

ANALYSIS & PDE

Volume 13

No. 8

2020

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We demonstrate a parametrix construction, together with associated pseudodifferential operator calculus, for an operator of sum-of-squares type with semiclassical parameter. The form of operator we consider includes the generator of kinetic Brownian motion on the cosphere bundle of a Riemannian manifold. Regularity estimates in semiclassical Sobolev spaces are proven by establishing mapping properties for the parametrix.

1. Introduction

We deal in this paper with a class of second order, subelliptic partial differential operators of the sum-of-squares form

$$P_h = X_0 - h \sum_{j=1}^d X_j^2 - h \sum_{j=1}^d c_j X_j, \quad h \in (0, 1], \quad (1-1)$$

where the X_j for $0 \leq j \leq d$ are smooth vector fields, the c_j are smooth functions, and $h > 0$ is considered as a semiclassical parameter. We work in $2d + 1$ dimensions, either on a compact manifold or an open subset of \mathbb{R}^{2d+1} , and make the following assumptions throughout this paper.

Assumption 1. • The collection of $2d + 1$ vectors $\{X_0, X_1, \dots, X_d, [X_0, X_1], \dots, [X_0, X_d]\}$ spans the tangent space at each base point.

• The collection $\{X_1, \dots, X_d\}$ is involutive (closed under commutation of vector fields).

For each $h > 0$ the operator P_h is subelliptic by a result of [Hörmander 1967], and by [Rothschild and Stein 1976] the operator P_h controls $\frac{2}{3}$ -derivatives in the Sobolev space sense. In the semiclassical setting it is natural to work with a semiclassical notion of Sobolev spaces; we refer to [Zworski 2012] for a treatment of semiclassical analysis. The question of interest in this paper is the dependence on h of the various constants in a priori inequalities for P_h , both in L^2 and semiclassical Sobolev spaces.

Our work is motivated by that of Alexis Drouot [2017], who studied such an operator on the cosphere bundle $S^*(M)$ of a $(d+1)$ -dimensional Riemannian manifold M . The paper [Drouot 2017] considers the operator $P_h = H + h\Delta_{\mathbb{S}}$, with H the generator of the Hamiltonian/geodesic flow and $\Delta_{\mathbb{S}}$ the nonnegative Laplace–Beltrami operator along the fibers of the cosphere bundle. In local coordinate charts this operator can be represented in the form (1-1), where $X_0 = H$, and $\{X_j\}_{j=1}^d$ is any local orthonormal frame for the

This material is based upon work supported by the National Science Foundation under Grant DMS-1500098.
MSC2010: primary 35H20; secondary 35S05.
Keywords: subelliptic equations, semiclassical analysis, resonance.

tangent space of the fibers of $S^*(M)$. In [Drouot 2017] it is shown that, if M is negatively curved, then as $h \rightarrow 0$ the eigenvalues of $-iP_h$ converge to the Pollicott–Ruelle resonances of M . The analogous result was proven in [Dyatlov and Zworski 2015] for $P_h = H + h\Delta$, where Δ is the Laplacian on $S^*(M)$. The interest in taking $P_h = H + h\Delta_{\mathbb{S}}$ is that this operator generates what is known as kinetic Brownian motion on M . For a treatment of this process we refer to [Franchi and Le Jan 2007; Grothaus and Stilgenbauer 2013; Angst, Bailleul, and Tardif 2015; Li 2016].

A key step in the proof of convergence in [Drouot 2017] was controlling the subelliptic estimates for P_h as $h \rightarrow 0$. We emphasize that the estimates we prove are the same as in that paper, with an occasional improvement in the remainder terms. The aim here is to obtain a finer microlocal understanding of the parametrix. We obtain a parametrix valid on the region $h\Delta \geq 1$, strictly larger than the semiclassical region $h^2\Delta \geq 1$. The restriction $h\Delta \geq 1$ arises from the largest region of phase space on which the uncertainty principle holds for the parametrix. The estimates in [Drouot 2017] were obtained through commutator methods, analogous to the work of [Hörmander 1967]. Our approach is more similar to that of [Rothschild and Stein 1976], in that we use an approximation to the operator at each point by a model nilpotent Lie group, and construct a parametrix from the inverse of the model operator on that group. Estimates are then obtained from mapping properties for the parametrix. In contrast to [Rothschild and Stein 1976], which lifted the operator to a higher-dimensional Lie group on which the parametrix is represented as a singular integral kernel, we construct the parametrix in pseudodifferential form on the space itself. This procedure is motivated by the author's work [Smith 1994] on the $\bar{\partial}_b$ problem on three-dimensional CR manifolds of finite type.

When constructing a parametrix for P_h of the form (1-1), it is more natural from the semiclassical viewpoint to consider $hP_h = hX_0 + \sum_{j=1}^d (hX_j)^2$, and quantize symbols in terms of $h\eta$. This leads to placing an extra factor of h on the variables η'' dual to X_j for $d+1 \leq j \leq 2d$, since $[hX_0, hX_j] \sim h^2X_{j+d}$. The quantization of symbols is naturally carried out using exponential coordinates with respect to an extension of $\{X_j\}_{j=0}^d$ to a frame $\{X_j\}_{j=0}^{2d}$. We will require that:

Assumption 2. If $1 \leq i \leq d$, then $[X_0, X_i] - 2X_{i+d} \in \text{span}(X_0, \dots, X_d)$.

This can of course be arranged by setting $X_{i+d} = 2[X_0, X_i]$. In the model nilpotent Lie group setting where all other commutators vanish, there is a natural nonisotropic dilation structure using powers (2, 1, 3). Precisely, we split $\eta \in \mathbb{R}^{2d+1} = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ into (η_0, η', η'') , and similarly use X' as abbreviation for the collection (X_1, \dots, X_d) , and $X'' = (X_{d+1}, \dots, X_{2d})$. Then the dilation that respects the fundamental solution for the model operator is

$$\delta_r(\eta) = (r^2\eta_0, r\eta', r^3\eta'').$$

We now summarize the main result of this paper, leaving details to be expanded upon in later sections. For simplicity consider an open set $U \subset \mathbb{R}^{2d+1}$. For a multi-index $\alpha \in \mathbb{N}^{2d+1}$, let

$$\text{order}(\alpha) = 2\alpha_0 + |\alpha'| + 3|\alpha''|.$$

We use $\exp_x(y)$ to denote the time-1 flow of x along $\sum_{j=0}^{2d} y_j X_j$.

Proposition 3. *Given $\rho(x) \in C_c^\infty(U)$, there is $\chi_0 \in C_c^\infty(\mathbb{R}^{2d+1})$ and an h -dependent family of symbols $a(x, \eta)$ satisfying*

$$|\partial_x^\beta \partial_\eta^\alpha a(x, \eta)| \leq C_{\alpha, \beta} h (h^{\frac{1}{2}} + |\eta_0|^{\frac{1}{2}} + |\eta'| + |\eta''|^{\frac{1}{3}})^{-2 - \text{order}(\alpha)},$$

with $C_{\alpha, \beta}$ independent of $h \in (0, 1]$, so that the operator $a_h(x, hD)$ defined by

$$a_h(x, hD) = \frac{1}{(2\pi)^{2d+1}} \int e^{-i\langle y, \eta \rangle} a(x, h\eta_0, h\eta', h^2\eta'') f(\exp_x(y)) \chi_0(y) dy d\eta$$

satisfies

$$a_h(x, hD) \circ P_h = \rho(x) + r_h(x, hD),$$

where $r_h(x, hD)$ is an operator that satisfies the following with C_{p_1, p_2} independent of $h \in (0, 1]$, for any polynomials $p_j(\eta)$ on \mathbb{R}^{2d+1} :

$$\|p_1(X_0, h^{\frac{1}{2}}X', h^{\frac{1}{2}}X'') \circ r_h(x, hD) \circ p_2(X_0, h^{\frac{1}{2}}X', h^{\frac{1}{2}}X'') f\|_{L^2} \leq C_{p_1, p_2} \|f\|_{L^2}.$$

For example, one can take p_1 or p_2 to yield the operator $(1 + X_0^* X_0)^{N_1} (1 + h\Delta)^{N_2}$, where Δ is the Laplacian on \mathbb{R}^{2d+1} . These bounds roughly say that the parametrix inverts P_h on the region

$$\{\Delta \geq h^{-1}\} \cup \{|X_0| \geq 1\}.$$

In particular, the remainder term r_h will be of order h^∞ if the solution is localized to a region where $\Delta \geq h^{-1-\epsilon}$ for some $\epsilon > 0$.

We remark that in the calculus developed here P_h is of order 2, and thus distinct from the standard semiclassical calculus where hP_h is of order 2. This is related to the fact that we are working on the region $|\eta| \geq h^{1/2}$ rather than $|\eta| \geq 1$. Symbols of order j are weighted by a factor $h^{-j/2}$ to ensure that symbols of negative order (but not necessarily their derivatives) remain bounded as $h \rightarrow 0$. With this accounting, X_0 is an operator of order 2, $h^{1/2}X_j$ is of order 1 for $1 \leq j \leq d$, and $h^{1/2}X_j$ is of order 3 for $d + 1 \leq j \leq 2d$.

Together with the composition calculus, pseudolocality arguments, and L^2 mapping bounds for operators, we deduce the regularity results on $S^*(M)$ for P_h that were established in [Drouot 2017]. These are stated in Theorems 20 and 21.

The outline of this paper is as follows. In Section 2 we introduce a model operator of P_h on a step-2 nilpotent group, and discuss the homogeneous fundamental solution in this setting. In Section 3 we study the degree to which the model operator, attached to M by exponential coordinates, approximates P_h . Careful estimates of the Taylor expansion of vector fields and exponential coordinates are needed to obtain uniform estimates as $h \rightarrow 0$. In Section 4 we prove that operators of the form $a_h(x, hD)$ form an algebra under composition. This allows for the construction of parametrices from the inverse for the model operator on the nilpotent Lie group. In Section 5 we establish L^2 boundedness of order-0 operators in local coordinates, using a nonisotropic Littlewood–Paley decomposition of the operator and the Cotlar–Stein lemma. Finally, in Section 6 we establish the main regularity estimates for P_h in h -Sobolev spaces, leading to the proof of the bounds in [Drouot 2017].

2. Operators on model domains

In this section we consider a nilpotent Lie group structure on \mathbb{R}^{2d+1} that captures the commutation relations of the vector fields X_j , and we introduce a left-invariant model of P_h . The top-order term in the parametrix for P_h at a point $x \in U$ will be given by the fundamental solution for the model operator, attached to U via exponential coordinates at x relative to the frame $\{X_j\}_{j=0}^{2d}$. In subsequent sections we show that the model operator agrees to leading order with the expression of P_h in exponential coordinates, and develop a graded pseudodifferential calculus that allows us to produce a parametrix for P_h modulo a smoothing operator. We start by considering $h = 1$, and then obtain the fundamental solution for all h by a suitable dilation.

We use the variables $y = (y_0, y', y'') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, with dual variables $\eta = (\eta_0, \eta', \eta'')$, and introduce the dilation structure

$$\delta_r(\eta) = (r^2\eta_0, r\eta', r^3\eta''), \quad \delta_{r^{-1}}(y) = (r^{-2}y_0, r^{-1}y', r^{-3}y'').$$

We also introduce a corresponding nonisotropic homogeneous weight $m \in C^\infty(\mathbb{R}^{2d+1} \setminus \{0\})$,

$$m(\eta) = (|\eta_0|^6 + |\eta'|^{12} + |\eta''|^4)^{\frac{1}{12}},$$

so that $m(\delta_r(\eta)) = rm(\eta)$, and $3^{-12/5} \leq m(\eta) \leq 1$ when $|\eta| = 1$.

Consider the frame of vector fields on \mathbb{R}^{2d+1} given by

- $Y_0 = \partial_0 - \sum_{j=1}^d y_j \partial_{j+d}$,
- $Y_j = \partial_j + y_0 \partial_{j+d}$ for $1 \leq j \leq d$,
- $Y_j = \partial_j$ for $j \geq d + 1$,

and observe that

$$[Y_0, Y_j] = 2Y_{j+d} \quad \text{if } 1 \leq j \leq d,$$

with all other commutators equal to 0. The collection $\{Y_j\}_{j=0}^{2d}$ forms a nilpotent (step-2) Lie algebra. These are left-invariant vector fields associated to the nilpotent Lie group structure on \mathbb{R}^{2d+1} with product

$$y \times w = (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y').$$

The exponential map at base point y , and corresponding exponential coordinates, are given by

$$\begin{aligned} \overline{\text{exp}}_y(w) &= (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y'), \\ \overline{\Theta}_y(z) &= (z_0 - y_0, z' - y', z'' - y'' - y_0 z' + z_0 y'), \end{aligned} \tag{2-1}$$

so in particular $\overline{\Theta}_0(w) = w$.

The vector field Y_0 is homogeneous of order 2 under δ_r in that

$$Y_0(f \circ r^{-1}) = r^{-2}(Y_0 f) \circ \delta_{r^{-1}},$$

which we summarize by writing $\text{order}(Y_0) = 2$. Similarly, $\text{order}(Y_j) = 1$ for $1 \leq j \leq d$, and $\text{order}(Y_j) = 3$ for $d + 1 \leq j \leq 2d$. More generally, if we define the order of a multi-index α by

$$\text{order}(\alpha) = 2\alpha_0 + \alpha_1 + \dots + \alpha_d + 3\alpha_{d+1} + \dots + 3\alpha_{2d} = 2\alpha_0 + |\alpha'| + 3|\alpha''|,$$

then the monomial differential operator $y^\beta \partial_y^\alpha$ will be homogeneous, with order given by

$$\text{order}(y^\beta \partial_y^\alpha) = \text{order}(\alpha) - \text{order}(\beta). \tag{2-2}$$

The left-invariant differential operator $Y_0 - \sum_{j=1}^d Y_j^2$ is subelliptic and homogeneous of order 2. By [Folland 1975], it has a unique homogeneous fundamental solution $K(y) \in C^\infty(\mathbb{R}^{2d+1} \setminus \{0\})$,

$$\left(Y_0 - \sum_{j=1}^d Y_j^2 \right) K(y) = \delta(y), \quad K(\delta_{r^{-1}}(y)) = r^{(2+4d)-2} K(y).$$

The homogeneous inverse for $Y_0 - \sum_{j=1}^d Y_j^2$ is given by convolution with K , which we will express in pseudodifferential form. Precisely, if we let $q_0(\eta) = \widehat{K}$, then $q_0(\delta_r \eta) = r^{-2} q_0(\eta)$, and the operator

$$q_0(D) f(y) = \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{2d+1}} e^{-i\langle \bar{\Theta}_y(z), \eta \rangle} q_0(\eta) f(z) dz d\eta$$

is a left and right inverse for $Y_0 - \sum_{j=1}^d Y_j^2$ on the space of Schwartz functions.

To conclude this section we consider the semiclassical subelliptic operator $hY_0 - \sum_{j=1}^d h^2 Y_j^2$. This is naturally associated to dilating y_0 and y' by h , and y'' by h^2 , in that

$$\left(Y_0 - \sum_{j=1}^d Y_j^2 \right) (f(hy_0, hy', h^2 y'')) = \left(hY_0 - \sum_{j=1}^d h^2 Y_j^2 \right) (f(hy_0, hy', h^2 y'')).$$

Consequently, if we introduce the operation on symbols

$$a_h(\eta) = a(\eta_0, \eta', h\eta''),$$

then the inverse for $hY_0 - \sum_{j=1}^d h^2 Y_j^2$ is given by the semiclassical quantization of q_h ,

$$\begin{aligned} q_{0,h}(hD) f(y) &= \frac{1}{(2\pi h)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle \bar{\Theta}_y(z), \xi \rangle / h} q_{0,h}(\eta) f(z) dz d\xi \\ &= \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle z, \xi \rangle} q_{0,h}(\eta) f(\bar{\text{exp}}_y(hz)) dz d\xi. \end{aligned}$$

3. Approximation by the model domain

Recall that we consider a spanning collection $\{X_0, X_1, \dots, X_{2d}\}$ of vector fields on an open subset U of \mathbb{R}^{2d+1} satisfying the following conditions:

- The collection $\{X_1, \dots, X_d\}$ is involutive (closed under commutation of vector fields).
- If $1 \leq i \leq d$, then $[X_0, X_i] - 2X_{i+d} \in \text{span}(X_0, \dots, X_d)$.

We will use x, \tilde{x} to denote variables in U and y, z to denote variables in \mathbb{R}^{2d+1} .

Let $\text{exp}_x(y)$ be exponential coordinates with base point x in the frame $\{X_0, \dots, X_{2d}\}$. That is, $\text{exp}_x(y) = \gamma(1)$, where $\gamma(0) = x$ and $\gamma'(t) = \sum_{j=0}^{2d} y_j X_j(\gamma(t))$. Define exponential coordinates Θ_x as the local inverse of exp_x in a neighborhood of x :

$$\Theta_x(\text{exp}_x(y)) = y, \quad \text{exp}_x(\Theta_x(\tilde{x})) = \tilde{x}.$$

Recall the definition (2-2) of the order of a monomial differential operator in y . Consistent with this we have $\text{order}(Y_0) = 2$, $\text{order}(Y_j) = 1$ if $1 \leq j \leq d$, and $\text{order}(Y_j) = 3$ if $d + 1 \leq j \leq 2d$.

Lemma 4. For $0 \leq j \leq 2d$, we can write

$$(X_j f)(\exp_x(y)) = Y_j(f(\exp_x(y))) + R_j(x, y, \partial_y)f(\exp_x(y)),$$

where $\text{order}(R_j) < \text{order}(Y_j)$, in the sense that the Taylor expansion

$$R_j(x, y, \partial_y) = \sum_{\alpha, k} c_{j, \alpha, k}(x) y^\alpha \partial_k$$

includes only terms with $\text{order}(y^\alpha \partial_k) < \text{order}(Y_j)$.

Additionally, $c_{0, \alpha, k}(x) \equiv 0$ unless there is at least one factor of y_j with $j \geq 1$ occurring in y^α .

Proof. Any term $y^\alpha \partial_k$ with $|\alpha| > 2$ is of order ≤ 0 , so we need examine the Taylor expansion of X_j in exponential coordinates only to second power in y . Additionally, $\text{order}(y_i y_j \partial_k) \leq 1$, and equals 1 only if $1 \leq i, j \leq d$ and $k \geq d + 1$. To see that such a term cannot arise in R_j for $1 \leq j \leq d$, the only case where $\text{order}(Y_j) \leq 1$, we use involutivity of $\{X_1, \dots, X_d\}$ and the Frobenius theorem to see that this collection remains tangent to the flowout of the subspace $y_0 = y'' = 0$, and hence we can write $X_j = \sum_{k=1}^d c_k(x, y) \partial_k$ if $y_0 = y'' = 0$ and $1 \leq j \leq d$.

Thus, we need show that in the expansion of R_j about $y = 0$ the terms linear in y are of order strictly less than $\text{order}(Y_j)$. For $j \geq d + 1$ this is immediate, since $X_j = \partial_j = Y_j$ at $y = 0$, and any vector field that vanishes at 0 includes terms of order at most 2. For $0 \leq j \leq d$ we expand

$$X_j = \partial_j + \sum_{i, k} c_{jik}(x) y_i + (y^2) \partial_y.$$

Since radial lines in exponential coordinates are integral curves of $\sum_{j=0}^{2d} y_j X_j$, we have

$$\sum_{j=0}^{2d} y_j X_j = \sum_{j=0}^{2d} y_j \partial_j, \tag{3-1}$$

from which we deduce

$$c_{ijk} = -c_{jik}.$$

Also, since $[X_0, X_j] - 2X_{j+d} \in \text{span}(X_0, \dots, X_d)$, we deduce for $j = 1, \dots, d$ that

$$c_{j0k} = \begin{cases} 1, & k = j + d, \\ 0, & k > d \text{ and } k \neq j + d. \end{cases}$$

Since $\text{order}(y_i \partial_k) < 2$ unless $k > d$, we deduce $\text{order}(R_0(x, y, \partial_y)) < 2$.

By involutivity of $\{X_1, \dots, X_d\}$, if $1 \leq i, j \leq d$ then $c_{jik} = 0$ unless also $1 \leq k \leq d$, in which case $\text{order}(y_i \partial_k) = 0$. And if $i > d$ then $\text{order}(y_i \partial_k) \leq 0$ for all k . So if $1 \leq j \leq d$ then all terms $c_{jik} y_i \partial_k$ for $i \neq 0$ have $\text{order} \leq 0$, and since $c_{j0k} = \delta_{k, j+d}$ we conclude $\text{order}(R_j(x, y, \partial_y)) \leq 0$ if $1 \leq j \leq d$.

To conclude the lemma, we note by (3-1) that if $y' = y'' = 0$ then $X_0 = \partial_{y_0}$, from which we obtain $R_0 \equiv 0$ along $y' = y'' = 0$. □

For $x \in U$, and y, z in a neighborhood of 0 in \mathbb{R}^{2d+1} , we introduce the functions

$$\Theta(x, y, z) = \Theta_{\exp_x(y)}(\exp_x(z)), \quad \tilde{\Theta}(x, y, w) = \Theta_x(\exp_{\exp_x(y)}(w)), \tag{3-2}$$

where we recall $\Theta_x(\tilde{x})$ denotes exponential coordinates in X_j centered at x . For fixed x and y these are inverse functions of each other on their domains:

$$z = \tilde{\Theta}(x, y, w) \iff w = \Theta(x, y, z).$$

To invert in the y -variable we note that $v = \tilde{\Theta}(x, y, w)$ implies $y = \tilde{\Theta}(x, v, -w)$.

Observe that $\Theta(x, y, z) = -\Theta(x, z, y)$, and $\Theta(x, y, z) = z - y + \mathcal{O}(y, z)^2$. For more precise estimates on Θ and $\tilde{\Theta}$ we consider their Taylor expansions in exponential coordinates at x . We first assign a notion of order to a smooth function $f(x, y, z)$. Consistent with (2-2), we make the following definition.

Definition 5. For a smooth function $f(x, y, z)$ defined on an open subset of $U \times \mathbb{R}^{2d+1} \times \mathbb{R}^{2d+1}$ containing $U \times \{0, 0\}$, we say that $\text{order}(f) < -j$ if for all $x \in U$

$$(\partial_y^\alpha \partial_z^\beta f)(x, 0, 0) = 0 \quad \text{for all } \alpha, \beta : \text{order}(\alpha + \beta) \leq j.$$

Equivalently, the Taylor expansion of f in y, z about $y = z = 0$ contains only monomials $y^\alpha z^\beta$ with $\text{order}(\alpha + \beta) > j$. We let $\text{order}(f)$ be the least $n \in \mathbb{Z}$ such that $\text{order}(f) < n + 1$.

Recalling the definition (2-1) of $\bar{\Theta}_y(z)$, which are exponential coordinates in the frame Y_j on the model domain, we have the following.

Lemma 6. We have $\Theta(x, y, z) = \bar{\Theta}_y(z) + R(x, y, z)$, where $\text{order}(R_j) < \text{order}(y_j)$ for each j . Similarly, $\tilde{\Theta}(x, y, w) = \bar{\Theta}_{-y}(w) + \tilde{R}(x, y, w)$, where $\text{order}(\tilde{R}_j) < \text{order}(y_j)$ for each j .

Proof. We work in exponential coordinates $y = \Theta_x(\cdot)$ centered at x , and use Lemma 4 to consider X_j as a vector field in y . Then $z = \tilde{\Theta}(x, y, w)$ means that $z = \gamma(1)$, where $\gamma(t)$ is the integral curve of $w \cdot X \equiv \sum w_k X_k$ with $\gamma(0) = y$. Taking the Taylor expansion of $\gamma(t)$ about $t = 0$ and evaluating at $t = 1$ gives the following expansion of $z = \tilde{\Theta}(x, y, w)$ in terms of w :

$$z_j = y_j + (w \cdot X)_j(y) + \sum_{k=1}^{\infty} \frac{1}{(k+1)!} (w \cdot X)^k (w \cdot X)_j(y), \tag{3-3}$$

where $(w \cdot X)(y, \partial_y)$ acts on y and $(w \cdot X)_j(y)$ is its ∂_j coefficient as a function of y . It is seen from Lemma 4 that $w \cdot X$ does not increase the order of a function $f(x, y, w)$, and $w \cdot X - w \cdot Y$ decreases the order of $f(x, y, w)$ by at least 1. Also, as functions of (y, w)

$$\text{order}((w \cdot Y)_j(y)) = \text{order}(y_j), \quad \text{order}((w \cdot X)_j(y) - (w \cdot Y)_j(y)) < \text{order}(y_j).$$

Thus, if we replace $w \cdot X$ by $w \cdot Y$ in the expansion (3-3) then the right-hand side is changed by terms of strictly lower order than y_j . It follows that we can write

$$(z_0, z', z'') = (y_0 + w_0, y' + w', y'' + w'' + y_0 w' - w_0 y') + (\tilde{R}_0, \tilde{R}', \tilde{R}''), \tag{3-4}$$

where $\text{order}(\tilde{R}_0) < -2$, $\text{order}(\tilde{R}') < -1$, and $\text{order}(\tilde{R}'') < -3$ as functions of (y, w) . Recalling the formula (2-1), this completes the second statement of the lemma.

We next invert the map $w \rightarrow z$ to express $w = w(y, z) = \Theta(x, y, z)$, and use (3-4) to write

$$w(y, z) = \bar{\Theta}_y(z) - (\tilde{R}_0, \tilde{R}', -y_0\tilde{R}' + y'\tilde{R}_0 + \tilde{R}'') \equiv \bar{\Theta}_y(z) - R(x, y, z),$$

where $\tilde{R} = \tilde{R}(y, w(y, z))$. Since w is equal to $z - y$ plus quadratic terms in (y, z) , we see that $R(x, y, z)$ has no linear terms in y or z , and hence $\text{order}(w_0) \leq -2$ and $\text{order}(w') \leq -1$, since quadratic terms are of order at most -2 . This also shows that $\text{order}(-y_0\tilde{R}' + y'\tilde{R}_0) \leq -3$.

To conclude the lemma it suffices to show that $\text{order}(\tilde{R}''(y, z)) \leq -3$, since together with the preceding this shows that $\text{order}(w_j(y, z)) \leq \text{order}(y_j)$ for all j , from which it follows that $\text{order}(\tilde{R}_j(y, w(y, z))) \leq \text{order}(\tilde{R}_j(y, w)) < \text{order}(y_j)$. We know that $\text{order}(w''(y, z)) \leq -2$ since quadratic terms are order ≤ -2 , and by the above that $\text{order}(w_j(y, z)) \leq \text{order}(y_j)$ for $j \leq d$. Since $\text{order}(\tilde{R}'') < -3$ as a function of (y, w) , it is easy to see by examining terms in (y, w) of order ≤ -4 that $\text{order}(\tilde{R}'') \leq -3$ as a function of (y, z) , concluding the proof. \square

We make a few important additional observations about the terms that can occur in the Taylor expansion of $\Theta(x, y, z)$ about $y = z = 0$ and $\tilde{\Theta}(x, y, w)$ about $y = w = 0$. First, we have

$$\begin{aligned} \Theta(x, y, z) &= z_0 - y_0 && \text{if } y' = z' = y'' = z'' = 0, \\ \tilde{\Theta}(x, y, w) &= y_0 + w_0 && \text{if } y' = w' = y'' = w'' = 0. \end{aligned}$$

Consequently, every nonvanishing term in the Taylor expansion of $R(x, y, z)$ about $y = z = 0$ must include a factor of either y', z', y'' , or z'' . Similarly, every nonvanishing term in the Taylor expansion of $\tilde{R}(x, y, w)$ about $y = w = 0$ must include a factor of either y', w', y'' , or w'' .

Additionally, since the collection $\{X_j\}_{j=1}^d$ is involutive it follows that R_0 and R'' vanish if $y_0 = z_0 = y'' = z'' = 0$, and hence every nonvanishing term in the Taylor expansions of R_0 and R'' must contain a factor other than (y', z') . Similarly \tilde{R}_0 and \tilde{R}'' must each contain a factor other than (y', w') . Combining this with the fact that $R(x, y, z) = 0$ if $z = y$, we can write

$$R_j(x, y, z) = \sum_{\substack{|\alpha|+|\beta|=2 \\ |\beta|\geq 1}} c_{j,\alpha,\beta}(x)y^\alpha(z-y)^\beta + \sum_{\substack{|\alpha|+|\beta|=3 \\ |\beta|\geq 1}} c_{j,\alpha,\beta}(x, y, z)y^\alpha(z-y)^\beta \tag{3-5}$$

for smooth functions $c_{j,\alpha,\beta}$, where $c_{j,\alpha,\beta} \equiv 0$ unless $\text{order}(y^\alpha z^\beta) < \text{order}(y_j)$, and also unless one of $\alpha', \beta', \alpha''$, or β'' is nonzero. Additionally, if $j = 0$ or $j \geq d + 1$ then $c_{j,\alpha,\beta} \equiv 0$ unless one of $\alpha_0, \beta_0, \alpha''$, or β'' is nonzero.

The same conditions also hold on $\tilde{c}_{j,\alpha,\beta}$ in the following expansion of $\tilde{R}(x, y, w)$:

$$\tilde{R}_j(x, y, w) = \sum_{\substack{|\alpha|+|\beta|=2 \\ |\beta|\geq 1}} \tilde{c}_{j,\alpha,\beta}(x)y^\alpha w^\beta + \sum_{\substack{|\alpha|+|\beta|=3 \\ |\beta|\geq 1}} \tilde{c}_{j,\alpha,\beta}(x, y, w)y^\alpha w^\beta.$$

4. The semiclassical calculus on U

In this section we introduce the nonisotropic semiclassical quantization and h -dependent symbol classes that we use to construct the parametrix for P_h . As seen for the model operator, the phase variables associated to X'' need to be scaled by h^2 , as opposed to the h -scaling for variables associated to X_0

and X' . The symbol classes are naturally associated to the nonisotropic dilation structure δ_r . We will define them using the nonisotropic norm

$$m(\eta) = (|\eta_0|^6 + |\eta'|^{12} + |\eta''|^4)^{\frac{1}{12}} \approx |\eta_0|^{\frac{1}{2}} + |\eta'| + |\eta''|^{\frac{1}{3}},$$

which is smooth for $\eta \neq 0$ and homogeneous of degree 1, in that $m(\delta_r(\eta)) = rm(\eta)$.

We assume that K is compactly contained in U , and choose r_1 so that the exponential map $y \rightarrow \exp_x(y)$ is a diffeomorphism on the ball $\{|y| \leq r_1\}$ for all $x \in K$. We also fix $r_0 < r_1$ such that

$$\bigcup_{\tilde{x} \in \exp_x(\bar{B}_{r_0})} \exp_{\tilde{x}}(\bar{B}_{r_0}) \subset \exp_x(B_{r_1}).$$

We fix functions $\chi_j \in C_c^\infty(B_{r_j})$ with $\chi_0(y) = 1$ for $|y| \leq \frac{1}{2}r_0$ and

$$\chi_1(\Theta_x(\cdot)) = 1 \text{ on a neighborhood of } \bigcup_{\tilde{x} \in \exp_x(\bar{B}_{r_0})} \exp_{\tilde{x}}(\bar{B}_{r_0}).$$

Given a symbol $a(x, \eta) \in C^\infty(U \times \mathbb{R}^{2d+1})$ supported where $x \in K$, we let

$$a_h(x, \eta) = a(x, \eta_0, \eta', h\eta''),$$

and define a nonisotropic semiclassical quantization of a by the rule

$$\begin{aligned} a_h(x, hD)f(x) &= \frac{1}{(2\pi h)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle y, \eta \rangle / h} a_h(x, \eta) \chi_0(y) f(\exp_x(y)) dy d\eta \\ &= \frac{1}{(2\pi)^{2d+1}} \int_{\mathbb{R}^{4d+2}} e^{-i\langle y, \eta \rangle} a_h(x, \eta) \chi_0(hy) f(\exp_x(hy)) dy d\eta. \end{aligned} \tag{4-1}$$

Thus the Schwartz kernel of $a_h(x, hD)$ is supported in $K \times K_{r_0}$, where K_{r_0} is the image of $K \times \bar{B}_{r_0}$ under $(x, y) \rightarrow \exp_x(y)$. In contrast to the usual semiclassical scaling $\eta \rightarrow h\eta$, the nonisotropic scaling $(h\eta_0, h\eta', h^2\eta'')$ arises from the missing directions X'' being obtained from commutators of X_0 and X' .

If $p(x, \eta) = \sum_{|\alpha| \leq n} c_\alpha(x) \eta^\alpha$ is a polynomial in η , then

$$(p_h(x, hD)f)(x) = p_h(x, (-i\partial_y))f(\exp_x(hy))\Big|_{y=0}.$$

In particular, we have the following correspondence of symbols to operators:

$$i\eta_j : hX_j, \quad 0 \leq j \leq d, \quad i\eta_j : h^2X_j, \quad d+1 \leq j \leq 2d. \tag{4-2}$$

Suppose that the symbol a satisfies homogeneous order-0 type estimates of the form

$$|\partial_x^\beta \partial_\eta^\alpha a(x, \eta)| \leq C_{\alpha, \beta} m(\eta)^{-\text{order}(\alpha)}.$$

The uncertainty principle, needed for example for proving L^2 continuity of $a_h(x, hD)$, requires uniform bounds on $\partial_x^\alpha (h\partial_\eta)^\alpha a_h(x, \eta)$. On the other hand,

$$\begin{aligned} |\partial_x^\alpha (h\partial_\eta)^\alpha a_h(x, \eta)| &= h^{|\alpha_0| + |\alpha'| + 2|\alpha''|} |(\partial_x^\alpha \partial_\eta^\alpha a)_h(x, \eta)| \\ &\leq C_\alpha h^{|\alpha_0| + |\alpha'| + 2|\alpha''|} m(\eta_0, \eta', h\eta'')^{-2|\alpha_0| - |\alpha'| - 3|\alpha''|}. \end{aligned}$$

To have uniform bounds as $h \rightarrow 0$ for every α would require truncating $a(x, \eta)$ to where $m(\eta) \geq h^{1/2}$. It is convenient to work with bounded symbols; hence for symbols of order n we will multiply by a factor of $h^{-n/2}$ to ensure that symbols of any order be of size $\lesssim 1$ when $m(\eta) \leq h^{1/2}$.

Definition 7. Let $m(h, \eta) = (h^{1/2} + m(\eta))$. A h -dependent family of symbols $a(x, \eta)$ belongs to $S^n(m)$ if, for all α, β , there is $C_{\alpha,\beta}$ independent of h such that, for $0 < h \leq 1$,

$$|\partial_x^\beta \partial_\eta^\alpha a(x, \eta)| \leq C_{\alpha,\beta} h^{-\frac{n}{2}} m(h, \eta)^{n - \text{order}(\alpha)}.$$

We let $\Psi_h^n(m)$ denote the collection of operators $a_h(x, hD)$ as in (4-1) with $a \in S_h^n(m)$.

We also define $S^{-\infty}(m) = \bigcap_{k \in \mathbb{N}} S^{-k}(m)$, and let $\Psi_h^{-\infty}(m)$ denote operators that can be written in the form (4-1) with χ_0 replaced by χ_1 , and $a \in S^{-\infty}(m)$.

Remark 8. We define $\Psi_h^{-\infty}(m)$ using χ_1 in the quantization rule (4-1) since the composition of operators defined using χ_0 need not have Schwartz kernel supported inside B_{r_0} (in local exponential coordinates). We also note that results below concerning continuity and composition of symbols are independent of the particular choice of χ_0 . We show in Lemma 13 that replacing χ_0 by another function in $C_c^\infty(B_{r_0})$ that equals 1 on $B_{r_0/2}$ changes $a_h(x, hD)$ by a term in $\Psi_h^{-\infty}(m)$.

For polynomial symbols we note that

$$h^{-\frac{1}{2} \text{order}(\alpha)} \eta^\alpha \in S^{\text{order}(\alpha)}(m). \tag{4-3}$$

By (4-2) we then have the following examples, which will show that $P_h \in \Psi_h^2(m)$:

$$\begin{aligned} X_0 &\in \Psi_h^2(m), \\ h^{\frac{1}{2}} X_j &\in \Psi_h^1(m), \quad 1 \leq j \leq d, \\ h^{\frac{1}{2}} X_j &\in \Psi_h^3(m), \quad d + 1 \leq j \leq 2d. \end{aligned} \tag{4-4}$$

A more general example of a symbol in $S^n(m)$ is $h^{-n/2} a(\eta)(1 - \phi(h^{-1/2} m(\eta)))$, where $\phi \in C_c^\infty(\mathbb{R}^{2d+1})$ equals 1 on a neighborhood of 0, and $a \in C^\infty(\mathbb{R}^{2d+1} \setminus \{0\})$ satisfies $a(\delta_r \eta) = r^n a(\eta)$.

It is easy to verify the following properties:

$$\begin{aligned} S^n(m) \cdot S^{n'}(m) &\subset S^{n+n'}(m), \\ S^n(m) \supset S^{n'}(m) &\quad \text{if } n' < n, \\ a \in S^n(m) &\implies h^{\frac{1}{2} \text{order}(\alpha)} \partial_\eta^\alpha \partial_x^\beta a \in S^{n - \text{order}(\alpha)}. \end{aligned} \tag{4-5}$$

Definition 9. Given a sequence of symbols $a_j \in S^{n-j}(m)$ we say that $a \sim \sum_j a_j$ if for all N

$$a - \sum_{j=0}^{N-1} a_j \in S^{n-N}(m).$$

Consequently, a is uniquely determined up to a symbol in $S^{-\infty}(m)$.

We note the following simple example of a symbol in $S^{-\infty}(m)$:

$$\text{If } \phi \in \mathcal{S}(\mathbb{R}) \text{ and } \phi(s) = 1 \text{ when } |s| \leq 1 \text{ then } \phi(h^{-\frac{1}{2}}m(\eta)) \in S^{-\infty}(m). \tag{4-6}$$

That this symbol belongs to $S^{-\infty}(m)$ is seen by noting that

$$(1 + h^{-\frac{1}{2}}m(\eta))^{-N} = h^{\frac{N}{2}}m(h, \eta)^{-N},$$

together with the bounds $|\partial_\eta^\alpha m(\eta)| \leq C_\alpha m(\eta)^{1-\text{order}(\alpha)}$, where we use that all derivatives vanish unless $m(\eta) \geq h^{1/2}$; hence $m(\eta) \approx m(h, \eta)$, since ϕ is assumed constant near 0.

Lemma 10. *Suppose that $a_j \in S^{n-j}(m)$, $j \in \mathbb{N}$. Then there exists $a \in S^n(m)$ with $a \sim \sum_j a_j$.*

Proof. Fix $\phi \in C_c^\infty((-2, 2))$ with $\phi = 1$ for $|s| < 1$. We will construct a sequence of real numbers $R_j \geq 1$ with $R_j \rightarrow \infty$ such that for all N

$$\sum_{j=N}^\infty (1 - \phi(R_j^{-1}h^{-\frac{1}{2}}m(\eta)))a_j(x, \eta) \text{ converges in } S^{n-N}(m). \tag{4-7}$$

Defining a to be this sum for $N = 0$ then gives the result since by (4-6), for each j ,

$$\phi(R_j^{-1}h^{-\frac{1}{2}}m(\eta))a_j(x, \eta) \in S^{-\infty}(m).$$

The proof of (4-6) shows that the $S^0(m)$ seminorms of $\phi(R^{-1}h^{-1/2}m(\eta))$ are uniformly bounded independent of R for $R \geq 1$. The result (4-7) follows if we choose R_j so that for all $|\alpha| + |\beta| \leq j$

$$|\partial_x^\beta \partial_\eta^\alpha (1 - \phi(R_j^{-1}h^{-\frac{1}{2}}m(\eta)))a_j(x, \eta)| \leq 2^{-j} h^{-\frac{n+1-j}{2}} m(h, \eta)^{n+1-j-\text{order}(\alpha)}.$$

Such R_j can be chosen by observing that on the support of $1 - \phi(R_j^{-1}h^{-1/2}m(\eta))$ we have the bound $h^{1/2}m(h, \eta)^{-1} \leq (1 + R_j)^{-1}$. □

We now turn to the composition result for operators. Due to support considerations of the Schwartz kernels involved, expressing the composition of two operators quantized using the cutoff χ_0 requires quantizing the symbol of the composition using the cutoff χ_1 , but we shall later see that the difference is an operator with symbol in $S^{-\infty}$. For simplicity we consider the case where the order of the composition is negative, which is the case needed to produce an inverse for P_h modulo $\Psi_h^{-\infty}(m)$.

In the proof we decompose an operator $a_h(x, hD)$ into a sum of nonisotropic dilates of unit-scale convolution kernels. This decomposition is also used in establishing L^2 bounds for order 0 operators. Let ϕ and ψ generate a smooth Littlewood–Paley decomposition of $[0, \infty)$:

$$1 = \phi(s) + \sum_{j=1}^\infty \psi(2^{-j}s), \quad \text{supp}(\phi) \subset [0, 2), \quad \text{supp}(\psi) \subset (\frac{1}{2}, 2). \tag{4-8}$$

Given a symbol $a \in S^n(m)$, we make the decomposition

$$a(x, \eta) = \phi(h^{-\frac{1}{2}}m(\eta))a(x, \eta) + \sum_{j=1}^\infty \psi(h^{-\frac{1}{2}}2^{-j}m(\eta))a(x, \eta) = \sum_{j=0}^\infty a_j(x, \eta). \tag{4-9}$$

Then a_j is supported where $m(h, \eta) \approx 2^j h^{1/2}$, and thus

$$|\partial_x^\beta \partial_\eta^\alpha a_j(x, \eta)| \leq C_{\alpha, \beta} 2^{jn} (2^j h^{1/2})^{-\text{order}(\alpha)}.$$

It follows that $a_0(x, \delta_{h^{1/2}}(\eta)) \in C_c^\infty(K \times \{|\eta| < 8\})$ with bounds uniform over h , and for $j \geq 1$ that $2^{-jn} a_j(x, \delta_{2^j h^{1/2}}(\eta))$ is uniformly bounded in $C_c^\infty(K \times \{\frac{1}{8} < |\eta| < 8\})$ over h and j .

Theorem 11. *Given $a \in S^n(m)$ and $b \in S^{n'}(m)$, with $n + n' < 0$, there is $c \in S^{n+n'}(m)$ so that*

$$a_h(x, hD) \circ b_h(x, hD) f(x) = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle / h} c_h(x, \xi) \chi_1(y) f(\exp_x(y)) dy d\xi. \tag{4-10}$$

Proof. For $x \in K$ and $h > 0$ we can write

$$\chi_1(\Theta_x(\tilde{x})) f(\tilde{x}) = \frac{1}{(2\pi h)^{2d+1}} \int e^{i\langle \Theta_x(\tilde{x}), \xi \rangle / h - i\langle y, \xi \rangle / h} \chi_1(y) f(\exp_x(y)) dy d\xi.$$

Since $a_h(x, hD) b_h(x, hD) f(x) = a_h(x, hD) b_h(x, hD) (\chi_1(\Theta_x(\cdot)) f)(x)$, we know (4-10) holds with

$$c_h(x, \xi) = (a_h(x, hD) b_h(x, hD) e^{i\langle \Theta_x(\cdot), \xi \rangle / h})(x).$$

We thus need to show that $c(x, \xi) = c_h(x, \xi_0, \xi', h^{-1}\xi'') \in S^{n+n'}(m)$.

Let a_i and b_j be the nonisotropic Littlewood–Paley decomposition of a and b as in (4-9), and define c_{ij} by

$$(c_{ij})_h(x, \xi) = ((a_i)_h(x, hD) (b_j)_h(x, hD) e^{i\langle \Theta_x(\cdot), \xi \rangle / h})(x),$$

so that $c = \sum_{ij} c_{ij}$. From (4-1) we can write $(c_{ij})_h(x, \xi)$ as

$$\frac{1}{(2\pi)^{4d+2}} \int e^{-i\langle y, \eta \rangle - i\langle w, \zeta \rangle + ih^{-1}\langle \tilde{\Theta}(x, hy, hw), \xi \rangle} (a_i)_h(x, \eta) (b_j)_h(\exp_x(hy), \zeta) \chi_0(hy) \chi_0(hw) dw d\zeta dy d\eta.$$

Consider first the case $i \geq j$. We substitute $w = h^{-1}\Theta(x, hy, hz)$, defined in (3-2), to write this as

$$\begin{aligned} & \frac{1}{(2\pi)^{4d+2}} \int e^{-i\langle y, \eta \rangle - ih^{-1}\langle \Theta(x, hy, hz), \zeta \rangle + i\langle z, \xi \rangle} a_i(x, \eta_0, \eta', h\eta'') b_j(\exp_x(hy), \zeta_0, \zeta', h\zeta'') \\ & \quad \times \chi_0(hy) \chi_0(\Theta(x, hy, hz)) |D_z \Theta|(x, hy, hz) dz d\zeta dy d\eta. \end{aligned}$$

By the comments following (4-9) applied to b_j , we write

$$\begin{aligned} & b_j(\exp_x(hy), \zeta_0, \zeta', h\zeta'') \chi_0(hy) \chi_0(\Theta(x, hy, hz)) |D_z \Theta|(x, hy, hz) \\ & \quad = 2^{jn'} \tilde{b}_j(x, hy, hz, \delta_{2^{-j}h^{-1/2}}(\zeta_0, \zeta', h\zeta'')), \end{aligned}$$

where $\tilde{b}_j \in C_c^\infty(K \times B_{r_0} \times B_{r_1} \times B_8)$, with bounds uniform over h and j , and a similar representation holds for a_i with 2^j replaced by 2^i and n' replaced by n . We make a nonisotropic dilation of ζ and η by the factors $(2^{2j}h, 2^j h^{1/2}, 2^{3j} h^{1/2})$, and of z and y by the reciprocal factors, to write

$$c_{ij}(x, \xi) = 2^{j(n+n')} \tilde{c}_{ij}(x, \delta_{2^{-j}h^{-1/2}}(\xi)),$$

where $\tilde{c}_{ij}(x, \xi)$ is given by

$$2^{(i-j)n} \frac{1}{(2\pi)^{4d+2}} \int e^{-i\langle y, \eta \rangle - i\langle \bar{\Theta}_y(z) + R(h, x, y, z), \xi \rangle + i\langle z, \xi \rangle} \tilde{a}_i(x, \delta_{2^{j-i}}(\eta)) \times \tilde{b}_j(x, \delta_{2^{-j}}(y_0, h^{\frac{1}{2}}y', h^{\frac{1}{2}}y''), \delta_{2^{-j}}(z_0, h^{\frac{1}{2}}z', h^{\frac{1}{2}}z''), \zeta) dz d\xi dy d\eta, \quad (4-11)$$

where, recalling Lemma 6,

$$\begin{aligned} \langle R(h, x, y, z), \zeta \rangle = & 2^{2j} R_0(x, \delta_{2^{-j}}(y_0, h^{\frac{1}{2}}y', h^{\frac{1}{2}}y''), \delta_{2^{-j}}(z_0, h^{\frac{1}{2}}z', h^{\frac{1}{2}}z'')) \zeta_0 \\ & + 2^j h^{-\frac{1}{2}} R'(x, \delta_{2^{-j}}(y_0, h^{\frac{1}{2}}y', h^{\frac{1}{2}}y''), \delta_{2^{-j}}(z_0, h^{\frac{1}{2}}z', h^{\frac{1}{2}}z'')) \cdot \zeta' \\ & + 2^{3j} h^{-\frac{1}{2}} R''(x, \delta_{2^{-j}}(y_0, h^{\frac{1}{2}}y', h^{\frac{1}{2}}y''), \delta_{2^{-j}}(z_0, h^{\frac{1}{2}}z', h^{\frac{1}{2}}z'')) \cdot \zeta''. \end{aligned}$$

By the support condition on \tilde{b}_j we have $|\zeta| \leq 8$. Also, if $i \geq 1$ then $\tilde{a}_i(x, \eta) = 0$ when $|\eta| \leq \frac{1}{8}$.

We next apply the expansion (3-5) to the right-hand side. The condition on order($y^\alpha z^\beta$) ensures that we bring out strictly more powers of 2^{-j} than needed to cancel the powers of 2^j in front, and since there is at least one factor of (y', z', y'', z'') we also bring out a factor $h^{1/2}$ to cancel off the $h^{-1/2}$ in front. We conclude that, on the support of the integrand,

$$|R(h, x, y, z)| \leq C2^{-j}|z - y|(|y| + |z - y| + |y|^2 + |z - y|^2),$$

and also

$$|\partial_x^\gamma \partial_y^\alpha \partial_z^\beta R(h, x, y, z)| \leq C_{\gamma, \alpha, \beta} 2^{-j} (1 + |y|^3 + |z - y|^3). \quad (4-12)$$

Additionally, if we let $w = \bar{\Theta}(x, y, z) + R(h, x, y, z)$, then with analogous notation we see from (3-2) and Lemma 6 that $z = \bar{\Theta}_{-y}(w) + \tilde{R}(h, x, y, w)$, where

$$|\tilde{R}(h, x, y, w)| \leq C2^{-j}|w|(|y| + |w| + |y|^2 + |w|^2).$$

Consequently, since $\tilde{\Theta}$ is the inverse function to Θ for fixed y , uniformly over j we have

$$\begin{aligned} |\bar{\Theta}_y(z) + R(h, x, y, z)| & \leq C|z - y|(1 + |y|^2 + |z - y|^2), \\ |z - y| & \leq C|\bar{\Theta}_y(z) + R(h, x, y, z)|(1 + |y|^2 + |\bar{\Theta}_y(z) + R(h, x, y, z)|^2), \end{aligned}$$

and hence

$$(1 + |y|^2)^{-1}|z - y| \leq C|\bar{\Theta}_y(z) + R(h, x, y, z)|(1 + |\bar{\Theta}_y(z) + R(h, x, y, z)|^2). \quad (4-13)$$

Considering the function

$$g_{ij}(x, y) = \frac{1}{(2\pi)^{4d+2}} \int e^{-i\langle y, \eta \rangle} \tilde{a}_i(x, \delta_{2^{j-i}}(\eta)) d\eta,$$

simple estimates show that

$$|\partial_x^\alpha g_{ij}(x, y)| \leq C_{N, \alpha, \beta} 2^{(4d+2)(i-j)} (1 + 2^{2(i-j)}|y_0| + 2^{i-j}|y'| + 2^{3(i-j)}|y''|)^{-N}. \quad (4-14)$$

Additionally, if $i > j$, hence $i \geq 1$, then $\tilde{a}_i(x, \eta)$ vanishes for $|\eta| \leq \frac{1}{8}$, and thus can be assumed to be of the form $|\eta|^{2k} \tilde{a}_i(x, \eta)$ for similar $\tilde{a}_i(x, \eta)$. Thus, if $i > j$ then for all $k \in \mathbb{N}$ we can write

$$g_{ij}(x, y) = \sum_{|\gamma|=2k} 2^{(j-i)\text{order}(\gamma)} \partial_y^\gamma g_{ij,\gamma}(x, y), \tag{4-15}$$

where $g_{ij,\gamma}(x, y)$ satisfies the same estimates (4-14) as $g_{ij}(x, y)$. On the other hand, if we set

$$f_j(x, y, z) = \int e^{-i(\bar{\Theta}_y(z) + R(h,x,y,z), \zeta)} \tilde{b}_j(x, \delta_{2^{-j}}(y_0, h^{\frac{1}{2}}y', h^{\frac{1}{2}}y''), \delta_{2^{-j}}(z_0, h^{\frac{1}{2}}z', h^{\frac{1}{2}}z''), \zeta) d\zeta,$$

then

$$f_j(x, y, z) = \rho(x, y, z, \bar{\Theta}_y(z) + R(h, x, y, z)),$$

where $\rho(x, y, z, w)$ is smooth in (x, y, z) and Schwartz in w . By (4-12) and (4-13) we have

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\theta f_j(x, y, z)| \leq C_{N,\alpha,\beta,\theta} (1 + |y| + |y - z|)^{3(|\alpha| + |\beta| + |\theta|)} (1 + (1 + |y|^2)^{-1} |y - z|)^{-N}.$$

Applying (4-15) and integrating by parts in y leads to the bound, for all N, α, β ,

$$\left| \partial_x^\alpha \partial_z^\beta \int g_{ij}(x, y) f_j(x, y, z) dy \right| \leq C_{N,\alpha,\beta} 2^{2k(j-i)} (1 + |z|)^{-N}.$$

Since $\tilde{c}_{ij}(x, \xi)$, defined in (4-11), is $2^{(i-j)n}$ times the Fourier transform in z of this integral, we obtain uniform (over i and j) Schwartz bounds on $2^{i-j} \tilde{c}_{ij}(x, \xi)$, with compact support in x .

In the case $j \geq i$, we can similarly write $c_{ij}(x, \xi) = 2^{i(n+n')} \tilde{c}_{ij}(x, \delta_{2^{-i}h^{-1/2}}(\xi))$, where we have uniform Schwartz bounds over i and j on $2^{j-i} \tilde{c}_{ij}(x, \xi)$. The analysis is similar to the case $i \geq j$, using instead the following representation for $c_{ij}(x, \xi)$:

$$\frac{1}{(2\pi)^{4d+2}} \int e^{-ih^{-1}(\tilde{\Theta}(x,hv,-hw), \eta) - i(w, \zeta) + i(v, \xi)} a_i(x, \eta_0, \eta', h\eta'') b_j(\exp_{\exp_x(hv)}(-hw), \zeta_0, \zeta', h\zeta'') \times \chi_0(\tilde{\Theta}(x, hv, -hw)) \chi_0(hw) |D_v \tilde{\Theta}|(x, hv, -hw) dw d\zeta dv d\eta.$$

It thus suffices to show that $\sum_{i \geq j} 2^{j(n+n')} \tilde{c}_{ij}(x, \delta_{2^{-j}h^{-1/2}}(\xi)) \in S^{n+n'}(m)$. We prove that

$$\left| \sum_{i \geq j} 2^{j(n+n')} \tilde{c}_{ij}(x, \delta_{2^{-j}h^{-1/2}}(\xi)) \right| \leq C(1 + h^{-\frac{1}{2}}m(\xi))^{n+n'}.$$

Estimates on derivatives will follow similarly since applying ∂_ξ^α has the effect of multiplying the j -th term by $(2^{-j}h^{-1/2})^{\text{order}(\alpha)}$. We use the uniform Schwartz bounds on \tilde{c}_{ij} to bound the sum by

$$C_N \sum_{i \geq j \geq 0} 2^{j(n+n')} 2^{j-i} (1 + 2^{-j}h^{-\frac{1}{2}}m(\xi))^{-N}.$$

The sum over i is trivial. Given ξ , take j_0 so that $2^{j_0} = h^{-1/2}m(\xi)$. We then split

$$\sum_{j \geq 0} 2^{j(n+n')} (1 + 2^{-j}h^{-\frac{1}{2}}m(\xi))^{-N} \leq \sum_{j \geq j_0} 2^{j(n+n')} + \sum_{j < j_0} 2^{j(n+n'+N)} (h^{-\frac{1}{2}}m(\xi))^{-N}.$$

Recall that we assume $n + n' < 0$. We take N so $N + n + n' > 0$. If $h^{-1/2}m(\xi) \leq 1$, we have only the first sum, which is bounded by a constant. If $h^{-1/2}m(\xi) > 1$, then the two terms are convergent geometric sums that both are bounded by $(h^{-1/2}m(\xi))^{n+n'}$. \square

Remark 12. The result of [Theorem 11](#) still holds if one replaces the function $\chi_0(y)$ used in quantizing a or b by any function $\chi(x, y) \in C_c^\infty(K \times B_{r_0})$, since this is harmlessly absorbed into \tilde{b}_j without changing the estimates for \tilde{b}_j nor the condition on the support of the Schwartz kernel.

Lemma 13. *Suppose that $\beta, \chi \in C_c^\infty(B_{r_1})$, and $\beta(y) = 0$ for $|y| \leq \delta$, where $\delta > 0$. Suppose also that $\chi = 1$ on $\text{supp}(\beta)$. Then if $a \in S^n(m)$ for some n , one can write*

$$\frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} a_h(x, \xi) \beta(y) f(\exp_x(y)) dy d\xi = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} r_h(x, \xi) \chi(y) f(\exp_x(y)) dy d\xi,$$

where $r \in S^{-\infty}(m)$.

Proof. We write $\beta(y) = |y|^{2N} \beta_N(y)$ for $\beta_N \in C_c^\infty(B_{r_1})$. Since $\chi \beta_N = \beta_N$, following the first part of [Theorem 11](#) we have equality of the two sides if r_h is the symbol

$$r_h(x, \xi) = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi - \eta \rangle/h} ((h^2 \Delta_{\eta_0, \eta'} + h^4 \Delta_{\eta''})^N a)_h(x, \eta) \beta_N(y) dy d\eta.$$

By (4-5), $a_N = (h^2 \Delta_{\eta_0, \eta'} + h^4 \Delta_{\eta''})^N a \in S^{n-2N}(m)$. We then write

$$r(x, \xi) = \frac{1}{(2\pi)^{2d+1} h^{3d+1}} \int \hat{\beta}_N \left(\frac{\xi_0 - \eta_0}{h}, \frac{\xi' - \eta'}{h}, \frac{\xi'' - \eta''}{h^2} \right) a_N(x, \eta) d\eta.$$

We have $|a_N(x, \eta)| \leq C_N (1 + m(\delta_{h^{-1/2}}(\eta)))^{n-2N}$, and Peetre’s inequality yields

$$(1 + |\delta_{h^{-1/2}}(\xi - \eta)|)^{-|n|-2N} (1 + m(\delta_{h^{-1/2}}(\eta)))^{n-2N} \leq C_N (1 + m(\delta_{h^{-1/2}}(\xi)))^{n-2N},$$

which shows that $|r(x, \xi)| \leq C_N h^N m(h, \xi)^{-2N}$ for all N . The term $\partial_x^\beta \partial_\xi^\alpha r(x, \xi)$ comes from the same convolution applied to $\partial_x^\beta \partial_\eta^\alpha a_N(x, \eta)$, and we conclude $r \in S^{-\infty}$. \square

Corollary 14. *Suppose P_h is as in (1-1). Given $\rho \in C_c^\infty(K^o)$, there is a symbol $q(x, \xi) \in S^{-2}(m)$, with principal symbol $h\rho(x)(1 - \phi(h^{-1/2}m(\xi)))q_0(\xi)$, so that*

$$q_h(x, hD) \circ P_h = \rho(x) + R, \quad R \in \Psi_h^{-\infty}(m).$$

Proof. Fix $\tilde{\rho}(x) \in C_c^\infty(K)$ with $\tilde{\rho} = 1$ on a neighborhood of $\text{supp}(\rho)$. Define

$$\tilde{q}_0(\xi) = h(1 - \phi(h^{-\frac{1}{2}}m(\xi)))q_0(\xi),$$

where $q_0(\xi)$ is the Fourier transform of the fundamental solution for $Y_0 - \sum_{j=1}^d Y_j^2$, as defined in [Section 2](#), and ϕ is as in (4-8). Then $\tilde{\rho}(x)\tilde{q}_0(\xi) \in S^{-2}(m)$. We first show that

$$\tilde{\rho}(x)\tilde{q}_{0,h}(hD) \circ P_h = \tilde{\rho}(x) - r_h^1(x, hD), \quad r^1 \in S^{-1}(m).$$

By the construction of $q_0(\xi)$ we have

$$\frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} h q_{0,h}(\xi) \left(Y_0 - h \sum_{j=1}^d Y_j^2 \right) \chi_0(y) f(\exp_x(y)) dy d\xi = f(x),$$

where the Y_j are the model vector fields acting in the y -variable. Replacing hq_0 by \tilde{q}_0 changes the composition by an order-0 symbol supported where $m(\xi) \leq 2h^{1/2}$, hence by a symbol in $S^{-\infty}$.

Generally, we see that for $f(x, y)$ compactly supported in y we can write

$$\int e^{-i\langle y, \xi \rangle/h} \tilde{q}_{0,h}(\xi) y^\alpha \partial_y^\beta f(x, y) dy d\xi = \int e^{-i\langle y, \xi \rangle/h} b_h(\xi) f(x, y) dy d\xi,$$

where

$$b(x, \xi) = i^{|\beta|-|\alpha|} h^{\alpha_0+|\alpha'|+2|\alpha''|-|\beta|} \partial_\xi^\alpha \xi^\beta \tilde{q}_0(\xi).$$

By (4-3) and (4-5), we know that

$$h^{\frac{1}{2} \text{order}(\alpha) - \frac{1}{2} \text{order}(\beta)} \partial_\xi^\alpha \xi^\beta \tilde{q}_0(\xi) \in S^{\text{order}(\beta) - \text{order}(\alpha) - 2}(m).$$

Recall that

$$(X_j f)(\exp_x(y)) = (Y_j + R_j(x, y, \partial_y)) f(\exp_x(y)),$$

where the Taylor expansion of R_j contains terms $y^\alpha \partial_y^\beta$ of order strictly less than $\text{order}(Y_j)$, and where $|\beta| = 1$. Since commutators of X_j with χ_0 lead to terms of order $-\infty$, we need show that

$$\frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} h q_{0,h}(\xi) R_0(x, y, \partial_y) \chi_0(y) f(\exp_x(y)) dy d\xi$$

is an operator of order -1 in f . For the terms that arise in the Taylor expansion of R_0 we have $\text{order}(\beta) - \text{order}(\alpha) \leq 1$, so we need check for such terms we also have

$$\alpha_0 + |\alpha'| + 2|\alpha''| - |\beta| \geq \frac{1}{2} \text{order}(\alpha) - \frac{1}{2} \text{order}(\beta),$$

in order to match up the powers of h . Since $|\beta| = 1$ and $\text{order}(\alpha) = 2\alpha_0 + |\alpha'| + 3|\alpha''|$, this holds provided that $|\alpha'| + |\alpha''| \geq 1$, which is the case for R_0 by Lemma 4.

We similarly need check that this is an operator of order -2 if R_0 is replaced by $h^{1/2} R_j$ with $1 \leq j \leq d$. Since $\text{order}(\beta) - \text{order}(\alpha) \leq 0$ in this case, this reduces to verifying that

$$\alpha_0 + |\alpha'| + 2|\alpha''| - |\beta| + \frac{1}{2} \geq \frac{1}{2} \text{order}(\alpha) - \frac{1}{2} \text{order}(\beta),$$

which always holds if $|\beta| = 1$.

We note that the remainder term in the Taylor expansion will also be of the desired order, but with $\chi_0(y)$ replaced by $c_{j,\alpha,k}(x, y) \chi_0(y)$. By Remark 12 this does not affect the conclusion of the corollary, since the form for q_h will involve composition with $r_h^1(x, hD)$.

By Theorem 11 we can recursively define symbols $r^j \in S^{-j}(m)$ for $j \geq 2$ by the rule

$$r_h^j(x, hD) \circ r_h^1(x, hD) f(x) = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} r_h^{j+1}(x, \xi) \chi_1(y) f(\exp_x(y)) dy d\xi,$$

where we recall that $r_h^j(x, hD)$ is quantized using χ_0 . Let $r \sim \sum_{j=0}^\infty r^j$, so $r \in S^{-1}(m)$. Also define $q \in S^{-2}(m)$ so that

$$\rho(x)(I + r_h(x, hD))\tilde{\rho}(x)\tilde{q}_{0,h}(hD)f(x) = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} q_h(x, \xi) \chi_1(y) f(\exp_x(y)) dy d\xi.$$

By the above and Lemma 13, the following operator is in $\Psi_h^{-\infty}(m)$:

$$Rf(x) = \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} q_h(x, \xi) (\chi_1(y) - \chi_0(y)) (P_h f)(\exp_x(y)) dy d\xi.$$

Thus, modulo $\Psi_h^{-\infty}(m)$ we have

$$q_h(x, hD) \circ P_h = \rho(x)(I + r_h(x, hD))(\tilde{\rho}(x) - r_h^1(x, hD)).$$

Next we choose $\delta > 0$ so that $\tilde{\rho}(\exp_x(y)) = 1$ if $x \in \text{supp}(\rho)$ and $|y| \leq \delta$, and take $\chi_\delta \in C_c^\infty(B_\delta)$ with $\chi_\delta = 1$ on $B_{\delta/2}$. Then

$$\begin{aligned} &\rho(x)r_h(x, hD)((1 - \tilde{\rho})f)(x) \\ &= \frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} \rho(x)r_h(x, \xi) (\chi_0(y) - \chi_\delta(y)) ((1 - \tilde{\rho})f)(\exp_x(y)) dy d\xi, \end{aligned}$$

so by Lemma 13 we have, modulo $\Psi_h^{-\infty}(m)$,

$$q_h(x, hD) \circ P_h = \rho(x)(I + r_h(x, hD))(I - r_h^1(x, hD)).$$

Finally, since the difference between using χ_1 instead of χ_0 in the quantization of r^{j+1} gives a term in $\Psi_h^{-\infty}(m)$, we see that $q_h(x, hD) \circ P_h = \rho(x)$ modulo $\Psi_h^{-\infty}(m)$. □

Remark 15. The above proof shows the following composition result concerning partial differential operators. Suppose

$$P_h = \sum_{\text{order}(\alpha) \leq n'} c_\alpha(x) X_0^{\alpha_0} (h^{\frac{1}{2}} X')^{\alpha'} (h^{\frac{1}{2}} X'')^{\alpha''}, \quad c_\alpha(x) \in C_c^\infty(K).$$

Then if $a \in S^n(m)$, we can write $a_h(x, hD) \circ P_h f$ and $P_h \circ a_h(x, hD) f$ in the form

$$\frac{1}{(2\pi h)^{2d+1}} \int e^{-i\langle y, \xi \rangle/h} b_h(x, \xi) \chi(y) f(\exp_x(y)) dy d\xi$$

for $\chi \in C_c^\infty(B_{r_0})$ and $b \in S^{n+n'}(m)$.

5. L^2 boundedness for order-0 operators

Given a symbol in $S^n(m)$ we decompose $a = \sum_j a_j$ as in (4-9). The operator $a_{j,h}(x, hD)$ is given by the following integral kernel on $U \times U$ with respect to the measure $dm(\tilde{x})$, where $w(x, \tilde{x}) dm(\tilde{x}) = \exp_x^*(dy)$:

$$K_j(x, \tilde{x}) = w(x, \tilde{x}) \chi_0(\Theta_x(\tilde{x})) \int e^{-i\langle \Theta_x(\tilde{x}), \eta \rangle} a_{j,h}(x, h\eta) d\eta.$$

We can write $a_{j,h}(x, h\eta) = 2^{jn} \tilde{a}_j(x, 2^{-2j}\eta_0, 2^{-j}h^{1/2}\eta', 2^{-3j}h^{1/2}\eta'')$, where $\tilde{a}_j(x, \eta) \in C_c^\infty(K \times B_8)$, with uniform bounds over j . Furthermore, \tilde{a}_j vanishes for $|\eta| \leq \frac{1}{8}$ if $j \geq 1$.

Consequently, there are Schwartz functions $\rho_j(x, y)$, supported for $x \in K$ with Schwartz norms independent of j , so that

$$(w^{-1}K_j)(x, \exp_x(y)) = 2^{jn}2^{j(2+4d)}h^{-d}\rho_j(x, 2^{2j}y_0, 2^j h^{-\frac{1}{2}}y', 2^{3j}h^{-\frac{1}{2}}y'')\chi_0(y), \tag{5-1}$$

and in particular, for all N ,

$$|K_j(x, \tilde{x})| \leq C_N 2^{jn}2^{j(2+4d)}h^{-d}(1 + 2^{2j}|\Theta_x(\tilde{x})_0| + 2^j h^{-\frac{1}{2}}|\Theta_x(\tilde{x})'| + 2^{3j}h^{-\frac{1}{2}}|\Theta_x(\tilde{x})''|)^{-N}. \tag{5-2}$$

If $a \in S^{-\infty}$ then (5-1) holds for all $n \in \mathbb{Z}$, and summing over j we obtain the following.

Corollary 16. *If $a \in S^{-\infty}(m)$, then $a_h(x, hD)$ is given by a smooth integral kernel $K(x, \tilde{x})$ in the measure $dm(\tilde{x})$, so that for some Schwartz function $\rho(x, y)$, supported for $x \in \text{supp}(a)$,*

$$(w^{-1}K)(x, \exp_x(y)) = h^{-d}\rho(x, y_0, h^{-\frac{1}{2}}y', h^{-\frac{1}{2}}y'')\chi_0(y).$$

We next observe that the vector fields $2^{-2j}Y_0$, $2^{-j}h^{1/2}Y'$, and $2^{-3j}h^{1/2}Y''$ acting as differential operators in y all preserve the form (5-1) of $w^{-1}K_j$; that is, they give an expression of the same form with ρ_j uniformly bounded over j in each Schwartz seminorm.

The same holds for the operators $2^{-2j}X_0$, $2^{-j}h^{1/2}X'$, and $2^{-3j}h^{1/2}X''$, acting on $K_j(x, \tilde{x})$ as differential operators in either the x - or \tilde{x} -variable. For action in the \tilde{x} -variable, this follows by Lemma 4, where we use that there is at least one factor of y' or y'' in the expansion of $R_0(x, y, \partial_y)$ to compensate for the factor of $h^{-1/2}$ coming from the $\partial_{y'}$ and $\partial_{y''}$ terms in the expansion of X_0 . For action in the x -variable we work in coordinates $x = \exp_{\tilde{x}}(y)$, hence $\tilde{x} = \exp_x(-y)$, to write

$$(w^{-1}K_j)(\exp_{\tilde{x}}(y), \tilde{x}) = 2^{jn}2^{j(2+4d)}h^{-d}\rho_j(\exp_{\tilde{x}}(y), -2^{2j}y_0, -2^j h^{-\frac{1}{2}}y', -2^{3j}h^{-\frac{1}{2}}y'')\chi_0(-y). \tag{5-3}$$

To summarize, for $a \in S^n(m)$, we can write

$$\begin{aligned} &(2^{-2j}X_0)^{\alpha_0}(2^{-j}h^{\frac{1}{2}}X')^{\alpha'}(2^{-3j}h^{\frac{1}{2}}X'')^{\alpha''}K_j(x, \tilde{x}) \\ &= 2^{jn}2^{j(2+4d)}h^{-d}\rho_{j,\alpha}(x, 2^{2j}\Theta_x(\tilde{x})_0, 2^j h^{-\frac{1}{2}}\Theta_x(\tilde{x})', 2^{3j}h^{-\frac{1}{2}}\Theta_x(\tilde{x})'')\chi_\alpha(x, \tilde{x}), \end{aligned} \tag{5-4}$$

where the functions $\rho_{j,\alpha}$ and χ_α satisfy seminorm bounds that depend on α , but are uniform over j and h . This holds with any given vector X in the product acting as a vector field in x or \tilde{x} .

Conversely, suppose that $j \geq 1$, so that $\tilde{a}_j(x, \eta) \in C_c^\infty(K \times \{\frac{1}{8} \leq |\eta| \leq 8\})$. Then for any ℓ , dividing \tilde{a}_j by $|\eta|^{2\ell}$ shows that we can write

$$\begin{aligned} &(w^{-1}K_j)(x, \exp_x(y)) \\ &= 2^{jn}2^{j(2+4d)}h^{-d} \sum_{|\alpha|=2\ell} \chi_\alpha(x, y)(2^{-2j}\partial_{y_0})^{\alpha_0}(2^{-j}h^{\frac{1}{2}}\partial_{y'})^{\alpha'}(2^{-3j}h^{\frac{1}{2}}\partial_{y''})^{\alpha''}\rho_{j,\alpha}(x, 2^{2j}y_0, 2^j h^{-\frac{1}{2}}y', 2^{3j}h^{-\frac{1}{2}}y'') \end{aligned}$$

for Schwartz functions $\rho_{j,\alpha}$ that are uniformly bounded over j , and $\chi_\alpha \in C_c^\infty(K \times B_{r_0})$.

Using Lemma 4, we write

$$\begin{aligned} \partial_{y_0} &= X_0 + y' \cdot X'' - R_0(x, y, \partial_y) - y' \cdot R''(x, y, \partial_y), \\ \partial_{y'} &= X' - y_0 X'' - R'(x, y, \partial_y) + y_0 R''(x, y, \partial_y), \\ \partial_{y''} &= X'' - R''(x, y, \partial_y), \end{aligned}$$

where the X_j act in y . Substituting this into $R(x, y, \partial_y)$, and using that the X_j form a smooth frame, we can expand each ∂_{y_j} as a finite sum over $2 \leq |\alpha| \leq 3$:

$$\begin{aligned} \partial_{y_0} &= X_0 + y' \cdot X'' + \sum_{\alpha,k} c_{0,\alpha,k}(x, y) y^\alpha X_k, & \text{order}(Y_k) - \text{order}(\alpha) < 2, \\ \partial_{y_j} &= X_j - y_0 X_{j+d} + \sum_{\alpha,k} c_{j,\alpha,k}(x, y) y^\alpha X_k, & \text{order}(Y_k) - \text{order}(\alpha) < 1, \quad 1 \leq j \leq d, \\ \partial_{y_j} &= X_j + \sum_{\alpha,k} c_{j,\alpha,k}(x, y) y^\alpha X_k, & \text{order}(Y_k) - \text{order}(\alpha) < 3, \quad d + 1 \leq j \leq 2d. \end{aligned}$$

Additionally, $c_{0,\alpha,k} \equiv 0$ unless either $\alpha' \neq 0$ or $\alpha'' \neq 0$.

Let \bar{X}_j denote the transpose of the differential operator X_j with respect to dy . Taking the transpose of the above identities, it follows that, with the \bar{X}_j acting on y , we can write

$$\begin{aligned} &(w^{-1}K_j)(x, \exp_x(y)) \\ &= 2^{jn} 2^{j(2+4d)} h^{-d} \sum_{|\alpha|=2\ell} \chi_\alpha(x, y) (2^{-2j} \bar{X}_0)^{\alpha_0} (2^{-j} h^{\frac{1}{2}} \bar{X}')^{\alpha'} (2^{-3j} h^{\frac{1}{2}} \bar{X}'')^{\alpha''} \rho_{j,\alpha}(x, 2^{2j} y_0, 2^j h^{-\frac{1}{2}} y', 2^{3j} h^{-\frac{1}{2}} y''), \end{aligned}$$

where the $\rho_{j,\alpha}$ may depend on h , but with uniform Schwartz bounds over $0 \leq h \leq 1$ and $j \in \mathbb{N}$. Expressing the action of \bar{X} in terms of \tilde{x} , this leads to the expansion

$$K_j(x, \tilde{x}) = \sum_{|\alpha|=2\ell} \sum_{\beta \leq \alpha} 2^{-j \text{order}(\alpha)} (\bar{X}_0)^{\beta_0} (h^{\frac{1}{2}} \bar{X}')^{\beta'} (h^{\frac{1}{2}} \bar{X}'')^{\beta''} K_{j,\alpha,\beta}(x, \tilde{x})$$

for kernels $K_{j,\alpha,\beta}$ satisfying (5-2) with C_N depending on ℓ but uniform over j, α, β . Here we can take \bar{X}_j to be the transpose of X_j with respect to $dm(\tilde{x})$, since that differs from the transpose with respect to dy by a smooth function.

Theorem 17. *If $a \in S^0(m)$, then $a_h(x, hD)$ is a bounded linear operator on $L^2(U)$, with operator norm depending on only a finite number of seminorm bounds for $a(x, \xi)$. In particular, the operator norm is uniformly bounded over $0 < h \leq 1$.*

Proof. We decompose $a_h(x, hD) = \sum_{j=0}^\infty a_{j,h}(x, hD)$. Using (5-1) and (5-3) it is easily verified that the kernel $K_j(x, \tilde{x})$ of $a_{j,h}(x, hD)$ satisfies the Schur test,

$$\sup_x \int K_j(x, \tilde{x}) dm(\tilde{x}) \leq C, \quad \sup_{\tilde{x}} \int K_j(x, \tilde{x}) dm(x) \leq C.$$

We deduce L^2 boundedness from the Cotlar–Stein lemma (see [Knapp and Stein 1971; Stein 1993]), by showing that, for any $N \in \mathbb{N}$,

$$\|a_{i,h}(x, hD)^* a_{j,h}(x, hD)\|_{L^2 \rightarrow L^2} + \|a_{i,h}(x, hD) a_{j,h}(x, hD)^*\|_{L^2 \rightarrow L^2} \leq C 2^{-N|i-j|} \tag{5-5}$$

for a constant C uniform over h and j . If $i = j$ this follows from L^2 boundedness of each term, so without loss of generality we consider $j > i \geq 0$, and in particular $j \geq 1$. Given $\ell \in \mathbb{N}$ we then write the integral kernel of $a_{i,h}(x, hD) a_{j,h}(x, hD)^*$ as

$$\begin{aligned} & \int K_i(x, w) \overline{K_j(\tilde{x}, w)} dm(w) \\ &= \int K_i(x, w) \sum_{|\alpha|=2\ell} \sum_{\beta \leq \alpha} 2^{-j \operatorname{order}(\alpha)} (\overline{X_0})^{\beta_0} (h^{\frac{1}{2}} \overline{X}')^{\beta'} (h^{\frac{1}{2}} \overline{X}'')^{\beta''} \overline{K_{j,\alpha,\beta}(x, w)} dm(w) \\ &= \sum_{|\alpha|=2\ell} \sum_{\beta \leq \alpha} 2^{i \operatorname{order}(\beta) - j \operatorname{order}(\alpha)} \int K_{i,\beta}(x, w) \overline{K_{j,\alpha,\beta}(x, w)} dm(w), \end{aligned}$$

where $K_{i,\beta}(x, w) = (2^{-2i} X_0)^{\beta_0} (2^{-i} h^{1/2} X')^{\beta'} (2^{-3i} h^{1/2} X'')^{\beta''} K_i(x, w)$, and in all cases X acts on w . Since $i \operatorname{order}(\beta) - j \operatorname{order}(\alpha) \leq 2\ell(i - j)$, by using (5-4) and the Schur test on the composition we obtain the bound (5-5) with $N = 2\ell$ for the term $a_{i,h} a_{j,h}^*$. To handle the term $a_{i,h}^* a_{j,h}$ we use the same argument, together with symmetry of the derivative estimates in x and \tilde{x} . \square

We note the following result for $a \in S^n(m)$, which holds since $2^{-jn} a_j(x, \eta) \in S^0(m)$,

$$\sup_{j \geq 0} 2^{-jn} \|a_{j,h}(x, hD) f\|_{L^2(U)} \leq C \|f\|_{L^2(U)}, \quad a \in S^n(m). \tag{5-6}$$

6. Estimates on $S^*(M)$

Let (M, g) be a compact Riemannian manifold of dimension $d + 1$, and $S^*(M) \subset T^*(M)$ its unit cosphere bundle. We consider the Hamiltonian function $\frac{1}{2} |\zeta|_{g(z)}^2 = \frac{1}{2} \sum_{i,k=1}^{d+1} g^{ik}(z) \zeta_i \zeta_k$, and recall that $S^*(M)$ is the level set $|\zeta|_{g(z)} = 1$. We use $X_0 = H$ to denote the Hamiltonian field for $\frac{1}{2} |\zeta|_{g(z)}^2$,

$$X_0 = \sum_{i,k=1}^{d+1} g^{ik}(z) \zeta_i \partial_{z_k} - \frac{1}{2} \sum_{i,j,k=1}^{d+1} \partial_{z_j} g^{ik}(z) \zeta_i \zeta_k \partial_{\zeta_j},$$

which is tangent to $S^*(M)$.

We cover $S^*(M)$ by a finite collection of open coordinate charts as follows. Let $\{V_\alpha\}$ form a finite covering of M by coordinate charts, over which we can identify $T^*(M)$ with $V_\alpha \times \mathbb{R}^{d+1}$ and $S^*(M)$ with $V_\alpha \times \mathbb{S}^d$. We cover \mathbb{S}^d by two coordinate charts W^\pm over each of which there is a section of the frame bundle. We thus obtain a cover of $S^*(M)$ by open charts $\{V_\alpha \times W^\pm\}$, which by counting each V_α twice we can label as U_α , such that on U_α there is an orthonormal collection $\{X_j\}_{j=1}^d$ of vertical vector fields that span the tangent space to $S_z^*(M)$ over each $z \in V_\alpha$. The collection $\{X_j\}_{j=1}^d$ is involutive, since it spans the vertical vector fields on U_α .

There is a natural isometric identification $T_\zeta(T_z^*(M)) \sim T_z(M)$, which identifies $\{X_j|_{(z,\zeta)}\}_{j=1}^d$ with an orthonormal collection of vectors $\{\tilde{X}_j\}_{j=1}^d \subset T_z(M)$, which are also orthogonal to $\pi_*(X_0|_{(z,\zeta)})$. We let $X_{j+d}|_{(z,\zeta)}$ be $-\frac{1}{2}$ times the horizontal lift of \tilde{X}_j . We observe that

$$\pi_*[X_j, X_0] = \sum_{i,k=1}^{d+1} g^{ik}(z) X_j(\zeta_i) \partial_{z_k} = \tilde{X}_j,$$

so that $[X_0, X_j] - 2X_{j+d} \in \text{span}\{X_j\}_{j=1}^d$. Thus the assumptions of the [Introduction](#) are satisfied for the collection $\{X_j\}_{j=0}^{2d}$.

Let $\Delta_{\mathbb{S}}$ be the induced nonnegative Laplacian acting on the fibers $S_z^*(M)$ of the bundle, and let Δ be the nonnegative Laplacian on $S^*(M)$. See for example [\[Drouot 2017, Section 2.1\]](#) for details, where it is shown that Δ and $\Delta_{\mathbb{S}}$ commute. One verifies that, over each U_α , one has

$$\Delta_{\mathbb{S}} = - \sum_{j=1}^d X_j^2 + \sum_{j=1}^d c_j(z, \zeta) X_j.$$

We now use $x \in \mathbb{R}^{2d+1}$ to denote the variables on U_α , and define

$$P_h = H + h\Delta_{\mathbb{S}} = X_0 - \sum_{j=1}^d hX_j^2 + \sum_{j=1}^d c_j(x)hX_j.$$

Thus on each U_α , the operator P_h differs from the sum of squares considered previously by an operator in $h^{1/2}\Psi_h^1(m)$, and the pseudodifferential calculus shows that, given $\chi_\alpha \in C_c^\infty(U_\alpha)$, there exists a symbol $q_\alpha \in S^{-2}(m)$, the quantization of which depends on χ_α through the choice of χ_0 in (4-1), so that on U_α we have

$$q_{\alpha,h}(x, hD) \circ P_h u = \chi_\alpha(x)u + R_\alpha u, \quad R_\alpha \in \Psi_h^{-\infty}(m).$$

Note that both $q_{\alpha,h}(x, hD)$ and R_α are properly supported in U_α . We now take a partition of unity χ_α subordinate to the cover U_α , and define

$$Q_h v = \sum_\alpha q_{\alpha,h}(x, hD)v, \quad Rv = \sum_\alpha R_\alpha v.$$

Then $Q_h \circ P_h = I + R$, and for all N_1, N_2 we have

$$\|(h\Delta)^{N_1} R(h\Delta)^{N_2} u\|_{L^2(S^*(M))} \leq C_{N_1, N_2} \|u\|_{L^2(S^*(M))}. \tag{6-1}$$

This follows from Theorems 11 and 17 and the fact that $h\Delta \in \Psi_h^6(U_\alpha)$ for each α , which follows from (4-4).

More generally, we define $\Psi_h^\sigma(m)$ on $S^*(M)$ as sums $\sum_\alpha a_{\alpha,h}(x, hD)$ with $a_\alpha \in S^\sigma(m)$ on U_α . The function χ_0 in the quantization (4-1) depends on the x -support of $a_\alpha(x, \eta)$, which is always assumed to be a compact subset of U_α .

The semiclassical Sobolev spaces are defined on $S^*(M)$ using the spectral decomposition of Δ , with norm

$$\|f\|_{H_h^\sigma} = \|(1 + h^2\Delta)^{\sigma/2} f\|_{L^2}.$$

We will consider cutoffs $\rho(s)$ satisfying, for some $c' > c > 0$,

$$\rho(s) \in C^\infty(\mathbb{R}), \quad \rho(s) = 0 \quad \text{if } s \leq c, \quad \rho(s) = 1 \quad \text{if } s \geq c'. \tag{6-2}$$

The operator $\rho(h^2\Delta)$ is then defined as a spectral multiplier. We observe the following simple result for $R \in \Psi_h^{-\infty}(m)$ on $S^*(M)$. For all N and σ we have

$$\|\rho(h^2\Delta)Ru\|_{H_h^\sigma} + \|R\rho(h^2\Delta)u\|_{H_h^\sigma} \leq C_{N,\sigma}h^N\|u\|_{L^2}. \tag{6-3}$$

This follows by writing $\rho(h^2\Delta)(1+h^2\Delta)^\sigma = f(h^2\Delta) \circ (h^2\Delta)^N$, where the function $f(s)$ is a bounded function provided $N > \sigma$, and using (6-1).

Theorem 18. *Suppose that $\sigma \leq 0$, that $A_h \in \Psi_h^\sigma(m)$, and that ρ satisfies (6-2). Then*

$$\|\rho(h^2\Delta)A_hu\|_{H_h^{-\sigma/3}} + \|A_h\rho(h^2\Delta)u\|_{H_h^{-\sigma/3}} \leq Ch^{-\sigma/6}\|u\|_{L^2}.$$

Proof. Choose k so $6k + \sigma > 0$. For each $h \in (0, 1]$, we show that $A_h = A_{0,h} + A_{1,h}$, where

$$\|(h^2\Delta)^k A_{0,h}u\|_{L^2} + \|A_{0,h}(h^2\Delta)^k u\|_{L^2} + \|A_{1,h}u\|_{H_h^{-\sigma/3}} \leq Ch^{-\sigma/6}\|u\|_{L^2}.$$

The result then follows since $\rho(s) \leq \min(s^k, 1)$. Using the Littlewood–Paley decomposition as in the proof of Theorem 17, applied to each a_α in the sum defining a , we let

$$A_{0,h} = \sum_{2^j \leq h^{-1/6}} a_{j,h}(x, hD), \quad A_{1,h} = \sum_{2^j > h^{-1/6}} a_{j,h}(x, hD).$$

Recalling the form (5-4), we see that applying $h^2\Delta$ to $a_{j,h}(x, hD)$ is equivalent to multiplying it by at most $2^{6j}h$. As in the proof of (5-5) we conclude that

$$\|(1+h^2\Delta)^k a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h^2\Delta)^k\|_{L^2 \rightarrow L^2} \leq (1+2^{6i}h)^k(1+2^{6j}h)^k 2^{\sigma(i+j)-|i-j|}.$$

For $2^j, 2^i \geq h^{-1/6}$, we interpolate with the L^2 bounds (5-6) to obtain

$$\|(1+h^2\Delta)^{-\sigma/6} a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h^2\Delta)^{-\sigma/6}\|_{L^2 \rightarrow L^2} \leq Ch^{-\sigma/3} 2^{-|i-j|}.$$

This estimate also holds for the transposed operators. The Cotlar–Stein lemma then implies the bounds for $A_{1,h}$.

Similarly, we have

$$\|(h^2\Delta)^k a_{j,h}(x, hD)\|_{L^2 \rightarrow L^2} + \|a_{j,h}(x, hD)(h^2\Delta)^k\|_{L^2 \rightarrow L^2} \leq C(2^{6j}h)^k 2^{\sigma j},$$

which we may sum over $2^j \leq h^{-1/6}$ to conclude the bounds involving $A_{0,h}$. □

Corollary 19. *Suppose that $\sigma \leq 0$ and $A_h \in \Psi_h^\sigma(m)$. Then*

$$\|(1+h\Delta)^{-\sigma/6} A_hu\|_{L^2} \leq C\|u\|_{L^2}.$$

Proof. As in the proof of Theorem 18 we observe that, for $k = 0, 1, 2, \dots$,

$$\|(1+h\Delta)^k a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h\Delta)^k\|_{L^2 \rightarrow L^2} \leq 2^{6k(i+j)} 2^{\sigma(i+j)-|i-j|}.$$

We interpolate between $k = 0$ and any $k > -\sigma/6$ to obtain

$$\|(1+h\Delta)^{-\sigma/6} a_{j,h}(x, hD)a_{i,h}(x, hD)^*(1+h\Delta)^{-\sigma/6}\|_{L^2 \rightarrow L^2} \leq C2^{-|i-j|}.$$

This estimate also holds for the transposed operators. The Cotlar–Stein lemma then implies the result. □

Theorem 20. *The following bound holds for $h \in (0, 1]$ and all $N \in \mathbb{N}$:*

$$\|Hu\|_{L^2} + h\|\Delta_{\mathbb{S}}u\|_{L^2} + \|(1 + h\Delta)^{\frac{1}{3}}u\|_{L^2} \leq C\|P_hu\|_{L^2} + C_N\|(1 + h\Delta)^{-N}u\|_{L^2}.$$

Proof. Write $u = Q_h P_h u + Ru$, where $Q_h \in \Psi_h^{-2}(m)$, and note that $HQ_h, h\Delta_{\mathbb{S}}Q_h \in \Psi_h^0(m)$ by Remark 15. Also, for all N we have $HR(1 + h\Delta)^N, h\Delta_{\mathbb{S}}R(1 + h\Delta)^N \in \Psi_h^0(m)$; hence

$$\|HRu\|_{L^2} + h\|\Delta_{\mathbb{S}}Ru\|_{L^2} \leq C_N\|(1 + h\Delta)^{-N}u\|_{L^2}.$$

Since $Q_h, R(1 + h\Delta)^N \in \Psi_h^{-2}(m)$, the result then follows by Corollary 19. □

Theorem 21. *Suppose that ρ_1 and ρ_2 satisfy (6-2), and $\rho_2 = 1$ on a neighborhood of $\text{supp}(\rho_1)$. Given $\lambda_0 > 0$, the following holds for all N , and all $|\lambda| \leq \lambda_0$ and $h \in (0, 1]$:*

$$h^{-\frac{1}{3}}\|\rho_1(h^2\Delta)u\|_{H_h^{2/3}} + h^{\frac{1}{3}}\sum_{j=1}^d\|X_j\rho_1(h^2\Delta)u\|_{H_h^{1/3}} + \|X_0\rho_1(h^2\Delta)u\|_{L^2} + \|h\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u\|_{L^2} \leq C_{N,\lambda_0}(\|\rho_2(h^2\Delta)(P_h - \lambda)u\|_{L^2} + h^N\|u\|_{L^2}).$$

Proof. We follow the scheme of the proof of Theorem 2 of [Drouot 2017], using the parametrix Q_h of P_h to replace the positive commutator arguments. Write

$$\rho_1(h^2\Delta)u = Q_h\rho_1(h^2\Delta)(P_h - \lambda)u + Q_h[P_h, \rho_1(h^2\Delta)]u + \lambda Q_h\rho_1(h^2\Delta)u + R\rho_1(h^2\Delta)u.$$

To handle the commutator term, we use that $[\Delta_{\mathbb{S}}, \rho_1(h^2\Delta)] = 0$; hence $[P_h, \rho_1(h^2\Delta)] = [X_0, \rho_1(h^2\Delta)]$. Now let $\tilde{\rho}_1(s)$ be any function satisfying (6-2) which equals 1 on a neighborhood of $\text{supp}(\rho_1)$. Then following [Drouot 2017], we use that the essential support of $[X_0, \rho_1(h^2\Delta)]$ is contained within the elliptic set of $\tilde{\rho}(h^2\Delta)$, and we can thus bound

$$\|[P_h, \rho_1(h^2\Delta)]u\|_{L^2} \leq C\|\tilde{\rho}_1(h^2\Delta)u\|_{L^2} + C_Nh^N\|u\|_{L^2}.$$

Applying Theorem 18 and (6-3) we obtain

$$h^{-\frac{1}{3}}\|\rho_1(h^2\Delta)u\|_{H_h^{2/3}} + h^{-\frac{1}{6}}\sum_{j=1}^d\|h^{\frac{1}{2}}X_j\rho_1(h^2\Delta)u\|_{H_h^{1/3}} + \|X_0\rho_1(h^2\Delta)u\|_{L^2} + \|h\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u\|_{L^2} \leq C(\|\rho_1(h^2\Delta)(P_h - \lambda)u\|_{L^2} + \|\tilde{\rho}_1(h^2\Delta)u\|_{L^2} + (1 + |\lambda|)\|\rho_1(h^2\Delta)u\|_{L^2}) + C_Nh^N\|u\|_{L^2}.$$

For h bounded away from 0 we can absorb the term $(1 + |\lambda|)\|\rho_1(h^2\Delta)u\|_{L^2}$ into $C_Nh^N\|u\|_{L^2}$, and for h small we can subtract it from both sides.

From this we deduce the following bound for any such $\tilde{\rho}_1$:

$$\|\rho_1(h^2\Delta)u\|_{L^2} \leq C_{N,\lambda_0}(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h - \lambda)u\|_{L^2} + h^{\frac{1}{3}}\|\tilde{\rho}_1(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}).$$

We now choose a sequence of cutoffs $\tilde{\rho}_j$ for $1 \leq j \leq 3N$, satisfying (6-2), such that for all j we have $\tilde{\rho}_{j+1} = 1$ on a neighborhood of $\text{supp}(\tilde{\rho}_j)$, and $\rho_2 = 1$ on a neighborhood of $\text{supp}(\tilde{\rho}_j)$. Then replacing ρ_1 by $\tilde{\rho}_j$, the preceding estimate shows that

$$\|\tilde{\rho}_j(h^2\Delta)u\|_{L^2} \leq C_{N,\lambda_0}(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h - \lambda)u\|_{L^2} + h^{\frac{1}{3}}\|\tilde{\rho}_{j+1}(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}).$$

We conclude by iteration that

$$\begin{aligned}\|\tilde{\rho}_1(h^2\Delta)u\|_{L^2} &\leq C_{N,\lambda_0}(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^N\|\rho_2(h^2\Delta)u\|_{L^2} + h^N\|u\|_{L^2}) \\ &\leq C_{N,\lambda_0}(h^{\frac{1}{3}}\|\rho_2(h^2\Delta)(P_h-\lambda)u\|_{L^2} + h^N\|u\|_{L^2}).\end{aligned}$$

Together with the above this yields the statement of the theorem. \square

Acknowledgements

The author would like to thank Maciej Zworski for his encouragement to pursue the topic of this paper. The author would also like to thank the referee for their careful and detailed review, which led to improvements in the exposition of the results.

References

- [Angst, Bailleul, and Tardif 2015] J. Angst, I. Bailleul, and C. Tardif, “Kinetic Brownian motion on Riemannian manifolds”, *Electron. J. Probab.* **20** (2015), art. id. 110, 40 pp. [MR](#) [Zbl](#)
- [Drouot 2017] A. Drouot, “Stochastic stability of Pollicott–Ruelle resonances”, *Comm. Math. Phys.* **356**:2 (2017), 357–396. [MR](#) [Zbl](#)
- [Dyatlov and Zworski 2015] S. Dyatlov and M. Zworski, “Stochastic stability of Pollicott–Ruelle resonances”, *Nonlinearity* **28**:10 (2015), 3511–3533. [MR](#) [Zbl](#)
- [Folland 1975] G. B. Folland, “Subelliptic estimates and function spaces on nilpotent Lie groups”, *Ark. Mat.* **13**:2 (1975), 161–207. [MR](#) [Zbl](#)
- [Franchi and Le Jan 2007] J. Franchi and Y. Le Jan, “Relativistic diffusions and Schwarzschild geometry”, *Comm. Pure Appl. Math.* **60**:2 (2007), 187–251. [MR](#) [Zbl](#)
- [Grothaus and Stilgenbauer 2013] M. Grothaus and P. Stilgenbauer, “Geometric Langevin equations on submanifolds and applications to the stochastic melt-spinning process of nonwovens and biology”, *Stoch. Dyn.* **13**:4 (2013), art. id. 1350001, 34 pp. [MR](#) [Zbl](#)
- [Hörmander 1967] L. Hörmander, “Hypoelliptic second order differential equations”, *Acta Math.* **119** (1967), 147–171. [MR](#) [Zbl](#)
- [Knapp and Stein 1971] A. W. Knap and E. M. Stein, “Intertwining operators for semisimple groups”, *Ann. of Math. (2)* **93** (1971), 489–578. [MR](#) [Zbl](#)
- [Li 2016] X.-M. Li, “Random perturbation to the geodesic equation”, *Ann. Probab.* **44**:1 (2016), 544–566. [MR](#) [Zbl](#)
- [Rothschild and Stein 1976] L. P. Rothschild and E. M. Stein, “Hypoelliptic differential operators and nilpotent groups”, *Acta Math.* **137**:3-4 (1976), 247–320. [MR](#) [Zbl](#)
- [Smith 1994] H. F. Smith, “A calculus for three-dimensional CR manifolds of finite type”, *J. Funct. Anal.* **120**:1 (1994), 135–162. [MR](#) [Zbl](#)
- [Stein 1993] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, 1993. [MR](#) [Zbl](#)
- [Zworski 2012] M. Zworski, *Semiclassical analysis*, Graduate Studies in Mathematics **138**, American Mathematical Society, Providence, RI, 2012. [MR](#) [Zbl](#)

Received 13 Oct 2018. Revised 27 Jun 2019. Accepted 25 Sep 2019.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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ANALYSIS & PDE

Volume 13 No. 8 2020

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