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FEDERER'S CHARACTERIZATION OF SETS OF FINITE PERIMETER IN METRIC SPACES

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Federer's characterization of sets of finite perimeter states (in Euclidean spaces) that a set is of finite perimeter if and only if the measure-theoretic boundary of the set has finite Hausdorff measure of codimension 1. In complete metric spaces that are equipped with a doubling measure and support a Poincaré inequality, the "only if" direction was shown by Ambrosio (2002). By applying fine potential theory in the case $p = 1$, we prove that the "if" direction holds as well.

1. Introduction

In the past two decades, there has been great interest in studying problems of first-order analysis in the setting of general metric measure spaces; see, e.g., [Ambrosio 2002; Ambrosio et al. 2004; Björn and Björn 2011; Heinonen and Koskela 1998; Miranda 2003; Shanmugalingam 2000]. In particular, Sobolev functions (sometimes called Newton–Sobolev functions in the metric setting) and functions of bounded variation (BV functions) have been topics of central interest. In much of the literature (as well as in the current paper) one assumes that the space is complete, equipped with a doubling measure, and supports a Poincaré inequality; see Section 2 for definitions. Studying questions in such an abstract setting provides an opportunity to unify the theories developed in specific settings such as weighted Euclidean spaces, Riemannian manifolds, Carnot groups, etc. Moreover, without having the Euclidean structure available, one is forced to develop novel methods and proofs, giving new insight into various problems.

In the theory of BV functions in the Euclidean setting, a key result originally due to De Giorgi states that if E is a set of finite perimeter, then the perimeter measure $P(E,\cdot)$ coincides with the $(n-1)$-dimensional Hausdorff measure restricted to the measure-theoretic boundary $\partial^* E$. In particular, $P(E,\mathbb{R}^n)<\infty$ implies $\mathcal{H}^{n-1}(\partial^* E)<\infty$. By a deep result of [Federer 1969, Section 4.5.11], the converse holds as well, so in fact $P(E,\mathbb{R}^n)<\infty$ if and only if $\mathcal{H}^{n-1}(\partial^* E)<\infty$. This is known as Federer's characterization of sets of finite perimeter. In the metric setting, where it is natural to formulate this kind of result by means of the *codimension-1* Hausdorff measure $\mathcal{H}$, the "only if" direction of the characterization was shown in [Ambrosio 2002], but the "if" direction has remained an open problem. In the current paper, we show that this direction holds as well.

Theorem 1.1. *Let $\Omega\subset X$ be an open set, let $E\subset X$ be a μ-measurable set, and suppose $\mathcal{H}(\partial^* E\cap\Omega)<\infty$. Then $P(E,\Omega)<\infty$.*

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The "only if" direction of Federer's characterization is part of a more general structure theorem for sets of finite perimeter, which in the metric setting states that the perimeter measure is comparable to the Hausdorff measure of codimension 1 restricted to the measure-theoretic boundary. This structure theorem is an indispensable tool in analysis of sets of finite perimeter, and hence more general BV functions as well. While not equally essential, the "if" direction of Federer's characterization has a number of applications as well. For example, in [Kinnunen et al. 2012] the authors proved a characterization of Newton–Sobolev functions with zero boundary values by means of a natural Lebesgue point-type condition on the boundary. However, the proof relied on assuming that Federer's characterization holds; now we know that this is the case under the usual assumptions on the space. We will discuss other applications in Section 5.

Previously there have been some partial results toward a proof of the "if" direction. The paper [Korte et al. 2015] showed that if the metric space is assumed to contain a "thick" bundle of curves between each pair of points, then the "if" direction can be proved by mimicking the Euclidean proof. In the current paper we take a completely different approach, which relies on *fine potential theory*. In the case $1 < p < \infty$, fine potential theory deals with superharmonic functions as understood by means of the *fine topology*; see [Adams and Hedberg 1996; Heinonen et al. 1993; Malý and Ziemer 1997] for the theory and its history in the Euclidean setting, and the recent papers [Björn and Björn 2015; Björn et al. 2015; 2016; 2018] for similar results in the metric setting. In [Lahti 2017a], the author proved some analogous results in the case $p = 1$, by relying on certain continuity properties of BV functions proved earlier in [Lahti 2017b; Lahti and Shanmugalingam 2017]. An application of these results led to the following characterization of sets of finite perimeter, which is in the same vein as Federer's characterization. Below, $\partial^1 I_E$ denotes the fine boundary of E, or more precisely of its measure-theoretic interior; one always has $\partial^* E \subset \partial^1 I_E$.

Theorem 1.2 [Lahti 2017a, Theorem 1.1]. *For an open set $\Omega \subset X$ and a μ-measurable set $E \subset X$, we have $P(E, \Omega) < \infty$ if and only if $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$. Furthermore, $\mathcal{H}((\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0$.*

In the current paper, our main goal is to show that if $\mathcal{H}(\partial^* E \cap \Omega) < \infty$, then $\mathcal{H}((\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0$ and thus Theorem 1.1 follows from Theorem 1.2. The proofs will be given in Section 4, and they rely mostly on properties of the 1-fine topology proved in [Lahti 2017a; 2020], as well as boxing inequality-type arguments. Our methods and the underlying theory should be of interest already in Euclidean spaces, where Federer's original argument has remained (as far as we know) essentially the only known proof for the characterization.

2. Preliminaries

In this section we introduce the standard definitions, notation, and assumptions used in the paper.

Throughout this paper, (X, d, μ) is a complete metric space that is equipped with a metric d and a Borel regular outer measure μ satisfying a doubling property, meaning that there exists a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B(x,r) := \{y \in X : d(y,x) < r\}$. We assume that X consists of at least two points. Given a ball $B = B(x,r)$ and $\beta > 0$, we sometimes abbreviate $\beta B := B(x, \beta r)$. Note that in metric spaces, a ball (as a set) does not necessarily have a unique center point and radius, but we understand these to be prescribed for all balls that we consider. When we want to state that a constant C depends on the parameters $a, b, \dots$, we write $C = C(a, b, \dots)$.

All functions defined on X or its subsets will take values in $[-\infty, \infty]$. A complete metric space equipped with a doubling measure is proper; that is, closed and bounded sets are compact. For any open set $\Omega \subset X$, we define $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ as the set of functions that are in the class $\mathrm{Lip}(\Omega')$ for every open $\Omega' \Subset \Omega$; here $\Omega' \Subset \Omega$ means that $\overline{\Omega}'$ is a compact subset of Ω. Other local function spaces are defined analogously.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined as

$$\mathcal{H}_R(A) := \inf\left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \le R \right\}.$$

The codimension-1 Hausdorff measure of $A \subset X$ is then defined as

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$

In the Euclidean space $\mathbb{R}^n$ (which we always equip with the Euclidean metric and the Lebesgue measure $\mathcal{L}^n$, unless otherwise specified) this is comparable to the $(n-1)$-dimensional Hausdorff measure.

By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval of the real line into X. A nonnegative Borel function g on X is an upper gradient of a function u on X if for all curves γ, we have

$$|u(x) - u(y)| \le \int_\gamma g\, ds, \tag{2.1}$$

where x and y are the endpoints of γ and the curve integral is defined by using an arc-length parametrization; see [Heinonen and Koskela 1998, Section 2], where upper gradients were originally introduced. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite. An upper gradient of a function $u \in C^1(\mathbb{R}^n)$ is given by $|\nabla u|$.

Let $1 \le p < \infty$ (we will work almost exclusively with $p = 1$). The p-modulus of a family of curves Γ is defined as

$$\mathrm{Mod}_p(\Gamma) := \inf \int_X \rho^p\, d\mu,$$

where the infimum is taken over all admissible test functions ρ, which are nonnegative Borel functions such that $\int_\gamma \rho\, ds \ge 1$ for every $\gamma \in \Gamma$. A property is said to hold for p-almost every curve if it fails only for a curve family with zero p-modulus. If g is a nonnegative μ-measurable function on X and (2.1) holds for p-almost every curve, we say that g is a p-weak upper gradient of u. By only considering curves γ in a set $A \subset X$, we can talk about a function g being a (p-weak) upper gradient of u in A.

Given an open set $\Omega \subset X$, we let

$$\|u\|_{N^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \inf \|g\|_{L^p(\Omega)},$$

where the infimum is taken over all p-weak upper gradients g of u in Ω. The Newton–Sobolev space is defined as

$$N^{1,p}(\Omega) := \{u : \|u\|_{N^{1,p}(\Omega)} < \infty\}.$$

In $\mathbb{R}^n$ this coincides, up to a choice of pointwise representatives, with the usual Sobolev space $W^{1,p}(\Omega)$; this is shown in Theorem 4.5 of [Shanmugalingam 2000], where the Newton–Sobolev space was originally introduced.

We understand a Newton–Sobolev function to be defined at every $x \in \Omega$ (even though $\|\cdot\|_{N^{1,p}(\Omega)}$ is then only a seminorm). It is known that for any $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ there exists a minimal p-weak upper gradient of u in Ω, always denoted by g_u, satisfying $g_u \le g$ almost everywhere in Ω, for any p-weak upper gradient $g \in L^p_{\mathrm{loc}}(\Omega)$ of u in Ω; see [Björn and Björn 2011, Theorem 2.25]. In $\mathbb{R}^n$, the minimal p-weak upper gradient coincides (a.e.) with $|\nabla u|$; see [Björn and Björn 2011, Corollary A.4].

The space of Newton–Sobolev functions with zero boundary values is defined as

$$N^{1,p}_0(\Omega) := \{u|_\Omega : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus \Omega\}.$$

This class can be understood to be a subclass of $N^{1,p}(X)$ in a natural way.

The p-capacity of a set $A \subset X$ is defined as

$$\mathrm{Cap}_p(A) := \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u \ge 1$ in A.

The variational 1-capacity of a set $A \subset \Omega$ with respect to an open set $\Omega \subset X$ is defined as

$$\mathrm{cap}_1(A, \Omega) := \inf \int_X g_u \, d\mu,$$

where the infimum is taken over functions $u \in N^{1,1}_0(\Omega)$ such that $u \ge 1$ on A, and g_u is the minimal 1-weak upper gradient of u (in X). For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see [Björn and Björn 2011].

We will assume throughout the paper that X supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \ge 1$ such that for every ball $B(x,r)$, every $u \in L^1_{\mathrm{loc}}(X)$, and every upper gradient g of u, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \le C_P r \fint_{B(x,\lambda r)} g \, d\mu,$$

where

$$u_{B(x,r)} := \fint_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

The standard example of a complete metric space equipped with a doubling measure and supporting a $(1,1)$-Poincaré inequality is the (unweighted) Euclidean space. Other examples include certain weighted Euclidean spaces (see, e.g., [Heinonen et al. 1993, Section 15]), complete Riemannian manifolds with nonnegative Ricci curvature (see [Saloff-Coste 2002, Section 3.3.5]), as well as Carnot groups which we will discuss briefly in Section 5.

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [Miranda 2003]. See also, e.g., [Ambrosio et al. 2000; Evans and Gariepy 1992; Federer 1969; Giusti 1984; Ziemer 1989] for the classical theory in the Euclidean setting. Let $\Omega \subset X$ be an open set. Given a function $u \in L^1_{\mathrm{loc}}(\Omega)$, we define the total variation of u in Ω as

$$\|Du\|(\Omega) := \inf\left\{\liminf_{i\to\infty} \int_\Omega g_{u_i}\, d\mu : u_i \in N^{1,1}_{\mathrm{loc}}(\Omega),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\},$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i in Ω. In $\mathbb{R}^n$ this agrees with the usual Euclidean definition involving distributional derivatives; see, e.g., [Ambrosio et al. 2000, Proposition 3.6, Theorem 3.9]. (In [Miranda 2003], local Lipschitz constants were used instead of upper gradients, but the properties of the total variation can be proved similarly with either definition.) We say that a function $u \in L^1(\Omega)$ is of bounded variation, and write $u \in \mathrm{BV}(\Omega)$ if $\|Du\|(\Omega) < \infty$. For an arbitrary set $A \subset X$, we define

$$\|Du\|(A) := \inf\{\|Du\|(W) : A \subset W,\ W \subset X \text{ is open}\}.$$

If $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$, then $\|Du\|(\cdot)$ is a Radon measure on Ω by [Miranda 2003, Theorem 3.4]. A μ-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where χ_E is the characteristic function of E. The perimeter of E in Ω is also denoted by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

Sets of finite perimeter include for example bounded domains with a Lipschitz boundary in $\mathbb{R}^n$, and in this case the perimeter is simply the $(n-1)$-dimensional Hausdorff measure of the boundary. However, sets of finite perimeter can also be highly irregular, as demonstrated by Example 2.6 below.

Applying the Poincaré inequality to sequences of approximating locally Lipschitz functions in the definition of the total variation, we get the following BV version: for every ball $B(x,r)$ and every $u \in L^1_{\mathrm{loc}}(X)$, we have

$$\fint_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \le C_P r\, \frac{\|Du\|(B(x,\lambda r))}{\mu(B(x,\lambda r))}.$$

For a μ-measurable set $E \subset X$, this implies the relative isoperimetric inequality

$$\min\{\mu(B(x,r)\cap E),\ \mu(B(x,r)\setminus E)\} \le 2C_P r\, P(E, B(x,\lambda r)); \tag{2.2}$$

see, e.g., [Korte and Lahti 2014, equation (3.1)].

The measure-theoretic interior of a set $E \subset X$ is defined as

$$I_E := \left\{x \in X : \lim_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} = 0\right\},$$

and the measure-theoretic exterior as

$$O_E := \left\{x \in X : \lim_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} = 0\right\}.$$

The measure-theoretic boundary $\partial^* E$ is defined as the set of points $x \in X$ at which both E and its complement have strictly positive upper density; i.e.,

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} > 0. \tag{2.3}$$

Then $X = I_E \cup O_E \cup \partial^* E$.

For an open set $\Omega \subset X$ and a μ-measurable set $E \subset X$ with $P(E,\Omega) < \infty$, we know that for any Borel set $A \subset \Omega$,

$$P(E,A) = \int_{\partial^* E\cap A} \theta_E \, d\mathcal{H}, \tag{2.4}$$

where $\theta_E \colon \partial^* E \to [\alpha, C_d]$, with $\alpha = \alpha(C_d, C_P, \lambda) > 0$; see [Ambrosio 2002, Theorem 5.3] and [Ambrosio et al. 2004, Theorem 4.6].

If $\Omega \subset X$ is an open set and $u, v \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\|D\min\{u,v\}\|(\Omega) + \|D\max\{u,v\}\|(\Omega) \le \|Du\|(\Omega) + \|Dv\|(\Omega); \tag{2.5}$$

for a proof see, e.g., [Lahti 2018, Lemma 3.1].

The measure-theoretic boundary $\partial^* E$ is always a subset of the topological boundary ∂E but the boundaries can be quite different, as shown by the following example of the so-called enlarged rationals.

Example 2.6. Let $X = \mathbb{R}^2$ (unweighted). Let $\{q_j\}_{j=1}^\infty$ be an enumeration of $\mathbb{Q}^2$. Define

$$E := \bigcup_{j=1}^{\infty} B(q_j, 2^{-j}).$$

Then by (2.5) and the fact that the perimeter is clearly lower semicontinuous with respect to convergence in $L^1(\mathbb{R}^2)$, we get

$$P(E,\mathbb{R}^2) \le \sum_{j=1}^{\infty} P(B(q_j,2^{-j}),\mathbb{R}^2) = 2\pi \sum_{j=1}^{\infty} 2^{-j} = 2\pi.$$

Thus E is of finite perimeter, and so by (2.4) we get $\mathcal{H}^1(\partial^* E) < \infty$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure. On the other hand, E is dense in $\mathbb{R}^2$, so that $\partial E = \mathbb{R}^2 \setminus E$ and thus $\mathcal{L}^2(\partial E) = \infty$. This illustrates that the measure-theoretic boundary is the natural boundary to consider.

The lower and upper approximate limits of a function u on X are defined respectively by

$$u^\wedge(x) := \sup\left\{ t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x,r)\cap\{u<t\})}{\mu(B(x,r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf\left\{ t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x,r)\cap\{u>t\})}{\mu(B(x,r))} = 0 \right\}.$$

Unlike Newton–Sobolev functions, we understand BV functions to be μ-equivalence classes. To consider fine properties, we need to consider the pointwise representatives $u^\wedge$ and $u^\vee$. We note that for $u = \chi_E$ with $E \subset X$, we have $x \in I_E$ if and only if $u^\wedge(x) = u^\vee(x) = 1$, $x \in O_E$ if and only if $u^\wedge(x) = u^\vee(x) = 0$, and $x \in \partial^* E$ if and only if $u^\wedge(x) = 0$ and $u^\vee(x) = 1$.

Throughout this paper we assume that (X, d, μ) is a complete metric space that is equipped with the doubling measure μ and supports a $(1, 1)$-Poincaré inequality.

3. The 1-fine topology

In this section we have gathered all the results concerning the 1-fine topology that our argument will rely on. For these, we refer to [Lahti 2017a; 2017b; 2020]. Most of the results are analogous to those that hold in the case $1 < p < \infty$, which has been studied in the metric setting in [Björn and Björn 2015; Björn et al. 2018; 2015].

Definition 3.1. We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r\to 0} r \frac{\operatorname{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.$$

We also say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on X.

We denote the 1-fine interior of a set $H \subset X$, i.e., the largest 1-finely open set contained in H, by fine-int H. We denote the 1-fine closure of a set $H \subset X$, i.e., the smallest 1-finely closed set containing H, by $\overline{H}^1$. The 1-fine boundary of H is $\partial^1 H := \overline{H}^1 \setminus \text{fine-int}\, H$. Finally, the 1-base $b_1 H$ is defined as the set of points where H is *not* 1-thin.

See [Lahti 2017b, Section 4] for discussion on this definition, and for a proof of the fact that the 1-fine topology is indeed a topology. By [Björn and Björn 2011, Proposition 6.16], a set $A \subset X$ is 1-thin at $x \in X$ if and only if

$$\lim_{r\to 0} \frac{\operatorname{cap}_1(A \cap B(x, r), B(x, 2r))}{\operatorname{cap}_1(B(x, r), B(x, 2r))} = 0,$$

and so it is clear that $W \subset b_1 W$ for any open set $W \subset X$.

Now we collect some facts concerning the 1-fine topology proved in [Lahti 2017a]. According to Corollary 3.5 of that work, the 1-fine closure of $A \subset X$ can be characterized in the following way:

$$\overline{A}^1 = A \cup b_1 A. \tag{3.2}$$

From this it easily follows that for any $A \subset X$ and any ball $B(x, r)$, we have $\overline{A}^1 \cap B(x, r) \subset \overline{A \cap B(x, r)}^1$, and then by [Lahti 2017a, Proposition 3.3] we get

$$\operatorname{cap}_1(\overline{A}^1 \cap B(x, r), B(x, 2r)) = \operatorname{cap}_1(A \cap B(x, r), B(x, 2r)). \tag{3.3}$$

By Lemma 4.6 of the same work the 1-fine boundary of a measure-theoretic interior can be characterized as follows: for any μ-measurable set $E \subset X$,

$$\partial^1 I_E = b_1 I_E \cap b_1(X \setminus I_E). \tag{3.4}$$

By [Lahti 2017a, Lemma 3.1] we know that for any μ-measurable set $E \subset X$,

$$\partial^* E \subset \partial^1 I_E. \tag{3.5}$$

Conversely, if $\Omega \subset X$ is open and $E \subset X$ is μ-measurable such that $P(E, \Omega) < \infty$, then by Theorem 1.2,

$$\mathcal{H}((\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0.$$

Combining this with (2.4) gives

$$\alpha \mathcal{H}(\partial^1 I_E \cap \Omega) \le P(E, \Omega) \le C_d \mathcal{H}(\partial^1 I_E \cap \Omega). \tag{3.6}$$

In fact this holds for every μ-measurable $E \subset X$; to see this we can assume that $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$, and then $P(E, \Omega) < \infty$ by Theorem 1.2.

We also have the following version of the relative isoperimetric inequality: for every ball $B(x, r)$ and every μ-measurable $E \subset X$,

$$\min\{\mu(B(x, r) \cap E),\ \mu(B(x, r) \setminus E)\} \le 2C_P C_d r \mathcal{H}(\partial^1 I_E \cap B(x, \lambda r)); \tag{3.7}$$

this follows from the ordinary relative isoperimetric inequality (2.2) and (3.6).

Remark 3.8. It may seem strange to talk about $\partial^1 I_E$, as it seems that we are first taking the interior in one topology and then the boundary in another. However, if we define the measure topology more axiomatically, then I_E is actually *not* the interior of E in the measure topology, and should be seen as a measure-theoretic quantity rather than a topological one; see [Lahti 2017a, Remark 4.9]. Moreover, $\partial^* E$ is actually the boundary of I_E in the measure topology; let us denote it by $\partial^0 I_E$. Thus $\partial^1 I_E$ is a natural set to consider as well. Finally, we can note that $\partial^1 I_E = \partial^1 O_E$; see [Lahti 2017a, Lemma 4.8].

The following *weak Cartan property* in the case $p = 1$ was proved in [Lahti 2020, Theorem 5.2].

Theorem 3.9. *Let $A \subset X$ and let $x \in X \setminus A$ be such that A is 1-thin at x. Then there exist $R > 0$ and $E_0, E_1 \subset X$ such that $\chi_{E_0}, \chi_{E_1} \in \mathrm{BV}(X)$, $\max\{\chi_{E_0}^{\wedge}, \chi_{E_1}^{\wedge}\} = 1$ in $A \cap B(x, R)$, $\chi_{E_0}^{\vee}(x) = 0 = \chi_{E_1}^{\vee}(x)$, $\{\max\{\chi_{E_0}^{\vee}, \chi_{E_1}^{\vee}\} > 0\}$ is 1-thin at x, and*

$$\lim_{r \to 0} r \frac{P(E_0, B(x, r))}{\mu(B(x, r))} = 0, \quad \lim_{r \to 0} r \frac{P(E_1, B(x, r))}{\mu(B(x, r))} = 0. \tag{3.10}$$

The following simpler formulation will be sufficient for our purposes.

Corollary 3.11. *Let $A \subset X$ and let $x \in X \setminus A$ be such that A is 1-thin at x. Then there exist $R > 0$ and $F \subset X$ such that $\chi_F \in \mathrm{BV}(X)$, $A \cap B(x, R) \subset I_F$, I_F is 1-thin at x, and*

$$\lim_{r \to 0} r \frac{P(F, B(x, r))}{\mu(B(x, r))} = 0. \tag{3.12}$$

Proof. Take $E_0, E_1 \subset X$ as given by Theorem 3.9, and set $F := E_0 \cup E_1$. By (2.5) we obtain $\chi_F \in \mathrm{BV}(X)$, and (2.5) and (3.10) together give (3.12). From the fact that $\max\{\chi_{E_0}^{\wedge}, \chi_{E_1}^{\wedge}\} = 1$ in $A \cap B(x, R)$ we obtain

$$A \cap B(x, R) \subset I_{E_0} \cup I_{E_1} \subset I_F.$$

Finally, since $\{\max\{\chi_{E_0}^{\vee}, \chi_{E_1}^{\vee}\} > 0\}$ is 1-thin at x, so is

$$I_{E_0} \cup I_{E_1} \cup \partial^* E_0 \cup \partial^* E_1 \supset I_F. \qquad \square$$

In [Lahti 2020, Lemma 4.4] it was also shown that if $A \subset X$ is 1-thin at a point $x \in X \setminus A$, then there exists an open set that contains A and is also 1-thin at x; that is:

$$\text{If } x \notin A \cup b_1 A, \text{ then there exists an open } W \supset A \text{ such that } x \notin b_1 W. \tag{3.13}$$

4. Proof of the characterization

In [Korte and Lahti 2014, Theorem 3.11] it was shown that for any μ-measurable set $E \subset X$, we have $\overline{\partial^* E} = \partial I_E$; that is, the closure of the measure-theoretic boundary (in the metric topology) is the whole topological boundary of a suitable representative of E (namely the measure-theoretic interior I_E). Now we prove the analogous result with the metric topology replaced by the 1-fine topology. This will be the crux of our proof of Federer's characterization.

Theorem 4.1. *For any μ-measurable set $E \subset X$, we have $\overline{\partial^* E}^1 = \partial^1 I_E$.*

Note that by Remark 3.8, the above can be written as $\overline{\partial^0 I_E}^1 = \partial^1 I_E$, showing that the result describes the interplay between the measure topology and the 1-fine topology. It is natural to ask which other sets and topologies would satisfy an analogous property, but we will not pursue this problem here. Previously, properties of the measure topology and fine topologies have been studied in [Lukeš et al. 1986].

Proof. By (3.5) we have

$$\overline{\partial^* E}^1 \subset \overline{\partial^1 I_E}^1 = \partial^1 I_E,$$

where the last equality follows from the fact that boundaries are closed sets in every topology. Thus we only need to show that $\overline{\partial^* E}^1 \supset \partial^1 I_E$. Let $x_0 \in \partial^1 I_E$ and let $U \ni x_0$ be a 1-finely open set. We need to show that $\partial^* E \cap U \neq \varnothing$. By (3.13) there exists an open set $W \supset X \setminus U$ that is 1-thin at x_0. Since $\overline{W}^1 = W \cup b_1 W = b_1 W$ by (3.2), we have $x_0 \notin \overline{W}^1$. We will show that $\partial^* E \setminus W \neq \varnothing$; suppose that instead $\partial^* E \setminus W = \varnothing$.

Claim. *Let $x \in \partial^1 I_E \setminus \overline{W}^1$ and $s_1 > 0$. Then there exists $y \in B(x, s_1) \cap \partial^1 I_E \setminus \overline{W}^1$ and $0 < s_2 \le s_1/2$ such that*

$$\frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil + 1}} \le \frac{\mu(E \cap B(y, s_2))}{\mu(B(y, s_2))} \le 1 - \frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil}},$$

where $\lceil a \rceil$ is the smallest integer at least $a \in \mathbb{R}$.

Proof of claim. Step 1: We can assume that $x \in O_E$; the case $x \in I_E$ is handled analogously (recall that $\partial^1 I_E = \partial^1 O_E$ from Remark 3.8). Since $x \in \partial^1 I_E$, by (3.4) we have

$$\limsup_{r \to 0} r \frac{\operatorname{cap}_1(I_E \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} > 0. \tag{4.2}$$

Since x belongs to the 1-finely open set $X \setminus \overline{W}^1$, we have

$$\lim_{r\to 0} r\frac{\mathrm{cap}_1(\overline{W}^1 \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0.$$

We apply Corollary 3.11 to find $R > 0$ and $F \subset X$ such that $I_F \supset \overline{W}^1 \cap B(x,R)$ and I_F is 1-thin at x. Then by (3.3), also

$$\lim_{r\to 0} r\frac{\mathrm{cap}_1(\overline{I_F}^1 \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0.$$

Combining this with (4.2), by subadditivity of the variational 1-capacity we get

$$\limsup_{r\to 0} r\frac{\mathrm{cap}_1((I_E \setminus \overline{I_F}^1) \cap B(x,r), B(x,2r))}{\mu(B(x,r))} > 0. \tag{4.3}$$

According to Corollary 3.11, the set F also satisfies

$$\lim_{r\to 0} r\frac{P(F, B(x,r))}{\mu(B(x,r))} = 0.$$

Thus by (3.6) and the doubling property of μ,

$$\lim_{r\to 0} r\frac{\mathcal{H}(\partial^1 I_F \cap B(x,2r))}{\mu(B(x,r))} = 0. \tag{4.4}$$

Combining the fact that $x \in O_E$ with (4.3) and (4.4), we find a number $a > 0$ and a radius

$$0 < r_f \le \frac{\min\{R, s_1\}}{2} \tag{4.5}$$

such that

$$\frac{\mu(E \cap B(x,2r_f))}{\mu(B(x,2r_f))} \le \frac{1}{2C_d^{\lceil \log_2(60\lambda)\rceil}} \tag{4.6}$$

and

$$r_f\frac{\mathrm{cap}_1((I_E \setminus \overline{I_F}^1) \cap B(x,r_f), B(x,2r_f))}{\mu(B(x,r_f))} > a \tag{4.7}$$

and

$$r_f\frac{\mathcal{H}(\partial^1 I_F \cap B(x,2r_f))}{\mu(B(x,r_f))} < \frac{a}{16C_d^{\lceil \log_2(10\lambda)\rceil+2}C_P}. \tag{4.8}$$

Step 2: Let D consist of all points $z \in B(x,r_f) \setminus \overline{I_F}^1$ for which there exists a radius $0 < t \le (10\lambda)^{-1} r_f$ such that

$$\frac{\mu(F \cap B(z,t))}{\mu(B(z,t))} > \frac{1}{4C_d}.$$

Consider $z \in D$ and the corresponding radius t. Since $z \notin \overline{I_F}^1 \supset I_F \cup \partial^* F$ (recall (3.5)), we have

$$\lim_{r\to 0} \frac{\mu(F \cap B(z,r))}{\mu(B(z,r))} = 0.$$

Take the smallest $k = 0, 1, \ldots$ such that

$$\frac{\mu(F \cap B(z,2^{-k}t))}{\mu(B(z,2^{-k}t))} \le \frac{1}{2}. \tag{4.9}$$

If $k = 0$, let $r_z := t$ so that

$$\frac{1}{4C_d} < \frac{\mu(F \cap B(z,r_z))}{\mu(B(z,r_z))} \le \frac{1}{2}.$$

If $k \ge 1$, let $r_z := 2^{-k+1}t$, and then

$$\frac{\mu(F \cap B(z,r_z))}{\mu(B(z,r_z))} > \frac{1}{2}$$

and

$$\begin{aligned}
\frac{\mu(F \cap B(z,r_z))}{\mu(B(z,r_z))} &= \frac{\mu(F \cap B(z,2^{-k+1}t))}{\mu(B(z,2^{-k+1}t))} \\
&\le \frac{\mu(B(z,2^{-k+1}t)) - \mu(B(z,2^{-k}t) \setminus F)}{\mu(B(z,2^{-k+1}t))} \\
&\le \frac{\mu(B(z,2^{-k+1}t)) - \mu(B(z,2^{-k}t))/2}{\mu(B(z,2^{-k+1}t))} \quad \text{by (4.9)} \\
&\le 1 - \frac{1}{2C_d}.
\end{aligned}$$

Thus in both cases, we have $r_z \le (10\lambda)^{-1} r_f$ and

$$\frac{1}{4C_d} < \frac{\mu(F \cap B(z,r_z))}{\mu(B(z,r_z))} \le 1 - \frac{1}{2C_d}.$$

By the relative isoperimetric inequality (3.7) we now obtain

$$\mu(B(z,r_z)) \le 8C_d^2 C_P r_z \mathcal{H}(\partial^1 I_F \cap B(x,\lambda r_z)). \tag{4.10}$$

Performing the same for every $z \in D$, we obtain a covering $\{B(z,\lambda r_z)\}_{z\in D}$. By the 5-covering theorem, we can extract a countable collection $\{B_j = B(z_j, r_j)\}_{j=1}^{\infty}$ such that the balls λB_j are pairwise disjoint and $D \subset \bigcup_{j=1}^{\infty} 5\lambda B_j$. For each $j \in \mathbb{N}$, define the Lipschitz function

$$\eta_j := \max\left\{0, 1 - \frac{\operatorname{dist}(\,\cdot\,, 5\lambda B_j)}{5\lambda r_j}\right\},$$

so that $\eta_j = 1$ on $5\lambda B_j$, $\eta_j = 0$ outside $10\lambda B_j$, and the minimal 1-weak upper gradient satisfies $g_{\eta_j} \le (5\lambda r_j)^{-1}\chi_{10\lambda B_j}$; see [Björn and Björn 2011, Corollary 2.21]. Moreover, $r_j \le (10\lambda)^{-1} r_f$ and so $\eta_j \in N_0^{1,1}(B(x, 2r_f))$ for all $j \in \mathbb{N}$. Now we have

$$\begin{aligned}\operatorname{cap}_1(D, B(x, 2r_f)) &\le \operatorname{cap}_1\Big(\bigcup_{j=1}^{\infty} 5\lambda B_j, B(x, 2r_f)\Big) \le \sum_{j=1}^{\infty} \operatorname{cap}_1(5\lambda B_j, B(x, 2r_f))\\ &\le \sum_{j=1}^{\infty} \int_X g_{\eta_j}\, d\mu \le \sum_{j=1}^{\infty} \frac{\mu(10\lambda B_j)}{5\lambda r_j} \le C_d^{\lceil \log_2(10\lambda)\rceil} \sum_{j=1}^{\infty} \frac{\mu(B_j)}{r_j}\\ &\le 8C_d^{\lceil \log_2(10\lambda)\rceil+2} C_P \sum_{j=1}^{\infty} \mathcal{H}(\partial^1 I_F \cap \lambda B_j) \quad \text{by (4.10)}\\ &\le 8C_d^{\lceil \log_2(10\lambda)\rceil+2} C_P \mathcal{H}(\partial^1 I_F \cap B(x, 2r_f)).\end{aligned}$$

Thus by (4.8),

$$r_f \frac{\operatorname{cap}_1(D, B(x, 2r_f))}{\mu(B(x, r_f))} < \frac{a}{2}$$

and so

$$\begin{aligned}&r_f \frac{\operatorname{cap}_1((I_E \setminus (\overline{I_F}^1 \cup D)) \cap B(x, r_f), B(x, 2r_f))}{\mu(B(x, r_f))}\\ &\qquad\ge r_f \frac{\operatorname{cap}_1((I_E \setminus \overline{I_F}^1) \cap B(x, r_f), B(x, 2r_f)) - \operatorname{cap}_1(D, B(x, 2r_f))}{\mu(B(x, r_f))} > \frac{a}{2} \qquad (4.11)\end{aligned}$$

by (4.7).

<u>Step 3</u>: Now consider $z \in (I_E \setminus (\overline{I_F}^1 \cup D)) \cap B(x, r_f)$. We have

$$\lim_{r\to 0} \frac{\mu(E \cap B(z, r))}{\mu(B(z, r))} = 1,$$

and so we can choose $0 < t \le (20\lambda)^{-1} r_f$ such that

$$\frac{\mu(E \cap B(z, t))}{\mu(B(z, t))} > \frac{1}{2}.$$

Note also that for any $r \in [(20\lambda)^{-1} r_f, (10\lambda)^{-1} r_f]$, we have $B(x, 2r_f) \subset B(z, 60\lambda r)$ and so

$$\frac{\mu(E \cap B(z, r))}{\mu(B(z, r))} \le C_d^{\lceil \log_2(60\lambda)\rceil} \frac{\mu(E \cap B(x, 2r_f))}{\mu(B(x, 2r_f))} \le \frac{1}{2}$$

by (4.6). Set $r_z := 2^k t$ for the smallest $k \in \mathbb{N}$ such that

$$\frac{\mu(E \cap B(z, 2^k t))}{\mu(B(z, 2^k t))} \le \frac{1}{2}.$$

Then we have $0 < r_z \le (10\lambda)^{-1} r_f$ and

$$\frac{1}{2C_d} < \frac{\mu(E \cap B(z, r_z))}{\mu(B(z, r_z))} \le \frac{1}{2} \qquad (4.12)$$

and since $z \notin D$,

$$\frac{1}{4C_d} \le \frac{\mu((E \setminus F) \cap B(z, r_z))}{\mu(B(z, r_z))} \le \frac{1}{2}.$$

Then by the relative isoperimetric inequality (3.7), we have

$$\frac{\mu(B(z, r_z))}{r_z} \le 8C_d^2 C_P \mathcal{H}(\partial^1 I_{E\setminus F} \cap B(z, \lambda r_z)). \tag{4.13}$$

(Note that the right-hand side could be infinity.) Let

$$A := \bigcup_{z\in(I_E\setminus(\overline{I_F}^1\cup D))\cap B(x,r_f)} B(z, \lambda r_z) \subset B(x, 2r_f). \tag{4.14}$$

Consider the covering $\{B(z, \lambda r_z)\}_{z\in(I_E\setminus(\overline{I_F}^1\cup D))\cap B(x,r_f)}$. By the 5-covering theorem, we can extract a countable collection $\{B_j = B(z_j, r_j)\}_{j=1}^{\infty}$ such that the balls λB_j are pairwise disjoint and

$$(I_E \setminus (\overline{I_F}^1 \cup D)) \cap B(x, r_f) \subset \bigcup_{j=1}^{\infty} 5\lambda B_j.$$

Just as in the previous step, for each $j \in \mathbb{N}$ define the Lipschitz function

$$\eta_j := \max\left\{0, 1 - \frac{\operatorname{dist}(\,\cdot\,, 5\lambda B_j)}{5\lambda r_j}\right\},$$

so that $\eta_j = 1$ on $5\lambda B_j$, $\eta_j = 0$ outside $10\lambda B_j$, and the minimal 1-weak upper gradient satisfies $g_{\eta_j} \le (5\lambda r_j)^{-1}\chi_{10\lambda B_j}$. Moreover, $r_j \le (10\lambda)^{-1} r_f$ and so $\eta_j \in N_0^{1,1}(B(x, 2r_f))$ for all $j \in \mathbb{N}$. Now we have

$$\begin{aligned}
\operatorname{cap}_1((I_E\setminus(\overline{I_F}^1\cup D))\cap B(x,r_f), B(x,2r_f)) &\le \operatorname{cap}_1\left(\bigcup_{j=1}^{\infty} 5\lambda B_j, B(x,2r_f)\right) \le \sum_{j=1}^{\infty} \operatorname{cap}_1(5\lambda B_j, B(x,2r_f)) \\
&\le \sum_{j=1}^{\infty}\int_X g_{\eta_j}\,d\mu \le \sum_{j=1}^{\infty}\frac{\mu(10\lambda B_j)}{5\lambda r_j} \le C_d^{\lceil\log_2(10\lambda)\rceil}\sum_{j=1}^{\infty}\frac{\mu(B_j)}{r_j} \\
&\le 8C_d^{\lceil\log_2(10\lambda)\rceil+2} C_P \sum_{j=1}^{\infty}\mathcal{H}(\partial^1 I_{E\setminus F}\cap\lambda B_j) \quad \text{by (4.13)} \\
&\le 8C_d^{\lceil\log_2(10\lambda)\rceil+2} C_P \mathcal{H}(\partial^1 I_{E\setminus F}\cap A).
\end{aligned} \tag{4.15}$$

Step 4: Next we show that

$$\partial^1 I_{E\setminus F} \subset (\partial^1 I_E \setminus \overline{I_F}^1) \cup \partial^1 I_F. \tag{4.16}$$

To see this, note that $X \setminus \overline{I_F}^1 \subset O_F$ (recall (3.5)) and so $I_{E\setminus F} \setminus \overline{I_F}^1 = I_E \setminus \overline{I_F}^1$. Since $X \setminus \overline{I_F}^1$ is a 1-finely open set, it follows that $\partial^1 I_{E\setminus F} \setminus \overline{I_F}^1 = \partial^1 I_E \setminus \overline{I_F}^1$. Moreover, $I_{E\setminus F} \cap \text{fine-int}\, I_F = \varnothing$ and fine-int I_F is 1-finely open, and so $\partial^1 I_{E\setminus F} \cap \text{fine-int}\, I_F = \varnothing$. From these, (4.16) follows.

By (4.11) and (4.15),

$$r_f \frac{\mathcal{H}(\partial^1 I_{E\setminus F} \cap A)}{\mu(B(x, r_f))} \ge \frac{a}{16C_d^{\lceil\log_2(10\lambda)\rceil+2} C_P}.$$

Now by first using (4.16) and the fact that $A \subset B(x, 2r_f)$ (recall (4.14)), and then (4.8) and the above inequality, we get

$$r_f \frac{\mathcal{H}((\partial^1 I_E \setminus \overline{I_F}^1) \cap A)}{\mu(B(x, r_f))} \geq r_f \frac{\mathcal{H}(\partial^1 I_{E \setminus F} \cap A) - \mathcal{H}(\partial^1 I_F \cap B(x, 2r_f))}{\mu(B(x, r_f))} > 0.$$

In particular, there exists a point $y \in (\partial^1 I_E \setminus \overline{I_F}^1) \cap A$. Recall from (4.5) that $0 < r_f \leq \min\{R, s_1\}/2$, and so $\overline{W}^1 \cap B(x, 2r_f) \subset I_F \cap B(x, 2r_f)$. Thus

$$y \in (\partial^1 I_E \setminus \overline{W}^1) \cap B(x, 2r_f) \subset (\partial^1 I_E \setminus \overline{W}^1) \cap B(x, s_1),$$

as desired. By the definition of A (recall (4.12), (4.14)) there is a point $z \in X$ and a radius $0 < r_z \leq (10\lambda)^{-1} r_f$ such that $y \in B(z, \lambda r_z)$ and

$$\frac{1}{2C_d} \leq \frac{\mu(E \cap B(z, r_z))}{\mu(B(z, r_z))} \leq \frac{1}{2}.$$

For $s_2 := 2\lambda r_z \leq s_1/2$ we then have $B(z, r_z) \subset B(y, s_2) \subset B(z, 3\lambda r_z)$, and so

$$\frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil + 1}} \leq \frac{\mu(E \cap B(y, s_2))}{\mu(B(y, s_2))} \leq \frac{\mu(B(y, s_2)) - \mu(B(z, r_z))/2}{\mu(B(y, s_2))} \leq 1 - \frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil}}.$$

This completes the proof of the claim. $\square$

Define $r_0 = 1$. We use the claim repeatedly, first with the choice $x = x_0$ and $s_1 = r_0$, to find a sequence of points $x_j \in B(x_{j-1}, r_{j-1}) \cap \partial^1 I_E \setminus \overline{W}^1$ and a sequence of numbers $0 < r_j \leq r_{j-1}/2$ such that

$$\min\left\{ \frac{\mu(B(x_j, r_j) \cap E)}{\mu(B(x_j, r_j))}, \frac{\mu(B(x_j, r_j) \setminus E)}{\mu(B(x_j, r_j))} \right\} \geq \frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil + 1}}$$

for all $j \in \mathbb{N}$. By completeness of the space and the fact that W is open, we find $x \in X \setminus W$ such that $x_j \to x$. For each $j \in \mathbb{N}$ we have

$$d(x, x_j) \leq \sum_{k=j}^{\infty} d(x_k, x_{k+1}) \leq \sum_{k=j}^{\infty} r_k \leq 2r_j.$$

Thus $B(x_j, r_j) \subset B(x, 3r_j) \subset B(x_j, 5r_j)$ for all $j \in \mathbb{N}$, and so

$$\frac{\mu(B(x, 3r_j) \cap E)}{\mu(B(x, 3r_j))} \geq \frac{\mu(B(x_j, r_j) \cap E)}{\mu(B(x, 3r_j))} \geq \frac{1}{C_d^3} \frac{\mu(B(x_j, r_j) \cap E)}{\mu(B(x_j, r_j))} \geq \frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil + 4}}$$

and similarly

$$\frac{\mu(B(x, 3r_j) \setminus E)}{\mu(B(x, 3r_j))} \geq \frac{\mu(B(x_j, r_j) \setminus E)}{\mu(B(x, 3r_j))} \geq \frac{1}{C_d^3} \frac{\mu(B(x_j, r_j) \setminus E)}{\mu(B(x_j, r_j))} \geq \frac{1}{2C_d^{\lceil \log_2(3\lambda) \rceil + 4}}.$$

Thus

$$\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0,$$

and so $x \in \partial^* E \setminus W$, which proves the theorem by the discussion in the first paragraph of the proof. $\square$

By using another argument involving Lipschitz cutoff functions, it is easy to see that, for any $A \subset X$ and any ball $B(x,r)$,

$$\operatorname{cap}_1(A \cap B(x,r), B(x,2r)) \le C_d \mathcal{H}(A \cap B(x,r)). \tag{4.17}$$

Theorem 4.18. *Let $\Omega \subset X$ be open and let $E \subset X$ be μ-measurable with $\mathcal{H}(\partial^* E \cap \Omega) < \infty$. Then $\mathcal{H}((\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0$.*

Proof. By a standard covering argument (see, e.g., the proof of [Kinnunen et al. 2014, Lemma 2.6]) we find that

$$\lim_{r \to 0} r \frac{\mathcal{H}(\partial^* E \cap B(x,r))}{\mu(B(x,r))} = 0$$

for all $x \in \Omega \setminus (\partial^* E \cup N)$, with $\mathcal{H}(N) = 0$. Then by (4.17), also

$$\begin{aligned}\limsup_{r \to 0} r \frac{\operatorname{cap}_1((\partial^* E \cup N) \cap B(x,r), B(x,2r))}{\mu(B(x,r))} &\le C_d \limsup_{r \to 0} r \frac{\mathcal{H}((\partial^* E \cup N) \cap B(x,r))}{\mu(B(x,r))} \\ &= C_d \limsup_{r \to 0} r \frac{\mathcal{H}(\partial^* E \cap B(x,r))}{\mu(B(x,r))} = 0\end{aligned}$$

for all $x \in \Omega \setminus (\partial^* E \cup N)$. Thus $\Omega \setminus (\partial^* E \cup N)$ is a 1-finely open set. Now by Theorem 4.1,

$$\partial^1 I_E \cap (\Omega \setminus (\partial^* E \cup N)) = \varnothing$$

and the result follows. □

Now we can prove our main theorem.

Proof of Theorem 1.1. By Theorem 4.18 we have $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$. Then by Theorem 1.2 we have $P(E, \Omega) < \infty$. □

5. Some consequences and discussion

Theorem 1.1 is, in particular, new in Carnot groups, which are a class of connected and simply connected Lie groups. We point out that the metric space theory we consider is consistent with the Carnot group theory involving vector fields, since the so-called horizontal Sobolev spaces defined in Carnot groups by means of distributional derivatives coincide with the Newton–Sobolev spaces defined in metric spaces. See [Hajłasz and Koskela 2000, Theorems 11.6 and 11.7]. The Lebesgue measure on a Carnot group is doubling and supports a (1, 1)-Poincaré inequality; see [Hajłasz and Koskela 2000, Proposition 11.17] as well as [Heinonen et al. 2015, Section 14.2].

We state Federer's characterization in metric spaces as follows.

Corollary 5.1. *Let $\Omega \subset X$ be open and let $E \subset X$ be μ-measurable. Then $P(E, \Omega) < \infty$ if and only if $\mathcal{H}(\partial^* E \cap \Omega) < \infty$.*

Proof. This follows by combining Theorem 1.1 and (2.4). □

In general, the sets $\partial^* E$ and $\partial^1 I_E$ can be quite different.

Example 5.2. Let $X = \mathbb{R}$ (unweighted). Let $\{q_j\}_{j=1}^\infty$ be an enumeration of all rational numbers and let $E := \bigcup_{j=1}^\infty B(q_j, 2^{-j})$. Then $\mathcal{L}^1(I_E) \le 2$ and $\mathcal{L}^1(\partial^* E) = 0$ by Lebesgue's differentiation theorem. On the other hand, it is straightforward to check that for any $A \subset \mathbb{R}$, we have $\partial^1 A = \partial A$. Thus $\partial^1 I_E = \partial I_E \supset \mathbb{R} \setminus I_E$ and so $\mathcal{L}^1(\partial^1 I_E) = \infty$.

In the Euclidean setting, the "if" direction of Federer's characterization is proved by first showing that almost every coordinate line intersecting I_E and O_E also intersects $\partial^* E$; see [Federer 1969, Section 4.5.11] or [Evans and Gariepy 1992, pp. 222–226]. Proving this fact relies heavily on the Euclidean structure, but now we have the following generalization to metric spaces.

Proposition 5.3. *Let $E \subset X$ be μ-measurable and suppose that $\mathcal{H}(\partial^* E) < \infty$. Then* 1*-almost every curve intersecting I_E and O_E also intersects $\partial^* E$.*

Proof. By Theorem 1.1, $P(E, X) < \infty$. Then the result follows from [Lahti and Shanmugalingam 2017, Corollary 6.4]. □

If the above property were true also in the case $\mathcal{H}(\partial^* E) = \infty$, this could be useful when proving generalizations of Federer's characterization, e.g., to noncomplete spaces. One might expect that when the measure-theoretic boundary is very large, then it should be "easier" for curves to intersect it. However, the intersection property turns out to be false in this case.

Example 5.4. Let $X = \mathbb{R}^2$ equipped with the Euclidean metric and the weighted Lebesgue measure $d\mu := w\, d\mathcal{L}^2$, with $w(x) = |x|^{-3/2}$. It is straightforward to check that w is a Muckenhoupt A_1-weight, and thus μ is doubling and supports a (1, 1)-Poincaré inequality; see, e.g., [Heinonen et al. 1993, Chapter 15] for these concepts. Let

$$E := \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 \le 2,\ -x_1^{3/2} < x_2 < x_1^{3/2}\}.$$

It is easy to check that

$$\partial^* E = \{x \in \mathbb{R}^2 : x_1 > 0,\ |x_2| = x_1^{3/2}\}$$

and then that $\mathcal{H}(\partial^* E) = \infty$. Now consider the curve family Γ consisting of the curves

$$\gamma_{r,t}(s) := (s, t s^{3/2}), \quad 0 < r \le 1,\ -1 < t < 1,\ s \in [0, r].$$

For all $0 < r \le 1$ and $-1 < t < 1$, clearly $\gamma_{r,t}(0) \in O_E$ and $\gamma_{r,t}(r) \in I_E$, but $\gamma_{r,t}$ does not intersect $\partial^* E$.

Denote the image of $\gamma_{r,t}$ in X by the same symbol. Let ρ be an admissible function for Γ. This means that $\int_{\gamma_{r,t}} \rho\, d\mathcal{H}^1 \ge 1$ for all $0 < r \le 1$ and $-1 < t < 1$, where $\mathcal{H}^1$ is the 1-dimensional Lebesgue measure. It follows that for every $-1 < t < 1$ and every $k \in \mathbb{N}$,

$$\sum_{j=k}^{\infty} \int_{\gamma_{1,t}|_{[2^{-j}, 2^{-j+1}]}} \rho\, d\mathcal{H}^1 = \int_{\gamma_{2^{-k+1},t}} \rho\, d\mathcal{H}^1 \ge 1,$$

and so

$$\sum_{j=1}^{\infty} \int_{\gamma_{1,t}|_{[2^{-j}, 2^{-j+1}]}} \rho\, d\mathcal{H}^1 = \infty. \tag{5.5}$$

Define the sets

$$A_j := \{x \in E : 2^{-j} < x_1 \le 2^{-j+1}\}, \quad j \in \mathbb{N}.$$

By the classical coarea formula we have

$$\begin{aligned}\int_{\mathbb{R}^2} \rho\, d\mu \ge \sum_{j=1}^{\infty} 2^{3j/2-3} \int_{A_j} \rho\, d\mathcal{L}^2 &\ge \sum_{j=1}^{\infty} 2^{3j/2-3} 2^{-3j/2} \int_{-1}^{1} \int_{\gamma_{1,t}|_{[2^{-j},2^{-j+1}]}} \rho\, d\mathcal{H}^1\, dt \\ &= 2^{-3} \sum_{j=1}^{\infty} \int_{-1}^{1} \int_{\gamma_{1,t}|_{[2^{-j},2^{-j+1}]}} \rho\, d\mathcal{H}^1\, dt = \infty\end{aligned}$$

by (5.5). It follows that $\mathrm{Mod}_1(\Gamma) = \infty$.

It is reasonable to expect Federer's characterization to find various applications especially in the metric setting, where certain tools of Euclidean BV theory, such as the Gauss–Green theorem, are not available. One likely application is in the study of images of sets of finite perimeter under quasiconformal mappings (see [Kelly 1973] for the Euclidean case), since such mappings are known to preserve the measure-theoretic boundary (see [Korte et al. 2012, Theorem 6.1]).

Now we discuss some existing applications. From the characterization it follows that the space supports the following *strong relative isoperimetric inequality* introduced in [Kinnunen et al. 2012]; compare this with (2.2) and (3.7).

Corollary 5.6. *For every ball $B(x, r)$ and every μ-measurable $E \subset X$, we have*

$$\min\{\mu(B(x,r) \cap E),\ \mu(B(x,r) \setminus E)\} \le 2C_P C_d r \mathcal{H}(\partial^* E \cap B(x, \lambda r)).$$

Proof. We can assume that the right-hand side is finite. By Theorem 1.1 we know that $P(E, B(x, \lambda r)) < \infty$, and now the result follows by combining the relative isoperimetric inequality (2.2) and (2.4). □

In [Kinnunen et al. 2012] the authors worked with the same standing assumptions as we do in the current paper, but additionally they assumed that the space supports the above strong relative isoperimetric inequality. Now we know that this does not need to be separately assumed, and the following theorem [Kinnunen et al. 2012, Theorem 1.1] holds under our standing assumptions (completeness, doubling, and Poincaré).

Theorem 5.7. *Let $\Omega \subset X$ be a bounded open set and let $u \in N^{1,p}(\Omega)$ with $1 \le p < \infty$. Then $u \in N_0^{1,p}(\Omega)$ if and only if*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{\Omega \cap B(x,r)} |u|\, d\mu = 0$$

for Cap_p*-almost every $x \in \partial\Omega$.*

Theorem 6.1 in [Lahti and Shanmugalingam 2018] considered an analogous characterization of a class of BV functions with zero boundary values, also under the additional assumption of a strong relative isoperimetric inequality. Such a class and the characterization are needed in an ongoing study of new fine properties of BV functions and capacities, see [Lahti 2019; 2018], and this was in fact a key motivation for the current paper. The strong relative isoperimetric inequality was also used in proving approximation results for BV functions; see [Lahti and Shanmugalingam 2018, Corollary 6.7, Theorem 6.9], as well as [Lahti and Shanmugalingam 2017, Corollary 7.6] and the comment after it. Now we know that all of these results hold in every complete metric space equipped with a doubling measure and supporting a Poincaré inequality.

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