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THE EINSTEIN-NULL DUST SYSTEM WITH AN INNER MIRROR**





# A PROOF OF THE INSTABILITY OF ADS FOR THE EINSTEIN-NULL DUST SYSTEM WITH AN INNER MIRROR

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In 2006, Dafermos and Holzegel formulated the so-called AdS instability conjecture, stating that there exist *arbitrarily small* perturbations to AdS initial data which, under evolution by the Einstein vacuum equations for  $\Lambda < 0$  with reflecting boundary conditions on conformal infinity  $\mathcal{I}$ , lead to the formation of black holes. The numerical study of this conjecture in the simpler setting of the spherically symmetric Einstein-scalar field system was initiated by Bizón and Rostworowski (*Phys. Rev. Lett.* **107**:3 (2011), art. id. 031102), followed by a vast number of numerical and heuristic works by several authors.

In this paper, we provide the first rigorous proof of the AdS instability conjecture in the simplest possible setting, namely for the spherically symmetric Einstein-massless Vlasov system, in the case when the Vlasov field is moreover supported only on radial geodesics. This system is equivalent to the Einstein-null dust system, allowing for both ingoing and outgoing dust. In order to overcome the breakdown of this system occurring once the null dust reaches the center  $r = 0$ , we place an inner mirror at  $r = r_0 > 0$  and study the evolution of this system on the exterior domain  $\{r \geq r_0\}$ . The structure of the maximal development and the Cauchy stability properties of general initial data in this setting are studied in our companion paper (2017, arXiv: 1704.08685).

The statement of the main theorem is as follows: We construct a family of mirror radii  $r_{0\varepsilon} > 0$  and initial data  $\mathcal{S}_\varepsilon$ ,  $\varepsilon \in (0, 1]$ , converging, as  $\varepsilon \rightarrow 0$ , to the AdS initial data  $\mathcal{S}_0$  in a suitable norm, such that, for any  $\varepsilon \in (0, 1]$ , the maximal development  $(\mathcal{M}_\varepsilon, g_\varepsilon)$  of  $\mathcal{S}_\varepsilon$  contains a black hole region. Our proof is based on purely physical space arguments and involves the arrangement of the null dust into a large number of beams which are successively reflected off  $\{r = r_{0\varepsilon}\}$  and  $\mathcal{I}$ , in a configuration that forces the energy of a certain beam to increase after each successive pair of reflections. As  $\varepsilon \rightarrow 0$ , the number of reflections before a black hole is formed necessarily goes to  $+\infty$ . We expect that this instability mechanism can be applied to the case of more general matter fields.

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## 1. Introduction

Anti-de Sitter spacetime  $(\mathcal{M}_{\text{AdS}}^{n+1}, g_{\text{AdS}})$ ,  $n \geq 3$ , is the simplest solution of the *Einstein vacuum equations*

$$\text{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (1-1)$$

with a negative cosmological constant  $\Lambda$ . In the standard polar coordinate chart on  $\mathcal{M}_{\text{AdS}}$ , the AdS metric takes the form

$$g_{\text{AdS}} = -\left(1 - \frac{2}{n(n-1)}\Lambda r^2\right)dt^2 + \left(1 - \frac{2}{n(n-1)}\Lambda r^2\right)^{-1}dr^2 + r^2g_{\mathbb{S}^{n-1}}, \quad (1-2)$$

where  $g_{\mathbb{S}^{n-1}}$  is the round metric on the  $(n-1)$ -dimensional sphere; see [Hawking and Ellis 1973].

Despite being geodesically complete,  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  fails to be globally hyperbolic. In particular, it can be conformally identified with the interior of  $(\mathbb{R} \times \mathbb{S}_+^n, g_E)$ , where  $\mathbb{S}_+^n$  is the closed upper hemisphere of  $\mathbb{S}^n$  and  $g_E$  is the metric

$$g_E = -d\tilde{t}^2 + \left(-\frac{n(n-1)}{2\Lambda}\right)g_{\mathbb{S}^n}. \quad (1-3)$$

Through this identification, the time-like boundary

$$\mathcal{I}^n = \mathbb{R} \times \partial\mathbb{S}_+^n \simeq \mathbb{R} \times \mathbb{S}^{n-1} \quad (1-4)$$

of  $(\mathbb{R} \times \mathbb{S}_+^n, g_E)$  is naturally attached to  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  as a “conformal boundary at infinity”; see [Hawking and Ellis 1973].

Maldacena [1998], Gubser, Klebanov, and Polyakov [Gubser et al. 1998] and Witten [1998] proposed the *AdS/CFT conjecture*, suggesting a correspondence between certain conformal field theories defined on  $\mathcal{I}^n$  (in the strongly coupled regime) and supergravity on spacetimes asymptotically of the form  $(\mathcal{M}_{\text{AdS}}^{n+1} \times S^k, g_{\text{AdS}} + g_{S^k})$ , where  $(S^k, g_{S^k})$  is a suitable compact Riemannian manifold of dimension  $k$ . Following the introduction of this conjecture, asymptotically AdS spacetimes (i.e., spacetimes  $(\mathcal{M}, g)$  with an asymptotic region with geometry resembling that of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  in the vicinity of  $\mathcal{I}$ ) became a subject of intense study in the high-energy physics literature; see, e.g., [Aharony et al. 2000; Hartnoll 2009; Ammon and Erdmenger 2015].

The correct setting for the study of the dynamics of asymptotically AdS solutions  $(\mathcal{M}, g)$  to (1-1) is that of an *initial value problem* with appropriate *boundary conditions* prescribed asymptotically on  $\mathcal{I}$ . The issue of the right boundary conditions on  $\mathcal{I}$  leading to well-posedness for the resulting initial-boundary value problem for (1-1) was first addressed by Friedrich [1995]. Well-posedness for more general boundary conditions and matter fields in the spherically symmetric case was obtained in [Holzegel and Smulevici 2012; Holzegel and Warnick 2015]; see also [Holzegel et al. 2020; Friedrich 2014]. In general, most physically interesting boundary conditions on  $\mathcal{I}$  leading to a well-posed initial-boundary value problem can be classified as either *reflecting* (for which an appropriate “energy flux” for  $g$  through  $\mathcal{I}$  vanishes) or *dissipative* (allowing for a nonvanishing outgoing “energy flux” for  $g$  through  $\mathcal{I}$ ), with substantially different global dynamics associated to each case; see the discussion in [Holzegel et al. 2020].

Dafermos and Holzegel [2006a], see also [Dafermos 2006], suggested the following conjecture:

**AdS instability conjecture.** *There exist arbitrarily small perturbations to the initial data of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for the vacuum Einstein equations (1-1) with a reflecting boundary condition on  $\mathcal{I}$  which lead to the development of trapped surfaces and, thus, black hole regions. In particular,  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is nonlinearly unstable.*

This conjecture was motivated in [Dafermos and Holzegel 2006a] by the study of asymptotically AdS solutions to (1-1) with biaxial Bianchi IX symmetry in  $4 + 1$  dimensions, a symmetry class in which the vacuum Einstein equations (1-1) reduce to a  $1 + 1$  hyperbolic system with nontrivial dynamics. This model was introduced in [Bizoń et al. 2005]. In this setting, it was observed in [Dafermos and Holzegel 2006a] that perturbations of the initial data of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  (which, if not trivial, necessarily have strictly positive ADM mass  $M_{\text{ADM}}$ , in view of [Gibbons et al. 1983]) cannot settle down to a horizonless static spacetime, since  $M_{\text{ADM}}$  is conserved along  $\mathcal{I}$  under reflecting boundary conditions and no static asymptotically AdS solution of (1-1) with  $M_{\text{ADM}} > 0$  exists (according to [Boucher et al. 1984]). This picture was supported by results of [Anderson 2006].

The following remarks should be made regarding the statement of the AdS instability conjecture:

- The perturbations referred to in the conjecture are assumed to be small with respect to a norm for which (1-1) is well-posed and  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is *Cauchy stable* as a solution to (1-1) (otherwise, the conjecture is trivial).<sup>1</sup> For such perturbations, Cauchy stability implies that the “time” elapsed before the formation of a trapped surface tends to  $+\infty$  as the size of the initial perturbation shrinks to 0.
- The AdS instability conjecture stands in contrast to the nonlinear stability of Minkowski space  $(\mathbb{R}^{3+1}, \eta)$ , in the case  $\Lambda = 0$  (see [Christodoulou and Klainerman 1993]), or de Sitter space  $(\mathcal{M}_{\text{dS}}, g_{\text{dS}})$ , in the case  $\Lambda > 0$  (see [Friedrich 1986]). The proof of the nonlinear stability of  $(\mathbb{R}^{3+1}, \eta)$  and  $(\mathcal{M}_{\text{dS}}, g_{\text{dS}})$  is based on a stability mechanism related to the fact that linear fields on those spacetimes satisfy sufficiently strong decay rates. The decay rates are, however, borderline in the case  $\Lambda = 0$ , and thus the stability of  $(\mathbb{R}^{3+1}, \eta)$  is a deep fact depending on the precise nonlinear structure of the system (1-1), whereas, in the case  $\Lambda > 0$ , the decay is exponential and stability can be inferred relatively easily. In contrast, on  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ , it can be shown that linear fields satisfying a reflecting boundary condition on  $\mathcal{I}$  remain bounded, but do *not* decay in time. It is precisely the lack of a sufficiently fast decay rate at the linear level which is associated to the possibility of nonlinear instability.
- The prescription of a reflecting boundary condition on  $\mathcal{I}$  is essential for the conjecture: for maximally dissipative boundary conditions, it is expected that  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is nonlinearly *stable*, in view of the quantitative decay rates obtained for the linearized vacuum Einstein equations (and other linear fields) around  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  by Holzegel, Luk, Smulevici, and Warnick [Holzegel et al. 2020].
- In the biaxial Bianchi IX symmetry class, all perturbations of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  leading to the formation of a trapped surface can be shown to possess a complete conformal infinity  $\mathcal{I}$  and are expected to settle down to a member of the Schwarzschild-AdS family; see [Dafermos and Holzegel 2006a; 2006b]. However,

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<sup>1</sup>Here, Cauchy stability of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  refers to Cauchy stability of the conformal compactification of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  (including, therefore, the time-like boundary  $\mathcal{I}$ ); see the discussion in the next section.

in the absence of any symmetry, the picture regarding the end state of the evolution of general vacuum perturbations of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is complicated; see the discussion in the next section.

Starting from the pioneering work [Bizoń and Rostworowski 2011], a plethora of numerical and heuristic results have been obtained in the direction of establishing the AdS instability conjecture, mainly in the context of the spherically symmetric Einstein-scalar field system. See the discussion in Section 1A.

In this paper, we will prove the AdS instability conjecture in the simplest possible setting, namely for the Einstein-massless Vlasov system in spherical symmetry, further reduced to the case when the Vlasov field  $f$  is supported only on radial geodesics. We will call this system the *spherically symmetric Einstein-radial massless Vlasov system*. In fact, this is a singular reduction; the resulting system is equivalent to the *spherically symmetric Einstein-null dust system*, allowing for both ingoing and outgoing dust. This system has been studied in the  $\Lambda = 0$  case in [Poisson and Israel 1990].

A serious problem with the spherically symmetric Einstein-null dust system is that it suffers from a severe breakdown when the null dust reaches the center  $r = 0$ . In particular, in any reasonable initial-data topology, the spherically symmetric Einstein-null dust system is *not* well-posed and  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is *not* a Cauchy stable solution of it. One way to restore the well-posedness of this system (a necessary step for the study of the AdS instability conjecture in this setting) is to place an *inner mirror* at some radius sphere  $\{r = r_0\}$  with  $r_0 > 0$  and study the evolution of the system in the exterior region  $\{r \geq r_0\}$ . However, fixing the mirror radius  $r_0$  results in a trivial global stability statement for  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ , as initial data perturbations with total ADM mass  $\tilde{m}_{\text{ADM}} < \frac{1}{2}r_0$  cannot form a black hole. Thus, it is necessary to allow the radius  $r_0$  to shrink to 0 as the total ADM mass of the initial data shrinks to 0, in order to address the AdS instability conjecture in this setting. See the discussion in Section 1B.

A nontechnical statement of our result is the following:

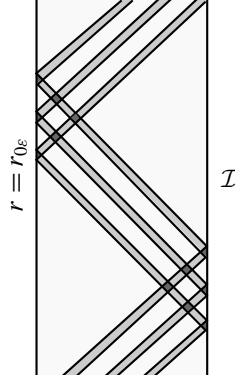
**Theorem 1** (rough version). *The AdS spacetime  $(\mathcal{M}_{\text{AdS}}^{3+1}, g_{\text{AdS}})$  is nonlinearly unstable under evolution by the spherically symmetric Einstein-radial massless Vlasov system with a reflecting boundary condition on  $\mathcal{I}$  and an inner mirror, in the following sense.*

*There exists a one-parameter family of spherically symmetric initial data  $\mathcal{S}_\varepsilon$ ,  $\varepsilon \in (0, 1]$ , and a family of inner mirror radii  $r = r_{0\varepsilon}$  (with  $r_{0\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$ ) satisfying the following properties:*

- (1) *As  $\varepsilon \rightarrow 0$ ,  $\|\mathcal{S}_\varepsilon\|_{\mathcal{CS}} \rightarrow 0$ ; i.e., the  $\mathcal{S}_\varepsilon$  converge to the initial data  $\mathcal{S}_0$  of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ .*
- (2) *For any  $\varepsilon > 0$ , the maximal future development  $(\mathcal{M}_\varepsilon, g_\varepsilon)$  of  $\mathcal{S}_\varepsilon$  contains a trapped surface and, thus, a black hole region. Moreover,  $(\mathcal{M}_\varepsilon, g_\varepsilon)$  possesses a complete conformal infinity  $\mathcal{I}$ .*

*The norm  $\|\cdot\|_{\mathcal{CS}}$  in 1 measures the concentration of the energy of  $\mathcal{S}_\varepsilon$  in annuli of width  $\sim r_{0\varepsilon}$  and has the property that the radial Einstein-massless Vlasov system is well-posed and  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is Cauchy stable with respect to  $\|\cdot\|_{\mathcal{CS}}$  independently of the precise value of  $r_{0\varepsilon}$ . See Figure 1.*

For progressively more detailed statements of Theorem 1, see Sections 1C and 4. For further discussion on the need of an inner mirror at  $r \sim r_{0\varepsilon}$  and its relation to natural dispersive mechanisms appearing in other matter models, see Section 1B.



**Figure 1.** The family of initial data  $\mathcal{S}_\epsilon$  that we construct for the proof of Theorem 1 gives rise to a large number of Vlasov beams, which are successively reflected off  $\mathcal{I}$  and the inner mirror at  $r = r_{0\epsilon}$ . The Cauchy stability statement for  $\|\cdot\|_{\mathcal{CS}}$  implies that the number of reflections necessarily goes to  $+\infty$  as  $\epsilon \rightarrow 0$  (see the remark after the statement of the AdS instability conjecture).

We should also note the following:

- Except for the condition  $r_{0\epsilon} < 2(M_{\text{ADM}})_\epsilon$  referred to earlier, where  $(M_{\text{ADM}})_\epsilon$  is the ADM mass of  $\mathcal{S}_\epsilon$ , there is considerable flexibility in the choice of the mirror radii  $r_{0\epsilon}$  in the statement of Theorem 1 and this can be exploited to one's advantage. For simplicity, we choose  $r_{0\epsilon}$  to satisfy  $r_{0\epsilon} \sim (M_{\text{ADM}})_\epsilon$  (see also the discussion in Section 1C).
- While we do not address the issue of the end state of the evolution of  $\mathcal{S}_\epsilon$ , it can be easily inferred from our proof of Theorem 1 that the spacetimes  $(\mathcal{M}_\epsilon, g_\epsilon)$  settle down to a member of the Schwarzschild-AdS family; see also [Moschidis 2017].

The trivial instability at  $r = 0$  occurring for the spherically symmetric Einstein-null dust system is absent in the case of smooth solutions to the general spherically symmetric Einstein-massless Vlasov system (not reduced to the radial case). In particular, the smooth initial value problem for the spherically symmetric Einstein-massless Vlasov system is well-posed, and placing an inner mirror at  $r = r_0 > 0$  is not necessary.<sup>2</sup> For a proof of the AdS instability in this setting, see our forthcoming [Moschidis 2018].

**1A. Earlier numerical and heuristic works.** Restricted under spherical symmetry, all solutions to the Einstein vacuum equations (1-1) are locally isometric to a member of the Schwarzschild-AdS family; see [Eiesland 1925]. Thus, any attempt to search for unstable vacuum perturbations of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for (1-1) in 3 + 1 dimensions cannot be reduced to a problem for a 1 + 1 hyperbolic system (where the wide variety of available tools would make the problem more tractable).<sup>3</sup> For this reason, instead of (1-1), numerical

<sup>2</sup>In fact, well-posedness for the smooth initial value problem for the Einstein–Vlasov system also holds outside spherical symmetry; see [Choquet-Bruhat 1971]. In the case  $\Lambda = 0$ , the stability of Minkowski spacetime for the Einstein-massless Vlasov system without any symmetry assumptions was recently established in [Taylor 2017].

<sup>3</sup>This problem is circumvented in 4 + 1 dimensions by the biaxial Bianchi IX symmetry class referred to earlier; see [Bizoń et al. 2005].

and heuristic works on the AdS instability have so far mainly focused on the *Einstein-scalar field system*

$$\begin{cases} \text{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}[\varphi], \\ \square_g \varphi = 0, \\ T_{\mu\nu}[\varphi] \doteq \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \varphi \partial_\alpha \varphi. \end{cases} \quad (1-5)$$

The system (1-5), whose mathematical study in the case  $\Lambda = 0$  was pioneered in [Christodoulou 1999], admits nontrivial dynamics in spherical symmetry, and spherically symmetric solutions to (1-5) share many qualitative properties with general solutions of (1-1). Reduced under spherical symmetry in a double null gauge  $(u, v)$  in  $3 + 1$  dimensions, i.e., a gauge where

$$g = -\Omega^2 du dv + r^2 g_{\mathbb{S}^2}, \quad (1-6)$$

the system (1-5) takes the form

$$\begin{cases} \partial_u \partial_v (r^2) = -\frac{1}{2} (1 - \Lambda r^2) \Omega^2, \\ \partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2} (1 + 4\Omega^{-2} \partial_u r \partial_v r) - 8\pi \partial_u \varphi \partial_v \varphi, \\ \partial_v (\Omega^{-2} \partial_v r) = -4\pi r \Omega^{-2} (\partial_v \varphi)^2, \\ \partial_u (\Omega^{-2} \partial_u r) = -4\pi r \Omega^{-2} (\partial_u \varphi)^2, \\ \partial_u \partial_v (r\varphi) = -\frac{\Omega^2 - 4 \partial_u r \partial_v r}{4r^2} \cdot r\varphi. \end{cases} \quad (1-7)$$

The well-posedness of the asymptotically AdS initial-boundary value problem for the system (1-7) with reflecting boundary conditions on  $\mathcal{I}$  was established by [Holzegel and Smulevici 2012].

Numerical results in the direction of establishing the AdS instability conjecture were first obtained by Bizoń and Rostworowski [2011], who studied the evolution of spherically symmetric perturbations of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for (1-5) in Schwarzschild-type coordinates. More precisely, [Bizoń and Rostworowski 2011] numerically simulated the evolution of initial data for (1-5) with  $\varphi$  initially arranged into small amplitude wave packets. It was found that, for certain families of initial arrangements of this form (of “size”  $\varepsilon$ ), after a finite number of reflections on  $\mathcal{I}$  (proportional to  $\varepsilon^{-2}$ ), the energy of the wave packets becomes substantially concentrated, leading to a breakdown of the coordinate system associated with the threshold of trapped surface formation.

Following [Bizoń and Rostworowski 2011], a vast amount of numerical and heuristic works have been dedicated to the understanding of the global dynamics of perturbations of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for the system (1-5); see, e.g., [Dias et al. 2012a; 2012b; Buchel et al. 2012; Maliborski and Rostworowski 2013; Balasubramanian et al. 2014; Craps et al. 2014; 2015; Bizoń et al. 2015; Dimitrakopoulos et al. 2015; 2016; 2018; Green et al. 2015; Horowitz and Santos 2015; Dimitrakopoulos and Yang 2015]. In these works, the picture that arises regarding the long-time dynamics of *generic* spherically symmetric perturbations is rather complicated: Apart from perturbations that lead to instability and trapped surface formation [Dias et al. 2012b; Buchel et al. 2012], it appears that there exist certain types of perturbations (dubbed “islands of stability”) which remain close to  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for long



times; see [Dias et al. 2012a; Maliborski and Rostworowski 2013; Balasubramanian et al. 2014; Dimitrakopoulos and Yang 2015]. Perturbations of the latter type might in fact occupy an open set in the moduli space of spherically symmetric initial data for (1-5); see [Balasubramanian et al. 2014; Dimitrakopoulos and Yang 2015]. The question of existence of open “corners” of initial data around  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  leading to trapped surface formation has also been studied; see, e.g., [Dimitrakopoulos et al. 2015].

Another interesting problem in this context is the characterization of the possible end states of the evolution of unstable perturbations of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ . Holzegel and Smulevici [2013b] established that the Schwarzschild-AdS spacetime  $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$  is an asymptotically stable solution of the system (1-5) in spherical symmetry, with perturbations decaying at an exponential rate.<sup>4</sup> This result supports the expectation that all spherically symmetric perturbations of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for the system (1-5) leading to the formation of a trapped surface eventually settle down to a member of the Schwarzschild-AdS family; see [Dafermos and Holzegel 2006a; 2006b]. However, beyond spherical symmetry, Holzegel and Smulevici [2013a; 2014] showed that solutions to the linear scalar wave equation

$$\square_{g_{\text{Sch}}} \varphi = 0 \tag{1-8}$$

on  $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$  (and, more generally, on Kerr-AdS) decay at a slow (logarithmic) rate, which is insufficient in itself to yield the nonlinear stability of  $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$  (see our remark below the statement of the AdS instability conjecture). Thus, [Holzegel and Smulevici 2014] conjectured that  $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$  is nonlinearly unstable. On the other hand, based on a detailed analysis of quasinormal modes on  $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$ , Dias, Horowitz, Marolf, and Santos [Dias et al. 2012a] suggested that sufficiently regular, nonlinear perturbations of  $(\mathcal{M}_{\text{Sch}}, g_{\text{Sch}})$  still remain small, at least for long times. As a result, the picture regarding the end state of the evolution of generic perturbations of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  outside spherical symmetry remains unclear; see also [Horowitz and Santos 2015; Dias and Santos 2016; Rostworowski 2017].

Following [Bizoń and Rostworowski 2011], the bulk of heuristic works have implemented a frequency space analysis in the study of the AdS instability conjecture. A notable exception is the work of Dimitrakopoulos, Freivogel, Lippert, and Yang [Dimitrakopoulos et al. 2015], where a physical space mechanism possibly leading to instability for the system (1-7) is suggested. We will revisit the mechanism of [Dimitrakopoulos et al. 2015] and compare it with the results of this paper at the end of Section 1D.

**1B. The Einstein-null dust system in spherical symmetry.** A spherically symmetric model for (1-1) which is even simpler than (1-5) is the *Einstein-massless Vlasov* system; see [Andréasson 2011; Rein 1995]. The case where the Vlasov field is supported only on radial geodesics is a singular reduction of this system which is equivalent to the *Einstein-null dust* system, allowing for both ingoing and outgoing dust; see [Rendall 1997]. This system was studied in the seminal work [Poisson and Israel 1990] on mass

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<sup>4</sup>A similar result can presumably also be deduced for the vacuum Einstein equations (1-1) reduced under the biaxial Bianchi IX symmetry in  $4 + 1$  dimensions, following by an amalgamation of the proofs of [Holzegel 2010] and [Holzegel and Smulevici 2013b].

inflation. In  $3 + 1$  dimensions, it takes the form (in double null coordinates  $(u, v)$ )

$$\begin{cases} \partial_u \partial_v (r^2) = -\frac{1}{2}(1 - \Lambda r^2)\Omega^2, \\ \partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2}(1 + 4\Omega^{-2} \partial_u r \partial_v r), \\ \partial_v (\Omega^{-2} \partial_v r) = -4\pi r^{-1} \Omega^{-2} \bar{\tau}, \\ \partial_u (\Omega^{-2} \partial_u r) = -4\pi r^{-1} \Omega^{-2} \tau, \\ \partial_u \bar{\tau} = 0, \\ \partial_v \tau = 0. \end{cases} \quad (1-9)$$

In certain cases, the Einstein-null dust system (1-9) can be formally viewed as a high-frequency limit of the Einstein-scalar field system (1-7) (as was already discussed in [Poisson and Israel 1990]): Setting

$$\tau \doteq r^2(\partial_u \varphi)^2, \quad \bar{\tau} \doteq r^2(\partial_v \varphi)^2$$

in (1-5) and dropping all lower-order terms from the wave equation for  $\varphi$ , one formally obtains (1-9) in the region where  $\partial_u \varphi \partial_v \varphi$  is negligible, i.e., outside the intersection of the supports of  $\tau, \bar{\tau}$ . While this formal limiting procedure can be rigorously justified away from  $r = 0$ , the dynamical similarities between (1-7) and (1-9) break down close to  $r = 0$ . A fundamental difference between these systems is the fact that, while small-data asymptotically AdS solutions to (1-7) satisfying a reflecting boundary condition at  $\mathcal{I}$  remain regular (and “small”) for large times, all nontrivial solutions to the system (1-9) break down once the support of  $\bar{\tau}$  reaches the axis  $\gamma$  (i.e., the time-like portion of  $\{r = 0\}$ ), independently of the boundary conditions imposed at  $\mathcal{I}$ . This is an ill-posedness statement for (1-9), which needs to be addressed before any attempt to study the AdS instability conjecture in the setting of (1-9).

We will now proceed to discuss this difference of (1-7) and (1-9) in more detail.

*Cauchy stability for the Einstein-scalar field system.* The following Cauchy stability result holds for the system (1-7):

**Proposition 1** (Cauchy stability for (1-7); see [Holzegel and Smulevici 2012]). *For a suitable initial-data norm  $\|\cdot\|_{\text{initial}}$ ,  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is Cauchy stable as a solution of the system (1-7) with reflecting boundary conditions on  $\mathcal{I}$ . That is to say, for all fixed times  $T_* > 0$ , any perturbation of the initial data of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  which is small enough (when measured in terms of  $\|\cdot\|_{\text{initial}}$ ) with respect to  $T_*$  gives rise to a solution of (1-7) which is regular and close to  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for times up to  $T_*$ .*

**Remark.** In the statement of Proposition 1, Cauchy stability of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  refers to stability over fixed compact subsets of the *conformal compactification* of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ , such as subsets of the form  $\{0 \leq t \leq T_*\}$  in the  $(t, r, \vartheta, \varphi)$  coordinate chart. Any such subset contains, in particular, a compact subset of the time-like boundary  $\mathcal{I}$ .

The initial-data norm  $\|\cdot\|_{\text{initial}}$ , for which the Cauchy stability of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  follows from [Holzegel and Smulevici 2012], is a higher-order, suitably weighted  $C^k$  norm. However, this is not the only norm for which  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  can be shown to be Cauchy stable: An additional, highly nontrivial example of such a norm is the bounded variation norm of [Christodoulou 1993] (modified with suitable  $r$ -weights near  $r = \infty$ ). Similar low-regularity norms will also play an important role in this paper (see Section 1B).

Assuming, for simplicity, that initial data are prescribed on the outgoing null hypersurface corresponding to  $u = 0$ , for  $0 \leq v \leq v_*$ , a necessary condition for Cauchy stability of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for the system (1-7) with respect to an initial-data norm  $\|\cdot\|_{\text{initial}}$  is that, for any given  $R_0 > 0$ ,  $\|\cdot\|_{\text{initial}}$  controls the quantity

$$\mathcal{M} \doteq \sup_{\substack{0 \leq v_1 < v_2 \leq v_* \\ r(0, v_2)/r(0, v_1) < 3/2 \\ r(0, v_2) \leq R_0}} \frac{\tilde{m}(0, v_2) - \tilde{m}(0, v_1)}{(r(0, v_2) - r(0, v_1))|\log(r(0, v_2)/r(0, v_1) - 1)|}, \quad (1-10)$$

where  $\tilde{m}$  is the *renormalized Hawking mass*, defined in terms of the Hawking mass  $m$ ,

$$m \doteq \frac{1}{2}r(1 - 4\Omega^{-2} \partial_u r \partial_v r), \quad (1-11)$$

by the relation

$$\tilde{m} \doteq m - \frac{1}{6}\Lambda r^3. \quad (1-12)$$

This is a consequence of the fact that, when  $\mathcal{M}$  exceeds a certain threshold (depending on  $R_0$ ), there exists a point  $p = (u_+, v_+)$  in the development of the initial data, with  $0 < u_+ < v_*$ , such that

$$\frac{2m}{r}(u_+, v_+) > 1, \quad (1-13)$$

a result proven in [Christodoulou 1991].<sup>5</sup> The bound (1-13) implies that

$$\partial_u r(u_+, v_+) < 0, \quad \partial_v r(u_+, v_+) < 0, \quad (1-14)$$

i.e., that the symmetry sphere associated to  $(u_+, v_+)$  is a *trapped surface*. In particular,  $(u_+, v_+)$  is contained in a black hole.<sup>6</sup>

As a corollary, it follows that the total ADM mass of the initial data, though expressible as a coercive functional on the space of initial data of (1-7), does not yield a norm for which  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is Cauchy stable for (1-7), since the ADM mass manifestly fails to control (1-10).

*Break down at  $r = 0$  and “trivial” Cauchy instability for the Einstein-null dust system.* The following instability result holds for the system (1-9) (see [Moschidis 2017]):

**Proposition 2** (Cauchy instability for (1-9)). *Any globally hyperbolic spherically symmetric solution  $(\mathcal{M}, g; \tau, \bar{\tau})$  of (1-9) with nonempty axis  $\gamma$  “breaks down” at the first point when a radial geodesic in the support of  $\bar{\tau}$  reaches  $\gamma$ : Beyond that point,  $(\mathcal{M}, g; \tau, \bar{\tau})$  is  $C^0$  inextendible as a spherically symmetric solution to (1-9). As a result,  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is not a Cauchy stable solution of (1-9) for any “reasonable” initial-data topology.*

For the precise definition of the notion of  $C^0$  inextendibility as a spherically symmetric solution to (1-9), see [Moschidis 2017]. Note that this is a *stronger* statement than  $(\mathcal{M}, g; \tau, \bar{\tau})$  breaking down as a smooth solution of (1-9). We should also note the following regarding Proposition 2:

<sup>5</sup>The result of [Christodoulou 1991] was restricted to the case  $\Lambda = 0$ , but the proof can be readily modified to include the case  $\Lambda < 0$ .

<sup>6</sup>We should remark that (1-14) follows from (1-13) under the assumption that  $\partial_u r < 0$  (which always holds provided, initially,  $\partial_u r|_{u=0} < 0$ , see [Christodoulou 1993]).

- Proposition 2 holds independently of the value of the cosmological constant  $\Lambda$ . In particular, Minkowski spacetime  $(\mathbb{R}^{3+1}, \eta)$  is not Cauchy stable for (1-9) with  $\Lambda = 0$  for any “reasonable” initial-data topology.
- Proposition 2 yields a uniform upper bound on the time of existence of solutions  $(\mathcal{M}, g)$  to (1-9) for any initial data set for which  $\bar{\tau}$  is not identically equal to 0, depending only on the distance of the initial support of  $\bar{\tau}$  from the axis and, thus, independent of the proximity of the initial data to the trivial data (in any reasonable initial-data norm). We should also highlight that the instability of Proposition 2 has nothing to do with trapped surface formation: Up to the first retarded time when a radial geodesic in the support of  $\bar{\tau}$  reaches  $\gamma$ , any solution  $(\mathcal{M}, g)$  to (1-9) arising from smooth initial data close to  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  remains smooth and close to  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ , and  $(\mathcal{M}, g)$  *contains no trapped surface*. In fact, in this case, despite being  $C^0$  inextendible as a globally hyperbolic spherically symmetric solution to (1-9),  $(\mathcal{M}, g)$  is globally  $C^\infty$ -extendible as a spherically symmetric Lorentzian manifold; see [Moschidis 2017].
- The Cauchy stability statement for  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for the system (1-7) stated in Proposition 1 can be informally interpreted as the result of a natural dispersive mechanism close to the axis  $\gamma$  displayed by the system (1-7), which does not allow the energy of  $\varphi$  to concentrate on scales smaller than  $\tilde{m}$  in  $O(1)$  time, provided a suitable initial norm of  $\varphi$  (controlling at least (1-10)) is small enough. No such mechanism is present for the system (1-9), as is illustrated by Proposition 2.

*Resolution of the “trivial” instability of (1-9) through an inner mirror.* In order to turn the spherically symmetric Einstein-null dust system (1-9) into a well-posed, Cauchy-stable system (a necessary step for converting (1-9) into an effective model of the vacuum Einstein equations (1-1)), it is necessary to explicitly add to (1-9) a mechanism that prevents the breakdown at  $r = 0$  described by Proposition 2, so that, moreover, an analogue of Proposition 1 holds for (1-9). This can be achieved by placing an *inner mirror* at  $r = r_0 > 0$ , i.e., by restricting (1-9) on  $\{r \geq r_0\}$ , for some  $r_0 > 0$ , and imposing a reflecting boundary condition on the portion  $\gamma_0$  of the set  $\{r = r_0\}$  which is time-like.

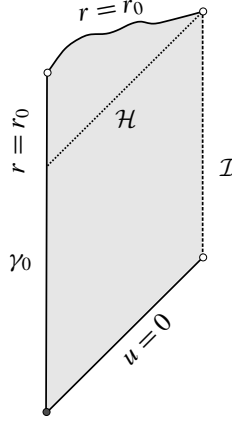
**Remark.** The reflecting boundary condition on  $\gamma_0$  can be motivated by the fact that, for smooth spherically symmetric solutions  $(\mathcal{M}, g; \varphi)$  to (1-5), the function  $\varphi$ , viewed as a function on the quotient of  $(\mathcal{M}, g)$  by the spheres of symmetry, satisfies a reflecting boundary condition on the axis.

The well-posedness and the properties of the maximal development for the system (1-9) with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$  are addressed in the companion paper [Moschidis 2017]. The following result is established in that paper:

**Theorem 2** (well-posedness for (1-9) with an inner mirror). *For any  $r_0 > 0$  and any smooth asymptotically AdS initial data set  $(r, \Omega^2, \tau, \bar{\tau})|_{u=0}$  on  $u = 0$ , there exists a unique smooth maximal future development  $(r, \Omega^2, \tau, \bar{\tau})$  on  $\{r \geq r_0\}$ , solving (1-9) with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$ , where  $r|_{\gamma_0} = r_0$  and  $\gamma_0$  coincides with the portion of the curve  $\{r = r_0\}$  which is time-like (fixing the gauge freedom by imposing a reflecting gauge condition on both  $\mathcal{I}$  and  $\gamma_0$ ). For this development,  $\mathcal{I}$  is complete and  $\{r = r_0\}$  is time-like in the past of  $\mathcal{I}$  (see Figure 2).*

*In the case when the future event horizon  $\mathcal{H}^+$  is nonempty, it is smooth and future complete. A necessary condition for  $\mathcal{H}^+$  to be nonempty is the existence of a point  $(u_+, v_+)$  where (1-13) holds. If the total mass*





**Figure 2.** Schematic depiction of the domain on which the maximal future development  $(r, \Omega^2, \tau, \bar{\tau})$  of a smooth initial data set on  $u = 0$  (with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$ ) is defined. A gauge condition ensures that  $\mathcal{I}$  and  $\gamma_0$  are straight vertical lines. Conformal infinity  $\mathcal{I}$  is always complete in this setting. In the case when the future event horizon  $\mathcal{H}^+$  is nonempty, it is smooth and has infinite affine length. In this case, apart from the mirror  $\gamma_0$ , the boundary of the domain has a space-like portion on which  $\{r = r_0\}$ .

$\tilde{m}|_{\mathcal{I}}$  and the mirror radius  $r_0$  satisfy

$$\frac{2\tilde{m}|_{\mathcal{I}}}{r_0} \leq 1 - \frac{1}{3}\Lambda r_0^2, \quad (1-15)$$

then necessarily  $\mathcal{H}^+ = \emptyset$ .

For a more detailed statement of Theorem 2, see Section 3 and [Moschidis 2017].

In view of the fact that  $\mathcal{H}^+ = \emptyset$  in the case when the total mass  $\tilde{m}|_{\mathcal{I}}$  and the mirror radius  $r_0$  satisfy (1-15), in order to address the AdS instability conjecture for the system (1-9) with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$ , it is necessary to allow  $r_0$  to shrink to 0 with the size of the data. Thus, addressing the AdS instability conjecture in this setting requires establishing a Cauchy stability statement for  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  which is *independent of the precise value of the mirror radius*  $r_0$ . This is the statement of the following result, proved in our companion paper [Moschidis 2017]:

**Theorem 3** (Cauchy stability for (1-9) uniformly in  $r_0$ ). *Given  $\varepsilon > 0$ ,  $u_* > 0$ , there exists a  $\delta > 0$  such that the following statement holds: for any  $r_0 > 0$  and any initial data set  $(r, \Omega^2, \tau, \bar{\tau})|_{u=0}$  satisfying*

$$\begin{aligned} & \| (r, \Omega^2, \tau, \bar{\tau}) \|_{u=0} \\ & \doteq \sup_{\bar{v}} \int_{u=0} \frac{\bar{\tau}(0, v)}{|\rho(0, v) - \rho(0, \bar{v})| + \tan^{-1}(\sqrt{-\Lambda}r_0)} \frac{\sqrt{-\Lambda} dv}{\partial_v \rho(0, v)} + \sup_{u=0} \left( \left| \left( 1 - \frac{2\tilde{m}}{r} \right)^{-1} - 1 \right| + \sqrt{-\Lambda} \tilde{m} \right) \\ & \leq \varepsilon, \end{aligned} \quad (1-16)$$

where

$$\rho(0, v) \doteq \tan^{-1}(\sqrt{-\Lambda}r)(0, v), \quad (1-17)$$

the corresponding solution  $(r, \Omega^2, \tau, \bar{\tau})$  to (1-9) with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$  will satisfy

$$\sup_{0 \leq \bar{u} \leq u_*} \|(r, \Omega^2, \tau, \bar{\tau})\|_{u=\bar{u}} \leq \delta. \quad (1-18)$$

For a more detailed statement of Theorem 3, see Section 3 and [Moschidis 2017].

Notice that the norm (1-16) vanishes only for the trivial initial data  $(r, \Omega^2, 0, 0)$ . Informally, Theorem 3 implies that, if the energy of the initial data concentrated on scales proportional to the mirror radius  $r_0$  is small enough, then the energy of the solution to (1-9) (with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$ ) will remain similarly dispersed for times less than any given constant. In particular, no trapped surface can form in this timescale if  $\varepsilon$  is chosen sufficiently small.

In Section 3, we will also present a Cauchy stability statement for general solutions of (1-9) with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$ , which will be used in the proof of Theorem 1 (see Theorem 3.6).

**1C. Statement of Theorem 1: the nonlinear instability of AdS.** According to Theorem 3, a Cauchy stability statement holds for  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  for time intervals which are independent of the precise value of the mirror radius  $r_0$ , depending only on the smallness of the initial-data norm (1-16). As a result, it is possible to study the AdS instability conjecture for the system (1-9) with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$ , for perturbations which are small with respect to (1-16), *allowing the mirror radius  $r_0$  to shrink with the size of the data*. In this paper, we will prove the following result:

**Theorem 1** (more precise version). *There exists a family of positive numbers  $r_{0\varepsilon}$  (satisfying  $r_{0\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0$ ) and smooth initial data  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$  for the system (1-9) satisfying the following properties:*

- (1) *In the norm  $\|\cdot\|_{u=0}$  defined by (1-16),*

$$\|(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}\|_{u=0} \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (1-19)$$

- (2) *For any  $\varepsilon > 0$ , the maximal development  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$  of  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$  for the system (1-9) with reflecting boundary conditions on  $\mathcal{I}$  and  $\gamma_0$ ,  $r|_{\gamma_0} = r_{0\varepsilon}$  contains a trapped sphere; i.e., there exists a point  $(u_\varepsilon, v_\varepsilon)$  such that*

$$\frac{2m^{(\varepsilon)}}{r^{(\varepsilon)}}(u_\varepsilon, v_\varepsilon) > 1. \quad (1-20)$$

*Thus, in view of Theorem 2,  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$  contains a nonempty, smooth and future complete event horizon  $\mathcal{H}^+$  and a complete conformal infinity  $\mathcal{I}$ .*

For the definitive statement of Theorem 1, see Section 4. The following remarks should be made concerning Theorem 1:

- In view of the Cauchy stability of  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  with respect to (1-16) (see Theorem 3), the time<sup>7</sup> required to elapse before  $(1 - 2m^{(\varepsilon)}/r^{(\varepsilon)})$  becomes negative necessarily tends to  $+\infty$  as  $\varepsilon \rightarrow 0$ . However, our proof yields a quantitative upper bound (increasing to  $+\infty$  as  $\varepsilon \rightarrow 0$ ) on the time of trapped sphere formation for the development of  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$  (see (4-6)).

<sup>7</sup>Where time is measured with respect to the (dimensionless) coordinate function  $\bar{t} = \sqrt{-\Lambda}(u + v)$ .

- In view of the fact that  $\mathcal{H}^+ = \emptyset$  when (1-15) holds, in order for  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$  to satisfy both (1-19) and (1-20), it is necessary that  $r_{0\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , at a rate which is at least as fast as that of  $2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$ ,<sup>8</sup> i.e.,

$$r_{0\varepsilon} \leq 2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}. \quad (1-21)$$

In fact, we choose the family  $r_{0\varepsilon}, (r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$  of Theorem 1 to saturate the bound (1-21) in the limit  $\varepsilon \rightarrow 0$ ; i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{r_{0\varepsilon}}{2\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}} = 1. \quad (1-22)$$

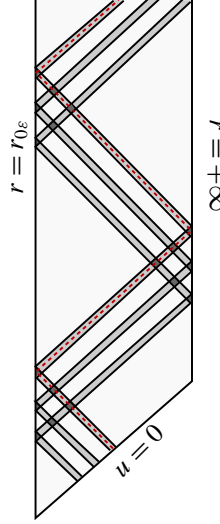
For the proof of Theorem 1, (1-22) is not essential and can be relaxed; however, it is fundamental for our proof that  $r_{0\varepsilon}$  is bounded from below by some small multiple of  $\tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$ .

- It follows from the proof of Theorem 2 that, in the case  $\Lambda = 0$ , Minkowski spacetime  $(\mathbb{R}^{3+1}, \eta)$  is *globally stable* (for the system (1-9) with reflecting boundary conditions on the inner mirror  $\{r = r_0\}$ ) to initial data perturbations which are small with respect to the norm (1-16), independently of the precise choice of  $r_0$ . This fact further justifies the choice of the matter model and the norm (1-16) as a setting for establishing the AdS instability conjecture.
- The proof of Theorem 1 also applies in the case  $\Lambda = 0$  when placing an *outer mirror* at  $r = R_0 \gg r_0$  (in addition to the inner mirror at  $r = r_0$ ), i.e., restricting the solutions of (1-9) in the region  $\{r_0 \leq r \leq R_0\}$  and imposing reflecting boundary conditions on both  $\{r = r_0\}$  and  $\{r = R_0\}$ . This is in accordance with the numerical results of [Buchel et al. 2012] for the system (1-5).
- It can be readily inferred by Cauchy stability (see Theorem 3.6 in Section 3D) that, for any  $y_\varepsilon \doteq (r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$  in the family of initial data of Theorem 1, there exists an open neighborhood  $\mathcal{W}_\varepsilon$  of initial data around  $y_\varepsilon$  such that, for all  $y \in \mathcal{W}_\varepsilon$ , the maximal future development of  $y$  also contains a trapped surface. In particular, the set of initial data leading to trapped surface formation is open. An even stronger genericity statement would be the existence of an *open instability corner* in the space of initial data around  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  (see [Dimitrakopoulos et al. 2015]), i.e., the existence of a  $c_1 > 0$  such that  $\{y : \text{dist}(y, y_\varepsilon) \leq c_1 \|y_\varepsilon\|_{u=0}\} \subset \mathcal{W}_\varepsilon$  for all  $\varepsilon > 0$  (with  $\text{dist}(\cdot, \cdot)$  being the distance function associated to (1-16) for  $r_0 = r_{0\varepsilon}$ ). While we have not addressed the issue of genericity of the unstable initial data in this paper, we expect that the proof of Theorem 1 can be adapted to yield the existence of an instability corner around  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ .
- A plethora of numerical works, see, e.g., [Bizoń and Rostworowski 2011; Buchel et al. 2012; Bizoń et al. 2015; Dimitrakopoulos et al. 2015], suggest that, in the case of the Einstein-scalar field system (1-5), for families of initial data  $(\varphi_\varepsilon^{(0)}, \varphi_\varepsilon^{(1)})$  for the scalar field  $\varphi$  of the form

$$(\varphi_\varepsilon^{(0)}, \varphi_\varepsilon^{(1)}) = (\varepsilon\varphi^{(0)}, \varepsilon\varphi^{(1)}) \quad (1-23)$$

(where  $(\varphi^{(0)}, \varphi^{(1)})$  is a fixed initial profile), trapped surface formation occurs at time  $\sim \varepsilon^{-2}$ . However, any rigorous formulation of this statement for general families of initial data, not necessarily of the form (1-23), requires fixing an initial-data norm for which the initial data size is measured, with different

<sup>8</sup>Note that the renormalized Hawking mass  $\tilde{m}^{(\varepsilon)}$  is constant on  $\mathcal{I}$  when imposing a reflecting boundary condition.



**Figure 3.** The initial data  $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}|_{u=0}$  give rise to a bundle of ingoing beams which are successively reflected off  $\{r = r_{0\epsilon}\}$  and  $\mathcal{I} = \{r = +\infty\}$ . While the number of beams goes to infinity as  $\epsilon \rightarrow 0$ , for simplicity, we only depict here a bundle of three beams. As long as the total width of the bundle of beams remains small, the interaction set naturally splits into a part which lies close to  $\{r = r_{0\epsilon}\}$  and a part near  $\mathcal{I}$ . We have also marked with a red dashed line the beam lying (initially) to the future of the rest.

choices of (scale-invariant) norms possibly leading to different time scales of trapped surface formation for initial data of size  $\sim \epsilon$ . For this reason, given that the initial data  $(\tau, \bar{\tau})^{(\epsilon)}|_{u=0}$  in Theorem 1 cannot be viewed as a rescaling of a fixed profile of the form (1-23), we have not tried to optimize the time required for trapped surface formation in Theorem 1 in terms of the initial norm (1-16).

**1D. Sketch of the proof and remarks on Theorem 1.** We will now proceed to sketch the main arguments involved in the proof of Theorem 1.

*Construction of the initial data.* The family of initial data  $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}|_{u=0}$  in Theorem 1 is chosen so that its total ADM mass  $\tilde{m}^{(\epsilon)}|_{\mathcal{I}}$  and the mirror radius  $r_{0\epsilon}$  satisfy (for  $\epsilon \ll 1$ )

$$r_{0\epsilon}, \tilde{m}^{(\epsilon)}|_{\mathcal{I}} \sim \epsilon(-\Lambda)^{-1/2}. \quad (1-24)$$

In particular, fixing a function  $h(\epsilon)$  in terms of  $\epsilon$  such that

$$\epsilon \ll h(\epsilon) \ll 1,$$

the initial data  $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}|_{u=0}$  are constructed so that the null dust initially forms a bundle of narrow ingoing beams emanating from the region  $r \sim 1$ ; see Figure 3. The number of the beams is chosen to be large, i.e., of order  $\sim (h(\epsilon))^{-1}$ , and the beams are initially separated by gaps of  $r$ -width  $\sim (h(\epsilon))^{-1}\epsilon(-\Lambda)^{-1/2}$ . The large number of beams and their initial separation are chosen so that

$$\|(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}\|_{u=0} \sim h(\epsilon) \xrightarrow{\epsilon \rightarrow 0} 0. \quad (1-25)$$



*Remarks on the configuration of the null dust beams.* As the solution  $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}$  arising from the initial data set  $(r, \Omega^2, \tau, \bar{\tau})^{(\epsilon)}|_{u=0}$  evolves according to (1-9), the null dust beams are reflected successively off  $\gamma_0 = \{r = r_{0\epsilon}\}$  and  $\mathcal{I}$ , as depicted in Figure 3. The beams separate the spacetime into vacuum regions (the larger rectangular regions between the beams in Figure 3), where the renormalized Hawking mass  $\tilde{m}^{(\epsilon)}$  is constant (recall the definitions of the Hawking mass and the renormalized Hawking mass by (1-11) and (1-12), respectively). The *interaction set* of the beams consists of all the points in the spacetime where two different beams intersect (depicted in Figure 3 as the union of all the smaller dark rectangles, lying in the intersection of any two beams). As long as the total width of the bundle of beams remains small, the interaction set can be split into two sets, one consisting of the intersections occurring close to the mirror  $\gamma_0$  and one consisting of the intersections near  $\mathcal{I}$  (see Figure 3).

Every beam is separated by the interaction set into several components. To each such component, we can associate the *mass difference*  $\mathfrak{D}\tilde{m}$  between the two vacuum regions which are themselves separated by that beam component. The mass difference  $\mathfrak{D}\tilde{m}$  measures the energy content of each beam component and, in view of the nonlinearity of the system (1-9), it is not necessarily conserved along the beam after an intersection with another beam. Precisely determining the resulting change in the mass difference after the interaction of two beams will be the crux of the proof of Theorem 1.

*Beam interactions and change in mass difference.* In Figure 4, the region around the intersection of an incoming null dust beam  $\zeta_{\text{in}}$  and an outgoing null dust beam  $\zeta_{\text{out}}$  is depicted. This region is separated by the beams into four vacuum subregions  $\mathcal{R}_1, \dots, \mathcal{R}_4$  with associated renormalized Hawking masses  $\tilde{m}_1, \dots, \tilde{m}_4$  (see Figure 4). Before the intersection of the two beams, the mass difference of the incoming beam  $\zeta_{\text{in}}$  is

$$\bar{\mathfrak{D}}_-\tilde{m} = \tilde{m}_3 - \tilde{m}_4, \quad (1-26)$$

while the mass difference of the outgoing beam  $\zeta_{\text{out}}$  is

$$\mathfrak{D}_-\tilde{m} = \tilde{m}_4 - \tilde{m}_2. \quad (1-27)$$

After the intersection of the beams, the mass differences associated to  $\zeta_{\text{in}}$  and  $\zeta_{\text{out}}$  become

$$\bar{\mathfrak{D}}_+\tilde{m} = \tilde{m}_1 - \tilde{m}_2, \quad (1-28)$$

$$\mathfrak{D}_+\tilde{m} = \tilde{m}_3 - \tilde{m}_1, \quad (1-29)$$

respectively.

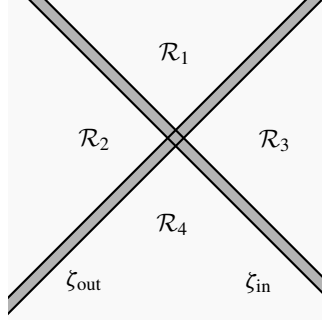
Assuming that

$$\frac{2m}{r} < 1, \quad (1-30)$$

$$\partial_u r < 0 < \partial_v r, \quad (1-31)$$

we can readily obtain the following differential relations for  $r$  and  $\tilde{m}$  from (1-9):

$$\begin{aligned} \partial_u \log \left( \frac{\partial_v r}{1 - 2m/r} \right) &= -\frac{4\pi}{r} \frac{\tau}{-\partial_u r}, \\ \partial_v \log \left( \frac{-\partial_u r}{1 - 2m/r} \right) &= \frac{4\pi}{r} \frac{\bar{\tau}}{\partial_v r} \end{aligned} \quad (1-32)$$



**Figure 4.** The region in the  $(u, v)$ -plane around the intersection of an incoming beam  $\zeta_{\text{in}}$  and an outgoing beam  $\zeta_{\text{out}}$ . The regions  $\mathcal{R}_i$ ,  $i = 1, \dots, 4$ , are vacuum and the renormalized Hawking mass  $\tilde{m}$  is constant (and equal to  $\tilde{m}_i$ ) on each of the  $\mathcal{R}_i$ 's.

and

$$\begin{aligned}\partial_u \tilde{m} &= -2\pi \left( \frac{1 - 2m/r}{-\partial_u r} \right) \tau, \\ \partial_v \tilde{m} &= 2\pi \left( \frac{1 - 2m/r}{\partial_v r} \right) \bar{\tau}.\end{aligned}\tag{1-33}$$

We will also assume that:

- The null dust beams  $\zeta_{\text{in}}$  and  $\zeta_{\text{out}}$  are sufficiently narrow so that, on their intersection  $\zeta_{\text{in}} \cap \zeta_{\text{out}}$ ,  $r$  can be considered nearly constant:<sup>9</sup>

$$\sup_{\zeta_{\text{in}} \cap \zeta_{\text{out}}} r - \inf_{\zeta_{\text{in}} \cap \zeta_{\text{out}}} r \ll \varepsilon (-\Lambda)^{-1/2}.\tag{1-34}$$

- $\bar{\mathfrak{D}}_+ \tilde{m} - \bar{\mathfrak{D}}_- \tilde{m}$  and  $\mathfrak{D}_+ \tilde{m} - \mathfrak{D}_- \tilde{m}$  are relatively small.<sup>10</sup>

Then, (1-32)–(1-33), combined with the conservation laws

$$\partial_u \bar{\tau} = 0, \quad \partial_v \tau = 0,$$

yield the following relations for the change in the mass difference associated to  $\zeta_{\text{in}}$  and  $\zeta_{\text{out}}$  after their intersection:

$$\bar{\mathfrak{D}}_+ \tilde{m} = \bar{\mathfrak{D}}_- \tilde{m} \cdot \exp \left( \frac{2}{r} \frac{\mathfrak{D}_- \tilde{m}}{1 - 2m_2/r} + \mathfrak{Err}_{\text{in}} \right),\tag{1-35}$$

$$\mathfrak{D}_+ \tilde{m} = \mathfrak{D}_- \tilde{m} \cdot \exp \left( -\frac{2}{r} \frac{\bar{\mathfrak{D}}_- \tilde{m}}{1 - 2m_2/r} + \mathfrak{Err}_{\text{out}} \right),\tag{1-36}$$

where the error terms  $\mathfrak{Err}_{\text{in}}$ ,  $\mathfrak{Err}_{\text{out}}$  are negligible compared to the other terms in (1-35), (1-36) (see also the relations (6-51) and (6-52) in Section 6A2). In particular, whenever an ingoing and an outgoing null dust beam intersect, *the mass difference of the ingoing beam increases, while that of the outgoing beam decreases*.

<sup>9</sup>This is possible in view of the fact that, for solutions  $(r, \Omega^2, \tau, \bar{\tau})$  to (1-9),  $r$  remains uniformly continuous in the limit when  $\tau, \bar{\tau}$  tend to  $\delta$ -functions in the  $u$ -,  $v$ -variables, respectively.

<sup>10</sup>Note that, necessarily,  $\bar{\mathfrak{D}}_+ \tilde{m} - \bar{\mathfrak{D}}_- \tilde{m} = -(\mathfrak{D}_+ \tilde{m} - \mathfrak{D}_- \tilde{m})$ .

**Remark.** Notice that, according to (1-35) and (1-36), the change in the mass difference of each of the beams  $\zeta_{\text{in}}$ ,  $\zeta_{\text{out}}$  after their intersection can be estimated in terms of the mass difference of the other beam and the value of  $r$  and  $\inf(1 - 2m/r)$  in the region of intersection. A relation for the change of the mass difference of two infinitely thin, intersecting null dust beams was also obtained in [Poisson and Israel 1990].

*The instability mechanism.* Let us now consider, among the null dust beams arising from the initial data  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$ , the beam  $\zeta_0$  which initially lies to the future of the rest (this is the beam marked with a dashed line in Figure 3). Defining

$$\mathcal{E}_{\zeta_0}[t_*] \doteq \text{mass difference associated to } \zeta_0 \text{ at } \zeta_0 \cap \{u + v = t_*\}, \quad (1-37)$$

we will examine how  $\mathcal{E}_{\zeta_0}$  changes along  $\zeta_0$ , after each successive intersection of  $\zeta_0$  with the rest of the beams:

(1) Starting from  $u = 0$  up to the first reflection of  $\zeta_0$  off the inner mirror  $\gamma_0$ , the beam  $\zeta_0$  is ingoing and intersects all the other beams *after* they are reflected off  $\gamma_0$ . Thus, applying (1-35) successively at each intersection of  $\zeta_0$  with an outgoing beam, we infer that  $\mathcal{E}_{\zeta_0}$  *increases* at this step by a multiplicative factor

$$A_{\text{in}} \geq \exp\left(\frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\gamma_0}}(1 - \varepsilon)\right), \quad (1-38)$$

where  $r_{\gamma_0}$  is the value of  $r$  at the region of intersection of  $\zeta_0$  with the first beam which is reflected off  $\{r = r_{0\varepsilon}\}$  (note that  $r_{\gamma_0}$  is also the  $r$ -width of the bundle of beams when  $\zeta_0$  first reaches the mirror  $\gamma_0$ ). In obtaining (1-38), we have assumed that

$$r_{0\varepsilon} \ll r_{\gamma_0} \ll (-\Lambda)^{-1/2}, \quad \tilde{m}^{(\varepsilon)}|_{\mathcal{I}} \sim r_{0\varepsilon}, \quad \mathcal{E}_{\zeta_0}|_{u=0} \ll \tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$$

(where the latter holds in view of the way the initial data were chosen).

(2) The mass difference  $\mathcal{E}_{\zeta_0}$  right before and right after the reflection of  $\zeta_0$  off  $\gamma_0$  is the same, in view of the reflecting boundary conditions on  $\gamma_0$ .

(3) From its first reflection off  $\gamma_0$  up to its first reflection off  $\mathcal{I}$ , the beam  $\zeta_0$  is outgoing and intersects (again) the rest of the beams in the region close to  $\mathcal{I}$  (*after* these beams are reflected off  $\mathcal{I}$ ). Applying (1-36) successively at each intersection, we infer that  $\mathcal{E}_{\zeta_0}$  *decreases* at this step, being multiplied by a factor

$$1 > A_{\text{out}} \geq \exp\left(-\frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}} - \mathcal{E}_{\zeta_0}|_{u=0})}{r_{\mathcal{I}}}\left(\frac{1}{(1 - \varepsilon - \frac{1}{3}\Lambda r_{\mathcal{I}}^2)} + \varepsilon\right)\right), \quad (1-39)$$

where  $r_{\mathcal{I}}$  is the value of  $r$  at the region of intersection of  $\zeta_0$  with the first beam which is reflected off  $\mathcal{I}$ . In obtaining (1-39), we have assumed that  $r_{\mathcal{I}} \gg (-\Lambda)^{-1/2}$  (which holds in view of the way the initial data were chosen).

(4) The mass difference  $\mathcal{E}_{\zeta_0}$  right before and right after the reflection of  $\zeta_0$  off  $\mathcal{I}$  is the same, in view of the reflecting boundary conditions on  $\mathcal{I}$ .

Therefore, provided  $r_{\gamma_0} \ll (-\Lambda)^{-1/2} \ll r_{\mathcal{I}}$ , we infer that, after the first reflection of  $\zeta_0$  off  $\gamma_0$  and  $\mathcal{I}$ , the mass difference  $\mathcal{E}_{\zeta_0}$  increases by a factor

$$\begin{aligned} A_{\text{tot}} = A_{\text{in}} \cdot A_{\text{out}} &\geq \exp\left(\frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}-\mathcal{E}_{\zeta_0}|_{u=0}})}{r_{\gamma_0}}(1-\varepsilon) - \frac{2(\tilde{m}^{(\varepsilon)}|_{\mathcal{I}-\mathcal{E}_{\zeta_0}|_{u=0}})}{r_{\mathcal{I}}}\left(\frac{1}{(1-\varepsilon-\frac{1}{3}\Lambda r_{\mathcal{I}}^2)} + \varepsilon\right)\right) \\ &\geq \exp\left(\frac{\tilde{m}^{(\varepsilon)}|_{\mathcal{I}-\mathcal{E}_{\zeta_0}|_{u=0}}}{r_{\gamma_0}}\right). \end{aligned} \quad (1-40)$$

The steps (1)–(4) in the above procedure can then be repeated for each successive reflection of  $\zeta_0$  off  $\gamma_0$  and  $\mathcal{I}$ , as long as

$$r_{0\varepsilon} \ll r_{\gamma_0;n} \ll (-\Lambda)^{-1/2} \ll r_{\mathcal{I};n}, \quad (1-41)$$

where  $r_{\gamma_0;n}$ ,  $r_{\mathcal{I};n}$  are the values of  $r_{\gamma_0}$ ,  $r_{\mathcal{I};n}$  after the  $n$ -th reflection of  $\zeta_0$  on  $\gamma_0$  and  $\mathcal{I}$  (note that  $r_{\gamma_0;n}$  is also the  $r$ -width of the bundle of beams at the  $n$ -th reflection of  $\zeta_0$  off  $\gamma_0$ ). Thus, as long as (1-41) holds, denoting with  $\mathcal{E}_{\zeta_0;n}$  the value of  $\mathcal{E}_{\zeta_0}$  at the  $n$ -th reflection of  $\zeta_0$  off  $\gamma_0$ , the following inductive bound holds:

$$\mathcal{E}_{\zeta_0;n} \geq A_{\text{tot};n} \cdot \mathcal{E}_{\zeta_0;n-1}, \quad (1-42)$$

where the multiplicative factor

$$A_{\text{tot};n} \doteq \exp\left(\frac{\tilde{m}^{(\varepsilon)}|_{\mathcal{I}-\mathcal{E}_{\zeta_0;n}}}{r_{\gamma_0;n}}\right) \quad (1-43)$$

is always greater than 1, since  $\mathcal{E}_{\zeta_0;n} < \tilde{m}^{(\varepsilon)}|_{\mathcal{I}}$  (see also the relation (6-13) in Section 6A). *This is the main mechanism driving the instability*, and the proof of Theorem 1 is aimed at showing that, for some large enough  $n(\varepsilon)$  depending on  $\varepsilon$ ,

$$\prod_{n=0}^{n(\varepsilon)} A_{\text{tot};n} > \frac{r_{0\varepsilon}}{2\mathcal{E}_{\zeta_0}|_{u=0}}. \quad (1-44)$$

Inequality (1-44) implies (in view of (1-42)) that

$$\frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} > 1, \quad (1-45)$$

i.e., that, after the  $n(\varepsilon)$ -th successive reflection of  $\zeta_0$  on  $\gamma_0$  and  $\mathcal{I}$ , the mass difference  $\mathcal{E}_{\zeta_0}$  has become so large that a trapped surface (in particular, a point where  $2m/r > 1$ ) necessarily forms before  $\zeta_0$  reaches the mirror  $\gamma_0 = \{r = r_{0\varepsilon}\}$  for the  $(n(\varepsilon)+1)$ -th time (provided  $\zeta_0$  was initially chosen sufficiently “narrow”).<sup>11</sup>

*Control of  $r_{\gamma_0;n}$  and the final step before trapped surface formation.* The main obstacle to establishing (1-44) (and, thus, Theorem 1) is the following: Once  $\mathcal{E}_{\zeta_0}$  exceeds  $c \cdot r_{0\varepsilon}$  for some fixed (small)  $c > 0$ , the total  $r$ -width of the bundle of beams close to  $\gamma_0$ , i.e.,  $r_{\gamma_0;n}$  in (1-43), increases after each successive

<sup>11</sup>We should remark that, once a trapped surface  $\mathcal{S}$  has formed,  $\{r = r_{0\varepsilon}\} \cap J^+(\mathcal{S})$  (where  $J^+(\mathcal{S})$  is the future of  $\mathcal{S}$ ) will be space-like and we will not study the evolution of the spacetime beyond  $\{r = r_{0\varepsilon}\} \cap J^+(\mathcal{S})$ . In particular, no more reflections of the beams will occur in the future of  $\mathcal{S}$ . See Theorem 2.



reflection off  $\gamma_0$  and  $\mathcal{I}$ . Thus, the multiplicative factor (1-43) decreases as  $n$  grows. The increase in  $r_{\gamma_0;n}$  is more dramatic when the spacetime is close to having a trapped surface, i.e., when  $2m/r$  is close to 1.<sup>12</sup>

Controlling the growth of  $r_{\gamma_0;n}$  is achieved by establishing an inductive bound of the form

$$r_{\gamma_0;n} \leq r_{\gamma_0;n-1} \cdot \left( 1 + C_0 \frac{r_{0\varepsilon}}{r_{\gamma_0;n-1}} \left( \left| \log \left( 1 - \frac{2\mathcal{E}_{\zeta_0;n-1}}{r_{0\varepsilon}} \right) \right| + 1 \right) \right) \quad (1-46)$$

(see also the relation (6-14) in Section 6A). Obtaining the bound (1-46) is one of the most demanding parts in the proof of Theorem 1 and requires controlling the  $r$ -distance  $r_{\gamma_0;n}^{(1)}$  of  $\zeta_0$  from the second-to-top beam  $\zeta_1$  at the  $n$ -th reflection off  $\gamma_0$  for all  $n \leq n(\varepsilon)$ , i.e., establishing a bound of the form

$$\frac{r_{\gamma_0;n}^{(1)}}{r_{0\varepsilon}} \geq 1 + c_0 \left( \frac{\mathcal{E}_{\zeta_0;0}}{r_{0\varepsilon}} \right). \quad (1-47)$$

(see (6-12) in Section 6A). The bound (1-47) is in turn obtained by establishing an inductive bound of the form

$$\log \left( \frac{r_{\gamma_0;n-1}^{(1)}}{r_{\gamma_0;n}^{(1)}} \right) \leq C_0 \log \left( \frac{\mathcal{E}_{\zeta_0;n}}{\mathcal{E}_{\zeta_0;n-1}} \right), \quad (1-48)$$

estimating the decrease of  $r_{\gamma_0;n}^{(1)}$  by the increase of  $\mathcal{E}_{\zeta_0;n}$  at each reflection (see (6-129) in Section 6A2). The bound (1-47) is inferred from (1-48), in view of the fact that  $\mathcal{E}_{\zeta_0;n} \geq \mathcal{E}_{\zeta_0;n-1}$  and

$$\sum_{n=1}^{n(\varepsilon)} \log \left( \frac{\mathcal{E}_{\zeta_0;n}}{\mathcal{E}_{\zeta_0;n-1}} \right) = \log \left( \frac{\mathcal{E}_{\zeta_0;n(\varepsilon)}}{\mathcal{E}_{\zeta_0;0}} \right) \leq \log \left( \frac{r_{0\varepsilon}}{2\mathcal{E}_{\zeta_0;0}} \right). \quad (1-49)$$

At the level of the initial data, obtaining (1-46) and (1-48) requires introducing a certain hierarchy for the scales of the  $r$ -distances and mass differences associated to the beams initially (see (5-2) and (5-3) in Section 5A).

Combining (1-42) and (1-46), we can show that there exists a large  $n(\varepsilon)$  such that, after  $n(\varepsilon)$  reflections of  $\zeta_0$  off  $\gamma_0$  (but not earlier!), we have

$$\frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} > 1 - c(\varepsilon), \quad (1-50)$$

where  $c(\varepsilon) \ll h(\varepsilon)$  is a fixed function of  $\varepsilon$ . Note that, compared to (1-45), (1-50) is a slightly weaker bound, which just stops short of implying that a trapped surface is formed. In order to complete the proof of Theorem 1, we therefore have to consider two different scenarios for  $\mathcal{E}_{\zeta_0;n(\varepsilon)}$ :

**Case 1:** In the case when (1-45) holds, the proof of Theorem 1 follows readily, since (1-45) implies that, before  $\zeta_0$  reaches  $\{r = r_{0\varepsilon}\}$  for the  $(n(\varepsilon)+1)$ -th time, a point arises where  $2m/r > 1$ .

<sup>12</sup>The example of two outgoing null rays in the exterior of Schwarzschild-AdS, with mass  $M \ll (-\Lambda)^{-1/2}$ , serves to illustrate this phenomenon: The  $r$ -separation of two rays emanating from the region close to the future event horizon  $\mathcal{H}^+$ , where  $2m/r \sim 1$ , increases dramatically by the time they reach the region  $r \sim (-\Lambda)^{-1/2}$ . This is, of course, nothing other than the celebrated *red-shift* effect.

Case 2: In the case when (1-50) holds but (1-45) is violated, we have the bound

$$1 - c(\varepsilon) < \frac{2\mathcal{E}_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} \leq 1. \quad (1-51)$$

In this case,  $\zeta_0$  reaches  $\{r = r_{0\varepsilon}\}$  for the  $(n(\varepsilon)+1)$ -th time before a trapped surface has formed. One would be tempted to repeat the above procedure for one more reflection, in an attempt to establish that a trapped surface has formed before the  $(n(\varepsilon)+2)$ -th reflection of  $\zeta_0$  off  $\gamma_0$ . However, the bound (1-51) implies that most of the bootstrap assumptions needed for the proof of Theorem 1 (which we have suppressed in this sketch for the sake of simplicity) are violated beyond the  $(n(\varepsilon)+1)$ -th reflection and, thus, the above procedure cannot be repeated. For this reason, we choose a different path: Applying a Cauchy stability statement backwards in time (see Theorem 3.6), we show that there exists a small perturbation  $(r', (\Omega')^2, \tau', \bar{\tau}')^{(\varepsilon)}|_{u=0}$  of the initial data  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$  (satisfying (1-25)), such that the perturbed solution  $(r', (\Omega')^2, \tau', \bar{\tau}')^{(\varepsilon)}$  to (1-9) satisfies (1-45) and, furthermore,

$$\frac{2\mathcal{E}'_{\zeta_0;n(\varepsilon)}}{r_{0\varepsilon}} > 1 \quad (1-52)$$

(where  $\mathcal{E}'_{\zeta_0}$  is similarly defined by the relation (1-37) for  $(r', (\Omega')^2, \tau', \bar{\tau}')^{(\varepsilon)}$  in place of  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}$ ). Thus, we end up in the scenario of Case 1, and the proof of Theorem 1 follows readily.

*Further remarks on the proof of Theorem 1.* The proof of Theorem 1 involves many technical issues related to the final step of the evolution before a trapped surface is formed. Most of these technical issues simplify considerably in the case when one restricts to showing a weaker instability statement for  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$ , e.g., by replacing (1-20) with

$$\left(1 - \frac{2m}{r}\right)^{(\varepsilon)} \Big|_{(u_\varepsilon, v_\varepsilon)} < \frac{1}{2}. \quad (1-53)$$

See Sections 4 and 6A1 for more details.

The mechanism leading to trapped surface formation in the proof of Theorem 1 only made use of the fact that we chose the initial data  $(r, \Omega^2, \tau, \bar{\tau})^{(\varepsilon)}|_{u=0}$  so that the matter was supported in narrow null beams, successively reflected off  $\gamma_0$  and  $\mathcal{I}$ , while the matter model satisfied the condition

$$T_{uv} = \Omega^2 g^{AB} T_{AB} = 0. \quad (1-54)$$

Thus, we expect that the same mechanism can be adapted to the case of more general matter fields, which allow for matter to be arranged into narrow and sufficiently localised null beams, satisfying (in a region around the set of intersection of the beams)

$$T_{uv}, |\Omega^2 g^{AB} T_{AB}| \ll T_{uu} + T_{vv}, \quad (1-55)$$

with such a configuration arising moreover from initial data which are small in a norm for which  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  is Cauchy stable. For an application of this mechanism in the case of the spherically symmetric Einstein-massless Vlasov system (without reducing to the radial case and without an inner mirror), see our forthcoming [Moschidis 2018].

Finally, let us remark that the general mechanism of instability suggested by the proof of Theorem 1 can be summarized as follows: In a configuration consisting of a relatively narrow bundle of nearly null beams of matter that are successively reflected on  $\mathcal{I}$  and  $r = 0$  (on an approximately  $(\mathcal{M}_{\text{AdS}}, g_{\text{AdS}})$  background), the energy content of the “top” beam will increase after each pair of reflections. A similar physical space mechanism was described for the Einstein-scalar field system (1-7) in [Dimitrakopoulos et al. 2015], where it was suggested that, on a nearly null scalar field beam successively reflected off  $\mathcal{I}$  and the center  $r = 0$ , the energy density on the top part of the beam tends to increase.

**1E. Outline of the paper.** This paper is organized as follows:

In Section 2, we will introduce the spherically symmetric Einstein-radial massless Vlasov system in double null coordinates. We will also formulate the notion of reflecting boundary conditions for this system on  $\mathcal{I}$  and on time-like hypersurfaces of the form  $\{r = r_0\}$ .

In Section 3, we will formulate the asymptotically AdS characteristic initial-boundary value problem for the spherically symmetric Einstein-radial massless Vlasov system. We will then recall the main results established in [Moschidis 2017] regarding the structure of the maximal development and the Cauchy stability properties for this system.

In Section 4, we will provide a technical statement of the main result of this paper, namely the instability of AdS for the Einstein-radial massless Vlasov system with reflecting boundary conditions on  $\{r = r_0\}$  and  $\mathcal{I}$ . The proof of this result will occupy Sections 5 and 6.

## 2. The Einstein-massless Vlasov system in spherical symmetry

In this section, we will review the basic properties of the spherically symmetric Einstein-massless Vlasov system in  $3 + 1$  dimensions, expressed in double null coordinates, following the conventions introduced in [Dafermos and Rendall 2016]. We will also introduce the notion of the reflecting boundary condition on time-like hypersurfaces for the radial massless Vlasov equation. To this end, we will follow the conventions adopted in our companion paper [Moschidis 2017].

**2A. Spherically symmetric spacetimes in double null coordinates.** Let  $(\mathcal{M}^{3+1}, g)$  be a smooth Lorentzian manifold such that  $\mathcal{M}$  is of the form

$$\mathcal{M} \simeq \mathcal{U} \times \mathbb{S}^2, \quad (2-1)$$

where  $\mathcal{U}$  is an open domain of  $\mathbb{R}^2$  with piecewise Lipschitz boundary  $\partial\mathcal{U}$  and, in the standard  $(u, v)$ -coordinates on  $\mathcal{U}$ ,  $g$  takes the form

$$g = -\Omega^2(u, v) du dv + r^2(u, v) g_{\mathbb{S}^2}, \quad (2-2)$$

where  $g_{\mathbb{S}^2}$  is the standard round metric on  $\mathbb{S}^2$  and  $\Omega, r : \mathcal{U} \rightarrow (0, +\infty)$  are smooth functions. In addition, we will assume that

$$\inf_{\mathcal{U}} r > 0. \quad (2-3)$$

We will also fix a time orientation on  $\mathcal{M}$  by requiring that the time-like vector field  $N = \partial_u + \partial_v$  is future-directed.

**Remark.** Notice that the action of  $\text{SO}(3)$  on  $(\mathcal{M}, g)$  through rotations of the  $\mathbb{S}^2$  factor of (2-1) is an isometric action.

We will also define the *Hawking mass*  $m : \mathcal{M} \rightarrow \mathbb{R}$  by the expression

$$m = \frac{1}{2}r(1 - g(\nabla r, \nabla r)). \quad (2-4)$$

Viewed as a function on  $\mathcal{U}$ ,  $m$  takes the form

$$m = \frac{1}{2}r(1 + 4\Omega^{-2} \partial_u r \partial_v r). \quad (2-5)$$

Equivalently, we have

$$\Omega^2 = 4 \frac{(-\partial_u r) \partial_v r}{1 - 2m/r}. \quad (2-6)$$

In any local coordinate chart  $(y^1, y^2)$  on  $\mathbb{S}^2$ , the nonzero Christoffel symbols of (2-2) in the  $(u, v, y^1, y^2)$  local coordinate chart on  $\mathcal{M}$  are computed as

$$\begin{aligned} \Gamma_{uu}^u &= \partial_u \log(\Omega^2), & \Gamma_{vv}^v &= \partial_v \log(\Omega^2), \\ \Gamma_{AB}^u &= \Omega^{-2} \partial_v(r^2)(g_{\mathbb{S}^2})_{AB}, & \Gamma_{AB}^v &= \Omega^{-2} \partial_u(r^2)(g_{\mathbb{S}^2})_{AB}, \\ \Gamma_{uB}^A &= r^{-1} \partial_u r \delta_B^A, & \Gamma_{vB}^A &= r^{-1} \partial_v r \delta_B^A, \\ \Gamma_{BC}^A &= (\Gamma_{\mathbb{S}^2})_{BC}^A, \end{aligned} \quad (2-7)$$

where the indices  $A, B, C$  are associated to the spherical coordinates  $y^1, y^2$ ,  $\delta_B^A$  is the Kronecker delta and  $\Gamma_{\mathbb{S}^2}$  are the Christoffel symbols of the round sphere in the  $(y^1, y^2)$  coordinate chart.

For any pair of smooth functions  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  with  $f_1', f_2' \neq 0$ , the coordinate transformation

$$(\bar{u}, \bar{v}) = (f_1(u), f_2(v)), \quad (2-8)$$

mapping  $\mathcal{U}$  to  $\bar{\mathcal{U}} \subset \mathbb{R}^2$ , can be used to diffeomorphically identify  $\mathcal{M}$  with  $\bar{\mathcal{U}} \times \mathbb{S}^2$ . In these new coordinates, the metric  $g$  takes the form

$$g = -\bar{\Omega}^2(\bar{u}, \bar{v}) d\bar{u} d\bar{v} + r^2(\bar{u}, \bar{v}) g_{\mathbb{S}^2}, \quad (2-9)$$

where

$$\bar{\Omega}^2(\bar{u}, \bar{v}) = \frac{1}{f_1' f_2'} \Omega^2(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})), \quad (2-10)$$

$$r(\bar{u}, \bar{v}) = r(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})). \quad (2-11)$$

We will frequently make use of such coordinate transformations, without renaming the coordinates each time.

Note that  $m$  is invariant under coordinate transformations of the form  $(u, v) \rightarrow (f_1(u), f_2(v))$ ; i.e.,

$$m(\bar{u}, \bar{v}) = m(f_1^{-1}(\bar{u}), f_2^{-1}(\bar{v})). \quad (2-12)$$



**2B. The radial massless Vlasov equation.** Let  $(\mathcal{M}, g)$  be as in Section 2A. Let  $f \geq 0$  be a measure on  $T\mathcal{M}$  which is constant along the geodesic flow, that is to say, in any local coordinate chart  $(x^0, x^1, x^2, x^3)$  on  $\mathcal{M}$  with associated momentum coordinates  $(p^0, p^1, p^2, p^3)$  on the fibers of  $T\mathcal{M}$ ,  $f$  satisfies (as a distribution) the first-order equation

$$p^\alpha \partial_{x^\alpha} f - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \partial_{p^\alpha} f = 0, \quad (2-13)$$

where  $\Gamma_{\beta\gamma}^\alpha$  are the Christoffel symbols of  $g$  in the chart  $(x^0, x^1, x^2, x^3)$ . We will call  $f$  a *massless Vlasov field* if it is supported on the set  $P \subset T\mathcal{M}$  of future-directed null vectors; any such vector  $(x; p) \in T\mathcal{M}$  will satisfy

$$g_{\alpha\beta}(x) p^\alpha p^\beta = 0. \quad (2-14)$$

Associated to  $f$  is a symmetric  $(0, 2)$ -form on  $\mathcal{M}$  (possibly defined only in the sense of distributions), the *energy momentum* tensor of  $f$ , given by the expression

$$T_{\alpha\beta}(x) = \int_{\pi^{-1}(x)} p_\alpha p_\beta f, \quad (2-15)$$

where  $\pi^{-1}(x)$  denotes the fiber of  $T\mathcal{M}$  over  $x \in \mathcal{M}$  and the indices of the momentum coordinates are lowered with the use of the metric  $g$ , i.e.,

$$p_\gamma = g_{\gamma\delta}(x) p^\delta. \quad (2-16)$$

**Remark.** In this paper, we will only consider distributions  $f$  for which the expression (2-15) is finite for all  $x \in \mathcal{M}$  and depends smoothly on  $x \in \mathcal{M}$ .

We will consider only distributions  $f$  which are spherically symmetric, i.e., invariant under the action of  $\text{SO}(3)$  on  $\mathcal{M}$ . In that case, in any  $(u, v, y^1, y^2)$  local coordinate chart as in Section 2A, the energy-momentum tensor  $T$  is of the form

$$T = T_{uu}(u, v) du^2 + 2T_{uv}(u, v) du dv + T_{vv}(u, v) dv^2 + T_{AB}(u, v) dy^A dy^B. \quad (2-17)$$

Furthermore, we will restrict to *radial* Vlasov fields  $f$ , i.e., fields supported only on radial null vectors which are normal to the orbits of the action of  $\text{SO}(3)$  on  $\mathcal{M}$ . In any  $(u, v, y^1, y^2)$  local coordinate chart as in Section 2A (with associated momentum coordinates  $(p^u, p^v, p^1, p^2)$ ), a spherically symmetric, radial massless Vlasov field  $f$  has the form

$$f(u, v, y^1, y^2; p^u, p^v, p^1, p^2) = (\bar{f}_{\text{in}}(u, v; p^u) + \bar{f}_{\text{out}}(u, v; p^v)) \delta(\sqrt{(g_{\mathbb{S}^2})_{AB}} p^A p^B) \delta(\Omega^2 p^u p^v), \quad (2-18)$$

where  $\bar{f}_{\text{in}}, \bar{f}_{\text{out}} \geq 0$  and  $\delta$  is the Dirac delta function on  $\mathbb{R}$ . In this case, the only nonzero components of the energy momentum tensor (2-15) are the  $T_{uu}$  and  $T_{vv}$  components. In particular, in terms of  $\bar{f}_{\text{in}}, \bar{f}_{\text{out}}$ , we (formally) compute

$$T_{uu}(u, v) = \int_0^{+\infty} \Omega^4 (p^v)^2 \bar{f}_{\text{out}}(u, v; p^v) r^2 \frac{dp^v}{p^v}, \quad (2-19)$$

$$T_{vv}(u, v) = \int_0^{+\infty} \Omega^4 (p^u)^2 \bar{f}_{\text{in}}(u, v; p^u) r^2 \frac{dp^u}{p^u}. \quad (2-20)$$

**Remark.** In this paper, we will only consider the case when  $\bar{f}_{\text{in}}, \bar{f}_{\text{out}}$  are smooth and compactly supported in the  $p^u$ -,  $p^v$ -variables, respectively.

In the case when  $f$  is of the form (2-18), equation (2-13) is equivalent to the following system for  $\bar{f}_{\text{in}}$  and  $\bar{f}_{\text{out}}$ :

$$\partial_u(\Omega^4 r^4 p^u \bar{f}_{\text{in}}) + \partial_{p^u}(\Omega^2 \partial_u(\Omega^2) r^4 (p^u)^2 \bar{f}_{\text{in}}) = 0, \quad (2-21)$$

$$\partial_v(\Omega^4 r^4 p^v \bar{f}_{\text{out}}) + \partial_{p^v}(\Omega^2 \partial_v(\Omega^2) r^4 (p^v)^2 \bar{f}_{\text{out}}) = 0. \quad (2-22)$$

The equations (2-21)–(2-22) readily yield the following transport equations for  $T_{uu}, T_{vv}$ :

$$\partial_v(r^2 T_{uu}) = 0, \quad (2-23)$$

$$\partial_u(r^2 T_{vv}) = 0. \quad (2-24)$$

**Remark.** Under a coordinate transformation of the form (2-8),  $\bar{f}_{\text{in}}, \bar{f}_{\text{out}}$  transform as

$$\bar{f}_{\text{in}}^{(\text{new})}(f_1(u), f_2(v); f'_1(u)p) = \bar{f}_{\text{in}}(u, v; p), \quad (2-25)$$

$$\bar{f}_{\text{out}}^{(\text{new})}(f_1(u), f_2(v); f'_2(v)p) = \bar{f}_{\text{out}}(u, v; p). \quad (2-26)$$

**2C. The spherically symmetric Einstein-radial massless Vlasov system.** Let  $(\mathcal{M}, g)$  be a smooth Lorentzian manifold and let  $\Lambda < 0$ . Let also  $f$  be a nonnegative measure on  $T\mathcal{M}$ . The *Einstein–Vlasov* system for  $(\mathcal{M}, g; f)$  with cosmological constant  $\Lambda$  is

$$\begin{cases} \text{Ric}_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \\ p^\alpha \partial_{x^\alpha} f - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \partial_{p^\alpha} f = 0, \end{cases} \quad (2-27)$$

where  $T_{\mu\nu}$  is expressed in terms of  $f$  by (2-15).

Restricting to the case where  $(\mathcal{M}, g)$  is a spherically symmetric spacetime as in Section 2A and  $f$  is a radial massless Vlasov field (i.e., has the form (2-18)), the system (2-27) is equivalent to the following system for  $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ :

$$\partial_u \partial_v(r^2) = -\frac{1}{2}(1 - \Lambda r^2)\Omega^2, \quad (2-28)$$

$$\partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2}(1 + 4\Omega^{-2} \partial_u r \partial_v r), \quad (2-29)$$

$$\partial_v(\Omega^{-2} \partial_v r) = -4\pi r T_{vv} \Omega^{-2}, \quad (2-30)$$

$$\partial_u(\Omega^{-2} \partial_u r) = -4\pi r T_{uu} \Omega^{-2}, \quad (2-31)$$

$$\partial_u(\Omega^4 r^4 p^u \bar{f}_{\text{in}}) = -\partial_{p^u}(\Omega^2 \partial_u(\Omega^2) r^4 (p^u)^2 \bar{f}_{\text{in}}), \quad (2-32)$$

$$\partial_v(\Omega^4 r^4 p^v \bar{f}_{\text{out}}) = -\partial_{p^v}(\Omega^2 \partial_v(\Omega^2) r^4 (p^v)^2 \bar{f}_{\text{out}}), \quad (2-33)$$

where  $T_{uu}, T_{vv}$  are expressed in terms of  $\bar{f}_{\text{out}}, \bar{f}_{\text{in}}$  by (2-19), (2-20), respectively. Notice that the system (2-28)–(2-33) reduces to the following system for  $(r, \Omega^2, T_{uu}, T_{vv})$ :

$$\partial_u \partial_v(r^2) = -\frac{1}{2}(1 - \Lambda r^2)\Omega^2, \quad (2-34)$$

$$\partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2}(1 + 4\Omega^{-2} \partial_u r \partial_v r), \quad (2-35)$$

$$\partial_v(\Omega^{-2} \partial_v r) = -4\pi r T_{vv} \Omega^{-2}, \quad (2-36)$$

$$\partial_u(\Omega^{-2} \partial_u r) = -4\pi r T_{uu} \Omega^{-2}, \quad (2-37)$$

$$\partial_u(r^2 T_{vv}) = 0, \quad (2-38)$$

$$\partial_v(r^2 T_{uu}) = 0. \quad (2-39)$$

**Remark.** The system (2-34)–(2-39) is the Einstein-null dust system with both ingoing and outgoing dust (used as a model for self-gravitating radiation already in [Poisson and Israel 1990]). In the notation of Section 1B of the Introduction,

$$r^2 T_{vv} = \bar{\tau}, \quad (2-40)$$

$$r^2 T_{uu} = \tau. \quad (2-41)$$

Defining the renormalized Hawking mass as

$$\tilde{m} \doteq m - \frac{1}{6} \Lambda r^3, \quad (2-42)$$

and using the relation (2-5), equations (2-34)–(2-39) formally give rise to the following system for  $(r, \tilde{m}, T_{uu}, T_{vv})$  (valid in the region of  $\mathcal{U}$  where  $\partial_v r > 0$ ,  $\partial_u r < 0$  and  $1 - 2m/r > 0$ ):

$$\partial_u \log\left(\frac{\partial_v r}{1 - 2m/r}\right) = -4\pi r^{-1} \frac{r^2 T_{uu}}{-\partial_u r}, \quad (2-43)$$

$$\partial_v \log\left(\frac{-\partial_u r}{1 - 2m/r}\right) = 4\pi r^{-1} \frac{r^2 T_{vv}}{\partial_v r}, \quad (2-44)$$

$$\partial_u \partial_v r = -\frac{2\tilde{m} - \frac{2}{3} \Lambda r^3}{r^2} \frac{(-\partial_u r) \partial_v r}{1 - 2m/r}, \quad (2-45)$$

$$\partial_u \tilde{m} = -2\pi \frac{(1 - 2m/r)}{-\partial_u r} r^2 T_{uu}, \quad (2-46)$$

$$\partial_v \tilde{m} = 2\pi \frac{(1 - 2m/r)}{\partial_v r} r^2 T_{vv} \quad (2-47)$$

$$\partial_u(r^2 T_{vv}) = 0, \quad (2-48)$$

$$\partial_v(r^2 T_{uu}) = 0. \quad (2-49)$$

**2D. The reflecting boundary condition for the radial Vlasov equation.** Let  $(\mathcal{M}, g)$  be as in Section 2A. Recall that  $\mathcal{M}$  splits topologically as the product

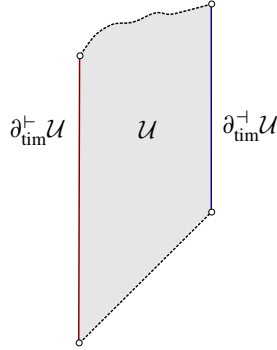
$$\mathcal{M} \simeq \mathcal{U} \times \mathbb{S}^2.$$

Let  $\partial_{\text{tim}} \mathcal{U}$  be the subset of the boundary  $\partial \mathcal{U}$  of  $\mathcal{U} \subset \mathbb{R}^2$  consisting of a union of connected, time-like Lipschitz curves with respect to the comparison metric

$$g_{\text{comp}} = -du dv \quad (2-50)$$

on  $\mathbb{R}^2$ . Recall that a connected Lipschitz curve  $\gamma$  in  $\mathbb{R}^2$  is said to be time-like with respect to (2-50) if, for every point  $p = (u_*, v_*) \in \gamma$ , we have

$$\gamma \setminus p \subset I^+(p) \cup I^-(p) \doteq (\{u > u_*\} \cap \{v > v_*\}) \cup (\{u < u_*\} \cap \{v < v_*\}). \quad (2-51)$$



**Figure 5.** For a domain  $\mathcal{U} \subset \mathbb{R}^2$  as depicted above, the time-like portion  $\partial_{\text{tim}}\mathcal{U}$  of the boundary  $\partial\mathcal{U}$  splits as the union of a “left” component  $\partial_{\text{tim}}^+\mathcal{U}$  and a “right” component  $\partial_{\text{tim}}^-\mathcal{U}$ . In general,  $\partial_{\text{tim}}^+\mathcal{U}$  and  $\partial_{\text{tim}}^-\mathcal{U}$  need not necessarily be straight line segments as depicted above. However, in the following sections, we will impose a gauge condition on the domains under consideration that will indeed fix  $\partial_{\text{tim}}^+\mathcal{U}$  and  $\partial_{\text{tim}}^-\mathcal{U}$  to be vertical line segments (see Definitions 3.2 and 3.3).

Let us fix  $w : \mathcal{U} \cup \partial_{\text{tim}}\mathcal{U} \rightarrow \mathbb{R}$  to be a smooth boundary-defining function of  $\partial_{\text{tim}}\mathcal{U}$ ; i.e.,

$$w|_{\partial_{\text{tim}}\mathcal{U}} = 0, \quad dw|_{\partial_{\text{tim}}\mathcal{U}} \neq 0, \quad w|_{\mathcal{U}} > 0.$$

We can split  $\partial_{\text{tim}}\mathcal{U}$  into its “left” and “right” components as

$$\partial_{\text{tim}}\mathcal{U} = \partial_{\text{tim}}^+\mathcal{U} \cup \partial_{\text{tim}}^-\mathcal{U} \quad (2-52)$$

(see Figure 5), where

$$\begin{aligned} \partial_{\text{tim}}^+\mathcal{U} &= \{(u_0, v_0) \in \partial_{\text{tim}}\mathcal{U} : \partial_v w(u_0, v_0) > 0\}, \\ \partial_{\text{tim}}^-\mathcal{U} &= \{(u_0, v_0) \in \partial_{\text{tim}}\mathcal{U} : \partial_v w(u_0, v_0) < 0\}. \end{aligned}$$

**Remark.** Notice that any future-directed radial null geodesic of  $\mathcal{M} = \mathcal{U} \times \mathbb{S}^2$  with a future-limiting point on  $\partial_{\text{tim}}^+\mathcal{U} \times \mathbb{S}^2$  (in the ambient  $\mathbb{R}^2 \times \mathbb{S}^2$  topology of  $\bar{\mathcal{U}} \times \mathbb{S}^2$ ) is necessarily ingoing. Similarly, future-directed radial null geodesics “terminating” at  $\partial_{\text{tim}}^-\mathcal{U} \times \mathbb{S}^2$  are necessarily outgoing.

In the next sections, we will only consider the reflection of radial null geodesics on parts of  $\partial_{\text{tim}}\mathcal{U}$  for which either  $r - r_0$  (for some constant  $r_0 > 0$ ) or  $1/r$  is a boundary-defining function.

Following [Moschidis 2017], we will define the reflecting boundary condition on  $\partial_{\text{tim}}\mathcal{U}$  for the radial massless Vlasov equation as follows:

**Definition.** A radial massless Vlasov field  $f$  on  $T\mathcal{M}$  will be said to satisfy the *reflecting boundary condition* on  $\partial_{\text{tim}}\mathcal{U} \times \mathbb{S}^2$  if and only if

- for any  $(u_0, v_0) \in \partial_{\text{tim}}^+\mathcal{U}$  and any  $p > 0$

$$\lim_{h \rightarrow 0^+} \left( \frac{\bar{f}_{\text{out}}(u_0, v_0 + h; (-\partial_u w / \partial_v w)(u_0, v_0) \cdot \Omega^{-2}(u_0, v_0 + h) \cdot p)}{\bar{f}_{\text{in}}(u_0 - h, v_0; \Omega^{-2}(u_0 - h, v_0) \cdot p)} \right) = 1, \quad (2-53)$$

- for any  $(u_1, v_1) \in \partial_{\text{tim}}^{-1}\mathcal{U}$  and any  $p > 0$

$$\lim_{h \rightarrow 0^+} \left( \frac{\bar{f}_{\text{in}}(u_1 + h, v_1; (-\partial_v w / \partial_u w)(u_1, v_1) \cdot \Omega^{-2}(u_1 + h, v_1) \cdot p)}{\bar{f}_{\text{out}}(u_1, v_1 - h; \Omega^{-2}(u_1, v_1 - h) \cdot p)} \right) = 1. \quad (2-54)$$

**Remark.** In the above definition, the relations (2-53)–(2-54) are equivalent to the condition that the radial Vlasov field  $f$  is constant along the *reflection* of radial null geodesics off  $\partial_{\text{tim}}\mathcal{U}$ ; for a precise definition of the notion of a reflected radial null geodesic, see [Moschidis 2017].

Note that the relations (2-53) and (2-54) for  $\bar{f}_{\text{in}}$ ,  $\bar{f}_{\text{out}}$  imply the following boundary relations for the components (2-19)–(2-20) of the energy momentum tensor  $T$ :

- For any  $(u_0, v_0) \in \partial_{\text{tim}}^{\perp}\mathcal{U}$ ,

$$\lim_{h \rightarrow 0^+} \frac{r^2 T_{uu}(u_0, v_0 + h)}{r^2 T_{vv}(u_0 - h, v_0)} = \left( \frac{-\partial_u w}{\partial_v w}(u_0, v_0) \right)^2. \quad (2-55)$$

- For any  $(u_1, v_1) \in \partial_{\text{tim}}^{-1}\mathcal{U}$ ,

$$\lim_{h \rightarrow 0^+} \frac{r^2 T_{vv}(u_1 + h, v_1)}{r^2 T_{uu}(u_1, v_1 - h)} = \left( \frac{-\partial_v w}{\partial_u w}(u_0, v_0) \right)^2. \quad (2-56)$$

### 3. The boundary-characteristic initial value problem: well-posedness and Cauchy stability

In this section, we will formulate the asymptotically AdS initial value problem for the system (2-28)–(2-33) with reflecting boundary conditions on  $\{r = r_0\}$  and  $\mathcal{I}$  for some  $r_0 > 0$ . We will then recall the main results established in [Moschidis 2017] regarding the well-posedness and the structure of the maximal development for this system.

**3A. Asymptotically AdS characteristic initial data.** The following was introduced in [Moschidis 2017]:

**Definition 3.1** [Moschidis 2017, Definition 3.1]. For any  $v_1 < v_2$  and any  $r_0 > 0$ , let  $r_{\text{f}} : [v_1, v_2) \rightarrow [r_0, +\infty)$ ,  $\Omega_{\text{f}} : [v_1, v_2) \rightarrow (0, +\infty)$  and  $\bar{f}_{\text{in}/}, \bar{f}_{\text{out}/} : [v_1, v_2) \times (0, +\infty) \rightarrow [0, +\infty)$  be  $C^\infty$  functions, such that

$$r_{\text{f}}(v_1) = r_0, \quad (3-1)$$

$$\lim_{v \rightarrow v_2} r_{\text{f}}(v) = +\infty. \quad (3-2)$$

Let us define  $(\partial_u r)_{\text{f}} : [v_1, v_2) \rightarrow (-\infty, 0)$  by the relation

$$(\partial_u r)_{\text{f}}(v) = \frac{1}{r_{\text{f}}(v)} \left( -r_{\text{f}} \partial_v r_{\text{f}}(v_1) - \frac{1}{4} \int_{v_1}^v (1 - \Lambda r_{\text{f}}^2(\bar{v})) \Omega_{\text{f}}^2(\bar{v}) d\bar{v} \right). \quad (3-3)$$

We will call  $(r_{\text{f}}, \Omega_{\text{f}}^2, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  an *asymptotically AdS boundary-characteristic initial data set* on  $[v_1, v_2)$  for the system (2-28)–(2-33) satisfying the reflecting gauge condition at  $r = r_0, +\infty$  if:

- $(r_{\text{f}}, \Omega_{\text{f}})$  satisfies the constraint equation

$$\partial_v (\Omega_{\text{f}}^{-2} \partial_v r_{\text{f}}) = -4\pi r_{\text{f}} (T_{vv})_{\text{f}} \Omega_{\text{f}}^{-2}, \quad (3-4)$$

where

$$(T_{vv})_{\text{f}}(v) \doteq \int_0^{+\infty} \Omega_{\text{f}}^4(v) (p^u)^2 \bar{f}_{\text{in}/}(v; p^u) r_{\text{f}}^2(v) \frac{dp^u}{p^u}. \quad (3-5)$$

- $\bar{f}_{\text{out}/}$  solves the massless radial Vlasov equation

$$\partial_v (\Omega_{/}^4(v) r_{/}^4(v) p^v \bar{f}_{\text{out}/}(v, p^v)) + \partial_{p^v} (\Omega_{/}^2 \partial_v (\Omega_{/}^2) r_{/}^4(v) (p^v)^2 \bar{f}_{\text{out}/}(v, p^v)) = 0. \quad (3-6)$$

- $(\partial_u r)_{/}$  satisfies

$$\lim_{v \rightarrow v_2^-} \frac{(\partial_u r)_{/}}{\partial_v r_{/}} = 1. \quad (3-7)$$

- $\bar{f}_{\text{out}/}, \bar{f}_{\text{in}/}$  satisfy the following compatibility conditions at  $v = v_1, v_2$  for any  $p > 0$ :

$$\frac{\bar{f}_{\text{out}/}(v_1; (-\partial_u r)_{/} / \partial_v r_{/})(v_1) \cdot \Omega_{/}^{-2}(v_1) \cdot p)}{\bar{f}_{\text{in}/}(v_1; \Omega_{/}^{-2}(v_1) \cdot p)} = 1, \quad (3-8)$$

and

$$\lim_{h \rightarrow 0^+} \left( \frac{\bar{f}_{\text{in}/}(v_2 - h; (\partial_v r_{/} - (\partial_u r)_{/})(v_2 - h) \cdot \Omega_{/}^{-2}(v_2 - h) \cdot p)}{\bar{f}_{\text{out}/}(v_2 - h; \Omega_{/}^{-2}(v_2 - h) \cdot p)} \right) = 1. \quad (3-9)$$

**Remark.** Notice that the constraint equation (3-4) implies

$$\partial_v (\Omega_{/}^{-2} \partial_v r_{/}) \leq 0. \quad (3-10)$$

Thus, (3-2) yields

$$\partial_v r_{/} > 0 \quad (3-11)$$

everywhere on  $[v_1, v_2)$ .

Given any asymptotically AdS boundary-characteristic initial data set  $(r_{/}, \Omega_{/}^2, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  on  $[v_1, v_2)$  with reflecting gauge conditions at  $r = r_0, +\infty$ , we will also define the initial Hawking mass  $m_{/}$  and initial renormalized Hawking mass  $\tilde{m}_{/}$  on  $[v_1, v_2)$  by the relations

$$m_{/} \doteq \frac{1}{2} r_{/} (1 - 4 \Omega_{/}^{-2} (\partial_u r)_{/} \partial_v r_{/}), \quad (3-12)$$

$$\tilde{m}_{/} \doteq m_{/} - \frac{1}{6} \Lambda r_{/}^3, \quad (3-13)$$

in accordance with (2-6), (2-42).

**3B. Developments with reflecting boundary conditions on  $r = r_0, +\infty$ .** We will only consider solutions  $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  to (2-28)–(2-33) satisfying a reflecting gauge condition on  $\partial_{\text{tim}} \mathcal{U}$ , which fixes  $\partial_{\text{tim}} \mathcal{U}$  to be a union of vertical straight lines in the  $(u, v)$ -plane. This motivates defining the following class of domains  $\mathcal{U}$  in the plane (see [Moschidis 2017]):

**Definition 3.2** [Moschidis 2017, Definition 3.3]. For any  $v_0 > 0$ , let  $\mathcal{U}_{v_0}$  be the set of all connected open domains  $\mathcal{U}$  of the  $(u, v)$ -plane with piecewise Lipschitz boundary  $\partial \mathcal{U}$ , with the property that  $\partial \mathcal{U}$  splits as the union

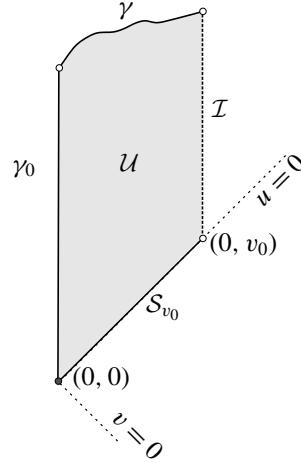
$$\partial \mathcal{U} = \gamma_0 \cup \mathcal{I} \cup \mathcal{S}_{v_0} \cup \text{clos}(\gamma), \quad (3-14)$$

where, for some  $0 < u_{\gamma_0}, u_{\mathcal{I}} \leq +\infty$ ,

$$\gamma_0 = \{u = v\} \cap \{0 \leq u < u_{\gamma_0}\}, \quad (3-15)$$

$$\mathcal{I} = \{u = v - v_0\} \cap \{0 \leq u < u_{\mathcal{I}}\}, \quad (3-16)$$

$$\mathcal{S}_{v_0} = \{0\} \times [0, v_0], \quad (3-17)$$



**Figure 6.** A typical domain  $\mathcal{U} \in \mathcal{U}_{v_0}$  would be as depicted above. In the case when the boundary set  $\gamma$  is empty, it is necessary that both  $\gamma_0$  and  $\mathcal{I}$  are unbounded (i.e., extend all the way to  $u + v = \infty$ ).

and  $\gamma : (x_1, x_2) \rightarrow \mathbb{R}^2$  is a Lipschitz, achronal (with respect to the reference Lorentzian metric (2-50)) curve, which is allowed to be empty (the closure  $\text{clos}(\gamma)$  of  $\gamma$  in (3-14) is considered with respect to the standard topology of  $\mathbb{R}^2$ ). See Figure 6.

**Remark.** It follows readily from Definition 3.2 that  $\mathcal{U}$  is necessarily contained in the future domain of dependence of  $\mathcal{S}_{v_0} \cup \gamma_0 \cup \mathcal{I}$  (with respect to the comparison metric (2-50)). In the case when  $\gamma = \emptyset$  in (3-14), it is necessary that both  $\gamma_0$  and  $\mathcal{I}$  extend all the way to  $u + v = +\infty$ .

A development of an asymptotically AdS boundary-characteristic initial data set for the system (2-28)–(2-33) with reflecting boundary conditions on  $r = r_0, +\infty$  is defined as follows (see [Moschidis 2017]):

**Definition 3.3** [Moschidis 2017, Definition 3.4]. For any  $v_0 > 0$  and  $r_0 > 0$ , let  $(r_/, \Omega_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  be a smooth asymptotically AdS boundary-characteristic initial data set on  $[0, v_0)$  for the system (2-28)–(2-33) satisfying the reflecting gauge condition at  $r = r_0, +\infty$ , according to Definition 3.1. A *future development* of  $(r_/, \Omega_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  will consist of an open set  $\mathcal{U} \in \mathcal{U}_{v_0}$  (see Definition 3.2) and smooth functions  $r : \mathcal{U} \rightarrow (r_0, +\infty)$ ,  $\Omega^2 : \mathcal{U} \rightarrow (0, +\infty)$  and  $\bar{f}_{\text{in}}, \bar{f}_{\text{out}} : \mathcal{U} \times (0, +\infty) \rightarrow [0, +\infty)$  satisfying the following properties:

- (1) The functions  $r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}}$  solve the system (2-28)–(2-33) on  $\mathcal{U}$ .
- (2) The functions  $r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}}$  satisfy the given initial conditions on  $\mathcal{S}_{v_0} = \{0\} \times [0, v_0)$ , i.e.,

$$(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})|_{\mathcal{S}_{v_0}} = (r_/, \Omega_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/}). \quad (3-18)$$

- (3) The functions  $(r, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  satisfy on  $\gamma_0$  the boundary conditions

$$r|_{\gamma_0} = r_0, \quad (3-19)$$

$$\bar{f}_{\text{out}}(u_*, v_*; p) = \bar{f}_{\text{in}}(u_*, v_*; p), \quad (3-20)$$

for all  $(u_*, v_*) \in \gamma_0$  and  $p > 0$ , and on  $\mathcal{I}$  the boundary conditions

$$(1/r)|_{\mathcal{I}} = 0, \quad (3-21)$$

$$\lim_{h \rightarrow 0^+} \left( \frac{\bar{f}_{\text{in}}(u_* + h, v_*; \Omega^{-2}(u_* + h, v_*) \cdot p)}{\bar{f}_{\text{out}}(u_*, v_* - h; \Omega^{-2}(u_*, v_* - h) \cdot p)} \right) = 1, \quad (3-22)$$

for all  $(u_*, v_*) \in \mathcal{I}$  and  $p > 0$ .

(4) The following are satisfied on  $\gamma_0$  and  $\mathcal{I}$ :

$$\partial_u r|_{\gamma_0} = -\partial_v r|_{\gamma_0}, \quad (3-23)$$

$$\partial_u (1/r)|_{\mathcal{I}} = -\partial_v (1/r)|_{\mathcal{I}}. \quad (3-24)$$

**Remark.** Notice that the boundary conditions (3-19) and (3-21), combined with the form (3-15) and (3-16) of  $\gamma_0$  and  $\mathcal{I}$ , respectively, imply the relations (3-23) and (3-24). However, the relations (3-23) and (3-24) should be viewed as *gauge conditions* fixing, in conjunction with (3-19) and (3-21), the form (3-15) and (3-16) of  $\gamma_0$  and  $\mathcal{I}$ .

If  $\mathcal{D} = (\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  and  $\mathcal{D}' = (\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$  are two future developments of the same initial data  $(r/, \Omega^2/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ , we will say that  $\mathcal{D}'$  is an extension of  $\mathcal{D}$ , writing  $\mathcal{D} \subseteq \mathcal{D}'$ , if  $\mathcal{U} \subseteq \mathcal{U}'$  and the restriction of  $(r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$  on  $\mathcal{U}$  coincides with  $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ .

**Remark.** If  $\mathcal{D} = (\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  and  $\mathcal{D}' = (\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$  are two future developments of the same initial data  $(r/, \Omega^2/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ , then

$$(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})|_{\mathcal{U} \cap \mathcal{U}'} = (r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})|_{\mathcal{U} \cap \mathcal{U}'}; \quad (3-25)$$

see [Moschidis 2017].

**3C. The maximal development.** The following result was established in [Moschidis 2017]:

**Theorem 3.4** [Moschidis 2017, Theorem 1]. *For any  $v_0 > 0$  and  $r_0 > 0$ , let  $(r/, \Omega^2/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  be a smooth asymptotically AdS boundary-characteristic initial data set on  $[0, v_0)$  for the system (2-28)–(2-33) satisfying the reflecting gauge condition at  $r = r_0, +\infty$ , according to Definition 3.1, such that the quantities  $\Omega_j^2/(1 - \frac{1}{3}\Lambda r_j^2)$ ,  $r_j^2(T_{vv})/$ , and  $\tan^{-1} r_j$  extend smoothly on  $v = v_0$ . Then, there exists a unique, smooth future development  $(\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  of  $(r/, \Omega^2/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  which is maximal; i.e., any other future development  $(\mathcal{U}'; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$  of  $(r/, \Omega^2/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  with  $r' \geq r_0$  everywhere on  $\mathcal{U}'$  satisfies  $\mathcal{U}' \subseteq \mathcal{U}$  and  $r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}}$  are the restrictions of  $r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}}$  on  $\mathcal{U}'$ .*

*The maximal future development  $(\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  satisfies the following properties (for the definition of the curves  $\gamma_0, \mathcal{I}, \gamma$ , see Definition 3.2):*

(1) *The renormalized Hawking mass  $\tilde{m}$  is conserved on  $\gamma_0$  and  $\mathcal{I}$ , i.e.,*

$$\tilde{m}|_{\gamma_0} = \tilde{m}|_{\gamma_0 \cap \{u=0\}}, \quad (3-26)$$

$$\tilde{m}|_{\mathcal{I}} = \tilde{m}|_{\mathcal{I} \cap \{u=0\}}. \quad (3-27)$$



(2) The curve  $\mathcal{I}$  is conformally complete; i.e.,  $\Omega^2/(1 - \frac{1}{3}\Lambda r^2)$  has a finite limit on  $\mathcal{I}$  and

$$\int_{\mathcal{I}} \sqrt{\frac{\Omega^2}{1 - \frac{1}{3}\Lambda r^2}} \Big|_{\mathcal{I}} du = +\infty. \quad (3-28)$$

(3) We have

$$\partial_u r < 0, \quad (3-29)$$

$$\left(1 - \frac{2m}{r}\right) \Big|_{J^-(\mathcal{I}) \cup J^-(\gamma_0)} > 0 \quad (3-30)$$

$$\partial_v r|_{J^-(\mathcal{I}) \cup J^-(\gamma_0)} > 0, \quad (3-31)$$

where

$$J^-(\mathcal{I}) = \{0 \leq u < \sup_{\mathcal{I}} u\} \cap \mathcal{U} \quad (3-32)$$

is the causal past of  $\mathcal{I}$  and

$$J^-(\gamma_0) = \{0 \leq v < \sup_{\gamma_0} v\} \cap \mathcal{U} \quad (3-33)$$

is the causal past of  $\gamma_0$  (with respect to the reference Lorentzian metric (2-50)).

(4) In the case  $\mathcal{U} \setminus J^-(\mathcal{I}) \neq \emptyset$ , the future event horizon

$$\mathcal{H}^+ \doteq \mathcal{U} \cap \partial J^-(\mathcal{I}) = \{u = \sup_{\mathcal{I}} u\} \cap \mathcal{U} \quad (3-34)$$

has the following properties:

(a)  $\mathcal{H}^+$  has infinite affine length, i.e.,

$$\int_{\mathcal{H}^+} \Omega^2 dv = +\infty. \quad (3-35)$$

(b) We have

$$\sup_{\mathcal{H}^+} r = r_S \quad (3-36)$$

and

$$\inf_{\mathcal{H}^+} \left(1 - \frac{2m}{r}\right) = 0, \quad (3-37)$$

where  $r_S$  defined by the relation

$$1 - 2 \frac{\lim_{v \rightarrow v_0^-} \tilde{m}(v)}{r_S} - \frac{1}{3} \Lambda r_S^2 = 0. \quad (3-38)$$

(5) In the case  $\mathcal{H}^+ \neq \emptyset$ , the curve  $\gamma_0$  is bounded and satisfies

$$\gamma_0 \not\subseteq J^-(\mathcal{I}); \quad (3-39)$$

i.e.,  $\gamma_0$  contains points lying to the future of  $\mathcal{H}^+$ .

(6) In the case  $\mathcal{H}^+ \neq \emptyset$ , the curve  $\gamma$  is nonempty, piecewise smooth and  $r$  extends continuously on  $\gamma$  with  $r|_{\gamma_0} = r_0$ . Furthermore, for any point  $(u_1, v_1) \in \gamma$ , the line  $\{v = v_1\}$  intersects  $\mathcal{I}$ .<sup>13</sup>

<sup>13</sup>In other words, there is no point in  $\gamma$  which lies on the curve  $\{v = v_{\mathcal{I}}\}$ , where  $(u_{\mathcal{I}}, v_{\mathcal{I}})$  is the future limit point of  $\mathcal{I}$ .

**Remark.** In the case when  $\mathcal{U} \setminus J^-(\mathcal{I}) \neq \emptyset$  (and thus  $\mathcal{H}^+ \neq \emptyset$ ), in view of (3-36), (3-38) and the fact that  $r > r_0$  on  $\mathcal{U}$ , it is necessary that

$$2 \frac{\lim_{v \rightarrow v_0^-} \tilde{m}_/(v)}{r_0} > 1 - \frac{1}{3} \Lambda r_0^2. \quad (3-40)$$

In a similar way, we can uniquely define the *maximal past development*  $(\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  of  $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$ , satisfying the properties outlined by Theorem 3.4 after performing a “time reversal” transformation  $(u, v) \rightarrow (-v, -u)$ . Notice that such a coordinate transformation turns an asymptotically AdS boundary-characteristic initial data set on  $u = 0$  into an asymptotically AdS boundary-characteristic initial data set on  $v = 0$ . However, Theorem 3.4 also holds (with exactly the same proof) for such initial data sets.

**3D. Cauchy stability in a rough norm, uniformly in  $r_0$ .** In [Moschidis 2017], the following “norm” was introduced for smooth asymptotically AdS boundary-characteristic initial data sets  $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  on  $[0, v_0)$  for the system (2-28)–(2-33):

$$\begin{aligned} \|(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})\|_{\mathcal{CS}} \\ \doteq \sqrt{-\Lambda} \sup_{0 \leq v < v_0} |\tilde{m}_/(v)| + (-\Lambda) \sup_{0 \leq v < v_0} \int_0^{v_0} \frac{1}{\rho_/(v) - \rho_/(\bar{v}) + \rho_/(0)} \left( \frac{r^2_/(T_{vv})_/}{\partial_v \rho_/} \right) (\bar{v}) d\bar{v} \\ + \sup_{0 \leq v < v_0} \max \left\{ \frac{2m_/}{r_/}, 0 \right\}, \end{aligned} \quad (3-41)$$

where

$$\rho_/(v) \doteq \tan^{-1}(\sqrt{-\Lambda} r_/(v)). \quad (3-42)$$

**Remark.** Note that, in (3-41),

$$\rho_/(0) = \tan^{-1}(\sqrt{-\Lambda} r_0).$$

The expression (3-41) is invariant under gauge transformations, as well as scale transformations of the form  $(u, v) \rightarrow (\lambda u, \lambda v)$ ,  $(r, \tilde{m}, \Lambda) \rightarrow (\lambda r, \lambda \tilde{m}, \lambda^{-2} \Lambda)$ ,  $r_0 \rightarrow \lambda r_0$ ,  $(\bar{f}_{\text{in}}, \bar{f}_{\text{out}}) \rightarrow (\lambda^{-4} \bar{f}_{\text{in}}, \lambda^{-4} \bar{f}_{\text{out}})$ . Moreover,  $\|(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})\|_{\mathcal{CS}} = 0$  if and only if  $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  are the initial data for pure AdS spacetime on  $\{r \geq r_0\}$ , i.e., if  $\bar{f}_{\text{in}/} = \bar{f}_{\text{out}/} = 0$  and  $\tilde{m} = 0$ .

The following Cauchy stability result for the trivial initial data was established in [Moschidis 2017]:

**Proposition 3.5** [Moschidis 2017, Corollary 1]. *For any (possibly large)  $l_* > 0$ , there exists a (small)  $\varepsilon_0 > 0$  and a constant  $C_{l_*} > 0$  depending only on  $l_*$ , so that the following statement holds: for any  $v_0 > 0$  and  $0 < r_0 < (-\Lambda)^{-1/2}$ , if  $(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})$  is a smooth asymptotically AdS boundary-characteristic initial data set on  $[0, v_0)$  for the system (2-28)–(2-33) satisfying the reflecting gauge condition at  $r = r_0, +\infty$ , according to Definition 3.1, such that the quantities  $\Omega^2_/(1 - \frac{1}{3} \Lambda r^2_/)$ ,  $r^2_/(T_{vv})_/$ , and  $\tan^{-1} r_/$  extend smoothly on  $v = v_0$  and moreover*

$$\|(r_/, \Omega^2_/, \bar{f}_{\text{in}/}, \bar{f}_{\text{out}/})\|_{\mathcal{CS}} < \varepsilon \quad (3-43)$$

for some  $0 < \varepsilon \leq \varepsilon_0$ , then the maximal development  $(\mathcal{U}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  satisfies

$$\mathcal{W}_{l_*} \doteq \{0 < u \leq l_* v_0\} \cap \{u < v < u + v_0\} \subset \mathcal{U} \quad (3-44)$$

and

$$\begin{aligned} \sqrt{-\Lambda} \sup_{\mathcal{W}_{l_*}} |\tilde{m}| + \sup_{\mathcal{W}_{l_*}} \log \left( \frac{1 - \frac{1}{3} \Lambda r^2}{1 - \max\{2m/r, 0\}} \right) \\ + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{l_*}} \frac{r T_{vv}}{\partial_v r} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{l_*}} \frac{r T_{uu}}{(-\partial_u r)} du < C_{l_*} \varepsilon. \end{aligned} \quad (3-45)$$

**Remark.** Proposition 3.5 should be interpreted as a Cauchy stability statement for the pure AdS initial data set with respect to the topology defined by (3-41) which is independent of the radius  $r_0$  of the reflecting boundary.

Considering the spherically symmetric Einstein-scalar field system (1-7) with an inner mirror placed at  $\{r = r_0\}$ , the analogue of the initial-data norm (3-41) (obtained using the substitution  $(T_{vv})_{/} \rightarrow (\partial_v \varphi)|_{u=0}$ ) is rougher compared to the bounded variation norm of Christodoulou [1993]. It is not known whether (1-7), restricted to the exterior of an inner mirror at  $\{r = r_0\}$ , satisfies a Cauchy stability estimate with respect to the analogue of the initial-data norm (3-41) which is independent of  $r_0$  (although local existence and uniqueness follow trivially in this case for fixed  $r_0$ ).

In fact, Proposition 3.5 is a special case of the following Cauchy stability estimate established in [Moschidis 2017]:

**Theorem 3.6** [Moschidis 2017, Theorem 2]. *For any  $v_1 < v_2$  and  $0 < r_0 < (-\Lambda)^{-1/2}$ , let  $(r_{/i}, \Omega_{/i}^2, \bar{f}_{\text{in}/i}, \bar{f}_{\text{out}/i})$ ,  $i = 1, 2$ , be two smooth asymptotically AdS boundary-characteristic initial data sets on  $[v_1, v_2)$  for the system (2-28)–(2-33) satisfying the reflective gauge condition at  $r = r_0, +\infty$ , according to Definition 3.1, such that the quantities  $\Omega_{/i}^2/(1 - \frac{1}{3} \Lambda r_{/i}^2)$ ,  $r_{/i}^2 (T_{vv})_{/i}$ , and  $\tan^{-1} r_{/i}$  extend smoothly on  $v = v_2$ . Assume, also, the following conditions:*

(1) *For some  $u_0 > 0$ , the maximal future development  $(\mathcal{U}_1; r_1, \Omega_1^2, \bar{f}_{\text{in}1}, \bar{f}_{\text{out}1})$  of  $(r_{/1}, \Omega_{/1}^2, \bar{f}_{\text{in}/1}, \bar{f}_{\text{out}/1})$  satisfies*

$$\mathcal{W}_{u_0} \doteq \{0 < u < u_0\} \cap \{u + v_1 < v < u + v_2\} \subset \mathcal{U}_1 \quad (3-46)$$

and

$$\begin{aligned} \sup_{\mathcal{W}_{u_0}} \left\{ \left| \log \left( \frac{\Omega_{/1}^2}{1 - \frac{1}{3} \Lambda r_{/1}^2} \right) \right| + \left| \log \left( \frac{2\partial_v r_1}{1 - 2m_1/r_1} \right) \right| + \left| \log \left( \frac{1 - 2m_1/r_1}{1 - \frac{1}{3} \Lambda r_{/1}^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}_1| \right\} \\ + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_0}} r_1 \frac{(T_{vv})_1}{\partial_v r_1} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_0}} r_1 \frac{(T_{uu})_1}{-\partial_u r_1} du = C_0 < +\infty. \end{aligned} \quad (3-47)$$

(2) *The  $(r_{/i}, \Omega_{/i}^2, \bar{f}_{\text{in}/i}, \bar{f}_{\text{out}/i})$ ,  $i = 1, 2$ , are  $\delta$ -close in the following sense:*

$$\begin{aligned} \sup_{v \in [v_1, v_2)} \left\{ \left| \log \left( \frac{\Omega_{/1}^2}{1 - \frac{1}{3} \Lambda r_{/1}^2} \right) - \log \left( \frac{\Omega_{/2}^2}{1 - \frac{1}{3} \Lambda r_{/2}^2} \right) \right| \right. \\ + \left| \log \left( \frac{2\partial_v r_1}{1 - 2m_{/1}/r_{/1}} \right) - \log \left( \frac{2\partial_v r_2}{1 - 2m_{/2}/r_{/2}} \right) \right| + \\ \left. + \left| \log \left( \frac{1 - 2m_{/1}/r_{/1}}{1 - \frac{1}{3} \Lambda r_{/1}^2} \right) - \log \left( \frac{1 - 2m_{/2}/r_{/2}}{1 - \frac{1}{3} \Lambda r_{/2}^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}_{/1} - \tilde{m}_{/2}| \right\} (v) \leq \delta \end{aligned} \quad (3-48)$$

and

$$\sup_{v \in [v_1, v_2]} (-\Lambda) \int_{v_1}^{v_2} \left| r_{/1}^2 \frac{(T_{vv})_{/1}}{\partial_v \rho_1}(\bar{v}) - r_{/2}^2 \frac{(T_{vv})_{/2}}{\partial_v \rho_2}(\bar{v}) \right| (|\rho_/(v) - \rho_/(\bar{v})| + \rho_/(v_1))^{-1} d\bar{v} \leq \delta, \quad (3-49)$$

where  $C_1$  is a large fixed absolute constant,  $\delta$  satisfies

$$0 \leq \delta \leq \delta_0 \doteq \exp\left(-\exp\left(C_1(1+C_0)\frac{u_0}{v_2-v_1}\right)\right) \quad (3-50)$$

and  $\rho_/$  is defined by the relation

$$\rho_/(v) \doteq \tan^{-1}\left(\sqrt{-\frac{\Lambda}{3}} r_/(v)\right). \quad (3-51)$$

Then, the maximal development  $(\mathcal{U}_2; r_2, \Omega_2^2, \bar{f}_{\text{in}2}, \bar{f}_{\text{out}2})$  of  $(r_{/2}, \Omega_{/2}^2, \bar{f}_{\text{in}/2}, \bar{f}_{\text{out}/2})$  satisfies

$$\mathcal{W}_{u_0} \subset \mathcal{U}_2 \quad (3-52)$$

and

$$\begin{aligned} & \sup_{\mathcal{W}_{u_0}} \left\{ \left| \log\left(\frac{\Omega_1^2}{1-\frac{1}{3}\Lambda r_1^2}\right) - \log\left(\frac{\Omega_2^2}{1-\frac{1}{3}\Lambda r_2^2}\right) \right| + \left| \log\left(\frac{2\partial_v r_1}{1-2m_1/r_1}\right) - \log\left(\frac{2\partial_v r_2}{1-2m_2/r_2}\right) \right| \right. \\ & \quad \left. + \left| \log\left(\frac{1-2m_1/r_1}{1-\frac{1}{3}\Lambda r_1^2}\right) - \log\left(\frac{1-2m_2/r_2}{1-\frac{1}{3}\Lambda r_2^2}\right) \right| + \sqrt{-\Lambda} |\tilde{m}_1 - \tilde{m}_2| \right\} \\ & \quad + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_0}} |r_1(T_{vv})_1 - r_2(T_{vv})_2| dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_0}} |r_1(T_{uu})_1 - r_2(T_{uu})_2| du \\ & \leq \exp\left(\exp(C_1(1+C_0))\frac{u_0}{v_2-v_1}\right) \delta. \end{aligned} \quad (3-53)$$

**Remark.** By repeating the proof of Theorem 3.6, the Cauchy stability estimate (3-53) also holds in the case when  $(\mathcal{U}_i; r_i, \Omega_i^2, \bar{f}_{\text{in};i}, \bar{f}_{\text{out};i})$ ,  $i = 1, 2$ , are the maximal *past* developments of  $(r_{/i}, \Omega_{/i}^2, \bar{f}_{\text{in}/i}, \bar{f}_{\text{out}/i})$ , i.e., when  $\mathcal{W}_{u_0}$  is replaced by

$$\mathcal{W}_{u_0}^{(-)} \doteq \{-u_0 \leq u < 0\} \cap \{u + v_1 < v < u + v_2\} \quad (3-54)$$

and (3-47) holds on  $\mathcal{W}_{u_0}^{(-)}$  in place of  $\mathcal{W}_{u_0}$ .

#### 4. Final statement of Theorem 1: the nonlinear instability of AdS

The main result of this paper is the following:

**Theorem 1** (final version). *For any  $\varepsilon \in (0, 1]$ , there exist  $r_{0\varepsilon}, v_{0\varepsilon}$  depending smoothly on  $\varepsilon$  such that*

$$r_{0\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (4-1)$$

$$\sqrt{-\Lambda} v_{0\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \sqrt{3}\pi, \quad (4-2)$$

as well as a family  $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\text{in}/}^{(\varepsilon)}, \bar{f}_{\text{out}/}^{(\varepsilon)})$  of smooth asymptotically AdS boundary-characteristic initial data sets for the system (2-28)–(2-33) satisfying the reflecting gauge condition at  $r = r_0, +\infty$ , such that the following hold:

(1) The family  $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\text{in}/}^{(\varepsilon)}, \bar{f}_{\text{out}/}^{(\varepsilon)})$  satisfies

$$\|(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\text{in}/}^{(\varepsilon)}, \bar{f}_{\text{out}/}^{(\varepsilon)})\|_{\mathcal{CS}} \xrightarrow{\varepsilon \rightarrow 0} 0, \quad (4-3)$$

where  $\|\cdot\|_{\mathcal{CS}}$  is the norm defined by (3-41).

(2) There exists a trapped sphere, i.e., a point  $(u_{\dagger}, v_{\dagger})$  in the maximal future development  $(\mathcal{U}_{\varepsilon}; r_{\varepsilon}, \Omega_{\varepsilon}^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$  of  $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\text{in}/}^{(\varepsilon)}, \bar{f}_{\text{out}/}^{(\varepsilon)})$  such that

$$\frac{2m}{r}(u_{\dagger}, v_{\dagger}) > 1. \quad (4-4)$$

In particular, in view of Theorem 3.4,  $(\mathcal{U}_{\varepsilon}; r_{\varepsilon}, \Omega_{\varepsilon}^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$  has a nonempty future event horizon  $\mathcal{H}^+$  (defined by (5-30)), satisfying the properties (4a) and (4b) of Theorem 3.4, and a complete conformal infinity  $\mathcal{I}$  (satisfying (3-28)).

**Remark.** If

$$\delta(\varepsilon) \doteq \|(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\text{in}/}^{(\varepsilon)}, \bar{f}_{\text{out}/}^{(\varepsilon)})\|_{\mathcal{CS}}, \quad (4-5)$$

the point  $(u_{\dagger}, v_{\dagger})$  satisfies the upper bound

$$u_{\dagger} \leq \exp(\exp(\delta^6(\varepsilon)))v_0. \quad (4-6)$$

On the other hand, in view of Proposition 3.5, we necessarily have

$$u_{\dagger} \xrightarrow{\varepsilon \rightarrow 0} +\infty. \quad (4-7)$$

In the simpler case when one is interested in a weaker instability statement, such as the existence of a point  $(u_{\dagger}, v_{\dagger})$  where

$$\frac{2m}{r} \Big|_{(u_{\dagger}, v_{\dagger})} > \frac{1}{2} \quad (4-8)$$

(instead of the stronger bound (4-4)), the proof of Theorem 1 can be substantially simplified. In the case of (4-8), the upper bound (4-6) can be improved into a polynomial bound

$$u_{\dagger} \leq (\delta(\varepsilon))^{-C_1} v_0 \quad (4-9)$$

for some fixed  $C_1 > 0$ .

## 5. Construction of the initial data and notation

As described already in Section 1D of the Introduction, the initial data family in Theorem 1 will be such that their development consists of a large number of initially ingoing Vlasov beams. In this section, we will construct such a family  $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  of asymptotically AdS boundary-characteristic initial data for (2-28)–(2-33). The family  $(r_{/}^{(\varepsilon)}, (\Omega_{/}^{(\varepsilon)})^2, \bar{f}_{\text{in}/}^{(\varepsilon)}, \bar{f}_{\text{out}/}^{(\varepsilon)})$  in the statement of Theorem 1 will be eventually obtained from  $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  after possibly adding a suitable perturbation (see Section 6).

This section is organized as follows: In Section 5A, we will introduce a certain hierarchy of parameters that will be necessary for the construction of  $(r/\varepsilon, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  in Section 5B. In Section 5C, we will introduce some basic notation related to the maximal future development  $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$  of  $(r/\varepsilon, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ . Finally, in Section 5D, we will perform some basic geometric constructions on  $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$ , related to the separation of  $\mathcal{U}_\varepsilon$  into various subregions by the Vlasov beams arising from the initial data.

**5A. Parameters and auxiliary functions.** Let us fix some smooth and strictly increasing functions  $h_0, h_1, h_2 : (0, 1) \rightarrow (0, 1)$ , so that

$$\lim_{\varepsilon \rightarrow 0^+} h_0(\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} h_1(\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} h_2(\varepsilon) = 0, \quad (5-1)$$

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon \cdot \exp\left(\frac{1}{(h_1(\varepsilon))^6}\right) = \lim_{\varepsilon \rightarrow 0^+} h_1(\varepsilon) \cdot \exp\left(\exp\left(\frac{1}{(h_0(\varepsilon))^6}\right)\right) = 0, \quad (5-2)$$

$$\lim_{\varepsilon \rightarrow 0^+} h_2(\varepsilon) \cdot \exp(\varepsilon^{-2}) = 0. \quad (5-3)$$

In particular, the following relations hold for  $\varepsilon \ll 1$ :

$$h_2(\varepsilon) \ll \varepsilon \ll h_1(\varepsilon) \ll h_0(\varepsilon) \ll 1. \quad (5-4)$$

Let  $\chi : \mathbb{R} \rightarrow [0, 1]$  be a smooth cut-off function, satisfying  $\chi|_{[-1, 1]} = 1$ ,  $\chi|_{\mathbb{R} \setminus [-2, 2]} = 0$  and

$$\chi|_{(-2, 2)} > 0, \quad (5-5)$$

and let  $\varepsilon_0 \ll 1$  be a small enough absolute constant. For any  $0 < \varepsilon < \varepsilon_0$  and any  $r_0 > 0$  satisfying

$$1 - \exp(-2(h_0(\varepsilon))^{-4}) < \frac{r_0}{(2/\sqrt{-\Lambda})\varepsilon - \frac{1}{3}\Lambda r_0^3} < 1 - \frac{1}{2}\exp(-2(h_0(\varepsilon))^{-4}) \quad (5-6)$$

(note that (5-6) implies that  $r_0\sqrt{-\Lambda}/(2\varepsilon) = 1 + O(\exp(-2(h_0(\varepsilon))^{-4}))$  as  $\varepsilon \rightarrow 0$ ), we will define the following function on  $[0, +\infty) \times (0, +\infty)$ :

$$\bar{f}_\varepsilon(v, p^u) \doteq C_{\varepsilon r_0} \sum_{j=0}^{\lceil 1/h_1(\varepsilon) \rceil} \chi(p^u - 3) \cdot \frac{1}{h_2(\varepsilon)} \chi\left(\frac{(v - v^{(j)})\sqrt{-\Lambda} - 2h_2(\varepsilon)}{h_2(\varepsilon)}\right) \cdot h_{(j)}(\varepsilon) \cdot \varepsilon \quad (5-7)$$

for some constant  $C_{\varepsilon r_0}$  to be specified in terms of  $\varepsilon, r_0$  later, where  $\lceil 1/h_1(\varepsilon) \rceil$  denotes the least integer greater than or equal to  $1/h_1(\varepsilon)$

$$v^{(j)} = \frac{\pi}{\sqrt{-\Lambda}} - j \frac{\varepsilon}{h_1(\varepsilon)\sqrt{-\Lambda}} \quad (5-8)$$

for any  $0 \leq j \leq \lceil 1/h_1(\varepsilon) \rceil$ ,

$$h_{(0)} = h_0, \quad (5-9)$$

$$h_{(j)} = h_1 \quad (5-10)$$

for all  $1 \leq j \leq \lceil 1/h_1(\varepsilon) \rceil$ .

**5B. Construction of the initial data family.** For any  $0 < \varepsilon < \varepsilon_0$  and any  $r_0$  satisfying (5-6), we will define the following asymptotically AdS boundary-characteristic initial data set according to Definition 3.1:

**Definition 5.1.** For any  $0 < \varepsilon < \varepsilon_0$  and any  $r_0$  satisfying (5-6), we define  $v_0 = v_0(r_0, \varepsilon) > 0$  and the set of smooth functions  $r_{/\varepsilon} : [0, v_0) \rightarrow [r_0, +\infty)$ ,  $\Omega_{/\varepsilon}^2 : [0, v_0) \rightarrow (0, +\infty)$ ,  $\bar{f}_{\text{in}/\varepsilon} : [0, v_0) \times (0, +\infty) \rightarrow [0, +\infty)$ , and  $\bar{f}_{\text{out}/\varepsilon} : [0, v_0) \times (0, +\infty) \rightarrow [0, +\infty)$  by the requirement that  $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  is an asymptotically AdS boundary-characteristic initial data set on  $[0, v_0)$  for the system (2-28)–(2-33) satisfying the reflecting gauge condition at  $r = r_0, +\infty$ , so that

$$\begin{cases} \frac{\partial_v r_{/\varepsilon}}{1 - 2m_{/\varepsilon}/r_{/\varepsilon}} = \frac{\Omega_{/\varepsilon}^2}{4\partial_v r_{/\varepsilon}}, \\ \left. \frac{\partial_v r_{/\varepsilon}}{1 - \frac{1}{3}\Lambda r_{/\varepsilon}} \right|_{v=0} = \frac{1}{2}, \end{cases} \quad (5-11)$$

$$\bar{f}_{\text{out}/\varepsilon} = 0, \quad (5-12)$$

$$\bar{f}_{\text{in}/\varepsilon}(v, p^u) = \bar{f}_\varepsilon(v, p^u) \quad (5-13)$$

for all  $0 \leq v \leq v_0$  and  $p^u > 0$ . The constant  $C_{\varepsilon r_0}$  in (5-7) is fixed in terms of  $\varepsilon, r_0$  by the requirement that

$$\lim_{v \rightarrow v_0} \tilde{m}_{/\varepsilon} = \frac{\varepsilon}{\sqrt{-\Lambda}} \quad (5-14)$$

(in particular, there exists some fixed (large)  $C_0 > 1$ , independent of  $\varepsilon, r_0$ , so that  $C_{\varepsilon r_0} \in [C_0^{-1}, C_0]$  for any  $0 < \varepsilon < \varepsilon_0$ , any  $r_0$  satisfying (5-6)).<sup>14</sup> See Figure 7.

**Remark.** The conditions (5-11)–(5-13) determine (implicitly)  $v_0$  and  $r_{/\varepsilon}, \Omega_{/\varepsilon}^2$  uniquely in terms of  $\varepsilon, r_0$ , provided  $\varepsilon_0$  has been fixed small enough: in view of the fact that (5-11) is equivalent to

$$\begin{cases} \frac{\partial_v r_{/\varepsilon}}{1 - 2m_{/\varepsilon}/r_{/\varepsilon}} = \frac{-\partial_u r_{/\varepsilon}}{1 - 2m_{/\varepsilon}/r_{/\varepsilon}}, \\ \left. \frac{\partial_v r_{/\varepsilon}}{1 - \frac{1}{3}\Lambda r_{/\varepsilon}} \right|_{v=0} = \frac{1}{2} \end{cases} \quad (5-15)$$

(where  $m_{/\varepsilon}$  is defined in terms of  $r_{/\varepsilon}, \Omega_{/\varepsilon}^2$  by (3-12)), from (5-14) and (2-44) (which is equivalent to the constraint equation (3-4)) we infer that

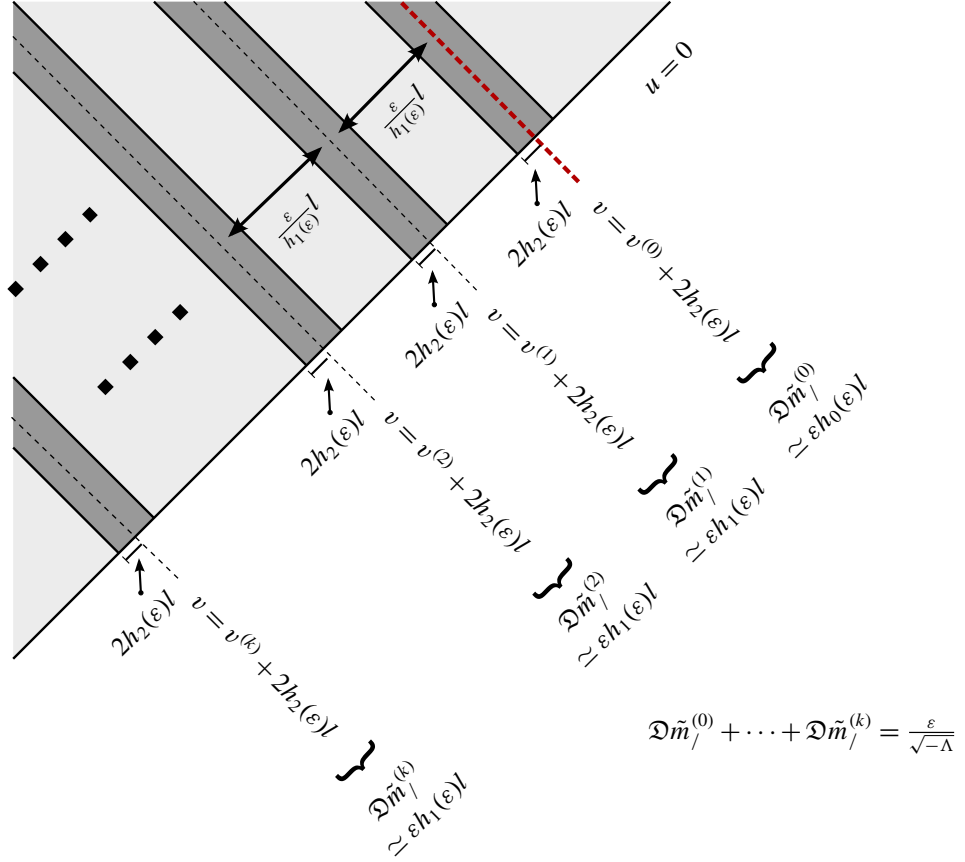
$$\frac{\partial_v r_{/\varepsilon}}{1 - \frac{1}{3}\Lambda r_{/\varepsilon}} = \frac{1}{2} + O(\varepsilon) \quad (5-16)$$

and, therefore,

$$v_0 = \sqrt{-\frac{3}{\Lambda}}\pi + O(\varepsilon). \quad (5-17)$$

While  $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  depend on both  $\varepsilon$  and  $r_0$ , we will only use the subscript  $\varepsilon$  in their notation, since most of the estimates that we will later establish for their maximal development will depend only on  $\varepsilon$ .

<sup>14</sup>In fact, it suffices to choose  $C_0 = 50$ .



**Figure 7.** The initial data  $(r_{\epsilon/\epsilon}, \Omega_{\epsilon/\epsilon}^2, \bar{f}_{in/\epsilon}, \bar{f}_{out/\epsilon})$  give rise to  $k+1$  Vlasov beams, initially arranged as depicted above (using, for convenience, the abbreviation  $k = \lceil 1/h_1(\epsilon) \rceil$  and  $l = (-\Lambda)^{-1/2}$ ). For any  $0 \leq j \leq k$ , we can associate to the beam centered around  $v = v^{(j)} + 2h_2(\epsilon)l$  the initial mass difference  $\mathfrak{D}\tilde{m}_j^{(j)}$ , defined as the renormalized Hawking mass difference between the two vacuum regions surrounding the beam. The top beam (centered initially around  $v = v^{(0)} + 2h_2(\epsilon)l$ ) has a greater initial mass difference than the rest of the beams.

The initial data  $(r_{\epsilon/\epsilon}, (\Omega_{\epsilon/\epsilon}^{(\epsilon)})^2, \bar{f}_{in/\epsilon}^{(\epsilon)}, \bar{f}_{out/\epsilon}^{(\epsilon)})$  in the statement of Theorem 1 will eventually be chosen to be small perturbations of  $(r_{\epsilon/\epsilon}, \Omega_{\epsilon/\epsilon}^2, \bar{f}_{in/\epsilon}, \bar{f}_{out/\epsilon})$  (see Section 6).

**5C. Notational conventions and basic computations.** For any  $0 < \epsilon < \epsilon_0$  and any  $r_0$  satisfying (5-6), let  $(r_{\epsilon/\epsilon}, \Omega_{\epsilon/\epsilon}^2, \bar{f}_{in/\epsilon}, \bar{f}_{out/\epsilon})$  be the initial data set defined by Definition 5.1. Assuming that  $\epsilon_0$  is fixed small enough, for any  $0 < \epsilon < \epsilon_0$  and any  $r_0$  satisfying (5-6), the initial data set  $(r_{\epsilon/\epsilon}, \Omega_{\epsilon/\epsilon}^2, \bar{f}_{in/\epsilon}, \bar{f}_{out/\epsilon})$  satisfies the following estimate depending only on  $\epsilon$ :

$$\|(r_{\epsilon/\epsilon}, \Omega_{\epsilon/\epsilon}^2, \bar{f}_{in/\epsilon}, \bar{f}_{out/\epsilon})\|_{CS} \leq Ch_0(\epsilon), \quad (5-18)$$

where  $\|\cdot\|_{CS}$  is defined by (3-41) and  $C > 0$  is a fixed constant.



Let  $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$  be the maximal future development of  $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  (see Theorem 3.4). In view of Proposition 3.5, the bound (5-18) implies that, for any fixed  $u_* > 0$  and any  $\delta > 0$ , there exists an  $\varepsilon_{\delta u_*} > 0$  sufficiently small depending only on  $\delta$  and  $u_*$  so that, for any  $0 \leq \varepsilon < \varepsilon_{\delta u_*}$ ,

$$\mathcal{W}_{u_*} \doteq \{0 < u < u_*\} \cap \{u < v < u + v_0\} \subset \mathcal{U} \quad (5-19)$$

and

$$\begin{aligned} \sqrt{-\Lambda} \sup_{\mathcal{W}_{u_*}} |\tilde{m}_\varepsilon| + \sup_{\mathcal{W}_{u_*}} \log \left( \frac{1 - \frac{1}{3} \Lambda r_\varepsilon^2}{1 - \max\{2m_\varepsilon/r_\varepsilon, 0\}} \right) \\ + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_*}} \frac{r_\varepsilon (T_{vv})_\varepsilon}{\partial_v r_\varepsilon} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_*}} \frac{r_\varepsilon (T_{uu})_\varepsilon}{(-\partial_u r_\varepsilon)} du < \delta, \end{aligned} \quad (5-20)$$

where  $m_\varepsilon, \tilde{m}_\varepsilon, (T_{uu})_\varepsilon, (T_{vv})_\varepsilon$  are defined in terms of  $r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon}$  by (2-5), (2-42), (2-19) and (2-20). In particular, if

$$g_{\text{AdS}} = -\Omega_{\text{AdS}, r_0, v_0}^2 du dv + r_{\text{AdS}, r_0, v_0} g_{\mathbb{S}^2} \quad (5-21)$$

is the pure AdS metric in a spherically symmetric coordinate chart  $(u, v)$  such that  $r_{\text{AdS}, r_0, v_0} = r_0$  on  $\{u = v\}$  and  $r_{\text{AdS}, r_0, v_0} = +\infty$  on  $\{u = v - v_0\}$ ,<sup>15</sup> then  $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \bar{f}_{\text{in}\varepsilon}, \bar{f}_{\text{out}\varepsilon})$ , when restricted on  $\mathcal{W}_{u_*}$ , is  $\delta$ -close to  $(\mathcal{W}_{u_*}; r_{\text{AdS}, r_0, v_0}, \Omega_{\text{AdS}, r_0, v_0}^2, 0, 0)$  with respect to the (gauge-invariant) distance defined by (3-41). Notice also that (5-20) implies that, provided  $\delta$  is small enough, the spacetime  $(\mathcal{W}_{u_*} \times \mathbb{S}^2, g_\varepsilon)$  does not contain any trapped surface, where

$$g_\varepsilon = -\Omega_\varepsilon^2 du dv + r_\varepsilon^2 g_{\mathbb{S}^2}. \quad (5-22)$$

Notice that, in view of the conservation of  $\tilde{m}$  on  $\gamma_0$  and  $\mathcal{I}$  (see (3-26) and (3-27)), we have

$$\tilde{m}_\varepsilon|_{\gamma_0} = 0, \quad (5-23)$$

$$\tilde{m}_\varepsilon|_{\mathcal{I}} = \lim_{v \rightarrow v_0} \tilde{m}_{/\varepsilon}(v) = \frac{\varepsilon}{\sqrt{-\Lambda}}. \quad (5-24)$$

For each  $0 \leq j \leq \lceil 1/h_1(\varepsilon) \rceil$ , we can associate to the beam centered at

$$v = v^{(j)} + \frac{2}{\sqrt{-\Lambda}} h_2(\varepsilon)$$

the mass difference

$$\mathfrak{D}\tilde{m}_{/}^{(j)} \doteq \tilde{m}_{/\varepsilon} \left( v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) - \tilde{m}_{/\varepsilon}(v^{(j)}). \quad (5-25)$$

Notice that

$$\mathfrak{D}\tilde{m}_{/}^{(0)} \simeq \frac{h_0(\varepsilon)\varepsilon}{\sqrt{-\Lambda}} \quad (5-26)$$

and, for all  $1 \leq j \leq \lceil 1/h_1(\varepsilon) \rceil$ ,

$$\mathfrak{D}\tilde{m}_{/}^{(j)} \simeq \frac{h_1(\varepsilon)\varepsilon}{\sqrt{-\Lambda}}. \quad (5-27)$$

<sup>15</sup>Note that such a coordinate chart is not unique.

Furthermore

$$\sum_{j=0}^{\lceil 1/h_1(\varepsilon) \rceil} \mathcal{D}\tilde{m}_j^{(j)} = \lim_{v \rightarrow v_0} \tilde{m}_{/\varepsilon}(v) = \frac{\varepsilon}{\sqrt{-\Lambda}}. \quad (5-28)$$

**5D. Some geometric constructions on  $\mathcal{U}_\varepsilon$ .** For any  $0 < \varepsilon < \varepsilon_0$  and any  $r_0$  satisfying (5-6), we will define some special subsets of the domain  $\mathcal{U}_\varepsilon$  of the maximal future development  $(\mathcal{U}_\varepsilon; r_\varepsilon, \Omega_\varepsilon^2, \tilde{f}_{\text{in}\varepsilon}, \tilde{f}_{\text{out}\varepsilon})$  of the initial data set  $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \tilde{f}_{\text{in}/\varepsilon}, \tilde{f}_{\text{out}/\varepsilon})$ .

**Remark.** In the rest of this section, we will adopt the convention that the boundary  $\partial A$  of a subset  $A \subseteq \mathcal{U}_\varepsilon$  is the boundary of  $A$  as a subset of  $\mathbb{R}^2$  (with respect to the ambient topology of  $\mathbb{R}^2$ ).

Let us define the domain of outer communications  $\mathcal{D}_\varepsilon$  of  $\mathcal{U}_\varepsilon$  as

$$\mathcal{D}_\varepsilon \doteq J^-(\mathcal{I}) \cap \mathcal{U}_\varepsilon, \quad (5-29)$$

where  $J^-(\mathcal{I})$  is the causal past of  $\mathcal{I}$  with respect to the reference metric (2-50) (see (3-32)). In accordance with Theorem 3.4, we will also define the future event horizon  $\mathcal{H}_\varepsilon^+$  of  $\mathcal{U}_\varepsilon$  as

$$\mathcal{H}_\varepsilon^+ \doteq \partial \mathcal{D}_\varepsilon \cap \mathcal{U}_\varepsilon. \quad (5-30)$$

Note that we allow  $\mathcal{H}_\varepsilon^+$  to be empty. In view of Theorem 3.4, in the case when  $\mathcal{H}_\varepsilon^+$  is nonempty, it is necessarily of the form

$$\mathcal{H}_\varepsilon^+ = \{u = u_{\mathcal{H}_\varepsilon^+}\} \cap \mathcal{U}_\varepsilon \quad (5-31)$$

and has infinite affine length.

We will also define

$$\mathcal{J}_\varepsilon \doteq J^-(\gamma_0) \cap \mathcal{U}_\varepsilon. \quad (5-32)$$

Notice that, as a consequence of Theorem 3.4, on  $\mathcal{J}_\varepsilon \cup \mathcal{D}_\varepsilon$  we have

$$1 - \frac{2m}{r} > 0; \quad (5-33)$$

i.e., trapped spheres can only appear in the region  $\mathcal{U}_\varepsilon \setminus (\mathcal{J}_\varepsilon \cup \mathcal{D}_\varepsilon)$ . In the case  $\mathcal{H}_\varepsilon^+ \neq \emptyset$ , Theorem 3.4 also implies that  $\mathcal{J}_\varepsilon \setminus \mathcal{D}_\varepsilon \neq \emptyset$ .

For any  $v_* \in [0, v_0]$  and any integer  $n \geq 1$ , we will define

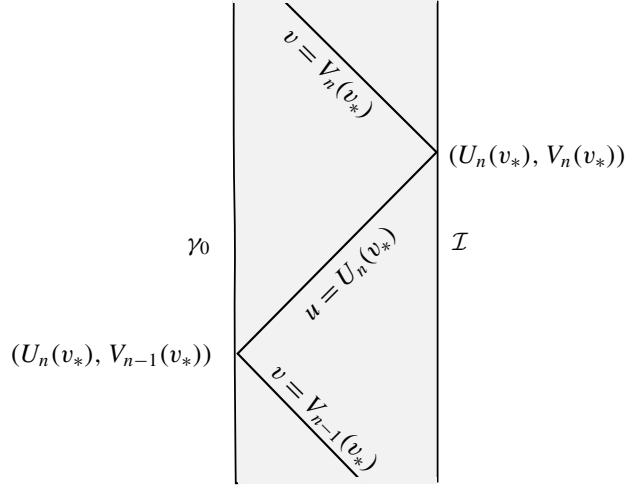
$$U_n(v_*) \doteq v_* + (n-1)v_0, \quad (5-34)$$

$$V_n(v_*) = v_* + nv_0. \quad (5-35)$$

We will also set

$$V_0(v_*) \doteq v_*. \quad (5-36)$$

Notice that the segment  $\{u = U_n(v_*)\} \cap \mathcal{U}_\varepsilon$  is the image of the ingoing null geodesic of  $\mathcal{U}_\varepsilon$  emanating from the point  $(0, v_*)$  after  $n$  reflections off  $\gamma_0$  and  $n-1$  reflections off  $\mathcal{I}$ , while the segment  $\{v = V_n(v_*)\} \cap \mathcal{U}_\varepsilon$  is the image of the same null geodesic after  $n$  reflections off  $\gamma_0$  and  $n$  reflections off  $\mathcal{I}$ ; see Figure 8.



**Figure 8.** Schematic depiction of the lines  $v = V_{n-1}(v_*)$ ,  $u = U_n(v_*)$ , and  $v = V_n(v_*)$ .

Let us define the domains  $\mathcal{R}_{\varepsilon n}^{(i,j)} \subset \mathcal{U}_\varepsilon$  for any  $n \in \mathbb{N}$ ,  $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$ , and  $i \leq j \leq \lceil 1/h_1(\varepsilon) \rceil + i + 1$  by the relation

$$\mathcal{R}_{\varepsilon n}^{(i,j)} = \left\{ U_n \left( v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) < u < U_n(v^{(i-1)}) \right\} \cap \left\{ V_n \left( v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) < v < V_n(v^{(j-1)}) \right\} \cap \mathcal{U}_\varepsilon, \quad (5-37)$$

where we have used the following conventions in the expression (5-37):

- (1)  $U_n(v^{(-1)}) \doteq U_{n+1}(v^{(\lceil 1/h_1(\varepsilon) \rceil)})$ .
- (2)  $V_n(v^{(\lceil 1/h_1(\varepsilon) \rceil + l)} + c) \doteq V_{n-1}(v^{(l-1)} + c)$  for any integer  $1 \leq l \leq \lceil 1/h_1(\varepsilon) \rceil$  and any  $c \geq 0$ .

**Remark.** The boundary of the domains  $\mathcal{R}_{\varepsilon n}^{(i,i)}$ ,  $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$ , contains a segment  $\mathcal{I}$ , while the boundary of the domains  $\mathcal{R}_{\varepsilon n}^{(i, \lceil 1/h_1(\varepsilon) \rceil + 1 + i)}$ ,  $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$ , contains a segment in  $\gamma_0$ . See Figure 9.

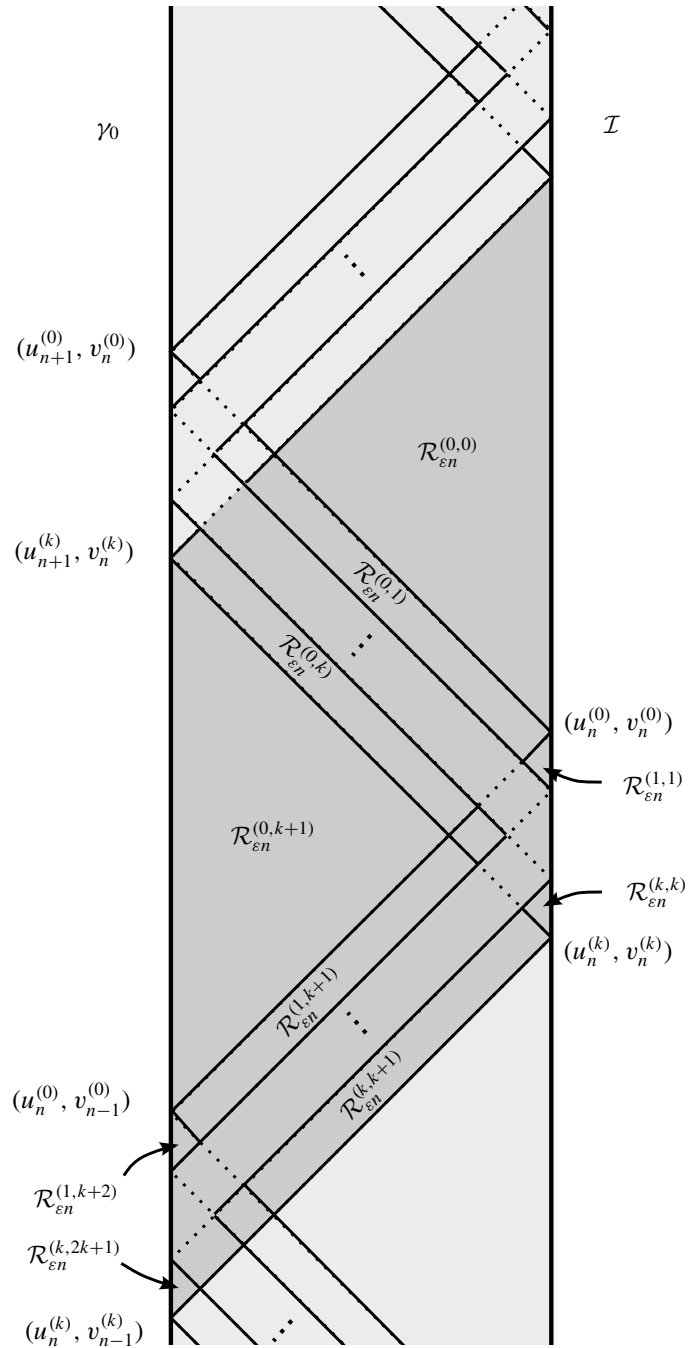
Notice that  $T_{uu} = T_{vv} = 0$  in  $\mathcal{R}_{\varepsilon n}^{(i,j)}$ . In particular, all the domains  $(\mathcal{R}_{\varepsilon n}^{(i,j)} \times \mathbb{S}^2, g_\varepsilon)$  are isometric to a region of a member of the Schwarzschild-AdS family (or to a region of pure AdS spacetime), and the renormalized mass function  $\tilde{m}_\varepsilon$  is constant on them. We will define for any  $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$ ,  $i \leq j \leq \lceil 1/h_1(\varepsilon) \rceil + i + 1$  and  $n \in \mathbb{N}$  such that  $\mathcal{R}_{\varepsilon n}^{(i,j)} \neq \emptyset$ ,

$$\tilde{m}_{\varepsilon n}^{(i,j)} \doteq \tilde{m}_\varepsilon|_{\mathcal{R}_{\varepsilon n}^{(i,j)}}. \quad (5-38)$$

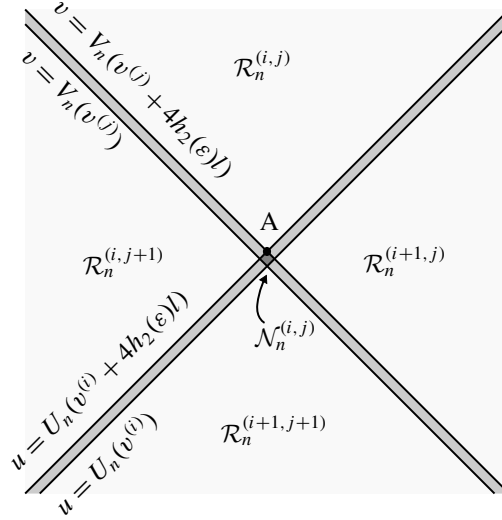
In view of (5-23) and (5-24), we immediately calculate that for all  $0 \leq i \leq \lceil 1/h_1(\varepsilon) \rceil$  and all  $n \in \mathbb{N}$  such that  $\mathcal{R}_{\varepsilon n}^{(i,i)}$ ,  $\mathcal{R}_{\varepsilon n}^{(i, \lceil 1/h_1(\varepsilon) \rceil + 1 + i)} \neq \emptyset$ ,

$$\tilde{m}_{\varepsilon n}^{(i, \lceil 1/h_1(\varepsilon) \rceil + 1 + i)} = 0, \quad (5-39)$$

$$\tilde{m}_{\varepsilon n}^{(i,i)} = \tilde{m}_{\varepsilon 0}^{(i,i)} = \frac{\varepsilon}{\sqrt{-\Lambda}}. \quad (5-40)$$



**Figure 9.** Schematic depiction of the domains  $\mathcal{R}_{\varepsilon n}^{(i,j)}$  for  $0 \leq i \leq k$  and  $i \leq j \leq k+i+1$  (where  $k = \lceil 1/h_1(\varepsilon) \rceil$ ). We have used the shorthand notation  $u_n^{(j)} = U_n(v^{(j)})$  and  $v_n^{(j)} = V_n(v^{(j)})$ . Having assumed that  $h_2(\varepsilon) \ll \varepsilon$ , all the beams of the form  $\{U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$  and  $\{V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$ , which separate the domains  $\mathcal{R}_{\varepsilon n}^{(i,j)}$ , are depicted as straight line segments.



**Figure 10.** Typical arrangement of neighboring vacuum domains not intersecting  $\gamma_0$  or  $\mathcal{I}$ . The point  $A$  at the lower corner of  $\mathcal{R}_n^{(i,j)}$  satisfies  $r(A) = r_n^{(i,j)}$ . For simplicity, we have used the shorthand notation  $l = (-\Lambda)^{-1/2}$ .

For any  $n \in \mathbb{N}$ ,  $0 \leq i \leq \lceil 1/h_1(\epsilon) \rceil$  and  $i+1 \leq j \leq \lceil 1/h_1(\epsilon) \rceil + i$ , we will define the interaction regions

$$\mathcal{N}_{\epsilon n}^{(i,j)} \doteq \left\{ U_n(v^{(i)}) \leq u \leq U_n\left(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon)\right) \right\} \cap \left\{ V_n(v^{(j)}) \leq u \leq V_n\left(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon)\right) \right\} \cap \mathcal{U}_\epsilon, \quad (5-41)$$

where the conventions stated below (5-37) hold regarding indices smaller than 0 or larger than  $\lceil 1/h_1(\epsilon) \rceil$ . See Figure 10.

Let us define for any  $0 \leq i \leq \lceil 1/h_1(\epsilon) \rceil$ ,  $i \leq j \leq \lceil 1/h_1(\epsilon) \rceil + i + 1$ , and  $n \in \mathbb{N}$  such that  $\mathcal{R}_{\epsilon n}^{(i,j)} \neq \emptyset$

$$r_{\epsilon n}^{(i,j)} \doteq r_\epsilon \left( U_n\left(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon)\right), V_n\left(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon)\right) \right). \quad (5-42)$$

Note that  $r_{\epsilon n}^{(i,i)} = +\infty$  and  $r_{\epsilon n}^{(i, \lceil 1/h_1(\epsilon) \rceil + i + 1)} = r_{0\epsilon}$ .

Finally, let us remark that, in view of property (6-204) of the cut-off used in the construction of the initial data and (2-48)–(2-49), for any  $1 \leq n \leq n_f$ ,  $0 \leq i \leq \lceil 1/h_1(\epsilon) \rceil$ ,  $i \leq j \leq \lceil 1/h_1(\epsilon) \rceil + i + 1$ , we have

$$T_{uu} > 0 \quad \text{on} \quad \left\{ U_n(v^{(i)}) < u < U_n\left(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon)\right) \right\}, \quad (5-43)$$

$$T_{vv} > 0 \quad \text{on} \quad \left\{ V_n(v^{(j)}) < u < V_n\left(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\epsilon)\right) \right\}. \quad (5-44)$$

## 6. Proof of Theorem 1

In this section, we will prove Theorem 1. In order to simplify our notation, from now on, we will often drop the subscripts  $\epsilon$  in notation related to the maximal future development  $(\mathcal{U}_\epsilon; r_\epsilon, \Omega_\epsilon^2, \bar{f}_{\text{in}\epsilon}, \bar{f}_{\text{out}\epsilon})$  of the initial data  $(r_{/\epsilon}, \Omega_{/\epsilon}^2, \bar{f}_{\text{in}/\epsilon}, \bar{f}_{\text{out}/\epsilon})$  (see Definition 5.1).

For any  $0 < \varepsilon < \varepsilon_0$  (provided  $\varepsilon_0$  is fixed sufficiently small) and any  $r_0 > 0$  satisfying (5-6), let  $(\mathcal{U}_\varepsilon; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  be the maximal future development of  $(r/\varepsilon, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ , and let us define

$$u_+ \doteq \sup \left\{ u_* > 0 : 1 - \frac{2m}{r} > h_3(\varepsilon) \text{ on } \mathcal{U}_\varepsilon \cap \{u < u_*\} \right\} \quad (6-1)$$

and

$$\mathcal{U}_\varepsilon^+ = \mathcal{U}_\varepsilon \cap \{u < \min\{u_+, (h_1(\varepsilon))^{-2} v_{0\varepsilon}\}\}, \quad (6-2)$$

where

$$h_3(\varepsilon) = \exp\{-\exp((h_1(\varepsilon))^{-5} \exp(-2(h_0(\varepsilon))^{-4}))\}. \quad (6-3)$$

Let us also set

$$k \doteq \lceil 1/h_1(\varepsilon) \rceil, \quad (6-4)$$

$$n_f \doteq \lfloor (u_+ - v^{(0)})/v_0 \rfloor, \quad (6-5)$$

where  $\lceil x \rceil$  denotes the least integer greater than or equal to  $x$ , while  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ .

The proof of Theorem 1 will follow in two steps: First, in Section 6B, we will show that

$$\sup_{\mathcal{U}_\varepsilon^+} \left( 1 - \frac{2m}{r} \right) = h_3(\varepsilon), \quad (6-6)$$

i.e., that  $\mathcal{U}_\varepsilon^+$  contains a nearly trapped sphere. Then, in Section 6C, we will show that, at the final step of the evolution, either a trapped sphere is formed, or there exists a small perturbation of the initial data  $(r/\varepsilon, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  giving rise to a trapped sphere.

Before proving (6-6), we need to establish some necessary bounds for the evolution of  $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  in the region  $\mathcal{U}_\varepsilon^+$ . These bounds, which will be obtained in Section 6A, will be used both in Section 6B and in Section 6C.

**6A. Inductive bounds for the evolution in the region  $\mathcal{U}_\varepsilon^+$ .** In this section, we will establish a number of useful bounds for  $(\mathcal{U}_\varepsilon^+; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ . These bounds will include a number of inductive bounds for the quantities  $\tilde{m}_n^{(1,k+1)}$ ,  $r_n^{(k,k+1)}$ , and  $r_n^{(1,k+1)}$  (with  $k$  defined by (6-4)) that will be of fundamental significance in the proof of Theorem 1.

In particular, we will prove the following result:

**Proposition 6.1.** *For any  $0 < \varepsilon < \varepsilon_0$ , the following bounds hold for  $(\mathcal{U}_\varepsilon^+; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ :*

(1) *On  $\mathcal{U}_\varepsilon^+$ , we can estimate*

$$\left| \log \left( \frac{-\partial_u r}{1 - \frac{1}{3} \Lambda r^2} \right) \right| + \left| \log \left( \frac{\partial_v r}{1 - 2m/r} \right) \right| \leq (h_1(\varepsilon))^{-4} \log((h_3(\varepsilon))^{-1}). \quad (6-7)$$

(2) *For any  $1 \leq n \leq n_f$ ,*

$$r_n^{(0,k)} \geq \frac{\varepsilon^{-1/2}}{\sqrt{-\Lambda}}, \quad (6-8)$$

$$r_n^{(k,k+1)} \leq \frac{\varepsilon^{1/2}}{\sqrt{-\Lambda}}, \quad (6-9)$$

$$\frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})}{r_0} \geq \exp(-2(h_0(\varepsilon))^{-4}), \quad (6-10)$$

$$\frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})}{r_0} \leq 1 - \frac{1}{C_0} h_0(\varepsilon), \quad (6-11)$$

$$\frac{r_n^{(1,k+1)}}{r_0} - 1 \geq \exp(-(h_0(\varepsilon))^{-4}), \quad (6-12)$$

where  $C_0 > 1$  is a large fixed constant (independent of all the parameters).

(3) For any  $2 \leq n \leq n_f$ ,

$$\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \geq 1 + \frac{1}{4} \exp(-2(h_0(\varepsilon))^{-4}) \frac{r_0}{r_n^{(k,k+1)}}, \quad (6-13)$$

$$\frac{r_n^{(k,k+1)} - r_0}{r_{n-1}^{(k,k+1)} - r_0} \leq 1 + 2C_0 \frac{r_0}{r_{n-1}^{(k,k+1)}} \left( \left| \log \left( 1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0} \right) \right| + (h_0(\varepsilon))^{-4} \right). \quad (6-14)$$

Before presenting the proof of Proposition 6.1 (in Section 6A2), we will briefly comment on the nature of the bounds (6-7)–(6-14) and their relation with the specific choice of the parameters (5-2)–(5-3).

**6A1. Remarks on Proposition 6.1.** The bounds (6-7)–(6-14) in Proposition 6.1 lie at the heart of the proof of Theorem 1. The precise form of the initial data (5-7), the range (5-6) for the mirror radius  $r_0$  and the asymptotic bounds (5-2)–(5-3) on the parameters  $h_0, h_1, h_2$  were carefully chosen so that (6-7)–(6-14) can be obtained. We will now proceed to briefly comment on the role of the bounds (6-7)–(6-14) in the proof of Theorem 1. The reader is advised to review first the sketch of the proof in Section 1D of the Introduction. Let us remark that, in the notation of Section 1D,

$$\mathcal{E}_{\zeta_0;n} = \tilde{m}_n^{(1,k+1)}, \quad (6-15)$$

$$r_{\gamma_0;n} = r_n^{(k,k+1)}, \quad (6-16)$$

$$r_{\gamma_0;n}^{(1)} = r_n^{(1,k+1)}. \quad (6-17)$$

The bound (6-7) is a “trivial” bound controlling quantities related to the chosen gauge. The right-hand side of (6-7), upon integration across any specific beam (in a direction transversal to the beam), will yield a small quantity, in view of the fact that the width of the null beams emanating from  $u = 0$ ,  $v \sim v^{(j)}$  was chosen to be  $\sim h_2(\varepsilon)$  and, moreover,  $h_2(\varepsilon)$  was chosen in (5-3) to be small compared to the right-hand side of (6-7). This fact will prove convenient for the proof of Proposition 6.1 and Theorem 1, as it will enable us to “ignore” the variation of certain quantities across the width of any specific beam. That is to say, the bound (6-7) will enable us to frequently treat the null beams as line segments having negligible width.

The bounds (6-8)–(6-9) are quantitative expressions of the fact that the set of interactions of the beams splits into two portions, one close to  $r = r_0$  and one close to  $\mathcal{I}$ .

The lower bound (6-10) is necessary in order to establish (6-13). In order to obtain (6-10), it is necessary that  $r_0$  satisfies the upper bound of (5-6).

The upper bound (6-11) implies that a trapped sphere (i.e., a sphere where  $2m/r > 1$ ) cannot be formed at  $\mathcal{R}_{\varepsilon n}^{(i,j)}$  for any  $j > k + 1$ , since one can also show that  $\tilde{m} \leq \tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)}$  in those regions. In order to obtain (6-11), it is necessary that the mirror radius  $r_0$  satisfy the lower bound of (5-6).

In the language of Section 1D of the Introduction, the bound (6-12) states that, when  $\zeta_0$  reaches  $\{r = r_0\}$  for the  $n$ -th time, the  $r$ -distance of the top beam  $\zeta_0$  from the second-to-top beam  $\zeta_1$ , i.e.,  $r_n^{(1,k+1)} - r_0$ , can be bounded from below by a small multiple of  $r_0$  which is large compared to  $h_1(\varepsilon)r_0$ . As a consequence of (6-12) and the bound (5-6) for  $r_0$ , for any  $i \neq 1$ ,  $\mathcal{R}_{\varepsilon n}^{(i,k+1)}$  does not contain a trapped sphere. As a result, combining (6-11), (6-8) and (6-12), we infer that, among all regions  $\mathcal{R}_{\varepsilon n}^{(i,j)}$ , a trapped sphere can only appear for  $i = 1$ ,  $j = k + 1$ . This fact serves to simplify the proof of Theorem 1, by avoiding considering multiple scenarios of trapped surface formation. Furthermore, it is crucial in obtaining (6-14).

Establishing (6-12) is the most demanding part of the proof of Proposition 6.1. It requires obtaining a lower bound in the rate of decrease of  $r_n^{(1,k+1)}$  in terms of the rate of increase of  $\tilde{m}_n^{(1,k+1)}$ , using also the fact that  $\tilde{m}_n^{(1,k+1)} \lesssim r_0$  before a trapped sphere is formed (see the relations (6-129) and (6-154) in the next section).

The bound (6-13) is a technical version of the bound (1-42), and its proof follows from the ideas outlined in Section 1D. In obtaining (6-13), the lower bound of (6-10) is necessary.

Finally, the bound (6-14) is a technical version of the bound (1-46) in Section 1D and provides an estimate for the decrease of the multiplicative factor in the right-hand side of (6-13). In obtaining (6-14) when  $2m/r \simeq 1$ , the fact that  $2m/r$  is bounded away from 1 everywhere but on  $\mathcal{R}_{\varepsilon n}^{(i,j)}$  is crucially used (in particular, the bound (6-12) is necessary for (6-14)).

**Remark.** As is evident from the above discussion, most of the technical difficulties in the proof of Proposition 6.1 are associated to issues related with the near-trapped regime  $2m/r \simeq 1$ . In the case when, instead of the stronger bound (4-4), one is merely interested in establishing the weaker instability estimate (4-8), the proof of Proposition 6.1 simplifies substantially: In that case, it is not necessary to demand that the worst instability scenario take place in  $\mathcal{R}_{\varepsilon n}^{(1,k+1)}$ . In particular, the bounds (6-11) and (6-12) can be omitted from the proof. Moreover, the lower bound for  $r_0$  in (5-6) can be relaxed, and the exponentials in the relations (5-2) between  $h_0(\varepsilon)$ ,  $h_1(\varepsilon)$  can be replaced by polynomial functions.

**6A2. Proof of Proposition 6.1.** In this section, we will make use of the  $O(\cdot)$  convention: for any pair of functions  $\mathcal{F}, \mathcal{G}$  defined on the same domain, with  $\mathcal{G} \geq 0$ , the notation

$$\mathcal{F} = O(\mathcal{G})$$

will imply that

$$|\mathcal{F}| \leq C \cdot \mathcal{G}$$

for some universal constant  $C > 0$  which is independent of all the parameters in the statement of Theorem 1. We should also remark that, throughout this proof, we will adopt the convention on the indices stated under (5-37); i.e.,

$$(1) \quad U_n(v^{(-1)}) \doteq U_{n+1}(v^{(k)}),$$

$$(2) \quad V_n(v^{(k+l)} + c) \doteq V_{n-1}(v^{(l-1)} + c) \text{ for any integer } 1 \leq l \leq k \text{ and any } c \geq 0.$$



In view of (6-1), on  $\mathcal{U}_\varepsilon^+$  we have

$$\partial_u r < 0 < \partial_v r, \quad (6-18)$$

$$\partial_u \tilde{m} \leq 0 \leq \partial_v \tilde{m}. \quad (6-19)$$

We will split the proof of Theorem 1 into two parts: In the first (and shortest) part, we will establish the bound (6-7) through a standard continuity argument. The proof of (6-7) will also yield (6-8) and (6-9). In the second (and more extended) part, we will establish the bounds (6-10)–(6-14) by induction on  $n$ .

Part I: proof of (6-7)–(6-9). Let  $u_* > 0$  be such that on

$$\mathcal{U}_\varepsilon^* \doteq \mathcal{U}_\varepsilon^+ \cap \{u < u_*\}, \quad (6-20)$$

we can bound

$$\left| \log \left( \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right) \right| + \left| \log \left( \frac{\partial_v r}{1 - 2m/r} \right) \right| \leq 2(h_1(\varepsilon))^{-4} \log((h_3(\varepsilon))^{-1}). \quad (6-21)$$

By showing that (6-7) holds on  $\mathcal{U}_\varepsilon^*$ , it will follow (by applying a standard continuity argument) that (6-7) holds on the whole of  $\mathcal{U}_\varepsilon^+$ .

*Inductive formulas for  $\partial_u r$  and  $\partial_v r$  and proof of (6-7).* From (2-45), we can readily derive the following renormalized equation:

$$\partial_v \partial_u \left\{ \sqrt{-\frac{3}{\Lambda}} \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} r \right) \right\} = -2 \frac{\tilde{m}}{r^2} \frac{(1 - \Lambda r^2)}{(1 - \frac{1}{3}\Lambda r^2)} \left( \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right) \left( \frac{\partial_v r}{1 - 2m/r} \right). \quad (6-22)$$

Let  $n \geq 1$ ,  $0 \leq i \leq k$ ,  $i \leq j \leq k + i$ ,  $\bar{u} < u_*$  and  $v_b$  be such that

$$U_n \left( v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) \leq \bar{u} \leq U_n(v^{(i-1)})$$

and

$$V_n(v^{(j)}) \leq v_b \leq V_n \left( v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right).$$

Integrating (6-22) for  $u = \bar{u}$  from  $v = V_n(v^{(j)})$  up to  $v = v_b$ , using also the fact that

$$\partial_u \left\{ \sqrt{-\frac{3}{\Lambda}} \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} r \right) \right\} = \frac{\partial_u r}{1 - \frac{1}{3}\Lambda r^2},$$

we obtain

$$\begin{aligned} & \left. \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(\bar{u}, V_n(v^{(j)}))} \\ &= \left. \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(\bar{u}, v_b)} + O \left( \sup_{\{u=\bar{u}\} \cap \{V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq v \leq V_n(v^{(j-1)})\}} \left| \frac{\tilde{m}}{r^2} \left( \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right) \left( \frac{\partial_v r}{1 - 2m/r} \right) \right| h_2(\varepsilon) \right). \end{aligned} \quad (6-23)$$

Using the bootstrap bound (6-21), combined with the trivial bounds

$$\tilde{m} \leq \tilde{m}|_{\mathcal{I}} = \frac{\varepsilon}{\sqrt{-\Lambda}}, \quad (6-24)$$

$$r \geq r_0 \quad (6-25)$$

(following from (2-46), (2-47) and (6-18)), as well as the relation (5-3) for  $h_2(\varepsilon)$ , the relation (6-23) yields

$$\left. \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(\bar{u}, V_n(v^{(j)}))} = \left. \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(\bar{u}, v_b)} + O((h_2(\varepsilon))^{1/2}). \quad (6-26)$$

Similarly, integrating (6-22) for  $v = \bar{v}$  from  $u = U_n(v^{(i)})$  up to  $u = u_b$  for any

$$V_n\left(v^{(j)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right) \leq \bar{v} \leq V_n(v^{(j-1)})$$

and any

$$U_n(v^{(i)}) \leq u_b \leq U_n\left(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right)$$

(assuming that  $u_b < u_*$ ), we infer

$$\left. \frac{\partial_v r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(U_n(v^{(i)}), \bar{v})} = \left. \frac{\partial_v r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(u_b, \bar{v})} + O((h_2(\varepsilon))^{1/2}). \quad (6-27)$$

By multiplying and dividing each factor by

$$1 - \frac{2m}{r} = 1 - \frac{2\tilde{m}}{r} - \frac{1}{3}\Lambda r^2,$$

the relations (6-26) and (6-27) are equivalent to

$$\left. \frac{-\partial_u r}{1 - 2m/r} \right|_{(\bar{u}, V_n(v^{(j)}))} = \left. \frac{-\partial_u r}{1 - 2m/r} \right|_{(\bar{u}, v_b)} \cdot \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))} \right)_{(\bar{u}, V_n(v^{(j)}))} + O((h_2(\varepsilon))^{1/2}) \quad (6-28)$$

and

$$\left. \frac{\partial_v r}{1 - 2m/r} \right|_{(U_n(v^{(i)}), \bar{v})} = \left. \frac{\partial_v r}{1 - 2m/r} \right|_{(u_b, \bar{v})} \cdot \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))} \right)_{(U_n(v^{(i)}), \bar{v})} + O((h_2(\varepsilon))^{1/2}). \quad (6-29)$$

**Remark.** In the vacuum case, where  $\tilde{m}$  is constant, the factors in the right-hand side of (6-28) and (6-29) become identically 1. In our case, however, where matter is present, by relaxing our definition of  $h_2$  and considering the limit  $h_2 \rightarrow 0$  for fixed  $\varepsilon$ , the dominant terms in the factors in the right-hand side of (6-28) and (6-29), i.e., the first summands, do *not* converge to 1. This is because, in this limit, while the function  $r$  remains  $C^1$ , the renormalized Hawking mass  $\tilde{m}$  has a jump discontinuity across the beam.

Since  $T_{uu} = T_{vv} = 0$  on  $\mathcal{R}_{\varepsilon n}^{(i,j)}$  for any  $n \geq 1$ , any  $0 \leq i \leq k$  and any  $i \leq j \leq k + i + 1$ , the relations (2-43)–(2-44) imply

$$\partial_v \left( \frac{-\partial_u r}{1 - 2m/r} \right) \Big|_{\mathcal{R}_{\varepsilon n}^{(i,j)}} = \partial_u \left( \frac{\partial_v r}{1 - 2m/r} \right) \Big|_{\mathcal{R}_{\varepsilon n}^{(i,j)}} = 0. \quad (6-30)$$

In particular, along lines of the form  $\{u = \bar{u}\}$ , the quantity  $-\partial_u r/(1 - 2m/r)$  remains constant on  $\{u = \bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}$  for any  $\bar{u} < u_*$  such that  $\{u = \bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}$  is nontrivial. In view of (6-28), the quantities

$-\partial_u r/(1-2m/r)|_{\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j+1)}}$  and  $-\partial_u r/(1-2m/r)|_{\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}}$  (for any  $\bar{u} < u_*$  such that  $\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}$  is nontrivial) are related by

$$\begin{aligned} & \frac{-\partial_u r}{1-2m/r} \Big|_{\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}} \\ &= \frac{-\partial_u r}{1-2m/r} \Big|_{\{u=\bar{u}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j+1)}} \cdot \left( \frac{1 - (2\tilde{m}/r(1 - \frac{1}{3}\Lambda r^2))|_{(\bar{u}, V_n(v^{(j)}))}}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(\bar{u}, V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}} + O((h_2(\varepsilon))^{1/2}) \right). \end{aligned} \quad (6-31)$$

Similarly, the quantity  $\partial_v r/(1-2m/r)$  remains constant along segments of the form  $\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}$ , and  $\partial_v r/(1-2m/r)|_{\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i+1,j)}}$  and  $\partial_v r/(1-2m/r)|_{\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}}$  are related (in view of (6-29)) by

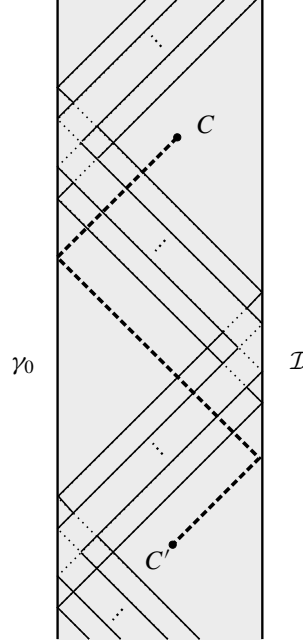
$$\begin{aligned} & \frac{\partial_v r}{1-2m/r} \Big|_{\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i,j)}} = \frac{\partial_v r}{1-2m/r} \Big|_{\{v=\bar{v}\} \cap \mathcal{R}_{\varepsilon n}^{(i+1,j)}} \\ & \cdot \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(U_n(v^{(i)}, \bar{v}))}}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v}))}} + O((h_2(\varepsilon))^{1/2}) \right). \end{aligned} \quad (6-32)$$

We infer, therefore, that for any point  $(\bar{u}, \bar{v}) \in \mathcal{R}_{\varepsilon, n}^{(i,j)}$  for some  $n \geq 2$ ,  $0 \leq i \leq k$  and  $i \leq j \leq k+i+1$  such that  $\bar{u} < u_*$ , the following relations hold between  $(\bar{u}, \bar{v})$  and  $(\bar{u}-v_0, \bar{v}-v_0) \in \mathcal{R}_{\varepsilon, n-1}^{(i,j)}$ :

$$\begin{aligned} & \frac{-\partial_u r}{1-2m/r} \Big|_{(\bar{u}, \bar{v})} = \frac{-\partial_u r}{1-2m/r} \Big|_{(\bar{u}-v_0, \bar{v}-v_0)} \\ & \times \prod_{\bar{j}=j}^{k+i} \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(\bar{u}, V_n(v^{(\bar{j})}))}}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(\bar{u}, V_n(v^{(\bar{j})} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}} + O((h_2(\varepsilon))^{1/2}) \right) \\ & \times \prod_{\bar{i}=i}^{k+i} \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(U_n(v^{(\bar{i})}, \bar{u}))}}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(U_n(v^{(\bar{i})} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{u}))}} + O((h_2(\varepsilon))^{1/2}) \right) \\ & \times \prod_{\bar{j}=k+i+1}^{k+j} \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(\bar{u}-v_0, V_n(v^{(\bar{j})}))}}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(\bar{u}-v_0, V_n(v^{(\bar{j})} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}} + O((h_2(\varepsilon))^{1/2}) \right) \end{aligned} \quad (6-33)$$

and

$$\begin{aligned} & \frac{\partial_v r}{1-2m/r} \Big|_{(\bar{u}, \bar{v})} = \frac{\partial_v r}{1-2m/r} \Big|_{(\bar{u}-v_0, \bar{v}-v_0)} \\ & \times \prod_{\bar{i}=i}^{j-1} \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(U_n(v^{(\bar{i})}, \bar{v}))}}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(U_n(v^{(\bar{i})} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v}))}} + O((h_2(\varepsilon))^{1/2}) \right) \\ & \times \prod_{\bar{j}=j}^{k+j} \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(\bar{u}-v_0, V_n(v^{(\bar{j})}))}}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(\bar{u}-v_0, V_n(v^{(\bar{j})} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}} + O((h_2(\varepsilon))^{1/2}) \right) \\ & \times \prod_{\bar{i}=j}^{k+i} \left( \frac{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(U_n(v^{(\bar{i})}, \bar{v}-v_0))}}{1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))|_{(U_n(v^{(\bar{i})} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v}-v_0)}} + O((h_2(\varepsilon))^{1/2}) \right). \end{aligned} \quad (6-34)$$



**Figure 11.** In order to obtain the formula (6-33) relating  $-\partial_u r/(1 - 2m/r)$  at the point  $C = (\bar{u}, \bar{v})$  with the same quantity at the point  $C' = (\bar{u} - v_0, \bar{v} - v_0)$ , we apply the relations (6-31) and (6-32) along the dashed path depicted above, using also the reflecting gauge condition on  $\gamma_0$  and  $\mathcal{I}$ .

The relation (6-33) is obtained as follows (see also Figure 11): First, (6-31) determines the evolution of  $-\partial_u r/(1 - 2m/r)$  (according to (6-31)) along the line  $\{u = \bar{u}\}$  in the past direction, from  $(\bar{u}, \bar{v})$  up to  $\gamma_0$ . Then, using the boundary relation

$$\left. \frac{-\partial_u r}{1 - 2m/r} \right|_{\gamma_0} = \left. \frac{\partial_v r}{1 - 2m/r} \right|_{\gamma_0}, \quad (6-35)$$

one repeats the same procedure for  $\partial_v r/(1 - 2m/r)$  along  $\{v = \bar{u}\}$  from  $\gamma_0$  up to  $\mathcal{I}$ . Finally, using

$$\left. \frac{-\partial_u r}{1 - 2m/r} \right|_{\mathcal{I}} = \left. \frac{\partial_v r}{1 - 2m/r} \right|_{\mathcal{I}}, \quad (6-36)$$

and following the evolution of  $-\partial_u r/(1 - 2m/r)$  along  $\{u = \bar{u} - v_0\}$  from  $\mathcal{I}$  up to  $(\bar{u} - v_0, \bar{v} - v_0)$ , one arrives at (6-33). The relation (6-34) is similarly obtained by following the same procedure along the lines  $\{v = \bar{v}\}$  (up to  $\mathcal{I}$ ),  $\{u = \bar{v} - v_0\}$  (from  $\mathcal{I}$  up to  $\gamma_0$ ) and  $\{v = \bar{v} - v_0\}$  (from  $\mathcal{I}$  up to  $(\bar{u} - v_0, \bar{v} - v_0)$ ).

In view of the bound

$$1 - \frac{2m}{r} \geq h_3(\varepsilon) \quad (6-37)$$

on  $\mathcal{U}_\varepsilon^+$  (see (6-1)), we can estimate in the region  $\{r \leq \varepsilon^{1/2}(-\Lambda)^{1/2}\} \cap \mathcal{U}_\varepsilon^+$

$$1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} = \frac{1 - 2m/r}{1 - \frac{1}{3}\Lambda r^2} \geq \frac{1}{2}h_1(\varepsilon). \quad (6-38)$$

On the other hand, in the region  $\{r \geq \varepsilon^{1/2}(-\Lambda)^{1/2}\} \cap \mathcal{U}_\varepsilon^+$ , using (5-24) to bound  $\tilde{m}$  we can trivially estimate (in view also of (5-2))

$$1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} \geq 1 - \frac{2\varepsilon}{\varepsilon^{1/2}} \geq h_1(\varepsilon). \quad (6-39)$$

Combining (6-38) and (6-39), using also that  $\tilde{m} \geq \tilde{m}|_{\gamma_0} = 0$  on  $\mathcal{U}_\varepsilon^+$ , we can bound  $1 - 2\tilde{m}/(r(1 - \frac{1}{3}\Lambda r^2))$  from above and below everywhere on  $\mathcal{U}_\varepsilon^+$  as

$$\frac{1}{2}h_3(\varepsilon) \leq 1 - \frac{2\tilde{m}}{r(1 - \frac{1}{3}\Lambda r^2)} \leq 1. \quad (6-40)$$

Thus, by considering the logarithm of the relations (6-33)–(6-34) and noting that the resulting right-hand side contains  $\sim k = \lceil 1/h_1(\varepsilon) \rceil$  summands, each controlled with the help of (6-40), we readily obtain for any  $n \geq 2$ ,  $0 \leq i \leq k$ , and  $i \leq j \leq k + i + 1$  and any point  $(\bar{u}, \bar{v}) \in \mathcal{R}_{\varepsilon n}^{(i,j)}$  with  $\bar{u} < u_*$

$$\left| \log\left(\frac{-\partial_u r}{1 - 2m/r}\right) \right|_{(\bar{u}, \bar{v})} - \log\left(\frac{-\partial_u r}{1 - 2m/r}\right) \Big|_{(\bar{u}-v_0, \bar{v}-v_0)} \Big| \leq \frac{C}{h_1(\varepsilon)} \log((h_3(\varepsilon))^{-1}) \quad (6-41)$$

and

$$\left| \log\left(\frac{\partial_v r}{1 - 2m/r}\right) \right|_{(\bar{u}, \bar{v})} - \log\left(\frac{\partial_v r}{1 - 2m/r}\right) \Big|_{(\bar{u}-v_0, \bar{v}-v_0)} \Big| \leq \frac{C}{h_1(\varepsilon)} \log((h_3(\varepsilon))^{-1}). \quad (6-42)$$

In view of (6-28)–(6-29), the bounds (6-41) and (6-42) (stated in the case when  $(\bar{u}, \bar{v})$  belongs to a vacuum region  $\mathcal{R}_{\varepsilon n}^{(i,j)}$ ) also hold when  $(\bar{u}, \bar{v})$  belongs to a beam, i.e., when

$$U_n(v^{(i)}) \leq \bar{u} \leq U_n\left(v^{(i-1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right)$$

or

$$V_n(v^{(j)}) \leq \bar{v} \leq V_n\left(v^{(i-1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right)$$

for some  $n \geq 2$ ,  $0 \leq i \leq k$ , and  $i \leq j \leq k + i + 1$ . Therefore, for any  $n \geq 2$ , the bounds (6-28)–(6-29) hold on the whole of

$$\mathcal{U}_{\varepsilon;n}^* \doteq \{U_n(v^{(k)}) \leq u \leq U_{n+1}(v^{(k)})\} \cap \mathcal{U}_\varepsilon^*. \quad (6-43)$$

From (6-2) and the definition (6-5), it follows that

$$n_f \leq (h_1(\varepsilon))^{-2}. \quad (6-44)$$

Since  $n \leq n_f$  (because  $\mathcal{U}_\varepsilon^* \subset \mathcal{U}_\varepsilon^+$ ), by substituting  $(\bar{u}, \bar{v}) \rightarrow (\bar{u} - v_0, \bar{v} - v_0)$  in (6-41)–(6-42)  $n - 2$  times and using (6-44), (5-16), (6-40) as well as the Cauchy stability estimate of Proposition 3.5 for the region  $\{0 \leq u \leq 2v_0\}$ , we readily obtain

$$\sup_{\mathcal{U}_\varepsilon^*} \left\{ \left| \log\left(\frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2}\right) \right| + \left| \log\left(\frac{\partial_v r}{1 - 2m/r}\right) \right| \right\} \leq \frac{C}{(h_1(\varepsilon))^3} \log((h_3(\varepsilon))^{-1}). \quad (6-45)$$

Thus, (6-7) holds on  $\mathcal{U}_\varepsilon^*$  in view of the relation (5-2) for the parameter  $h_1(\varepsilon)$  (provided  $\varepsilon_0$  is small enough). Therefore (as explained in the beginning of the proof), a standard continuity argument yields that (6-7) actually holds on the whole of  $\mathcal{U}_\varepsilon^+$ .

*Proof of (6-8) and (6-9).* For any  $1 \leq n \leq n_f$ , we can bound, in view of the definition (5-8) of  $v^{(j)}$  and the bound (6-7),

$$\begin{aligned}
 & \left| \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} r \right) \right|_{(U_n(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), V_n(v^{(k)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} r \right) \Big|_{\mathcal{I} \cap \{v = V_n(v^{(k)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}} \\
 &= \sqrt{-\frac{\Lambda}{3}} \int_{U_n(v^{(k)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))}^{U_n(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \Big|_{(u, V_n(v^{(k)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} du \\
 &\leq \frac{C\sqrt{-\Lambda}}{(h_1(\varepsilon))^4} \log((h_3(\varepsilon))^{-1}) |v^{(k)} - v^{(0)}| \\
 &\leq \frac{C\varepsilon}{(h_1(\varepsilon))^6} \log((h_3(\varepsilon))^{-1}) \tag{6-46}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left| \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} r \right) \right|_{(U_n(v^{(k)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), V_n(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \tan^{-1} \left( \sqrt{-\frac{\Lambda}{3}} r \right) \Big|_{\gamma_0 \cap \{v = V_n(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}} \\
 &= \sqrt{-\frac{\Lambda}{3}} \int_{U_n(v^{(k)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))}^{U_n(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u r}{1 - \frac{1}{3}\Lambda r^2} \Big|_{(u, V_n(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} du \\
 &\leq \frac{C\sqrt{-\Lambda}}{(h_1(\varepsilon))^4} \log((h_3(\varepsilon))^{-1}) |v^{(k)} - v^{(0)}| \\
 &\leq \frac{C\varepsilon}{(h_1(\varepsilon))^6} \log((h_3(\varepsilon))^{-1}). \tag{6-47}
 \end{aligned}$$

From (6-46) and (6-47) we readily obtain (6-8) and (6-9), respectively, in view of the relations (5-2) and (6-3) for  $h_1$ ,  $h_3$ , respectively, and the fact that  $r|_{\gamma_0} = r_0$ ,  $r|_{\mathcal{I}} = +\infty$ .

**Part II:** proof of (6-10)–(6-14). We will now proceed to establish the bounds (6-10)–(6-14). To this end, we will first derive some useful estimates for the differences of the renormalized masses  $\tilde{m}_n^{(i,j)}$  associated to the vacuum regions around each interaction region  $\mathcal{N}_{\varepsilon n}^{(i,j)}$ .<sup>16</sup>

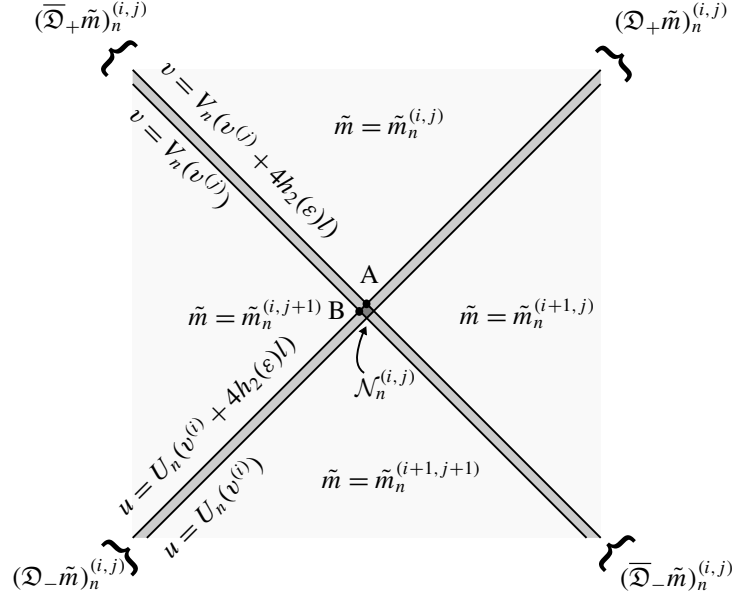
*Relations for the change in the mass difference of the beams.* Let us introduce the notion of the mass difference for the beams  $\{U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$  and  $\{V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$  around their interaction region  $\mathcal{N}_{\varepsilon n}^{(i,j)}$ : For any  $1 \leq n \leq n_f$ ,  $0 \leq i \leq k$ , and  $i+1 \leq j \leq k+i$ , we define the initial mass differences

$$\begin{aligned}
 (\mathfrak{D}_- \tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j+1)} - \tilde{m}_n^{(i,j+1)}, \\
 (\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i+1,j+1)} \tag{6-48}
 \end{aligned}$$

and the final mass differences

$$\begin{aligned}
 (\mathfrak{D}_+ \tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i,j)}, \\
 (\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)} &\doteq \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)}. \tag{6-49}
 \end{aligned}$$

<sup>16</sup>A relation for the change the mass differences of two intersecting, infinitely thin null dust beams was also obtained in [Poisson and Israel 1990].



**Figure 12.** Schematic depiction of two intersecting beams, with associated incoming and outgoing mass differences  $(\mathfrak{D}_- \tilde{m})_n^{(i,j)}$ ,  $(\mathfrak{D}_- \tilde{m})_n^{(i,j)}$  and  $(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}$ ,  $(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}$ , respectively. The point  $A$  satisfies  $r(A) = r_n^{(i,j)}$ , while the point  $B$  satisfies  $r(B) = \tilde{r}_n^{(i,j)}$ . For simplicity, we have used the shorthand notation  $l = (-\Lambda)^{-1/2}$ .

Note that  $(\mathfrak{D}_- \tilde{m})_n^{(i,j)}$  and  $(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}$  are the mass differences around the outgoing beam  $\{U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$  before and after crossing the region  $\mathcal{N}_{\varepsilon n}^{(i,j)}$ , respectively, while  $(\mathfrak{D}_- \tilde{m})_n^{(i,j)}$  and  $(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}$  are the mass differences around the ingoing beam  $\{V_n(v^{(j)}) \leq v \leq U_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\}$  before and after crossing the region  $\mathcal{N}_{\varepsilon n}^{(i,j)}$ . See Figure 12. Note the trivial identity

$$(\mathfrak{D}_- \tilde{m})_n^{(i,j)} + (\mathfrak{D}_- \tilde{m})_n^{(i,j)} = (\mathfrak{D}_+ \tilde{m})_n^{(i,j)} + (\mathfrak{D}_+ \tilde{m})_n^{(i,j)}. \quad (6-50)$$

We will establish the following bounds for any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$  and  $i+1 \leq j \leq k+i$ :

$$(\mathfrak{D}_+ \tilde{m})_n^{(i,j)} = (\mathfrak{D}_- \tilde{m})_n^{(i,j)} \exp\left(\frac{2}{\tilde{r}_n^{(i,j)}} \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}{1 - 2\tilde{m}_n^{(i+1,j)}/\tilde{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\tilde{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)})(1 - \mathfrak{Err}_{/n}^{(i,j)})\right), \quad (6-51)$$

$$(\mathfrak{D}_+ \tilde{m})_n^{(i,j)} = (\mathfrak{D}_- \tilde{m})_n^{(i,j)} \exp\left(-\frac{2}{\tilde{r}_n^{(i,j)}} \frac{(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}}{1 - 2\tilde{m}_n^{(i+1,j)}/\tilde{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\tilde{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)})(1 - \mathfrak{Err}_{/n}^{(i,j)})\right), \quad (6-52)$$

where the terms  $\mathfrak{Err}_{1,n}^{(i,j)}$  in (6-51) and (6-52) are allowed to be different from each other, but they both satisfy the bound

$$0 \leq \mathfrak{Err}_{1,n}^{(i,j)} \leq 1 - \frac{\tilde{r}_n^{(i,j)} - 2\tilde{m}_n^{(i+1,j)} - \frac{1}{3}\Lambda(\tilde{r}_n^{(i,j)})^3}{r|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - 2\tilde{m}_n^{(i,j+1)} - \frac{1}{3}\Lambda r^3|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}} \quad (6-53)$$

and  $\mathfrak{Err}_{\backslash n}^{(i,j)}$ ,  $\mathfrak{Err}_{/n}^{(i,j)}$  satisfy the bounds

$$0 \leq \mathfrak{Err}_{\backslash n}^{(i,j)} \leq 1 - \frac{(\mathfrak{D}_+ \tilde{m})_n^{(i,j)}}{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}, \quad (6-54)$$

$$0 \leq \mathfrak{Err}_{/n}^{(i,j)} \leq 1 - \frac{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}}{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}}. \quad (6-55)$$

Moreover, the following estimate will be useful in the proof of (6-12): for any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$ , and  $k+1 \leq j \leq k+i$ ,

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)} \geq (\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} \cdot \exp\left(\frac{1}{5C_0} \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}}\right). \quad (6-56)$$

**Remark.** Notice that, as a consequence of (6-51) and (6-52), during the interaction of the two beams at  $\mathcal{N}_{\varepsilon n}^{(i,j)}$ , the mass difference  $\bar{\mathfrak{D}}\tilde{m}$  of the ingoing beam increases, while the mass difference  $\mathfrak{D}\tilde{m}$  of the outgoing beam decreases.

*Proof of (6-51) and (6-52).* By differentiating (2-47) in  $u$  and using (2-43) and (2-48), we readily obtain the wave-type equation for  $\tilde{m}$

$$\partial_u \partial_v \tilde{m} = -F(r, \tilde{m}) \partial_u \tilde{m} \partial_v \tilde{m}, \quad (6-57)$$

where

$$F(r, \tilde{m}) \doteq \frac{2}{r - 2\tilde{m} - \frac{1}{3}\Lambda r^3}. \quad (6-58)$$

Note that, formally, (6-57) can be rewritten as

$$\partial_v \log(-\partial_u \tilde{m}) = -F(r, \tilde{m}) \partial_v \tilde{m} \quad (6-59)$$

or

$$\partial_u \log(\partial_v \tilde{m}) = F(r, \tilde{m})(-\partial_u \tilde{m}) \quad (6-60)$$

(note, however, that  $\log(-\partial_u \tilde{m})$ ,  $\log(\partial_v \tilde{m})$  will not be well-defined when  $\partial_u \tilde{m} = 0$  or  $\partial_v \tilde{m} = 0$ ).

For any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$ , and  $i+1 \leq j \leq k+i$ , integrating (6-57) first in  $u$ , for  $U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ , and then in  $v$ , for  $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ , we obtain

$$\begin{aligned} & \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} \\ &= \int_{V_n(v^{(j)})}^{V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \partial_v \tilde{m}|_{(U_n(v^{(i)}), v)} \cdot \exp\left(2 \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r - 2m} \Big|_{(u, v)} du\right) dv. \end{aligned} \quad (6-61)$$

**Remark.** Note that, at the formal level, the derivation of (6-61) is easiest seen by integrating (6-60) first in  $u$ , then exponentiating, and then integrating in  $v$ . This procedure can actually be done rigorously, since  $\partial_u \tilde{m} < 0 < \partial_v \tilde{m}$  in the interior of  $\mathcal{N}_{\varepsilon n}^{(i,j)}$ , in view of (2-46)–(2-47) and (5-43)–(5-44).

In view of (6-18)–(6-19), we can bound for any  $U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$  and any  $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$

$$\begin{aligned} & \bar{r}_n^{(i,j)} - 2\tilde{m}_n^{(i+1,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^3 \\ & \leq (r - 2m)|_{(u, v)} \\ & \leq r|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - 2\tilde{m}_n^{(i,j+1)} - \frac{1}{3}\Lambda r^3|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}, \end{aligned} \quad (6-62)$$



where

$$\bar{r}_n^{(i,j)} \doteq r|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), V_n(v^{(j)}))}.$$

Therefore, using (6-62) to estimate  $1/(r - 2m)$ , from (6-61) we readily infer that

$$\begin{aligned} \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} &= \int_{V_n(v^{(j)})}^{V_n(v^{(j)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \partial_v \tilde{m}|_{(U_n(v^{(i)}), v)} \\ &\quad \cdot \exp\left(\frac{2}{\bar{r}_n^{(i,j)}} \frac{\tilde{m}|_{(U_n(v^{(i)}), v)} - \tilde{m}|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v)}}{1 - 2\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)}(v))\right) dv, \end{aligned} \quad (6-63)$$

where, for any  $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ ,  $\mathfrak{Err}_{1,n}^{(i,j)}(v)$  satisfies the bound (6-53).

Equations (2-46), (2-44) and (2-23) imply that, for any  $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ ,

$$\partial_v(\tilde{m}|_{(U_n(v^{(i)}), v)} - \tilde{m}|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v)}) \leq 0 \quad (6-64)$$

and, therefore, for any  $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ ,

$$(\mathfrak{D}_+ \tilde{m})_n^{(i,j)} \leq \tilde{m}|_{(U_n(v^{(i)}), v)} - \tilde{m}|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} \leq (\mathfrak{D}_- \tilde{m})_n^{(i,j)}. \quad (6-65)$$

The bound (6-65) implies that (6-63) can be expressed as

$$\begin{aligned} \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} &= (\tilde{m}_n^{(i+1,j)} - \tilde{m}_n^{(i+1,j+1)}) \\ &\quad \cdot \exp\left(\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}{1 - 2\tilde{m}_n^{(i+1,j)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,j)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,j)})(1 - \mathfrak{Err}_n^{(i,j)})\right), \end{aligned} \quad (6-66)$$

where  $\mathfrak{Err}_{1,n}^{(i,j)}$  satisfies the bound (6-53) and  $\mathfrak{Err}_n^{(i,j)}$  satisfies the bound (6-54). In view of (6-48) and (6-49), (6-66) is equivalent to (6-51).

Similarly, integrating (6-57) first in  $v$ , for  $V_n(v^{(j)}) \leq v \leq V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ , and then in  $u$ , for  $U_n(v^{(i)}) \leq u \leq U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$  (see also (6-59)), we obtain (6-52).  $\square$

*Proof of (6-56).* Recall  $F$  defined by (6-58) and let us define the function  $\bar{F} : \mathcal{D}_{\bar{F}} \rightarrow (0, +\infty)$ , where

$$\mathcal{D}_{\bar{F}} = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } x - y - \frac{2}{3}\Lambda x^2 > 0\}, \quad (6-67)$$

by the relation

$$\bar{F}(x, y) \doteq \frac{2}{x - y - \frac{2}{3}\Lambda x^2}. \quad (6-68)$$

Note that, in view of (6-7), (6-9), (5-6) and (5-3), for any  $\mu \geq 0$  for which

$$\inf_{(u,v) \in \mathcal{N}_{\varepsilon n}^{(i,j)}} \left\{ r(u, v) - 2\mu - \frac{1}{3}\Lambda r^2(u, v) \right\} > h_3(\varepsilon), \quad (6-69)$$

we can readily bound

$$\max_{(u,v) \in \mathcal{N}_{\varepsilon n}^{(i,j)}} \bar{F}(r(u, v), \mu) < \min_{(u,v) \in \mathcal{N}_{\varepsilon n}^{(i,j)}} F(r(u, v), \mu) \quad (6-70)$$

and

$$\partial_\mu \bar{F}(r(u, v), \mu), \partial_\mu F(r(u, v), \mu) > 0 \quad (6-71)$$

(note that  $F(r|_{\mathcal{N}_{\varepsilon n}^{(i,j)}}, \mu)$  and  $\bar{F}(r|_{\mathcal{N}_{\varepsilon n}^{(i,j)}}, \mu)$  are well-defined and positive under the condition (6-69)).

For any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$ , and  $k+1 \leq j \leq k+i$ , let us consider the following characteristic initial value problem on  $\mathcal{N}_{\varepsilon n}^{(i,j)}$ :

$$\begin{cases} \partial_u \partial_v \bar{m} = -\bar{F}(r, \bar{m}) \partial_u \bar{m} \partial_v \bar{m} & \text{on } \mathcal{N}_{\varepsilon n}^{(i,j)}, \\ \bar{m} = \tilde{m} & \text{on } \left[ U_n(v^{(i)}), U_n\left(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\right) \right] \times \{V_n(v^{(j)})\} \cup \{U_n(v^{(i)})\} \\ & \times \left[ V_n(v^{(j)}), V_n\left(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)\right) \right]. \end{cases} \quad (6-72)$$

Note that  $\tilde{m}$  satisfies the same characteristic initial value problem with  $F(r, \tilde{m})$  in place of  $\bar{F}(r, \bar{m})$ . Notice also that, in view of (2-46)–(2-47) and (5-43)–(5-44), the initial data for  $\tilde{m}$  and  $\bar{m}$  satisfy

$$\partial_u \tilde{m} < 0 \quad \text{on } \left\{ U_n(v^{(i)}) < u < U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right\}, \quad (6-73)$$

$$\partial_v \tilde{m} > 0 \quad \text{on } \left\{ V_n(v^{(j)}) < u < V_n(v^{(j)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon)) \right\}. \quad (6-74)$$

Therefore, in view of (6-70)–(6-71) and (6-73)–(6-74), an application of Lemma A.1 (see Section A1 of the Appendix) with  $\tilde{m}, \bar{m}$  in place of  $z_2, z_1$ , respectively, yields the following a priori bounds for a solution  $\bar{m}$  of (6-72):

$$\bar{m} \leq \tilde{m} \quad \text{on } \mathcal{N}_{\varepsilon n}^{(i,j)}, \quad (6-75)$$

$$\partial_u \bar{m} < 0 < \partial_v \bar{m} \quad \text{in the interior of } \mathcal{N}_{\varepsilon n}^{(i,j)}. \quad (6-76)$$

Notice that the a priori bound (6-75) and the initial data in (6-72) imply that  $\bar{m} \geq 0$  and that (6-69) holds for  $\mu = \tilde{m}$  and  $\mu = \bar{m}$ ; in particular,  $\bar{F}(r, \bar{m})$  is well-defined and positive on  $\mathcal{N}_{\varepsilon n}^{(i,j)}$ . Thus, it readily follows (using standard arguments) that (6-72) indeed has a unique smooth solution  $\bar{m}$  satisfying (6-75).

With  $\bar{m}$  defined on  $\mathcal{N}_{\varepsilon n}^{(i,j)}$  as above for any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$ , and  $k+1 \leq j \leq k+i$ , we will define the following modified versions of (6-48) and (6-49):

$$\begin{aligned} (\mathfrak{D}_- \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)}))} - \bar{m}|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), V_n(v^{(j)}))}, \\ (\bar{\mathfrak{D}}_- \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)}))} \end{aligned} \quad (6-77)$$

and

$$\begin{aligned} (\mathfrak{D}_+ \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}, \\ (\bar{\mathfrak{D}}_+ \bar{m})_n^{(i,j)} &\doteq \bar{m}|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \bar{m}|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), V_n(v^{(j)}))}. \end{aligned} \quad (6-78)$$

Note that, in view of the initial data for (6-72),

$$\begin{aligned} (\mathfrak{D}_- \bar{m})_n^{(i,j)} &= (\mathfrak{D}_- \tilde{m})_n^{(i,j)}, \\ (\bar{\mathfrak{D}}_- \bar{m})_n^{(i,j)} &= (\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}, \end{aligned} \quad (6-79)$$

while, in view of the bound (6-75) (and the initial data for (6-72)),

$$\begin{aligned} (\mathfrak{D}_+ \bar{m})_n^{(i,j)} &\geq (\mathfrak{D}_+ \tilde{m})_n^{(i,j)}, \\ (\bar{\mathfrak{D}}_+ \bar{m})_n^{(i,j)} &\leq (\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}. \end{aligned} \quad (6-80)$$

By repeating exactly the same steps that led to (6-51) and (6-52) but using (6-72) instead of (6-57), we obtain for any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$ , and  $k+1 \leq j \leq k+i$ ,

$$(\bar{\mathcal{D}}_+ \bar{m})_n^{(i,j)} = (\bar{\mathcal{D}}_- \bar{m})_n^{(i,j)} \cdot \exp\left(\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\bar{\mathcal{D}}_- \bar{m})_n^{(i,j)}}{1 - \tilde{m}_n^{(i+1,j)} / \bar{r}_n^{(i,j)} - \frac{2}{3} \Lambda (\bar{r}_n^{(i,j)})^2} (1 - \bar{\mathfrak{Err}}_{1,n}^{(i,j)})(1 - \bar{\mathfrak{Err}}_{/n}^{(i,j)})\right), \quad (6-81)$$

$$(\mathcal{D}_+ \bar{m})_n^{(i,j)} = (\mathcal{D}_- \bar{m})_n^{(i,j)} \cdot \exp\left(-\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\bar{\mathcal{D}}_+ \bar{m})_n^{(i,j)}}{1 - \tilde{m}_n^{(i+1,j)} / \bar{r}_n^{(i,j)} - \frac{2}{3} \Lambda (\bar{r}_n^{(i,j)})^2} (1 - \bar{\mathfrak{Err}}_{1,n}^{(i,j)})(1 - \bar{\mathfrak{Err}}_{/n}^{(i,j)})\right), \quad (6-82)$$

where

$$0 \leq \bar{\mathfrak{Err}}_{1,n}^{(i,j)} \leq 1 - \frac{\bar{r}_n^{(i,j)} - \tilde{m}_n^{(i+1,j)} - \frac{2}{3} \Lambda (\bar{r}_n^{(i,j)})^3}{r|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} - \tilde{m}_n^{(i,j+1)} - \frac{2}{3} \Lambda r^3|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}}, \quad (6-83)$$

and

$$0 \leq \bar{\mathfrak{Err}}_{/n}^{(i,j)} \leq 1 - \frac{(\mathcal{D}_+ \bar{m})_n^{(i,j)}}{(\mathcal{D}_- \bar{m})_n^{(i,j)}}, \quad (6-84)$$

$$0 \leq \bar{\mathfrak{Err}}_{/n}^{(i,j)} \leq 1 - \frac{(\bar{\mathcal{D}}_- \bar{m})_n^{(i,j)}}{(\bar{\mathcal{D}}_+ \bar{m})_n^{(i,j)}} \quad (6-85)$$

(and, as before, we allow the terms  $\bar{\mathfrak{Err}}_{1,n}^{(i,j)}$  in (6-81) and (6-82) to be different).

Because

$$\tilde{m}_n^{(i+1,j)} \leq \tilde{m}|_I \leq \frac{2}{3} r_0 \leq \frac{2}{3} \bar{r}_n^{(i,j)} \quad (6-86)$$

(in view of (5-24), (5-6), (6-18) and (6-19)), from (6-82) (using also (6-9) and that  $\bar{\mathfrak{Err}}_{1,n}^{(i,j)}, \bar{\mathfrak{Err}}_{/n}^{(i,j)} \geq 0$ ) we can estimate for any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$ , and  $k+1 \leq j \leq k+i$

$$\begin{aligned} (\mathcal{D}_+ \bar{m})_n^{(i,j)} &= (\mathcal{D}_- \bar{m})_n^{(i,j)} \cdot \exp\left(-\frac{2}{\bar{r}_n^{(i,j)}} \frac{(\bar{\mathcal{D}}_+ \bar{m})_n^{(i,j)}}{1 - \tilde{m}_n^{(i+1,j)} / \bar{r}_n^{(i,j)} - \frac{2}{3} \Lambda (\bar{r}_n^{(i,j)})^2} (1 - \bar{\mathfrak{Err}}_{1,n}^{(i,j)})(1 - \bar{\mathfrak{Err}}_{/n}^{(i,j)})\right) \\ &\geq (\mathcal{D}_- \bar{m})_n^{(i,j)} \cdot \exp\left(-\frac{8(\bar{\mathcal{D}}_+ \bar{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}}\right). \end{aligned} \quad (6-87)$$

In view of the fact that

$$(\bar{\mathcal{D}}_+ \bar{m})_n^{(i,j)} \leq (\bar{\mathcal{D}}_+ \tilde{m})_n^{(i,j)} = \tilde{m}_n^{(i,j)} - \tilde{m}_n^{(i,j+1)} \leq \tilde{m}|_I - 0 \leq \frac{2}{3} r_0 \leq \frac{2}{3} \bar{r}_n^{(i,j)}$$

(following from (6-80)), (6-87) yields

$$(\mathcal{D}_+ \bar{m})_n^{(i,j)} \geq e^{-16/3} (\mathcal{D}_- \bar{m})_n^{(i,j)}. \quad (6-88)$$

In view of (6-84), (6-88) implies

$$1 - \bar{\mathfrak{Err}}_{/n}^{(i,j)} \geq \frac{1}{C_0}. \quad (6-89)$$

Using (6-89) in (6-81), we obtain

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)} \geq (\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} \cdot \exp\left(\frac{2}{C_0 \bar{r}_n^{(i,j)}} \frac{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}}{1 - \tilde{m}_n^{(i+1,j)} / \bar{r}_n^{(i,j)} - \frac{2}{3} \Lambda(\bar{r}_n^{(i,j)})^2} (1 - \bar{\mathfrak{Err}}_{1,n}^{(i,j)})\right). \quad (6-90)$$

In view of (6-86) and (6-9), we can also estimate

$$\frac{1 - \bar{\mathfrak{Err}}_{1,n}^{(i,j)}}{1 - \tilde{m}_n^{(i+1,j)} / \bar{r}_n^{(i,j)} - \frac{2}{3} \Lambda(\bar{r}_n^{(i,j)})^2} \geq \frac{1}{10} \quad (6-91)$$

and, thus, (6-90) yields

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)} \geq (\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} \cdot \exp\left(\frac{1}{5C_0} \frac{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}}\right). \quad (6-92)$$

From (6-92) and the relations (6-79) and (6-80), we readily obtain (6-56).  $\square$

*Proof of (6-11).* For any  $1 \leq n \leq n_f$ , from (6-51) we readily obtain that, for any  $1 \leq i \leq k$ ,

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)} \geq (\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,k+1)}. \quad (6-93)$$

Applying (6-93) successively for  $i = 1, 2, \dots, k$ , using also the identity

$$(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)} = (\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i+1,j)} \quad (6-94)$$

(which follows from the fact that  $\tilde{m}$  is constant over each  $\mathcal{R}_{\varepsilon n}^{(i,j)}$ ), we thus infer that, for any  $1 \leq i \leq k$ ,

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)} \geq (\bar{\mathfrak{D}}_- \tilde{m})_n^{(k,k+1)} = (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}). \quad (6-95)$$

Since

$$\tilde{m}_{n-1}^{(0,0)} = \tilde{m}_{n-1}^{(1,1)} = \tilde{m}|_{\mathcal{I}} \quad (6-96)$$

and

$$(\bar{\mathfrak{D}}_+ \tilde{m})_{n-1}^{(0,1)} = \tilde{m}_{n-1}^{(1,1)} - \tilde{m}_{n-1}^{(0,1)} = (\bar{\mathfrak{D}}_- \tilde{m})_n^{(k,k+1)}, \quad (6-97)$$

from (6-95) we infer that, for any  $1 \leq i \leq k$ ,

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)} \geq (\bar{\mathfrak{D}}_+ \tilde{m})_{n-1}^{(0,1)}. \quad (6-98)$$

Similarly as for the derivation of (6-93), applying the relation (6-52) successively for  $i = 0$  and  $j = 1, 2, \dots, k$  (with  $n-1$  in place of  $n$ ), we infer

$$\begin{aligned} & (\bar{\mathfrak{D}}_+ \tilde{m})_{n-1}^{(0,1)} \\ &= (\bar{\mathfrak{D}}_- \tilde{m})_{n-1}^{(0,k)} \cdot \exp\left(-\sum_{j=1}^k \frac{2}{\bar{r}_{n-1}^{(0,j)}} \frac{(\bar{\mathfrak{D}}_- \tilde{m})_{n-1}^{(0,j)}}{1 - 2\tilde{m}_{n-1}^{(0,j)} / \bar{r}_{n-1}^{(0,j)} - \frac{1}{3} \Lambda(\bar{r}_{n-1}^{(0,j)})^2} (1 - \bar{\mathfrak{Err}}_{1,n-1}^{(0,j)})(1 - \bar{\mathfrak{Err}}_{/n-1}^{(0,j)})\right). \end{aligned} \quad (6-99)$$

In view of the bound (5-24) for the total mass  $\tilde{m}|_{\mathcal{I}}$ , the lower bound (6-8) for  $r_n^{(0,k)}$  and the fact that

$$\bar{r}_{n-1}^{(0,j)} \leq r_{n-1}^{(0,k)} \leq \bar{r}_{n-1}^{(0,j)} (1 + (h_2(\varepsilon))^{1/2})$$

for  $1 \leq j \leq k$  (following from (6-7), and (5-3)), we can estimate

$$\sum_{j=1}^k \frac{2}{\bar{r}_{n-1}^{(0,j)}} \frac{(\bar{\mathfrak{D}}_- \tilde{m})_{n-1}^{(0,j)}}{1 - 2\tilde{m}_{n-1}^{(0,j)} / \bar{r}_{n-1}^{(0,j)} - \frac{1}{3}\Lambda(\bar{r}_{n-1}^{(0,j)})^2} (1 - \mathfrak{Err}_{1,n-1}^{(0,j)})(1 - \mathfrak{Err}_{/n-1}^{(0,j)}) \leq \varepsilon^{3/2}. \quad (6-100)$$

Therefore, (6-99) yields

$$\log \frac{(\mathfrak{D}_+ \tilde{m})_{n-1}^{(0,1)}}{(\mathfrak{D}_- \tilde{m})_{n-1}^{(0,k)}} \geq -\varepsilon^{3/2}. \quad (6-101)$$

From (6-98) and (6-101) we thus infer that, for any  $1 \leq i \leq k$  and any  $2 \leq n \leq n_f$ ,

$$\log \frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \geq -\varepsilon^{3/2}. \quad (6-102)$$

From (6-102) for  $i = 1$  and the fact that, for any  $1 \leq n \leq n_f$ ,

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(1,k+1)} = (\mathfrak{D}_- \tilde{m})_n^{(0,k)} = \tilde{m}_n^{(1,k+1)}, \quad (6-103)$$

we thus infer that, for all  $2 \leq n \leq n_f$ ,

$$\log \frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \geq -\varepsilon^{3/2}. \quad (6-104)$$

Applying (6-104) successively  $n - 1$  times, we thus infer, for any  $2 \leq n \leq n_f$ ,

$$\log \frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_1^{(1,k+1)}} \geq -\varepsilon^{3/2}(n - 1). \quad (6-105)$$

The bound (5-6) for  $r_0$  and the form (5-7) of the initial data imply

$$\frac{2(\tilde{m}_/(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) - \tilde{m}_/(v^{(0)}))}{r_0} \geq \frac{4}{C_0}h_0(\varepsilon). \quad (6-106)$$

Therefore, from (6-95) and (6-106) we infer that, for all  $1 \leq i \leq 1$ ,

$$\frac{2(\bar{\mathfrak{D}}_+ \tilde{m})_1^{(i,k+1)}}{r_0} \geq \frac{4}{C_0}h_0(\varepsilon). \quad (6-107)$$

From (6-105) and (6-107) for  $i = 1$  (when  $(\bar{\mathfrak{D}}_+ \tilde{m})_1^{(1,k+1)} = \tilde{m}_1^{(1,k+1)}$ ), using also the fact that  $n_f \leq (h_1(\varepsilon))^{-2}$ , we thus deduce that, for all  $1 \leq n \leq n_f$ ,

$$\frac{2\tilde{m}_n^{(1,k+1)}}{r_0} \geq \frac{2}{C_0}h_0(\varepsilon). \quad (6-108)$$

The relations (5-24) and (6-108) readily yield (6-11).  $\square$

*Proof of (6-10).* In view of the bound (6-37), we infer that, for any  $1 \leq n \leq n_f$ ,

$$1 - \frac{2\tilde{m}_n^{(1,k+1)}}{r|_{(U_n(v^{(1)}), V_n(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}} - \frac{1}{3}\Lambda r^2|_{(U_n(v^{(1)}), V_n(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} \geq h_3(\varepsilon). \quad (6-109)$$

Using the bounds

$$\frac{r|_{(U_n(v^{(1)}), V_n(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}}{r_0} \leq 1 + (h_2(\varepsilon))^{1/2} \quad (6-110)$$

(derived from (6-7), (6-9) and (5-3)) and

$$\frac{r_0}{(2/\sqrt{-\Lambda})\varepsilon - \frac{1}{3}\Lambda r_0^3} < 1 - \frac{1}{2} \exp(-2(h_0(\varepsilon))^{-4}) \quad (6-111)$$

(from (5-6)), as well as the relation (5-24) for  $\tilde{m}|_{\mathcal{I}}$ , we can readily derive from (6-109) that

$$\frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})}{r_0} \geq 2(1 + (h_2(\varepsilon))^{1/2})^{-1} \exp(-2(h_0(\varepsilon))^{-4}) + \frac{1}{3}\Lambda r_0^2(1 + (h_2(\varepsilon))^{1/2}). \quad (6-112)$$

The bound (6-10) follows readily from (6-111) and (6-112).  $\square$

*Proof of (6-13).* For any  $2 \leq n \leq n_f$ , applying the relation (6-51) successively for  $j = k + 1$  and  $i = 1, 2, \dots, k$ , using also the identity (6-94) and the trivial bound

$$(\mathfrak{D}_- \tilde{m})_n^{(i,j)} (1 - \mathfrak{Err}_n^{(i,k+1)}) \geq (\mathfrak{D}_+ \tilde{m})_n^{(i,k+1)} \quad (6-113)$$

(following directly from (6-54)), we obtain

$$\begin{aligned} & (\bar{\mathfrak{D}}_+ \tilde{m})_n^{(1,k+1)} \\ &= (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp \left( \sum_{i=1}^k \frac{2}{\bar{r}_n^{(i,k+1)}} \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,k+1)}}{1 - 2\tilde{m}_n^{(i+1,k+1)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,k+1)})(1 - \mathfrak{Err}_n^{(i,k+1)}) \right) \\ &\geq (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp \left( \sum_{i=1}^k \frac{2}{\bar{r}_n^{(i,k+1)}} \frac{(\mathfrak{D}_+ \tilde{m})_n^{(i,k+1)}}{1 - 2\tilde{m}_n^{(i+1,k+1)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,k+1)}) \right). \end{aligned} \quad (6-114)$$

In view of (6-7), (5-3), (6-53), (6-9) and the fact that

$$r_0 \leq \bar{r}_n^{(i,k+1)} \leq \bar{r}_n^{(k,k+1)},$$

for  $1 \leq i \leq k$  (following from (6-18)), we can bound, for any  $1 \leq i \leq k$ ,

$$\begin{aligned} & \frac{2}{\bar{r}_n^{(i,k+1)}} \frac{1}{1 - 2\tilde{m}_n^{(i+1,k+1)}/\bar{r}_n^{(i,j)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^2} (1 - \mathfrak{Err}_{1,n}^{(i,k+1)}) \\ &\geq 2 \min \left\{ \frac{1}{r|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}} - 2\tilde{m}_n^{(i,j+1)} - \frac{1}{3}\Lambda r^3|_{(U_n(v^{(i)}), V_n(v^{(j)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))}}, \right. \\ &\quad \left. \frac{1}{\bar{r}_n^{(i,j)} - 2\tilde{m}_n^{(i+1,k+1)} - \frac{1}{3}\Lambda(\bar{r}_n^{(i,k+1)})^3} \right\} \\ &\geq \frac{2 - O(\varepsilon)}{r_n^{(k,k+1)}}. \end{aligned} \quad (6-115)$$

Furthermore,

$$\sum_{i=1}^k (\mathfrak{D}_+ \tilde{m})_n^{(i,k+1)} = \sum_{i=1}^k (\tilde{m}_n^{(i+1,k+1)} - \tilde{m}_n^{(i,k+1)}) = \tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)}. \quad (6-116)$$

Therefore, in view of (6-10), (6-115), (6-116) and the fact that

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(1,k+1)} = \tilde{m}_n^{(1,k+1)} - \tilde{m}_n^{(1,k+2)} = \tilde{m}_n^{(1,k+1)}, \quad (6-117)$$

the bound (6-114) yields

$$\begin{aligned} \tilde{m}_n^{(1,k+1)} &\geq (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp\left(\frac{2 - O(\varepsilon)}{r_n^{(k,k+1)}} (\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(1,k+1)})\right) \\ &\geq (\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \cdot \exp\left(\frac{r_0}{2r_n^{(k,k+1)}} \exp(-2(h_0(\varepsilon))^{-4})\right). \end{aligned} \quad (6-118)$$

Using the bound (6-101) and the fact that

$$\begin{aligned} (\mathfrak{D}_+ \tilde{m})_{n-1}^{(0,1)} &= \tilde{m}_{n-1}^{(1,1)} - \tilde{m}_{n-1}^{(0,1)} = \tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}, \\ (\mathfrak{D}_- \tilde{m})_{n-1}^{(0,k)} &= \tilde{m}_{n-1}^{(1,k+1)}, \end{aligned}$$

we can estimate

$$(\tilde{m}_{n-1}^{(0,0)} - \tilde{m}_{n-1}^{(0,1)}) \geq e^{-\varepsilon^{3/2}} \tilde{m}_{n-1}^{(1,k+1)}. \quad (6-119)$$

From (6-118) and (6-119) we thus obtain (in view also of (6-9) and the properties (5-2) of  $h_0(\varepsilon)$ )

$$\tilde{m}_n^{(1,k+1)} \geq \tilde{m}_{n-1}^{(1,k+1)} \exp\left(\frac{r_0}{4r_n^{(k,k+1)}} \exp(-2(h_0(\varepsilon))^{-4})\right). \quad (6-120)$$

In particular, (6-13) holds for all  $2 \leq n \leq n_f$ .  $\square$

*Proof of (6-12).* Combining (6-102) and (6-11) (using also (6-107) in the case  $n = 1$ , as well as (5-24) and (5-6) for  $\tilde{m}|_{\mathcal{I}}, r_0$ ), we can readily estimate for any  $1 \leq n \leq n_f$  and any  $1 \leq i \leq k$

$$\frac{2(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)}}{r_0} \geq \frac{1}{C_0} h_0(\varepsilon). \quad (6-121)$$

Similarly, in view of (6-94), (6-97) and (6-101), we can bound for any  $1 \leq n \leq n_f$  and any  $1 \leq i \leq k$

$$\frac{2(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,k+1)}}{r_0} \geq \frac{1}{C_0} h_0(\varepsilon). \quad (6-122)$$

Using the relation

$$(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)} = \tilde{m}_n^{(i,k+1)} - \tilde{m}_n^{(i,k+2)} \quad (6-123)$$

and the trivial bounds

$$\tilde{m}_n^{(i,k+1)} \leq \tilde{m}|_{\mathcal{I}} \quad (6-124)$$

and

$$\max_{k+2 \leq j \leq k+i+1} \tilde{m}_n^{(i,j)} = \tilde{m}_n^{(i,k+2)} \quad (6-125)$$

(following from the monotonicity properties (6-19) of  $\tilde{m}$ ), from (6-121) we obtain for any  $1 \leq n \leq n_f$

$$\min_{\substack{1 \leq i \leq k \\ k+2 \leq j \leq k+i+1}} \frac{2(\tilde{m}|_{\mathcal{I}} - \tilde{m}_n^{(i,j)})}{r_0} \geq \frac{1}{C_0} h_0(\varepsilon). \quad (6-126)$$

In view of (5-24) and (5-6) and the properties (5-2) of  $h_0(\varepsilon)$ , from (6-126) we infer that, for any  $1 \leq n \leq n_f$ ,

$$\max_{\substack{1 \leq i \leq k \\ k+2 \leq j \leq k+i+1}} \frac{2\tilde{m}_n^{(i,j)}}{r_0} \leq 1 - \frac{1}{3C_0} h_0(\varepsilon). \quad (6-127)$$

In particular, for any  $1 \leq n \leq n_f$ , (6-127) implies

$$\sup_{\{u \leq v \leq V_n(v^{(k+1)})\} \cap \{u \geq U_n(v^{(k)})\}} \frac{1}{1 - 2m/r} \leq 4C_0(h_0(\varepsilon))^{-1}. \quad (6-128)$$

The main estimate that will be used in the proof of (6-12) is the following bound: for any  $1 \leq n_1 < n_2 \leq n_f$  and any  $V_{n_2}(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq v \leq V_{n_2}(v^{(k+1)})$

$$\begin{aligned} \partial_v r|_{(U_{n_2}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} &\geq \partial_v r|_{(U_{n_1-1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v - (n_2 - n_1 - 1)v_0)} \\ &\times \exp\left(-C_0^5(h_0(\varepsilon))^{-3} \max_{n_1 \leq n \leq n_2} \left\{ \frac{r_n^{(1,k+1)}}{r_0} \right\} \log\left(\frac{\tilde{m}_{n_2}^{(1,k+1)}}{\tilde{m}_{n_1}^{(1,k+1)}}\right) - 2\varepsilon^{1/2}\right). \end{aligned} \quad (6-129)$$

*Proof of (6-129).* For any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$ , and  $k+1 \leq j \leq k+i$ , integrating (2-43) from  $u = U_n(v^{(i)})$  up to  $U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$  (and using (2-46)), we infer that, for all  $V_n(v^{(j+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq \bar{v} \leq V_n(v^{(j)})$ ,

$$\begin{aligned} \log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{(U_n(v^{(i)}), \bar{v})} - \log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \\ = 4\pi \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{r T_{uu}}{-\partial_u r}\Big|_{(u, \bar{v})} du \\ = 2 \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r(1 - 2m/r)}\Big|_{(u, \bar{v})} du. \end{aligned} \quad (6-130)$$

In view of the monotonicity properties (6-18), (6-19) of  $r$  and  $\tilde{m}$ , we can estimate

$$\begin{aligned} &2 \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r(1 - 2m/r)}\Big|_{(u, \bar{v})} du \\ &\leq \sup_{U_n(v^{(i)}) \leq \bar{u} \leq U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \left(\frac{2}{r(1 - 2m/r)}\right)\Big|_{(u, \bar{v})} \int_{U_n(v^{(i)})}^{U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} (-\partial_u \tilde{m})\Big|_{(u, \bar{v})} du \\ &\leq \frac{2}{r(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \sup_{U_n(v^{(i)}) \leq \bar{u} \leq U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \left(\frac{1}{1 - 2m/r|_{(\bar{u}, \bar{v})}}\right) (\mathfrak{D}_- \tilde{m})_n^{(i,j)}. \end{aligned} \quad (6-131)$$



From (6-130), (6-131) and the bound (6-128) for  $1 - 2m/r$ , we infer that

$$\begin{aligned} \log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{(U_n(v^{(i)}), \bar{v})} - \log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \\ \leq 8C_0(h_0(\varepsilon))^{-1} \frac{\bar{r}_n^{(i,j)}}{r(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \cdot \frac{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}}. \end{aligned} \quad (6-132)$$

Notice that, in view of the bound (6-56), we can estimate

$$\frac{(\mathfrak{D}_- \tilde{m})_n^{(i,j)}}{\bar{r}_n^{(i,j)}} \leq 5C_0 \log\left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}}{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}}\right). \quad (6-133)$$

Thus, from (6-132) and (6-133), we deduce that, for any  $1 \leq n \leq n_f$ ,  $1 \leq i \leq k$ , and  $k+1 \leq j \leq k+i$  and any  $V_n(v^{(j+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq \bar{v} \leq V_n(v^{(j)})$ ,

$$\begin{aligned} \log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{(U_n(v^{(i)}), \bar{v})} - \log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \\ \leq 40C_0^2(h_0(\varepsilon))^{-1} \frac{\bar{r}_n^{(i,j)}}{r(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \log\left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,j)}}{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,j)}}\right). \end{aligned} \quad (6-134)$$

Applying the relation (6-134) successively for  $i = 1, \dots, k$  and  $\bar{v} = v$ , using also the fact that

$$\partial_u \left( \frac{\partial_v r}{1 - 2m/r} \right) = 0$$

on each  $\mathcal{R}_{\varepsilon n}^{(i,j)}$ , we obtain

$$\begin{aligned} \frac{\partial_v r}{1 - 2m/r}\Big|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \\ \geq \frac{\partial_v r}{1 - 2m/r}\Big|_{(U_{n-1}(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \\ \times \exp\left(-40C_0^2(h_0(\varepsilon))^{-1} \sum_{i=1}^k \frac{\bar{r}_n^{(i,k+1)}}{r(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \log\left(\frac{(\bar{\mathfrak{D}}_+ \tilde{m})_n^{(i,k+1)}}{(\bar{\mathfrak{D}}_- \tilde{m})_n^{(i,k+1)}}\right)\right). \end{aligned} \quad (6-135)$$

For any  $i = 1, \dots, k$ , integrating (2-43) in  $u$  from  $u = U_n(v^{(i)})$  up to  $U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$  for  $v = v_* \in [\bar{v}, V_n(v^{(j)})]$  and using (2-46) and the fact that  $\partial_u \tilde{m} = 0$  on  $\mathcal{R}_{\varepsilon n}^{(i,j)}$ , we infer

$$\begin{aligned} \frac{\partial_v r}{1 - 2m/r}\Big|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)} \\ = \frac{\partial_v r}{1 - 2m/r}\Big|_{(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)} \exp\left(-2 \int_{U_n(v^{(i)})}^{U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u \tilde{m}}{r - 2\tilde{m} - \frac{1}{3}\Lambda r^3} \Big|_{(u, v_*)} du\right). \end{aligned} \quad (6-136)$$

Using the fact that  $r \geq r_0$ , from (6-136) we infer (in view of the monotonicity property (6-19) for  $\tilde{m}$ ) that

$$\begin{aligned}
& \left. \frac{\partial_v r}{1-2m/r} \right|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)} \\
& \geq \left. \frac{\partial_v r}{1-2m/r} \right|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)} \exp \left( -2 \int_{U_n(v^{(i)})}^{U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{-\partial_u \tilde{m}|_{(u, v_*)}}{r_0 - 2\tilde{m}|_{(u, v_*)} - \frac{1}{3}\Lambda r_0^3} du \right) \\
& = \left. \frac{\partial_v r}{1-2m/r} \right|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)} \exp \left( - \int_{U_n(v^{(i)})}^{U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))} \partial_u \left( \log \left( 1 - \frac{2\tilde{m}|_{(u, v_*)}}{r_0} - \frac{1}{3}\Lambda r_0^2 \right) \right) du \right) \\
& = \left. \frac{\partial_v r}{1-2m/r} \right|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)} \cdot \left( \frac{1 - 2\tilde{m}|_{(U_n(v^{(i)}), v_*)}/r_0 - \frac{1}{3}\Lambda r_0^2}{1 - 2\tilde{m}|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)}/r_0 - \frac{1}{3}\Lambda r_0^2} \right). \tag{6-137}
\end{aligned}$$

In view of the bounds (6-9) and (6-127), (6-137) yields

$$\left. \frac{\partial_v r}{1-2m/r} \right|_{(U_n(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)} \geq \left. \frac{\partial_v r}{1-2m/r} \right|_{(U_n(v^{(i)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)), v_*)} \frac{h_0(\varepsilon)}{4C_0}. \tag{6-138}$$

Integrating (6-138) in  $v_* \in [\bar{v}, V_n(v^{(j)})]$  and using (6-128) (and (6-9)) for the  $1/(1-2m/r)$  factors, we thus obtain

$$\bar{r}_n^{(1,k+1)} - r_0 \geq \left( \bar{r}_n^{(i,k+1)} - r(U_n(v^{(i)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)), \bar{v}) \right) \frac{(h_0(\varepsilon))^2}{16C_0^2} \tag{6-139}$$

and, thus (in view of (6-7), (5-3) and the fact that  $r_0 \leq \min\{r_n^{(1,k+1)}, r(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})\}$ )

$$\frac{\bar{r}_n^{(i,k+1)}}{r(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), \bar{v})} \leq 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0}. \tag{6-140}$$

From (6-140) and (6-94), it follows that

$$\begin{aligned}
& \sum_{i=1}^k \frac{\bar{r}_n^{(i,k+1)}}{r(U_n(v^{(i)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} \log \left( \frac{(\bar{\mathcal{D}}_+ \tilde{m})_n^{(i,k+1)}}{(\bar{\mathcal{D}}_- \tilde{m})_n^{(i,k+1)}} \right) \\
& \leq 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \sum_{i=1}^k \log \left( \frac{(\bar{\mathcal{D}}_+ \tilde{m})_n^{(i,k+1)}}{(\bar{\mathcal{D}}_- \tilde{m})_n^{(i,k+1)}} \right) \\
& = 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \left\{ \sum_{i=1}^{k-1} \log \left( \frac{(\bar{\mathcal{D}}_+ \tilde{m})_n^{(i,k+1)}}{(\bar{\mathcal{D}}_+ \tilde{m})_n^{(i+1,k+1)}} \right) + \log \left( \frac{(\bar{\mathcal{D}}_+ \tilde{m})_n^{(k,k+1)}}{\tilde{m}|_{\mathcal{I}} - \tilde{m}_{n-1}^{(0,1)}} \right) \right\} \\
& = 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \log \left( \frac{(\bar{\mathcal{D}}_+ \tilde{m})_n^{(1,k+1)}}{\tilde{m}|_{\mathcal{I}} - \tilde{m}_{n-1}^{(0,1)}} \right) \\
& = 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \log \left( \frac{(\bar{\mathcal{D}}_+ \tilde{m})_n^{(1,k+1)}}{(\mathcal{D}_+ \tilde{m})_{n-1}^{(0,1)}} \right) \\
& \leq 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \left\{ \log \left( \frac{(\bar{\mathcal{D}}_+ \tilde{m})_n^{(1,k+1)}}{(\mathcal{D}_- \tilde{m})_{n-1}^{(0,k)}} \right) + \varepsilon^{3/2} \right\} \\
& = 16C_0^2(h_0(\varepsilon))^{-2} \frac{r_n^{(1,k+1)}}{r_0} \left\{ \log \left( \frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}} \right) + \varepsilon^{3/2} \right\} \tag{6-141}
\end{aligned}$$

(where the inequality at the sixth line of (6-141) follows from (6-101)). Therefore, (6-135) and (6-141) yield, for any  $1 \leq n \leq n_f$  and any  $V_n(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq v \leq V_n(v^{(k+1)})$ ,

$$\begin{aligned} & \left. \frac{\partial_v r}{1 - 2m/r} \right|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} \\ & \geq \left. \frac{\partial_v r}{1 - 2m/r} \right|_{(U_{n-1}(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} \exp\left(-C_0^5(h_0(\varepsilon))^{-3} \frac{r_n^{(1,k+1)}}{r_0} \left\{ \log\left(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}}\right) + \varepsilon^{3/2} \right\}\right). \end{aligned} \quad (6-142)$$

In view of (2-44), we can bound for any  $U_{n-1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq u \leq U_{n-1}(v^{(0)})$

$$\left. \frac{-\partial_u r}{1 - 2m/r} \right|_{(u, V_{n-1}(v^{(1)}))} \geq \left. \frac{-\partial_u r}{1 - 2m/r} \right|_{(u, V_{n-1}(v^{(k+1)}))}. \quad (6-143)$$

Hence, using that

- from (5-39), (6-9),

$$\begin{aligned} \left. \frac{\partial_v r}{1 - 2m/r} \right|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} &= \left. \frac{\partial_v r}{1 - \frac{1}{3}\Lambda r^2} \right|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} \\ &= (1 + O(\varepsilon)) \left. \partial_v r \right|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)}, \end{aligned} \quad (6-144)$$

- from (2-45), (5-24), (6-8) and (6-7)

$$\left. \frac{\partial_v r}{1 - 2m/r} \right|_{(U_{n-1}(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} = \left. \frac{\partial_v r}{1 - 2m/r} \right|_{(U_{n-1}(v^{(0)}), v)} (1 + O(\varepsilon)), \quad (6-145)$$

- from (2-44), (2-43) and the gauge condition (3-24),

$$\left. \frac{\partial_v r}{1 - 2m/r} \right|_{\mathcal{R}_{n-1}^{(1,1)} \cap \{v=\bar{v}\}} = \left. \frac{-\partial_u r}{1 - 2m/r} \right|_{\mathcal{R}_{n-1}^{(1,1)} \cap \{u=\bar{v}-v_0\}}, \quad (6-146)$$

the bounds (6-142) and (6-143) yield for any  $V_n(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq v \leq V_n(v^{(k+1)})$

$$\begin{aligned} & \left. \partial_v r \right|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} \\ & \geq \left. \frac{-\partial_u r}{1 - 2m/r} \right|_{(v-v_0, V_{n-1}(v^{(k+1)}))} \exp\left(-C_0^5(h_0(\varepsilon))^{-3} \frac{r_n^{(1,k+1)}}{r_0} \log\left(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}}\right) - \varepsilon^{1/2}\right). \end{aligned} \quad (6-147)$$

In view of (5-39), (6-9) and the fact that

$$\left. \frac{-\partial_u r}{1 - 2m/r} \right|_{\mathcal{R}_{n-1}^{(1,k+2)} \cap \{u=\bar{u}\}} = \left. \frac{\partial_v r}{1 - 2m/r} \right|_{\mathcal{R}_{n-1}^{(1,k+2)} \cap \{v=\bar{u}\}} \quad (6-148)$$

(following from the gauge condition (3-23)), from (6-147) we infer for any  $V_n(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq v \leq V_n(v^{(k+1)})$

$$\begin{aligned} & \left. \partial_v r \right|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} \\ & \geq \left. \partial_v r \right|_{(U_{n-1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v-v_0)} \exp\left(-C_0^5(h_0(\varepsilon))^{-3} \frac{r_n^{(1,k+1)}}{r_0} \log\left(\frac{\tilde{m}_n^{(1,k+1)}}{\tilde{m}_{n-1}^{(1,k+1)}}\right) - 2\varepsilon^{1/2}\right). \end{aligned} \quad (6-149)$$

Iterating (6-149) for  $n_1 < n \leq n_2$ , we thus obtain (6-129).  $\square$

Let  $2 \leq n_1 \leq n_f$  be such that

$$r_{n_1}^{(1,k+1)} \leq 2r_0, \quad (6-150)$$

$$r_{n_1-1}^{(1,k+1)} > 2r_0. \quad (6-151)$$

Note that if no such  $n_1$  exists, then (6-12) is automatically true ((6-151) holds for  $r_1^{(1,k+1)}$  as a corollary of the Cauchy stability estimates of Proposition 3.5 and the choice of the initial data).

Let us also define

$$n_2 = \max\{n_1 \leq n \leq n_f : r_l^{(1,k+1)} \leq 2r_0 \text{ for all } n_1 \leq l \leq n\}. \quad (6-152)$$

In order to establish (6-12), it suffices to establish that, for all  $n_1 \leq n \leq n_2$ ,

$$\frac{r_n^{(1,k+1)}}{r_0} - 1 \geq \exp(-C_0^7(h_0(\varepsilon))^{-3} \log((h_0(\varepsilon))^{-1})). \quad (6-153)$$

In view of the fact that

$$\sup_{1 \leq l_1 < l_2 \leq n_*} \frac{\tilde{m}_{l_2}^{(1,k+1)}}{\tilde{m}_{l_1}^{(1,k+1)}} \leq C_0(h_0(\varepsilon))^{-1} \quad (6-154)$$

(following from (5-7), (6-86) and the fact that the sequence  $\tilde{m}_n^{(1,k+1)}$  is increasing in  $n$  as a consequence of (6-120)), from (6-129) we infer that, for any  $n_1 \leq n \leq n_2$  and any  $V_{n_2}(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq v \leq V_{n_2}(v^{(k+1)})$ ,

$$\begin{aligned} \partial_v r|_{(U_n(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v)} \\ \geq \partial_v r|_{(U_{n_1-1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)), v - (n - n_1)v_0)} \exp(-C_0^6(h_0(\varepsilon))^{-3} \log((h_0(\varepsilon))^{-1})). \end{aligned} \quad (6-155)$$

Thus, integrating (6-155) from  $v = V_n(v^{(k+2)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$  up to  $v = V_n(v^{(k+1)})$  and using (6-151), we immediately infer (6-153).  $\square$

*Proof of (6-14).* In view of (5-3), (6-32), (6-31), as well as the boundary condition (3-24) and the bounds (6-8) and (6-9), the following one-sided bound holds for all  $2 \leq n \leq n_f$ :

$$\begin{aligned} r_n^{(k,k+1)} - r_0 &= \int_{V_n(v^{(2k)})}^{V_n(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \partial_v r|_{(U_n(v^{(k)}), v)} dv \\ &\leq \int_{V_n(v^{(2k)})}^{V_n(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \left. \frac{\partial_v r}{1 - 2m/r} \right|_{(U_n(v^{(k)}), v)} (1 + O(\varepsilon)) dv \\ &= \int_{U_{n-1}(v^{(k)})}^{U_{n-1}(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \left. \frac{-\partial_u r}{1 - 2m/r} \right|_{(u, V_{n-1}(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)))} (1 + O(\varepsilon)) du. \end{aligned} \quad (6-156)$$

We can readily compute (using also (5-3), (6-9) and (6-45))

$$\begin{aligned}
 & \int_{U_{n-1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}^{U_{n-1}(v^{(0)})} \frac{-\partial_u r}{1-2m/r} \Big|_{(u, V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} du \\
 &= \int_{U_{n-1}(v^{(1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon))}^{U_{n-1}(v^{(0)})} \left( 1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r|_{(u, V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))}} - \frac{1}{3}\Lambda r^2|_{(u, V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} \right)^{-1} \\
 & \quad \times (-\partial_u r)|_{(u, V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} du \\
 &= \int_{r_0+O((h_2(\varepsilon))^{1/2})}^{r_{n-1}^{(1,k+1)}} \left( 1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r} - \frac{1}{3}\Lambda r^2 \right)^{-1} dr \\
 &\leq r_{n-1}^{(1,k+1)} - r_0 + C_0 \tilde{m}_{n-1}^{(1,k+1)} \left| \log \left( 1 - \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0} \right) \right|. \tag{6-157}
 \end{aligned}$$

From (5-6) and (6-12) we can similarly estimate

$$\begin{aligned}
 \int_{U_{n-1}(v^{(k)})}^{U_{n-1}(v^{(1)})} \frac{-\partial_u r}{1-2m/r} \Big|_{(u, V_{n-1}(v^{(k+1)}+(4/\sqrt{-\Lambda})h_2(\varepsilon)))} du &\leq \int_{r_{n-1}^{(1,k+1)}+O((h_2(\varepsilon))^{1/2})}^{r_{n-1}^{(k,k+1)}} \left( 1 - \frac{2\tilde{m}|_{\mathcal{I}}}{r} + O(\varepsilon) \right)^{-1} dr \\
 &\leq r_{n-1}^{(k,k+1)} + C_0 \tilde{m}|_{\mathcal{I}} |\log(\exp(h_0(\varepsilon))^{-4}) + 1| \\
 &\leq r_{n-1}^{(k,k+1)} - r_{n-1}^{(1,k+1)} + C_0 \tilde{m}|_{\mathcal{I}} (h_0(\varepsilon))^{-4}. \tag{6-158}
 \end{aligned}$$

From (5-3), (6-45), (6-156), (6-157) and (6-158) one readily obtains the bound (6-14).  $\square$

**6B. Formation of a nearly trapped sphere.** In this section, we will establish (6-6), using the bounds (6-7)–(6-14) of Proposition 6.1.

Let us set

$$n_{\max} \doteq \max\{n_* \in \mathbb{N} : \mathcal{R}_n^{(1,k+1)} \subset \mathcal{U}_\varepsilon^+ \text{ for all } n \leq n_*\}. \tag{6-159}$$

Note that, in view of (6-2) and (6-5),  $n_{\max}$  satisfies

$$n_f \leq n_{\max} \leq n_f + 1. \tag{6-160}$$

Thus, (6-1) implies

$$n_{\max} \leq (h_1(\varepsilon))^{-2}. \tag{6-161}$$

Notice that (6-160) and the definition (6-2) imply that, if

$$n_{\max} < \frac{1}{2}(h_1(\varepsilon))^{-2}, \tag{6-162}$$

then, necessarily, (6-6) holds. Thus, in order to establish (6-6), it suffices to show (6-162).

We will show (6-162) by applying Lemma A.2 (see Section A1.1 of the Appendix). In particular, setting for any  $1 \leq n \leq n_{\max} + 1$

$$\mu_n \doteq \frac{2\tilde{m}_{n-1}^{(1,k+1)}}{r_0}, \tag{6-163}$$

$$\rho_n \doteq \frac{r_{n-1}^{(k,k+1)}}{r_0}, \tag{6-164}$$

the inductive bounds (6-13) and (6-14) imply that  $\mu_n, \rho_n$  satisfy

$$\begin{aligned}\rho_{n+1} &\leq \rho_n + C_1 \log((1 - \mu_n)^{-1} + 1), \\ \mu_{n+1} &\geq \mu_n \exp\left(\frac{c_1}{\rho_{n+1}}\right)\end{aligned}\tag{6-165}$$

for any  $1 \leq n \leq n_{\max} + 1$ , with

$$C_1 = (h_0(\varepsilon))^{-4},\tag{6-166}$$

$$c_1 = \frac{1}{16} \exp(-2(h_0(\varepsilon))^{-4}).\tag{6-167}$$

Furthermore, setting

$$\delta = h_3(\varepsilon)\tag{6-168}$$

(where  $h_3(\cdot)$  is defined by (6-3)), the definition (6-159) immediately implies

$$\max_{0 \leq n \leq n_{\max}} \mu_n < 1 - \delta.\tag{6-169}$$

Note that, in view of Definition 5.1 and Proposition 3.5, we have

$$\mu_0 \sim h_0(\varepsilon),\tag{6-170}$$

$$\rho_0 \sim (h_1(\varepsilon))^{-1}.\tag{6-171}$$

Hence, as a consequence of (5-2) and (6-3),

$$\left(\frac{C_1}{c_1}\right)^8 \ll \frac{\rho_0}{\mu_0}\tag{6-172}$$

and

$$\delta < \left(\frac{\mu_0}{\rho_0}\right)^{(C_1/c_1)^4}.\tag{6-173}$$

The relations (6-165)–(6-173) allow us to apply Lemma A.2 (see Section A1.1 of the Appendix) with  $n_* = n_{\max} + 1$  for the sequence  $\rho_n, \mu_n$ . Thus, in view of Lemma A.2, we obtain the following upper bound for  $n_{\max}$ :

$$n_{\max} + 1 \leq \exp(\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^{-1}.\tag{6-174}$$

In particular, (6-162) (and, thus, (6-6)) holds.

**6C. The final step of the evolution.** In this section, we will complete the proof of Theorem 1, using the near-trapping bound (6-6), the bounds (6-7)–(6-14) of Proposition 6.1, as well a backwards-in-time Cauchy stability estimate (see Lemma A.3 in Section A1.2).

The bound (6-6), combined with the estimates (6-11) and (6-12) of Proposition 6.1, imply that, necessarily (in view also of (5-2), (5-3), (6-7), (6-13) and (6-159))

$$\frac{2\tilde{m}_{n_{\max}+1}^{(1,k+1)}}{r_0} \geq 1 - 2h_3(\varepsilon).\tag{6-175}$$

Therefore, applying again Lemma A.2 for  $\mu_n, \rho_n$  (defined by (6-163), (6-164)) and  $n_* = n_{\max} + 1$  yields, in view of (6-175), that, either

$$\mu_{n_{\max}+1} > 1 + h_3(\varepsilon), \quad (6-176)$$

or

$$1 - 2h_3(\varepsilon) \leq \mu_{n_{\max}+1} \leq 1 + h_3(\varepsilon) \quad (6-177)$$

and

$$\mu_{n_{\max}} \leq 1 - \exp(-\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^2 \quad (6-178)$$

$$\max\{\rho_{n_{\max}+1}, \rho_{n_{\max}}\} \leq \exp(\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^{-1} \log((h_1(\varepsilon))^{-1}). \quad (6-179)$$

Let us set

$$\bar{v}_* \doteq V_{n_{\max}+1} \left( v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) \quad (6-180)$$

(recall that (6-180) equals  $V_{n_{\max}}(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$ , in view of our conventions on the indices). The proof of Theorem 1 will follow by showing that

• either

$$\inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \left( 1 - \frac{2m}{r} \right) < 0 \quad (6-181)$$

(in which case  $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon}) = (r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  in the statement of Theorem 1),

• or

$$\inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \left( 1 - \frac{2m'}{r'} \right) < 0, \quad (6-182)$$

where  $(r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$  is a (possibly different) smooth solution to the system (2-28)–(2-33) arising as a future development of an asymptotically AdS boundary-characteristic initial data set  $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$  on  $\{u=0\} \cap \{0 \leq v \leq v_{0\varepsilon}\}$  (satisfying the reflecting gauge condition at  $r=r_0, +\infty$ ) which is  $(h_1(\varepsilon))^2$  close to  $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  with respect to the norm (3-41), i.e., satisfies, in particular, (A-68) and (A-69) (in which case  $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon}) = (r_{/\varepsilon}, (\Omega_{/\varepsilon})^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  in the statement of Theorem 1).

Notice that, in both cases, (4-3) follows readily from (5-18) and (5-2). To this end, we will proceed to treat the cases (6-176) and (6-177) separately.

Case I: Assume that (6-176) holds. Then, we will show that (6-181) also holds. We will argue by contradiction, assuming that

$$\inf_{\mathcal{U}_\varepsilon \cap \{v=\bar{v}_*\}} \left( 1 - \frac{2m}{r} \right) \geq 0. \quad (6-183)$$

Let us set

$$\mathcal{C}_* \doteq \left\{ U_{n_{\max}+1} \left( v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) \leq u < U_{n_{\max}+1} (v^{(0)}) \right\} \cap \{v = \bar{v}_*\} \cap \mathcal{U}_\varepsilon. \quad (6-184)$$

The renormalized mass  $\tilde{m}$  is constant on  $\mathcal{C}_*$ , satisfying in particular

$$\tilde{m}|_{\mathcal{C}_*} = \tilde{m}_{n_{\max}+1}^{(1,k+1)}. \quad (6-185)$$

Since  $\partial_u r < 0$  on  $\mathcal{U}_\varepsilon$  (see (3-29)), from (6-183) and the fact that  $\mathcal{C}_*$  does not contain its future endpoint, we infer the following stronger bound:

$$1 - \frac{2m}{r} \Big|_{\mathcal{C}_*} > 0. \quad (6-186)$$

Thus, we also have

$$\partial_v r|_{\mathcal{C}_*} > 0. \quad (6-187)$$

We will now show that the future endpoint of  $\mathcal{C}_*$  is exactly  $(U_{n_{\max}+1}(v^{(0)}), \bar{v}_*)$ . If there existed some  $(u_b, \bar{v}_*) \in (\partial\mathcal{U}_\varepsilon \setminus \mathcal{I})$  such that  $U_{n_{\max}+1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon)) \leq u_b < U_{n_{\max}+1}(v^{(0)})$ , then Theorem 3.4 on the structure of the maximal future development would imply that  $r$  extends continuously on  $(u_b, \bar{v}_*)$  with

$$r(u_b, \bar{v}_*) = r_{0\varepsilon}. \quad (6-188)$$

However, in that case, (6-176), (6-185) and (6-188) would imply that, for some  $u_{b*}$  close enough to  $u_b$ ,

$$1 - \frac{2m}{r} \Big|_{(u_{b*}, \bar{v}_*)} < 0, \quad (6-189)$$

which is a contradiction in view of (6-183). Therefore,

$$\left\{ U_{n_{\max}+1} \left( v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) \leq u < U_{n_{\max}+1}(v^{(0)}) \right\} \cap \{v = \bar{v}_*\} \cap (\partial\mathcal{U}_\varepsilon \setminus \mathcal{I}) = \emptyset,$$

and, thus

$$\mathcal{C}_* = \left\{ U_{n_{\max}+1} \left( v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) \leq u < U_{n_{\max}+1}(v^{(0)}) \right\} \cap \{v = \bar{v}_*\}. \quad (6-190)$$

In order to complete the proof in the case when (6-176) holds, it suffices to establish that

$$\limsup_{\bar{u} \rightarrow U_{n_{\max}+1}(v^{(0)})} \frac{r|_{(\bar{u}, \bar{v}_*)}}{r_0} \leq 1 + O((h_2(\varepsilon))^{1/2}). \quad (6-191)$$

Assuming that (6-191) holds, from (6-176), (6-185) and (6-191) (in view also of (5-3), (6-3)) we readily obtain

$$\liminf_{\bar{u} \rightarrow U_{n_{\max}+1}(v^{(0)})} \left( 1 - \frac{2m}{r} \right) \Big|_{(\bar{u}, \bar{v}_*)} < -\frac{1}{2} h_3(\varepsilon) < 0, \quad (6-192)$$

which is a contradiction in view of (6-183).

Let us set

$$\mathcal{B}_* \doteq \left\{ U_{n_{\max}+1} \left( v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) \leq u < U_{n_{\max}+1}(v^{(0)}) \right\} \cap \{V_{n_{\max}+1}(v^{(k)}) \leq v \leq \bar{v}_*\}. \quad (6-193)$$

From (6-190) and the structure of the maximal future development of general initial data sets for (2-28)–(2-33) (see Theorem 3.4), we infer that

$$\mathcal{B}_* \subset \mathcal{U}_\varepsilon.$$

Furthermore, in view of (2-30) and (6-187), we infer that

$$\partial_v r|_{\mathcal{B}_*} > 0 \quad (6-194)$$



and, thus (in view of (3-29))

$$1 - \frac{2m}{r}|_{\mathcal{B}_*} > 0. \quad (6-195)$$

In view of (6-159) and the bounds (6-11) and (6-12), we have

$$\{u \leq U_{n_{\max}+1}(v^{(1)})\} \cap \mathcal{U}_\varepsilon \subset \mathcal{U}_\varepsilon^+. \quad (6-196)$$

Therefore, as a consequence of (6-7), we can estimate

$$\log\left(\frac{\partial_v r}{1 - 2m/r}\right)\Big|_{\{u=U_{n_{\max}+1}(v^{(1)})\} \cap \mathcal{U}_\varepsilon} \leq (h_1(\varepsilon))^{-4} \log((h_3(\varepsilon))^{-1}). \quad (6-197)$$

Since (2-43) implies

$$\partial_u \log\left(\frac{\partial_v r}{1 - 2m/r}\right) \leq 0, \quad (6-198)$$

from (6-197) (and (6-9)) we infer the one sided bound

$$\partial_v r|_{\mathcal{B}_*} \leq 2(h_1(\varepsilon))^{-4} \log((h_3(\varepsilon))^{-1}). \quad (6-199)$$

Integrating (6-199) from  $v = V_{n_{\max}+1}(v^{(k)})$  up to  $V_{n_{\max}+1}(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))$  using (5-3), we finally obtain (6-191). Thus, the proof in the case when (6-176) holds is complete.

Case II: Assume that (6-177) holds. Then, (6-178) and (6-179) also hold.

As a consequence of (6-11), (6-12) and (6-13), the bound (6-178) implies

$$\inf_{\{u \leq U_{n_{\max}}(v^{(0)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\} \cap \mathcal{U}_\varepsilon} \left(1 - \frac{2\tilde{m}}{r}\right) \geq \frac{1}{2} \exp(-\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^2. \quad (6-200)$$

Therefore, using (6-11), (6-12) and (6-13) to estimate  $1 - 2m/r$  in the region

$$\left\{U_{n_{\max}}\left(v^{(0)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right) \leq u \leq U_{n_{\max}+1}(v^{(0)})\right\} \setminus \mathcal{R}_{n_{\max}+1}^{(1,k+1)},$$

we infer that

$$\inf_{\{u \leq U_{n_{\max}+1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\} \cap \mathcal{U}_\varepsilon} \left(1 - \frac{2\tilde{m}}{r}\right) \geq \frac{1}{2} \exp(-\exp(2(h_0(\varepsilon))^{-4}))(h_1(\varepsilon))^2. \quad (6-201)$$

**Remark.** Notice that, while  $1 - 2\tilde{m}/r$  becomes  $\sim h_3(\varepsilon)$  in  $\{u \leq U_{n_{\max}+1}(v^{(0)})\} \cap \mathcal{U}_\varepsilon$  (in view of (6-177)), when restricting to the subregion  $\{u \leq U_{n_{\max}+1}(v^{(1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))\} \cap \mathcal{U}_\varepsilon$ , the improved bound (6-201) holds.

Let us set

$$u_* \doteq U_{n_{\max}+1}\left(v^{(1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right), \quad (6-202)$$

noticing that

$$\text{supp}(r^2 T_{vv}) \cap \{u = u_*\} \subset \{r \leq \varepsilon^{1/2}\} \quad (6-203)$$

as a consequence of (6-8). Let us also fix a smooth cut-off function  $\chi_\varepsilon : [u_*, u_* + v_{0\varepsilon}] \rightarrow [0, 1]$  such that

$$\chi_\varepsilon(v) = 1 \quad \text{for } v \in \left[V_{n_{\max}+1}(v^{(k+1)}), V_{n_{\max}+1}\left(v^{(k+1)} + \frac{4}{\sqrt{-\Lambda}}h_2(\varepsilon)\right)\right] \quad (6-204)$$

and

$$\chi_\varepsilon(v) = 0 \quad \text{for } v \in [u_*, u_* + v_{0\varepsilon}) \setminus \left[ V_{n_{\max}+1} \left( v^{(k+1)} - \frac{1}{\sqrt{-\Lambda}} h_2(\varepsilon) \right), V_{n_{\max}+1} \left( v^{(k+1)} + \frac{5}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) \right].$$

We will then define the function  $\tilde{T}_{vv} : [u_*, u_* + v_{0\varepsilon}) \rightarrow \mathbb{R}$  by the relation

$$\tilde{T}_{vv}(v) \doteq \exp\left(-2C_\varepsilon^2 \frac{u_*}{v_0}\right) (h_1(\varepsilon))^2 \chi_\varepsilon(v) T_{vv}(u_*, v), \quad (6-205)$$

where  $C_\varepsilon$  is defined by (A-62). Notice that, since

$$2\pi \int_{V_{n_{\max}+1}(v^{(k+1)})}^{V_{n_{\max}+1}(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{(1-2m/r)}{\partial_v r} r^2 T_{vv} \Big|_{(u_*, v)} dv = \tilde{m}_{n_{\max}+1}^{(1,k+1)} - \tilde{m}_{n_{\max}+1}^{(1,k+2)} = \tilde{m}_{n_{\max}+1}^{(1,k+1)}, \quad (6-206)$$

we can readily bound in view of (6-205), (6-12), (6-177) and (6-206)

$$\sup_{u_* \leq \bar{v} \leq u_* + v_{0\varepsilon}} (-\Lambda) \int_{u_*}^{u_* + v_{0\varepsilon}} \frac{r^2(u_*, v) |\tilde{T}_{vv}(v)| |\partial_v \rho(u_*, v)|}{|\rho(u_*, v) - \rho(u_*, \bar{v})| + \rho(u_*, u_*)} dv \leq \exp\left(-C_\varepsilon^2 \frac{u_*}{v_{0\varepsilon}}\right) (h_1(\varepsilon))^2, \quad (6-207)$$

where  $\rho$  is defined in terms of  $r$  by the relation

$$\rho \doteq \tan^{-1}\left(\sqrt{-\frac{\Lambda}{3}} r\right). \quad (6-208)$$

Applying the backwards-in-time Cauchy stability lemma, Lemma A.3, (see Section A1.2 of the Appendix), for  $u_*$  given by (6-202) and  $\tilde{T}_{vv}$  given by (6-205) (in view of (6-201), (6-203) and (6-207)), we infer that there exists a smooth asymptotically AdS boundary-characteristic initial data set  $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$  on  $\{u=0\}$  for the system (2-28)–(2-33) satisfying the reflecting gauge condition at  $r = r_{0\varepsilon}, +\infty$  with the following properties:

- (1) The initial data sets  $(r'_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  and  $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$  satisfy (A-68) and (A-69).
- (2) The maximal development  $(\mathcal{U}'_\varepsilon; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$  of  $(r'_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  satisfies (A-70), (A-71) and (A-72).

Using primes to denote quantities associated to  $(r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$ , we can readily estimate in view of (A-70), (A-71), (A-72) and (6-205)

$$\begin{aligned} \tilde{m}'|_{(u_*, \bar{v}_*)} &= \int_{V_{n_{\max}+1}(v^{(k+1)})}^{V_{n_{\max}+1}(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{(1-2m'/r')}{\partial_v r'} (r')^2 T'_{vv} \Big|_{(u_*, v)} dv \\ &= \int_{V_{n_{\max}+1}(v^{(k+1)})}^{V_{n_{\max}+1}(v^{(k+1)} + (4/\sqrt{-\Lambda})h_2(\varepsilon))} \frac{(1-2m'/r)}{\partial_v r} r^2 (T_{vv} + \tilde{T}_{vv}) \Big|_{(u_*, v)} dv \\ &\geq \left(1 + \exp\left(-2C_\varepsilon^2 \frac{u_*}{v_0}\right) (h_1(\varepsilon))^2\right) \tilde{m} \Big|_{(u_*, \bar{v}_*)}. \end{aligned} \quad (6-209)$$

Therefore, since  $\tilde{m}|_{(u_*, \bar{v}_*)} = \tilde{m}_{n_{\max}+1}^{(1,k+1)}$ , the bound (6-177) (in view also of (6-3) and (A-62)) implies

$$\frac{2\tilde{m}'|_{(u_*, \bar{v}_*)}}{r_0} \geq \left(1 + \exp\left(-2C_\varepsilon^2 \frac{u_*}{v_0}\right) (h_1(\varepsilon))^2\right) (1 - 2h_3(\varepsilon)) \geq 1 + h_3(\varepsilon). \quad (6-210)$$

Since  $\tilde{m}|_{(u_*, \bar{v}_*)}$  is constant on

$$\mathcal{C}'_* \doteq \left\{ U_{n_{\max}+1} \left( v^{(1)} + \frac{4}{\sqrt{-\Lambda}} h_2(\varepsilon) \right) \leq u < U_{n_{\max}+1}(v^{(0)}) \right\} \cap \{v = \bar{v}_*\} \cap \mathcal{U}'_\varepsilon$$

and satisfies (6-210), we can now repeat the same arguments as in Case I (i.e., the case when (6-176) holds) in order to infer that (6-182) holds.

Thus, the proof of Theorem 1 is complete.  $\square$

### Appendix: A maximum principle and Cauchy stability backwards in time

In this section, we will prove some lemmas which are necessary for the proof of Proposition 6.1 and Theorem 1.

**A1. A maximum principle for 1 + 1 wave-type equations.** The following lemma provides a comparison inequality for certain 1 + 1 equations of wave type, and is used in the proof of Proposition 6.1.

**Lemma A.1.** *For any  $u_0 < u_1$ ,  $v_0 < v_1$  and  $a \in \mathbb{R}$ , let  $F_1, F_2 : [u_0, u_1] \times [v_0, v_1] \times (-\infty, a] \rightarrow (0, +\infty)$  be smooth functions so that*

$$\max_{(u,v) \in [u_0, u_1] \times [v_0, v_1]} F_1(u, v, z) < \min_{(u,v) \in [u_0, u_1] \times [v_0, v_1]} F_2(u, v, z) \quad (\text{A-1})$$

for any  $z \in (-\infty, a]$  and

$$\partial_z F_1(u, v, z), \partial_z F_2(u, v, z) \geq 0 \quad (\text{A-2})$$

for any  $(u, v, z) \in [u_0, u_1] \times [v_0, v_1] \times (-\infty, a]$ . Suppose also  $z_1, z_2 : [u_0, u_1] \times [v_0, v_1] \rightarrow (-\infty, a]$  are smooth solutions to the equations

$$\partial_v \partial_u z_1 = -F_1(u, v, z_1) \partial_u z_1 \partial_v z_1, \quad (\text{A-3})$$

$$\partial_v \partial_u z_2 = -F_2(u, v, z_2) \partial_u z_2 \partial_v z_2, \quad (\text{A-4})$$

satisfying the same characteristic initial data

$$z_1(u, v_0) = z_2(u, v_0) = z_\backslash(u), \quad (\text{A-5})$$

$$z_1(u_0, v) = z_2(u_0, v) = z_/(v), \quad (\text{A-6})$$

where  $z_/: [v_0, v_1] \rightarrow (-\infty, a]$  and  $z_\backslash : [u_0, u_1] \rightarrow (-\infty, a]$  are smooth functions so that

$$z_/(v_0) = z_\backslash(v_1), \quad (\text{A-7})$$

$$\partial_v z_/|_{(v_0, v_1)} > 0, \quad (\text{A-8})$$

$$\partial_u z_\backslash|_{(u_0, u_1)} < 0. \quad (\text{A-9})$$

Then, the functions  $z_1, z_2$  satisfy

$$\partial_u z_i < 0 < \partial_v z_i, \quad i = 1, 2, \quad (\text{A-10})$$

in  $(u_0, u_1) \times (v_0, v_1)$  and

$$z_1 \leq z_2 \quad (\text{A-11})$$

everywhere on  $[u_0, u_1] \times [v_0, v_1]$ .

*Proof.* We will first establish (A-10). By applying a standard continuity argument, rewriting (A-3) as

$$\partial_v \log(-\partial_u z_1) = -\partial_v z_1 F_1(u, v, z_1) \quad (\text{A-12})$$

and integrating in  $v$ , using also the property (A-9) of the initial data, we obtain that

$$\partial_u z_1 < 0 \quad (\text{A-13})$$

everywhere on  $(u_0, u_1) \times (v_0, v_1)$ . Similarly, rewriting (A-3) as

$$\partial_u \log(\partial_v z_1) = -\partial_u z_1 F_1(u, v, z_1) \quad (\text{A-14})$$

and integrating in  $u$ , using (A-9), and then repeating the same procedure for  $z_2$ , we finally obtain (A-10).

In order to establish (A-11), we will argue by continuity: Let  $u_* \in [u_0, u_1]$  be such that (A-11),

$$\partial_v z_1 \leq \partial_v z_2, \quad (\text{A-15})$$

$$\partial_u z_1 \leq \partial_u z_2 \quad (\text{A-16})$$

hold on  $[u_0, u_*] \times [v_0, v_1]$ . Note that  $u_* = u_0$  satisfies this condition: in this case, (A-11) and (A-15) follow directly from (A-6), while (A-16) follows by integrating (A-12) (and its analogue for  $z_2$ ) and using (A-1). We will show that there exists a  $\delta > 0$ , such that (A-11), (A-15) and (A-16) hold on  $[u_0, u_* + \delta] \times [v_0, v_1]$ .

For any  $\bar{v} \in (v_0, v_1]$ , integrating (A-3) and (A-4) in  $v$  along  $\{u_*\} \times [v_0, \bar{v}]$ , we obtain

$$\log(-\partial_u z_1)(u_*, \bar{v}) = - \int_{v_0}^{\bar{v}} F_1(u_*, v, z_1) \partial_v z_1 dv + \log(-\partial_u z_1)(u_*), \quad (\text{A-17})$$

$$\log(-\partial_u z_2)(u_*, \bar{v}) = - \int_{v_0}^{\bar{v}} F_2(u_*, v, z_2) \partial_v z_2 dv + \log(-\partial_u z_2)(u_*). \quad (\text{A-18})$$

Let us define the auxiliary functions  $F_{1;u_*\bar{v}}, F_{2;u_*\bar{v}} : (-\infty, a] \rightarrow (0, +\infty)$  by the relations

$$F_{1;u_*\bar{v}}(z) = \max_{v \in [v_0, \bar{v}]} F_1(u_*, v, z), \quad (\text{A-19})$$

$$F_{2;u_*\bar{v}}(z) = \min_{v \in [v_0, \bar{v}]} F_2(u_*, v, z). \quad (\text{A-20})$$

In view of (A-1), (A-2) and the fact that (A-11) holds on  $\{u_*\} \times [v_0, \bar{v}]$ ,<sup>17</sup> we can bound for any  $v \in [v_0, \bar{v}]$

$$F_{1;u_*\bar{v}}(z_1(u_*, v)) < F_{2;u_*\bar{v}}(z_1(u_*, v)) \leq F_{2;u_*\bar{v}}(z_2(u_*, v)). \quad (\text{A-21})$$

Thus, subtracting (A-17) and (A-18) and using (A-21) and (A-15) (and the fact that  $\partial_v z_2 > 0$ ,  $\bar{v} > v_0$ ), we readily infer that

$$\begin{aligned} & \log(-\partial_u z_1)(u_*, \bar{v}) - \log(-\partial_u z_2)(u_*, \bar{v}) \\ & \geq \int_{v_0}^{\bar{v}} F_{2;u_*\bar{v}}(z_2(u_*, v)) \partial_v z_2(u_*, v) dv - \int_{v_0}^{\bar{v}} F_{1;u_*\bar{v}}(z_1(u_*, v)) \partial_v z_1(u_*, v) dv \\ & > 0. \end{aligned}$$

<sup>17</sup>Note that we can immediately restrict from  $[u_0, u_1] \times [v_0, v_1]$  to  $\{u_*\} \times [v_0, \bar{v}]$  in (A-1).

From (A-22) we thus infer that, for any  $v_0 < \bar{v} \leq v_1$ ,

$$\partial_u z_1(u_*, \bar{v}) < \partial_u z_2(u_*, \bar{v}). \quad (\text{A-22})$$

Therefore, since  $z_1, z_2$  are smooth, there exists a continuous function  $\delta_u : [v_0, v_1] \rightarrow [0, 1]$ , with  $\delta_u|_{(v_0, v_1]} > 0$ , such that

$$\partial_u z_1(u, v) \leq \partial_u z_2(u, v) \quad \text{for } \{v_0 \leq v \leq v_1\} \cap \{u_* \leq u \leq u_* + \delta_u(v)\}. \quad (\text{A-23})$$

Similarly, by integrating (A-3) and (A-4) in  $u$  along  $[u_0, u_1] \times \{v_0\}$  and repeating a similar procedure (using (A-5)), we also obtain that there exists a continuous function  $\delta_v : [u_0, u_1] \rightarrow [0, 1]$ , with  $\delta_v|_{(u_0, u_1]} > 0$ , such that

$$\partial_v z_1(\bar{u}, v_0) \leq \partial_v z_2(\bar{u}, v_0) \quad \text{for } \{u_0 \leq u \leq u_1\} \cap \{v_0 \leq v \leq v_0 + \delta_v(u)\}. \quad (\text{A-24})$$

From (A-22) and (A-24), we infer that there exists some  $\delta > 0$ , such that

$$z_1 \leq z_2 \quad \text{on } (u_*, u_* + \delta) \times [v_0, v_1]. \quad (\text{A-25})$$

In particular, (A-11) holds on  $[u_0, u_* + \delta) \times [v_0, v_1]$ . Furthermore, for any  $\bar{u} \in (u_*, u_* + \delta)$  and any  $\bar{v} \in (v_0, v_0 + \delta_v(\bar{u}))$ , repeating the procedure leading to (A-22) with  $\bar{u}$  in place of  $u_*$  and using (A-24) and (A-25) in place of (A-15) and (A-11), respectively, we infer that

$$\partial_u z_1(\bar{u}, \bar{v}) \leq \partial_u z_2(\bar{u}, \bar{v}). \quad (\text{A-26})$$

Thus, combining (A-23) and (A-26), we infer that (A-16) holds on  $[u_0, u_* + \delta') \times [v_0, v_1]$  for some  $0 < \delta' \leq \delta$ . Finally, the bound (A-15) on  $[u_0, u_* + \delta') \times [v_0, v_1]$  follows in a way similar to the proof of (A-22), by integrating (A-3) and (A-4) in  $u \in [u_0, u_* + \delta')$  for any  $\bar{v} \in (v_0, v_1)$  and using (A-1), (A-11) and (A-16) (which we have shown hold on  $[u_0, u_* + \delta') \times [v_0, v_1]$ ). We will omit the details.  $\square$

**A1.1. A lemma for a system of inductive inequalities.** The following lemma is used to show that the inductive bounds (6-13) and (6-14) for  $\tilde{m}_n^{(1, k+1)}$  and  $r_n^{(k, k+1)}$  indeed lead to the formation of an almost-trapped surface.

**Lemma A.2.** *Let  $0 < c_1 \ll 1 \ll C_1$ , and  $0 < \mu_0 \ll 1 \ll \rho_0$ ,  $0 < \delta \ll 1$  be given variables, such that*

$$\left(\frac{C_1}{c_1}\right)^8 \ll \frac{\rho_0}{\mu_0} \quad (\text{A-27})$$

and

$$\delta < \left(\frac{\mu_0}{\rho_0}\right)^{(C_1/c_1)^4}. \quad (\text{A-28})$$

*Let also  $\mu_n, \rho_n > 0$  be sequences of positive numbers, with  $\mu_n$  increasing in  $n$ , such that for  $0 \leq n \leq n_*$  they satisfy*

$$\rho_{n+1} \leq \rho_n + C_1 \log((1 - \mu_n)^{-1} + 1), \quad (\text{A-29})$$

$$\mu_{n+1} \geq \mu_n \exp\left(\frac{c_1}{\rho_{n+1}}\right), \quad (\text{A-30})$$

and

$$\max_{0 \leq n \leq n_*-1} \mu_n < 1 - \delta. \quad (\text{A-31})$$

Then,

$$n_* \leq \left(\frac{C_1}{c_1}\right)^3 \rho_0 \mu_0^{-(C_1/c_1)^2}. \quad (\text{A-32})$$

Furthermore, if  $1 - \delta \leq \mu_{n_*} \leq 1 + \delta$ , we can bound

$$\mu_{n_*-1} \leq 1 - \left(\frac{c_1}{C_1}\right)^3 \rho_0^{-2} \mu_0^{2(C_1/c_1)^2} \quad (\text{A-33})$$

and

$$\max\{\rho_{n_*}, \rho_{n_*-1}\} \leq \left(\frac{C_1}{c_1}\right)^4 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\rho_0}{\mu_0}\right). \quad (\text{A-34})$$

**Remark.** Notice that the right-hand side of (A-32) is independent of  $\delta$ .

*Proof.* Let us define for any integer  $k \geq 1$

$$n_k = \max\left\{0 \leq n \leq n_* : \mu_l \leq 1 - \frac{1}{2^k} \text{ for all } 0 \leq l \leq n\right\}, \quad (\text{A-35})$$

using the convention

$$n_0 = 0. \quad (\text{A-36})$$

Notice that, in view of the fact that the sequence  $\mu_n$  is increasing, for all  $k \geq 1$  and all  $n_{k-1} < n \leq n_k$  we can estimate

$$1 - \frac{1}{2^{k-1}} \leq \mu_n \leq 1 - \frac{1}{2^k} \quad (\text{A-37})$$

(note that, in the case  $n_{k-1} = n_k$ , there is no  $n$  satisfying  $n_{k-1} < n \leq n_k$  and (A-37)).

Using (A-37), from (A-29) we can bound for any  $k \geq 1$  such that  $n_{k-1} < n_k$  and any  $n_{k-1} < n \leq n_k$

$$\rho_n \leq \rho_{n_{k-1}} + 2C_1(\log 2)k(n - n_{k-1}) \quad (\text{A-38})$$

and, therefore, for any  $0 \leq n \leq n_k$  we have

$$\rho_n \leq 2C_1(\log 2) \left( \sum_{l=1}^{k-1} l(n_l - n_{l-1}) + k(n - n_{k-1}) \right) + \rho_0 \quad (\text{A-39})$$

(note that (A-39) holds for all  $0 \leq n \leq n_k$ , while the bounds (A-37) and (A-38) are nontrivial only for those values of  $k$  for which  $n_k > n_{k-1}$ ).

Let us set

$$k_1 \doteq 32 \left\lceil \log \frac{C_1}{c_1} \right\rceil. \quad (\text{A-40})$$

Then, (A-38) implies that, for all  $0 \leq n \leq n_{k_1}$ ,

$$\rho_n \leq \rho_0 + 2C_1(\log 2)k_1 n \quad (\text{A-41})$$

and, thus, (A-30) implies

$$\begin{aligned} \log\left(\frac{\mu_{n_{k_1}}}{\mu_0}\right) &\geq c_1 \sum_{n=1}^{n_{k_1}} \rho_n^{-1} \geq c_1 \sum_{n=1}^{n_{k_1}} \frac{1}{\rho_0 + 2C_1(\log 2)k_1 n} \\ &\geq c_1 \log\left(\frac{\rho_0 + 2C_1(\log 2)k_1 n_{k_1}}{\rho_0 + 2C_1(\log 2)k_1}\right) (4C_1(\log 2)k_1)^{-1}. \end{aligned} \quad (\text{A-42})$$

From (A-35) and (A-42) we readily infer that

$$c_1 \log\left(\frac{\rho_0 + 2C_1(\log 2)k_1 n_{k_1}}{\rho_0 + 2C_1(\log 2)k_1}\right) (4C_1(\log 2)k_1)^{-1} \leq -\log(\mu_0) \quad (\text{A-43})$$

and, therefore (using also (A-27))

$$n_{k_1} \leq \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}}. \quad (\text{A-44})$$

For any  $k \geq 2$  such that  $n_k > n_{k-1} + 1$ , from (A-30), (A-35), (A-37) and (A-39) we readily infer

$$\begin{aligned} \frac{1}{2^{k-2}} &\geq \log \frac{\mu_{n_k}}{\mu_{n_{k-1}+1}} \geq c_1 \sum_{n=n_{k-1}+2}^{n_k} \rho_n^{-1} \\ &\geq \frac{c_1}{4C_1(\log 2)} \frac{1}{(k-1) + \sum_{l=2}^{k-1} (l-1)(n_l - n_{l-1})/(n_k - n_{k-1} - 1)} \end{aligned} \quad (\text{A-45})$$

and, hence

$$\begin{aligned} n_k - n_{k-1} - 1 &\leq \frac{\sum_{l=2}^{k-1} (l-1)(n_l - n_{l-1})}{2^{k-2} c_1 / (4C_1(\log 2)) - (k-1)} \\ &\leq \frac{k(k-1)}{2} \frac{\max_{2 \leq l \leq k-1} (n_l - n_{l-1})}{2^{k-2} c_1 / (4C_1(\log 2)) - (k-1)} \\ &\leq \frac{k(k-1)}{2} \frac{1}{2^{k-2} c_1 / (4C_1(\log 2)) - (k-1)} n_{k-1}. \end{aligned} \quad (\text{A-46})$$

The relation (A-46) also holds (trivially) when  $n_k \leq n_{k-1} + 1$ . Thus, for any  $k \geq k_1$ , the bound (A-46) yields

$$n_k \leq \left(1 + \frac{C_1}{c_1} 2^{-(k-2)/4}\right) n_{k-1} + 1 \quad (\text{A-47})$$

and, therefore, for any  $k \geq 2$ ,

$$n_k \leq 16 \frac{C_1}{c_1} (n_{k_1} + \max\{k - k_1, 0\}). \quad (\text{A-48})$$

In view of (A-44), we thus obtain for any  $k \geq 2$

$$n_k \leq 16 \frac{C_1}{c_1} \left( \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} + \max\{k - k_1, 0\} \right). \quad (\text{A-49})$$

Let us set

$$k_2 \doteq 4k_1 + 2 \frac{\log(\rho_0/\mu_0^{(C_1/c_1)^2})}{\log 2}. \quad (\text{A-50})$$

Note that (A-28), (A-31) and (A-35) imply

$$n_{k_2} \leq n_* - 1. \quad (\text{A-51})$$

In view of (A-49), we have

$$n_{k_2} \leq \left( \frac{C_1}{c_1} \right)^3 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \quad (\text{A-52})$$

and, for all  $k \geq k_2$  (in view of (A-46) and (A-49))

$$n_k - n_{k-1} \leq 1. \quad (\text{A-53})$$

Furthermore, (A-39) implies (in view of (A-50) and (A-52)) that

$$\max\{\rho_{n_{k_2}+1}, \rho_{n_{k_2}}\} \leq \left( \frac{C_1}{c_1} \right)^3 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\rho_0}{\mu_0}\right). \quad (\text{A-54})$$

In view of (A-35), we have

$$\mu_{n_{k_2}} \leq 1 - 2^{-k_2} < \mu_{n_{k_2}+1}. \quad (\text{A-55})$$

We will consider two cases, depending on whether  $\mu_{n_{k_2}+1}$  is larger than  $1 - \delta$  or not.

(1) In the case  $\mu_{n_{k_2}+1} \geq 1 - \delta$ , (A-31) implies that  $n_{k_2} + 1 = n_*$ . Thus, (A-32) follows from (A-52). Furthermore, (A-34) follows from (A-54), while (A-33) follows from (A-55).

(2) In the case  $\mu_{n_{k_2}+1} < 1 - \delta$ , we can assume without loss of generality that  $n_{k_2} \leq n_* - 2$  (otherwise, (A-32) follows from (A-52)). From (A-29), (A-54) and (A-55), we thus infer that

$$\rho_{n_{k_2}+2} \leq \left( \frac{C_1}{c_1} \right)^3 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\rho_0}{\mu_0}\right) + C_1 \log((1 - \mu_{n_{k_2}+1})^{-1}) \quad (\text{A-56})$$

Hence, setting

$$M \doteq \left( \frac{C_1}{c_1} \right)^3 \frac{\rho_0}{\mu_0^{(C_1/c_1)^2}} \log\left(\frac{\rho_0}{\mu_0}\right), \quad (\text{A-57})$$

from (A-30) and (A-55) we calculate

$$\begin{aligned} \mu_{n_{k_2}+2} &\geq \mu_{n_{k_2}+1} \exp\left(\frac{c_1}{\rho_{n_{k_2}+2}}\right) \\ &\geq \begin{cases} (1 - 2^{-k_2})e^{c_1/(2M)} & \text{if } \log((1 - \mu_{n_{k_2}+1})^{-1}) \leq M/C_1, \\ \mu_{n_{k_2}+1} \left(1 + \frac{c_1}{C_1 \log((1 - \mu_{n_{k_2}+1})^{-1}) + 1}\right) & \text{if } \log((1 - \mu_{n_{k_2}+1})^{-1}) > M/C_1. \end{cases} \end{aligned} \quad (\text{A-58})$$

If  $\log((1 - \mu_{n_{k_2}+1})^{-1}) \leq M/C_1$ , in view of (A-50) and (A-57) we can bound (using also (A-27))

$$(1 - 2^{-k_2})e^{c_1/(2M)} \geq \left(1 - \frac{\mu_0^{2(C_1/c_1)^2}}{\rho_0^2}\right) \left(1 + \frac{c_1}{2} \left(\frac{c_1}{C_1}\right)^3 \frac{\mu_0^{(C_1/c_1)^2}}{\rho_0 \log(\rho_0/\mu_0)}\right) > 1 + \delta. \quad (\text{A-59})$$



If  $\log((1 - \mu_{n_{k_2}} + 1)^{-1}) > M/C_1 \gg (C_1/c_1)^3$ , we can also estimate

$$\begin{aligned} \mu_{n_{k_2}+1} \left( 1 + \frac{c_1}{C_1 \log((1 - \mu_{n_{k_2}} + 1)^{-1} + 1)} \right) &\geq (1 - e^{-\log((1 - \mu_{n_{k_2}} + 1)^{-1})}) \left( 1 + \frac{c_1}{C_1 \log((1 - \mu_{n_{k_2}} + 1)^{-1} + 1)} \right) \\ &> 1 + \delta. \end{aligned} \quad (\text{A-60})$$

Therefore, (A-58) implies

$$\mu_{n_{k_2}+2} > 1 + \delta \quad (\text{A-61})$$

and, hence,  $n_{k_2+2} = n_*$ . Thus, (A-32) follows again from (A-52).  $\square$

**A1.2. Cauchy stability backwards in time.** The following lemma, which is essentially a backwards-in-time Cauchy stability estimate for late time perturbations of  $(\mathcal{U}_\varepsilon; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$ , is an easy corollary of Theorem 3.6.

**Lemma A.3.** *For any  $0 < \varepsilon < \varepsilon_0$  (provided  $\varepsilon_0$  is sufficiently small) and any  $r_0 > 0$  satisfying (5-6), let  $(\mathcal{U}_\varepsilon; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  be the maximal future development of  $(r/\varepsilon, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$ , and let us set*

$$C_\varepsilon \doteq \exp(\exp(-2(h_0(\varepsilon))^{-4})(h_1(\varepsilon))^{-4}). \quad (\text{A-62})$$

Then, for any  $0 \leq u_* \leq (h_1(\varepsilon))^{-2}v_{0\varepsilon}$  such that

$$\mathcal{W}_{u_*} \doteq \{0 < u \leq u_*\} \cap \{u < v < u + v_{0\varepsilon}\} \subset \mathcal{U}_\varepsilon, \quad (\text{A-63})$$

$$\sup_{\mathcal{W}_{u_*}} \left( 1 - \frac{2\tilde{m}}{r} \right)^{-1} \leq C_\varepsilon, \quad (\text{A-64})$$

$$u_* + v_{0\varepsilon} \notin \text{supp}(r^2 T_{vv}(u_*, \cdot)), \quad (\text{A-65})$$

and for any  $\tilde{T}_{vv} : (u_*, u_* + v_{0\varepsilon}) \rightarrow \mathbb{R}$  smooth and compactly supported satisfying  $\tilde{T}_{vv}(\cdot) \geq -T_{vv}(u_*, \cdot)$  and

$$\sup_{u_* \leq \bar{v} \leq u_* + v_{0\varepsilon}} (-\Lambda) \int_{u_*}^{u_* + v_{0\varepsilon}} \frac{r^2(u_*, v) |\tilde{T}_{vv}(v)| / \partial_v \rho(u_*, v)}{|\rho(u_*, v) - \rho(u_*, \bar{v})| + \rho(u_*, u_*)} dv \leq \exp\left(-C_\varepsilon^2 \frac{u_*}{v_{0\varepsilon}}\right) (h_1(\varepsilon))^2, \quad (\text{A-66})$$

with

$$\rho(u, v) \doteq \tan^{-1}\left(\sqrt{-\frac{\Lambda}{3}}r\right), \quad (\text{A-67})$$

the following statement holds: There exists a smooth asymptotically AdS boundary-characteristic initial data set  $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$  on  $\{u = 0\}$  for the system (2-28)–(2-33) satisfying the reflecting gauge condition at  $r = r_{0\varepsilon}, +\infty$  with the following properties:

- (1) *The initial data sets  $(r_{/\varepsilon}, \Omega_{/\varepsilon}^2, \bar{f}_{\text{in}/\varepsilon}, \bar{f}_{\text{out}/\varepsilon})$  and  $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$  are  $(h_1(\varepsilon))^2$  close in the (3-41) norm, and in particular*

$$\begin{aligned} \sup_{v \in [0, v_{0\varepsilon})} &\left\{ \left| \log\left(\frac{\Omega_{/\varepsilon}^2}{1 - \frac{1}{3}\Lambda r_{/\varepsilon}^2}\right) - \log\left(\frac{(\Omega'_{/\varepsilon})^2}{1 - \frac{1}{3}\Lambda (r'_{/\varepsilon})^2}\right) \right| \right. \\ &+ \left| \log\left(\frac{2\partial_v r_{/\varepsilon}}{1 - 2m_{/\varepsilon}/r_{/\varepsilon}}\right) - \log\left(\frac{2\partial_v r'_{/\varepsilon}}{1 - 2m'_{/\varepsilon}/r'_{/\varepsilon}}\right) \right| \\ &\left. + \left| \log\left(\frac{1 - 2m_{/\varepsilon}/r_{/\varepsilon}}{1 - \frac{1}{3}\Lambda r_{/\varepsilon}^2}\right) - \log\left(\frac{1 - 2m'_{/\varepsilon}/r'_{/\varepsilon}}{1 - \frac{1}{3}\Lambda (r'_{/\varepsilon})^2}\right) \right| + \sqrt{-\Lambda} |\tilde{m}_{/\varepsilon} - \tilde{m}'_{/\varepsilon}| \right\} (v) \leq (h_1(\varepsilon))^2 \end{aligned} \quad (\text{A-68})$$

and

$$\sup_{\bar{v} \in [0, v_{0\varepsilon}]} \int_0^{v_{0\varepsilon}} \left| r_{/\varepsilon}^2 \frac{(T_{vv})_{/\varepsilon}}{\partial_v \rho_{/\varepsilon}}(v) - (r'_{/\varepsilon})^2 \frac{(T_{vv})'_{/\varepsilon}}{\partial_v \rho'_{/\varepsilon}}(v) \right| \left( |\rho_{/\varepsilon}(\bar{v}) - \rho_{/\varepsilon}(v)| + \rho_{/\varepsilon}(0) \right)^{-1} dv \leq (h_1(\varepsilon))^2. \quad (\text{A-69})$$

(2) The maximal future development  $(\mathcal{U}'_\varepsilon; r', (\Omega')^2, \bar{f}'_{\text{in}}, \bar{f}'_{\text{out}})$  of  $(r'_{/\varepsilon}, (\Omega'_{/\varepsilon})^2, \bar{f}'_{\text{in}/\varepsilon}, \bar{f}'_{\text{out}/\varepsilon})$  satisfies

$$\mathcal{W}_{u_*} \subset \mathcal{U}'_\varepsilon, \quad (\text{A-70})$$

$$r'|_{\{u=u_*\} \cap \text{supp}(T_{vv})} = r|_{\{u=u_*\} \cap \text{supp}(T_{vv})}, \quad (\text{A-71})$$

$$T'_{vv}|_{\{u=u_*\}} = T_{vv}|_{\{u=u_*\}} + \tilde{T}_{vv}. \quad (\text{A-72})$$

*Proof.* In view of (5-6), (5-24), (6-11) and (6-12), we can readily estimate

$$\sup_{\mathcal{W}_{u_*} \setminus \bigcup_n \mathcal{R}_n^{(1,k+1)}} \left( 1 - \frac{2\tilde{m}}{r} \right)^{-1} \leq 2 \exp((h_0(\varepsilon))^{-4}). \quad (\text{A-73})$$

Therefore, using (A-64) for  $\bigcup_n \mathcal{R}_n^{(1,k+1)}$  and (A-73) for  $\mathcal{W}_{u_*} \setminus \bigcup_n \mathcal{R}_n^{(1,k+1)}$ , the relations (6-33) and (6-34) imply (in view of (5-3), (6-8) and the fact that  $u_* \leq (h_1(\varepsilon))^{-2} v_{0\varepsilon}$ ) that

$$\sup_{\mathcal{W}_{u_*}} \left( \left| \log \left( \frac{-\partial_u r}{1 - 2m/r} \right) \right| + \left| \log \left( \frac{\partial_v r}{1 - 2m/r} \right) \right| \right) \leq (h_1(\varepsilon))^{-3} \exp((h_0(\varepsilon))^{-4}). \quad (\text{A-74})$$

Similarly, (2-43) and (2-44), in view of the relations (6-33), (6-34) (using again the bounds (A-64) (A-73)) imply

$$\sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_*}} r T_{vv} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_*}} r T_{uu} du \leq (h_1(\varepsilon))^{-1} \exp((h_0(\varepsilon))^{-4}). \quad (\text{A-75})$$

Let us fix a set of smooth functions  $r^*_j, (\Omega^*_j)^2 : [u_*, u_* + v_{0\varepsilon}] \rightarrow (0, +\infty)$  and

$$\bar{f}^*_{\text{in}/}, \bar{f}^*_{\text{out}/} : [u_*, u_* + v_{0\varepsilon}] \times (0, +\infty) \rightarrow [0, +\infty)$$

satisfying the following requirements:

- (1)  $(r^*_j, (\Omega^*_j)^2, \bar{f}^*_{\text{in}/}, \bar{f}^*_{\text{out}/})$  is a smooth asymptotically AdS boundary-characteristic initial data set for the system (2-28)–(2-33) on  $\{u_*\} \times [u_*, u_* + v_{0\varepsilon}]$  satisfying the reflecting gauge condition at  $r^* = r_{0\varepsilon}, +\infty$ .
- (2) The function  $r^*_j$  satisfies for any  $v$  such that  $(u_*, v) \in \text{supp}(T_{vv})$

$$r^*_j(v) = r|_{\text{supp}(T_{vv}) \cap \{u=u_*\}}. \quad (\text{A-76})$$

- (3) The function  $\bar{f}^*_{\text{in}/}$  satisfies, for all  $v \in [u_*, u_* + v_{0\varepsilon}]$ ,

$$\int_0^{+\infty} ((\Omega^*_j)^2(v) p^v)^2 \bar{f}^*_{\text{in}/}(v; p^v) (r^*_j)^2(v) \frac{dp^v}{p^v} = T_{vv}(u_*, v) + \tilde{T}_{vv}(v). \quad (\text{A-77})$$

(4) The function  $\bar{f}_{\text{out}/}^*$  satisfies, for all  $(v, p^v) \in [u_*, u_* + v_{0\varepsilon}) \times (0, +\infty)$ ,

$$\bar{f}_{\text{out}/}^*(v; p^v) = \bar{f}_{\text{out}}(u_*, v; p^v). \quad (\text{A-78})$$

(5) The initial data sets  $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})|_{u=u_*}$  and  $(r_{\text{f}}^*, (\Omega_{\text{f}}^*)^2, \bar{f}_{\text{in}/}^*, \bar{f}_{\text{out}/}^*)$  satisfy

$$\begin{aligned} & \sup_{v \in [u_*, u_* + v_{0\varepsilon})} \left\{ \left| \log \left( \frac{\Omega^2}{1 - \frac{1}{3} \Lambda r^2} \right) \right|_{u=u_*} - \log \left( \frac{(\Omega_{\text{f}}^*)^2}{1 - \frac{1}{3} \Lambda (r_{\text{f}}^*)^2} \right) \right| \\ & \quad + \left| \log \left( \frac{2\partial_v r}{1 - 2m/r} \right) \right|_{u=u_*} - \log \left( \frac{2\partial_v r_{\text{f}}^*}{1 - 2m_{\text{f}}^*/r_{\text{f}}^*} \right) \right| \\ & \quad + \left| \log \left( \frac{1 - 2m/r}{1 - \frac{1}{3} \Lambda r^2} \right) \right|_{u=u_*} - \log \left( \frac{1 - 2m_{\text{f}}^*/r_{\text{f}}^*}{1 - \frac{1}{3} \Lambda (r_{\text{f}}^*)^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}|_{u=u_*} - \tilde{m}_{\text{f}}^* \Big\} (v) \\ & \leq \exp \left( -\frac{1}{2} C_{\varepsilon}^2 \frac{u_*}{v_0} \right) (h_1(\varepsilon))^2 \end{aligned} \quad (\text{A-79})$$

and

$$\sup_{v \in [v_1, v_2]} \int_{v_1}^{v_2} \frac{|r^2 T_{vv}(u_*, \bar{v}) - (r_{\text{f}}^*)^2 (T_{vv})_{\text{f}}^*(\bar{v})|}{|\rho_{\text{f}}(v) - \rho_{\text{f}}(\bar{v})| + \rho_{\text{f}}(v_1)} d\bar{v} \leq \exp \left( -\frac{1}{2} C_{\varepsilon}^2 \frac{u_*}{v_0} \right) (h_1(\varepsilon))^2. \quad (\text{A-80})$$

**Remark.** As a consequence of (A-65), by suitably deforming  $r_{\text{f}}^*$  near  $v = u_* + v_{0\varepsilon}$ , we can always arrange that (3-7) and (A-76) are satisfied simultaneously. Furthermore, since  $\tilde{T}_{vv}$  is compactly supported in  $(u_*, u_* + v_{0\varepsilon})$ , we can always choose  $\bar{f}_{\text{in}/}^* = f_{\text{in}}|_{\{u=u_*\}}$  in a neighborhood of  $v = u_*, u_* + v_{0\varepsilon}$ , so that (3-8) and (3-9) are satisfied. Finally,  $(r_{\text{f}}^*, (\Omega_{\text{f}}^*)^2, \bar{f}_{\text{in}/}^*, \bar{f}_{\text{out}/}^*)$  can be chosen so that (A-79) and (A-80) are satisfied because of (A-66) and the relations (2-6), (2-44) and (2-47).

Let us now consider the two sets of initial data  $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})|_{u=u_*}$  and  $(r_{\text{f}}^*, (\Omega_{\text{f}}^*)^2, \bar{f}_{\text{in}/}^*, \bar{f}_{\text{out}/}^*)$  on  $\{u = u_*\} \cap \{u_* \leq v < u_* + v_{0\varepsilon}\}$ . The maximal *past* development of  $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})|_{u=u_*}$  (see the remark below Theorem 3.4) coincides with  $(\mathcal{W}_{u_*}; r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})$  when restricted on  $\{u \geq 0\}$  and, in view of (A-74) and (A-75), satisfies

$$\begin{aligned} & \sup_{\mathcal{W}_{u_*}} \left\{ \left| \log \left( \frac{\Omega^2}{1 - \frac{1}{3} \Lambda r^2} \right) \right| + \left| \log \left( \frac{2\partial_v r}{1 - 2m/r} \right) \right| + \left| \log \left( \frac{1 - 2m/r}{1 - \frac{1}{3} \Lambda r^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m}| \right\} \\ & \quad + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_*}} r T_{vv} dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_*}} r T_{uu} du \\ & \leq 4(h_1(\varepsilon))^{-3} \exp(-(h_0(\varepsilon))^{-4}). \end{aligned} \quad (\text{A-81})$$

Therefore, in view of (A-81), (A-79) and (A-80), Theorem 3.6 applied for the past development of  $(r, \Omega^2, \bar{f}_{\text{in}}, \bar{f}_{\text{out}})|_{u=u_*}$  on  $\mathcal{W}_{u_*}$  (see the remark below Theorem 3.6) implies that the maximal past development  $(\mathcal{U}^*; r^*, (\Omega^*)^2, \bar{f}_{\text{in}}^*, \bar{f}_{\text{out}}^*)$  of  $(r_{\text{f}}^*, (\Omega_{\text{f}}^*)^2, \bar{f}_{\text{in}/}^*, \bar{f}_{\text{out}/}^*)$  satisfies

$$\mathcal{W}_{u_*} \subset \mathcal{U}^* \quad (\text{A-82})$$

and

$$\begin{aligned}
& \sup_{\mathcal{W}_{u_*}} \left\{ \left| \log \left( \frac{\Omega^2}{1 - \frac{1}{3} \Lambda r^2} \right) - \log \left( \frac{(\Omega^*)^2}{1 - \frac{1}{3} \Lambda (r^*)^2} \right) \right| + \left| \log \left( \frac{2\partial_v r}{1 - 2m/r} \right) - \log \left( \frac{2\partial_v r^*}{1 - 2m^*/r^*} \right) \right| \right. \\
& \quad \left. + \left| \log \left( \frac{1 - 2m/r}{1 - \frac{1}{3} \Lambda r^2} \right) - \log \left( \frac{1 - 2m^*/r^*}{1 - \frac{1}{3} \Lambda (r^*)^2} \right) \right| + \sqrt{-\Lambda} |\tilde{m} - \tilde{m}^*| \right\} \\
& \quad + \sup_{\bar{u}} \int_{\{u=\bar{u}\} \cap \mathcal{W}_{u_*}} |r T_{vv} - r^* (T_{vv})^*| dv + \sup_{\bar{v}} \int_{\{v=\bar{v}\} \cap \mathcal{W}_{u_*}} |r T_{uu} - r^* (T_{uu})^*| du \\
& \leq (h_1(\varepsilon))^3.
\end{aligned} \tag{A-83}$$

Thus, the proof of the lemma concludes by setting

$$(r'_j, (\Omega'_j)^2, \bar{f}'_{\text{in}/j}, \bar{f}'_{\text{out}/j}) \doteq (r^*, (\Omega^*)^2, \bar{f}^*_{\text{in}}, \bar{f}^*_{\text{out}})|_{u=0}. \quad \square$$

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
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