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ON THE SHARP UPPER BOUND RELATED TO THE WEAK MUCKENHOUPT–WHEEDEN CONJECTURE

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We construct an example showing that the upper bound $[w]_{A_1}\log(\mathrm{e}+[w]_{A_1})$ for the $L^1(w)\to L^{1,\infty}(w)$ norm of the Hilbert transform cannot be improved in general.

1. Introduction

Define the Hardy–Littlewood maximal operator on $\mathbb{R}$ by

$$Mf(x)=\sup_{I\ni x}\frac{1}{|I|}\int_I |f(y)|\,dy,$$

where the supremum is taken over all intervals $I\subset\mathbb{R}$ containing the point x.

C. Fefferman and E. Stein [1971] established the following weighted weak-type inequality for M: there exists an absolute constant $C>0$ such that, for every weight w,

$$\sup_{\alpha>0}\alpha w\{x\in\mathbb{R}:Mf(x)>\alpha\}\le C\int_{\mathbb{R}}|f(x)|Mw(x)\,dx \tag{1-1}$$

(here by a weight we mean any nonnegative locally integrable function on $\mathbb{R}$, and we use the standard notation $w(E)=\int_E w$ for a measurable set $E\subset\mathbb{R}$).

Inequality (1-1) is important for several reasons. First, it is the key ingredient in extending the Hardy–Littlewood maximal theorem to a vector-valued case. Second, this result was a precursor of the weighted theory, which had started to develop rapidly in the beginning of the 70's. In particular, if we define the $[w]_{A_1}$ constant of the weight w by $[w]_{A_1}=\|Mw/w\|_{L^\infty}$, then, assuming $[w]_{A_1}<\infty$, (1-1) implies immediately that

$$\|Mf\|_{L^{1,\infty}(w)}\le C[w]_{A_1}\|f\|_{L^1(w)}. \tag{1-2}$$

Consider now the Hilbert transform,

$$Hf(x)=\mathrm{P.V.}\int_{\mathbb{R}}\frac{f(y)}{x-y}\,dy.$$

Inequality (1-1) with the maximal operator on the left-hand side replaced by the Hilbert transform has become known as the Muckenhoupt–Wheeden conjecture. Only recently this conjecture has been disproved by M. Reguera and C. Thiele [2012] (see also [Caldarelli et al. 2017; Criado and Soria 2016;

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Reguera 2011] for some complements and extensions). Their result, however, left open the question of whether a weaker form of the Muckenhoupt–Wheeden conjecture holds, with M replaced by H on the left-hand side of (1-2).

In [Lerner et al. 2009], it was proved that

$$\|Hf\|_{L^{1,\infty}(w)} \le C[w]_{A_1}\log(\mathrm{e}+[w]_{A_1})\|f\|_{L^1(w)}. \tag{1-3}$$

This improved a previous result in [Lerner et al. 2008], where the right-hand side contained an additional factor double logarithmic in $[w]_{A_1}$. Notice also that actually (1-3) in [Lerner et al. 2009] was proved for every Calderón–Zygmund operator on $\mathbb{R}^n$ with sufficiently smooth kernel.

On the other hand, in [Nazarov et al. 2015], it was shown for the martingale transform (and explained how to transfer the result to the Hilbert transform case) that the dependence of $[w]_{A_1}$ in the weighted weak-type-(1, 1) inequality cannot in general be made better than $[w]_{A_1}\log^{1/5}(\mathrm{e}+[w]_{A_1})$, thus disproving the weak Muckenhoupt–Wheeden conjecture. Later, in [Nazarov et al. 2018], the power of the logarithm was improved to $\frac{1}{3}$ (this was again done for the martingale transform).

Summarizing the results in [Lerner et al. 2009; Nazarov et al. 2015; 2018], if we denote by α_H the best possible exponent for which the inequality

$$\|Hf\|_{L^{1,\infty}(w)} \le C[w]_{A_1}\log^{\alpha_H}(\mathrm{e}+[w]_{A_1})\|f\|_{L^1(w)}$$

holds, then we have that $\frac{1}{3} \le \alpha_H \le 1$.

The main result of this paper shows, in particular, that $\alpha_H = 1$. For $t \ge 1$, define

$$\varphi_H(t) = \sup_{[w]_{A_1}\le t} \|H\|_{L^1(w)\to L^{1,\infty}(w)}.$$

Then (1-3) implies $\varphi_H(t) \le Ct\log(\mathrm{e}+t)$. We will show that actually $\varphi_H(t) \eqsim t\log(\mathrm{e}+t)$. Our main result is the following.

Theorem 1.1. *There exists $c' > 0$ such that, for all $t \ge 1$,*

$$\varphi_H(t) \ge c't\log(\mathrm{e}+t).$$

As a trivial corollary we obtain that the inequality

$$\|Hf\|_{L^{1,\infty}(w)} \le \psi([w]_{A_1})\|f\|_{L^1(w)}$$

fails in general for every function ψ increasing on $[1,\infty)$ satisfying

$$\lim_{t\to\infty}\frac{\psi(t)}{t\log(\mathrm{e}+t)} = 0.$$

2. Proof of Theorem 1.1

An overview of the proof. At the first step we show that the definition of φ_H along with the standard extrapolation and dualization arguments yields

$$\|H(w\chi_{[0,1)})\|_{L^2(\sigma)} \le 4\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\left(\int_0^1 w\right)^{1/2}, \tag{2-1}$$

where $\sigma = w^{-1}$. Notice that $\|M\|_{L^2(\sigma)\to L^2(\sigma)} < \infty$ if and only if $w \in A_2$, that is, if

$$\sup_{I\subset\mathbb{R}} \frac{w(I)\sigma(I)}{|I|^2} < \infty.$$

Therefore, we assume here that $w \in A_2$.

The key step is to show that there exist $C_1, C_2, C_3 > 0$ such that for every $t > C_3$, there is an A_2-weight w satisfying

$$\int_0^1 w = 1, \quad \|M\|_{L^2(\sigma)\to L^2(\sigma)} \le C_1 t, \quad \|H(w\chi_{[0,1)})\|_{L^2(\sigma)} \ge C_2 t \log t. \tag{2-2}$$

Plugging these estimates into (2-1), we finish the proof.

Extrapolation and dualization. First, we apply the standard Rubio de Francia extrapolation trick. Given $g \ge 0$ with $\|g\|_{L^2(\sigma)} = 1$, define

$$\mathcal{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})^k}.$$

Then $g \le \mathcal{R}g$, $\|\mathcal{R}g\|_{L^2(\sigma)} \le 2$, and $[\mathcal{R}g]_{A_1} \le 2\|M\|_{L^2(\sigma)\to L^2(\sigma)}$. These estimates along with the definition of φ_H and Hölder's inequality imply

$$\begin{aligned}\alpha \int_{\{x:|Hf(x)|>\alpha\}} g \le \alpha \int_{\{x:|Hf(x)|>\alpha\}} \mathcal{R}g &\le \varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^1(\mathcal{R}g)} \\ &\le \varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^2(w)}\|\mathcal{R}g\|_{L^2(\sigma)} \\ &\le 2\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^2(w)}.\end{aligned}$$

Taking here the supremum over all $g \ge 0$ with $\|g\|_{L^2(\sigma)} = 1$ yields

$$\|Hf\|_{L^{2,\infty}(w)} \le 2\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)})\|f\|_{L^2(w)}. \tag{2-3}$$

We now use the elementary estimate

$$\int_0^1 |Hf| w \le 2\|Hf\|_{L^{2,\infty}(w)} \left(\int_0^1 w\right)^{1/2}, \tag{2-4}$$

which along with (2-3) implies

$$\left|\int_{\mathbb{R}} (H(w\chi_{[0,1)}))f\right| = \left|\int_0^1 (Hf)w\right| \le 4\varphi_H(2\|M\|_{L^2(\sigma)\to L^2(\sigma)}) \left(\int_0^1 w\right)^{1/2} \|f\|_{L^2(w)}.$$

Taking here the supremum over all f with $\|f\|_{L^2(w)} = 1$ proves (2-1).

To show (2-4), notice that, for every $\lambda > 0$,

$$\int_0^1 |Hf|w \le \int_\lambda^\infty w\{x : |Hf(x)| > \alpha\}\, d\alpha + \lambda \int_0^1 w \le \frac{1}{\lambda}\|Hf\|^2_{L^{2,\infty}(w)} + \lambda \int_0^1 w.$$

Optimizing this estimate with respect to λ, we obtain (2-4).

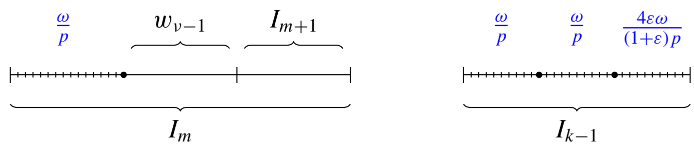

Figure 1. $w_\nu(\omega, \sigma, I)$ on intervals $I_m^{(i)}$ for $i = 1, 2$ and $0 \le m \le k-2$ and on $I_{k-1}^{(i)}$ for $i = 1, 2, 3$.

Construction of the weight. Fix $t \gg 1$. Take $k \in \mathbb{N}$ such that $t \le 3^k \le 3t$. Let $\varepsilon = 3^{-k}$ and

$$p = \frac{1}{3\varepsilon}\left(\frac{1+\varepsilon}{2} + \frac{4\varepsilon^2}{1+\varepsilon}\right).$$

The reason for this definition of p will be clarified a bit later. Note that we will frequently use the obvious estimate $1/(6\varepsilon) \le p \le 2/\varepsilon$.

For every two positive numbers ω and σ such that $\omega\sigma = p$ and any interval $I \subset \mathbb{R}$, we define inductively the sequence of weights $w_\nu = w_\nu(\omega, \sigma, I)$ $(\nu = 0, 1, 2, \dots)$ supported on I as follows.

Let $u = \sqrt{p} + \sqrt{p-1}$ be the larger root of $u + 1/u = 2\sqrt{p}$. Define

$$w_0(\omega, \sigma, I) = \frac{\omega}{\sqrt{p}}\left(u\chi_{I_-} + \frac{1}{u}\chi_{I_+}\right),$$

where I_- and I_+ are the left and the right halves of I respectively.

Suppose that $w_{\nu-1}(\omega, \sigma, I)$ is already defined for all ω, σ with $\omega\sigma = p$ and all $I \subset \mathbb{R}$. To construct $w_\nu(\omega, \sigma, I)$, first denote by I_m, $m = 0, 1, \dots, k-1$, the interval with the same right endpoint as I of length $3^{-m}|I|$, so

$$I_{k-1} \subset I_{k-2} \subset \cdots \subset I_0 = I$$

and $|I_{k-1}| = 3\varepsilon|I|$.

Given an interval J, denote by $J^{(i)}$, $i = 1, 2, 3$, the i-th from the left subinterval of J in the partition of J into three equal intervals.

Define $w_\nu(\omega, \sigma, I)$ by

$$w_\nu(\omega, \sigma, I) = \frac{\omega}{p}\left(\sum_{m=0}^{k-2}\chi_{I_m^{(1)}} + \chi_{I_{k-1}^{(1)} \cup I_{k-1}^{(2)}} + \frac{4\varepsilon}{1+\varepsilon}\chi_{I_{k-1}^{(3)}}\right) + \sum_{m=0}^{k-2} w_{\nu-1}\left(2\omega, \frac{\sigma}{2}, I_m^{(2)}\right). \tag{2-5}$$

See Figure 1.

Note that the interval $I_{k-1}^{(3)}$ plays a rather special role in the final step of this recursive construction. We shall call any interval of this type arising at any step in the construction of the weight $w_\nu(\omega, \sigma, I)$ a “tail interval”, so within I we shall have one tail interval $I_{k-1}^{(3)}$ arising at the final stage of the construction, $k-1$ tail intervals $(I_m^{(2)})_{k-1}^{(3)}$ arising in the construction of the weights $w_{\nu-1}(2\omega, \sigma/2, I_m^{(2)})$, and so on.

Finally, we define w as the 1-periodization of $w_n(1, p, [0, 1))$ with $n = 3^{k-1}$.

For $l = 0, 1, \dots, n$, we shall say that an interval I “carries w_{n-l}” if $w = w_{n-l}(2^l, 2^{-l}p, I)$ on I. Denote by $\operatorname{supp} w_{n-l}$ the union of all intervals carrying w_{n-l}. For example, $\operatorname{supp} w_n = \bigcup_{k\in\mathbb{Z}}[k, k+1) = \mathbb{R}$ as $[k, k+1)$ carries w_n for every $k \in \mathbb{Z}$.

Let us now establish several useful properties of $w_\nu(\omega, \sigma, I)$.

Proposition 2.1. *For every* $\nu \geq 0$,

$$\frac{1}{|I|}\int_I w_\nu(\omega,\sigma,I)\,dx = \omega \quad \text{and} \quad \frac{1}{|I|}\int_I w_\nu^{-1}(\omega,\sigma,I)\,dx = \sigma. \tag{2-6}$$

Proof. The proof is by induction on ν. For $\nu = 0$,

$$\frac{1}{|I|}\int_I w_0(\omega,\sigma,I)\,dx = \frac{\omega}{\sqrt{p}}\frac{1}{2}\left(u+\frac{1}{u}\right) = \omega,$$
$$\frac{1}{|I|}\int_I w_0^{-1}(\omega,\sigma,I)\,dx = \frac{\sqrt{p}}{\omega}\frac{1}{2}\left(\frac{1}{u}+u\right) = \frac{p}{\omega} = \sigma.$$

Assume that the statement holds for $\nu-1$ and let us prove it for ν. Observe that $w_\nu(\omega,\sigma,I)$ equals ω/p on a subset of I of total measure

$$\frac{1-3\varepsilon}{2}|I| + 2\varepsilon|I| = \frac{1+\varepsilon}{2}|I|,$$

it equals

$$\frac{4\varepsilon}{1+\varepsilon}\frac{\omega}{p}$$

on a set of measure $\varepsilon|I|$, and the average of $w_{\nu-1}(2\omega,\cdot,\cdot)$ over the remaining set of measure $\frac{1}{2}(1-3\varepsilon)|I|$ is 2ω by the inductive assumption. Thus

$$\begin{aligned}\frac{1}{|I|}\int_I w_\nu(\omega,\sigma,I)\,dx &= \frac{\omega}{p}\left(\frac{1+\varepsilon}{2}+\frac{4\varepsilon^2}{1+\varepsilon}\right)+\omega(1-3\varepsilon)\\ &= \omega + \left(\frac{1}{p}\left(\frac{1+\varepsilon}{2}+\frac{4\varepsilon^2}{1+\varepsilon}\right)-3\varepsilon\right)\omega = \omega\end{aligned}$$

(it is this equation that was used to determine p).

On the other hand, $w_\nu^{-1}(\omega,\sigma,I)$ equals $p/\omega = \sigma$ on a subset of I of measure $\frac{1}{2}(1+\varepsilon)|I|$ (the same set on which w_ν is defined as ω/p), it equals

$$\frac{1+\varepsilon}{4\varepsilon}\sigma$$

on a set of measure $\varepsilon|I|$, and its average over the remaining set of measure $\frac{1}{2}(1-3\varepsilon)|I|$ equals $\frac{1}{2}\sigma$. Thus

$$\frac{1}{|I|}\int_I w_\nu^{-1}(\omega,\sigma,I)\,dx = \frac{1+\varepsilon}{2}\sigma + \frac{1+\varepsilon}{4}\sigma + \frac{(1-3\varepsilon)}{2}\frac{\sigma}{2} = \sigma,$$

which completes the proof. □

In particular, it follows from Proposition 2.1 that for the constructed weight w,

$$\int_0^1 w\,dx = \int_0^1 w_n(1,p,[0,1))\,dx = 1.$$

Proposition 2.2. *Let* $I = [a, a+h)$. *Then, for every* $\nu \geq 0$ *and for all* $0 < \tau < h$,

$$\frac{1}{\tau}\int_a^{a+\tau} w_\nu(\omega,\sigma,I) \leq 3\omega \quad \text{and} \quad \frac{1}{\tau}\int_{a+h-\tau}^{a+h} w_\nu(\omega,\sigma,I) \leq \frac{9}{2}\omega. \tag{2-7}$$

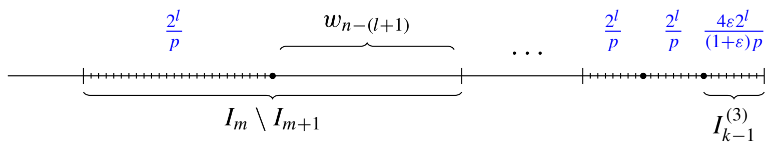

Figure 2. w on some interval I carrying w_{n-l} for $l < n$.

Proof. For $\nu = 0$ the statement is obvious since $w_0(\omega, \sigma, I) \le 2\omega$ on I. Assume that $\nu \ge 1$.

Since $w_\nu(\omega, \sigma, I) = \omega/p$ on $I^{(1)}$, we have

$$\frac{1}{\tau}\int_a^{a+\tau} w_\nu(\omega, \sigma, I) = \frac{\omega}{p}$$

for $0 < \tau < \frac{1}{3}h$. But if $\tau \ge \frac{1}{3}h$, then, by Proposition 2.1,

$$\frac{1}{\tau}\int_a^{a+\tau} w_\nu(\omega, \sigma, I)\,dx \le \frac{3}{|I|}\int_I w_\nu(\omega, \sigma, I)\,dx = 3\omega.$$

We now turn to the proof of the second estimate in (2-7). Let $I_m,\ m = 0, 1, \dots, k-1$, be the intervals appearing in the definition of $w_\nu(\omega, \sigma, I)$. Since $w_\nu(\omega, \sigma, I) \le \omega/p$ on I_{k-1}, the estimate is trivial if $a + h - \tau \in I_{k-1}$. Assume that $a + h - \tau \in I_m \setminus I_{m+1},\ m = 0, \dots, k-2$. Then

$$\frac{1}{\tau}\int_{a+h-\tau}^{a+h} w_\nu(\omega, \sigma, I)\,dx \le \frac{1}{|I_{m+1}|}\int_{I_m} w_\nu(\omega, \sigma, I)\,dx. \tag{2-8}$$

Next, by Proposition 2.1,

$$\begin{aligned}\int_{I_m} w_\nu(\omega, \sigma, I)\,dx &= \sum_{j=m}^{k-2}\left(\frac{\omega}{p}|I_j^{(1)}| + \int_{I_j^{(2)}} w_{\nu-1}\left(2\omega, \frac{\sigma}{2}, I_j^{(2)}\right)dx\right) + \int_{I_{k-1}} w_\nu(\omega, \sigma, I)\,dx \\ &\le \omega\sum_{j=m}^{k-1}|I_j| \le \frac{3\omega}{2}\frac{|I|}{3^m} = \frac{9\omega}{2}|I_{m+1}|, \end{aligned} \tag{2-9}$$

which along with (2-8) completes the proof. □

Assume that I carries w_{n-l}; see Figure 2. Consider the corresponding tail intervals contained in I, that is, the intervals on which

$$w = \frac{4\varepsilon}{1+\varepsilon}\frac{2^j}{p}, \qquad j = l, \dots, n-1.$$

These intervals will play the central role in the estimate of the Hilbert transform of $w\chi_{[0,1)}$. There is only one tail interval in $I \setminus \operatorname{supp} w_{n-(l+1)}$, and its measure equals $(1/3^k)|I|$. Next, there are $k-1$ tail intervals in

$$I \cap (\operatorname{supp} w_{n-(l+1)} \setminus \operatorname{supp} w_{n-(l+2)})$$

of total measure

$$\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\frac{1}{3^k}|I|.$$

Similarly, the measure of the union of tail intervals in

$$I \cap (\operatorname{supp} w_{n-(l+j)} \setminus \operatorname{supp} w_{n-(l+j+1)}), \qquad j = 0, \dots, n-l-1,$$

is

$$\left(\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\right)^j \frac{1}{3^k}|I|.$$

In particular, if we denote by A_l the union of tail intervals in

$$[0, 1) \cap (\operatorname{supp} w_{n-l} \setminus \operatorname{supp} w_{n-(l+1)}),$$

then

$$|A_l| = \left(\frac{1}{2}\left(1 - \frac{1}{3^{k-1}}\right)\right)^l \frac{1}{3^k}, \quad l = 0, \dots, n-1. \tag{2-10}$$

Estimate of the maximal operator. In this section, we will prove the first inequality in (2-2). We start with the reduction of this estimate to its triadic version.

Let $\mathcal{T}$ be the standard triadic lattice; that is,

$$\mathcal{T} = \{[3^j n, 3^j (n+1)) : j, n \in \mathbb{Z}\}.$$

Denote by $\mathcal{J}$ the family of all unions of two adjacent triadic intervals of equal length.

Our key tool will be the estimate

$$\|M\|_{L^2(\sigma)\to L^2(\sigma)} \le 24 \sup_{J\in\mathcal{J}} \left(\frac{1}{w(J)} \int_J (M(w\chi_J))^2 \sigma\right)^{1/2}. \tag{2-11}$$

This estimate is fairly standard and well known. For the reader's convenience, we supply the proof in the Appendix.

Combining (2-11) with the inequality $p \le 2/\varepsilon \le 6t$, we see that in order to prove the first estimate in (2-2), it suffices to show that there exists $C > 0$ such that, for every interval $J \in \mathcal{J}$,

$$\int_J (M(w\chi_J))^2 \sigma \le Cp^2 w(J). \tag{2-12}$$

Define an auxiliary 1-periodic function $\tilde{w}$ by

$$\tilde{w}(x) = \sum_{l=1}^{n} 2^l \chi_{\operatorname{supp} w_{n-(l-1)} \setminus \operatorname{supp} w_{n-l}}(x) + 2^{n+1} \chi_{\operatorname{supp} w_0}(x).$$

The role of this function is clarified in the following two propositions.

Proposition 2.3. *For all $x \in \mathbb{R}$,*

$$Mw(x) \le \tfrac{9}{2}\tilde{w}(x). \tag{2-13}$$

Proof. First, notice that for $x \in \operatorname{supp} w_0$ the statement is trivial. Indeed, $w \le 2^{n-1}/p$ on the complement of $\operatorname{supp} w_0$, and if I carries w_0, then on I

$$w_0 = \frac{2^n}{\sqrt{p}}\left((\sqrt{p} + \sqrt{p-1})\chi_{I_-} + \frac{1}{\sqrt{p} + \sqrt{p-1}}\chi_{I_+}\right).$$

Hence,

$$\|w\|_{L^\infty} \le \frac{2^n(\sqrt{p}+\sqrt{p-1})}{\sqrt{p}} \le 2^{n+1},$$

and therefore $\|Mw\|_{L^\infty} \le 2^{n+1}$.

On the other hand, for $x \in \operatorname{supp} w_{n-(l-1)} \setminus \operatorname{supp} w_{n-l}$, the estimate (2-13) follows immediately from the facts that $w \le 2^{l-1}/p$ on the complement of $\operatorname{supp} w_{n-l}$ and, by Proposition 2.2, the average of w over the intersection of any interval J carrying w_{n-l} with an interval not contained in J is at most $\frac{9}{2}\cdot 2^l$. $\square$

Recall that, given an interval I, we denoted by I_m, $m=0,\dots,k-1$, the interval with the same right endpoint as I of length $|I_m| = (1/3^m)|I|$. These intervals have already appeared in the definition of $w_\nu(\omega,\sigma,I)$.

Proposition 2.4. *Assume that I carries w_{n-l}. Then*

$$\int_{I_m} (\tilde{w})^2\sigma \le 30p^2 w(I_m), \quad l=0,\dots,n,\ m=0,\dots,k-2.$$

Proof. First, notice that the case when $l=n$ is trivial, since $2^{n+1} \le 4pw(x)$ on any interval I carrying w_0, and hence,

$$\int_J (\tilde{w})^2\sigma \le 16p^2 w(J) \quad \text{for every } J \subset \operatorname{supp} w_0. \tag{2-14}$$

Suppose now that $l \le n-1$ and consider first the case $m=0$. Assume that I carries w_{n-l}. For $j=0,\dots,n-l-1$, define

$$F_j = I \cap (\operatorname{supp} w_{n-(l+j)} \setminus \operatorname{supp} w_{n-(l+j+1)})$$

and let E_j be the union of the tail intervals contained in F_j. Observe that $w = 2^{l+j}/p$ on $F_j \setminus E_j$, and hence, $\tilde{w}(x) = 2pw(x)$ for $x \in F_j \setminus E_j$, which implies

$$\int_{\bigcup_j (F_j\setminus E_j)} (\tilde{w})^2\sigma \le 4p^2 w(I). \tag{2-15}$$

On the other hand,

$$w = \frac{4\varepsilon}{1+\varepsilon}\frac{2^{l+j}}{p}$$

on E_j and, as we have seen in the previous section,

$$|E_j| = \left(\frac{1}{2}\left(1-\frac{1}{3^{k-1}}\right)\right)^j \frac{1}{3^k}|I| \le \frac{1}{2^j}\frac{1}{3^k}|I|.$$

Combining this with Proposition 2.1 yields

$$\int_{\bigcup_j E_j} (\tilde{w})^2\sigma \le 4\sum_{j=0}^{n-l-1} 2^{2(l+j)}\frac{(1+\varepsilon)p}{4\varepsilon 2^{l+j}}\frac{1}{2^j}\frac{1}{3^k}|I|$$
$$\le \frac{2p}{\varepsilon}\frac{n}{3^k}2^l|I| \le 4p^2 2^l|I| = 4p^2 w(I). \tag{2-16}$$

Further, by (2-14),

$$\int_{I\cap \operatorname{supp} w_0} (\tilde{w})^2\sigma \le 16p^2 w(I\cap \operatorname{supp} w_0) \le 16p^2 w(I).$$

Combining this estimate with (2-15) and (2-16), we obtain

$$\int_I (\tilde{w})^2\sigma = \int_{\bigcup_j (F_j\setminus E_j)} (\tilde{w})^2\sigma + \int_{\bigcup_j E_j} (\tilde{w})^2\sigma + \int_{I\cap \operatorname{supp} w_0} (\tilde{w})^2\sigma \le 24p^2 w(I), \tag{2-17}$$

and this completes the proof in the case $m = 0$.

Assume now that $1 \le m \le k-2$. Notice that $I_m \setminus I_{m+1} = I_m^{(1)} \cup I_m^{(2)}$, where $I_m^{(2)}$ carries $w_{n-(l+1)}$, and $\tilde{w}(x) = 2pw(x)$ on $I_m^{(1)}$. Thus, by (2-17),

$$\begin{aligned}\int_{I_m\setminus I_{m+1}} (\tilde{w})^2\sigma &\le 4p^2 w(I_m^{(1)}) + 24p^2 w(I_m^{(2)}) \\ &\le 24p^2 w(I_m \setminus I_{m+1}).\end{aligned} \tag{2-18}$$

Further,

$$\int_{I_{k-1}} (\tilde{w})^2\sigma \le 4(2^l)^2 \frac{(1+\varepsilon)p}{4\varepsilon 2^l} |I_{k-1}| \le 6p2^l |I|.$$

On the other hand, by Proposition 2.1,

$$w(I_m) \ge w(I_m^{(2)}) = 2^{l+1}|I_m^{(2)}| = \frac{2^{l+1}}{3^{m+1}}|I|,$$

which, combined with the previous estimate, implies

$$\int_{I_{k-1}} (\tilde{w})^2\sigma \le p3^{m+2} w(I_m) \le \frac{p}{\varepsilon} w(I_m) \le 6p^2 w(I_m).$$

Therefore, using (2-18), we obtain

$$\int_{I_m} (\tilde{w})^2\sigma = \sum_{j=m}^{k-2} \int_{I_j\setminus I_{j+1}} (\tilde{w})^2\sigma + \int_{I_{k-1}} (\tilde{w})^2\sigma \le 30p^2 w(I_m),$$

which completes the proof. □

We now turn to the proof of (2-12). Let $J \in \mathcal{J}$. First consider the simple case when $|J| \ge 1$. In this case, $|J| = k$ for some $k \in \mathbb{N}$. Using that w and $\tilde{w}$ are 1-periodic along with the fact that $\int_0^1 w = 1$, and combining Propositions 2.3 and 2.4, we obtain

$$\frac{1}{w(J)} \int_J (M(w\chi_J))^2\sigma \le 25 \int_0^1 (\tilde{w})^2\sigma \le 25\cdot 30p^2.$$

Suppose that $|J| < 1$. We can represent J as the union of two triadic intervals $J = J_- \cup J_+$, where $J_-, J_+ \in \mathcal{T}$ are the left and the right halves of J respectively. Since J_- is triadic, we must have $|J_-| \le \frac{1}{3}$. Also, by the 1-periodicity of w, one can assume that $J_- \subset [0, 1)$.

Consider the case when J contains an interval carrying $\operatorname{supp} w_{n-(l+1)}$ for some l. Out of all such intervals choose the longest one. Note that since $|J| \le \frac{2}{3}$, we must have $l \ge 0$ in this case. Thus, the interval in question must be of the kind $R_m^{(2)}$, where R is an interval carrying w_{n-l}. Since neither $R = R_0$ nor $R_{m-1}^{(2)}$ (if $m \geq 1$) is contained in J, there are only three possible options:

- $J_- = R_m^{(2)},\ 0 \leq m \leq k-2$;
- $J_+ = R_m^{(2)},\ 0 \leq m \leq k-2$;
- $J_- = R_m,\ 1 \leq m \leq k-2$.

Suppose first that $J_- = R_m^{(2)}$ or $J_+ = R_m^{(2)}$. Then $J \subset R_m$. By (2-9), $w(R_m) \leq 3 \cdot 2^{l-1}|R_m|$. On the other hand, since $R_m^{(2)}$ carries $w_{n-(l+1)}$, by Proposition 2.1,

$$w(J) \geq w(R_m^{(2)}) = 2^{l+1}|R_m^{(2)}| = 2^{l+1}\frac{|R_m|}{3}, \tag{2-19}$$

which implies $w(R_m) \leq \frac{9}{4}w(J)$. Therefore, by Propositions 2.3 and 2.4,

$$\int_J (M(w\chi_J))^2\sigma \leq 25\int_{R_m}(\tilde{w})^2\sigma \leq 25\cdot 30p^2 w(R_m) \leq 25\cdot 75p^2 w(J).$$

Assume now that $J_- = R_m,\ 1 \leq m \leq k-2$. Then $w \equiv 2^{l-1}/p$ on J_+ if $l > 0$ and $w \equiv 2^l/p$ on J_+ if $l = 0$. In either case, $\tilde{w} = 2pw$ on J_+, so

$$\int_{J_+}(\tilde{w})^2\sigma = 4p^2 w(J_+),$$

and thus, by Propositions 2.3 and 2.4,

$$\begin{aligned}\int_J (M(w\chi_J))^2\sigma &\leq 25\biggl(\int_{R_m}(\tilde{w})^2\sigma + \int_{J_+}(\tilde{w})^2\sigma\biggr)\\ &\leq 25(30p^2 w(R_m) + 4p^2 w(J_+)) \leq 25\cdot 30p^2 w(J).\end{aligned}$$

It remains to consider the case when J does not contain an interval carrying $w_{n-(l+1)}$ for any $0 \leq l \leq n-1$. Denote by E the union of all tail intervals appearing in the definition of w. Notice that if $x \notin E$, then

$$\sum_{l=1}^{n} 2^l \chi_{\operatorname{supp} w_{n-(l-1)}\setminus \operatorname{supp} w_{n-l}}(x) = 2pw(x)\chi_{\mathbb{R}\setminus \operatorname{supp} w_0}.$$

Also, $2^{n+1} \leq 4pw(x)\chi_{\operatorname{supp} w_0}$. From this and from Proposition 2.3,

$$Mw(x) \leq 18pw(x) \quad (x \notin E).$$

Therefore, if $J \cap E = \varnothing$,

$$\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma \leq 18^2 p^2. \tag{2-20}$$

Suppose that $J \cap E \neq \varnothing$. Then there exists R carrying w_{n-l} for some $0 \leq l \leq n-1$ such that $J \cap R_{k-1}^{(3)} \neq \varnothing$. Since J_-, J_+ and $R_{k-1}^{(3)}$ are triadic, we have that either one half of J is contained in $R_{k-1}^{(3)}$

or $R^{(3)}_{k-1} \subset J$. Since J cannot contain any interval carrying $\operatorname{supp} w_{n-(l+1)}$, in both cases we obtain that w can take only three possible values

$$\frac{2^l}{p}, \quad \frac{4\varepsilon 2^l}{(1+\varepsilon)p}, \quad \frac{2^{l-1}}{p}$$

on J and therefore,

$$\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma \le \left(\frac{\sup_J w}{\inf_J w}\right)^2 \le \left(\frac{1+\varepsilon}{4\varepsilon}\right)^2 \le 9p^2,$$

which along with (2-20) implies

$$\frac{1}{w(J)}\int_J (M(w\chi_J))^2\sigma \le 18^2p^2.$$

This completes the proof of (2-12), and therefore the first estimate in (2-2) is proved.

Estimate of the Hilbert transform. The goal of this section is to prove the second estimate in (2-2).

Denote by A^*_l, $l = 0, \ldots, n-1$, the union of all intervals $\frac{1}{2}I$, where I is a tail interval contained in

$$[0, 1) \cap (\operatorname{supp} w_{n-l} \setminus \operatorname{supp} w_{n-(l+1)}).$$

In other words, A^*_l is the union of all intervals $\frac{1}{2}J^{(3)}_{k-1}$, where $J \subset [0, 1)$ carries w_{n-l}. Then, by (2-10),

$$|A^*_l| = \frac{1}{2}|A_l| = \frac{1}{2}\left(\frac{1}{2}\left(1-\frac{1}{3^{k-1}}\right)\right)^l \frac{1}{3^k}, \quad l = 0, \ldots, n-1. \tag{2-21}$$

The sets A^*_l plays the central role in establishing the lower bound for $H(w\chi_{[0,1)})$, as the following proposition shows.

Proposition 2.5. *There exists an absolute $C > 0$ such that for all $l = 0, \ldots, n-1$ and for every $x \in A^*_l$*

$$|H(w\chi_{[0,1)})(x)| \ge Ck2^l. \tag{2-22}$$

Let us first show how to derive the second estimate in (2-2) from here. By (2-21) and (2-22),

$$\int_{A^*_l} |H(w\chi_{[0,1)})|^2\sigma \ge C^2k^2 2^{2l}\frac{1+\varepsilon}{4\varepsilon}\frac{p}{2^l}\frac{1}{2}\left(\frac{1}{2}\left(1-\frac{1}{3^{k-1}}\right)\right)^l\frac{1}{3^k}.$$

Therefore,

$$\begin{aligned}\|H(w\chi_{[0,1)})\|^2_{L^2(\sigma)} &\ge \sum_{l=0}^{n-1}\int_{A^*_l}|H(w\chi_{[0,1)})|^2\sigma \\ &\ge \frac{C^2}{8}k^2p\sum_{l=0}^{n-1}\left(1-\frac{1}{3^{k-1}}\right)^l = \frac{C^2}{24}k^2 3^k p\left(1-\left(1-\frac{1}{3^{k-1}}\right)^n\right).\end{aligned}$$

Since $n = 3^{k-1}$ and $(1-1/n)^n < 1/\mathrm{e}$, we obtain

$$\|H(w\chi_{[0,1)})\|^2_{L^2(\sigma)} \ge \frac{C^2(1-1/\mathrm{e})}{24}k^2 3^k p \ge \frac{C^2(1-1/\mathrm{e})}{144(\log 3)^2}t^2\log^2 t.$$

Let us now turn to the proof of Proposition 2.5. Let $J=[a,b)\subset[0,1)$ be an interval carrying w_{n-l}. Assume that $x\in\frac{1}{2}J^{(3)}_{k-1}$. Write

$$\begin{aligned}&H(w\chi_{[0,1)})(x)\\&\quad=H(w\chi_{[0,a)})(x)+\sum_{m=0}^{k-2}H(w\chi_{J_m\setminus J_{m+1}})(x)+H(w\chi_{J_{k-1}\setminus J^{(3)}_{k-1}})(x)+H(w\chi_{J^{(3)}_{k-1}})(x)+H(w\chi_{[b,1)})(x)\\&\quad\equiv A(x)+B(x)+C(x)+D(x)+E(x).\end{aligned}$$

We will show that there are absolute constants C_1 and C_2 such that, for all $x\in\frac{1}{2}J^{(3)}_{k-1}$,

$$|B(x)|\ge C_1k2^l\quad\text{and}\quad\max\{|D(x)|,|E(x)|\}\le C_22^l. \tag{2-23}$$

Since $A(x)$, $B(x)$ and $C(x)$ are positive for all $x\in\frac{1}{2}J^{(3)}_{k-1}$, we obtain from (2-23) that

$$\begin{aligned}|H(w\chi_{[0,1)})(x)|&\ge|A(x)+B(x)+C(x)|-|D(x)|-|E(x)|\\&\ge|B(x)|-|D(x)|-|E(x)|\ge\frac{C_1}{2}k2^l\end{aligned}$$

for $k>4C_2/C_1$.

Now let us prove the first estimate in (2-23). If $y\in J_m\setminus J_{m+1}$ and $x\in\frac{1}{2}J^{(3)}_{k-1}$, then $0\le x-y\le|J_m|$. Using also that $J^{(2)}_m\subset J_m\setminus J_{m+1}$, by Proposition 2.1 we obtain

$$H(w\chi_{J_m\setminus J_{m+1}})(x)=\int_{J_m\setminus J_{m+1}}\frac{w(y)}{x-y}\,dy\ge\frac{w(J_m\setminus J_{m+1})}{|J_m|}\ge\frac{w(J^{(2)}_m)}{|J_m|}=\frac{2}{3}2^l.$$

Therefore,

$$B(x)>\tfrac{2}{3}(k-1)2^l.$$

Turn to the second part of (2-23). Let $J^{(3)}_{k-1}=[\alpha,b)$. Then, for all $x\in\frac{1}{2}J^{(3)}_{k-1}$,

$$\left|\int_{J^{(3)}_{k-1}}\frac{w(y)}{x-y}\,dy\right|=\frac{4\varepsilon}{(1+\varepsilon)}\frac{2^l}{p}\left|\log\left|\frac{x-\alpha}{x-b}\right|\right|\le4(\log3)\varepsilon\frac{2^l}{p}\le4(\log3)2^l.$$

It remains to estimate $|E(x)|$. Take the intervals $J^i=[a_i,b_i),\ i=0,\dots,l$, such that J^i carries w_{n-l+i} and

$$J=J^0\subset J^1\subset\cdots\subset J^l=[0,1).$$

We claim that, for every $i=1,\dots,l$ and for all x such that $0<x\le b_{i-1}-|J^{i-1}|/(4\cdot3^k)$,

$$|H(w\chi_{[b_{i-1},b_i)})(x)|\le13\cdot2^{l-i}. \tag{2-24}$$

Notice first that this claim immediately implies the desired estimate for $E(x)$. Indeed, let $x\in\frac{1}{2}J^{(3)}_{k-1}$. Then $0<x\le b-|J|/(4\cdot3^k)$, and hence (2-24) holds for $i=1$. But since $x\notin(J^i)_{k-1}$ for all $i=1,\dots,l$, we obviously obtain that $0<x\le b_{i-1}-|J^{i-1}|/(4\cdot3^k)$ for all $i\le l$. Therefore, by (2-24),

$$|H(w\chi_{[b,1)})(x)|\le\sum_{i=1}^{l}|H(w\chi_{[b_{i-1},b_i)})(x)|\le13\sum_{i=1}^{l}2^{l-i}\le13\cdot2^l.$$

It remains to prove the claim. Set $x_i = b_{i-1} - |J^{i-1}|/(4\cdot 3^k)$. Observe that $|H(w\chi_{[b_{i-1},b_i)})(x)|$ is an increasing function for $x < b_{i-1}$. Therefore, it suffices to prove that

$$|H(w\chi_{[b_{i-1},b_i)})(x_i)| \le 13\cdot 2^{l-i}. \tag{2-25}$$

There exists $0\le m\le k-2$ such that $J^{i-1} = (J^i_m)^{(2)}$. Then $[b_{i-1},b_i) = J^i_{m+1}$. Let $h = |J^i_{m+1}|$. Split the integral in (2-25) as follows:

$$\int_{b_{i-1}}^{b_i} \frac{w(y)}{y-x_i}\,dy = \int_{b_{i-1}}^{b_{i-1}+h/3} \frac{w(y)}{y-x_i}\,dy + \int_{b_{i-1}+h/3}^{b_i} \frac{w(y)}{y-x_i}\,dy.$$

Using that $w\equiv 2^{l-i}/p$ on $[b_{i-1}, b_{i-1}+h/3)$, we obtain

$$\int_{b_{i-1}}^{b_{i-1}+h/3} \frac{w(y)}{y-x_i}\,dy \le \frac{2^{l-i}}{p}\frac{h}{3}\frac{4\cdot 3^k}{h} \le 8\cdot 2^{l-i}.$$

Next, applying (2-9) yields

$$\int_{b_{i-1}+h/3}^{b_i} \frac{w(y)}{y-x_i}\,dy \le \frac{3}{h}\frac{3}{2}2^{l-i}|J^i_{m+1}| = \frac{9}{2}\cdot 2^{l-i},$$

which along with the previous estimate proves (2-25).

This proves the claim and so Proposition 2.5. Thus, Theorem 1.1 is completely proved.

Appendix

In this section, we will show how to prove (2-11). Let us show first that, for every interval $I\subset\mathbb{R}$, there exists an interval $J\in\mathcal{J}$ containing I and such that $|J|\le 6|I|$. Indeed, let $I = [a, a+h)$. Fix $j\in\mathbb{Z}$ such that $3^{j-1}\le h < 3^j$ and take $n\in\mathbb{Z}$ such that

$$3^j n \le a < 3^j(n+1).$$

Then $I\subset J = [3^j n, 3^j(n+2))$, and

$$\frac{|J|}{|I|} \le \frac{2\cdot 3^j}{3^{j-1}} = 6.$$

It follows immediately from this property that

$$Mf(x) \le 6M^{\mathcal{J}} f(x), \tag{A-1}$$

where

$$M^{\mathcal{J}} f(x) = \sup_{J\ni x,\, J\in\mathcal{J}} \frac{1}{|J|}\int_J |f|\,dy.$$

Next, it is easy to see that the intervals from $\mathcal{J}$ can be split into two disjoint triadic lattices, $\mathcal{J} = \mathcal{T}^1\cup\mathcal{T}^2$ (see Figure 3 for a geometric illustration of this fact).

Therefore, by (A-1),

$$\|M\|_{L^2(\sigma)\to L^2(\sigma)} \le 6(\|M^{\mathcal{T}^1}\|_{L^2(\sigma)\to L^2(\sigma)} + \|M^{\mathcal{T}^2}\|_{L^2(\sigma)\to L^2(\sigma)}). \tag{A-2}$$

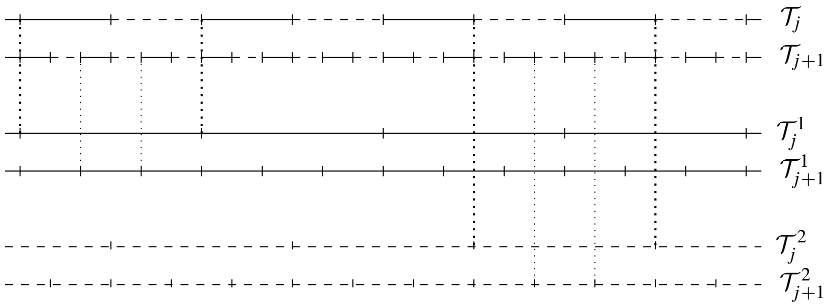

Figure 3. The lattices $\mathcal{T}$, $\mathcal{T}^1$ and $\mathcal{T}^2$ shown at two consecutive generations. The unions of solid and adjacent (from the right) dashed intervals from $\mathcal{T}_j$ form $\mathcal{T}_j^1$, and the unions of dashed and adjacent (from the right) solid intervals from $\mathcal{T}_j$ form $\mathcal{T}_j^2$. In turn, the unions of solid and dashed children from $\mathcal{T}_{j+1}$ form $\mathcal{T}_{j+1}^1$, and the unions of dashed and solid children from $\mathcal{T}_{j+1}$ form $\mathcal{T}_{j+1}^2$.

In order to estimate the right-hand side of (A-2), we invoke the following proposition.

Proposition A.1. *Let $\mathfrak{T}$ be a triadic lattice. Then*

$$\|M^{\mathfrak{T}}\|_{L^2(\sigma)\to L^2(\sigma)} \le 2 \sup_{R\in\mathfrak{T}} \left(\frac{1}{w(R)} \int_R (M^{\mathfrak{T}}(w\chi_R))^2\sigma\right)^{1/2}.$$

Remark A.2. For dyadic lattices this result can be found in [Moen 2009]. The proof there is closely related to the approach by E. Sawyer [1982] in his two-weighted characterization for the maximal operator. For triadic lattices the proof is essentially the same, and we give it for the sake of completeness.

Proof of Proposition A.1. Let $a > 1$. For $k \in \mathbb{Z}$ write the set $\Omega_k = \{M^{\mathfrak{T}} f > a^k\}$ as the union of pairwise disjoint triadic intervals I_j^k such that

$$\frac{1}{|I_j^k|} \int_{I_j^k} |f| > a^k.$$

Define $E_j^k = I_j^k \setminus \Omega_{k+1}$, and set $\alpha_{j,k} = (w(I_j^k)/|I_j^k|)^2 \sigma(E_j^k)$. We have

$$\begin{aligned} \|M^{\mathfrak{T}} f\|_{L^2(\sigma)}^2 &= \sum_{k\in\mathbb{Z}} \int_{\Omega_k\setminus\Omega_{k+1}} (M^{\mathfrak{T}} f)^2\sigma \le a^2 \sum_{k,j} \left(\frac{1}{|I_j^k|}\int_{I_j^k} |f|\right)^2 \sigma(E_j^k) \\ &= a^2 \sum_{k,j} \left(\frac{1}{w(I_j^k)} \int_{I_j^k} |f\sigma|w\right)^2 \alpha_{j,k}. \end{aligned} \tag{A-3}$$

Notice that for every $R \in \mathfrak{T}$,

$$\sum_{j,k:I_j^k\subset R} \alpha_{j,k} \le \int_R (M^{\mathfrak{T}}(w\chi_R))^2\sigma \le N^2 w(R), \tag{A-4}$$

where

$$N = \sup_{R \in \mathfrak{T}} \left(\frac{1}{w(R)} \int_J (M^{\mathfrak{T}}(w \chi_R))^2 \sigma \right)^{1/2}.$$

For $\lambda > 0$ set

$$E_\lambda = \left\{ (j,k) : \left(\frac{1}{w(I_j^k)} \int_{I_j^k} |f \sigma| w \right)^2 > \lambda \right\}.$$

Define the weighted maximal operator $M_w^{\mathfrak{T}}$ by

$$M_w^{\mathfrak{T}} f(x) = \sup_{J \ni x, J \in \mathfrak{T}} \frac{1}{w(J)} \int_J |f| w \, dy.$$

Writing the set $\{x : M_w^{\mathfrak{T}}(f\sigma)^2(x) > \lambda\}$ as the union of the maximal pairwise disjoint triadic intervals $\bigcup_i R_i$ and applying (A-4), we obtain

$$\sum_{(j,k) \in E_\lambda} \alpha_{j,k} \leq \sum_i \sum_{j,k : I_j^k \subset R_i} \alpha_{j,k} \leq N^2 w\{x : M_w^{\mathfrak{T}}(f\sigma)^2(x) > \lambda\}.$$

Therefore,

$$\begin{aligned} \sum_{k,j} \left(\frac{1}{w(I_j^k)} \int_{I_j^k} |f\sigma| w \right)^2 \alpha_{j,k} &= \int_0^\infty \left(\sum_{(j,k) \in E_\lambda} \alpha_{j,k} \right) d\lambda \\ &\leq N^2 \|M_w^{\mathfrak{T}}(f\sigma)\|_{L^2(w)}^2 \leq 4N^2 \|f\sigma\|_{L^2(w)}^2 = 4N^2 \|f\|_{L^2(\sigma)}^2, \end{aligned}$$

which, along with (A-3), completes the proof. □

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