side in the above display equation can be summed over all k, ℓ that satisfy $k > \ell(\alpha-1)+C$ and the standing assumption (4-11), and this gives the bound for (4-8) for such p, n, r for all k, ℓ as in Case 2.

Finally we estimate (4-8) for k, ℓ as in Case 3. Fix k, ℓ such that $k < \ell(\alpha-1)-C$ for some positive constant C. As in Case 2, if C is large enough, then the multiplier for $\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f$ is given by (4-13). For $u \in \mathbb{R}$, let $\tilde{m}_u(\xi)$ be the multiplier

$$\tilde{m}_u(\xi) = \chi(u)\psi(2^{-k}\xi)e(2^{\ell}\xi, 2^{\ell\alpha}u),$$

where $e \in S^{-\infty}(\mathbb{R}^{n+1})$ is as in (4-13). Then the multiplier of the operator $\chi(u)\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k}$ is precisely $\tilde{m}_u(2^{-j}\xi)$. For every $N \in \mathbb{N}$ we can write

$$\tilde{m}_u(\xi) = (2^{-\ell\alpha}u)^N \chi(u)\psi(2^{-k}\xi)\tilde{e}_N(2^{\ell}\xi, 2^{\ell\alpha}u)$$

for some symbol $\tilde{e}_N \in S^{-\infty}(\mathbb{R}^{n+1})$. Thus, applying Proposition 4.2 as in the proof of (4-23), we get

$$\| \| V_j^r \mathcal{H}^{(u)}_{\ell-j} P_{j+k} f \|_{\ell_j^r} \|_p \lesssim_N 2^{-\ell\alpha N} \| f \|_{L^p(\mathbb{R}^n)}$$

whenever one of the following two conditions is fulfilled: $n = 1$, $p \in [2, \infty)$ and $r \in [2, \infty)$, or $n \geq 2$, $p \in (2 + 4/n, \infty)$, and $r \in [2, \infty)$. The right-hand side in the above displayed equation can be summed over all k, ℓ that satisfy $k < \ell(\alpha-1)-C$ and the standing assumption (4-11). This gives the bound for (4-8) for such p, n, r for all k, ℓ as in Case 3.

We have thus completed the proof of (4-1) for all p, n, r satisfying (4-2) or (4-3).

5. Short jump estimates for $p \leq 2$

In this section, we establish

$$\left\| \left(\sum_{j \in \mathbb{Z}} |V_j^r(\mathcal{H}f)|^r \right)^{1/r} \right\|_p \lesssim \| f \|_p, \tag{5-1}$$

whenever $n \geq 2,\ 2n/(2n-1) < p \leq 2$, and $r \in [2, \infty)$. By complex interpolation (see [Bergh and Löfström 1976, Theorem 5.1.2]) with (4-1), we will then have (5-1) whenever $n \geq 2,\ p \in (2n/(2n-1), \infty)$, and $r \in [2, \infty)$, which concludes the proof of Proposition 1.5.

The key here is the following square function estimate.

Proposition 5.1. *Let $n \geq 2,\ 1 < p \leq 2$ and $\lambda > 1$. Then for any compact time interval I,*

$$\left\| \left(\int_I \left| \int_{\mathbb{R}^n} e^{ix\xi} \hat{f}(\xi) e^{it|\xi|^\lambda} \, d\xi \right|^2 dt \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{W^{\beta,p}(\mathbb{R}^n)},$$

with

$$\frac{\beta}{\lambda} = n\left(\frac{1}{p} - \frac{1}{2}\right).$$

The proof of this proposition is postponed to the end of this section.

Now let $2n/(2n-1) < p \leq 2,\ n \geq 2$, and $r \in [2, \infty)$. We proceed to establish (5-1). As in Section 4, we decompose $\mathcal{H}^{(u)} f$ as in (4-5) and (4-7), and estimate (4-8) for every $k, \ell \in \mathbb{Z}$. The inequalities (4-9) and (4-10) still hold under our current assumptions of p, n, r, and these estimates can be summed whenever $\ell \leq -k/(2(\alpha+1))$. Thus, it remains to consider pairs of (k, ℓ) for which (4-11) holds, and we still divide into Cases 1, 2, 3 as before. We will only treat Case 1 here which is the main case; an easy adaptation of this argument gives Cases 2 and 3.

So let $\ell \geq 0$ and $k = \ell(\alpha - 1) + O(1)$. By (4-14) with $q = 2$, the left-hand side of (5-1) is bounded by

$$2^{\ell\alpha/2} \| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f(x) \|_{L^2_u} \|_{\ell^2_j} \|_{L^p_x}. \tag{5-2}$$

We analyze the multiplier of $\mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k}$ as before, but in (4-19) we use Proposition 5.1 instead of Proposition 4.1 (since now $p \in (2n/(2n-1), 2)$). So instead of (4-20), we get

$$\| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f \|_{L^2_u} \|_{L^p(\mathbb{R}^n)} \lesssim 2^{\ell\alpha n(1/p - 1/2)} 2^{-\ell\alpha n/2} \|f\|_{L^p(\mathbb{R}^n)} = 2^{-\ell\alpha n/p'} \|f\|_{L^p(\mathbb{R}^n)} \tag{5-3}$$

uniformly in $j \in \mathbb{Z}$. Thus, we apply Proposition 4.2 with $A = 2^{-\ell\alpha n/p'}$ and $B = 2^{\ell N}$ for some large integer N depending only on α. This gives

$$\| \| \| \chi(u) \mathcal{H}^{(2^{j\alpha}u)}_{\ell-j} P_{j+k} f \|_{L^2_u} \|_{\ell^2_j} \|_{L^p(\mathbb{R}^n)} \lesssim_\epsilon 2^{-\ell\alpha n/p'} 2^{\ell\epsilon} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $\epsilon > 0$. Continuing from (5-2), we see that the left-hand side of (5-1) is bounded by

$$2^{\ell\alpha/2} 2^{-\ell\alpha n/p'} 2^{\ell\epsilon} \|f\|_{L^p(\mathbb{R}^n)}.$$

Since $p \in (2n/(2n-1), \infty)$, the above exponent of 2 is negative if ϵ is sufficiently small. Thus, we can sum over all $\ell \geq 0$ in this case, establishing the bound for (4-8) for all k, ℓ in Case 1. A similar argument establishes a bound for (4-8) for all k, ℓ in Cases 2 and 3. This completes the proof of (5-1), modulo the proof of Proposition 5.1.

Proof of Proposition 5.1. We will prove a slightly more general result. Let us write

$$T_u f(x) = \int_{\mathbb{R}^n} e^{ix\xi} \hat{f}(\xi) m_u(\xi) \, d\xi,$$

where

$$m_u(\xi) = e^{iu|\xi|^\lambda}(1+|\xi|^2)^{-(\beta+i\gamma)/2}.$$

Theorem 5.2. *Let I be a compact interval not containing 0. If $\lambda > 1$, $\beta = n\lambda/2$ and $\gamma \in \mathbb{R}$, then*

$$\left\| \left(\int_I |T_u f(x)|^2 \, du \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{H^1(\mathbb{R}^n)};$$

that is, T maps the Hardy space $H^1(\mathbb{R}^n)$ boundedly into $L^1_x(\mathbb{R}^n; L^2_u(I))$.

All implied constants may depend on λ, β, n, I, but not on f, γ, x, u.

Theorem 5.2 implies Proposition 5.1 via complex interpolation; see, for example, [Stein 1993, Chapter IV.6.17] for a discussion of interpolation between Hardy spaces. For the scalar theory of the multiplier m_u (for fixed u) we refer to [Miyachi 1981; Fefferman and Stein 1972; Fefferman 1970].

Recall that an H^1 atom of radius r is a bounded function a on $\mathbb{R}^n$ that is supported in a ball of radius r and satisfies $\|a\|_\infty \le r^{-n}$ and $\int_{\mathbb{R}^n} a = 0$. The Hardy space $H^1(\mathbb{R}^n)$ is a Banach space consisting of functions of the form $f = \sum_j c_j a_j$, with $\sum_j |c_j| < \infty$, where the $(a_j)_j$ are H^1 atoms. Its norm is defined as

$$\|f\|_{H^1(\mathbb{R}^n)} = \inf \sum_j |c_j|,$$

where the infimum is taken over all atomic decompositions of $f = \sum_j c_j a_j$.

To prove Theorem 5.2 it suffices to show that

$$\|Ta\|_{L^1_x(L^2_u)} \lesssim 1 \tag{5-4}$$

holds for every H^1 atom a of radius r. We may assume the support of a to be centered at the origin.

For $j > 0$, let P_j denote the usual Littlewood–Paley projection with $\widehat{P_j f} = \psi_j \hat{f}$, ψ_j supported on $|\xi| \simeq 2^j$. Let $\widehat{P_0 f} = \psi_0 \hat{f}$, where ψ_0 is such that

$$1 = \psi_0 + \sum_{j>0} \psi_j.$$

(Note that P_0 here is actually $P_{\le 0}$ from the previous section.) For $j \ge 0$ we denote by $\tilde{\psi}_j$ a smooth positive function that equals 1 on the support of ψ_j and whose support is contained in a small neighborhood of the support of ψ_j. Define

$$K_u^{(j)}(x) = \hat{m}_u * \hat{\psi}_j(x) = \int_{\mathbb{R}^n} e^{ix\xi + iu|\xi|^\lambda}(1+|\xi|^2)^{-\beta/2}\psi_j(\xi)\, d\xi.$$

Before we begin, we record the following pointwise estimates for $K_u^{(j)}(x)$. From estimating the second derivative of the phase we obtain

$$|K_u^{(j)}(x)| \lesssim 2^{-jn(\lambda-1)} \quad \text{for all } x \in \mathbb{R}^n,\ u \in I. \tag{5-5}$$

Here we used that $\beta = n\lambda/2$. Estimating the first derivative of the phase we obtain

$$|K_u^{(j)}(x)| \lesssim_N 2^{-j\beta} 2^{jn} (2^j|x|)^{-N} \quad \text{for } |x| \gtrsim 2^{j(\lambda-1)},\ u \in I \tag{5-6}$$

for all $N \ge 0$. Note that these estimates are uniform in $u \in I$.

Let us prove the main estimate (5-4). The first step is to apply the triangle inequality:

$$\|Ta\|_{L^1_x(L^2_u)} \le \sum_{j\ge 0} \|K^{(j)}_u * a\|_{L^1_x(L^2_u)}.$$

We will estimate the summand in two different ways: in particular, it will be shown below that

$$\|K^{(j)}_u * a\|_{L^1_x(L^2_u)} \lesssim (2^j r)^{-n/2} + 2^{-j\beta}, \tag{5-7}$$

$$\|K^{(j)}_u * a\|_{L^1_x(L^2_u)} \lesssim 2^j r. \tag{5-8}$$

These estimates immediately imply (5-4).

To prove (5-7) we first split up[1] the integral in x:

$$\|K^{(j)}_u * a\|_{L^1_x(L^2_u)} \le \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \|K^{(j)}_u * a\|_{L^1_x(\mathbb{R}^n \setminus B(C2^{j(\lambda-1)}+r);L^2_u(I))},$$

$$\mathrm{II} = \|K^{(j)}_u * a\|_{L^1_x(B(C2^{j(\lambda-1)}+r);L^2_u(I))}.$$

We claim that $\mathrm{I} \lesssim_N 2^{-jN}$. Indeed, we see from (5-6) that

$$\begin{aligned}
\mathrm{I} &\le \int_{|x|\ge C2^{j(\lambda-1)}+r} \left(\int_I \left(\int_{|y|\le r} |K^{(j)}_u(x-y)a(y)|\,dy \right)^2 du \right)^{1/2} dx \\
&\lesssim_N 2^{-j\beta} 2^{jn} 2^{-jN} \int_{|x|\ge C2^{j(\lambda-1)}+r} \int_{|y|\le r} |x-y|^{-N} |a(y)|\,dy\,dx \\
&\le 2^{-j\beta} 2^{jn} 2^{-jN} \|a\|_1 \int_{|x|\gtrsim 2^{j(\lambda-1)}} |x|^{-N}\,dx \lesssim 2^{-\beta j} 2^{jn\lambda} 2^{-jN\lambda},
\end{aligned}$$

which implies the claim (since N is arbitrary). To estimate the second part we use the Cauchy–Schwarz inequality:

$$\mathrm{II} \le (C2^{j(\lambda-1)} + r)^{n/2} \|K^{(j)}_u * a\|_{L^2_x(L^2_u)}.$$

Then we have by the Fubini and Plancherel theorems that

$$\|K^{(j)}_u * a\|_{L^2_x(L^2_u)} = \|\,\|K^{(j)}_u * a\|_2\|_{L^2_u(I)} \lesssim 2^{-j\beta}\|a\|_2 \lesssim 2^{-j\beta} r^{-n/2},$$

which implies

$$\mathrm{II} \lesssim (2^{j(\lambda-1)n/2} + r^{n/2}) 2^{-j\beta} r^{-n/2} = (2^j r)^{-n/2} + 2^{-j\beta},$$

as desired (we used that $\beta = n\lambda/2$). This proves (5-7).

It remains to show (5-8). Clearly we have

$$\|K^{(j)}_u * a\|_{L^1_x(L^2_u)} \lesssim \|\sup_{u\in I} |K^{(j)}_u * a|\|_1. \tag{5-9}$$

We claim that

$$\|\sup_{u\in I} |K^{(j)}_u * a|\|_1 \lesssim \|P_j a\|_1. \tag{5-10}$$

[1] C is a constant that may depend on the parameters λ, β, n.

To see this replace ψ_j by $\tilde{\psi}_j$ in the definition of $K_u^{(j)}$ and call the resulting kernel $\widetilde{K}_u^{(j)}$. Then we have

$$K_u^{(j)} * a = \widetilde{K}_u^{(j)} * P_j a.$$

It is clear that $\widetilde{K}_u^{(j)}$ satisfies the same pointwise estimates (5-5), (5-6) (possibly with larger constants). Thus, there exists a positive function w_j on $\mathbb{R}^n$ such that $\|w_j\|_1 \lesssim 1$ and

$$|\widetilde{K}_u^{(j)}(x)| \le w_j(x)$$

for all $x \in \mathbb{R}^n$ and $u \in I$. As a consequence,

$$\| \sup_{u\in I} |K_u^{(j)} * a| \|_1 \le \| \sup_{u\in I} |\widetilde{K}_u^{(j)}| * |P_j a| \|_1 \le \|w_j * |P_j a|\|_1 \lesssim \|P_j a\|_1,$$

which is our claim (5-10). But by the mean zero property of a and the mean value theorem we have

$$P_j a(x) = \int_{\mathbb{R}^n} (\hat{\psi}_j(x-y) - \hat{\psi}_j(x)) a(y)\, dy = -\int_{\mathbb{R}^n}\int_0^1 y \cdot \nabla\hat{\psi}_j(x-ty)\, dt\, a(y)\, dy.$$

This implies

$$\|P_j a\|_1 \le \int_{\mathbb{R}^n}\int_{|y|\le r}\int_0^1 |y||\nabla\hat{\psi}_j(x-ty)||a(y)|\, dt\, dy\, dx \lesssim 2^j r.$$

In view of (5-9) and (5-10), this establishes (5-8). □

6. A counterexample: the proof of Theorem 1.2

Let ϕ be a smooth test function supported in the annulus $\frac{1}{2} \le |\xi| \le 2$ and define f_k for $k \in \mathbb{Z}$ by

$$\hat{f}_k(\xi) = \phi(2^{-k}\xi).$$

On the one hand, clearly,

$$\|f_k\|_p = 2^{nk}\left(\int_{\mathbb{R}^n} |\hat{\phi}(2^k x)|^p\, dx\right)^{1/p} \approx 2^{nk/p'}.$$

On the other hand, we claim that

$$\|V^r\{\mathcal{H}^{(u)} f_k : u \in \mathbb{R}\}\|_p \gtrsim 2^{k(-n(\alpha-1)(1/p')+\alpha/r)}. \tag{6-1}$$

If (1-3) were to hold, then plugging in f_k into the estimate (1-3) and letting $k \to \infty$, we see that

$$-n(\alpha-1)\frac{1}{p'} + \frac{\alpha}{r} \le \frac{n}{p'},$$

which is equivalent to $p' \le nr$.

For simplicity we will verify this only in the case $n = 1$, $K(t) = \text{p.v.}(1/t)$, $\alpha = 2$. The general case can be treated in the same manner. In this case (6-1) takes the form

$$\|V^r\{\mathcal{H}^{(u)} f_k : u \in \mathbb{R}\}\|_p \gtrsim 2^{k(-1+1/p+2/r)}. \tag{6-2}$$

We can choose φ such that

$$\frac{1}{t} = \sum_{j\in\mathbb{Z}} \varphi_j(t)$$

for all $t \neq 0$, where $\varphi_j(t) = 2^{-j}\varphi(2^{-j}t)$. By Fourier inversion we have

$$\begin{aligned}\mathcal{H}^{(u)} f_k(x) &= \sum_{j\in\mathbb{Z}} \iint \phi(2^{-k}\xi) e^{ix\xi} e^{-it\xi+iut^2} \varphi_{j+k}(t)\, dt\, d\xi \\ &= \sum_{j\in\mathbb{Z}} 2^k \int e^{i2^k x\xi} \phi(\xi) \int e^{-i2^{2k}t\xi+i2^{2k}ut^2} \varphi_j(t)\, dt\, d\xi = \sum_{j\in\mathbb{Z}} \mathrm{I}_j.\end{aligned}$$

(Keep in mind that I_j also depends on k, x, u.) Let us take $u \in [1, 2]$. Then the phase of the oscillatory integral in t has no critical points if $|j| > 10$. This motivates us to set

$$\mathrm{I}^{\mathrm{main}} = \sum_{|j|\le 10} \mathrm{I}_j \quad \text{and} \quad \mathrm{I}^{\mathrm{err}} = \sum_{|j|>10} \mathrm{I}_j.$$

Write $B = [2^k, 2^{k+1}]$ and estimate

$$\begin{aligned}\|V^r\{\mathcal{H}^{(u)} f_k : u \in \mathbb{R}\}\|_p &\ge \|V^r\{\mathcal{H}^{(u)} f_k : u \in [1, 2]\}\|_{L^p(B)} \\ &\ge \|V^r\{\mathrm{I}^{\mathrm{main}} : u \in [1, 2]\}\|_{L^p(B)} - \|V^r\{\mathrm{I}^{\mathrm{err}} : u \in [1, 2]\}\|_{L^p(B)}.\end{aligned}$$

In order to verify (6-2) it suffices to show that

$$\|V^r\{\mathrm{I}^{\mathrm{main}} : u \in [1, 2]\}\|_{L^p([2^k,2^{k+1}])} \gtrsim 2^{k(-1+1/p+2/r)}, \tag{6-3}$$

$$\|V^r\{\mathrm{I}^{\mathrm{err}} : u \in [1, 2]\}\|_{L^p([2^k,2^{k+1}])} \lesssim 2^{-2k}. \tag{6-4}$$

We begin with the proof of (6-3). Write

$$\mathrm{I}^{\mathrm{main}} = 2^k \int e^{i2^k x\xi} \phi(\xi) \int e^{-i2^{2k}t\xi+i2^{2k}ut^2} \rho(t)\, dt\, d\xi,$$

where $\rho(t) = \sum_{|j|\le 10} \varphi_j(t)$. Note that the phase of the integral in t has a critical point at $t_c = \xi/(2u)$. By the principle of stationary phase [Stein 1993, Chapter VIII.5.7; Sogge 1993, Theorem 1.2.1] we have

$$\int e^{-i2^{2k}t\xi+i2^{2k}ut^2} \rho(t)\, dt = 2^{-k} c_0 e^{i2^{2k}c_1\xi^2u^{-1}} u^{-1/2} \rho\left(\frac{\xi}{2u}\right) + O(2^{-2k}).$$

Here c_0, c_1 are nonzero constants. To simplify the calculation, let us take $c_0 = c_1 = 1$. Thus, the main contribution to $\mathrm{I}^{\mathrm{main}}$ is

$$\int e^{i2^{2k}(\tilde{x}\xi+\xi^2u^{-1})} a(\xi, u)\, d\xi,$$

where $x = 2^k\tilde{x} \in [2^k, 2^{k+1}]$ and $a(\xi, u) = \phi(\xi)u^{-1/2}\rho(\xi/(2u))$. Note that the u-derivative of the error term coming from the stationary phase is also $O(2^{-2k})$. Therefore that term contributes only $O(2^{-2k})$ to the variation-norm and we can ignore it. From another application of the stationary phase principle we see that the previous integral can essentially be written in the form

$$2^{-k} e^{ix^2u} b(u) + O(2^{-2k}),$$

where $b(u) = \phi(\tilde{x}u/2)\rho(\tilde{x}/4)$. Let $u_\ell = \ell\pi/x^2$ for $x^2/\pi < \ell < 2x^2/\pi$. Then

$$\left(\sum_{x^2/\pi<\ell<2x^2/\pi} |e^{ix^2u_{\ell+1}} - e^{ix^2u_\ell}|^r\right)^{1/r} \approx 2^{2k/r},$$

which implies the claim (6-3) (the contribution of $b(u)$ is negligible).

It remains to treat $\mathrm{I}^{\mathrm{err}}$. Compute

$$\partial_u \mathrm{I}_j = i2^{3k+2j} \int \phi(\xi) e^{i2^k x\xi} \int e^{-i2^{2k+j}t\xi + i2^{2k+2j}ut^2} t^2 \varphi(t)\, dt\, d\xi$$

$$= i2^{3k+2j} \int \hat{\phi}(2^{2k+j}t - 2^k x) e^{i2^{2k+2j}ut^2} t^2 \varphi(t)\, dt.$$

Observe that if $x \in [2^k, 2^{k+1}]$, $|t| \in \left[\frac{1}{2}, 2\right]$ and $|j| > 10$ we have

$$|2^{2k+j}t - 2^k x| \approx 2^{2k} \max(1, 2^j).$$

Since $\hat{\phi}$ decays rapidly, we obtain

$$V^r\{\mathrm{I}_j : u \in [1,2]\} \lesssim \|\partial_u \mathrm{I}_j\|_{L^1_u([1,2])} \lesssim_N 2^{3k+2j-2Nk} \min(1, 2^{-Nj})$$

for every $N \geq 1$. Taking N large enough ($N = 3$ suffices) and summing over $|j| > 10$, we obtain (6-4).

7. Proof of Proposition 4.2

In this section we provide a proof of Proposition 4.2, which was stated in [Jones, Seeger, and Wright 2008] without proof. Indeed, Proposition 4.2 is a vector-valued analogue of Theorem 1 of [Seeger 1988]. The proof of Proposition 4.2 follows closely that of the scalar-valued case in [Seeger 1988]. On the other hand, at one point in the scalar-valued case, Seeger used a duality argument between L^p and $L^{p'}$, which is not available in the vector-valued setting. This is why we had to assume that hypothesis (4-21) holds not just for $s = p$, but also at $s = 2$.

To prove Proposition 4.2, one key tool is the Fefferman–Stein sharp function. Let B be a Banach space. For us we will only need the case $B = \ell^2(\mathbb{Z})L^2(I)$. For each measurable function $F : \mathbb{R}^n \to B$, define its Hardy–Littlewood maximal function $\mathcal{M}F$ by

$$\mathcal{M}F(x) = \sup_{x \in Q} \fint_Q |F(y)|_B\, dy$$

for each $x \in \mathbb{R}^n$, where the supremum is over all cubes Q containing x. Also define the sharp function $F^\sharp$ of F by

$$F^\sharp(x) = \sup_{x \in Q} \fint_Q |F(y) - F_Q|_B\, dy,$$

where $F_Q = \fint_Q F(y)\, dy$; again the supremum is over all cubes Q containing x. We have the following lemma about $F^\sharp$:

Lemma 7.1. *Suppose $0 < p < \infty$. Let $F \in L^{p_0}(\mathbb{R}^n, B)$ for some $0 < p_0 \leq p$. If $F^\sharp \in L^p(\mathbb{R}^n)$, then $\mathcal{M}F \in L^p(\mathbb{R}^n)$, and*

$$\|\mathcal{M}F\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|F^\sharp\|_{L^p(\mathbb{R}^n)}.$$

We give a proof of this lemma at the end.

Now given $f \in L^p \cap L^2(\mathbb{R}^n)$, define $Tf : \mathbb{R}^n \to B$ by

$$Tf(x) = (T_{u,j} f(x))_{u\in I, j\in\mathbb{Z}}.$$

Note $Tf \in L^2(\mathbb{R}^n, B)$. Then

$$\|\|\|T_{u,j} f\|_{L^2(I)}\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)} = \||Tf|_B\|_{L^p(\mathbb{R}^n)} \le \|\mathcal{M}(Tf)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|(Tf)^\sharp\|_{L^p(\mathbb{R}^n)},$$

where in the last inequality we invoked the lemma with $p_0 = 2$. Note that for a.e. $x \in \mathbb{R}^n$

$$(Tf)^\sharp(x) \simeq \fint_{Q_x} |Tf(y) - (Tf)_{Q_x}|_B \, dy$$

for some cube Q_x containing x; we may choose Q_x such that the side length of Q_x is $2^{r(x)}$ for some integer $r(x)$. Then we split

$$(Tf)^\sharp(x) \lesssim \sigma_1((T_{u,j} f), x) + \sigma_2((P_j f), x),$$

where N is a positive integer to be chosen; here

$$\sigma_1(G, x) = \fint_{Q_x} \Bigg(\sum_{|j+r(x)|\le N} \|G_{u,j}(y) - (G_{u,j})_{Q_x}\|^2_{L^2(I)} \Bigg)^{1/2} dy,$$

$$\sigma_2(H, x) = \fint_{Q_x} \Bigg(\sum_{|j+r(x)|> N} \|T_{u,j} H_j(y) - (T_{u,j} H_j)_{Q_x}\|^2_{L^2(I)} \Bigg)^{1/2} dy$$

for any functions $G = (G_{u,j}) : \mathbb{R}^n \to B$ and $H = (H_j) : \mathbb{R}^n \to \ell^2(\mathbb{Z})$. We claim that

$$\|\sigma_1(G, x)\|_{L^p(\mathbb{R}^n)} \lesssim N^{1/2-1/p} \|\|\|G_{u,j}(x)\|_{L^2(I)}\|_{\ell^p(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)}, \tag{7-1}$$

$$\|\sigma_2(H, x)\|_{L^p(\mathbb{R}^n)} \lesssim (A + B2^{-N}) \|\|H_j(x)\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)} \tag{7-2}$$

for any $G = (G_{u,j}) : \mathbb{R}^n \to B$ and $H = (H_j) : \mathbb{R}^n \to \ell^2(\mathbb{Z})$. But when $G_{u,j} = T_{u,j} f$, we have

$$\begin{aligned}
\|\|\|G_{u,j}(x)\|_{L^2(I)}\|_{\ell^p(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)} &= \|\|\|T_{u,j} f(x)\|_{L^2(I)}\|_{L^p(\mathbb{R}^n)}\|_{\ell^p(\mathbb{Z})} \\
&\lesssim A \|\|P_j f(x)\|_{L^p(\mathbb{R}^n)}\|_{\ell^p(\mathbb{Z})} \\
&\lesssim A \|\|P_j f(x)\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)} \\
&\lesssim A \|f\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

(we used assumption (4-21) in the second inequality, $p \in [2, \infty)$ in the third inequality, and Littlewood–Paley inequality in the last). Also, when $H_j = P_j f$, we have

$$\|\|H_j(x)\|_{\ell^2(\mathbb{Z})}\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Hence

$$\|(Tf)^\sharp\|_{L^p(\mathbb{R}^n)} \lesssim (AN^{1/2-1/p} + A + B2^{-N}) \|f\|_{L^p(\mathbb{R}^n)}.$$

Choosing $N \simeq \log(2 + B/A)$ gives the desired conclusion of the proposition. It remains to prove (7-1) and (7-2).

To prove (7-1), we interpolate between $p = 2$ and $p = \infty$. Indeed, we prove

$$\|\sigma_1(G, x)\|_{L^2(\mathbb{R}^n)} \lesssim \|\|\|G_{u,j}(x)\|_{L^2(I)}\|_{\ell^2(\mathbb{Z})}\|_{L^2(\mathbb{R}^n)}, \tag{7-3}$$

$$\|\sigma_1(G, x)\|_{L^\infty(\mathbb{R}^n)} \lesssim N^{1/2}\|\|\|G_{u,j}(x)\|_{L^2(I)}\|_{\ell^\infty(\mathbb{Z})}\|_{L^\infty(\mathbb{R}^n)}. \tag{7-4}$$

The desired estimate (7-1) then follows by complex interpolation and linearizing σ_1.

The estimate (7-3) follows since

$$\sigma_1(G, x) \le 2 \fint_{Q_x} \Big(\sum_{|j+r(x)|\le N} \|G_{u,j}(y)\|_{L^2(I)}^2 \Big)^{1/2} dy, \tag{7-5}$$

so

$$\sigma_1(G, x) \lesssim \fint_{Q_x} \|\|G_{u,j}(y)\|_{L^2(I)}\|_{\ell^2(\mathbb{Z})}\, dy \lesssim M\|\|G_{u,j}\|_{L^2(I)}\|_{\ell^2(\mathbb{Z})}(x),$$

where M is the standard (scalar-valued) Hardy–Littlewood maximal function on $\mathbb{R}^n$. Hence

$$\|\sigma_1(G, x)\|_{L^2(\mathbb{R}^n)} \lesssim \|\|\|G_{u,j}(x)\|_{L^2(I)}\|_{\ell^2(\mathbb{Z})}\|_{L^2(\mathbb{R}^n)}$$

as in (7-3).

To prove (7-4), note that for each $x \in \mathbb{R}^n$, we have, from (7-5), that

$$\sigma_1(G, x) \le 2 \sup_{y\in\mathbb{R}^n} \Big(\sum_{|j+r(x)|\le N} \|G_{u,j}(y)\|_{L^2(I)}^2 \Big)^{1/2} \lesssim N^{1/2} \sup_{y\in\mathbb{R}^n} \sup_{j\in\mathbb{Z}} \|G_{u,j}(y)\|_{L^2(I)},$$

with constants uniform in x. This gives (7-4).

Next, to prove (7-2), we will prove

$$\|\sigma_2(H, x)\|_{L^2(\mathbb{R}^n)} \lesssim A\|\|H_j(x)\|_{\ell^2(\mathbb{Z})}\|_{L^2(\mathbb{R}^n)}, \tag{7-6}$$

$$\|\sigma_2(H, x)\|_{L^\infty(\mathbb{R}^n)} \lesssim (A + B2^{-N})\|\|H_j(x)\|_{\ell^2(\mathbb{Z})}\|_{L^\infty(\mathbb{R}^n)}. \tag{7-7}$$

The desired estimate (7-2) then follows by complex interpolation and linearizing σ_2.

To prove (7-6), note that

$$\sigma_2(H, x) \le 2 \fint_{Q_x} \Big(\sum_{|j+r(x)|>N} \|T_{u,j}H_j\|_{L^2(I)}^2 \Big)^{1/2} dy,$$

so

$$\sigma_2(H, x) \lesssim \fint_{Q_x} \|\|T_{u,j}H_j\|_{L^2(I)}\|_{\ell^2(\mathbb{Z})}\, dy \lesssim M\|\|T_{u,j}H_j\|_{L^2(I)}\|_{\ell^2(\mathbb{Z})}(x).$$

Hence

$$\|\sigma_2(H, x)\|_{L^2(\mathbb{R}^n)} \lesssim \|\|\|T_{u,j}H_j(x)\|_{L^2(I)}\|_{\ell^2(\mathbb{Z})}\|_{L^2(\mathbb{R}^n)}.$$

We commute the $\ell^2(\mathbb{Z})$ norm outside. Since

$$\|\|T_{u,j}H_j(x)\|_{L^2(I)}\|_{L^2(\mathbb{R}^n)} \lesssim A\|H_j(x)\|_{L^2(\mathbb{R}^n)}, \tag{7-8}$$

one can conclude that

$$\|\sigma_2(H, x)\|_{L^2(\mathbb{R}^n)} \lesssim A\|\|H_j(x)\|_{L^2(\mathbb{R}^n)}\|_{\ell^2(\mathbb{Z})},$$

which gives (7-6) upon a further change in the order of the norms on the right-hand side.

Now we proceed to prove (7-7). For each $x \in \mathbb{R}^n$, we take the decomposition

$$H_j(y) = (\chi_{2Q_x} H_j)(y) + (\chi_{(2Q_x)^c} H_j)(y)$$

for all $y \in \mathbb{R}^n$. Then we plug this back into the formula for $\sigma_2(H, x)$. We find that

$$\sigma_2(H, x) \lesssim \mathrm{I}(x) + \mathrm{II}(x),$$

where

$$\mathrm{I}(x) = \fint_{Q_x} \left(\sum_{j\in\mathbb{Z}} \| T_{u,j}(\chi_{2Q_x} H_j)(y) - [T_{u,j}(\chi_{2Q_x} H_j)]_{Q_j} \|_{L^2(I)}^2 \right)^{1/2} dy,$$

$$\mathrm{II}(x) = \fint_{Q_x} \left(\sum_{|j+r(x)|>N} \| T_{u,j}(\chi_{(2Q_x)^c} H_j)(y) - [T_{u,j}(\chi_{(2Q_x)^c} H_j)]_{Q_j} \|_{L^2(I)}^2 \right)^{1/2} dy.$$

We estimate $\mathrm{I}(x)$ by

$$\begin{aligned}
\mathrm{I}(x) &\le 2 \fint_{Q_x} \left(\sum_{j\in\mathbb{Z}} \| T_{u,j}(\chi_{2Q_x} H_j)(y) \|_{L^2(I)}^2 \right)^{1/2} dy \\
&\lesssim \left(\fint_{Q_x} \sum_{j\in\mathbb{Z}} \| T_{u,j}(\chi_{2Q_x} H_j)(y) \|_{L^2(I)}^2 \, dy \right)^{1/2} \\
&\lesssim \frac{1}{|Q_x|^{1/2}} \| \| \| T_{u,j}(\chi_{2Q_x} H_j)(y) \|_{L^2(I)} \|_{L^2(\mathbb{R}^n)} \|_{\ell^2(\mathbb{Z})} \\
&\lesssim \frac{A}{|Q_x|^{1/2}} \| \| (\chi_{2Q_x} H_j)(y) \|_{L^2(\mathbb{R}^n)} \|_{\ell^2(\mathbb{Z})},
\end{aligned}$$

where in the last inequality we used the estimate (7-8). Then

$$\mathrm{I}(x) \lesssim A \left(\fint_{2Q_x} \| H_j(y) \|_{\ell^2(\mathbb{Z})} \, dy \right)^{1/2} \lesssim A \sup_{y\in\mathbb{R}^n} \| H_j(y) \|_{\ell^2(\mathbb{Z})},$$

which shows that

$$\| \mathrm{I}(x) \|_{L^\infty(\mathbb{R}^n)} \lesssim A \| \| H_j(x) \|_{\ell^2(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)}.$$

Next we estimate $\mathrm{II}(x)$. Let $K_{u,j}$ be the convolution kernel of $T_{u,j}$. Then

$$K_{u,j}(x) = 2^{jn} K_u(2^j x),$$

where

$$K_u(x) = \int_{\mathbb{R}^n} \tilde{m}_u(\xi) e^{2\pi i x\cdot\xi} \, d\xi.$$

Now by our assumption on $\partial_\xi^\tau \tilde{m}_u(\xi)$, we have

$$\sup_{u\in I} |K_u(x)| + \sup_{u\in I} |\nabla_x K_u(x)| \lesssim \frac{B}{(1+|x|)^{n+1}}.$$

We claim now

$$\sup_{y,z\in Q_x} \left(\sum_{|j+r(x)|>N} \left(\int_{(2Q_x)^c} \sup_{u\in I} |K_{u,j}(y-w) - K_{u,j}(z-w)| \, dw \right)^2 \right)^{1/2} \lesssim B 2^{-N} \tag{7-9}$$

uniformly for $x \in \mathbb{R}^n$. Indeed, suppose $y, z \in Q_x$, and $j + r(x) > -N$. Then we have the estimate

$$\begin{aligned}\int_{(2Q_x)^c} \sup_{u\in I} |K_{u,j}(y-w) - K_{u,j}(z-w)|\,dw &\lesssim \int_{(2Q_x)^c} \sup_{u\in I} (|K_{u,j}(y-w)| + |K_{u,j}(z-w)|)\,dw \\ &\lesssim 2^{jn} \int_{|w-x|\gtrsim 2^{r(x)}} \frac{B}{(2^j|w-x|)^{n+1}}\,dw \lesssim B2^{-j-r(x)}.\end{aligned}$$

On the other hand, if $y, z \in Q_x$, and $j + r(x) < -N$, then

$$\int_{(2Q_x)^c} \sup_{u\in I} |K_{u,j}(y-w) - K_{u,j}(z-w)|\,dw$$

is bounded by a constant times

$$\int_0^1 \int_{(2Q_x)^c} \sup_{u\in I} |y-z|\,|\nabla_x K_{u,j}((1-t)y+tz-w)|\,dw\,dt \lesssim 2^j 2^{r(x)} \int_{\mathbb{R}^n} 2^{jn} |(\nabla_x K_u)(2^j w)|\,dw \lesssim B2^{j+r(x)}.$$

Summing over j such that $j + r(x) > N$ and $j + r(x) < -N$ respectively, we see that (7-9) follows.

Finally, it suffices to observe that

$$\begin{aligned}\mathrm{II}(x) &\lesssim \fint_{Q_x} \fint_{Q_x} \Bigg(\sum_{|j+r(x)|>N} \|T_{u,j}(\chi_{(2Q_x)^c} H_j)(y) - T_{u,j}(\chi_{(2Q_x)^c} H_j)(z)\|_{L^2(I)}^2 \Bigg)^{1/2} dy\,dz \\ &= \sup_{y,z\in Q_x} \Bigg(\sum_{|j+r(x)|>N} \bigg\| \int_{(2Q_x)^c} [K_{u,j}(y-w) - K_{u,j}(z-w)] H_j(w)\,dw \bigg\|_{L^2(I)}^2 \Bigg)^{1/2} \\ &\lesssim \sup_{y,z\in Q_x} \Bigg(\sum_{|j+r(x)|>N} \bigg(\int_{(2Q_x)^c} \|K_{u,j}(y-w) - K_{u,j}(z-w)\|_{L^2(I)}\,dw \bigg)^2 \Bigg)^{1/2} \| \|H_j\|_{\ell^\infty(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \sup_{y,z\in Q_x} \Bigg(\sum_{|j+r(x)|>N} \bigg(\int_{(2Q_x)^c} \sup_{u\in I} |K_{u,j}(y-w) - K_{u,j}(z-w)|\,dw \bigg)^2 \Bigg)^{1/2} \| \|H_j\|_{\ell^2(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Invoking (7-9) yields

$$\|\mathrm{II}(x)\|_{L^\infty(\mathbb{R}^n)} \lesssim B2^{-N} \| \|H_j(x)\|_{\ell^2(\mathbb{Z})} \|_{L^\infty(\mathbb{R}^n)},$$

which together with our earlier estimate about $\|\mathrm{I}(x)\|_{L^\infty(\mathbb{R}^n)}$ gives (7-7).

Proof of Lemma 7.1. The key is a relative distribution inequality. Fix the Banach space B. For any $n \geq 1$, we claim that there exists $b_n \in (0, 1)$ such that for any $b, c > 0$ with $b \leq b_n$ we have

$$|\{x \in \mathbb{R}^n : \mathcal{M}F(x) > \alpha,\, F^\sharp(x) \leq c\alpha\}| \lesssim_n c|\{x \in \mathbb{R}^n : \mathcal{M}F(x) > b\alpha\}|.$$

If this is true, then by taking c sufficiently small, we can use Lemma 2 of Chapter IV.3.5 of [Stein 1993] (see also the remark on the bottom of page 152 there) and conclude the proof of Lemma 7.1.

To prove the above relative distributional inequality, let $b \in (0, 1)$ first. Let $F \in L^p(\mathbb{R}^n, B)$. Decompose the open set $\{x \in \mathbb{R}^n : \mathcal{M}F(x) > b\alpha\}$ into an essentially disjoint union of Whitney cubes $\{Q\}$, so that the distance of each Q from the complement of this set is bounded by 4 times the diameter of Q. Now since

$\{x \in \mathbb{R}^n : \mathcal{M}F(x) > \alpha,\ F^\sharp(x) \le c\alpha\}$ is a subset of $\{x \in \mathbb{R}^n : \mathcal{M}F(x) > b\alpha\}$, we just need to show that for each Whitney cube Q as above, we have

$$|\{x \in Q : \mathcal{M}F(x) > \alpha,\ F^\sharp(x) \le c\alpha\}| \lesssim_n c|Q|.$$

This inequality would be trivial if the set on the left-hand side were empty. So let's assume there exists a point $x_0 \in Q$ such that $F^\sharp(x_0) \le c\alpha$. Now let $\widetilde{Q}$ be any cube that intersects Q and that has diameter at least that of Q. Then $20\widetilde{Q}$ will contain a point y where $\mathcal{M}F(y) \le b\alpha$. Hence $⨍_{\widetilde{Q}} |F|_B \le 20^n b\alpha$ for all such cubes $\widetilde{Q}$. If $x \in Q$ and $\mathcal{M}F(x) > \alpha$, then by taking $b < 20^{-n}$, we see that $\mathcal{M}(F\chi_{3Q})(x) > \alpha$. We also have $⨍_{3Q} |F|_B \le 20^n b\alpha$. Thus,

$$\{x \in Q : \mathcal{M}F(x) > \alpha,\ F^\sharp(x) \le c\alpha\} \subset \left\{x \in Q : \mathcal{M}\left(F\chi_{3Q} - ⨍_{3Q} F\right)(x) > (1 - 20^n b)\alpha\right\},$$

where the measure of the right-hand side is bounded by

$$\frac{C_n}{(1-20^n b)\alpha} \int_Q |F\chi_{3Q}(y) - F_{3Q}|_B \, dy \le \frac{C_n}{(1-20^n b)\alpha} |3Q| F^\sharp(x_0) \le \frac{3^n C_n}{(1-20^n b)} c|Q|,$$

where C_n is the constant arising in the weak-type (1,1) bound of $\mathcal{M} : L^1(\mathbb{R}^n, B) \to L^{1,\infty}(\mathbb{R}^n)$. This proves the desired relative distributional inequality. □

Appendix A: An improved local smoothing estimate

In this section we prove Theorem 1.6. Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative smooth bump function supported on $1 \le |\xi| \le 2$. Define

$$E_0 f(x,t) := \int_{\mathbb{R}^n} f(\xi)\chi(\xi) e^{ix\cdot\xi + it|\xi|^\gamma} \, d\xi.$$

We will prove

$$\|E_0 f\|_{L^p(\mathbb{R}^n \times [-\lambda,\lambda])} \lesssim \lambda^{n(1/2-1/p)+\epsilon} \|\hat{f}\|_{L^p(\mathbb{R}^n)} \tag{A-1}$$

for every $\lambda \ge 1$. Once this is proved, a rescaling argument shows that

$$\left(\int_{\mathbb{R}^n \times I} \left| \int_{\mathbb{R}^n} e^{ix\xi} \hat{f}(\xi) e^{it|\xi|^\gamma} \, d\xi \right|^p dx\, dt\right)^{1/p} \lesssim 2^{k\gamma n(1/2-1/p) - k\gamma/p + k\epsilon} \|f\|_{L^p(\mathbb{R}^n)} \tag{A-2}$$

whenever $\hat{f}(\xi)$ is supported on the k-th annulus, that is, $|\xi| \approx 2^k$. As a consequence we obtain Theorem 1.6.

We will prove (A-1) for every elliptic phase. Let c_0 be a small positive real number. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with

$$|\phi(\xi)| + |\nabla\phi(\xi)| \lesssim 1, \quad c_0 I_n \le (\nabla^2\phi)(\xi) \le \frac{1}{c_0} I_n \quad \text{for every } |\xi| < 10, \tag{A-3}$$

where I_n is the identity matrix of order $n \times n$. Let $\chi_0 : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative smooth bump function supported on $|\xi| \le 2$. Define

$$E_\phi f(x,t) := \int_{\mathbb{R}^n} f(\xi)\chi_0(\xi) e^{ix\cdot\xi + it\phi(\xi)} \, d\xi.$$

We will prove

$$\sup_{\phi:\text{(A-3)}} \|E_\phi f\|_{L^p(\mathbb{R}^n\times[-\lambda,\lambda])} \lesssim \lambda^{n(1/2-1/p)+\epsilon}\|\hat{f}\|_{L^p(\mathbb{R}^n)} \tag{A-4}$$

for every $\lambda \geq 1$.

For a ball $B_\lambda \subset \mathbb{R}^{n+1}$ of radius λ, we will let B_λ^- denote its projection in the first n variables, i.e., spatial variables. That is, B_λ^- is a ball of radius λ in $\mathbb{R}^n$. We also define the associated weight

$$w_{B_\lambda^-}(x) := \frac{1}{(1+\|x-c\|/\lambda)^{100n}} \quad \text{for } x \in \mathbb{R}^n.$$

Here c denotes the center of B_λ^-. We remark that in the argument below, various implicit constants depend on this choice of weight. However, this dependence is not important, and to avoid unnecessary technicalities, we will not make these details explicit. We refer the interested reader to [Li 2020], which contains all the necessary details that are required to run the Bourgain–Guth argument [2011].

We prove (A-4) by an inductive argument. Denote by Q_λ the smallest constant such that

$$\sup_{\phi:\text{(A-3)}} \|E_\phi f\|_{L^p(B_\lambda)} \leq Q_\lambda \cdot \lambda^{n(1/2-1/p)}\|\hat{f}\|_{L^p(w_{B_\lambda^-})}$$

for every ball $B_\lambda \subset \mathbb{R}^{n+1}$ with $B_\lambda \subset B_\lambda^- \times [-\lambda, \lambda]$. Of course our goal is to prove that

$$Q_\lambda \lesssim_\epsilon \lambda^\epsilon \tag{A-5}$$

for every $\epsilon > 0$. In the following, for the sake of simplicity, we will always abbreviate $E_\phi f$ to Ef.

First, by translation invariance, we will assume that B_λ^- is centered at 0. We normalize f such that

$$\|\hat{f}\|_{L^p(w_{B_\lambda^-})}\lambda^{n(1/2-1/p)} = 1. \tag{A-6}$$

Next, let K_n be a large integer that is to be determined, satisfying

$$K_n \ll \lambda^\epsilon.$$

For a large dyadic integer K, let Col_K denote the collection of all dyadic cubes of length $1/K$. We write

$$Ef = \sum_{\alpha_n \in \mathrm{Col}_{K_n}} Ef_{\alpha_n}, \quad \text{with } f_{\alpha_n} := f \cdot \mathbb{1}_{\alpha_n}.$$

Here $\{\mathbb{1}_{\alpha_n}\}_n$ forms a smooth partition of unity, and $\mathbb{1}_{\alpha_n}$ is supported on $2\alpha_n$. On every ball $B_{K_n} \subset \mathbb{R}^{n+1}$ of radius K_n, by the uncertainty principle, we know that $|Ef_{\alpha_n}|$ is essentially a constant for every $\alpha_n \in \mathrm{Col}_{K_n}$. We let $|Ef_{\alpha_n}|(B_{K_n})$ denote this constant. Denote by α_n^* the cube that maximizes

$$\{|Ef_{\alpha_n}|(B_{K_n})\}_{\alpha_n \in \mathrm{Col}_{K_n}}.$$

Consider the collection

$$\mathrm{Col}_{K_n}^* := \{\alpha_n \in \mathrm{Col}_{K_n} : |Ef_{\alpha_n}|(B_{K_n}) \geq K_n^{-n}|Ef_{\alpha_n^*}|(B_{K_n})\}.$$

Here the choice of the coefficient K_n^{-n} is not strict. One can also use K_n^{-2n} or something even smaller.

There are three cases.

Case 1: There exists an integer $1 \leq j \leq n$, and cubes $\alpha_n^{(1)}, \ldots, \alpha_n^{(j)} \in \mathrm{Col}_{K_n}^*$ which are $(1/K_{n-1})$-separated such that every cube within $\mathrm{Col}_{K_n}^*$ is in the $1/K_{n-1}$ neighborhood of some $\alpha_n^{(j')}$.

Here $K_{n-1} \ll K_n^{\epsilon}$ is also to be determined. Next, we have:

Case 2: There exist cubes $\alpha_n^{(1)}, \dots, \alpha_n^{(n+1)} \in \mathrm{Col}_{K_n}^*$ that are $(1/K_{n-1})$-separated and do not lie in the $100/K_n$ neighborhood of any $(n-1)$-dimensional subspace.

If Case 1 and Case 2 are not satisfied, then we have:

Case 3: All cubes in $\mathrm{Col}_{K_n}^*$ lie in the $C(K_{n-1})/K_n$ neighborhood of a subspace of dimension $n-1$.

Here $C(K_{n-1})$ is a large constant depending on K_{n-1} which may change from line to line (it always suffices to take, say, $C(K_{n-1}) = K_{n-1}^{100n}$).

We deal with these three cases separately. In Case 1, we have

$$|Ef| \lesssim \max_{\alpha_n \in \mathrm{Col}_{K_n}} |Ef_{\alpha_n}| + \max_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} |Ef_{\alpha_{n-1}}|. \tag{A-7}$$

In Case 2, we use

$$|Ef| \lesssim K_n^{2n} \Big(\prod_{j=1}^{n+1} |Ef_{\alpha_n^{(j)}}| \Big)^{1/(n+1)}.$$

In Case 3, we use

$$|Ef| \lesssim \max_{\alpha_n \in \mathrm{Col}_{K_n}} |Ef_{\alpha_n}| + \max_{\substack{L_{n-1}\text{: subspace} \\ \text{of dimension } (n-1)}} \Big| \sum_{\alpha_{n-1}:\mathrm{dist}(\alpha_{n-1}, L_{n-1}) \le 1/K_{n-1}} Ef_{\alpha_{n-1}} \Big|. \tag{A-8}$$

In the last summation, we implicitly assumed that $\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}$, and that

$$\frac{C(K_{n-1})}{K_n} \le \frac{1}{K_{n-1}}.$$

Here we agree upon a convention: for $1 \le j \le n$, whenever the symbol α_j appears, we always assume that $\alpha_j \in \mathrm{Col}_{K_j}$, to keep notation simpler. Combining (A-7)–(A-8), we obtain

$$\begin{aligned} |Ef| \lesssim \max_{\alpha_n \in \mathrm{Col}_{K_n}} |Ef_{\alpha_n}| + \max_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} |Ef_{\alpha_{n-1}}| + K_n^{2n} \Big(\prod_{j=1}^{n+1} |Ef_{\alpha_n^{(j)}}| \Big)^{1/(n+1)} \\ + \max_{\substack{L_{n-1}\text{: subspace} \\ \text{of dimension } (n-1)}} \Big| \sum_{\alpha_{n-1}:\mathrm{dist}(\alpha_{n-1}, L_{n-1}) \le 1/K_{n-1}} Ef_{\alpha_{n-1}} \Big|. \end{aligned}$$

We raise both sides to the p-th power, then integrate over B_{K_n}, and in the end sum over balls B_{K_n} inside B_λ,

$$\begin{aligned} \int_{B_\lambda} |Ef|^p \lesssim & \sum_{\alpha_n \in \mathrm{Col}_{K_n}} \int_{B_\lambda} |Ef_{\alpha_n}|^p + \sum_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} \int_{B_\lambda} |Ef_{\alpha_{n-1}}|^p \\ & + \sum_{\alpha_n^{(1)}, \dots, \alpha_n^{(n+1)} \text{ in Case 2}} \int_{B_\lambda} K_n^{2pn} \Big(\prod_{j=1}^{n+1} |Ef_{\alpha_n^{(j)}}| \Big)^{p/(n+1)} \\ & + \sum_{B_{K_n} \subset B_\lambda} \max_{L_{n-1}} \int_{B_{K_n}} \Big| \sum_{\alpha_{n-1}:\mathrm{dist}(\alpha_{n-1}, L_{n-1}) \le 1/K_{n-1}} Ef_{\alpha_{n-1}} \Big|^p. \end{aligned} \tag{A-9}$$

There are four terms on the right-hand side. It is the contribution from the last term that gives us the ultimate constraint for the exponent p, as stated in (1-5).

Let us be more precise. The first and second summands on the right-hand side of (A-9) can be taken care of by parabolic rescaling. We will deal with the third summand using multilinear restriction estimates due to [Bennett, Carbery, and Tao 2006]. The last term requires further careful analysis.

For the first and second summands, we will apply rescaling. The argument is the same in both cases. Hence we will only write down the rescaling argument for the first summand.

$$\int_{B_\lambda} |Ef_{\alpha_n}|^p = \int_{B_\lambda} \left| \int f_{\alpha_n}(\xi) e^{i\xi x + it\phi(\xi)} d\xi \right|^p dx\, dt.$$

Here we apply the change of variable

$$\xi \to \frac{\xi}{K_n} + c_{\alpha_n}, \quad \text{with } c_{\alpha_n} \text{ being the center of } \alpha_n.$$

We obtain

$$K_n^{-np} \int_{B_\lambda} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i(\xi/K_n + c_{\alpha_n})x + it\phi(\xi/K_n + c_{\alpha_n})} d\xi \right|^p dx\, dt$$
$$= K_n^{-np} \int_{B_\lambda} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i\xi/K_n x + iK_n^2 \phi(\xi/K_n + c_{\alpha_n})(t/K_n^2)} d\xi \right|^p dx\, dt.$$

Next we apply the change of variables

$$x/K_n \to x \quad \text{and} \quad t/K_n^2 \to t$$

to obtain

$$K_n^{-np+n+2} \int_{\widetilde{B}_\lambda} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i\xi x + it\cdot K_n^2 \phi(\xi/K_n + c_{\alpha_n})} d\xi \right|^p dx\, dt. \tag{A-10}$$

Here $\widetilde{B}_\lambda \subset \mathbb{R}^{n+1}$ is a rectangular box of dimensions $\lambda/K_n \times \cdots \times \lambda/K_n \times \lambda/K_n^2$ centered at 0. The reason of writing it in this form is that

$$\tilde{\phi}(\xi) := K_n^2 \phi\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) - K_n \langle (\nabla\phi)(c_{\alpha_n}), \xi\rangle - K_n^2 \phi(c_{\alpha_n}) \text{ still satisfies (A-3).}$$

By a change of variable, (A-10) can be bounded by

$$K_n^{-np+n+2} \int_{2\widetilde{B}_\lambda} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i\xi x + it\tilde{\phi}(\xi)} d\xi \right|^p dx\, dt.$$

Next we split the rectangular box $2\widetilde{B}_\lambda$ into a union of cubes of side-length λ/K_n^2.

$$K_n^{-np} K_n^{n+2} \sum_{B_{\lambda/K_n^2} \subset 2\widetilde{B}_\lambda} \int_{B_{\lambda/K_n^2}} \left| \int f_{\alpha_n}\left(\frac{\xi}{K_n} + c_{\alpha_n}\right) e^{i\xi x + it\tilde{\phi}(\xi)} d\xi \right|^p dx\, dt$$
$$\lesssim K_n^{-np} K_n^{n+2} \sum_{B^-_{\lambda/K_n^2} \subset B^-_{\lambda/K_n}} \left(\frac{\lambda}{K_n^2}\right)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} \int_{\mathbb{R}^n} \left| \widehat{f_{\alpha_n}\left(\frac{\cdot}{K_n} + c_{\alpha_n}\right)}(x) \right|^p w_{B^-_{\lambda/K_n^2}}(x)\, dx.$$

Here we applied the induction hypothesis. We may now sum the weights over all $B^-_{\lambda/K_n^2} \subset B^-_{\lambda/K_n}$, and bound the above by

$$\lesssim K_n^{-np} K_n^{n+2} \left(\frac{\lambda}{K_n^2}\right)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} \int_{\mathbb{R}^n} \left| \widehat{f_{\alpha_n}\left(\frac{\cdot}{K_n} + c_{\alpha_n}\right)}(x) \right|^p w_{B^-_{\lambda/K_n}}(x)\, dx$$
$$\lesssim K_n^{-np} K_n^{n+2} K_n^{np-n} \left(\frac{\lambda}{K_n^2}\right)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} \int_{\mathbb{R}^n} |\hat{f}_{\alpha_n}(\xi)|^p w_{B^-_\lambda}(x)\, dx$$

Summing over α_n, we obtain

$$K_n^{-np} K_n^{n+2} \left(\frac{\lambda}{K_n^2}\right)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} K_n^{-n} K_n^{np} \sum_{\alpha_n} \int_{\mathbb{R}^n} |\hat{f}_{\alpha_n}|^p w_{B^-_\lambda}$$
$$\lesssim K_n^{-np} K_n^{n+2} \left(\frac{1}{K_n^2}\right)^{np(1/2-1/p)} Q^p_{\lambda/K_n^2} K_n^{-n} K_n^{np}.$$

In the last step, we applied the normalization condition (A-6). Let the exponent of K_n be equal to zero and we obtain

$$2 - 2np\left(\frac{1}{2} - \frac{1}{p}\right) = 0 \quad \Longrightarrow \quad p = \frac{2(n+1)}{n}.$$

This is exactly the exponent in the Fourier restriction conjecture. Moreover, the last display tells us that, for the contribution from the first and second terms in (A-9), the induction can be closed whenever $p > 2(n+1)/n$.

Now we deal with the third summand on the right-hand side of (A-9). When $p \geq 2(n+1)/n$, by multilinear restriction of [Bennett, Carbery, and Tao 2006] and by Bernstein's inequality and Hölder's inequality,

$$\sum_{\alpha_n^{(1)},\ldots,\alpha_n^{(n+1)} \text{ in Case 2}} \int_{B_\lambda} K_n^{2n} \left(\prod_{j=1}^{n+1} |Ef_{\alpha_n^{(j)}}|\right)^{p/(n+1)} \lesssim K_n^{2n} K_n^{100n!} \lambda^\epsilon.$$

Again we see that there is no problem for this term as K_n can be chosen to be much smaller compared with λ^ϵ.

In the end, we come to the last summand on the right-hand side of (A-9). Fix a ball $B_{K_n} \subset \mathbb{R}^{n+1}$. Assume that the maximum is attained at the $(n-1)$-dimensional subspace L_{n-1}. We need to consider

$$\int_{B_{K_n}} \left| \sum_{\alpha_{n-1}:\operatorname{dist}(\alpha_{n-1}, L_{n-1}) \leq 1/K_{n-1}} Ef_{\alpha_{n-1}} \right|^p.$$

Notice that each $|Ef_{\alpha_{n-1}}|$ is essentially a constant on $B_{K_{n-1}}$, a ball of radius K_{n-1} which is much smaller compared with K_n^ϵ. Hence tentatively we fix a ball $B_{K_{n-1}} \subset B_{K_n}$. Let α^*_{n-1} denote the cube that maximizes

$$\{|Ef_{\alpha_{n-1}}|(B_{K_{n-1}})\}_{\alpha_{n-1}:\operatorname{dist}(\alpha_{n-1}, L_{n-1}) \leq 1/K_{n-1}}.$$

Consider the collection

$$\mathrm{Col}^*_{K_{n-1}} := \left\{ \alpha_{n-1} : \operatorname{dist}(\alpha_{n-1}, L_{n-1}) \leq \frac{1}{K_{n-1}} \text{ and } |Ef_{\alpha_{n-1}}|(B_{K_{n-1}}) \geq K_{n-1}^{-n} |Ef_{\alpha^*_{n-1}}|(B_{K_{n-1}}) \right\}.$$

There are three further cases:

Case 3.1: There exists an integer $1 \le j \le n-1$, and cubes $\alpha_{n-1}^{(1)}, \dots, \alpha_{n-1}^{(j)} \in \mathrm{Col}_{K_{n-1}}^*$ which are $(1/K_{n-2})$-separated such that every cube within $\mathrm{Col}_{K_{n-1}}^*$ is in the $1/K_{n-2}$ neighborhood of some $\alpha_{n-1}^{(j')}$.

Here $K_{n-2} \ll K_{n-1}^{\epsilon}$ is also to be determined. Moreover, we have

Case 3.2: There exist cubes $\alpha_{n-1}^{(1)}, \dots, \alpha_{n-1}^{(n)} \in \mathrm{Col}_{K_{n-1}}^*$ that are $(1/K_{n-2})$-separated and do not lie in the $100/K_{n-1}$ neighborhood of any $(n-2)$-dimensional subspace.

If the above two cases are not satisfied, then we must have

Case 3.3: All cubes in $\mathrm{Col}_{K_{n-1}}^*$ lie in the $C(K_{n-2})/K_{n-1}$ neighborhood of a linear subspace of dimension $n-2$.

Here $C(K_{n-2})$ is a large constant depending on K_{n-2}. It suffices to take $C(K_{n-2}) = K_{n-2}^{100n}$.

Similarly to (A-7)–(A-8), we have that for every point $(x,t) \in B_{K_{n-1}}$

$$|Ef| \lesssim \max_{\alpha_n \in \mathrm{Col}_{K_n}} |Ef_{\alpha_n}| + \max_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} |Ef_{\alpha_{n-1}}| + \max_{\alpha_{n-2} \in \mathrm{Col}_{K_{n-2}}} |Ef_{\alpha_{n-2}}| + K_{n-1}^{2n} \Big(\prod_{j=1}^{n} |Ef_{\alpha_{n-1}^{(j)}}| \Big)^{1/n} + \max_{L_{n-2}} \Big| \sum_{\alpha_{n-2}: \mathrm{dist}(\alpha_{n-2}, L_{n-2}) \le 1/K_{n-2}} Ef_{\alpha_{n-2}} \Big|.$$

We first raise both sides to the p-th power, integrate over $B_{K_{n-1}}$, and then sum over all balls $B_{K_{n-1}} \subset B_{K_n}$, and in the end sum over all balls $B_{K_n} \subset B_\lambda$:

$$\begin{aligned} \int_{B_\lambda} |Ef|^p \lesssim & \sum_{\alpha_n \in \mathrm{Col}_{K_n}} \int_{B_\lambda} |Ef_{\alpha_n}|^p + \sum_{\alpha_{n-1} \in \mathrm{Col}_{K_{n-1}}} \int_{B_\lambda} |Ef_{\alpha_{n-1}}|^p \\ & + \sum_{\alpha_{n-2} \in \mathrm{Col}_{K_{n-2}}} \int_{B_\lambda} |Ef_{\alpha_{n-2}}|^p + \sum_{\alpha_{n-1}^{(1)}, \dots, \alpha_{n-1}^{(n)} \text{ in Case 3.2}} \int_{B_\lambda} K_{n-1}^{2n} \Big(\prod_{j=1}^{n} |Ef_{\alpha_{n-1}^{(j)}}| \Big)^{p/n} \\ & + \sum_{B_{K_n} \subset B_\lambda} \sum_{B_{K_{n-1}} \subset B_{K_n}} \max_{L_{n-2}} \int_{B_{K_{n-1}}} \Big| \sum_{\alpha_{n-2}: \mathrm{dist}(\alpha_{n-2}, L_{n-2}) \le 1/K_{n-2}} Ef_{\alpha_{n-2}} \Big|^p. \end{aligned}$$

There are five terms on the right-hand side of the last display. By the same scaling argument as above, we can handle the first three summands. For the fourth summand, we again apply multilinear restrictions due to [Bennett, Carbery, and Tao 2006]. However, notice that we are applying an n-linear restriction estimate in $\mathbb{R}^{n+1}$. This will not give us the restriction exponent $2(n+1)/n$, but something larger. To be precise, we have

$$\sum_{\alpha_{n-1}^{(1)}, \dots, \alpha_{n-1}^{(n)} \text{ in Case 3.2}} \int_{B_\lambda} K_{n-1}^{2n} \Big(\prod_{j=1}^{n} |Ef_{\alpha_{n-1}^{(j)}}| \Big)^{p/n} \lesssim K_{n-1}^{2n} K_{n-1}^{100n!} \lambda^{\epsilon}$$

for every

$$p \ge \frac{2n}{n-1},$$

which we assume.

Hence it remains to handle the last summand

$$\sum_{B_{K_{n-1}}\subset B_\lambda} \max_{L_{n-2}} \int_{B_{K_{n-1}}} \Big| \sum_{\alpha_{n-2}:\mathrm{dist}(\alpha_{n-2},L_{n-2})\le 1/K_{n-2}} Ef_{\alpha_{n-2}} \Big|^p.$$

We repeat this iteration until we reach

$$\sum_{B_{K_{n-k}}\subset B_\lambda} \max_{L_{n-k-1}} \int_{B_{K_{n-k}}} \Big| \sum_{\alpha_{n-k-1}:\mathrm{dist}(\alpha_{n-k-1},L_{n-k-1})\le 1/K_{n-k-1}} Ef_{\alpha_{n-k-1}} \Big|^p, \tag{A-11}$$

where k is the largest positive integer such that

$$k \le \frac{n-1}{3}; \tag{A-12}$$

in other words,

$$k = \begin{cases} (n-3)/3 & \text{if } n \equiv 0 \pmod 3, \\ (n-1)/3 & \text{if } n \equiv 1 \pmod 3, \\ (n-2)/3 & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{A-13}$$

Collecting all the constraints on the exponent p from applying the multilinear restriction estimate, we obtain

$$p \ge \frac{2(n-k+1)}{n-k}. \tag{A-14}$$

Instead of running the previous argument again on (A-11), we apply the decoupling inequalities of [Bourgain and Demeter 2015]:

$$\int_{B_{K_{n-k}}} \Big| \sum_{\alpha_{n-k-1}:\mathrm{dist}(\alpha_{n-k-1},L_{n-k-1})\le 1/K_{n-k-1}} Ef_{\alpha_{n-k-1}} \Big|^p \lesssim (K_{n-k-1})^{(n-k-1)(1/2-1/p)p+\epsilon} \sum_{\alpha_{n-k-1}} \int_{B_{K_{n-k}}} |Ef_{\alpha_{n-k-1}}|^p.$$

The above inequality will hold as long as

$$p \le \frac{2(n-k+1)}{n-k-1}. \tag{A-15}$$

In the end, we sum over all balls $B_{K_{n-k}}$ inside B_λ, and obtain

$$(K_{n-k-1})^{(n-k-1)(1/2-1/p)p+\epsilon} \sum_{\alpha_{n-k-1}} \int_{B_\lambda} |Ef_{\alpha_{n-k-1}}|^p.$$

It is clear now that we should apply parabolic rescaling. This gives us

$$(K_{n-k-1})^{(n-k-1)(1/2-1/p)p+\epsilon} \times (K_{n-k-1})^{2-2np(1/2-1/p)} \times Q^p_{\lambda/K^2_{n-k-1}}.$$

By equating the exponent of K_{n-k-1} with zero we obtain the constraint

$$p > \frac{2(n+k+3)}{n+k+1}. \tag{A-16}$$

This constraint is more restrictive than (A-14), by condition (A-12). By choosing k as in (A-13), and substituting that into (A-16), we obtain the constraint on p in (1-5). To summarize, we have shown that for p satisfying (1-5) and (A-15), we can close the induction. Thus, we have established (A-5) and therefore

(1-6) for such p's. Finally, by the result of [Rogers and Seeger 2010], we already know that (1-6) holds for all $p > 2+4/(n+1)$ (even with $\epsilon = 0$), which completes the proof of Theorem 1.6 for the claimed range in p.

Appendix B: A maximal multifrequency estimate of Krause and Lacey

Fix $\ell_0 \in \mathbb{Z}$ and $d \in \mathbb{N}$ with $d \geq 2$. Take the decomposition $1 = \sum_{\ell \in \mathbb{Z}} \varphi_\ell(t)$, where $\varphi_\ell(t) := \varphi_0(2^{-\ell}t)$ for some smooth even function φ_0 supported on $|t| \simeq 1$. For $u > 0$, let

$$T^{(u)} f(x) = \sum_{\ell \geq \ell_0} T_\ell^{(u)} f(x),$$

where

$$T_\ell^{(u)} f(x) := \int_{\mathbb{R}} f(x-t)\varphi_\ell(t) e^{iut^d} \frac{dt}{t}.$$

Following the method in [Bourgain 1989], we will prove the following maximal multifrequency estimate.

Theorem B.1. *There exists a constant C such that for any $\tau > 0$, $M \in \mathbb{N}$ and any $\theta_1, \ldots, \theta_M \in \mathbb{R}$ with $\min_{1 \leq i < j \leq M} |\theta_i - \theta_j| > 2\tau$, we have*

$$\left\| \sup_{u>0} \left| \sum_{m=1}^{M} \mathrm{Mod}_{\theta_m} T^{(u)} P_\tau f_m \right| \right\|_{L^2(\mathbb{R})} \leq C(\log M)^2 \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \right)^{1/2} \tag{B-1}$$

for all $f_1, \ldots, f_M \in L^2(\mathbb{R})$, where P_τ is the Littlewood–Paley projection onto the frequency interval $[-\tau, \tau]$.

Here $\mathrm{Mod}_\theta f(x)$ is the modulation $\mathrm{Mod}_\theta f(x) := e^{i\theta x} f(x)$.

As a corollary, we obtain Theorem 3.5 of [Krause and Lacey 2017]:

Corollary B.2 [Krause and Lacey 2017]. *There exists a constant C such that for any $\tau > 0$, $M \in \mathbb{N}$ and any $\theta_1, \ldots, \theta_M \in \mathbb{R}$ with $\min_{1 \leq i < j \leq M} |\theta_i - \theta_j| > 2\tau$, we have*

$$\left\| \sup_{u>0} \left| \sum_{m=1}^{M} \mathrm{Mod}_{\theta_m} T^{(u)} (P_\tau \, \mathrm{Mod}_{-\theta_m} f) \right| \right\|_{L^2(\mathbb{R})} \leq C(\log M)^2 \|f\|_{L^2(\mathbb{R})}$$

for any $f \in L^2(\mathbb{R})$.

Indeed, one can obtain the corollary by applying Theorem B.1 to $f_m := P_\tau \, \mathrm{Mod}_{-\theta_m} f$, and noting that then $\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \leq \|f\|_{L^2(\mathbb{R})}^2$.

The corollary is slightly stronger than Theorem 3.5 of [Krause and Lacey 2017] because it allows one to take supremum over all $u > 0$ (not just over $u \in (0, \tau^2)$).

To prove Theorem B.1, we use the following variant of our Theorem 1.1.

Theorem B.3. *There exists a constant C such that for all $p \in (2, 3)$ we have*

$$\|V^p\{T^{(u)} f : u > 0\}\|_{L^p(\mathbb{R})} \leq C(p-2)^{-1} \|f\|_{L^p(\mathbb{R})}. \tag{B-2}$$

Stein and Wainger proved that

$$\|\sup\{T^{(u)} f : u > 0\}\|_{L^q(\mathbb{R})} \leq C_q \|f\|_{L^q(\mathbb{R})} \tag{B-3}$$

for $1 < q < \infty$. By complex interpolation, we then get the following corollary:

Corollary B.4. *There exists a constant C such that for all $r \in (2,3)$ we have*

$$\|V^r\{T^{(u)}f : u>0\}\|_{L^2(\mathbb{R})} \le C(r-2)^{-1}\|f\|_{L^2(\mathbb{R})}. \tag{B-4}$$

Indeed, for any $r \in (2,\infty)$, one can obtain (B-4) by interpolating between (B-2) with $p=(r+6)/4$, and (B-3) with $q=\frac{3}{2}$.

Below we first prove Theorem B.3, and then use Corollary B.4 to prove Theorem B.1.

Proof of Theorem B.3. By the argument in Section 3, we have

$$\sup_{\lambda>0}\|\lambda\sqrt{N_\lambda\{T^{(2^{kd})}f : k\in\mathbb{Z}\}}\|_{L^p(\mathbb{R})} \le C_p\|f\|_{L^p(\mathbb{R})}$$

for $1<p<\infty$. By the real interpolation argument in Lemma 3.3 of [Bourgain 1989] (see also Lemma 2.1 of [Jones, Seeger, and Wright 2008]), we have

$$\|V^p\{T^{(2^{kd})}f : k\in\mathbb{Z}\}\|_{L^p(\mathbb{R})} \le C(p-2)^{-1}\|f\|_{L^p(\mathbb{R})} \tag{B-5}$$

for all $2<p<3$. Furthermore, by the argument in Section 4, we have

$$\|V_j^pT^{(u)}f\|_{\ell^p(j\in\mathbb{Z})}\|_{L^p(\mathbb{R})} \le C(p-2)^{-1}\|f\|_{L^p(\mathbb{R})} \tag{B-6}$$

for all $2<p<3$, where $V_j^pT^{(u)}f(x) := V^p\{T^{(u)}f(x) : u\in[2^{jd},2^{(j+1)d}]\}$. Indeed, the left-hand side above is bounded by

$$\sum_{k,\ell\in\mathbb{Z}}\|\|V_j^pT^{(u)}_{\ell-j}P_{j+k}f\|_{\ell^p\{j:\ell-j\ge\ell_0\}}\|_{L^p(\mathbb{R})},$$

and the arguments of Sections 4B and 4E show that

$$\sum_{\substack{k,\ell\in\mathbb{Z}\\ \ell\le -k/(2(d+1))}}\|\|V_j^pT^{(u)}_{\ell-j}P_{j+k}f\|_{\ell_j^p}\|_{L^p(\mathbb{R})} \le C_p\|f\|_{L^p(\mathbb{R})} \tag{B-7}$$

for $1<p<\infty$, and

$$\sum_{\substack{k,\ell\in\mathbb{Z}\\ \ell>-k/(2(d+1)),\,k>\ell(d-1)+C}} + \sum_{\substack{k,\ell\in\mathbb{Z}\\ \ell>-k/(2(d+1)),\,k<\ell(d-1)-C}}\|\|V_j^pT^{(u)}_{\ell-j}P_{j+k}f\|_{\ell_j^p}\|_{L^p(\mathbb{R})} \le C_p\|f\|_{L^p(\mathbb{R})} \tag{B-8}$$

for $2\le p<\infty$, where the constants C_p satisfy $\sup_{2\le p\le 3}C_p<\infty$. Furthermore, the arguments in Section 4D show that there exist absolute constants C and $\delta>0$ such that if $\ell\ge 0$ and $k=\ell(d-1)+O(1)$, then

$$\|\|V_j^pT^{(u)}_{\ell-j}P_{j+k}f\|_{\ell_j^p}\|_{L^p(\mathbb{R})} \le C2^{-\ell\delta d(p-2)}\|f\|_{L^p(\mathbb{R})}$$

for all $2<p<3$. Summing these up, we get

$$\sum_{\ell\ge0}\sum_{k=\ell(d-1)+O(1)}\|\|V_j^pT^{(u)}_{\ell-j}P_{j+k}f\|_{\ell_j^p}\|_{L^p(\mathbb{R})} \le C(p-2)^{-1}\|f\|_{L^p(\mathbb{R})} \tag{B-9}$$

for all $2<p<3$. Inequality (B-6) then follows from (B-7), (B-8) and (B-9). Since

$$V^p\{T^{(u)}f : u>0\} \le V^p\{T^{(2^{kd})}f : k\in\mathbb{Z}\} + \|V_j^pT^{(u)}f\|_{\ell^p(j\in\mathbb{Z})},$$

we obtain the desired conclusion (B-2) from (B-5) and (B-6). □

Proof of Theorem B.1. We will deduce Theorem B.1 from Theorem B.3, following [Bourgain 1989] closely (see Lemma 4.13 there). Suppose $\tau > 0$, and $\theta_1, \dots, \theta_M \in \mathbb{R}$ are such that $\min_{1\le i<j\le M} |\theta_i - \theta_j| > 2\tau$. First, to prove (B-1), it will suffice to show that

$$\left(\fint_{w\in[0,1/(100\tau)]} \left\| \sup_{u>0} \left| \sum_{m=1}^{M} e^{i\theta_m x} T^{(u)} P_\tau f_m(x+w) \right| \right\|_{L^2(\mathbb{R})}^2 dw \right)^{1/2} \le C(\log M)^2 \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \right)^{1/2}. \quad \text{(B-10)}$$

Morally speaking, this is the uncertainty principle at work: note that for every $u > 0$, the function $x \mapsto T^{(u)} P_\tau f_m(x)$ has Fourier support contained in $[-\tau, \tau]$, and hence $|\operatorname{Mod}_{\theta_m} T^{(u)} P_\tau f_m(x)|$ can be thought of as locally constant on an interval of length $\simeq 1/\tau$. To be precise, for each $w \in [0, 1/(100\tau)]$, we have, by Plancherel's identity, that

$$\|P_\tau f_m(\cdot) - P_\tau f_m(\cdot + w)\|_{L^2} \le \tfrac{1}{2} \|f_m\|_{L^2}$$

whenever $w \in [0, 1/(100\tau)]$. Thus if B is the best constant for which (B-1) holds, then for all $w \in [0, 1/(100\tau)]$, we have

$$\left\| \sup_{u>0} \left| \sum_{m=1}^{M} \operatorname{Mod}_{\theta_m} T^{(u)} P_\tau f_m(x) \right| \right\|_{L^2(\mathbb{R})} \le \left\| \sup_{u>0} \left| \sum_{m=1}^{M} e^{i\theta_m x} T^{(u)} P_\tau f_m(x+w) \right| \right\|_{L^2(\mathbb{R})} + \frac{B}{2} \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \right)^{1/2},$$

so taking L^2 average over all $w \in [0, 1/(100\tau)]$, and using (B-10), we have

$$B \le C(\log M)^2 + \frac{B}{2};$$

i.e., $B \le 2C(\log M)^2$ as desired. Thus, it remains to establish (B-10), which can be rewritten as

$$\left\| \left(\fint_{w\in[0,1/(100\tau)]} \sup_{u>0} \left| \sum_{m=1}^{M} e^{-i\theta_m w} e^{i\theta_m x} T^{(u)} P_\tau f_m(x) \right|^2 dw \right)^{1/2} \right\|_{L^2(\mathbb{R})} \le C(\log M)^2 \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2 \right)^{1/2} \quad \text{(B-11)}$$

by first changing variable $x \mapsto x - w$, and then interchanging the integrals in x and w.

To prove (B-11), for each $x \in \mathbb{R}$, consider the (bounded) set $A_x \subset \mathbb{R}^M$, given by

$$A_x := \{(T^{(u)} P_\tau f_1(x), \dots, T^{(u)} P_\tau f_M(x)) : u > 0\}.$$

If λ is bigger than the diameter of A_x, let $E_\lambda(x) = 0$; otherwise let $E_\lambda(x)$ be the minimal number of balls in $\mathbb{R}^M$ of radius λ that is required to cover A_x. ($E_\lambda(x)$ is sometimes called the entropy number.) One then observes that for every $s \in \mathbb{Z}$, there exists a finite subset $B_s(x) \subset A_x - A_x$ of cardinality at most $E_{2^s}(x)$ such that

$$|b_s| \le 2^{s+1} \quad \text{for every } b_s \in B_s(x)$$

and such that every element a of A_x admits a decomposition

$$a = \sum_{s\in\mathbb{Z}} b_s \quad \text{with } b_s \in B_s(x) \text{ for every } s \in \mathbb{Z}.$$

Then the left-hand side of (B-11) is bounded by

$$\Big\| \sum_{s\in\mathbb{Z}} \max_{b_s\in B_s(x)} \Big(\fint_{w\in[0,1/(100\tau)]} \Big| \sum_{m=1}^{M} e^{-i\theta_m w} e^{i\theta_m x} b_{s,m} \Big|^2 \, dw \Big)^{1/2} \Big\|_{L^2(\mathbb{R})};$$

here $b_s = (b_{s,1}, \dots, b_{s,m})$. Using Cauchy–Schwarz for the sum over m, the above displayed equation is further bounded by

$$\Big\| \sum_{s\in\mathbb{Z}} \min\Big\{ 2^{s+1} M^{1/2}, \Big(\sum_{b_s\in B_s(x)} \fint_{w\in[0,1/(100\tau)]} \Big| \sum_{m=1}^{M} e^{-i\theta_m w} e^{i\theta_m x} b_{s,m} \Big|^2 \, dw \Big)^{1/2} \Big\} \Big\|_{L^2(\mathbb{R})}. \tag{B-12}$$

To estimate the integral in w above, observe that from the separation of the $\theta_1, \dots, \theta_M$, we have

$$\Big(\fint_{w\in[0,1/(100\tau)]} \Big| \sum_{m=1}^{M} e^{-i\theta_m w} c_m \Big|^2 \, dw \Big)^{1/2} \lesssim \Big(\sum_{m=1}^{M} |c_m|^2 \Big)^{1/2};$$

indeed, the key is that if $\mathfrak{h} : \mathbb{R}^M \to \mathbb{R}^M$ is defined by

$$(\mathfrak{h} a)_m := \sum_{m=1}^{M} \frac{\tau}{\theta_m - \theta_n} a_n,$$

then the operator norm of $\mathfrak{h}$ is bounded independent of M, which can be deduced, for instance, by comparing it to the (continuous) Hilbert transform on $\mathbb{R}$. Thus (B-12) is bounded by

$$\Big\| \sum_{s\in\mathbb{Z}} \min\{2^{s+1} M^{1/2}, 2^{s+1} E_{2^s}(x)^{1/2}\} \Big\|_{L^2(\mathbb{R})} = 2 \Big\| \sum_{s\in\mathbb{Z}} 2^s \min\{M^{1/2}, E_{2^s}(x)^{1/2}\} \Big\|_{L^2(\mathbb{R})}. \tag{B-13}$$

Now let

$$F(x) := \Big(\sum_{m=1}^{M} \sup_{u>0} |T^{(u)} P_\tau f_m(x)|^2 \Big)^{1/2}.$$

Then the diameter of A_x is at most $2F(x)$. Hence the entropy number satisfies $E_{2^s}(x) = 0$ whenever $2^s > 2F(x)$. We are then led to sum

$$\sum_{2^s \le 2F(x)} 2^s \min\{M^{1/2}, E_{2^s}(x)^{1/2}\}.$$

We split this sum into two, one where $2^s \le M^{-1/2} F(x)$, and another where $M^{-1/2} F(x) \le 2^s \le 2F(x)$. The former sum is bounded by $F(x)$, while the latter sum is bounded by $(\log M) M^{1/2-1/r} \sup_{s\in\mathbb{Z}} 2^s E_{2^s}(x)^{1/r}$ for any $r \in (2, \infty)$. Now pick $r \in (2, \infty)$ such that

$$\frac{1}{2} - \frac{1}{r} = (\log M)^{-1},$$

so that $M^{1/2-1/r} \simeq 1$. Then (B-13) is bounded by

$$\big\| F(x) + (\log M) \sup_{s\in\mathbb{Z}} 2^s E_{2^s}(x)^{1/r} \big\|_{L^2(\mathbb{R})}.$$

But by Stein and Wainger's inequality (B-3), we have

$$\|F\|_{L^2(\mathbb{R})} \lesssim \left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2\right)^{1/2}.$$

Furthermore, one can relate the entropy $E_{2^s}(x)^{1/r}$ with the r-th variation norm pointwise:

$$\sup_{s\in\mathbb{Z}} 2^s E_{2^s}(x)^{1/r} \le \left(\sum_{m=1}^{M} |V^r\{T^{(u)} P_\tau f_m(x) : u > 0\}|^2\right)^{1/2}.$$

Hence

$$\Big\|(\log M)\sup_{s\in\mathbb{Z}} 2^s E_{2^s}(x)^{1/r}\Big\|_{L^2(\mathbb{R})} \le (\log M)\left(\sum_{m=1}^{M} \|V^r\{T^{(u)} P_\tau f_m : u > 0\}\|_{L^2(\mathbb{R})}^2\right)^{1/2}.$$

By Corollary B.4, the latter is bounded by

$$C(\log M)(r-2)^{-1}\left(\sum_{m=1}^{M} \|f_m\|_{L^2(\mathbb{R})}^2\right)^{1/2},$$

and since $(r-2)^{-1} \simeq \log M$ by our choice of r, this completes the proof of Theorem B.1. □

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