Geometry & Topology 24 (2020) 179–307

Contact handles, duality, and sutured Floer homology

ANDRÁS JUHÁSZ
IAN ZEMKE

We give an explicit construction of the Honda–Kazez–Matić gluing maps in terms of contact handles. We use this to prove a duality result for turning a sutured manifold cobordism around and to compute the trace in the sutured Floer TQFT. We also show that the decorated link cobordism maps on the hat version of link Floer homology defined by the first author via sutured manifold cobordisms and by the second author via elementary cobordisms agree.

57R58; 57M27, 57R17

1 Introduction

The purpose of this paper is to provide an explicit construction of the Honda–Kazez–Matić gluing map [12] in terms of contact handles, and use this to prove several results about the sutured Floer TQFT defined by the first author [15]. Additionally, we show that the decorated link cobordism maps on the hat version of link Floer homology defined via sutured manifold cobordisms by the first author [15] and the maps defined using elementary link cobordisms by the second author [30] agree.

Published: 25 March 2020 DOI: 10.2140/gt.2020.24.179

1.1 The contact gluing map

Sutured manifolds were introduced by Gabai [5] to construct taut foliations on 3–manifolds, and are also ubiquitous in contact topology. In this paper, a sutured manifold is a pair (M,γ), where M is a compact oriented 3–manifold with boundary, and the set of sutures $\gamma \subseteq \partial M$ is an oriented 1–manifold that divides ∂M into subsurfaces $R_+(\gamma)$ and $R_-(\gamma)$ that meet along γ. For example, if M carries a contact structure such that ∂M is convex with dividing set γ, then (M,γ) is a sutured manifold.

We say that (M,γ) is balanced if M has no closed components, each component of M contains a suture, and $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$. Sutured Floer homology, defined by the first author [14], assigns an $\mathbb{F}_2$–vector space $\mathrm{SFH}(M,\gamma)$ to a balanced sutured manifold (M,γ). It is a common extension of the hat version of Heegaard Floer homology of closed 3–manifolds and link Floer homology, both due to Ozsváth and Szabó [24; 23; 26], to 3–manifolds with boundary.

Let (M,γ) and (M',γ') be sutured manifolds such that $M \subseteq \mathrm{int}(M')$. Given a contact structure ξ on $M' \setminus \mathrm{int}(M)$ such that $\partial M \cup \partial M'$ is convex with dividing set $\gamma \cup \gamma'$, Honda, Kazez, and Matić [12] define a gluing map

$$\Phi_\xi\colon \mathrm{SFH}(-M,-\gamma) \to \mathrm{SFH}(-M',-\gamma')$$

using partial open book decompositions that satisfy a "contact compatibility" condition near the boundary. However, the contact compatibility condition makes working with and computing the gluing map impractical. In the first part of this paper, we give a new definition of the contact gluing map based on contact handle attachments, prove invariance via contact cell decompositions, and show that our map agrees with the Honda–Kazez–Matić gluing map. In particular, this allows us to give a simple diagrammatic description of the gluing map for a single contact handle attachment. For a precise statement about the gluing map associated to a contact handle attachment, see Proposition 5.6. Contact handles were introduced by Giroux [6]; see Definition 5.5.

We now describe the map C_{h^i} that we assign to attaching a contact i–handle h^i for $i \in \{0,1,2,3\}$. Let $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ be a diagram of (M,γ); then $(\overline{\Sigma},\boldsymbol{\alpha},\boldsymbol{\beta})$ is a diagram of $(-M,-\gamma)$. Attaching a contact 0–handle corresponds to taking the disjoint union of Σ with a disk. A contact 1–handle corresponds to attaching a 1–handle to $\partial\Sigma$; see Figure 1. Adding a disk or a 1–handle to $\partial\Sigma$ does not change the sutured Floer complex, and we define C_{h^i} to be the tautological map on intersection points. A contact 2–handle is attached to ∂M along a curve l that intersects γ in two points. Let $\lambda_\pm \subseteq \Sigma$

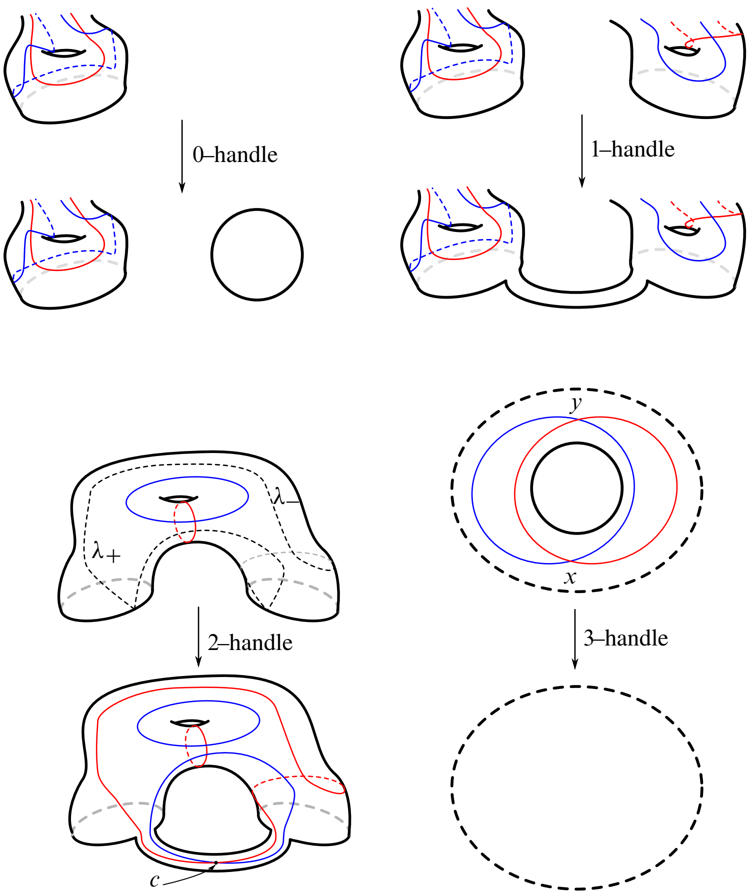

Figure 1: The diagrams used in the definition of the contact handle maps.

be the properly embedded arc corresponding to $l \cap R_\pm(\gamma)$. As in Figure 1, we glue a 1–handle H to Σ along $\partial\Sigma$, and add a curve α to $\boldsymbol{\alpha}$ and a curve β to $\boldsymbol{\beta}$ that intersect in H in a single point c and are such that $\alpha \cap \Sigma = \lambda_-$ and $\beta \cap \Sigma = \lambda_+$. Then, given a generator $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we let $C_{h^2}(\boldsymbol{x}) = \boldsymbol{x} \times \{c\}$. Finally, suppose that we attach a contact 3–handle h^3 along an S^2 component S of ∂M containing the suture $\gamma_S = \gamma \cap S$, giving rise to the sutured manifold (M', γ'). We then choose a diagram where γ_S is encircled by a curve $\alpha \in \boldsymbol{\alpha}$ and a curve $\beta \in \boldsymbol{\beta}$ such that $\alpha \cap \beta = \{x, y\}$ and such that there are no other $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ curves between α and γ_S or β and γ_S. Let Σ' be the result of gluing a disk to Σ along γ_S. Then $(\Sigma', \boldsymbol{\alpha} \setminus \{\alpha\}, \boldsymbol{\beta} \setminus \{\beta\})$ is a diagram of (M', γ'); see Figure 1. We let $C_{h^3}(\boldsymbol{x} \times \{x\}) = 0$ and $C_{h^3}(\boldsymbol{x} \times \{y\}) = \boldsymbol{x}$, where $\mu(\boldsymbol{x} \times \{x\}, \boldsymbol{x} \times \{y\}) = 1$ in $(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta})$.

Note that Zarev [28] has also defined a type of gluing map in sutured Floer homology, corresponding to a convex decomposition. Combining this with the EH invariant of Honda, Kazez, and Matić [13], one can define a map for gluing a contact structure to a sutured manifold. Zarev conjectured that this map agrees with the Honda–Kazez–Matić gluing map, though we will not address Zarev's construction in this paper.

1.2 The sutured Floer TQFT

The first author [15] defined the category of balanced sutured manifolds and sutured manifold cobordisms, and extended SFH to a functor on this category. A sutured manifold cobordism from (M_0,γ_0) to (M_1,γ_1) is a triple $\mathcal{W}=(W,Z,[\xi])$, where W is a 4–manifold with boundary and corners, Z is a codimension-0 compact submanifold of ∂W such that $\partial W\setminus \mathrm{int}(Z) = -M_0\sqcup M_1$, and $[\xi]$ is a certain equivalence class of a contact structure ξ on Z such that ∂Z is convex with dividing set $\gamma_0\sqcup\gamma_1$. The sutured cobordism map

$$F_{\mathcal{W}}\colon \mathrm{SFH}(M_0,\gamma_0)\to \mathrm{SFH}(M_1,\gamma_1)$$

is the composition of the contact gluing map for $-\xi$ and 4–dimensional handle maps.

Let $\xi_{I\times\partial M}$ denote the I–invariant contact structure on $-I\times\partial M$ that induces the dividing set γ on ∂M. Consider the trace cobordism

$$\Lambda_{(M,\gamma)} = (I\times M, \xi_{I\times\partial M})$$

from $(M,\gamma)\sqcup(-M,\gamma)$ to $\varnothing$, and the cotrace cobordism

$$V_{(M,\gamma)} = (I\times M, \xi_{I\times\partial M})$$

from $\varnothing$ to $(-M,\gamma)\sqcup(M,\gamma)$. In Theorem 8.1, we answer [15, Conjecture 11.13] positively:

Theorem 1.1 *The trace cobordism $\Lambda_{(M,\gamma)}$ induces the canonical trace map*

$$\mathrm{tr}\colon \mathrm{SFH}(M,\gamma)\otimes\mathrm{SFH}(-M,\gamma)\to\mathbb{F}_2,$$

obtained by evaluating cohomology on homology. The cotrace cobordism $V_{(M,\gamma)}$ induces the canonical cotrace map

$$\mathrm{cotr}\colon \mathbb{F}_2\to\mathrm{SFH}(-M,\gamma)\otimes\mathrm{SFH}(M,\gamma).$$

The proof relies on the following deep technical result, which is Theorem 7.1. Before stating it, recall that, given a sutured triple diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$, we can associate to it a sutured manifold cobordism $\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ from $(M_{\boldsymbol{\alpha},\boldsymbol{\beta}},\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \sqcup (M_{\boldsymbol{\beta},\boldsymbol{\gamma}},\gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}})$ to $(M_{\boldsymbol{\alpha},\boldsymbol{\gamma}},\gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$.

Theorem 1.2 *Let $\mathcal{T} = (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ be an admissible balanced sutured triple diagram. Then the cobordism map*

$$F_{\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}}\colon \mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}) \otimes \mathrm{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\gamma}) \to \mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\gamma})$$

is chain homotopic to the map $F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ defined in [15, Definition 5.13] *that counts holomorphic triangles on the triple diagram $\mathcal{T}$.*

One can obtain from Theorem 1.1 a positive answer to [15, Question 11.9]:

Theorem 1.3 *If $\mathcal{W}\colon (M,\gamma) \to (M',\gamma')$ is a balanced cobordism of sutured manifolds, and $\mathcal{W}^\vee$ is the cobordism obtained by turning around $\mathcal{W}$, then*

$$F_{\mathcal{W}^\vee} = (F_{\mathcal{W}})^\vee,$$

with respect to the trace pairing.

We also give a self-contained proof of this result, without invoking Theorem 1.1. As a special case, we obtain that the decorated link cobordism maps $F^J_{\mathcal{X}}$ of the first author satisfy an analogous duality property when we turn a decorated link cobordism $\mathcal{X}$ around; see Juhász and Marengon [17, Section 5.7]. Indeed, these maps are defined by assigning a sutured manifold cobordism to a decorated link cobordism, and applying the SFH functor.

The second author [30] later gave a different construction of link Floer cobordism maps $F^Z_{\mathcal{X}}$ by composing maps defined for elementary link cobordisms, and showing independence of the decomposition. Note that this construction makes sense for all versions of link Floer homology, not just the hat version. In the last section, we prove that the two maps agree:

Theorem 1.4 *Given a decorated link cobordism $\mathcal{X}$, we have $F^J_{\mathcal{X}} = \hat{F}^Z_{\mathcal{X}}$.*

A key technical lemma that we use throughout the paper gives a simple formula for the naturality map for a compound stabilization operation on a sutured diagram (called a $(k,0)$– or $(0,l)$–stabilization by Juhász, Thurston, and Zemke [18]), which consists of a simple stabilization, followed by handle sliding some α–curves over the new α–curve, or some β–curves over the new β–curve; see Proposition 2.2.

In an upcoming paper, we will use Theorem 1.1 to compute the effect of a generalization of the Fintushel–Stern knot surgery operation using a self-concordance of a knot, called concordance surgery by Akbulut [1, Section 2], on the Ozsváth–Szabó 4–manifold invariant. The formula involves the graded Lefschetz number of the concordance map on knot Floer homology. In another work, we will apply Theorem 1.1 to compute the invariant due to Juhász and Marengon [16] of a slice disk obtained by the deform-spinning construction of Litherland [19]. Hence, we will show that this invariant can effectively distinguish different slice disks of a knot, answering [16, Question 1.4].

1.3 Notation and conventions

Throughout this paper, if A and B are smooth manifolds, then we write $A \cong B$ if A and B are diffeomorphic. Given a submanifold A of C, we write $N(A)$ for a regular neighborhood of A in C. We denote Heegaard diagrams by $\mathcal{H}$, and handle decompositions by $\boldsymbol{H}$. If M is an oriented n–manifold, then we will denote the same manifold with its orientation reversed by $\overline{M}$ when n is even, and by $-M$ when n is odd. The closure of a set X is $\mathrm{cl}(X)$. If ξ is a co-oriented 2–plane field on M, we will write $-\xi$ for the co-oriented 2–plane field obtained by reversing the co-orientation of ξ.

We orient the boundary of a manifold using the "outward normal first" convention. To be consistent with this convention, all our cobordisms go from left to right.

Acknowledgements We would like to thank Ko Honda and Jacob Rasmussen for helpful discussions on the contact gluing maps.

Juhász was supported by a Royal Society Research Fellowship, and Zemke by an NSF Postdoctoral Research Fellowship (DMS-1703685). This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 674978).

2 1–handle and 3–handle maps and triangle maps; compound stabilizations

In this section, we describe several results about the interactions between holomorphic triangles and 1–handle, 3–handle, and stabilization maps. These will be used in later sections.

2.1 1–handle and 3–handle maps

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be an admissible sutured diagram, and let $p_1, p_2 \in \Sigma$ be a pair of points that are both in components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ that intersect $\partial\Sigma$. We construct the admissible sutured diagram $(\Sigma', \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$ by removing disks centered at p_1 and p_2, and adding an annulus A connecting the boundaries of the disks. Furthermore, α_0 and β_0 are homologically nontrivial curves in A that intersect transversely at two points $\theta^+_{\alpha_0,\beta_0}$ and $\theta^-_{\alpha_0,\beta_0}$ such that $\theta^+_{\alpha_0,\beta_0}$ has the larger relative Maslov grading. The first author [15, Section 7] defined the 1–handle map

$$F_1^{\alpha_0,\beta_0}\colon \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to \mathrm{CF}(\Sigma', \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\}), \quad \boldsymbol{x} \mapsto \boldsymbol{x} \times \theta^+_{\alpha_0,\beta_0},$$

and the 3–handle map

$$F_3^{\alpha_0,\beta_0}\colon \mathrm{CF}(\Sigma', \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\}) \to \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}),$$
$$\boldsymbol{x} \times \theta^+_{\alpha_0,\beta_0} \mapsto 0 \quad \text{and} \quad \boldsymbol{x} \times \theta^-_{\alpha_0,\beta_0} \mapsto \boldsymbol{x}.$$

If $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is an admissible sutured triple, then it induces a holomorphic triangle map

$$F_{\mathcal{T}}\colon \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \otimes \mathrm{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}) \to \mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}).$$

If $p_1, p_2 \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{\gamma})$ are distinct points that are in components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{\gamma})$ that intersect $\partial\Sigma$, then we can similarly form the admissible Heegaard triple

$$\mathcal{T}' := (\Sigma',\ \boldsymbol{\alpha}' = \boldsymbol{\alpha} \cup \{\alpha_0\},\ \boldsymbol{\beta}' = \boldsymbol{\beta} \cup \{\beta_0\},\ \boldsymbol{\gamma}' = \boldsymbol{\gamma} \cup \{\gamma_0\}),$$

where Σ' is obtained by adding a 1–handle A with feet at p_1 and p_2 and three new curves, α_0, β_0, and γ_0, that are homologically nontrivial in A and pairwise intersect

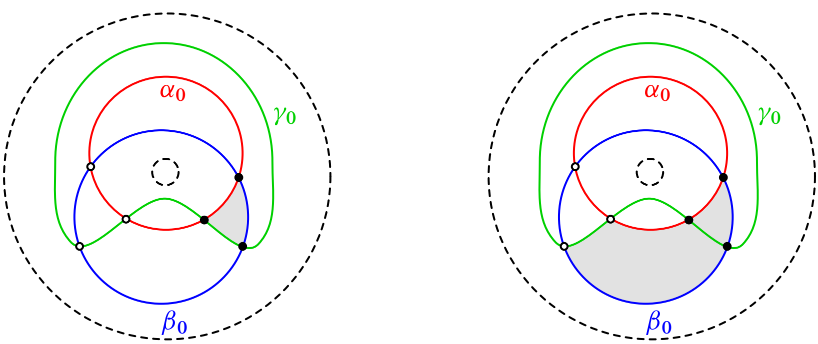

Figure 2: The annulus A is bounded by the two dashed circles. The intersection points $\theta^+_{\alpha_0,\beta_0}$, $\theta^+_{\beta_0,\gamma_0}$, and $\theta^+_{\alpha_0,\gamma_0}$ are marked by solid circles, and $\theta^-_{\alpha_0,\beta_0}$, $\theta^-_{\beta_0,\gamma_0}$, and $\theta^-_{\alpha_0,\gamma_0}$ by empty circles. On the left, the only index 0 triangle connecting $\theta^+_{\alpha_0,\beta_0}$, $\theta^+_{\beta_0,\gamma_0}$, and $\theta^+_{\alpha_0,\gamma_0}$ is shaded. On the right, the only index 0 triangle connecting $\theta^+_{\alpha_0,\beta_0}$, $\theta^-_{\beta_0,\gamma_0}$, and $\theta^-_{\alpha_0,\gamma_0}$ is shaded.

in two points; see Figure 2. If x and y are sets of attaching curves on some Heegaard surface S, then we denote the diagram (S,x,y) by $\mathcal{H}_{x,y}$.

Proposition 2.1 *With the above notation, the following diagrams are commutative:*

$$\begin{array}{ccc}
\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}})\otimes\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}) & \xrightarrow{F_{\mathcal{T}}} & \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\gamma}}) \\
\downarrow{\scriptstyle F_1^{\alpha_0,\beta_0}\otimes F_1^{\beta_0,\gamma_0}} & & \downarrow{\scriptstyle F_1^{\alpha_0,\gamma_0}} \\
\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'})\otimes\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta}',\boldsymbol{\gamma}'}) & \xrightarrow{F_{\mathcal{T}'}} & \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha}',\boldsymbol{\gamma}'})
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}})\otimes\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta}',\boldsymbol{\gamma}'}) & \xrightarrow{\mathrm{id}_{\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}})}\otimes F_3^{\beta_0,\gamma_0}} & \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}})\otimes\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}) & \xrightarrow{F_{\mathcal{T}}} & \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\gamma}}) \\
\downarrow{\scriptstyle F_1^{\alpha_0,\beta_0}\otimes\mathrm{id}_{\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta}',\boldsymbol{\gamma}'})}} & & & \nearrow{\scriptstyle F_3^{\alpha_0,\gamma_0}} & \\
\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'})\otimes\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta}',\boldsymbol{\gamma}'}) & \xrightarrow{F_{\mathcal{T}'}} & \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha}',\boldsymbol{\gamma}'}) & &
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'})\otimes\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}) & \xrightarrow{F_3^{\alpha_0,\beta_0}\otimes\mathrm{id}_{\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})}} & \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}})\otimes\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}) & \xrightarrow{F_{\mathcal{T}}} & \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\gamma}}) \\
\downarrow{\scriptstyle \mathrm{id}_{\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'})}\otimes F_1^{\beta_0,\gamma_0}} & & & \nearrow{\scriptstyle F_3^{\alpha_0,\gamma_0}} & \\
\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'})\otimes\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta}',\boldsymbol{\gamma}'}) & \xrightarrow{F_{\mathcal{T}'}} & \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha}',\boldsymbol{\gamma}'}) & &
\end{array}$$

Proof Consider the first diagram. The assumption that the points p_1 and p_2 are in components of $\Sigma\setminus(\boldsymbol{\alpha}\cup\boldsymbol{\beta}\cup\boldsymbol{\gamma})$ that intersect $\partial\Sigma$ allows one to reduce the claim to the model computation

$$F_{\alpha_0,\beta_0,\gamma_0}(\theta^+_{\alpha_0,\beta_0}\otimes\theta^+_{\beta_0,\gamma_0})=\theta^+_{\alpha_0,\gamma_0} \tag{2-1}$$

in the annulus A, which was established in the proof of [15, Theorem 7.6]; see the left-hand side of Figure 2.

We now show the claim for the second diagram. Let $\boldsymbol{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ and $\boldsymbol{y}\in\mathbb{T}_\beta\cap\mathbb{T}_\gamma$. Then

$$F_{\mathcal{T}}\circ(\mathrm{id}_{\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}})}\otimes F_3^{\beta_0,\gamma_0})(\boldsymbol{x}\otimes(\boldsymbol{y}\times\theta^-_{\beta_0,\gamma_0}))=F_{\mathcal{T}}(\boldsymbol{x}\otimes\boldsymbol{y}).$$

On the other hand,

$$\begin{aligned}
&F_3^{\alpha_0,\gamma_0}\circ F_{\mathcal{T}'}\circ(F_1^{\alpha_0,\beta_0}\otimes\mathrm{id}_{\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta}',\boldsymbol{\gamma}'})})(\boldsymbol{x}\otimes(\boldsymbol{y}\times\theta^-_{\beta_0,\gamma_0}))\\
&\qquad\qquad=F_3^{\alpha_0,\gamma_0}(F_{\mathcal{T}}(\boldsymbol{x},\boldsymbol{y})\times F_{\alpha_0,\beta_0,\gamma_0}(\theta^+_{\alpha_0,\beta_0}\otimes\theta^-_{\beta_0,\gamma_0})).
\end{aligned}$$

Hence, commutativity for the generator $\boldsymbol{x}\otimes(\boldsymbol{y}\times\theta^-_{\beta_0,\gamma_0})$ follows from

$$F_{\alpha_0,\beta_0,\gamma_0}(\theta^+_{\alpha_0,\beta_0}\otimes\theta^-_{\beta_0,\gamma_0})=\theta^-_{\alpha_0,\gamma_0},$$

which can be shown similarly to (2-1); see the right-hand side of Figure 2. Note that there is a unique index 0 pseudoholomorphic triangle in A connecting $\theta^+_{\alpha_0,\beta_0}$, $\theta^-_{\beta_0,\gamma_0}$, and $\theta^-_{\alpha_0,\gamma_0}$, and there is none connecting $\theta^+_{\alpha_0,\beta_0}$, $\theta^-_{\beta_0,\gamma_0}$, and $\theta^+_{\alpha_0,\gamma_0}$.

On a generator of the form $\boldsymbol{x} \otimes (\boldsymbol{y} \times \theta^+_{\beta_0,\gamma_0})$, we have

$$F_{\mathcal{T}} \circ (\mathrm{id}_{\mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}})} \otimes F_3^{\beta_0,\gamma_0})(\boldsymbol{x} \otimes (\boldsymbol{y} \times \theta^+_{\beta_0,\gamma_0})) = 0,$$

and, using (2-1),

$$\begin{aligned} F_3^{\alpha_0,\gamma_0} \circ F_{\mathcal{T}'} \circ (F_1^{\alpha_0,\beta_0} \otimes \mathrm{id}_{\mathrm{CF}(\mathcal{H}_{\boldsymbol{\beta}',\boldsymbol{\gamma}'})})(\boldsymbol{x} \otimes (\boldsymbol{y} \times \theta^+_{\beta_0,\gamma_0})) \\ = F_3^{\alpha_0,\gamma_0}(F_{\mathcal{T}}(\boldsymbol{x},\boldsymbol{y}) \times F_{\alpha_0,\beta_0,\gamma_0}(\theta^+_{\alpha_0,\beta_0} \otimes \theta^+_{\beta_0,\gamma_0})) \\ = F_3^{\alpha_0,\gamma_0}(F_{\mathcal{T}}(\boldsymbol{x},\boldsymbol{y}) \times \theta^+_{\alpha_0,\gamma_0}) \\ = 0. \end{aligned}$$

This establishes commutativity of the second diagram. Commutativity of the third diagram is analogous. □

2.2 Compound stabilization

In this section, we describe an elaboration of the usual stabilization operation on Heegaard diagrams. Suppose that $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible sutured diagram, and that λ is an embedded path on Σ between two distinct points on $\partial\Sigma$ that avoids the $\boldsymbol{\alpha}$ curves. We define the *compound stabilization* of $\mathcal{H}$ along λ, as follows.

First, construct a surface Σ' by pushing λ into the sutured compression body $U_{\boldsymbol{\alpha}}$, and add a tube that is the boundary of a regular neighborhood of λ. Let α_0 be a longitude of the tube, concatenated with a portion of the curve λ on Σ. Furthermore, let β_0 be a meridian of the tube. The curve α_0 may intersect other $\boldsymbol{\beta}$ curves; however, β_0 intersects only α_0. The construction is shown in Figure 3. Let us denote by $\mathcal{H}'$ the Heegaard diagram $(\Sigma', \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$. This is an instance of a $(k,0)$–stabilization, using the terminology of [18, Definition 6.26], where $k = |\alpha_0 \cap \boldsymbol{\beta}|$. If λ avoids $\boldsymbol{\beta}$, then we can perform an analogous operation, with the roles of α and β swapped, which is an instance of a $(0,l)$–stabilization. We also call this a compound stabilization. In the opposite direction, we say that $\mathcal{H}$ is obtained from $\mathcal{H}'$ by a *compound destabilization*.

We denote the unique intersection point of α_0 and β_0 by c_{α_0,β_0}. There is a map

$$\sigma^{\alpha_0,\beta_0}\colon \mathrm{SFH}(\mathcal{H}) \to \mathrm{SFH}(\mathcal{H}'),$$

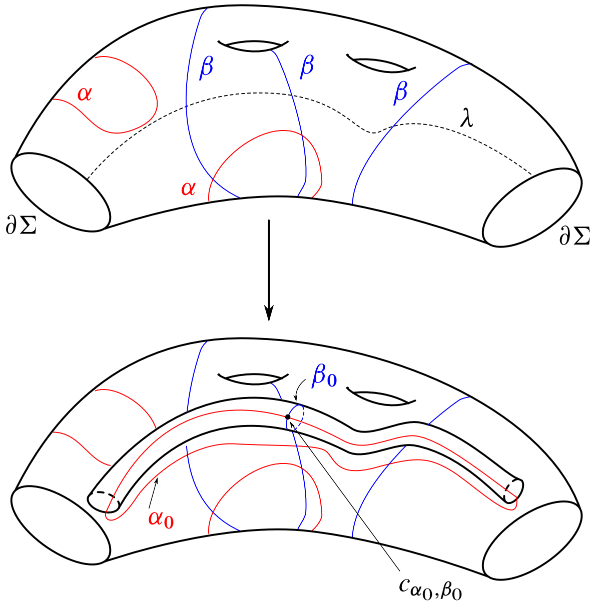

Figure 3: An example of the compound stabilization operation along a path λ.

defined by

$$\sigma^{\alpha_0,\beta_0}(\boldsymbol{x}) = \boldsymbol{x} \times c_{\alpha_0,\beta_0},$$

which is a chain isomorphism since the tube is added near $\partial\Sigma$.

On the other hand, there is also a naturality map $\Psi_{\mathcal{H}\to\mathcal{H}'}$. One would expect these to be equal. Indeed, we prove the following (compare [12, Proposition 3.7]):

Proposition 2.2 *The compound stabilization map $\sigma^{\alpha_0,\beta_0}$ is chain homotopic to the naturality map $\Psi_{\mathcal{H}\to\mathcal{H}'}$.*

One strategy to prove the above theorem would be to handle slide all the $\boldsymbol{\beta}$ curves that intersect α_0 across β_0. The map from naturality induced by these handle slides can be computed by counting holomorphic triangles. To prove Proposition 2.2, one could analyze how holomorphic triangles degenerate as one stretches two necks (one on each end of the tube we are adding). While this can be done, we will give a somewhat indirect argument that avoids performing a neck-stretching argument.

Suppose that $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$ is an admissible sutured triple with a path λ from $\partial\Sigma$ to itself that does not intersect any $\boldsymbol{\alpha}$ curves. Then we can perform the compound stabilization procedure on $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$ to obtain a Heegaard triple

$$\mathcal{T}' = (\Sigma',\ \boldsymbol{\alpha} \cup \{\alpha_0\},\ \boldsymbol{\beta} \cup \{\beta_0\},\ \boldsymbol{\beta}' \cup \{\beta_0'\}),$$

where $(\Sigma', \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$ is the compound stabilization of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ along λ. Furthermore, the curve β_0' is isotopic to β_0 and $|\beta_0 \cap \beta_0'| = 2$, while $|\alpha_0 \cap \beta_0| = |\alpha_0 \cap \beta_0'| = 1$. An example is shown in Figure 4. Let $\theta^+_{\beta_0,\beta_0'}$ be the point of $\beta_0 \cap \beta_0'$ with the higher relative Maslov grading, and write $\alpha_0 \cap \beta_0 = \{c_{\alpha_0,\beta_0}\}$ and $\alpha_0 \cap \beta_0' = \{c_{\alpha_0,\beta_0'}\}$.

Lemma 2.3 *If $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$ is an admissible sutured triple and*

$$\mathcal{T}' = (\Sigma',\ \boldsymbol{\alpha} \cup \{\alpha_0\},\ \boldsymbol{\beta} \cup \{\beta_0\},\ \boldsymbol{\beta}' \cup \{\beta_0'\})$$

is a compound stabilization of $\mathcal{T}$, as described in the previous paragraph, then

$$F_{\mathcal{T}'}(\boldsymbol{x} \times c_{\alpha_0,\beta_0},\ \boldsymbol{y} \times \theta^+_{\beta_0,\beta_0'}) = F_{\mathcal{T}}(\boldsymbol{x}, \boldsymbol{y}) \times c_{\alpha_0,\beta_0'}.$$

Proof Since the tube is added near $\partial\Sigma$, the result is obtained by a model computation inside the tube. This is shown in Figure 4. □

Remark 2.4 Despite the notation, the triangle map computation of Lemma 2.3 does not assume that the curves $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ appearing in the triple $\mathcal{T}$ are related by a sequence of handle slides or isotopies. However, we will only need the result for examples where that is the case.

Analogously, we need to consider moves of the α_0 curve appearing in a compound stabilization. To this end, suppose that $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a sutured triple with two paths, λ and λ', from $\partial\Sigma$ to itself such that λ and λ' have the same endpoints and disjoint interiors. Furthermore, suppose that λ avoids $\boldsymbol{\alpha}$ and λ' avoids $\boldsymbol{\alpha}'$. We can construct a compound stabilization of the triple $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta})$ to obtain

$$\mathcal{T}' = (\Sigma',\ \boldsymbol{\alpha}' \cup \{\alpha_0'\},\ \boldsymbol{\alpha} \cup \{\alpha_0\},\ \boldsymbol{\beta} \cup \{\beta_0\}),$$

where $(\Sigma', \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$ is the compound stabilization of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ along λ. Furthermore, the curve α_0' is a concatenation of a portion of the curve λ' on Σ with a longitude of the tube $\Sigma' \setminus \Sigma$ such that $|\alpha_0 \cap \alpha_0'| = 2$ and $|\alpha_0 \cap \beta_0| = |\alpha_0' \cap \beta_0| = 1$. In the tube, α_0, α_0', and β_0 are configured as in Figure 5. Let $\theta^+_{\alpha_0',\alpha_0}$ be the point of $\alpha_0' \cap \alpha_0$ with the larger relative Maslov grading, and write $\alpha_0 \cap \beta_0 = \{c_{\alpha_0,\beta_0}\}$ and $\alpha_0' \cap \beta_0 = \{c_{\alpha_0',\beta_0}\}$.

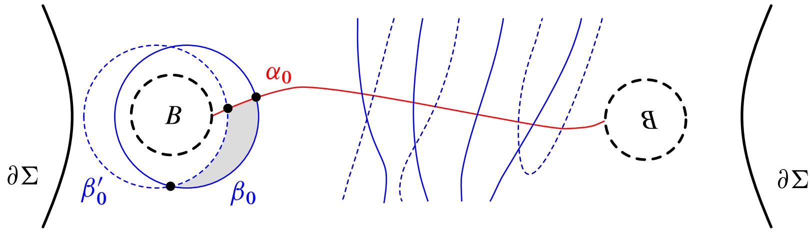

Figure 4: A compound stabilization of a sutured triple, and the model computation of Lemma 2.3. The $\boldsymbol{\alpha}$ curves are shown as red solid lines, the $\boldsymbol{\beta}$ curves are shown as blue solid lines, and the $\boldsymbol{\beta}'$ curves are shown as blue dashed lines. The circles marked B are identified.

Lemma 2.5 *If $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible sutured triple and*

$$\mathcal{T}' = (\Sigma', \boldsymbol{\alpha}' \cup \{\alpha_0'\}, \boldsymbol{\alpha} \cup \{\alpha_0\}, \boldsymbol{\beta} \cup \{\beta_0\})$$

is a compound stabilization, as described in the previous paragraph, then

$$F_{\mathcal{T}'}(\boldsymbol{x} \times \theta^+_{\alpha_0',\alpha_0}, \ \boldsymbol{y} \times c_{\alpha_0,\beta_0}) = F_{\mathcal{T}}(\boldsymbol{x}, \boldsymbol{y}) \times c_{\alpha_0',\beta_0}.$$

Proof As before, since the ends of the tube $\Sigma' \setminus \Sigma$ are near $\partial\Sigma$, we obtain constraints on the multiplicities of any homology class of triangles which has holomorphic representatives. An easy model computation shows that triangles with representatives have homology class $\psi \sqcup \psi_0$, where ψ is a homology class on $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta})$ and ψ_0 is a homology class supported entirely on the tube. The appropriate model computation is shown in Figure 5. □

Using the above two lemmas, we now prove Proposition 2.2.

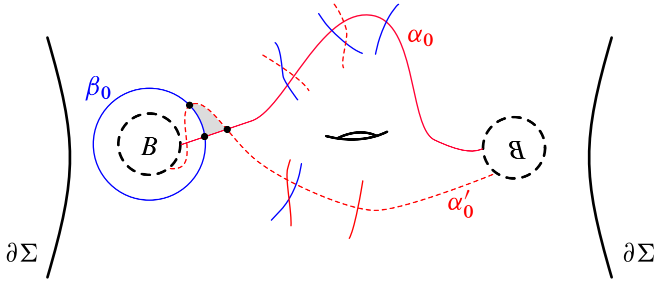

Figure 5: The model computation of Lemma 2.5. The $\boldsymbol{\alpha}'$ curves are shown as dashed red, the $\boldsymbol{\alpha}$ curves are shown as solid red, and the $\boldsymbol{\beta}$ curves are shown as solid blue.

Proof of Proposition 2.2 Let $\mathcal{H}'=(\Sigma',\boldsymbol{\alpha}\cup\{\alpha_0\},\boldsymbol{\beta}\cup\{\beta_0\})$ denote a compound stabilization of $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ using a path λ with ends on $\partial\Sigma$, and let D be the tube attached. Let $\widehat{\mathcal{H}}=(\widehat{\Sigma},\boldsymbol{\alpha}\cup\{\alpha_1\},\boldsymbol{\beta}\cup\{\beta_1\})$ be another compound stabilization of $\mathcal{H}$, along a path that is parallel to λ. Write B for the attached tube. Let $\widehat{\mathcal{H}}'=(\widehat{\Sigma}',\boldsymbol{\alpha},\cup\{\alpha_0,\alpha_1\},\boldsymbol{\beta}\cup\{\beta_0,\beta_1\})$ denote the two-fold compound stabilization of Σ along both paths. Write $\{c_0\}=\alpha_0\cap\beta_0$ and $\{c_1\}=\alpha_1\cap\beta_1$.

We claim that

$$\Psi_{\mathcal{H}'\to\widehat{\mathcal{H}}'}(\boldsymbol{x}\times c_0)=\Psi_{\mathcal{H}\to\widehat{\mathcal{H}}}(\boldsymbol{x})\times c_0. \tag{2-2}$$

To see this, we note that a sequence of diagrams from $\mathcal{H}'$ to $\widehat{\mathcal{H}}'$ can be constructed by starting with $\mathcal{H}'$, performing a simple stabilization near the boundary, and then moving one foot of the new tube along Σ, parallel to α_0. At various points, we will have to handle slide a $\boldsymbol{\beta}$ curve across β_1. It is not obvious what the holomorphic triangle count will be for each handle slide. However, by the holomorphic triangle count from Lemma 2.3, it is unchanged by the presence of the compound stabilization along α_0. In particular, the triangles counted by going from $\mathcal{H}'$ to $\widehat{\mathcal{H}}'$ are the same as the ones counted in the analogous sequence of diagrams from $\mathcal{H}$ to $\widehat{\mathcal{H}}$, so (2-2) follows.

We now consider the path of Heegaard diagrams from $\widehat{\mathcal{H}}'$ to $\widehat{\mathcal{H}}$ shown in Figure 6. The diagram $\widehat{\mathcal{H}}''$ is obtained by handle sliding β_1 over β_0. We let β_1' denote the curve resulting from this handle slide. The diagram $\widehat{\mathcal{H}}'''$ is obtained by isotoping the Heegaard surface by sliding the foot of the tube marked D inside β_1' over the tube marked B, carrying α_0 and β_0 along, and then handle sliding α_0 over α_1, giving rise to a new curve α_0'. Note that $\widehat{\mathcal{H}}'''$ is a simple stabilization of $\widehat{\mathcal{H}}$ (in [18], this type of stabilization was also referred to as a $(0,0)$–stabilization), and hence there is a destabilization from $\widehat{\mathcal{H}}'''$ to $\widehat{\mathcal{H}}$.

Using the presence of the boundary $\partial\Sigma$ to simplify the computation, one can see that the only holomorphic triangles contributing to the change of diagrams map $\Psi_{\widehat{\mathcal{H}}'\to\widehat{\mathcal{H}}''}$ have homology class $\psi\sqcup\psi_0$, where ψ is a holomorphic triangle on an unstabilized Heegaard triple $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}')$, where $\boldsymbol{\beta}'$ is a small isotopy of $\boldsymbol{\beta}$, and ψ_0 is the homology class shown in Figure 7 . Using an additional triangle map to move the $\boldsymbol{\beta}'$ back to $\boldsymbol{\beta}$ (and only isotoping β_0 and β_1' a small amount), which can be analyzed similarly, we have that

$$\Psi_{\widehat{\mathcal{H}}'\to\widehat{\mathcal{H}}''}(\boldsymbol{x}\times c_0\times c_1)=\boldsymbol{x}\times c_0\times c_1', \tag{2-3}$$

where $c_1'=\alpha_1\cap\beta_1'$.

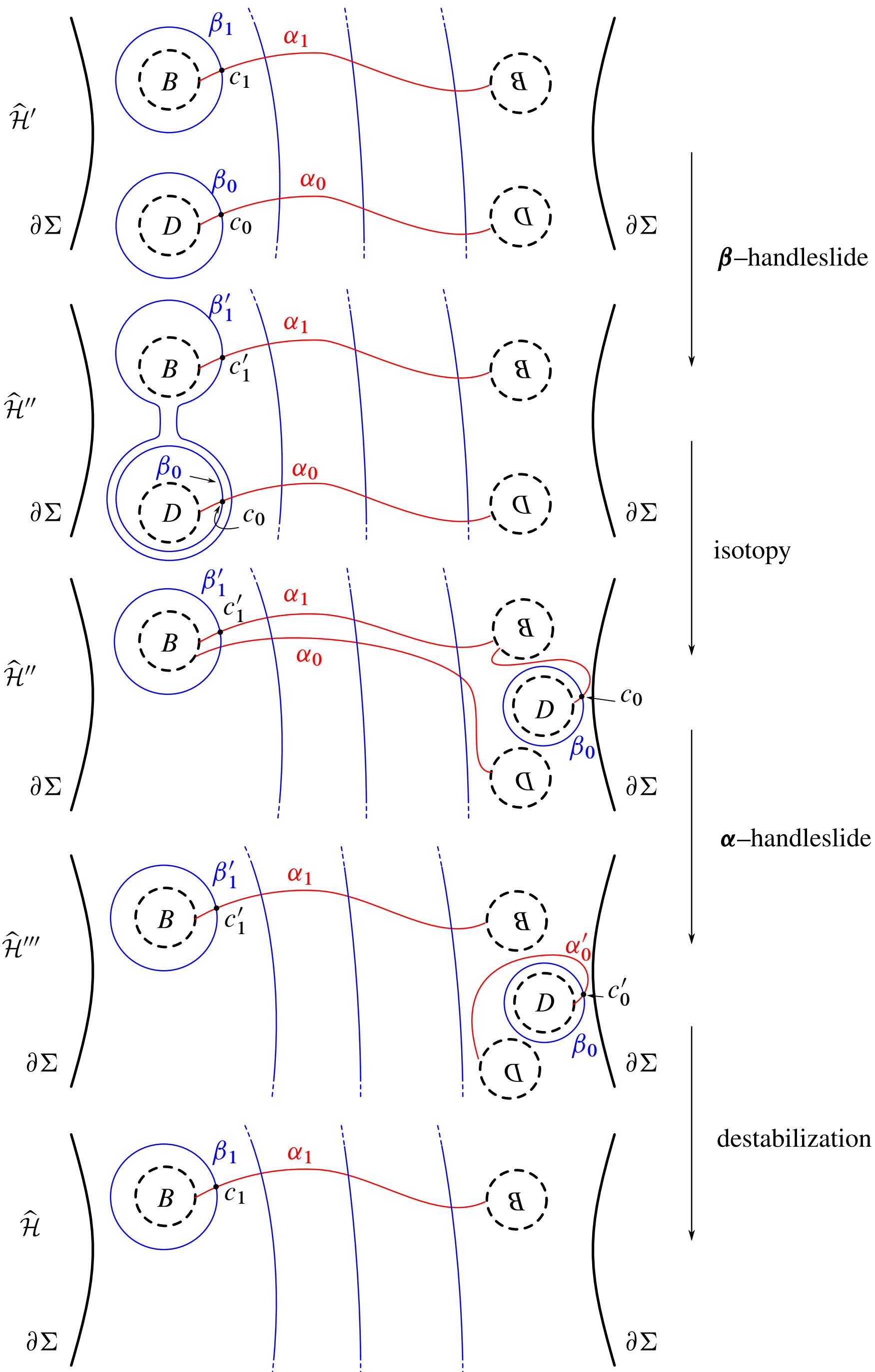

Figure 6: A sequence of Heegaard diagrams from $\widehat{\mathcal{H}}'$ to $\widehat{\mathcal{H}}$.

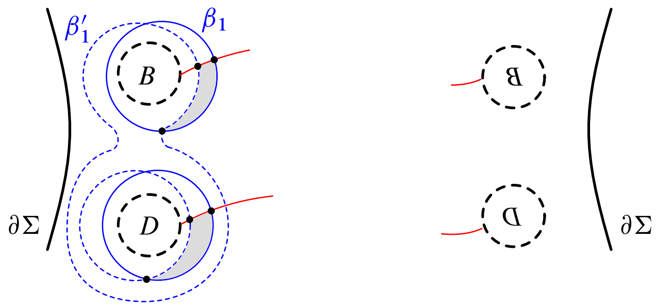

Figure 7: Computing $\Psi_{\widehat{\mathcal{H}}' \to \widehat{\mathcal{H}}''}$.

By Lemma 2.5, we have that

$$\Psi_{\widehat{\mathcal{H}}'' \to \widehat{\mathcal{H}}'''}(\boldsymbol{x} \times c_0 \times c_1') = \boldsymbol{x} \times c_0' \times c_1', \tag{2-4}$$

where $c_0' = \alpha_0' \cap \beta_0$. Equations (2-3) and (2-4) imply that

$$\Psi_{\widehat{\mathcal{H}}' \to \widehat{\mathcal{H}}'''}(\boldsymbol{x} \times c_0 \times c_1) = \boldsymbol{x} \times c_0' \times c_1'. \tag{2-5}$$

The diagrams $\widehat{\mathcal{H}}'''$ and $\widehat{\mathcal{H}}$ are related by a destabilization of the curves α_0' and β_0. Hence

$$\Psi_{\widehat{\mathcal{H}}''' \to \widehat{\mathcal{H}}}(\boldsymbol{x} \times c_0' \times c_1') = \boldsymbol{x} \times c_1. \tag{2-6}$$

In $\widehat{\mathcal{H}}$, the only α–curve that intersects β_1 is α_1, and $\alpha_1 \cap \beta_1 = \{c_1\}$, hence every generator in $\widehat{\mathcal{H}}$ is of the form $\boldsymbol{y} \times c_1$ for some $\boldsymbol{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$. So, there are constants $c_{\boldsymbol{x},\boldsymbol{y}} \in \mathbb{F}_2$ such that

$$\Psi_{\mathcal{H} \to \widehat{\mathcal{H}}}(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y} \times c_1).$$

If we substitute this into (2-2), we get that

$$\Psi_{\mathcal{H}' \to \widehat{\mathcal{H}}'}(\boldsymbol{x} \times c_0) = \sum_{\boldsymbol{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y} \times c_0 \times c_1).$$

Together with (2-5) and (2-6), we arrive at the equality

$$\begin{aligned}\Psi_{\mathcal{H}' \to \widehat{\mathcal{H}}}(\boldsymbol{x} \times c_0) &= (\Psi_{\widehat{\mathcal{H}}''' \to \widehat{\mathcal{H}}} \circ \Psi_{\widehat{\mathcal{H}}' \to \widehat{\mathcal{H}}'''} \circ \Psi_{\mathcal{H}' \to \widehat{\mathcal{H}}'})(\boldsymbol{x} \times c_0) \\ &= \sum_{\boldsymbol{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}} c_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y} \times c_1) = \Psi_{\mathcal{H} \to \widehat{\mathcal{H}}}(\boldsymbol{x}).\end{aligned}$$

Hence, we obtain that

$$\Psi_{\mathcal{H}' \to \mathcal{H}}(\boldsymbol{x} \times c_0) = (\Psi_{\widehat{\mathcal{H}} \to \mathcal{H}} \circ \Psi_{\mathcal{H}' \to \widehat{\mathcal{H}}})(\boldsymbol{x} \times c_0) = (\Psi_{\widehat{\mathcal{H}} \to \mathcal{H}} \circ \Psi_{\mathcal{H} \to \widehat{\mathcal{H}}})(\boldsymbol{x}) = \boldsymbol{x}.$$

We note that the last equality follows from naturality. Hence $\Psi_{\mathcal{H}'\to\mathcal{H}}(\boldsymbol{x}\times c_0)=\boldsymbol{x}$, completing the proof of Proposition 2.2. □

3 Contact cell decompositions and the gluing map

In this section, we give a definition of the contact gluing map using contact cell decompositions, and prove invariance. The construction is similar to the one due to Honda, Kazez, and Matić [12]. On a formal level, the gluing map is described as follows. Suppose that (M,γ) is a sutured submanifold of (M',γ') and that ξ is a co-oriented contact structure on $M'\setminus\operatorname{int}(M)$ such that ∂M is a convex surface with dividing set γ and $\partial M'$ is a convex surface with dividing set γ'. Note that this implies that a contact vector field positively transverse to ∂M lies on the positive side of ξ along $R_-(M,\gamma)$, and on the negative side of ξ along $R_+(M,\gamma)$. In this situation, there is an induced map

$$\Phi_\xi\colon \mathrm{SFH}(-M,-\gamma)\to\mathrm{SFH}(-M',-\gamma'),$$

called the gluing map.

3.1 Sutured cell decompositions

In order to discuss cell decompositions of contact 3–manifolds, we need the following notion of cell decomposition for surfaces with divides:

Definition 3.1 Let F be a closed, orientable surface, and $\gamma\subseteq F$ a dividing set. A *sutured cell decomposition* of (F,γ) consists of the following:

- **Fattened 0–cells** A collection of pairwise disjoint disks $B_1,\dots,B_n\subseteq F$ such that each $B_i\cap\gamma$ is an arc and each component of γ intersects some B_i.
- **1–cells** A collection of pairwise disjoint, properly embedded arcs

  $$\lambda_1,\dots,\lambda_m\subseteq F\setminus\bigcup_{i=1}^n\operatorname{int}(B_i)$$

  disjoint from γ such that each component of

  $$F\setminus(B_1\cup\dots\cup B_n\cup\lambda_1\cup\dots\cup\lambda_m)$$

  intersects γ nontrivially and is homeomorphic to an open disk.

A sutured cell decomposition of a torus with two parallel divides is illustrated in Figure 8.

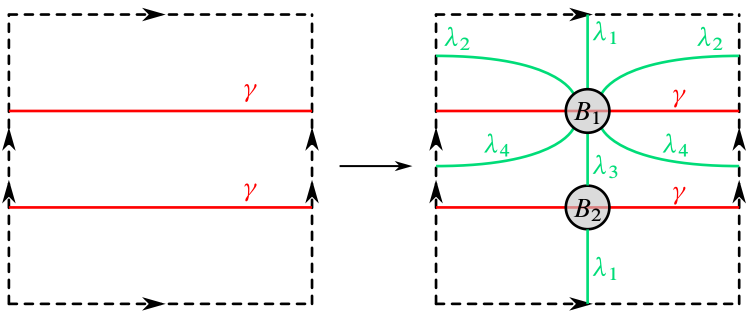

Figure 8: A torus F with two parallel divides γ (left) and a sutured cell decomposition (right).

Remark 3.2 Let $F_0 \subseteq F$ be the closure of a connected component of

$$F \setminus (B_1 \cup \cdots \cup B_n \cup \lambda_1 \cup \cdots \cup \lambda_m).$$

Since each component of γ intersects some B_i, the dividing set $\gamma \cap F_0$ contains no closed curves. Hence, according to Giroux's criterion [10, Theorem 3.5], if F is a convex surface in the contact manifold (M, ξ), and $\partial B_1 \cup \cdots \cup \partial B_n \cup \lambda_1 \cup \cdots \cup \lambda_m$ is a Legendrian graph, then F_0 has a tight neighborhood in M.

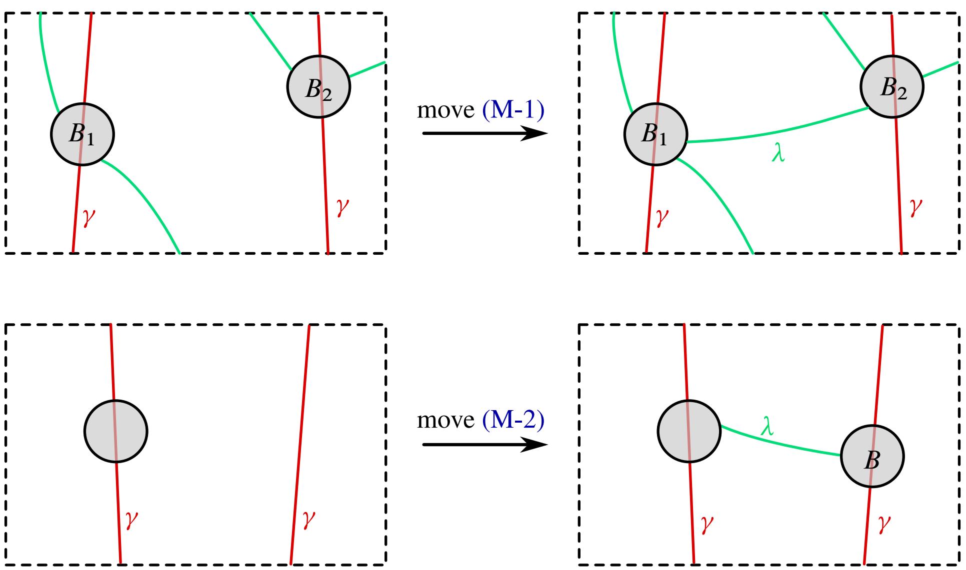

Figure 9: Moves (M-1) and (M-2) between sutured cell decompositions of (F, γ). The gray disks are the fattened 0–cells, and the green arcs marked λ are the Legendrians. The red arcs marked γ form the dividing set.

We now describe two moves between sutured cell decompositions $\mathcal{D}$ and $\mathcal{D}'$ of a surface F with dividing set γ:

($\mathcal{M}$-1) $\mathcal{D}'$ is obtained from $\mathcal{D}$ by adding or removing a 1–cell $\lambda \subseteq F \setminus \gamma$ that has both ends on boundaries of fattened 0–cells in $\mathcal{D}$, but is otherwise disjoint from the 0– and 1–cells of $\mathcal{D}$.

($\mathcal{M}$-2) $\mathcal{D}'$ is obtained from $\mathcal{D}$ by adding or removing a fattened 0–cell B and a 1–cell λ that connects B to another 0–cell in $\mathcal{D}$, but is otherwise disjoint from the 0– and 1–cells of $\mathcal{D}$.

Moves (M-1) and (M-2) are illustrated in Figure 9.

Lemma 3.3 *If F is a closed, orientable surface and $\gamma \subseteq F$ is a dividing set, then any two sutured cell decompositions of (F, γ) can be connected by a sequence of moves (M-1) and (M-2).*

To prove Lemma 3.3, it is convenient to consider the following notion of cell decomposition for surfaces with divides which is more general than Definition 3.1:

Definition 3.4 A *generalized sutured cell decomposition* $\mathcal{D}^\bullet$ of a surface F with divides γ consists of the following:

- **Fattened 0–cells** A collection of pairwise disjoint disks $B_1, \dots, B_n \subseteq F$ such that each $B_i \cap \gamma$ is an arc and each component of γ intersects some B_i.
- **1–cells** A collection of pairwise disjoint, properly embedded arcs
$$\lambda_1, \dots, \lambda_m \subseteq F \setminus \bigcup_{i=1}^{n} \operatorname{int}(B_i)$$
which are transverse to γ such that each component of
$$F \setminus (B_1 \cup \dots \cup B_n \cup \lambda_1 \cup \dots \cup \lambda_m)$$
is homeomorphic to an open disk (which may be disjoint from γ).
- **Splitting arcs** A collection of oriented, properly embedded, pairwise disjoint arcs
$$c_1, \dots, c_l \subseteq \bigcup_{i=1}^{n} B_i,$$
disjoint from each of the λ_i, such that $|c_i \cap \gamma| = 1$ for all i and such that each component of the closure of $F \setminus (B_1 \cup \dots \cup B_n \cup \lambda_1 \cup \dots \cup \lambda_m)$ which is disjoint from γ intersects the terminal endpoint of at least one c_i.

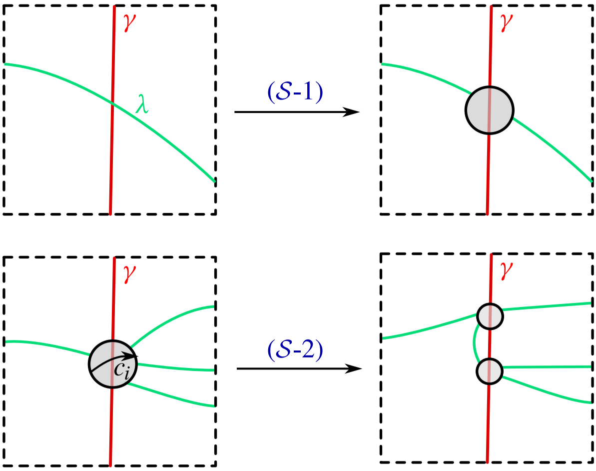

Figure 10: Splitting a generalized sutured cell decomposition $\mathcal{D}^\bullet$. On the left is $\mathcal{D}^\bullet$, and on the right is the split, $\mathcal{S}(\mathcal{D}^\bullet)$.

Note that a sutured cell decomposition can be viewed as a generalized sutured cell decomposition with no splitting arcs. In the other direction, if $\mathcal{D}^\bullet$ is a generalized sutured cell decomposition of a surface F with dividing set γ, we can define the *split* of $\mathcal{D}^\bullet$, denoted by $\mathcal{S}(\mathcal{D}^\bullet)$, as the (genuine) sutured cell decomposition obtained by performing the following two modifications to $\mathcal{D}^\bullet$:

($\mathcal{S}$-1) A small, fattened 0–cell is added at each intersection point between a 1–cell λ_i and the dividing set γ.

($\mathcal{S}$-2) The fattened 0–cells are split in half along the arcs $c_1, \dots, c_l$. A new 1–cell is added for each of the arcs c_i, as shown in Figure 10. The new 1–cell is on the initial side of c_i, with respect to the orientation of c_i.

Proof of Lemma 3.3 Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are two sutured cell decompositions of (F, γ). By move (M-2), we can increase the number of fattened 0–cells in both, and hence we can assume that they have the same number of fattened 0–cells. It is not hard to see that we can isotope or rescale a fattened 0–cell by a sequence of moves (M-1) and (M-2). The procedure is illustrated in Figure 11. So we can assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same fattened 0–cells.

We can view the fattened 0–cells as 0–handles of the surface, and the 1–cells $\lambda_1, \dots, \lambda_n$ as the cores of 1–handles. Therefore, a generalized sutured cell decomposition with

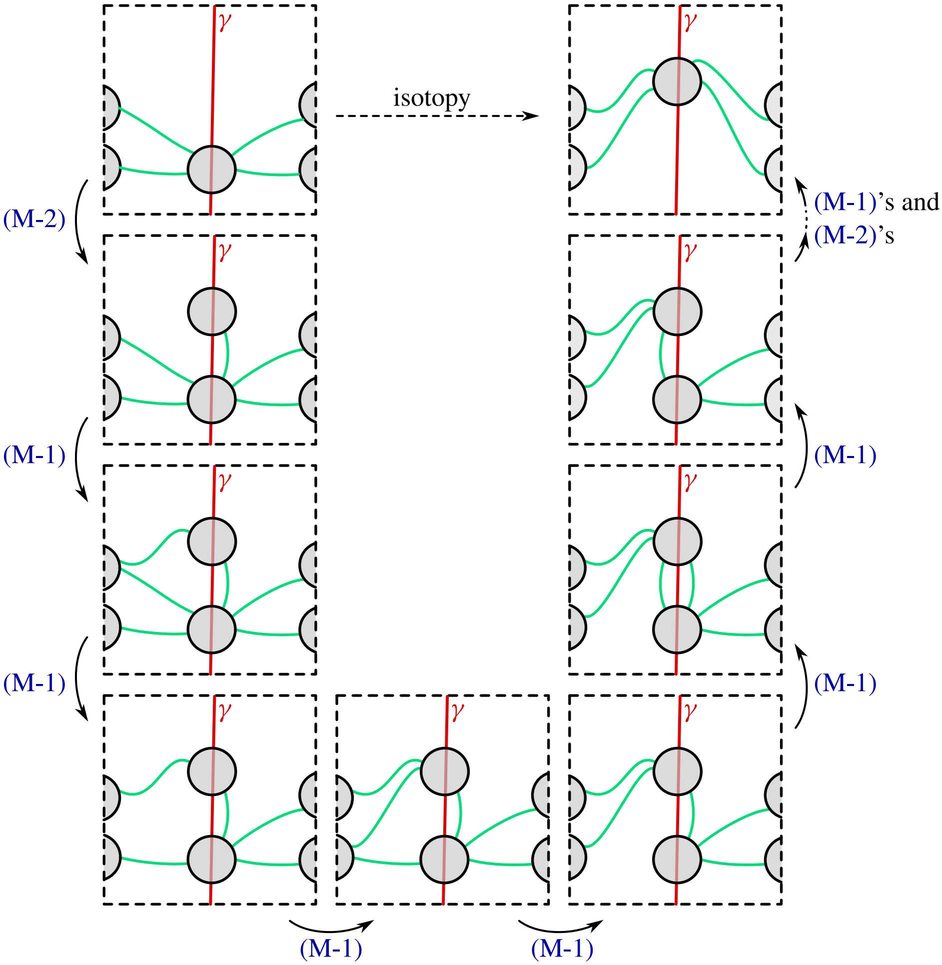

Figure 11: Isoting a fattened 0–cell via a sequence of moves (M-1) and (M-2).

fattened 0–cells $B_1, \dots, B_n$ corresponds to a Morse function

$$f\colon \Sigma \setminus \operatorname{int}(B_1 \cup \cdots \cup B_n) \to [0, \infty)$$

that has no index 0 critical points, achieves its minimal value of 0 on $\partial B_1 \cup \cdots \cup \partial B_n$, and is such that the union of the descending manifolds of the index 1 critical points is $\lambda_1 \cup \cdots \cup \lambda_m$. The collection of splitting arcs $c_1, \dots, c_l$ is some additional combinatorial data, which can always be constructed, given such a Morse function. Two such Morse functions can always be connected by a path of smooth functions with no index 0 critical points relative to $\partial B_1 \cup \cdots \cup \partial B_n$ (see Lemma 7.6 for a proof of a closely related result that adapts to our present setting). Consequently, two generalized sutured cell decompositions can be connected by the following moves:

($g\mathcal{M}$-1) Adding or removing a splitting arc disjoint from all the other splitting arcs and 1–cells.

($g\mathcal{M}$-2) Isotoping a 1–cell.

($g\mathcal{M}$-3) Adding or removing a 1–cell.

($g\mathcal{M}$-4) Arc sliding a 1–cell λ_i across another 1–cell λ_j.

We can view any sutured cell decomposition as a generalized sutured cell decomposition with no splitting arcs, and with no intersections between the 1–cells and the dividing set. Hence, to prove the main claim, it suffices to show that if two generalized sutured cell decompositions $\mathcal{D}_1^\bullet$ and $\mathcal{D}_2^\bullet$ differ by moves ($g\mathcal{M}$-1)–($g\mathcal{M}$-4), then their splits $\mathcal{S}(\mathcal{D}_1^\bullet)$ and $\mathcal{S}(\mathcal{D}_2^\bullet)$ differ by a sequence of moves (M-1) and (M-2).

We first address move ($g\mathcal{M}$-1). We illustrate in Figure 12 an example of how to connect the splits of two generalized sutured cell decompositions that differ by move ($g\mathcal{M}$-1).

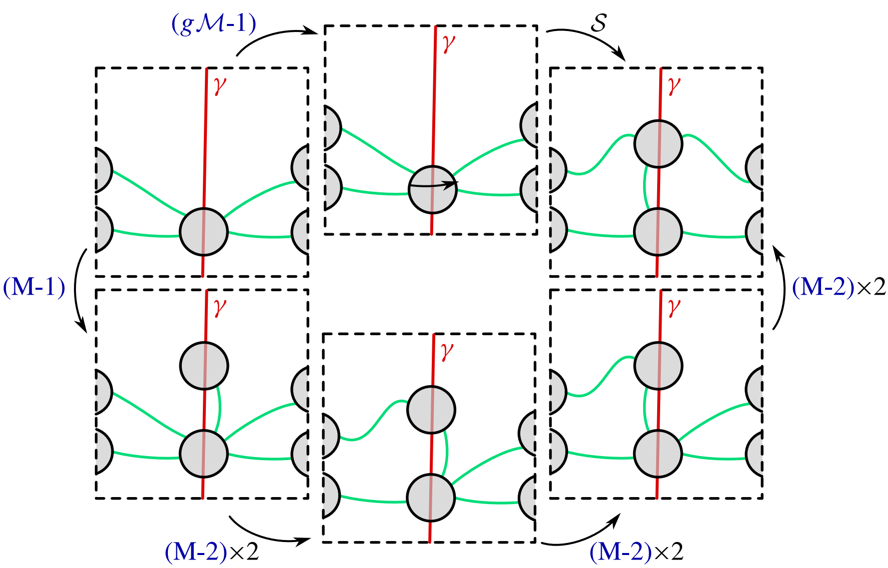

Figure 12: Connecting $\mathcal{S}(\mathcal{D}_1^\bullet)$ and $\mathcal{S}(\mathcal{D}_2^\bullet)$ by a sequence of moves (M-1) and (M-2) when $\mathcal{D}_1^\bullet$ and $\mathcal{D}_2^\bullet$ differ by move ($g\mathcal{M}$-1).

Next, we consider move ($g\mathcal{M}$-2), isotopies of a 1–cell λ_i. Moves of type ($g\mathcal{M}$-2) can be further broken down into three subtypes:

(1) Isotopies supported outside a neighborhood of γ.

(2) Isotopies supported in a neighborhood of γ that are fixed on the circles ∂B_j and which either create or cancel a pair of intersection points between γ and λ_i.

(3) Isotopies supported in a neighborhood of a single ∂B_i that isotope an end of a 1–cell λ_i across γ.

Isotopies of type (1) can be addressed using a manipulation similar to the one shown in Figure 11, so we leave this case to the reader. In case (2), a sequence of moves (M-1) and (M-2) suffice; an example is shown in Figure 13. We leave case (3) to the reader, since the manipulation is similar.

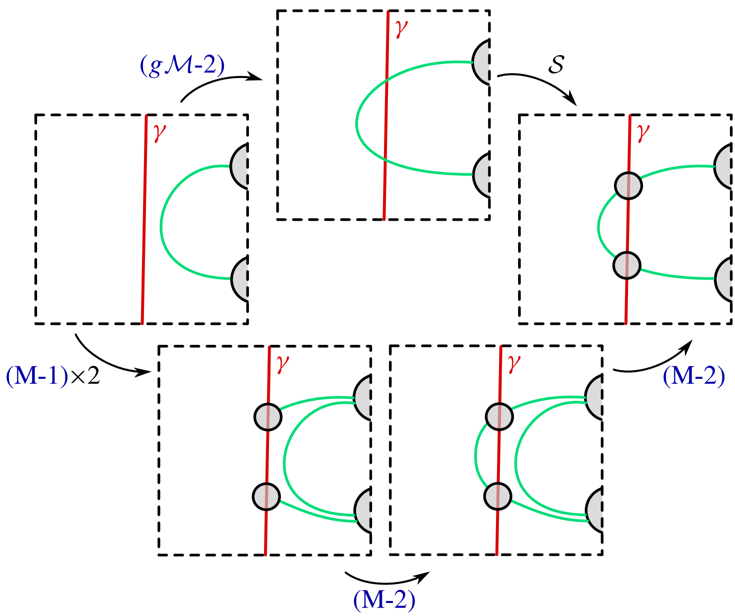

Figure 13: An example of connecting $\mathcal{S}(\mathcal{D}_1^\bullet)$ and $\mathcal{S}(\mathcal{D}_2^\bullet)$ by a sequence of moves (M-1) and (M-2) when $\mathcal{D}_1^\bullet$ and $\mathcal{D}_2^\bullet$ differ by move ($g\mathcal{M}$-2).

Sufficiency of moves (M-1) and (M-2) to connect splits of generalized sutured cell decompositions differing by moves ($g\mathcal{M}$-3) and ($g\mathcal{M}$-4) is proven in an analogous fashion. We leave the argument as an easy exercise for the reader. □

3.2 Contact cell decompositions

In this section, we describe some background on contact cell decompositions. The technical content is due to Honda, Kazez, and Matić [13, Section 1.1] and Giroux [8].

Definition 3.5 Suppose that (M, γ) is a sutured submanifold of (M', γ') and ξ is a contact structure on $Z := M' \setminus \operatorname{int}(M)$ such that ∂Z is convex with dividing set $\gamma \cup \gamma'$. A *contact cell decomposition of* (Z, ξ) consists of the following data:

(1) A nonvanishing contact vector field v defined on a neighborhood of ∂Z in Z and transverse to ∂Z such that it induces the dividing set $\gamma \cup \gamma'$. The flow of v induces a diffeomorphism of $\partial Z \times I$ with a collar neighborhood of ∂Z in Z. Under this diffeomorphism, v corresponds to $\partial/\partial t$, the boundary ∂M is identified with $\partial M \times \{0\}$, and $\partial M'$ is identified with $\partial M' \times \{1\}$. We let $\nu = v|_{\partial M \times I}$ and $\nu' = v|_{\partial M' \times I}$.

(2) "Barrier" surfaces $S \subseteq \partial M \times (0,1)$ and $S' \subseteq \partial M' \times (0,1)$ in Z that are isotopic to ∂M and $\partial M'$, respectively, and are transverse to v. Write N and N' for the collar neighborhoods of ∂M and $\partial M'$ in Z that are bounded by S and S', respectively, and set $Z' = Z \setminus \operatorname{int}(N \cup N')$.

(3) A Legendrian graph $\Gamma \subseteq Z'$ that intersects $\partial Z'$ transversely in a finite collection of points along the dividing set of $\partial Z'$ with respect to the vector field v. Furthermore, Γ is tangent to the vector field v near $\partial Z'$.

(4) A choice of regular neighborhood $N(\Gamma)$ of Γ such that $\xi|_{N(\Gamma)}$ is tight and $\partial N(\Gamma) \setminus \partial Z'$ is a convex surface. Furthermore, $N(\Gamma) \cap \partial Z'$ is a collection of disks D with Legendrian boundary such that $\operatorname{tb}(\partial D) = -1$. We assume that the edge rounding procedure of Honda [10, Section 3.3.2] has been performed so that $N(\Gamma)$ meets $\partial Z'$ tangentially along the Legendrian unknots forming $\partial N(\Gamma)$.

(5) A collection of convex 2–cells $D_1, \dots, D_n$ inside $Z' \setminus \operatorname{int}(N(\Gamma))$ with Legendrian boundary on $\partial\big(Z' \setminus \operatorname{int}(N(\Gamma))\big)$ and $\operatorname{tb}(\partial D_i) = -1$.

Furthermore, the following hold:

(a) The complement of $N(\Gamma) \cup D_1 \cup \dots \cup D_n$ in Z' is a finite collection of topological 3–balls, and ξ is tight on each.

(b) The disks in $N(\Gamma) \cap \partial Z'$ and the Legendrian arcs $\partial D_i \cap \partial Z'$ induce a sutured cell decomposition of $\partial Z'$ with dividing set $\{x \in \partial Z' : v_x \in \xi_x\}$ (Definition 3.1).

Remark 3.6 Given surfaces S and S' in Z and a transverse contact vector field v defined in a neighborhood of ∂Z that satisfy (1) and (2), it is not always possible to construct a contact cell decomposition of Z with S and S' as barrier surfaces. For example, the characteristic foliations on S and S' may obstruct the existence of sutured cell decompositions (Definition 3.1) such that the arcs λ_i and the curves ∂B_i are Legendrian.

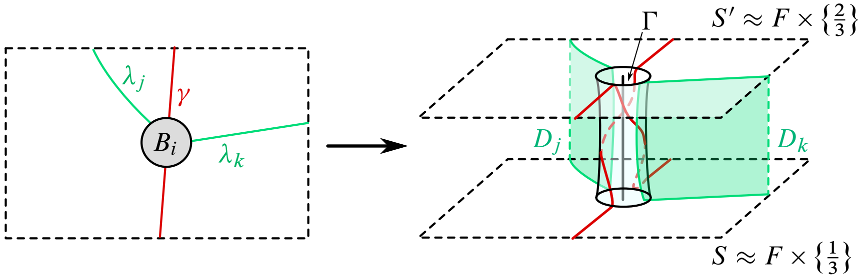

Figure 14: The left shows a sutured cell decomposition of a surface F with dividing set γ. The right shows the induced product contact cell decomposition of an I–invariant contact structure on $F \times I$ with dividing set γ from Example 3.7. The barrier surfaces are S and S', which are C^0 close to $F \times \{\frac{1}{3}\}$ and $F \times \{\frac{2}{3}\}$, respectively. The fattened 0–cell B_i is shown on the left in gray. The dividing set γ is shown in red. The Legendrian arcs λ_j and λ_k, as well as the corresponding convex 2–cells D_j and D_k, are shown in green.

We now describe an important example of contact cell decompositions:

Example 3.7 Suppose that (M, γ) is a sutured submanifold of (M', γ') and ξ is a compatible contact structure on $Z = M' \setminus \operatorname{int}(M)$. Furthermore, suppose that (Z, ξ) is contactomorphic to $F \times I$ with an I–invariant contact structure for a surface $F \cong \partial M \cong \partial M'$. Suppose that the disks $B_1, \dots, B_n \subseteq F$ and the (not necessarily Legendrian) properly embedded arcs $\lambda_1, \dots, \lambda_m \subseteq F \setminus \operatorname{int}(B_1 \cup \dots \cup B_n)$ give a sutured cell decomposition of $F \times \{0\}$, as in Definition 3.1.

There is an induced contact cell decomposition of Z, called the *product contact cell decomposition*, which we describe presently. For an illustration, see Figure 14. Let $G \subseteq F$ be the graph

$$G = \partial B_1 \cup \dots \cup \partial B_n \cup \lambda_1 \cup \dots \cup \lambda_m;$$

this graph is not necessarily Legendrian. However, using Giroux's flexibility theorem [6, Proposition 3.6], we can find a surface $S \subseteq F \times I$ that is the image of a C^0–small isotopy of $F \times \{\frac{1}{3}\}$ that is transverse to $\partial/\partial t$ and such that the image of $G \times \{\frac{1}{3}\}$ is Legendrian. We let S' be the translation of S by $\frac{1}{3}$ in the $\partial/\partial t$ direction. We use S and S' for the barrier surfaces of our contact cell decomposition. We will also write $B_1, \dots, B_n$ and $\lambda_1, \dots, \lambda_m$ for the images on S of the corresponding cells on $F \times \{\frac{1}{3}\}$, under the chosen C^0–small isotopy. Let N and N' be the collars of ∂M and $\partial M'$ bounded by S and S', respectively, and write $Z' = Z \setminus \operatorname{int}(N \cup N')$.

For every $i \in \{1, \dots, n\}$, choose a point $p_i \in \operatorname{int}(\gamma \cap B_i)$. We define the Legendrian graph Γ as

$$\bigcup_{i=1}^{n} (\{p_i\} \times I) \cap Z',$$

and set $N(\Gamma) = \bigcup_{i=1}^{n} (B_i \times I) \cap Z'$. The contact 2–cells of the decomposition are defined to be $D_i := (\lambda_i \times I) \cap Z'$. Using Giroux's flexibility theorem on $\partial N(\Gamma)$, we can assume that each D_i has Legendrian boundary. The disks D_i have $\operatorname{tb}(\partial D_i) = -1$ since their boundaries intersect the dividing set exactly twice. If C is a component of $S \setminus (B_1 \cup \dots \cup B_n \cup \lambda_1 \cup \dots \cup \lambda_m)$, then $(C \times I) \cap Z'$ is a tight contact ball in the complement of $N(\Gamma) \cup D_1 \cup \dots \cup D_n$ in Z'.

To show invariance of the gluing map, we need to describe how two contact cell decompositions are related. We have the following; cf [13, Theorem 1.2]:

Proposition 3.8 *Suppose that (M, γ) is a sutured submanifold of (M', γ'), and ξ is a contact structure on $Z = M' \setminus \operatorname{int}(M)$ such that ∂Z is convex with dividing set $\gamma \cup \gamma'$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are contact cell decompositions of Z, then there is a sequence $\mathcal{C}_{(1)}, \dots, \mathcal{C}_{(\ell)}$ of contact cell decompositions with $\mathcal{C}_1 = \mathcal{C}_{(1)}$ and $\mathcal{C}_2 = \mathcal{C}_{(\ell)}$ such that $\mathcal{C}_{(i+1)}$ is obtained from $\mathcal{C}_{(i)}$ by one of the following moves, or its inverse:*

(C-1) ***Isotopy*** *Replacing a contact cell decomposition $\mathcal{C}$ with a contact cell decomposition of the form $\phi(\mathcal{C})$, where $\phi \in \operatorname{Diff}(Z)$* (not necessarily a contactomorphism) *fixes ∂Z pointwise and is isotopic to the identity relative to ∂Z.*

(C-2) ***Index 0/1 cancellation*** *Subdividing a Legendrian edge of Γ or adding a Legendrian edge that has one endpoint on Γ, but which is otherwise disjoint from Γ and the 2–cells of $\mathcal{C}$.*

(C-3) ***Index 1/2 cancellation*** *Adding a Legendrian edge λ to Γ and adding a convex 2–cell D with $\operatorname{tb}(\partial D) = -1$ such that the interiors of λ and D are disjoint from all the other cells, and $\partial D = c' \cup c''$, where c' is a Legendrian arc on a neighborhood of λ, and $c'' \subseteq \partial(Z' \setminus \operatorname{int} N(\Gamma))$ is a Legendrian arc that is disjoint from the dividing set on $\partial(Z' \setminus \operatorname{int} N(\Gamma))$.*

(C-4) ***Index 2/3 cancellation*** *Adding a convex disk D with $\partial D \subseteq \partial(Z' \setminus \operatorname{int} N(\Gamma))$ and $\operatorname{tb}(\partial D) = -1$ disjoint from the other cells.*

Proof The result follows from an adaptation of the subdivision techniques due to Giroux [8] and Honda, Kazez, and Matić [13, Theorem 1.2]. Giroux's technique involves the following move: Replace a convex 2–cell D with a pair of convex 2–cells D_1 and D_2 with $\mathrm{tb}(\partial D_i) = -1$ that meet along a Legendrian arc λ such that $D_1 \cup D_2 = D$ and $\lambda \subseteq D$ is an arc that intersects the dividing set of D transversely at a single point.

Lemma 3.9 *The above subdivision of a 2–cell can be achieved using moves (C-1) through (C-4).*

Proof Let D' be a parallel copy of D, intersecting D only along ∂D. Let $D_1' \subseteq D'$, $D_2' \subseteq D'$, and $\lambda' \subseteq D'$ be the images of D_1, D_2, and λ, respectively. We can add D_1' and λ' to the decomposition using move (C-3). Then we can add D_2' to the decomposition using move (C-4). Then we remove D from the decomposition using the inverse of move (C-4). Finally, we use move (C-1) to move D_1', D_2', and λ' into the positions of D_1, D_2, and λ, while preserving all the other cells. $\square$

Step 1 *Using move (C-1), we can change the vector field v_1 of $\mathcal{C}_1$ to the vector field v_2 of $\mathcal{C}_2$.*

We will focus on changing the contact vector field $\nu_1 = v_1|_{\partial M \times I}$ to the vector field $\nu_2 = v_2|_{\partial M \times I}$, as changing $\nu_1' = v_1|_{\partial M' \times I}$ to the vector field $\nu_2' = v_2|_{\partial M' \times I}$ is analogous. The idea is that the space of germs of contact vector fields defined on open neighborhoods of ∂M that induce the dividing set γ is convex and hence contractible. There is an open neighborhood of ∂M where the convex combinations $\nu_t = (2-t)\nu_1 + (t-1)\nu_2$ are nonvanishing for $t \in [1,2]$. Then ν_t is transverse to ∂M for every $t \in [1,2]$. We can define an isotopy ψ_t of a neighborhood of ∂M, by flowing a point $p \in M$ backward along ν_1 until it hits ∂M (say, at a time $-T(p)$), then flowing along ν_t for time $T(p)$. This yields a 1–parameter family of contactomorphic embeddings of a neighborhood U of ∂M into Z. We can extend this to a family of contactomorphisms ψ_t of all of M for $t \in [1,2]$ by writing ψ_t as the integral of a time-dependent contact vector field X_t (ie X_t is a contact vector field for each t) defined over $\partial M \times I$ and then extending X_t to all of Z using time-dependent contact Hamiltonians that vanish outside a neighborhood of ∂M. Using move (C-1), we can isotope S_1 into U using move (C-1) and then push $\mathcal{C}_1$ forward under ψ_1.

Step 2 *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are contact cell decompositions with the same vector fields v and v', then we can achieve that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same barrier surfaces with identical induced sutured cell decompositions using moves (C-1) through (C-4).*

Let $\mathcal{D}_i$ and $\mathcal{D}_i'$ be the sutured cell decompositions induced on the barrier surfaces S_i and S_i' of the contact cell decomposition $\mathcal{C}_i$ for $i \in \{1,2\}$. Write $\pi\colon \partial M \times I \to \partial M$

and $\pi'\colon \partial M' \times I \to \partial M'$ for the projections. Note that $\pi(\mathcal{D}_1)$ and $\pi(\mathcal{D}_2)$ are sutured cell decompositions of ∂M, and $\pi'(\mathcal{D}_1')$ and $\pi'(\mathcal{D}_2')$ are sutured cell decompositions of $\partial M'$. Let us focus on the barrier surfaces S_1 and S_2 near ∂M, and their sutured cell decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$. By Lemma 3.3, it follows that $\pi(\mathcal{D}_1)$ and $\pi(\mathcal{D}_2)$ can be connected by moves (M-1) and (M-2).

We will show that, using moves (C-1)–(C-4), we can connect $\mathcal{C}_1$ to a contact cell decomposition $\widehat{\mathcal{C}}_2$ with the same contact vector field as $\mathcal{C}_1$ and $\mathcal{C}_2$, whose induced sutured cell decomposition of ∂M is C^0 close to $\pi(\mathcal{D}_2)$. It is sufficient to show the claim when $\pi(\mathcal{D}_2)$ is obtained from $\pi(\mathcal{D}_1)$ by a single instance of move (M-1) or (M-2).

Firstly, we can construct two contact automorphisms φ_1 and φ_2 of (Z,ξ) that are isotopic to the identity relative to ∂Z such that $\varphi_1(\mathcal{C}_1)$ and $\varphi_2(\mathcal{C}_2)$ are cell decompositions with Legendrian graphs $\varphi_1(\Gamma_1)$ and $\varphi_2(\Gamma_2)$, respectively, that intersect $\partial M \times I$ and $\partial M' \times I$ along arcs of the form $\{p\} \times [a,1]$ or $\{p'\} \times [0,a]$ for various $p \in \gamma$, $p' \in \gamma'$, and $a > 0$. Furthermore, by applying move (C-1), we may assume that each component of the intersection of every 2–cell with $\partial Z \times I$ is C^0 close to a set of the form $(\lambda \times I) \cap Z'$ for some arc λ in ∂Z.

Consider first the case when $\pi(\mathcal{D}_2)$ is obtained from $\pi(\mathcal{D}_1)$ by adding a 1–cell, as in move (M-1). Write $\lambda \subseteq \partial M$ for the new 1–cell, which is added to the complement of the other 1–cells of $\pi(\mathcal{D}_1)$. To construct $\widehat{\mathcal{C}}_2$, we add a new Legendrian edge e to the graph Γ_1 of $\mathcal{C}_1$, as well as a new 2–cell c, as follows. Let $\lambda_1 \subseteq S_1$ denote the preimage of λ under $\pi|_{S_1}$. Using Legendrian realization that only changes S_1 in the $\partial/\partial t$–direction, we may assume that the resulting arc $\lambda_1 \subseteq S_1$ is Legendrian, and projects C^0 close to λ under π. Note that the isotopy of S_1 used for Legendrian realization can be chosen to fix the 0– and 1–cells of $\mathcal{D}_1$, and hence can be realized by an instance of move (C-1) according to the following result:

Lemma 3.10 *Let $\mathcal{C}$ be a contact cell decomposition with barrier surface S and induced sutured cell decomposition $\mathcal{D}$ of S. Then any isotopy of S through surfaces that are transverse to $v = \partial/\partial t$ that fixes the 0– and 1–cells of $\mathcal{D}$ can be achieved using move (C-1). The same holds for S', v', and $\mathcal{D}'$.*

Proof As above, we can assume that the graph Γ is a union of Legendrians of the form $\{p\} \times [a,1]$ for various $p \in \gamma$ and $0 < a < 1$, and the 2–cells are small perturbations of sets of the form $(\lambda \times I) \cap Z'$ for arcs $\lambda \subseteq \partial M$. Suppose that $f\colon S \to \partial M \times I$ is an embedding that is the identity on the 0– and 1–cells of $\mathcal{D}$ and whose image is transverse to v. We can extend f to an automorphism of Z that is the identity outside $\partial M \times (0,1)$.

It is straightforward to see that $f(\mathcal{C})$ is a contact cell decomposition, and that $\mathcal{C}$ and $f(\mathcal{C})$ are related by move (C-1). □

Let $\varepsilon > 0$ be small, and let

$$\theta\colon S_1 \times (-\varepsilon, \varepsilon) \to \partial M \times I$$

denote the map induced by the flow of $v = \partial/\partial t$, ie translation in the I–direction. We obtain the new edge e by extending $\theta(\lambda_1 \times \{\varepsilon\})$ into $N(\Gamma_1)$. The new 2–cell c is the intersection of $\theta(\lambda_1 \times [0,\varepsilon])$ with the complement of $N(\Gamma_1 \cup e)$. A schematic is shown in Figure 15. Adding e and c to $\mathcal{C}_1$ is an instance of move (C-3).

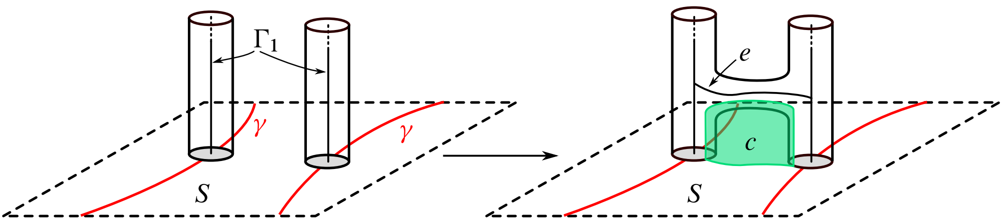

Figure 15: Adding a new edge e and a new 2–cell c to a contact cell decomposition to realize move (M-1) on the induced sutured cell decomposition of ∂M. Shown on the left is a neighborhood of the Legendrian graph Γ_1, as well as the barrier surface S. Shown on the right is a neighborhood of $\Gamma_1 \cup e$, as well as the new 2–cell c.

In the opposite direction, suppose that $\pi(\mathcal{D}_2)$ is obtained from $\pi(\mathcal{D}_1)$ by removing a 1–cell λ, as in move (M-1). Let c be the 2–cell of $\mathcal{C}_1$ corresponding to $\lambda_1 = (\pi|_{S_1})^{-1}(\lambda)$. Note that the 2–cells of $\pi(\mathcal{D}_1)$ neighboring λ are disjoint, since otherwise there would be a component of the complement of $\pi(\mathcal{D}_2)$ that were not a disk. We subdivide $\mathcal{C}_1$ away from $S \cup S'$ such that the 3–cells of $\mathcal{C}_1$ corresponding to the 2–cells of $\mathcal{D}_1$ on the two sides of λ_1 become distinct. We then subdivide c along $e := \theta(\lambda_1 \times \{\varepsilon\})$ for ε small using Lemma 3.9, and cancel the 2–cell $\theta(\lambda_1 \times [0,\varepsilon]) \setminus N(\Gamma_1 \cup e)$ with one of the neighboring 3–cells using move (C-4).

Next, we consider the case when $\pi(\mathcal{D}_2)$ is obtained from $\pi(\mathcal{D}_1)$ by adding a fattened 0–cell B and a 1–cell λ, as in move (M-2). To construct the new contact cell decomposition $\widehat{\mathcal{C}}_2$, we add two new Legendrian edges e_1 and e_2 to $\mathcal{C}_1$ that meet at a valence 2 vertex, as well as a new 2–cell c, as follows. Let $\lambda_1 \subseteq S_1$ denote the Legendrian realization of the preimage under $\pi|_{S_1}$ of an extension of the Legendrian λ to a point $p \in \operatorname{int}(B)$. The Legendrian realization can be achieved using move (C-1) according to Lemma 3.10. We define the edge e_1 to be $\theta(\lambda_1 \times \{\varepsilon\})$ for some small $\varepsilon > 0$ and e_2 as $\theta(\{p\} \times [0,\varepsilon])$. We let the 2–cell c be $\theta(\lambda_1 \times [0,\varepsilon]) \setminus N(\Gamma_1 \cup e_1 \cup e_2)$. Note that adding e_1, e_2, and c to $\mathcal{C}_1$ can be achieved by moves (C-2) and (C-3). Writing $\widehat{\mathcal{D}}_2$ for the sutured cell decomposition of S induced by $\widehat{\mathcal{C}}_2$, we note that $\pi(\widehat{\mathcal{D}}_2)$ is C^0 close to the one obtained from $\pi(\mathcal{D}_1)$ by move (M-2).

In the opposite direction, suppose that $\pi(\mathcal{D}_2)$ is obtained from $\pi(\mathcal{D}_1)$ by removing a fattened 0–cell B and a 1–cell λ, as in move (M-2). Then we split the 2–cell of $\mathcal{C}_1$ corresponding to $\lambda_1 = (\pi|_{S_1})^{-1}(\lambda)$ along $e_1 := \theta(\lambda_1 \times \{\varepsilon\})$, and cancel the 2–cell $\theta(\lambda_1 \times [0,\varepsilon]) \setminus N(\Gamma_1 \cup e_1)$ and the 1–cell $\theta(\{p\} \times [0,\varepsilon])$ for $p = B \cap \Gamma_1$ using move (C-3).

Having obtained $\widehat{\mathcal{C}}_2$ as above, we can construct an isotopy of M that is supported in a neighborhood of ∂M and moves $\widehat{\mathcal{C}}_2$ to a contact cell decomposition $\mathcal{C}_2''$ that shares the same barrier surface and induced sutured cell decomposition of ∂M as $\mathcal{C}_2$. It follows that $\widehat{\mathcal{C}}_2$ and $\mathcal{C}_2''$ are related by move (C-1). This completes Step 2.

Step 3 *Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are contact cell decompositions with the same barrier surfaces S and S' and the same contact vector fields v and v' such that $\mathcal{C}_1$ and $\mathcal{C}_2$ induce the same sutured cell decompositions $\mathcal{D}$ and $\mathcal{D}'$ of S and S' and such that the Legendrian 1–skeletons of $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect S and S' at the same points. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ can be connected by moves (C-1) through (C-4).*

The proof follows from the strategy of [13, Theorem 1.2], which we summarize using our present notation. As in Step 2, we can assume that the graph Γ is a union of Legendrians of the form $\{p\} \times [a,1]$ or $\{p'\} \times [0,a]$ for various $p \in \gamma$, $p' \in \gamma'$, and $0 < a < 1$, and the 2–cells are small perturbations of sets of the form $(\lambda \times I) \cap Z'$ for arcs $\lambda \subseteq \partial Z$.

Using Lemma 3.9, we subdivide each 2–cell D of $\mathcal{C}_1$ and $\mathcal{C}_2$ that intersects $\partial Z \times I$ by adding a Legendrian arc obtained by perturbing $D \cap (\partial M \times \{1\} \cup \partial M' \times \{0\})$ along v relative to its endpoints. Furthermore, using move (C-4), we add a new convex 2–cell D with Legendrian boundary and $\operatorname{tb}(\partial D) = -1$ near $\partial M \times \{1\}$ for each 2–cell of the sutured cell decomposition $\mathcal{D}$ and near $\partial M \times \{0\}$ for each 2–cell of $\mathcal{D}'$. Then we apply the subdivision procedure of Giroux [8] away from $\partial Z \times I$. □

3.3 Contact handle maps

We first recall the definition of contact handles in dimension 3 due to Giroux [6]; see also Ozbagci [21].

Definition 3.11 For $k \in \{0,1,2,3\}$, a 3–*dimensional contact handle of index* k attached to a sutured manifold (M,γ) is a tight contact ball (B_0,ξ_0) with convex boundary (possibly with corners) that is attached via a map $\phi\colon S \to \partial M$ for some subset $S \subseteq \partial B_0$. Furthermore, the dividing set of ξ_0 on S is mapped into γ under ϕ, and we have the following requirements, depending on the index:

Index 0 $B_0 = D^3$ has no corners, and $S = \varnothing$. The dividing set on ∂B_0 is a single circle.

Index 1 As a manifold with corners, B_0 is $I \times D^2$, and $S = \partial I \times D^2$. The dividing set on $\partial I \times D^2$ consists of one arc on each component. The dividing set on $I \times \partial D^2$ consists of two arcs, each connecting the two components of $\partial I \times \partial D^2$.

Index 2 As a manifold with corners, B_0 is $D^2 \times I$ and $S = \partial D^2 \times I$. The dividing set is the same as on a contact 1–handle.

Index 3 $B_0 = D^3$ has no corners, and $S = \partial D^3$. The dividing set on ∂B_0 is a single circle.

In Figure 16, we have drawn the dividing sets on contact handles. Note that, for handles of index 1 and 2, the curves $\partial I \times \partial D^2$ and $\partial D^2 \times \partial I$, respectively, are not Legendrian.

3.3.1 Contact 0–handle map Adding a contact 0–handle h^0 amounts to adding a copy of the product sutured manifold $(D^2 \times [-1,1], S^1 \times \{0\})$ to (M,γ). Noting that

$$\mathrm{SFH}(-D^2 \times [-1,1], -S^1 \times \{0\}) \cong \mathbb{F}_2,$$

the contact 0–handle map is simply the tautological map

$$\begin{aligned} C_{h^0}\colon \mathrm{SFH}(-M,-\gamma) &\xrightarrow{\cong} \mathrm{SFH}(-M,-\gamma) \otimes \mathbb{F}_2 \\ &\xrightarrow{\cong} \mathrm{SFH}(-M,-\gamma) \otimes \mathrm{SFH}(-D^2 \times [-1,1], -S^1 \times \{0\}) \\ &\xrightarrow{\cong} \mathrm{SFH}\big((-M,-\gamma) \sqcup (-D^2 \times [-1,1], -S^1 \times \{0\})\big). \end{aligned}$$

On the chain level, this map is given as follows. Choose a diagram $\mathcal{H} = (\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of $(-M,-\gamma)$. Since $\mathcal{H}_0 = (\overline{D}^2, \varnothing, \varnothing)$ is a diagram of $(-D^2 \times [-1,1], -S^1 \times \{0\})$, the disjoint union $\mathcal{H} \sqcup \mathcal{H}_0$ is a diagram of $(-M,-\gamma) \sqcup (-D^2 \times [-1,1], -S^1 \times \{0\})$. Let $i\colon \overline{\Sigma} \to \overline{\Sigma} \sqcup \overline{D}^2$ be the inclusion. Then, for $\boldsymbol{x} = (x_1,\dots,x_d) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we set $C_{h^0}(\boldsymbol{x}) = (i(x_1),\dots,i(x_d))$. This clearly induces C_{h^0} on homology.

3.3.2 Contact 1–handle map A contact 1–handle h^1 determines two points along the sutures. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard surface for (M,γ), glue a strip to $\partial\Sigma$ where the

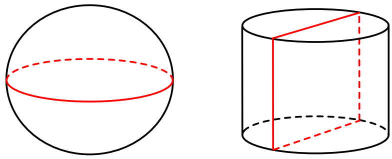

Figure 16: Contact handles. On the left is a picture of a contact 0–handle or 3–handle. On the right is a contact 1–handle or 2–handle.

feet of the 1–handle are attached; see Figure 17. Add no new $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ curves. The map on complexes is the tautological map induced by the inclusion of Heegaard surfaces. Let us call this map C_{h^1}. This is an isomorphism of chain complexes, since the domains of the curves counted by the boundary map on $\mathrm{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ have coefficient zero along $\partial\Sigma$, where the strip is attached; see [14, Lemma 9.13].

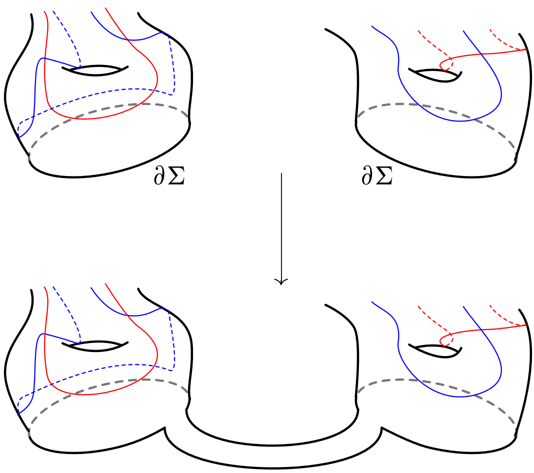

Figure 17: A contact 1–handle, on the level of diagrams.

Lemma 3.12 *The map C_{h^1} is natural; ie it commutes with change of diagrams maps.*

Proof The proof is straightforward, since the change of diagrams maps are either tautological (stabilization, isotopy) or involve counting holomorphic curves that do not intersect $\partial\Sigma \times \mathrm{Sym}^{k-1}(\Sigma) \subseteq \mathrm{Sym}^k(\Sigma)$ (continuation maps for changes of the almost complex structure, triangle maps for changes of the $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ curves). □

3.3.3 Contact 2–handle map We now define a map for contact 2–handles. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be an admissible diagram of (M, γ). Suppose that l is the curve on ∂M along which we add a 2–handle h^2, and l intersects γ (the sutures) exactly twice.

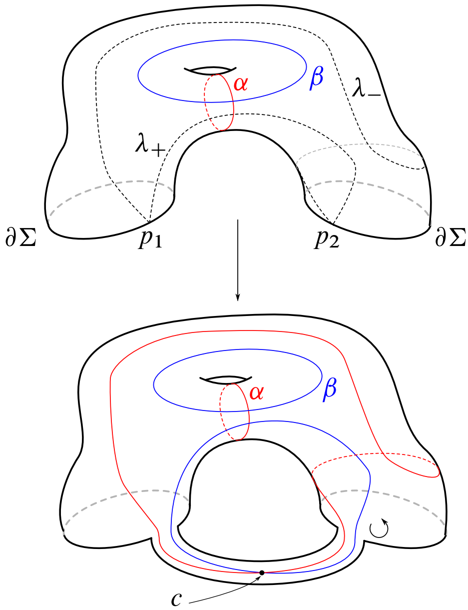

Figure 18: The contact 2–handle map. The orientation of $\overline{\Sigma}$ is shown on the bottom of the surface.

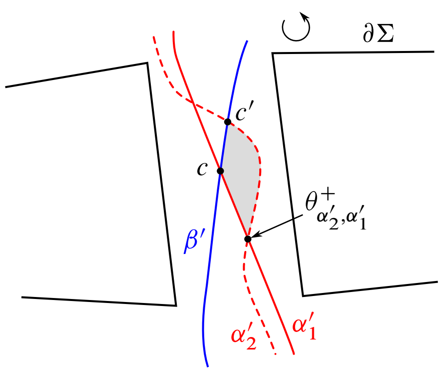

Figure 19: The local computation required to show that the contact 2–handle map is independent of handle slides of the α' curve used in the definition of the contact 2–handle map. A local computation using the vertex multiplicities (identical to the one in [13, Lemma 3.5]) shows that any holomorphic triangle with vertices at c and $\theta^+_{\alpha_2',\alpha_1'}$ also has a vertex at c', and has domain equal to the shaded region. The orientation of $\overline{\Sigma}$ is shown.

Let $p_1, p_2 \in \gamma$ be the two points of intersection and $l_\pm = l \cap R_\pm(\gamma)$. We denote the result of the 2–handle attachment by (M', γ').

We now construct a diagram of (M', γ'). Choose a sutured Morse function f and gradient-like vector field v on (M, γ) that induce $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, and follow the flow of v from l_+ and l_- onto Σ. Writing λ_+ and λ_- for the resulting arcs on Σ, we note that λ_+ avoids the $\boldsymbol{\beta}$ curves and λ_- avoids the $\boldsymbol{\alpha}$ curves. Hence, we can form a diagram $(\Sigma', \boldsymbol{\alpha} \cup \{\alpha'\}, \boldsymbol{\beta} \cup \{\beta'\})$, as in Figure 18. The surface Σ' is obtained by adding a band B to the boundary of Σ at p_1 and p_2. The curve α' is obtained by concatenating the curve λ_- with an arc in the band. The curve β' is obtained by concatenating the curve λ_+ with an arc in the band. We assume that α' and β' intersect in a single point in the band (and possibly other places outside the band). Furthermore, we assume that the intersection point of α' and β' has the configuration shown in Figure 18; ie locally, it looks like there is a holomorphic disk on $(\overline{\Sigma}', \boldsymbol{\alpha} \cup \{\alpha'\}, \boldsymbol{\beta} \cup \{\beta'\})$ going towards the intersection point that does not intersect $\partial\Sigma'$.

To see that $(\Sigma', \boldsymbol{\alpha} \cup \{\alpha'\}, \boldsymbol{\beta} \cup \{\beta'\})$ is indeed a diagram of (M', γ'), let

$$J = [-1, 1] \times \{0\} \subseteq D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

Consider the arc $a = J \times \left\{\frac{1}{2}\right\}$ in the 2–handle $D^2 \times I \subseteq M'$ connecting p_1 and p_2. Furthermore, consider the neighborhood

$$N(a) = \big(([-1, 1] \times (-\varepsilon, \varepsilon)) \cap D^2\big) \times I$$

of a, where $\varepsilon \in (0, 1)$. Then $N(a) \cap R_\pm(\gamma')$ is a regular neighborhood of

$$\gamma' \cap (D^2 \times I) = J \times \{0, 1\},$$

and $N(a)$ is a product 1–handle. On the sutured diagram level, attaching $N(a)$ corresponds to adding the band $B = J \times I$. Then $(D^2 \times I) \setminus N(a)$ is the disjoint union of two 3–dimensional 2–handles, attached to $(M \cup N(a), \gamma')$ along $l_\pm \cup s_\pm$, where $s_\pm$ is a properly embedded arc in $N(a) \cap R_\pm(\gamma')$. These attaching curves correspond to α' and β' on the Heegaard surface Σ'. The diagram $(\Sigma', \boldsymbol{\alpha} \cup \{\alpha'\}, \boldsymbol{\beta} \cup \{\beta'\})$ is also admissible.

The contact 2–handle map

$$C_{h^2}\colon \mathrm{SFH}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to \mathrm{SFH}(\overline{\Sigma}', \boldsymbol{\alpha} \cup \{\alpha'\}, \boldsymbol{\beta} \cup \{\beta'\})$$

is defined for $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ by the formula

$$C_{h^2}(\boldsymbol{x}) := \boldsymbol{x} \times c,$$

where $c \in \alpha' \cap \beta'$ is the intersection point in the band.

Lemma 3.13 (1) *The map C_{h^2} is a chain map.*

(2) *The map C_{h^2} is independent of the choices made in the construction, ie the choice of diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ and the choice of arcs λ_+ and λ_- obtained by projecting the two arcs l_+ and l_- onto Σ.*

Proof The claim that C_{h^2} is a chain map is straightforward. We wish to show that $\partial\circ C_{h^2}=C_{h^2}\circ\partial$, and hence we need to check that $\partial(\boldsymbol{x}\times c)=\partial\boldsymbol{x}\times c$. To this end, we note that any disk counted by $\partial(\boldsymbol{x}\times c)$ must have zero multiplicity around c, since the boundary $\partial\Sigma$ is nearby. Hence, the homology class of the disk is equal to a class on $(\overline{\Sigma},\boldsymbol{\alpha},\boldsymbol{\beta})$ with the constant class at c added.

We now show that C_{h^2} is independent of the choices made in the construction (ie commutes with the change of diagrams maps, appropriately). We note that there are two sources of ambiguity: the curves λ_+ and λ_- in Σ, and the diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$. Let us address the ambiguity of λ_+ and λ_-. Since they are gotten by flowing curves in ∂M under the gradient-like vector field of a Morse function, any two choices of λ_+ are related by handle slides over $\boldsymbol{\beta}$ curves (as well as any isotopy of λ_+, relative $\partial\Sigma$, such that it never intersects any $\boldsymbol{\beta}$ curves). Similarly, any two choices of λ_- are related by handle slides over the $\boldsymbol{\alpha}$ curves and isotopies within $\Sigma\setminus\boldsymbol{\alpha}$. To see that the change of diagrams map commutes with C_{h^2}, one realizes the handle slide and isotopy maps as triangle maps, and uses the local computation at the end of the proof of [13, Lemma 3.5], which is illustrated by [13, Figure 8]. For the reader's convenience, we have reproduced the relevant picture in Figure 19, since we will use the local computation later also. □

3.3.4 Contact 3–handle maps We will now define the contact 3–handle map. Let (M',γ') be the result of gluing a contact 3–handle h^3 to the balanced sutured manifold (M,γ), and suppose that (M',γ') is also balanced. The 3–handle is attached to a 2–sphere component $S\subseteq\partial M$ such that $\gamma_S=\gamma\cap S$ is a single closed curve. One defines the contact 3–handle map C_{h^3} to be equal to the composition of the 4–dimensional 3–handle map obtained by surgering $(-M,-\gamma)$ along the 2–sphere obtained by pushing S into $\operatorname{int}(M)$, followed by the inverse of the contact 0–handle map corresponding to removing the resulting copy of $(-D^2\times[-1,1],-S^1\times\{0\})$.

On the diagram level, we choose a diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ of (M,γ) such that there are curves $\alpha\in\boldsymbol{\alpha}$ and $\beta\in\boldsymbol{\beta}$ that are parallel to $\gamma_S\subseteq\partial\Sigma$ and intersect transversely in two points x and y, and there are no other $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ curves between α and γ_S or β and γ_S. Let Σ' be the result of gluing a disk to Σ along γ_S. Then $(\Sigma',\boldsymbol{\alpha}\setminus\{\alpha\},\boldsymbol{\beta}\setminus\{\beta\})$ is a diagram of (M', γ'). Suppose that x has larger relative grading than y in $(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta})$. For $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we set $C_{h^3}(\boldsymbol{x} \times \{x\}) = 0$ and $C_{h^3}(\boldsymbol{x} \times \{y\}) = \boldsymbol{x}$.

3.4 Definition of the contact gluing map

We now define the gluing map, in terms of a contact cell decomposition (Definition 3.5). Suppose that (M, γ) and (M', γ') are balanced sutured manifolds, (M, γ) is a sutured submanifold of (M', γ'), and ξ is contact structure on $Z = M' \setminus \operatorname{int}(M)$ that has dividing set $\gamma \sqcup \gamma'$. Let $\mathcal{C}$ be a contact cell decomposition of Z. Let ν and ν' be the chosen contact vector fields, and S and S' the chosen barrier surfaces near ∂M and $\partial M'$, respectively. Recall that we write N and N' for the collar neighborhoods of ∂M and $\partial M'$ bounded by S and S', respectively. Let $\Gamma \subseteq Z \setminus \operatorname{int}(N \cup N')$ denote the Legendrian 1–skeleton and $D_1, \dots, D_n$ the convex 2–cells.

A neighborhood of each vertex of Γ that is not contained in S or S' determines a contact 0–handle. A neighborhood of each edge of Γ is a contact 1–handle. A neighborhood of each convex 2–cell D_i is a contact 2–handle. Finally, after removing neighborhoods of the graph Γ and the disks D_i, we are left with a collection of tight contact 3–balls that we view as a collection of contact 3–handles. Write $h_1, \dots, h_n$ for these handles, ordered so that their indices are nondecreasing.

Let γ_0 be the dividing set on S with respect to ν and ξ, and write γ_0' for the dividing set on S' with respect to ν' and ξ. The flow of ν induces a diffeomorphism

$$\psi^\nu\colon (M, \gamma) \to (M \cup N, \gamma_0)$$

that is well defined up to isotopy. Note that $(N', \gamma_0' \cup \gamma')$ is a balanced sutured manifold, and ξ induces the dividing set $\gamma_0' \cup \gamma'$ on $\partial N'$. There is a canonical isomorphism

$$\mathrm{SFH}(-M \sqcup -N', -\gamma \cup -\gamma_0' \cup -\gamma') \cong \mathrm{SFH}(-M, -\gamma) \otimes \mathrm{SFH}(-N', -\gamma_0' \cup -\gamma').$$

Hence, we can define a map

$$\Phi_{\sqcup N'}\colon \mathrm{SFH}(-M, -\gamma) \to \mathrm{SFH}(-M \sqcup -N', -\gamma \cup -\gamma_0' \cup -\gamma')$$

via the formula

$$\Phi_{\sqcup N'}(\boldsymbol{x}) = \boldsymbol{x} \otimes \mathrm{EH}(N', \gamma_0' \cup \gamma', \xi|_{N'}),$$

where $\mathrm{EH}(N', \gamma_0' \cup \gamma', \xi|_{N'}) \in \mathrm{SFH}(-N', -\gamma_0' \cup -\gamma')$ is the contact invariant of $\xi|_{N'}$, constructed using a partial open book decomposition, as defined by Honda, Kazez, and Matić [13]. We describe the invariant $\mathrm{EH}(N', \gamma_0' \cup \gamma', \xi|_{N'})$ in more detail in Section 4, when we prove some properties of the gluing map. We will show in Lemma 4.3 that $\Phi_{\sqcup N'}$ can be written as a composition of contact handle maps.

We now define the contact gluing map as the composition

$$\Phi_{\xi,\mathcal{C}} := C_{h_n} \circ \cdots \circ C_{h_1} \circ (\psi_*^\nu \otimes \mathrm{id}) \circ \Phi_{\sqcup N'}, \tag{3-1}$$

where C_{h_i} is the map induced by the contact handle h_i, as in Section 3.3.

3.5 Invariance of the contact gluing map

In this section, we prove the following:

Theorem 3.14 *The contact gluing map $\Phi_{\xi,\mathcal{C}}$ is independent of the contact cell decomposition $\mathcal{C}$.*

As a first step, the following lemma is helpful:

Lemma 3.15 *If h_1 and h_2 are two contact handles* (of arbitrary index) *that are disjoint and are attached to (M,γ), then*

$$C_{h_1} \circ C_{h_2} = C_{h_2} \circ C_{h_1}.$$

Proof The proof follows by analyzing the formulas for the two maps. If one of h_1 and h_2 is a 0–handle or a 1–handle, then the statement is obvious. Let us consider the case when h_1 and h_2 are disjoint 2–handles. Let us recall how the maps C_{h_i} are defined for $i \in \{1,2\}$. We first attach a band B_i to the boundary of a sutured diagram for $(-M,-\gamma)$. The band B_i is attached where the attaching circle of h_i intersects $\gamma \subseteq \partial M$. We pick curves α_i and β_i, according to the attaching circle of h_i, that intersect at a single point c_i in B_i. The map C_{h_i} is defined by $C_{h_i}(\boldsymbol{x}) = \boldsymbol{x} \times c_i$. Since the handles h_1 and h_2 are disjoint, the bands B_1 and B_2 can be chosen to be disjoint, and the curves α_i and β_i can be assumed to not intersect the band B_j when $i \neq j$. Hence, it follows that we can use the curves α_1, β_1, α_2, and β_2 to compute both compositions $C_{h_2} \circ C_{h_1}$ and $C_{h_2} \circ C_{h_1}$, and since the formulas for the two compositions clearly agree, we conclude that $C_{h_2} \circ C_{h_1} = C_{h_1} \circ C_{h_2}$. In a similar manner, one can show that the same formula holds if at least one of h_1 and h_2 is a 3–handle. □

Proof of Theorem 3.14 Independence of the relative ordering of cells of the same index follows from Lemma 3.15, so it is sufficient to check invariance under the moves in Proposition 3.8.

First consider the case when $\mathcal{C}_1$ and $\mathcal{C}_2$ differ by move (C-1) (isotopy). The maps $\Phi_{\xi,\mathcal{C}_1}$ and $\Phi_{\xi,\mathcal{C}_2}$ differ by postcomposition with a map $\phi_*\colon \mathrm{SFH}(M',\gamma') \to \mathrm{SFH}(M',\gamma')$

for a diffeomorphism $\phi\colon M' \to M'$ that is isotopic to the identity relative to $\partial M'$. By naturality of sutured Floer homology [18, Theorem 1.9], the map ϕ_* is the identity.

We now consider move (C-2) (index $0/1$ cell cancellation). We need to check that subdividing an edge of Γ, or adding a Legendrian edge λ to Γ that meets Γ at a single vertex and intersects none of the other cells does not change the map. Let us first consider subdivision. The contact 0–handle map adds a disk to the Heegaard surface with no new α or β curves, and the contact 1–handle map attaches a band to the boundary of the Heegaard surface with no new α or β curves. When we subdivide a Legendrian edge into two Legendrian edges that meet at a single vertex, the contact handle map changes by first adding a disk to S (the 0–handle map), followed by two bands, each of which has one foot on the new disk, and another foot at a foot of the original band. Clearly, the induced maps agree. Invariance under adding a Legendrian edge λ that intersects Γ at a single vertex follows similarly.

We now consider the case when $\mathcal{C}_1$ and $\mathcal{C}_2$ differ by move (C-3) (index $1/2$ contact cell cancellation). The proof is a model computation, which we now describe. The computation is summarized in Figure 20. Note that the contact cell decompositions $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same barrier surfaces S and S' near ∂M and $\partial M'$, respectively. Recall that we write $N, N' \subseteq Z$ for the neighborhoods of ∂M and $\partial M'$ that are bounded by S and S', and $Z' := Z \setminus \operatorname{int}(N \cup N')$. By definition of an index $1/2$ contact cell cancellation, the graph Γ_2 is formed by adding a Legendrian edge λ to Γ_1, and a new 2–cell D is added that intersects $\partial(Z' \setminus \operatorname{int} N(\Gamma_1))$ along an arc that does not intersect the dividing set.

To demonstrate, let us assume that the new 2–cell D intersects $\partial(Z' \setminus \operatorname{int} N(\Gamma_1))$ along R_+. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a sutured diagram for $M \cup N \cup N(\Gamma_1) \cup N'$, the effect of adding the 1–handle h^1 corresponding to λ is to attach a band B to the boundary of $\overline{\Sigma}$. The attaching circle of the 2–handle h^2 corresponding to D intersects the dividing set of $\partial N(\lambda)$ at two points. On the level of diagrams, the effect is to attach a band B' to B, and add new curves α_0 and β_0, as in Figure 20. Note that the effect of adding B and B' to $\overline{\Sigma}$ is to attach a tube to two points along the interior of $\overline{\Sigma}$ near $\partial\overline{\Sigma}$, and the curve β_0 is a meridian of the tube, while the curve α_0 is the concatenation of a longitude of the tube with a path on $\overline{\Sigma}$ between the two ends of the tube. The curves α_0 and β_0 intersect at a single point c, and the composition of the contact 1–handle and 2–handle maps is

$$(C_{h^2} \circ C_{h^1})(\boldsymbol{x}) = \boldsymbol{x} \times c.$$

This is the compound stabilization map from Section 2.2. By Proposition 2.2, this is equal to the transition map from the naturality of sutured Floer homology.

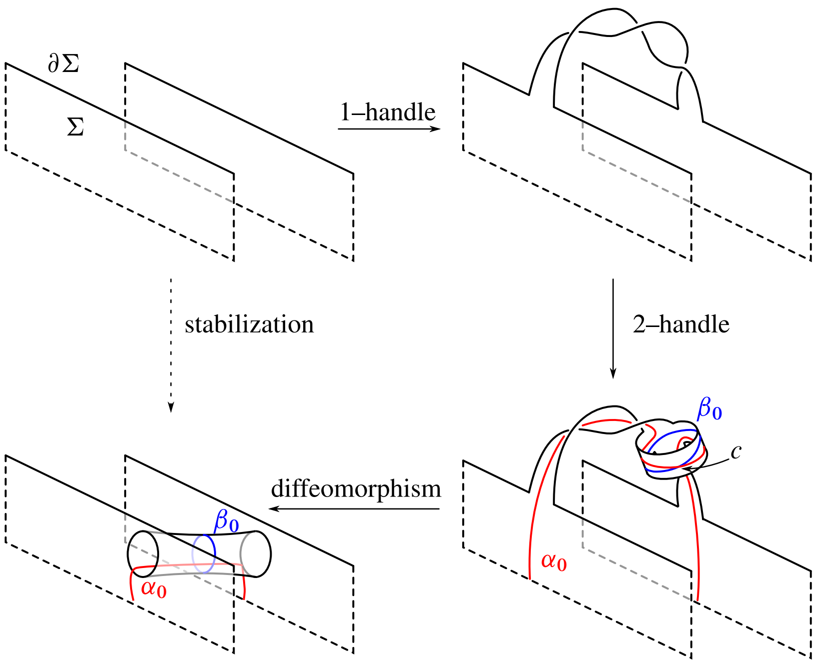

Figure 20: The composition of the contact 1–handle map followed by a contact 2–handle map, for a canceling pair of contact 1– and 2–cells. The composition is the compound stabilization map described in Section 2.2.

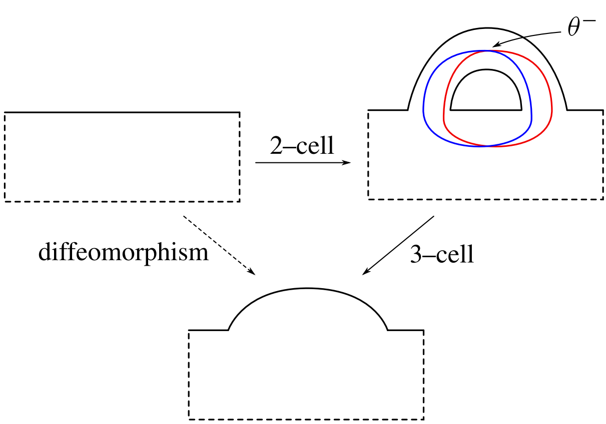

Figure 21: The composition of a contact 2–handle map followed by a contact 3–handle map, for a canceling pair of contact 2– and 3–cells.

Finally, we consider invariance under move (C-4) (index $2/3$ contact cell cancellation). In this move, we add a convex 2–cell D to the decomposition that has Legendrian boundary and $\mathrm{tb}(\partial D) = -1$ so that ∂D intersects the dividing set on $\partial(Z' \setminus \operatorname{int} N(\Gamma))$ exactly twice. By definition of a contact cell decomposition, the disk D cuts one of the contact 3–cells in our decomposition into two contact 3–cells. Hence, the effect on the contact gluing map from (3-1) is to insert a contact 2–handle map C_{h^2}, followed by a contact 3–handle map C_{h^3}. The composition is easily seen to be a diffeomorphism map, as demonstrated in Figure 21.

Indeed, for notational simplicity, assume that h^2 is attached to (M,γ). Let (M',γ') be the result of attaching h^2 to (M,γ), and (M'',γ'') the result of attaching h^3 to (M',γ'). Given a diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ of (M,γ), we get a diagram of (M',γ') by attaching a band B to a component of $\partial\Sigma$, and add curves α' to $\boldsymbol{\alpha}$ and β' to $\boldsymbol{\beta}$ parallel to the suture γ_S along which h^3 is attached and such that $\alpha' \cap \beta' \cap B = \{c\}$. For $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have $C_{h^2}(\boldsymbol{x}) = \boldsymbol{x} \times c$. We can choose α' and β' so that $|\alpha' \cap \beta'| = 2$, and we write $\alpha' \cap \beta' = \{\theta^+, \theta^-\}$, where θ^+ and θ^- are distinguished by the relative Maslov grading. By construction, on $(\overline{\Sigma}, \boldsymbol{\alpha} \cup \{\alpha'\}, \boldsymbol{\beta} \cup \{\beta'\})$ there are two bigons connecting the two points of $\alpha' \cap \beta'$, which we can use to determine that c has lower relative grading, so $c = \theta^-$. It follows that $(C_{h^3} \circ C_{h^2})(\boldsymbol{x}) = C_{h^3}(\boldsymbol{x} \times c) = \boldsymbol{x}$. □

Remark 3.16 It is possible to directly show invariance of the contact gluing map under subdividing a contact 2–cell into two contact 2–cells that meet along a Legendrian arc. Indeed, the topological manipulation described in the proof of Lemma 3.9 gives a recipe for doing so. Nonetheless, the model computations required to show invariance of the gluing map under moves (C-1)–(C-4) are simpler.

4 Contact handles and partial open book decompositions

4.1 Partial open book decompositions and sutured Heegaard diagrams

In order to prove basic properties of the gluing map, and to compare our construction to the one due to Honda, Kazez, and Matić [12], we need the following definition:

Definition 4.1 [13, Section 1.2] A *partial open book decomposition* is a triple (S, P, h) consisting of a compact, connected, oriented surface S with nonempty boundary, a compact subsurface $P \subseteq S$ (the *page*) such that S is obtained from $\mathrm{cl}(S \setminus P)$ by successively attaching 1–handles, and a smooth embedding $h\colon P \to S$ (the *monodromy*) such that $h|_{\partial S \cap P} = \mathrm{id}$.

An abstract (nonembedded) partial open book decomposition (S, P, h) defines a sutured manifold (M, γ) via the formula

$$M = \left(S \times \left[0, \tfrac{1}{2}\right] \cup P \times \left[\tfrac{1}{2}, 1\right]\right)/{\sim_h},$$

where $\sim_h$ is the equivalence relation defined as

$$\begin{aligned} &(x,t) \sim_h (x,t') && \text{if } x \in \partial S \text{ and } t, t' \in \left[0, \tfrac{1}{2}\right], \\ &(x,t) \sim_h (x,t') && \text{if } x \in \partial P \cap \partial S \text{ and } t, t' \in \left[\tfrac{1}{2}, 1\right], \\ &(x,1) \sim_h (h(x),0) && \text{if } x \in P. \end{aligned}$$

The manifold M contains the properly embedded surface

$$\Sigma := \left(S \times \left\{\tfrac{1}{4}\right\}\right) \cup \left(P \times \left\{\tfrac{3}{4}\right\}\right).$$

The curves $\gamma = \partial M \cap \Sigma$ divide ∂M into two subsurfaces that meet along γ. Furthermore, (M, γ) is a balanced sutured manifold and Σ is a sutured Heegaard surface for (M, γ). Using our orientation conventions, $R_+(M)$ deformation retracts onto $(S \setminus P) \times \left\{\tfrac{1}{2}\right\}$ and $R_-(M)$ deformation retracts onto $(S \setminus h(P)) \times \{0\}$.

Definition 4.2 If (S, P, h) is a partial open book decomposition, a *basis of arcs for P in S* is a collection of pairwise disjoint, properly embedded arcs $\boldsymbol{a} = \{a_1, \dots, a_k\}$ on P with ends on $P \cap \partial S$ such that $P \setminus (a_1 \cup \dots \cup a_k)$ deformation retracts onto $\partial P \setminus \partial S$ (or, equivalently, $S \setminus (a_1 \cup \dots \cup a_k)$ deformation retracts onto $\mathrm{cl}(S \setminus P)$).

Given a basis of arcs for P in S, we can construct attaching curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ on Σ that make $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ a Heegaard diagram for (M, γ). Let b_i be an isotopic copy of a_i obtained by moving the ends of a_i along $\partial\Sigma$ in the positive direction such that $a_i \cap b_j = \delta_{ij}$. We then set

$$\alpha_i = \left(a_i \times \left\{\tfrac{1}{4}\right\}\right) \cup \left(a_i \times \left\{\tfrac{3}{4}\right\}\right) \quad \text{and} \quad \beta_i = \left(b_i \times \left\{\tfrac{3}{4}\right\}\right) \cup \left(h(b_i) \times \left\{\tfrac{1}{4}\right\}\right).$$

A partial open book decomposition (S, P, h) determines a unique contact structure ξ on (M, γ), up to equivalence, as follows; see [2, Proposition 1.2]. Let

$$U_1 := \left(S \times \left[0, \tfrac{1}{2}\right]\right)/{\sim_1} \quad \text{and} \quad U_2 := \left(P \times \left[\tfrac{1}{2}, 1\right]\right)/{\sim_2}, \tag{4-1}$$

where $\sim_1$ and $\sim_2$ are the relations defined by

$$(x,t) \sim_1 (x,t') \quad \text{if } x \in \partial S, \qquad \text{and} \qquad (x,t) \sim_2 (x,t') \quad \text{if } x \in \partial P \cap \partial S.$$

Then $\xi|_{U_1}$ is the unique tight contact structure on U_1 with dividing set $\partial S \times \left\{\tfrac{1}{4}\right\}$, and $\xi|_{U_2}$ is the unique tight contact structure on U_2 with dividing set $\partial P \times \left\{\tfrac{3}{4}\right\}$. Hence, we say that (S, P, h) is a partial open book decomposition of the contact 3–manifold (M, γ, ξ) if we are given a contactomorphism between the contact 3–manifold defined by (S, P, h) and (M, γ, ξ).

Given a partial open book decomposition (S, P, h) of (M, γ, ξ) and a basis of arcs $\{a_1, \dots, a_k\}$, let $x_i = \big(a_i \times \{\tfrac{3}{4}\}\big) \cap \big(b_i \times \{\tfrac{3}{4}\}\big)$ for $i \in \{1, \dots, k\}$ and $\boldsymbol{x}_\xi = x_1 \times \cdots \times x_k$. Then Honda, Kazez, and Matić [13] showed that $\boldsymbol{x}_\xi$ is a cycle whose homology class $\mathrm{EH}(M, \gamma, \xi) \in \mathrm{SFH}(-M, -\gamma)$ is independent of the choice of partial open book and basis of arcs and is hence an invariant of ξ.

4.2 Partial open books and contact handles

Contact handle decompositions were defined by Giroux [6]. In this section, we describe some useful relations between contact handle decompositions and the Honda–Kazez–Matić definition of a partial open book; compare [2].

Given a partial open book decomposition (S, P, h) for the contact sutured manifold (M, γ, ξ), we can naturally construct a contact handle decomposition of (M, γ) with no 3–handles, as follows. Recall that the manifold M is obtained by gluing the contact handlebodies U_1 and U_2 together, as described in Section 4.1, along a portion of their boundaries, using the map h.

We start by constructing a contact handle decomposition of U_1 consisting of only 0–handles and 1–handles. Such a description is obtained by giving the surface S a decomposition into 2–dimensional 0–handles and 1–handles. Next, we extend this decomposition to a contact handle decomposition of all of M. To do this, pick a basis of arcs $\boldsymbol{a}$ for P in S. The closed curves

$$l_i := \big(a_i \times \{\tfrac{1}{2}\}\big) \cup (h(a_i) \times \{0\})$$

bound disks in U_2 that intersect the dividing set $\partial S \times \{0\}$ of ∂U_1 exactly twice. By perturbing U_1 and U_2 slightly inside M, we can assume that the curves l_i are Legendrian. As the curves l_i each intersect the dividing set of ∂U_1 exactly twice, it follows that they bound convex disks with $\mathrm{tb} = -1$, and that neighborhoods of these convex disks are contact 2–handles. Furthermore, after attaching these contact 2–handles, we obtain the sutured manifold M.

In the opposite direction, given a contact handle decomposition $\boldsymbol{H}$ of (M, γ, ξ) with handles ordered with nondecreasing index and no 3–handles, viewed as a cobordism from $\varnothing$ to ∂M, we can construct a partial open book decomposition (S, P, h), as follows. The handlebody U_1 is the union of the 0– and 1–handles. If γ_1 is the dividing set of ξ on ∂U_1, then (U_1, γ_1) is diffeomorphic to the product sutured manifold $\left((S \times [0, \frac{1}{2}])/{\sim}, \partial S \times \{\frac{1}{4}\}\right)$ where $S := R_+(\gamma_1) \subseteq \partial U_1$. Let U_2 be the union of the 2–handles, and let γ_2 be the dividing set of ξ on ∂U_2. Then (U_2, γ_2) is a product sutured manifold of the form $\left((P \times [\frac{1}{2}, 1])/{\sim}, \partial P \times \{\frac{3}{4}\}\right)$ for $P := U_2 \cap S$. We finally set h to be $\pi_1 \circ i_2\colon P \to S$, where $\pi_1\colon S \times [0, \frac{1}{2}] \to S$ is the projection and

$$i_2\colon P \to P \times \{1\} \subseteq R_-(\gamma_1) \approx S \times \{0\}$$

is the canonical embedding.

Lemma 4.3 *Let $h_1, \dots, h_n$ be the handles of a contact handle decomposition $\boldsymbol{H}$ of (M, γ, ξ), ordered so that their indices are nonincreasing. If there are no 3–handles, then*

$$\mathrm{EH}(M, \gamma, \xi) = (C_{h_1} \circ \cdots \circ C_{h_n})(1) \in \mathrm{SFH}(-M, -\gamma),$$

where $1 \in \mathbb{F}_2 \cong \mathrm{SFH}(\varnothing)$.

Proof Let (S, P, h) be the partial open book corresponding to $\boldsymbol{H}$, as above. Suppose that $h_1, \dots, h_k$ are the 2–handles. On the level of diagrams, $C_{h_{k+1}} \circ \cdots \circ C_{h_n}$ corresponds to adding a disk for each 0–handle, and a band for each 1–handle. The union of these is $S \times \{\frac{1}{4}\}$. Adding a 2–handle h_i for $i \in \{1, \dots, k\}$ corresponds to attaching a band B_i to S, and adding curves α_i and β_i. Then $B_1 \cup \cdots \cup B_k = P \times \{\frac{3}{4}\}$, and α_i and β_i are obtained from the basis of arcs $\{a_1, \dots, a_k\}$ as described in Section 4.1, where a_i is the core of B_i. Let $x_i = \alpha_i \cap \beta_i \cap B_i$ for $i \in \{1, \dots, k\}$. Then the element $(C_{h_1} \circ \cdots \circ C_{h_n})(1) = x_1 \times \cdots \times x_k$ tautologically agrees with the cycle $\boldsymbol{x}_\xi$ representing $\mathrm{EH}(M, \gamma, \xi)$, as defined by Honda, Kazez, and Matić [13] for the partial open book decomposition (S, P, h) and the basis of arcs $\{a_1, \dots, a_k\}$. □

We now describe the effect of attaching a single contact handle on the level of partial open books. We begin with the effect of attaching a contact 1–handle:

Lemma 4.4 *Suppose that we obtain the contact sutured manifold (M', γ', ξ') from (M, γ, ξ) by attaching a contact 1–handle h^1. Let (S, P, h) be a partial open book decomposition for the contact structure ξ on (M, γ). A partial open book decomposition (S', P', h') for the contact structure ξ' on (M', γ') can be obtained by setting*

(1) *$S' = S \cup B$, where B is a band attached to ∂S,*

(2) *$P' = P$, and*

(3) *$h' = \iota_S \circ h$, where $\iota_S\colon S \to S'$ is the embedding.*

Proof First, we isotope the partial open book (S, P, h) in M such that the attaching sphere of h^1 lies in $\partial S \setminus P$. We set $S' = S \cup B$, where the band $B \subseteq h^1$ has boundary equal to the dividing set on h^1. Then

$$\big(S' \times [0, \tfrac{1}{2}] \cup P \times [\tfrac{1}{2}, 1]\big)/{\sim_{h'}} = \big(S \times [0, \tfrac{1}{2}] \cup P \times [\tfrac{1}{2}, 1]\big)/{\sim_{h'}} \cup \big(B \times [0, \tfrac{1}{2}]\big)/{\sim_{h'}}.$$

Furthermore,

$$\big(S \times [0, \tfrac{1}{2}] \cup P \times [\tfrac{1}{2}, 1]\big)/{\sim_{h'}} = \big(S \times [0, \tfrac{1}{2}] \cup P \times [\tfrac{1}{2}, 1]\big)/{\sim_{h}} = M,$$

and $\big(B \times [0, \tfrac{1}{2}]\big)/{\sim_{h'}} = h^1$. □

We now consider the effect of attaching a contact 2–handle h^2 to (M, γ). Let ξ' denote the contact structure on $M \cup h^2$ with dividing set γ', obtained by gluing the tight contact structure ξ_2 on h^2 to ξ. Let $p_1, p_2 \in \gamma$ denote the two points of intersection of the attaching circle of h^2 with γ. Furthermore, the attaching circle of h^2 consists of a path l_+ in $R_+(\gamma)$ from p_1 to p_2, concatenated with the reverse of a path l_- in $R_-(\gamma)$ from p_1 to p_2. We can isotope the partial open book decomposition (S, P, h) so that p_1 and p_2 lie in $\partial S \setminus P$. Since we have identifications

$$S \setminus P \cong R_+(M) \quad \text{and} \quad S \setminus h(P) \cong R_-(M),$$

we can view l_+ as a path λ_+ in $S \setminus P$, and l_- as a path λ_- in $S \setminus h(P)$. Using the above notation, we are now prepared to describe a partial open book decomposition for the contact structure ξ' on $(M \cup h^2, \gamma')$.

Lemma 4.5 *Suppose h^2 is a contact 2–handle attached to (M, γ, ξ), and let ξ' be the resulting contact structure on $(M \cup h^2, \gamma')$. Given a partial open book decomposition (S, P, h) for ξ on (M, γ), a partial open book decomposition for ξ' on $(M \cup h^2, \gamma')$ is given by (S', P', h'), where*

(1) $S' = S$,

(2) $P' = P \cup N(\lambda_+)$,

(3) *$h'|_P = h$, and h' maps $N(\lambda_+)$ to $N(\lambda_-)$.*

Proof Using (4-1), we write M as the union of the subsets $U_1 = \big(S \times [0, \tfrac{1}{2}]\big)/{\sim_1}$ and $U_2 = \big(P \times [\tfrac{1}{2}, 1]\big)/{\sim_2}$. We note that

$$\big(P' \times [\tfrac{1}{2}, 1]\big)/{\sim_2} = \big(P \times [\tfrac{1}{2}, 1]\big)/{\sim_2} \cup \big(N(\lambda_+) \times [\tfrac{1}{2}, 1]\big)/{\sim_2}.$$

Let $U_2' = \left(P' \times \left[\frac{1}{2}, 1\right]\right)/{\sim_2}$ and $h^2 = \left(N(\lambda_+) \times \left[\frac{1}{2}, 1\right]\right)/{\sim_2}$. Then $(U_1 \cup U_2')/{\sim_{h'}}$ is obtained from $(U_1 \cup U_2)/{\sim_h}$ by attaching h^2 along a neighborhood of the circle $\left(\lambda_+ \times \left\{\frac{1}{2}\right\}\right) \cup (\lambda_- \times \{0\})$. By definition of a partial open book decomposition, it follows that (S', P', h') is a partial open book for ξ' on $(M \cup h^2, \gamma')$. □

4.3 Positive stabilizations and contact handle cancellations

Honda, Kazez, and Matić [13] extended the notion of positive stabilizations to partial open book decompositions, adapting Giroux's construction [8] for open books of closed manifolds. In this section, as an instructive example, we show how to interpret their construction in terms of canceling pairs of contact handles.

We begin with Honda, Kazez, and Matić's definition of a positive stabilization of a partial open book:

Definition 4.6 Suppose that (S, P, h) is a partial open book decomposition of the contact 3–manifold (M, γ, ξ), and suppose that c is a properly embedded arc on S. The arc c is allowed to intersect P; however, we require $\partial c \subseteq \partial S \setminus P$. The *positive stabilization* (S', P', h') of (S, P, h) along the arc c is defined as follows. Let $S' := S \cup B$, where B is a band that we attach along $\partial c \subseteq \partial S$. Furthermore, let $P' := P \cup B$ be the new page. Let $\tau \subseteq S'$ be the curve obtained by concatenating the arc c with a core of B. The new partial monodromy map $h'\colon P' \to S'$ is defined as

$$h' := R_\tau \circ (h \cup \mathrm{id}_B),$$

where R_τ is a right-handed Dehn twist along τ, with respect to the orientation of S.

We now wish to relate positive stabilizations to canceling pairs of contact handles. As described in Section 4.2, the partial open book decomposition (S, P, h), together with a choice of handle decomposition of the surface S into 0–handles and 1–handles, as well as a basis of arcs $\boldsymbol{a}$ for P, determines a contact handle decomposition of (M, γ). We now show that the contact handle decomposition arising from a positive stabilization (S', P', h') of (S, P, h) can be obtained from a contact handle decomposition arising from (S, P, h) by inserting a pair of canceling index 1 and 2 contact handles.

Lemma 4.7 *Suppose that (S, P, h) is a partial open book decomposition of (M, γ, ξ) and that (S', P', h') is a positive stabilization of (S, P, h) along a properly embedded arc $c \subseteq S$. Let U_1 and U_2 be the tight contact handlebodies defined in (4-1) associated to (S, P, h), whose union is M. Consider a contact handle decomposition $\boldsymbol{H}$*

of (M,γ,ξ) arising from the partial open book (S,P,h) as in Section 4.2. Then a handle decomposition $\boldsymbol{H}'$ arising from the positive stabilization (S',P',h') can be obtained from $\boldsymbol{H}$ by adding a pair of canceling index 1 and 2 contact handles between U_1 and U_2.

Proof We attach a pair of canceling contact 1– and 2–handles to U_1. Let h^1 denote a 1–handle, attached with feet along $\partial c \times \{\frac{1}{4}\}$. We then attach a contact 2–handle h^2 with attaching circle equal to the concatenation of $c \times \{0\} \subseteq \partial U_1$ with a longitude of the 1–handle h^1; see the middle row of Figure 22. The contact manifolds U_1 and $U_1 \cup h^1 \cup h^2$ are equivalent, since h^1 and h^2 are canceling contact handles.

Note that $(S,\varnothing,\varnothing)$ is a partial open book decomposition for U_1. By Lemma 4.4, if B denotes a band attached at $\partial c \subseteq \partial S$, then the surface $(S \cup B,\varnothing,\varnothing)$ is a partial open book decomposition for $U_1 \cup h^1$.

Let $h_1^2,\dots,h_n^2$ denote the contact 2–handles obtained by picking an arc basis of P. To build M from $U_1 \cup h^1$, we first attach the contact 2–handle h^2, followed by $h_1^2,\dots,h_n^2$.

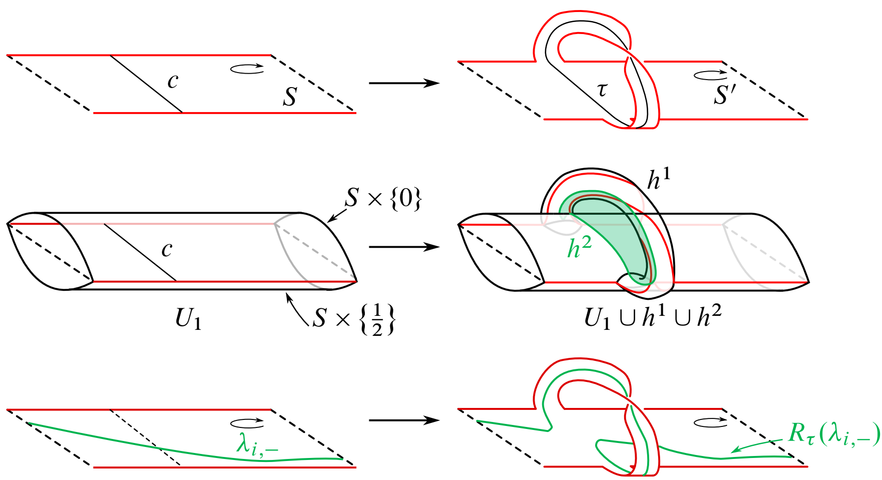

Figure 22: Positively stabilizing a partial open book is the same as inserting a pair of canceling contact 1– and 2–handles. On the top row, we show a schematic of the new partial open book. In the middle row, we show the contact handlebody $U_1 = S \times [0,\frac{1}{2}]/{\sim_1}$ (left), as well as the equivalent contact handlebody $U_1 \cup h^1 \cup h^2$ (right). On the bottom row, we show that the attaching circle of any contact 2–handle in the previous decomposition is changed by a positive Dehn twist along τ. Note that the Dehn twist looks negative; however, the picture is of $(S \times [0,\frac{1}{2}])/{\sim_1}$ turned "upside down", because we are drawing the images of the contact 2–handles on $R_-(U_1) = S \times \{0\}$.

We can determine the effect of this on the partial open book using Lemma 4.5. The partial monodromy map after attaching h^2 (but before attaching the other 2–handles) is determined by the attaching circle of the 2–handle h^2. Upon examining Figure 22, the partial monodromy map is the composition of the inclusion of B into S, followed by a right-handed Dehn twist along the curve τ obtained by concatenating c with the core of B.

We now need to compute the effect on the partial open book of attaching $h^2_1, \dots, h^2_n$. Since the contact 2–handles $h^2_1, \dots, h^2_n$ were obtained by picking a basis of arcs of P, it follows that the attaching circle of the 2–handle h^2_i is equal to the concatenation of properly embedded arcs $\lambda_{+,i} \times \{\frac{1}{2}\} \subseteq S \times \{\frac{1}{2}\}$ and $\lambda_{-,i} \times \{0\} \subseteq S \times \{0\}$, where $\lambda_{+,i}, \lambda_{-,i} \subseteq S$.

The monodromy map h is determined up to isotopy relative to $\partial P \cap \partial S$ by requiring $\lambda_{+,i}$ to be sent to $\lambda_{-,i}$ by h. It is clear that, after attaching h^1 and h^2, the attaching arcs $\lambda_{-,i}$ must be modified if they intersect the arc c on S, since now they are attached on top of h^1 and h^2. Indeed, by examining Figure 22, we see that the effect is to replace each $\lambda_{-,i}$ by $R_\tau(\lambda_{-,i})$. It follows that the new partial diffeomorphism map is simply

$$R_\tau \circ (h \cup \mathrm{id}_B),$$

completing the proof. □

5 Properties of the gluing map

5.1 The gluing map for I–invariant contact structures

In this section, we prove that gluing on a copy of $\partial M \times I$ induces the identity map, in a sense that we now describe. Suppose that (M, γ) is a sutured submanifold of (M', γ') and ξ is a contact structure on $M' \setminus \mathrm{int}(M)$. Furthermore, suppose that there is a Morse function f on $M' \setminus \mathrm{int}(M)$ such that $f|_{\partial M} \equiv 0$, $f|_{\partial M'} \equiv 1$, f has no critical points, and there is a contact vector field ν such that the derivative satisfies $\nu(f) > 0$. Furthermore, suppose that the dividing set of ξ on $\partial M \cup \partial M'$, with respect to ν, is equal to $\gamma \cup \gamma'$. The vector field ν induces a diffeomorphism

$$\phi^\nu \colon (-M, -\gamma) \to (-M', -\gamma'),$$

which is well defined up to isotopy, relative to ∂M. The induced diffeomorphism map

$$\phi^\nu_* \colon \mathrm{SFH}(-M, -\gamma) \to \mathrm{SFH}(-M', -\gamma')$$

has a simple description. If $(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a sutured Heegaard diagram for $(-M, -\gamma)$, then we can construct a Heegaard diagram for $(-M', -\gamma')$ as

$$(\overline{\Sigma} \cup \bar{A}, \boldsymbol{\alpha}, \boldsymbol{\beta}),$$

where A is the *characteristic surface* of ξ, with respect to ν; ie

$$A := \{p \in M' \setminus \mathrm{int}(M) : \nu_p \in \xi_p\}.$$

The surface A is a collection of annuli. With respect to these two diagrams, the diffeomorphism map takes the form

$$\phi^\nu_*(\boldsymbol{x}) = \boldsymbol{x}.$$

Our gluing map satisfies the following analogue of [12, Theorem 6.1].

Proposition 5.1 *Suppose that (M, γ) is a sutured submanifold of (M', γ') and ξ is a contact structure on $M' \setminus \mathrm{int}(M)$. Furthermore, suppose that there is a Morse function f on $M' \setminus \mathrm{int}(M)$ such that $f|_{\partial M} \equiv 0$, $f|_{\partial M'} \equiv 1$, and there is contact vector field ν such that $\nu(f) > 0$. Under the above assumptions, the contact gluing map Φ_ξ satisfies*

$$\Phi_\xi = \phi^\nu_*,$$

where $\phi^\nu\colon (-M, -\gamma) \to (-M', -\gamma')$ is the diffeomorphism described above.

Before we begin with the proof, we need the following definition regarding sutured cell decompositions of surfaces:

Definition 5.2 The sutured cell decompositions $\mathcal{D} = (B_1, \dots, B_n, \lambda_1, \dots, \lambda_m)$ and $\mathcal{D}^* = (B'_1, \dots, B'_n, \lambda'_1, \dots, \lambda'_m)$ of the surface with divides (F, γ) are *dual* if the following hold:

(1) $B_i \cap B'_j = \varnothing$ for all i and j.

(2) Each component of $F \setminus (B_1 \cup \dots \cup B_n \cup \lambda_1 \cup \dots \cup \lambda_m)$ intersects γ in a single arc and contains a single fattened 0–cell B'_k. The same statement holds with the roles of $\mathcal{D}$ and $\mathcal{D}^*$ reversed.

(3) $|\lambda_i \cap \lambda'_j| = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta.

A sutured cell decomposition $\mathcal{D}$ that admits a dual is called *dualizable*.

As an example, the sutured cell decomposition of a torus with two parallel sutures shown in Figure 8 is dualizable.

Remark 5.3 Not all sutured cell decompositions $\mathcal{D}$ admit dual cell decompositions. For example, if the intersection of a component of $F\setminus(B_1\cup\cdots\cup B_n\cup\lambda_1\cup\cdots\cup\lambda_m)$ with γ is disconnected, then $\mathcal{D}$ is not dualizable.

Remark 5.4 Every surface with divides (F,γ) such that $\pi_0(\gamma)\to\pi_0(F)$ is surjective admits a dualizable cell decomposition. To construct one, we pick sets of arcs A_0 and A_1 along γ such that the arcs in $A_0\cup A_1$ are pairwise disjoint and A_0 and A_1 each contain exactly one arc in every component of γ. We can then view $R_+(\gamma)$ as a cobordism with product boundary from A_0 to A_1. By picking a Morse function on $R_+(\gamma)$ which is minimized along A_0, maximized along A_1, and which only has index 1 critical points, we get a collection of arcs $\lambda_1,\dots,\lambda_k$ (the stable manifolds of the index 1 critical points) with boundary on A_0 which cut $R_+(\gamma)$ into a collection of disks, each of which contains exactly one component of A_1. By picking a similar Morse function on $R_-(\gamma)$, we obtain another collection of arcs $\lambda_{k+1},\dots,\lambda_m$ in $R_-(\gamma)$ with boundary on A_0 that cut $R_-(\gamma)$ into disks, each of which contains exactly one arc of A_1. We get a sutured cell decomposition with a fattened 0–cell B_i along each arc of A_0, and we use the collection of arcs $\lambda_1,\dots,\lambda_m$. The 0–cells $B_1,\dots,B_n$ and the arcs $\lambda_1,\dots,\lambda_m$ determine a handle decomposition of F, and the dual sutured cell decomposition is obtained by using the dual handle decomposition obtained by turning the Morse functions upside down.

Proof of Proposition 5.1 Let $\mathcal{D}$ and $\mathcal{D}^*$ be dual sutured cell decompositions of $(\partial M,\gamma)$. Let $\mathcal{C}$ be the product contact cell decomposition of $M'\setminus\operatorname{int}(M)$ constructed from $\mathcal{D}$ as in Example 3.7, with barrier surfaces S and S'. Write N and N' for the collar neighborhoods of ∂M and $\partial M'$ in $M'\setminus\operatorname{int}(M)$ that are bounded by S and S', respectively. We denote the dividing set on S' by γ_0'.

For each fattened 0–cell B of $\mathcal{D}$, there is a contact 1–handle h_B of $\mathcal{C}$. For each arc λ of $\mathcal{D}$, there is a contact 2–handle h_λ of $\mathcal{C}$. For each 2–cell c of $\mathcal{D}$, there is a contact 3–handle h_c of $\mathcal{C}$. Let $h_1,\dots,h_n$ be an enumeration of these handles with nondecreasing index. Using (3-1), the gluing map is defined as

$$\Phi_\xi(\boldsymbol{x}):=(C_{h_n}\circ\cdots\circ C_{h_1})(\psi_*^\nu(\boldsymbol{x})\otimes\mathrm{EH}(N',\gamma'\cup\gamma_0',\xi|_{N'})). \tag{5-1}$$

The element $\mathrm{EH}(N',\gamma'\cup\gamma_0,\xi|_{N'})$ is defined using a partial open book decomposition of $(N',\gamma'\cup\gamma_0,\xi|_{N'})$. In Section 4.2, we described how a contact handle decomposition of N' with no 3–handles, viewed as a cobordism from $\varnothing$ to $\partial N'$, can be used to construct a partial open book. By turning around our construction of a contact cell

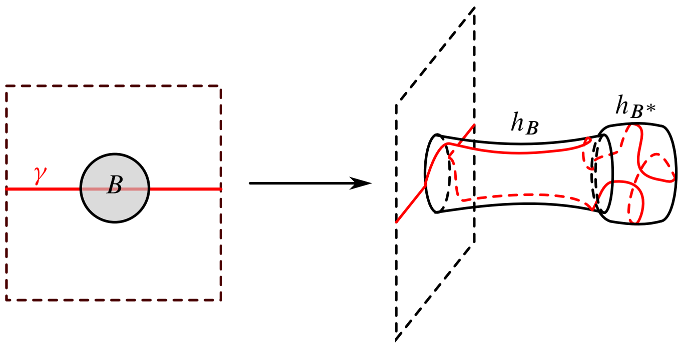

Figure 23: A fattened 0–cell B of a sutured cell decomposition $\mathcal{D}$ induces a pair of canceling index 0 and 1 contact handles. We label the handles as h_B and h_{B^*}. Here B^* denotes the 2–cell of $\mathcal{D}^*$ that is dual to B. A portion of the sutured cell decomposition $\mathcal{D}$ is shown on the left, and the corresponding contact handles are shown on the right.

decomposition $\mathcal{C}$ from $\mathcal{D}$, we can also construct a contact handle decomposition of N' from a sutured cell decomposition of $(\partial M, \gamma)$; however, the indices of the corresponding handles will be different. For our argument to work, we will actually consider the handle decomposition $\boldsymbol{H}'$ of N' induced by the dual sutured cell decomposition $\mathcal{D}^*$. For each fattened 0–cell B' of $\mathcal{D}^*$, there is a contact 2–handle $h_{B'}$ of $\boldsymbol{H}'$. For each arc λ' of $\mathcal{D}^*$, there is a contact 1–handle $h_{\lambda'}$ of $\boldsymbol{H}'$. For each 2–cell c' of $\mathcal{D}^*$, there is a contact 0–handle $h_{c'}$ of $\boldsymbol{H}'$.

By the above construction, there is a correspondence between the k–handles of $\boldsymbol{H}'$ and the $(2-k)$–cells of $\mathcal{D}^*$, which, in turn, correspond to the k–cells of $\mathcal{D}$. Finally, there is a correspondence between the k–cells of $\mathcal{D}$ and the $(k+1)$–handles of $\mathcal{C}$. Combining these, we get a correspondence between the k–handles of $\boldsymbol{H}'$ and the $(k+1)$–handles of $\mathcal{C}$. Let $h'_1, \dots, h'_n$ be an enumeration of the contact handles of $\boldsymbol{H}'$ such that h'_i corresponds to h_i under the above correspondence. By Lemma 4.3,

$$\mathrm{EH}(N', \gamma' \cup \gamma'_0, \xi) = (C_{h'_n} \circ \cdots \circ C_{h'_1})(1),$$

where 1 is the generator of $\mathrm{SFH}(\varnothing) \cong \mathbb{F}_2$. Using Lemma 3.15, we can commute contact handle maps for disjoint contact handles, so we can rearrange (5-1) as

$$\Phi_\xi(\boldsymbol{x}) = ((C_{h_n} \circ C_{h'_n}) \circ \cdots \circ (C_{h_1} \circ C_{h'_1}))(\psi^\nu_*(\boldsymbol{x})). \tag{5-2}$$

The contact handles h_i and h'_i do not always form a canceling pair in the sense of Proposition 3.8. However, they are close enough to a canceling pair to allow us to

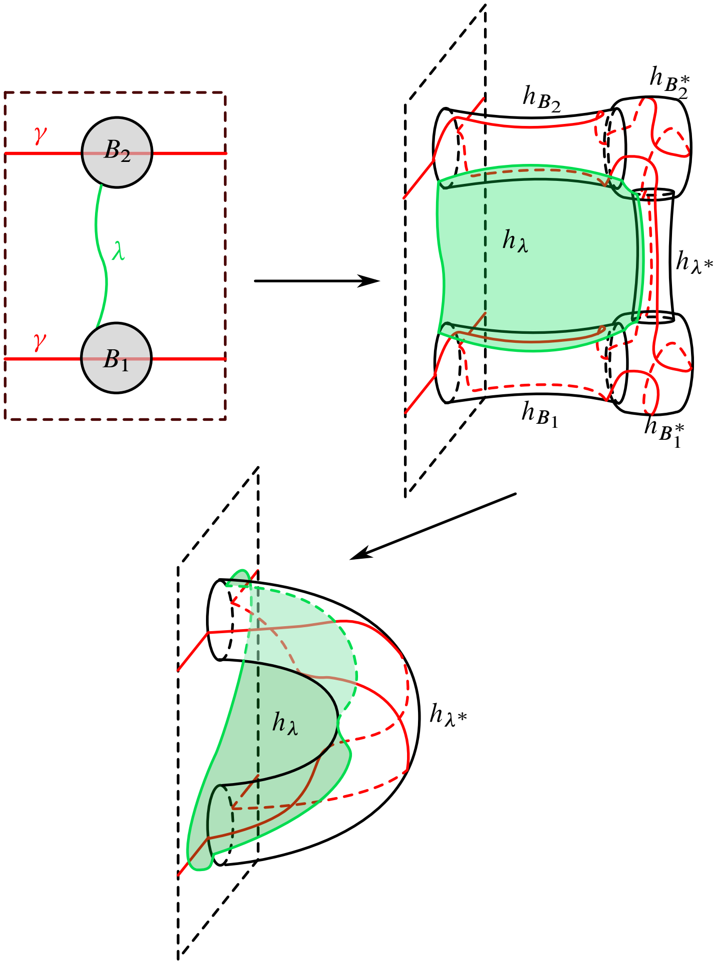

Figure 24: A canceling pair of index 1 and 2 contact handles induced by a 1–cell λ of $\mathcal{D}$, and the dual 1–cell λ^* of $\mathcal{D}^*$. The first picture shows the sutured cell decomposition $\mathcal{D}$ near λ. The second shows the contact handles associated to the cells B_1, B_2, B_1^*, B_2^*, λ, and λ^*. The third picture is obtained by canceling h_{B_i} against $h_{B_i^*}$ for $i \in \{1, 2\}$. After this cancellation and a small isotopy of h_λ, the handles h_λ and h_{λ^*} form a canceling pair of contact handles.

reduce the above composition to the diffeomorphism map ψ_*^ν by performing a sequence of handle cancellations and isotopies, as we now describe.

Let us first consider the case when h_i and h_i' correspond to a 0–cell B of $\mathcal{D}$. Under the previously described correspondence, there is a 2–cell B^* of $\mathcal{D}^*$ that B corresponds to. Using the previous notation, we have $h_i = h_B$ and $h'_i = h_{B^*}$. Then h_B and h_{B^*} form a pair of canceling index 0 and 1 contact handles; see Figure 23. Hence $C_{h_i} \circ C_{h'_i}$ is a diffeomorphism map by the computation in the proof of Theorem 3.14.

We now consider the case when h_i and h'_i correspond to a 1–cell λ of $\mathcal{D}$. Let λ^* denote the corresponding dual arc of $\mathcal{D}^*$. Let B_1 and B_2 be the fattened 0–cells of $\mathcal{D}$ at $\partial\lambda$ (note that we do not exclude the possibility that $B_1 = B_2$). Let h_{B_1} and h_{B_2} denote the corresponding 1–handles of $\mathcal{C}$. The arc λ corresponds to a 2–handle h_λ of $\mathcal{C}$. Let B_1^* and B_2^* denote the 2–cells of $\mathcal{D}^*$ that correspond to B_1 and B_2, respectively. The 2–cells B_1^* and B_2^* correspond to 0–handles $h_{B_1^*}$ and $h_{B_2^*}$ of $\boldsymbol{H}'$. The dual arc λ^* of $\mathcal{D}^*$ induces a contact 1–handle of $\boldsymbol{H}'$. As described above, the handles h_{B_i} and $h_{B_i^*}$ form a canceling pair of index 1 and 2 contact handles. After canceling these two handles, the handles h_λ and h_{λ^*} do not quite form a canceling pair of index 1 and 2 handles in the sense of Proposition 3.8, because the attaching circle of the 2–handle h_λ does not intersect the dividing set along h_{λ^*}. Instead, it intersects the dividing set near the feet of the 1–handle h_{λ^*}. This is shown in Figure 24. After performing an isotopy of h_λ, the handles h_λ and h_{λ^*} form a canceling pair of index 1 and 2 contact handles. Hence, the composition $C_{h_i} \circ C_{h'_i}$ induces a diffeomorphism map, by the computation in the proof Theorem 3.14.

Finally, we consider the case when h_i and h'_i correspond to a 2–cell c of $\mathcal{D}$. Corresponding to c, there is a fattened 0–cell c^* of $\mathcal{D}^*$. The 2–cell c induces a 3–handle $h_c = h_i$ of $\mathcal{C}$, as well as a 2–handle $h_{c^*} = h'_i$ in $\boldsymbol{H}'$. The handles h_c and h_{c^*} form a canceling pair of index 2 and 3 contact handles. Hence, the composition $C_{h_c} \circ C_{h_{c^*}}$ is equal to a diffeomorphism map, by the same computation as in the proof of Theorem 3.14.

We have shown that $C_{h_i} \circ C_{h'_i}$ is equal to a diffeomorphism map for every $i \in \{1, \dots, n\}$, induced by canceling the handles h_i and h'_i that are stacked horizontally in the ν–direction. It follows from (5-2) that Φ_ξ is equal to the diffeomorphism map ϕ_*^ν. □

5.2 Morse-type contact handles

In Section 3.3, we defined maps for gluing a contact handle h onto the boundary of a sutured manifold (M, γ). We note that M is *not* a sutured submanifold of $M \cup h$, so the Honda–Kazez–Matić framework does not assign a gluing map to the inclusion $M \subseteq M \cup h$. Nonetheless, there is a natural notion of contact handle that fits into the Honda–Kazez–Matić TQFT framework:

Definition 5.5 Suppose that (M,γ) is a sutured submanifold of (M',γ'), and ξ is a contact structure on $Z = M' \setminus \operatorname{int}(M)$ with dividing set $\gamma \cup \gamma'$. We say that (Z,ξ) is a *Morse-type contact handle of index* k if there is a contact vector field v on Z that points into Z on ∂M and out of Z on $\partial M'$ and a decomposition $Z = Z_0 \cup h$ such that

(1) Z_0 is diffeomorphic to $\partial M \times I$,

(2) v is nonvanishing on Z_0, points into Z_0 on $\partial M \times \{0\}$ and out of Z_0 on $\partial M \times \{1\}$, and each flow line of v is an arc from $\partial M \times \{0\}$ to $\partial M \times \{1\}$,

(3) h is a topological 3–ball with piecewise smooth boundary, and ξ is tight on h.

Furthermore, h is a contact k–handle attached to $M \cup Z_0$, as in Definition 3.11, with corners smoothed.

If (M,γ) is a sutured submanifold of (M',γ'), and $(Z,\xi) = (M' \setminus \operatorname{int}(M),\xi)$ is a Morse-type contact handle of index k attached to (M,γ), then we call a choice of contact vector field v and decomposition $Z = Z_0 \cup h$ a *parametrization* of (Z,ξ). Given a parametrization of (Z,ξ), there is a natural candidate for the contact gluing map Φ_ξ, namely

$$C_h \circ \phi_*^{v|_{Z_0}} \colon \mathrm{SFH}(-M,-\gamma) \to \mathrm{SFH}(-M',-\gamma').$$

In the above expression,

$$\phi_*^{v|_{Z_0}} \colon \mathrm{SFH}(-M,-\gamma) \to \mathrm{SFH}(-M \cup -Z_0,-\gamma_0)$$

is the diffeomorphism map induced by the vector field v, as discussed in Section 5.1, where γ_0 is the dividing set of ξ on $\partial(M \cup Z_0)$. Furthermore,

$$C_h \colon \mathrm{SFH}(-M \cup -Z_0,-\gamma_0) \to \mathrm{SFH}(-M',-\gamma')$$

is the contact handle map, as defined in Section 3.3. Indeed, we will prove the following:

Proposition 5.6 *Suppose (M,γ) is a sutured submanifold of (M',γ'), and $(Z,\xi) = (M' \setminus \operatorname{int}(M),\xi)$ is a Morse-type contact handle, with a parametrizing contact vector field v and decomposition $Z = Z_0 \cup h$. Then the contact gluing map Φ_ξ is equal to the composition $C_h \circ \phi_*^{v|_{Z_0}}$.*

Proof The proof is essentially the same for all handle indices, so for definiteness we will focus on 2–handles. By assumption, the contact vector field v is nonvanishing on Z_0, and on a collar neighborhood of $\partial M'$. Let $D \subseteq Z$ be a core of h. Pick an incoming barrier surface $S \subseteq Z_0$. Extend D down into Z_0 so that $\partial D \subseteq S$ is Legendrian with $\mathrm{tb}(\partial D) = -1$. Then we can perturb D while fixing ∂D so that it becomes convex. Let N denote the collar of ∂M bounded by S, and let

$$\widetilde{Z} := \mathrm{cl}\big(Z \setminus (N \cup N(D))\big).$$

We can pick $N(D)$ such that ν is nonvanishing on $\widetilde{Z}$. Using the flow of ν, one can construct a Morse function f on $\widetilde{Z}$ that is 0 on $\partial\widetilde{Z} \setminus \partial M'$ and 1 on $\partial M'$ and such that $\nu(f) > 0$.

The image of $\partial N(D) \setminus S$ in $\partial M'$ under the flow of ν consists of two disks, D_1 and D_2. Pick a dualizable sutured cell decomposition $\mathcal{D}$ of $(\partial M', \gamma')$ with no 0–cells or 1–cells that intersect D_1 or D_2. Let $B_1, \dots, B_m$ be the 0–cells of $\mathcal{D}$, and let $\lambda_1, \dots, \lambda_n$ be the 1–cells. Write $c_1, \dots, c_k$ for the 2–cells. Adapting Example 3.7, after performing a C^0–small isotopy of S, and picking a barrier surface S' that bounds a collar neighborhood N' of $\partial M'$, we can construct a contact cell decomposition $\mathcal{C}$ of Z that has barrier surfaces S and S'. The contact cell decomposition $\mathcal{C}$ has no 0–cells, one 1–cell for each 0–cell $B_1, \dots, B_m$ of $\mathcal{D}$, one 2–cell for each 1–cell $\lambda_1, \dots, \lambda_n$ of $\mathcal{D}$, one 3–cell for each 2–cell $c_1, \dots, c_k$ of $\mathcal{D}$, and also the 2–cell D. Let us write $h_1, \dots, h_n$ for the handles induced by $\mathcal{D}$, and write h_D for $N(D)$, viewed as a contact 2–handle. By definition

$$\Phi_\xi(\boldsymbol{x}) = (C_{h_n} \circ \cdots \circ C_{h_1} \circ C_{h_D})(\phi_*^{\nu|_N}(\boldsymbol{x}) \otimes \mathrm{EH}(N', \xi|_{N'})). \tag{5-3}$$

A dual sutured cell decomposition $\mathcal{D}^*$ of $(\partial M', \gamma')$ gives rise to a contact handle decomposition $\boldsymbol{H}'$ of $(N', \xi|_{N'})$ starting at the empty sutured manifold. We can compute $\mathrm{EH}(N', \xi|_{N'})$ by applying Lemma 4.3 to $\boldsymbol{H}'$. Exactly as in the proof of Proposition 5.1, the handles of $\boldsymbol{H}'$ cancel $h_1, \dots, h_n$ pairwise, and we can reduce (5-3) to

$$(\phi_*^{\nu|_{\widetilde{Z}}} \circ C_{h_D} \circ \phi_*^{\nu|_N})(\boldsymbol{x}).$$

Given the description of the diffeomorphism maps $\phi_*^{\nu|_{\widetilde{Z}}}$ and $\phi_*^{\nu|_N}$ from Section 5.1, the above expression is clearly equal to $C_h \circ \phi_*^{\nu|_{Z_0}}$. □

5.3 Functoriality of the gluing map

We now show that the gluing map defined in this paper satisfies the functoriality property of the Honda–Kazez–Matić construction [12, Proposition 6.2]. This property will be useful when we prove the equivalence of our construction with the Honda–Kazez–Matić construction.

Proposition 5.7 *Suppose that we have a chain of sutured submanifolds*

$$(M,\gamma)\subseteq(M',\gamma')\subseteq(M'',\gamma''),$$

as well as a contact structure ξ *on* $M''\setminus\operatorname{int}(M)$ *such that* ∂M*,* $\partial M'$*, and* $\partial M''$ *are convex with dividing sets* γ*,* γ'*, and* γ''*, respectively. Writing* ξ' *for* $\xi|_{M'\setminus\operatorname{int}(M)}$ *and* ξ'' *for* $\xi|_{M''\setminus\operatorname{int}(M')}$*, we have*

$$\Phi_\xi=\Phi_{\xi''}\circ\Phi_{\xi'}.$$

Proof Define $Z':=M'\setminus\operatorname{int}(M)$ and $Z'':=M''\setminus\operatorname{int}(M')$. Let $\mathcal{C}'$ and $\mathcal{C}''$ be contact cell decompositions of (Z',ξ') and (Z'',ξ''). Let us write $h'_1,\dots,h'_n$ for the contact handles of $\mathcal{C}'$, and $h''_1,\dots,h''_m$ for the contact handles of $\mathcal{C}''$, ordered so that their indices are nondecreasing. Let ν' and ν'' denote the contact vector fields chosen on the incoming ends of Z' and Z'', let N_1 and N_2 denote the incoming layers of Z' and Z'', and let N'_1 and N'_2 denote the outgoing layers, respectively, as described in Definition 3.5. By definition, the composition $\Phi_{\xi''}\circ\Phi_{\xi'}$ is equal to

$$(C_{h''_m}\circ\cdots\circ C_{h''_1})\circ\Phi_{\sqcup N'_2}\circ\phi_*^{\nu''|_{N_2}}\circ(C_{h'_n}\circ\cdots\circ C_{h'_1})\circ\Phi_{\sqcup N'_1}\circ\phi_*^{\nu'|_{N_1}}. \tag{5-4}$$

The map $\Phi_{\sqcup N'_i}$ is given by tensoring with $\operatorname{EH}(N'_i,\xi|_{N'_i})$ for $i\in\{1,2\}$. As in Lemma 4.3, the element $\Phi_{\sqcup N'_1}$ can be written as a composition of contact handle maps $C_{h_1}\circ\cdots\circ C_{h_\ell}$, for a sequence of contact 0–, 1–, and 2–handles $h_1,\dots,h_\ell$. Hence, the composition in (5-4) can be written as

$$(C_{h''_m}\circ\cdots\circ C_{h''_1})\circ\Phi_{\sqcup N'_2}\circ\phi_*^{\nu''|_{N_2}}\circ(C_{h'_n}\circ\cdots\circ C_{h'_1})\circ(C_{h_\ell}\circ\cdots\circ C_{h_1})\circ\phi_*^{\nu'|_{N_1}}. \tag{5-5}$$

By picking ν'' and N_2 appropriately, we can assume that the diffeomorphism

$$\phi^{\nu''|_{N_2}}\colon M'\to M'\cup N_2$$

is a contactomorphism on all of Z' and is the identity on ∂M. Write $\bar{h}_k=\phi^{\nu''|_{N_2}}(h_k)$ and $\bar{h}'_k=\phi^{\nu''|_{N_2}}(h'_k)$. Using the diffeomorphism invariance of the contact handle maps, we can rewrite (5-5) as

$$(C_{h''_m}\circ\cdots\circ C_{h''_1})\circ\Phi_{\sqcup N'_2}\circ(C_{\bar{h}'_n}\circ\cdots\circ C_{\bar{h}'_1})\circ(C_{\bar{h}_\ell}\circ\cdots\circ C_{\bar{h}_1})\circ\phi_*^{\nu'|_{N_1}}. \tag{5-6}$$

The map $\Phi_{\sqcup N'_2}$ can be commuted with all the contact handle maps to the right of it by Lemma 3.15. After possibly isotoping some of the remaining contact handles, we can apply Lemma 3.15 and reorder the handles so that they are attached with nondecreasing index. Furthermore, after isotoping some of the handles, we can assume that the handles in the above composition are induced by a contact cell decomposition (ie the 0–handles and 1–handles are induced by a Legendrian graph, and the 2–handles are induced by a sequence of convex disks with $\mathrm{tb}=-1$ attached to a neighborhood of the graph and $\partial((Z'\cup Z'')\setminus \mathrm{int}(N_1\cup N_2'))$. It follows that (5-6) is equal to $\Phi_{\xi,\mathcal{C}}$ for some contact cell decomposition $\mathcal{C}$ of (Z,ξ), completing the proof. □

5.4 Equivalence with the Honda–Kazez–Matić construction

In this section, we prove that our construction of the gluing map from Section 3.4 is equivalent to the original construction due to Honda, Kazez, and Matić [12]. We will write Φ_ξ^{HKM} for the map defined using their construction.

Theorem 5.8 *Suppose (M,γ) is a sutured submanifold of (M',γ') with no isolated component and that ξ is a contact structure on $M'\setminus\mathrm{int}(M)$ with convex boundary and dividing set $\gamma\cup\gamma'$. Then the Honda–Kazez–Matić gluing map Φ_ξ^{HKM} is equal to the gluing map Φ_ξ we defined in Section 3.4.*

Proof Using the composition law for both constructions of the gluing map (Proposition 5.7 and [12, Proposition 6.2]), it is sufficient to show the claim when $M'\setminus\mathrm{int}(M)$ consists of a single Morse-type contact handle of index 0, 1, or 2.

For a Morse-type index 0 handle, the claim is straightforward. Write $M'\setminus\mathrm{int}(M)$ as $Z_0\cup h^0$ where $Z_0\cong(I\times\partial M)$ and h^0 is a 3–ball. The contact structure ξ is the union of the I–invariant contact structure on Z_0 and the unique tight contact structure on h^0. Write γ_0' for the suture on $M\cup Z_0$, and suppose that v is a parametrizing contact vector field on Z_0.

Under the identification

$$\mathrm{SFH}(M\cup Z_0\cup h^0,\gamma_0'\cup\gamma_0)\cong\mathrm{SFH}(M\cup Z_0,\gamma_0')\otimes\mathrm{SFH}(h^0,\gamma_0),$$

where γ_0 consists of a single suture on h^0, both gluing maps

$$\Phi_\xi,\ \Phi_\xi^{\mathrm{HKM}}\colon \mathrm{SFH}(M,\gamma)\to\mathrm{SFH}(M\cup Z_0,\gamma_0')\otimes\mathrm{SFH}(h^0,\gamma_0)$$

take the form

$$\boldsymbol{x}\mapsto\phi_*^v(\boldsymbol{x})\otimes\mathrm{EH}(h^0,\gamma_0,\xi_0)=\phi_*^v(\boldsymbol{x})\otimes 1,$$

where 1 denotes the generator of $\mathrm{SFH}(h^0,\gamma_0)\cong\mathbb{F}_2$. For Φ_ξ^{HKM}, this follows from [12, Proposition 6.1], and for our map Φ_ξ, from Proposition 5.6.

Before we consider index 1 and 2 Morse-type contact handles, we must first give a more detailed description of the construction of Honda, Kazez, and Matić [12]. The definition of the map Φ_ξ^{HKM} uses the description of partial open books from [13]. A contact 3–manifold with convex boundary is called *product disk decomposable* if it is a union of handlebodies, and it contains a collection of pairwise disjoint compressing disks, each intersecting the dividing set in two points, whose complement is a union of standard contact balls. Given a contact sutured manifold (M,γ,ξ), one picks a properly embedded Legendrian graph $K\subseteq M$ that intersects ∂M along a collection of univalent vertices in γ such that $M\setminus \mathrm{int}(N(K))$ is product disk decomposable. Here $N(K)$ denotes a standard contact neighborhood, which is also product disk decomposable. It follows that $M\setminus\mathrm{int}(N(K))$ is contactomorphic to $(S\times I)/{\sim_1}$ for a compact surface S with boundary, and $N(K)$ is contactomorphic to $(P\times I)/{\sim_2}$ for a compact surface with boundary P (for the definition of $\sim_1$ and $\sim_2$, see Section 4.1). Here P has piecewise smooth boundary whose edges can naturally be divided into two types: those that intersect $\partial N(K)\setminus\partial M$, and those that intersect ∂M. By $(P\times I)/{\sim_2}$, we mean the space obtained by quotienting out the I direction along the edges that are contained in $\partial N(K)$. Since $(P\times I)/{\sim_2}$ meets $(S\times I)/{\sim_1}$ along $\partial N(K)\setminus\partial M$, the surfaces $P\times\{0\}$ and $P\times\{1\}$ give two embeddings of P into S. Using the projection of $S\times I$ onto S, we identify $P\times\{0\}\subseteq S\times\{1\}$ with a subset of S, for which we also write P. The surface $P\times\{1\}$ then gives another smooth embedding $h\colon P\to S$, which is the monodromy map.

The Honda–Kazez–Matić map is easiest to define if one picks a contact structure ζ on (M,γ) such that ∂M is convex with dividing set γ. To define the map Φ_ξ^{HKM}, one picks Legendrian graphs $K\subseteq M$ and $K'\subseteq M'\setminus\mathrm{int}(M)$ whose complements are product disk decomposable. After modifying K' in a neighborhood of ∂M, one extends K to a Legendrian graph on all of M' whose complement is product disk decomposable. Furthermore, outside a small neighborhood of ∂M, the extension agrees with K and K'. Let us write $\overline{K}$ for this Legendrian graph. The graph K is also required to satisfy a *contact compatibility* condition near ∂M, described in [12], though the specific form is not important for our present argument. The graph $K\subseteq M$ induces a partial open book (S,P,h) for (M,γ,ζ), which induces a diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ for (M,γ). The graph $\overline{K}$ induces a partial open book (S',P',h') for (M',γ'), which gives rise to the diagram $(\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}')$. Furthermore $\Sigma'=\Sigma\cup\Sigma''$, $\boldsymbol{\alpha}'=\boldsymbol{\alpha}\cup\boldsymbol{\alpha}''$, and $\boldsymbol{\beta}=\boldsymbol{\beta}\cup\boldsymbol{\beta}''$, for a surface Σ'' and a collection of curves $\boldsymbol{\alpha}''$ and $\boldsymbol{\beta}''$ on Σ'. There is a canonical intersection point $\boldsymbol{x}_\xi\in\mathbb{T}_{\boldsymbol{\alpha}''}\cap\mathbb{T}_{\boldsymbol{\beta}''}$, and the map Φ_ξ^{HKM} on the chain level is defined by the formula

$$\Phi_\xi^{\mathrm{HKM}}(\boldsymbol{x})=\boldsymbol{x}\times\boldsymbol{x}_\xi.$$

We now consider a Morse-type contact 1–handle addition. In this case,

$$(M' \setminus \operatorname{int}(M), \xi) \cong (Z_0 \cup h_1, \xi_0 \cup \xi_1),$$

where (Z_0, ξ_0) is an I–invariant contact structure on $I \times \partial M$, and (h^1, ξ_1) is a contact 1–handle. Let $\overline{K} \subseteq M \cup Z_0$ denote an extension that can be used to compute the gluing map. We note that if $K' \subseteq Z_0$ is a graph whose complement is product disk decomposable, then the complement of K' in $Z_0 \cup h^1$ is also product disk decomposable. It follows that we can use the same graph

$$\overline{K} \subseteq M \cup Z_0 \subseteq M \cup Z_0 \cup h^1$$

to compute the gluing map for $Z_0 \cup h^1$. Write (S, P, h) for the partial open book of (M, γ) induced by K. Write (S'_0, P'_0, h'_0) for the partial open book of $(M \cup Z_0, \gamma'_0)$ induced by $\overline{K} \subseteq M \cup Z_0$ and (S', P', h') for the partial open book of $(M \cup Z_0 \cup h^1, \gamma')$ induced by $\overline{K}$.

Because $(M \cup Z_0 \cup h^1) \setminus N(\overline{K})$ is obtained by attaching a contact 1–handle to $(M \cup Z_0) \setminus N(\overline{K})$, we can apply Lemma 4.4 to see that the partial open book (S', P', h') is obtained from (S'_0, P'_0, h'_0) by attaching a 1–handle to S'_0 along $\partial S'_0 \setminus \partial P'_0$, and setting $P' = P'_0$ and $h' = \iota_{S'_0} \circ h'_0$, where $\iota_{S'_0}\colon S'_0 \to S'$ is the embedding. The same basis of arcs $\boldsymbol{a}'$ for P'_0 in S'_0 can be used for P' in S', which we assume extends a basis of arcs $\boldsymbol{a}$ for P in S. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $(\Sigma'_0, \boldsymbol{\alpha}'_0, \boldsymbol{\beta}'_0)$, and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ be the diagrams induced by (S, P, h), (S'_0, P'_0, h'_0), and (S', P', h') with the bases $\boldsymbol{a}$, $\boldsymbol{a}'$, and $\boldsymbol{a}'$, respectively.

The Heegaard surface Σ' is thus obtained by attaching a 1–handle along the boundary of Σ'_0. Notice that, since $\boldsymbol{\alpha}'_0 = \boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'_0 = \boldsymbol{\beta}'$, and Σ' is obtained from Σ'_0 by attaching a band along $\partial \Sigma'_0$, the groups $\mathrm{SFH}(\Sigma'_0, \boldsymbol{\alpha}'_0, \boldsymbol{\beta}'_0)$ and $\mathrm{SFH}(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ are naturally isomorphic, and the isomorphism is given by the contact 1–handle map defined in this paper. Furthermore, we observe that

$$\Phi^{\mathrm{HKM}}_{\xi_0 \cup \xi_1} = C_{h^1} \circ \Phi^{\mathrm{HKM}}_{\xi_0}. \tag{5-7}$$

By [12, Proposition 6.1], the above expression is equal to $C_{h^1} \circ \phi^\nu_*$, where

$$\phi^\nu_*\colon \mathrm{SFH}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to \mathrm{SFH}(\Sigma'_0, \boldsymbol{\alpha}'_0, \boldsymbol{\beta}'_0)$$

is the composition of the tautological map induced by a diffeomorphism and the transition maps induced by naturality. By Proposition 5.6, we see that the expression in (5-7) agrees with the definition of $\Phi_{\xi_0 \cup \xi_1}$ in this paper.

The argument when

$$(M' \setminus \operatorname{int}(M), \xi) = (Z_0 \cup h^2, \xi_0 \cup \xi_2)$$

is a Morse-type contact 2–handle is similar. Suppose that $K \subseteq M$ is a Legendrian graph such that $M \setminus \operatorname{int}(N(K))$ is product disk decomposable and induces a partial open book that satisfies the contact compatibility condition near ∂M. We then let $\overline{K}_0$ denote a Legendrian extension to $M \cup Z_0$ whose complement is product disk decomposable and which can be used to compute the map $\Phi^{\mathrm{HKM}}_{\xi_0}$ for gluing (Z_0, ξ_0) to M. We can define an extension $\overline{K}$ of K into all of $M \cup Z_0 \cup h^2$ by setting

$$\overline{K} := \overline{K}_0 \cup c,$$

where c is a Legendrian cocore of the 2–handle h^2. We note that $(M \cup Z_0 \cup h^2) \setminus \operatorname{int}(N(\overline{K}))$ is product disk decomposable. Let (S, P, h), (S'_0, P'_0, h'_0), and (S', P', h') denote the partial open books induced by $K \subseteq (M, \gamma)$, $\overline{K}_0 \subseteq (M \cup Z_0, \gamma'_0)$, and $\overline{K} \subseteq (M \cup Z_0 \cup h^2, \gamma')$, respectively. Let $\pi\colon S \times I \to S$ be the projection. Writing the 2–handle h^2 as $(B \times I)/{\sim_2}$, where B is a square, we observe that P' is obtained by adding the 1–handle $\pi(B \times \{0\})$ to P'_0. The monodromy is extended to P' by mapping $\pi(B \times \{0\}) \subseteq S$ to $\pi(B \times \{1\}) \subseteq S$.

We start with a basis of arcs $\boldsymbol{a}$ for $P \subseteq S$, and extend $\boldsymbol{a}$ to a basis $\boldsymbol{a}'_0$ for $P'_0 \subseteq S'_0$. A basis of arcs $\boldsymbol{a}'$ for $P' \subseteq S'$ can then be obtained from $\boldsymbol{a}'_0$ by adding a new arc a' which is a cocore of the band $\pi(B \times \{0\})$. Write $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $(\Sigma'_0, \boldsymbol{\alpha}'_0, \boldsymbol{\beta}'_0)$, and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ for the diagrams obtained from the partial open books (S, P, h), (S'_0, P'_0, h'_0), and (S', P', h'), with bases $\boldsymbol{a}$, $\boldsymbol{a}'_0$, and $\boldsymbol{a}'$, respectively. Also, let us write $\boldsymbol{\alpha}'_0 = \boldsymbol{\alpha} \cup \boldsymbol{\alpha}''_0$ and $\boldsymbol{\beta}'_0 = \boldsymbol{\beta} \cup \boldsymbol{\beta}''_0$, where $\boldsymbol{\alpha}''_0$ and $\boldsymbol{\beta}''_0$ are the curves induced by the arcs in $\boldsymbol{a}'_0 \setminus \boldsymbol{a}$, and $\boldsymbol{x}_{\xi_0}$ for the canonical intersection point in $\mathbb{T}_{\boldsymbol{\alpha}''_0} \cap \mathbb{T}_{\boldsymbol{\beta}''_0}$. Finally, write α' and β' for the curves induced by the new basis arc a', and write c' for the canonical intersection point of $\alpha' \cap \beta'$. The map $\Phi^{\mathrm{HKM}}_{\xi_0 \cup \xi_2}$ on $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ is defined by the formula

$$\Phi^{\mathrm{HKM}}_{\xi_0 \cup \xi_2}(\boldsymbol{x}) = \boldsymbol{x} \times \boldsymbol{x}_{\xi_0} \times c'.$$

Noting that $\Phi^{\mathrm{HKM}}_{\xi_0}(\boldsymbol{x}) = \boldsymbol{x} \times \boldsymbol{x}_{\xi_0}$, we see that

$$\Phi^{\mathrm{HKM}}_{\xi_0 \cup \xi_2}(\boldsymbol{x}) = (C_{h^2} \circ \Phi^{\mathrm{HKM}}_{\xi_0})(\boldsymbol{x}).$$

By the same argument as for contact 1–handles, this is equal to $(C_{h^2} \circ \phi^{\nu}_{*})(\boldsymbol{x}) = \Phi_{\xi_0 \cup \xi_2}(\boldsymbol{x})$, completing the proof. □

6 Turning around cobordisms of sutured manifolds and duality

In this section, we compute the effect of turning around a cobordism of sutured manifolds, proving Theorem 1.3.

6.1 The canonical trace pairing

As described in [4, Proposition 2.14] and [15, Section 11.2], there is duality between $\mathrm{SFH}(M,\gamma)$ and $\mathrm{SFH}(-M,\gamma)$. If $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is a diagram for (M,γ), then $(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})$ is a diagram for $(-M,\gamma)$. Since $\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$ is equal to $\mathbb{T}_{\boldsymbol{\beta}}\cap\mathbb{T}_{\boldsymbol{\alpha}}$, we can define a map

$$\mathrm{tr}\colon \mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})\otimes\mathrm{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})\to\mathbb{F}_2 \tag{6-1}$$

by the formula

$$\mathrm{tr}(\boldsymbol{x}\otimes\boldsymbol{y})=\begin{cases}1 & \text{if } \boldsymbol{x}=\boldsymbol{y},\\ 0 & \text{otherwise.}\end{cases}$$

It is straightforward to see that tr is a chain map, because J–holomorphic discs on $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ from $\boldsymbol{x}$ to $\boldsymbol{y}$ are in bijection with J–holomorphic discs on $(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})$ from $\boldsymbol{y}$ to $\boldsymbol{x}$. Note that the trace pairing gives a natural isomorphism

$$\mathrm{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})\cong\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})^\vee:=\mathrm{Hom}_{\mathbb{F}_2}(\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}),\mathbb{F}_2).$$

In particular, tr is the usual pairing between homology and cohomology.

In the opposite direction, there is the cotrace map,

$$\mathrm{cotr}\colon \mathbb{F}_2\to\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})\otimes\mathrm{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha}).$$

The cotrace map is defined by the formula

$$\mathrm{cotr}(1)=\sum_{\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}}\boldsymbol{x}\otimes\boldsymbol{x}.$$

We note that if V is a finite-dimensional vector space over $\mathbb{F}_2$, then there are canonical isomorphisms

$$\mathrm{Hom}_{\mathbb{F}_2}(V,V)\cong\mathrm{Hom}_{\mathbb{F}_2}(V\otimes V^\vee,\mathbb{F}_2)\cong\mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2,V^\vee\otimes V).$$

Under these isomorphisms, the tr and cotr maps are identified with $\mathrm{id}_V\in\mathrm{Hom}_{\mathbb{F}_2}(V,V)$.

In [15], the first author defined a pairing

$$\langle\,,\rangle\colon \mathrm{SFH}(M,\gamma)\otimes\mathrm{SFH}(-M,-\gamma)\to\mathbb{F}_2,$$

which at first glance appears to have a different domain than the trace map defined in (6-1). However, there is a canonical isomorphism

$$\mathrm{SFH}(-M,\gamma) \cong \mathrm{SFH}(-M,-\gamma), \tag{6-2}$$

which we describe presently. The diagram $(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})$ represents $(-M,\gamma)$, while $(-\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ represents $(-M,-\gamma)$. If $\phi \in \pi_2(\boldsymbol{x},\boldsymbol{y})$ is a homology class of disks for $(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})$, then there is a uniquely determined homology class $\bar{\phi} \in \pi_2(\boldsymbol{x},\boldsymbol{y})$ for $(-\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$. Furthermore, by precomposing a holomorphic disk $u\colon \mathbb{D} \to \mathrm{Sym}^n(\Sigma)$ with the unique antiholomorphic involution of $\mathbb{D}$ that fixes $\pm i \in \partial\mathbb{D}$, we obtain a bijection

$$\mathcal{M}_J(\phi)/\mathbb{R} \cong \mathcal{M}_{-J}(\bar{\phi})/\mathbb{R},$$

establishing the isomorphism in (6-2). We note that tr agrees with $\langle\, ,\rangle$ under the isomorphism in (6-2).

6.2 Sutured manifold cobordisms and the induced maps

In this section, we review the definition of sutured manifold cobordisms, special cobordisms, boundary cobordisms, and the construction of the sutured cobordism maps. We finally give a simpler definition of the cobordism maps using our gluing map from Section 3.4. The following is [15, Definition 2.3].

Definition 6.1 The contact structures ξ_0 and ξ_1 on the sutured manifold (M,γ) are *equivalent* if they can be connected by a 1–parameter family $\{\xi_t : t \in I\}$ of contact structures on (M,γ) such that ∂M is convex with dividing set γ for each ξ_t. In this case, we write $\xi_1 \sim \xi_2$, and denote the equivalence class of ξ by $[\xi]$.

Sutured manifold cobordisms were defined in [15, Definition 2.4].

Definition 6.2 Let (M_0,γ_0) and (M_1,γ_1) be sutured manifolds. A *cobordism* from (M_0,γ_0) to (M_1,γ_1) is a triple $\mathcal{W} = (W, Z, [\xi])$ such that

- W is a compact, oriented 4–manifold with boundary and corners,
- Z is a codimension-0 submanifold of ∂W, and $\partial W \setminus \mathrm{int}(Z) = -M_0 \sqcup M_1$,
- ξ is a positive contact structure on $(Z, \gamma_0 \cup \gamma_1)$.

Note that equivalent contact structures can have different characteristic foliations on ∂M, which gives us enough flexibility to compose cobordisms. The sutured manifold cobordism $\mathcal{W}$ is *balanced* if (M_0,γ_0) and (M_1,γ_1) are balanced sutured manifolds. Furthermore, we say that Z_0 is an *isolated component* of Z if $Z_0 \cap M_1 = \varnothing$. The following is [15, Definition 5.1]:

Definition 6.3 The cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0,γ_0) to (M_1,γ_1) is *special* if

- $\mathcal{W}$ is balanced,
- $\partial M_0 = \partial M_1$, and $Z = -I \times \partial M_0$ is the trivial cobordism between them,
- ξ is an I–invariant contact structure on Z such that $\{t\} \times \partial M_0$ is convex with dividing set $\{t\} \times \gamma_0$ for every $t \in I$, with respect to the contact vector field $\partial/\partial t$.

Given a special cobordism $\mathcal{W}$ from (M_0,γ_0) to (M_1,γ_1), we define the map

$$F_{\mathcal{W}}\colon \mathrm{SFH}(M_0,\gamma_0) \to \mathrm{SFH}(M_1,\gamma_1)$$

by composing maps associated to 4–dimensional handle attachments along the interior of M_0; see [15, Section 8]. The following is equivalent to [15, Definition 10.4].

Definition 6.4 A sutured cobordism $(W, Z, [\xi])$ from (M,γ) to (M',γ') is called a *boundary cobordism* if $M \subseteq \mathrm{Int}(M')$, $W = I \times M'/\sim$, where $(t,x) \sim (t',x)$ for every $x \in \partial M'$ and $t \in I$, and ξ is a contact structure on $Z = -\{0\} \times (M' \setminus \mathrm{int}(M))$ inducing the sutures $\{0\} \times \gamma$ on $\{0\} \times \partial M$ and $\{0\} \times \gamma'$ on $\{0\} \times \partial M'$.

If $\mathcal{W} = (W, Z, [\xi])$ is a boundary cobordism from (M,γ) to (M',γ'), then we can view $(-M,-\gamma)$ as a sutured submanifold of $(-M',-\gamma')$, and $-\xi$ is a positive contact structure on $Z = -M' \setminus \mathrm{int}(-M)$ with dividing set $-\gamma_0 \cup -\gamma_1$. If Z has no isolated components, then the map $F_{\mathcal{W}}\colon \mathrm{SFH}(M,\gamma) \to \mathrm{SFH}(M',\gamma')$ is defined as the Honda–Kazez–Matić gluing map $\Phi_{-\xi}$.

Every balanced cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0,γ_0) to (M_1,γ_1) can uniquely be written as a composition $\mathcal{W}^s \circ \mathcal{W}^b$, where

$$\mathcal{W}^b = (I \times (M_0 \cup -Z)/\sim,\ \{0\} \times Z,\ [\xi])$$

is a boundary cobordism from (M_0,γ_0) to $(M_0 \cup -Z, \gamma_1)$. Furthermore,

$$\mathcal{W}^s = (W,\ -I \times \partial M_1,\ [\eta])$$

is a special cobordism from $(M_0 \cup -Z, \gamma_1)$ to (M_1,γ_1), where $-I \times \partial M_1$ is a collar of ∂M_1 in Z, and η is an I–invariant contact structure with dividing set $\{t\} \times \gamma_1$

on $\{t\} \times \partial M_1$ for every $t \in I$, with respect to $\partial/\partial t$. We call $\mathcal{W}^s$ the special part, and $\mathcal{W}^b$ the boundary part, of $\mathcal{W}$. If Z has no isolated components, then the cobordism map $F_{\mathcal{W}}$ is defined as $F_{\mathcal{W}^s} \circ F_{\mathcal{W}^b}$.

According to [15, Definition 10.1], in the general case, we choose a standard contact ball $B_0 \subseteq \mathrm{int}(Z_0)$ with convex boundary and dividing set δ_0 in each isolated component Z_0 of Z. We write (B, δ) for the union of the balls (B_0, δ_0), and consider the cobordism $\mathcal{W}' = (W, Z', [\xi'])$ from (M_0, γ_0) to $(M_1, \gamma_1) \sqcup (B, \delta)$, where $Z' = Z \setminus \mathrm{int}(B)$ and $\xi' = \xi|_{Z'}$. Since Z' has no isolated components and

$$\mathrm{SFH}((M_1, \gamma_1) \sqcup (B, \delta)) \cong \mathrm{SFH}(M_1, \gamma_1),$$

we can define $F_{\mathcal{W}} := F_{\mathcal{W}'}$. This is independent of the choice of B.

The gluing map that we defined in Section 3.4 also assigns maps to contact 3–handles, and hence $\Phi_{-\xi}$ makes sense even if Z has isolated components whenever $M_0 \cup -Z$ has no closed components, giving rise to an alternative definition of $F_{\mathcal{W}^b}$. We now show that the two constructions agree.

Proposition 6.5 *Let $\mathcal{W} = (W, Z, \xi)$ be a sutured manifold cobordism from (M_0, γ_0) to (M_1, γ_1), possibly with Z having isolated components, but such that $M_0 \cup -Z$ has no closed components. Then*

$$F_{\mathcal{W}} = F_{\mathcal{W}^s} \circ \Phi_{-\xi},$$

where $F_{\mathcal{W}}$ is the cobordism map defined in [15, Definition 10.1] *using the Honda–Kazez–Matić gluing map, and $\Phi_{-\xi}$ is the gluing map from Section 3.4.*

Proof By definition, $F_{\mathcal{W}} = i \circ F_{\mathcal{W}'} = i \circ F_{(\mathcal{W}')^s} \circ F_{(\mathcal{W}')^b}$, where

$$\mathcal{W}' = (W, Z', [\xi']) \colon (M, \gamma) \to (M', \gamma') \sqcup (B, \delta)$$

is the cobordism defined above, and

$$i \colon \mathrm{SFH}((M_1, \gamma_1) \sqcup (B, \delta)) \xrightarrow{\simeq} \mathrm{SFH}(M_1, \gamma_1)$$

is the canonical isomorphism. As Z' has no isolated components, $F_{(\mathcal{W}')^b}$ is the Honda–Kazez–Matić gluing map $\Phi_{-\xi'}$, which agrees with our gluing map from Section 3.4 by Theorem 5.8.

On the other hand, $\Phi_{-\xi} = \Phi_{-\xi_B} \circ \Phi_{-\xi'}$, where $\xi_B = \xi|_B$. Hence, it suffices to show

$$i \circ F_{(\mathcal{W}')^s} = F_{\mathcal{W}^s} \circ \Phi_{-\xi_B}.$$

Let $\mathcal{W}_B$ be the special cobordism from $(M_0 \cup -Z', \gamma_1 \cup \delta)$ to $(M_0 \cup -Z, \gamma_1) \sqcup (B, \delta)$ obtained by pushing ∂B slightly into Z', and attaching a 4–dimensional 3–handle along each component. Then

$$(\mathcal{W}')^s = (\mathcal{W}^s \sqcup \mathrm{id}_{(B,\delta)}) \circ \mathcal{W}_B.$$

By definition, the contact 3–handle map $\Phi_{-\xi_B}$ is equal to $j \circ F_{\mathcal{W}_B}$, where

$$j\colon \mathrm{SFH}((M_0 \cup -Z, \gamma_1) \sqcup (B, \delta)) \xrightarrow{\simeq} \mathrm{SFH}(M_0 \cup -Z, \gamma_1)$$

is the canonical isomorphism; see Section 3.3.4. Hence, it suffices to show that

$$i \circ F_{\mathcal{W}^s \sqcup \mathrm{id}_{(B,\delta)}} = F_{\mathcal{W}^s} \circ j.$$

This holds since $F_{\mathcal{W}^s \sqcup \mathrm{id}_{(B,\delta)}} = F_{\mathcal{W}^s} \otimes \mathrm{id}_{\mathrm{SFH}(B,\delta)}$ and $\mathrm{SFH}(B,\delta) \cong \mathbb{F}_2$. □

6.3 Turning around sutured cobordisms

In this section, we use our contact handle maps to prove a first result about duality in sutured Floer homology. If

$$\mathcal{W} = (W, Z, [\xi])\colon (M, \gamma) \to (M', \gamma')$$

is a cobordism of sutured manifolds, then we can form the cobordism

$$\mathcal{W}^\vee := (W, Z, [\xi])\colon (-M', \gamma') \to (-M, \gamma)$$

by reversing which ends of W are viewed as incoming or outgoing. The main result of this section is the following:

Theorem 6.6 *If $\mathcal{W}\colon (M,\gamma) \to (M',\gamma')$ is a balanced cobordism of sutured manifolds, and $\mathcal{W}^\vee$ is the cobordism obtained by turning around $\mathcal{W}$, then*

$$F_{\mathcal{W}^\vee} = (F_{\mathcal{W}})^\vee,$$

with respect to the trace pairing from Section 6.1.

Suppose that (M, γ) is a sutured submanifold of (M', γ'), and let ξ be a contact structure on $Z = -M' \setminus \mathrm{int}(M)$ with dividing set $\gamma \cup \gamma'$ on the convex surface ∂Z. Consider the boundary cobordism

$$\mathcal{W} := (I \times M'/\sim,\ \{0\} \times Z,\ [\xi])$$

from (M, γ) to (M', γ'). Then $\mathcal{W}^\vee$ is a sutured cobordism from $(-M', \gamma')$ to $(-M, \gamma)$. In general, $\mathcal{W}^\vee$ will be neither a special cobordism nor a boundary cobordism.

It is the product of a boundary cobordism

$$(\mathcal{W}^\vee)^b\colon (-M',\gamma') \to (-M' \cup -Z, \gamma)$$

and a special cobordism

$$(\mathcal{W}^\vee)^s\colon (-M' \cup -Z, \gamma) \to (-M, \gamma).$$

The 4–manifold underlying $(\mathcal{W}^\vee)^s$ is also $M' \times I/{\sim}$. We need the following topological description of the special cobordism $(W^\vee)^s$:

Lemma 6.7 *Suppose that (M,γ) is a sutured submanifold of (M',γ') and*

$$\mathcal{W}^\vee := (I \times M'/{\sim},\ \{0\} \times Z,\ [\xi])\colon (-M',\gamma') \to (-M,\gamma)$$

is the dual of the corresponding boundary cobordism, as described above, and suppose that (Z,ξ) has a contact handle decomposition relative to ∂M with an associated Morse function $f\colon Z \to I$. Then the special part $(W^\vee)^s$ of $\mathcal{W}^\vee$ has a Morse function F whose critical points are in bijective correspondence with the critical points of f. Furthermore, if p is a critical point of f and p' is the associated critical point of F, then

$$\operatorname{ind}_{p'}(F) = 4 - \operatorname{ind}_p(f).$$

The intersection of the descending manifold of a critical point of F with

$$-M' \cup_{\partial M'} -Z = -M \cup_{\partial M} Z \cup_{\partial M'} -Z$$

is equal to the union of the ascending flow lines of the corresponding critical point of f in Z, together with their images in $-Z$.

Proof We first define an auxiliary function

$$G\colon (I \times [-1,2])/(I \times \{2\}) \to I,$$

where $I = [0,1]$. We require G to satisfy the following:

- $G(t,s) = t$ for $s = -1$.
- $\nabla G \neq 0$ for all (t,s).
- If $(t,s) \in I \times I$, then $(\partial G/\partial t)(t,s) = 0$ if and only if $t = \frac{1}{2}$.
- If $(t,s) \in \left\{\frac{1}{2}\right\} \times I$, then $(\partial G/\partial s) < 0$.
- $G|_{\{0\}\times[-1,2]} \equiv G|_{I\times\{2\}} \equiv G|_{\{1\}\times[0,2]} \equiv 0$.

The graph of such a function $G(t,s)$ is shown in Figure 25.

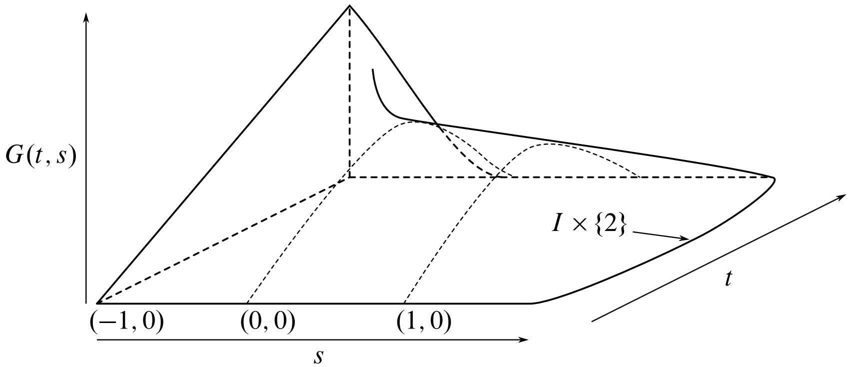

Figure 25: An example of a function $G(t,s)\colon (I \times [-1,2])/(I \times \{2\}) \to I$ considered in Lemma 6.7.

Let us view M' as

$$M' \cong M \cup (\partial M \times [-1,0]) \cup -Z \cup (\partial M' \times [1,2]),$$

and extend f over all of M' so that

- $f|_M \equiv -1$,
- $f|_{\partial M \times [-1,0]}(x,s) = s$, and
- $f|_{\partial M' \times [1,2]}(x,s) = s$.

We then consider the function $F\colon I \times M'/\!\sim\, \to I$ given by

$$F(t,x) := G(t, f(x)).$$

It is then straightforward to verify that F has the stated properties. $\square$

Remark 6.8 For an index 2 critical point of f, the attaching sphere of the corresponding critical point of F will be a knot K. The framing of K depends on some auxiliary choices, such as a choice of Riemannian metric, and the precise choice of G. However, up to isotopy, the framing is determined uniquely by the property that the framing of $K \cap Z$ is the mirror of the framing of $K \cap (-Z)$.

Proof of Theorem 6.6 The claim was shown for special cobordisms in [15, Theorem 11.8]. Hence, it only remains to verify it for boundary cobordisms. By the composition law for the gluing map, it is sufficient to prove the claim when the boundary cobordism $\mathcal{W} = (W, Z, [\xi])$ is formed by adding a single contact k–handle for $k \in \{0,1,2,3\}$. The cobordism map

$$F_{\mathcal{W}}\colon \mathrm{SFH}(M,\gamma) \to \mathrm{SFH}(M',\gamma')$$

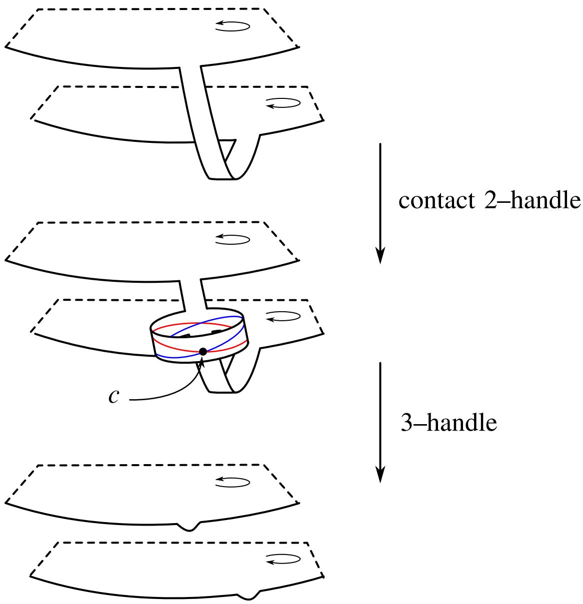

Figure 26: Computing the cobordism map for a turned-around contact 1–handle. Orientations on the Heegaard surface are shown.

is the gluing map $\Phi_{-\xi}$ corresponding to the sutured submanifold $(-M,-\gamma)$ of $(-M',-\gamma')$.

For a contact 0–handle, the map $\Phi_{-\xi}$ is the tautological one from $\mathrm{SFH}(M,\gamma)$ to

$$\mathrm{SFH}(M',\gamma') \cong \mathrm{SFH}(M,\gamma) \otimes \mathrm{SFH}(D^2 \times [-1,1], S^1 \times \{0\}) \cong \mathrm{SFH}(M,\gamma) \otimes \mathbb{F}_2.$$

On the other hand, consider $\mathcal{W}^\vee$ from $(-M',\gamma')$ to $(-M,\gamma)$. Here Z has an isolated component Z_0 corresponding to the contact 0–handle; ie $Z_0 \cap M = \varnothing$. Hence, by [15, Definition 10.1], the map $F_{\mathcal{W}^\vee}$ is defined by removing a standard contact ball B with connected dividing set δ on ∂B from the interior of Z_0, and adding (B,δ) to $(-M,\gamma)$. The resulting cobordism is a product from $(-M',\gamma') = (-M,\gamma) \sqcup (B,\delta)$ to $(-M',\gamma')$. Hence $F_{\mathcal{W}^\vee}$ is $\mathrm{id}_{\mathrm{SFH}(-M',\gamma')}$, followed by the canonical identification of $\mathrm{SFH}(-M',\gamma')$ with $\mathrm{SFH}(-M,\gamma)$, which is $\Phi^\vee_{-\xi}$.

Now suppose that $\mathcal{W}$ is formed by adding a contact 1–handle to (M,γ). In this case, $F_{\mathcal{W}} = \Phi_{-\xi}$, which is obtained by adding a strip to the boundary of a Heegaard diagram for $(-M,-\gamma)$. By Lemma 6.7, the cobordism $\mathcal{W}^\vee$ is obtained by gluing $(Z,-\xi)$ along $\partial M'$ to $M' = M \cup_{\partial M} -Z$, then attaching a 4–dimensional 3–handle. Note that we attach Z along $\partial M'$, not ∂M, so it becomes a contact 2–handle. The 4–dimensional

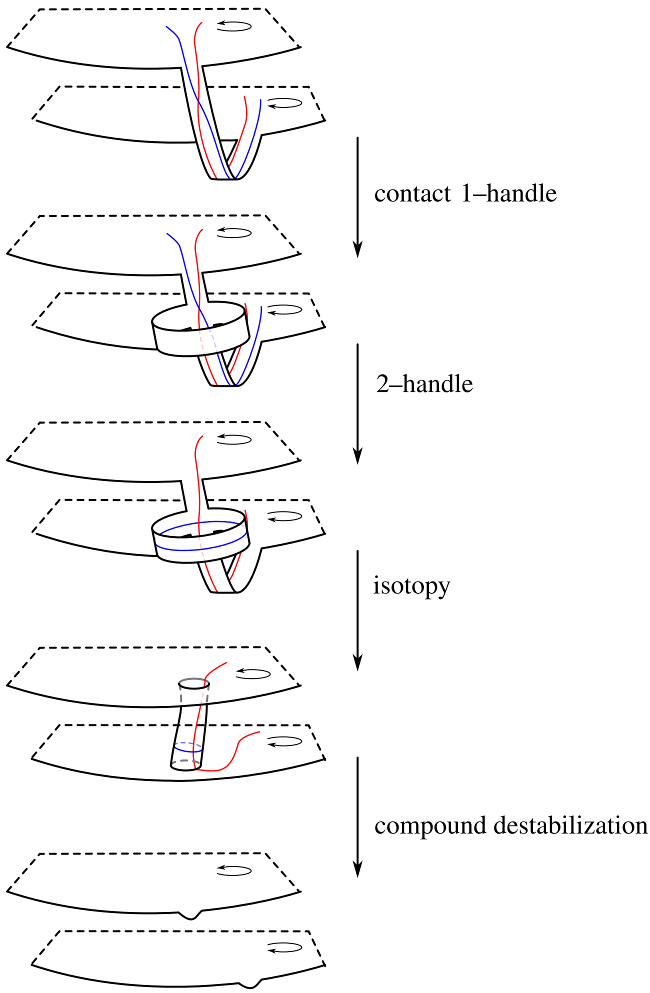

Figure 27: Computing the cobordism map for a turned-around contact 2–handle.

3–handle is attached to a 2–sphere in $Z \cup_{\partial M'} -Z$. As in Lemma 6.7, the 2–sphere is the union of the ascending flow lines of f in Z (ie the cocore of Z, viewed as a contact 1–handle attached to $-M$) together with its image in the copy of $-Z$ that we glue onto $-M'$. Diagrammatically, this is shown in Figure 26. An easy model computation shows that this is equal to the dual of the contact 1–handle map.

We now consider a sutured cobordism $\mathcal{W}$ formed by a contact 2–handle attachment to the sutured manifold (M, γ). In this case, the dual cobordism $\mathcal{W}^\vee$ is formed by gluing $(Z, -\xi)$ to $M' = M \cup -Z$ along $\partial M'$ and then attaching a 4–dimensional

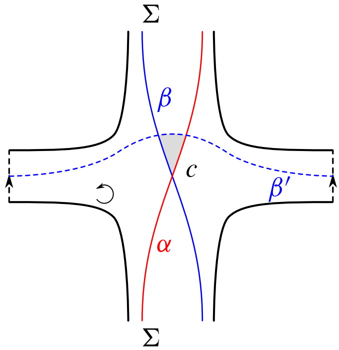

Figure 28: The triangle map computation for the 4–dimensional 2–handle map in the cobordism map for a turned-around contact 2–handle. The orientation of the surface is shown. The two dashed lines with arrows on the left and right are identified. The only homology classes of triangles that have a vertex at c have multiplicity 1 in the shaded region, and zero in the other regions shown.

2–handle. Gluing $(Z,-\xi)$ to M' is now a contact 1–handle attachment. As described in Lemma 6.7, the knot that we attach a 2–handle along is given as the union of the cocore of Z (viewed as a contact 2–handle) as well as its image in $-Z$. Let $\mathcal{H}'$ be an admissible diagram of $(-M',\gamma')$. After adding the contact 1–handle and the 4–dimensional 2–handle, we get a diagram that is a compound stabilization of $\mathcal{H}'$. After performing a compound destabilization, we get back to $\mathcal{H}'$. An easy triangle map computation for the 2–handle map shows that the composition of the triangle map and the compound destabilization map is dual to the contact 2–handle map. That the compound stabilization map agrees with the map from naturality is shown in Proposition 2.2. A schematic for the turned-around contact 2–handle cobordism is shown in Figure 27. The triangle map is shown in more detail in Figure 28.

Finally, we consider the case when $\mathcal{W}$ is formed by adding a contact 3–handle. In this case, Z has an isolated component, and the cobordism map $F_{\mathcal{W}}$ is obtained by removing a standard contact ball from the 3–ball we are adding. Then the map is computed as a trivial gluing map, followed by a 4–dimensional 3–handle map. The 4–dimensional 3–handle is attached along an embedded 2–sphere that is the push-off of the boundary 2–sphere on which we are attaching the contact 3–handle. The dual cobordism $\mathcal{W}^\vee$ is formed by adding a contact 0–handle, followed by a 4–dimensional 1–handle. Hence, the claim that $F_{\mathcal{W}}^\vee = F_{\mathcal{W}^\vee}$ follows from the fact that the 4–dimensional 1–handle and 3–handle maps are dual to each other, as in [25, Theorem 3.5]. □

7 Triangle cobordisms

If $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is a balanced sutured triple diagram, then there is a natural sutured cobordism

$$\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} = (W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}, Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}, \xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}),$$

as in [15, Section 5]. We note, however, that the construction of the contact structure $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ in [15, Section 5] was incorrect, as it involved gluing contact structures along annuli whose boundaries did not intersect the dividing set. In this section, we will provide a different description, which we will take as the definition.

Before describing the 4–manifold $W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$, we establish some notation. For $\boldsymbol{\tau} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, let $U_{\boldsymbol{\tau}}$ be the sutured compression body obtained from $\Sigma \times I$ by attaching 3–dimensional 2–handles along $\boldsymbol{\tau} \times \{0\} \subseteq \Sigma \times \{0\}$. We view Σ as being embedded into $\partial U_{\boldsymbol{\tau}}$ as $\Sigma \times \{1\}$. Similarly, we denote by $R_{\boldsymbol{\tau}}$ the result of surgering $\Sigma \times \{0\}$ along $\boldsymbol{\tau} \times \{0\}$. Using this notation,

$$\partial U_{\boldsymbol{\tau}} = \Sigma \cup (\partial \Sigma \times I) \cup \bar{R}_{\boldsymbol{\tau}},$$

where $\bar{R}_{\boldsymbol{\tau}}$ is the surface $R_{\boldsymbol{\tau}}$ with the opposite orientation.

For $\boldsymbol{\tau}, \boldsymbol{\tau}' \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, consider the 3–manifold

$$M_{\boldsymbol{\tau},\boldsymbol{\tau}'} := U_{\boldsymbol{\tau}} \cup_{\Sigma} -U_{\boldsymbol{\tau}'},$$

and let $R_{\boldsymbol{\tau},\boldsymbol{\tau}'}$ denote the surface

$$\bar{R}_{\boldsymbol{\tau}} \cup_{\partial \Sigma} R_{\boldsymbol{\tau}'}.$$

Here, we write $-U_{\boldsymbol{\tau}'}$ for the 3–manifold $U_{\boldsymbol{\tau}'}$ with the opposite orientation. Note that, using this orientation convention, we have

$$\partial M_{\boldsymbol{\tau},\boldsymbol{\tau}'} = R_{\boldsymbol{\tau},\boldsymbol{\tau}'}.$$

The oriented 1–manifold $\partial\Sigma$ has a natural embedding into $\partial M_{\boldsymbol{\tau},\boldsymbol{\tau}'}$, which we denote by $\gamma_{\boldsymbol{\tau},\boldsymbol{\tau}'}$, and $(M_{\boldsymbol{\tau},\boldsymbol{\tau}'}, \gamma_{\boldsymbol{\tau},\boldsymbol{\tau}'})$ is a sutured manifold with diagram $(\Sigma, \boldsymbol{\tau}, \boldsymbol{\tau}')$.

Let Δ be a regular triangle in $\mathbb{C}$ with edges $e_{\boldsymbol{\alpha}}$, $e_{\boldsymbol{\beta}}$, and $e_{\boldsymbol{\gamma}}$ appearing clockwise, and give Δ the complex orientation; see Figure 29. We define the 4–manifold

$$W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} := (\Delta \times \Sigma) \cup (e_{\boldsymbol{\alpha}} \times U_{\boldsymbol{\alpha}}) \cup (e_{\boldsymbol{\beta}} \times U_{\boldsymbol{\beta}}) \cup (e_{\boldsymbol{\gamma}} \times U_{\boldsymbol{\gamma}})/\sim,$$

where $\sim$ denotes gluing $\Delta \times \Sigma$ to $e_{\boldsymbol{\tau}} \times U_{\boldsymbol{\tau}}$ along $e_{\boldsymbol{\tau}} \times \Sigma$ for $\boldsymbol{\tau} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$. Also, let

$$Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} := (\Delta \times \partial\Sigma) \cup (e_{\boldsymbol{\alpha}} \times R_{\boldsymbol{\alpha}}) \cup (e_{\boldsymbol{\beta}} \times R_{\boldsymbol{\beta}}) \cup (e_{\boldsymbol{\gamma}} \times R_{\boldsymbol{\gamma}}) \subseteq \partial W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}},$$

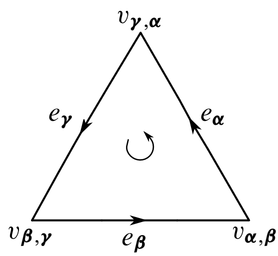

Figure 29: A triangle Δ, oriented as in the construction of $W_{\alpha,\beta,\gamma}$. The boundary orientations of the edges e_α, e_β, and e_γ are also shown.

which we orient as the boundary of $W_{\alpha,\beta,\gamma}$. We then have the decomposition

$$\partial W_{\alpha,\beta,\gamma} = -M_{\alpha,\beta} \cup -M_{\beta,\gamma} \cup M_{\alpha,\gamma} \cup Z_{\alpha,\beta,\gamma}.$$

Note that $M_{\alpha,\gamma} = -M_{\gamma,\alpha}$. Using our orientation conventions,

$$\partial Z_{\alpha,\beta,\gamma} = R_{\alpha,\beta} \sqcup R_{\beta,\gamma} \sqcup \bar{R}_{\alpha,\gamma}.$$

There is a natural collection of sutures $\kappa_{\alpha,\beta,\gamma}$ on $\partial Z_{\alpha,\beta,\gamma}$, defined as

$$\kappa_{\alpha,\beta,\gamma} := \{v_{\alpha,\beta}, v_{\beta,\gamma}, v_{\gamma,\alpha}\} \times \partial\Sigma = \gamma_{\alpha,\beta} \cup \gamma_{\beta,\gamma} \cup \gamma_{\gamma,\alpha},$$

where $v_{\sigma,\tau}$ is the vertex of Δ between e_σ and e_τ. Then

$$R_+(\kappa_{\alpha,\beta,\gamma}) = (\{v_{\alpha,\beta}\} \times R_\beta) \cup (\{v_{\beta,\gamma}\} \times R_\gamma) \cup (\{v_{\gamma,\alpha}\} \times R_\alpha).$$

There is a natural contact structure $\xi_{\alpha,\beta,\gamma}$ on $Z_{\alpha,\beta,\gamma}$ with dividing set $\kappa_{\alpha,\beta,\gamma}$, which is positive for the orientation of $Z_{\alpha,\beta,\gamma}$ we have described. We delay our description of $\xi_{\alpha,\beta,\gamma}$ until Section 7.1. The triple $\mathcal{W}_{\alpha,\beta,\gamma} = (W_{\alpha,\beta,\gamma}, Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})$ is a sutured manifold cobordism from $(M_{\alpha,\beta}, \gamma_{\alpha,\beta}) \sqcup (M_{\beta,\gamma}, \gamma_{\beta,\gamma})$ to $(M_{\alpha,\gamma}, \gamma_{\alpha,\gamma})$. The main goal of this section is to prove the following:

Theorem 7.1 *Let $(\Sigma, \alpha, \beta, \gamma)$ be an admissible balanced sutured triple diagram. Then the cobordism map*

$$F_{\mathcal{W}_{\alpha,\beta,\gamma}} \colon \mathrm{CF}(\Sigma, \alpha, \beta) \otimes \mathrm{CF}(\Sigma, \beta, \gamma) \to \mathrm{CF}(\Sigma, \alpha, \gamma)$$

is chain homotopic to the map $F_{\alpha,\beta,\gamma}$ defined in [15, Definition 5.13] *that counts holomorphic triangles on the triple diagram $(\Sigma, \alpha, \beta, \gamma)$.*

The proof occupies the remainder of this section. The first step is to define the contact structure $\xi_{\alpha,\beta,\gamma}$ in detail and describe some useful properties. Next, we introduce some special diagrams called "doubled diagrams" and "weakly conjugated diagrams" that appear when computing the cobordism map for $\mathcal{W}_{\alpha,\beta,\gamma}$. We then compute convenient formulas for the contact gluing map for $(Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})$, and for the 4–dimensional handle attachments of the cobordism $\mathcal{W}_{\alpha,\beta,\gamma}$. Finally, we put all the pieces together, using associativity of the triangle maps and some other relations to show that the cobordism map $F_{\mathcal{W}_{\alpha,\beta,\gamma}}$ is chain homotopic to the triangle map $F_{\alpha,\beta,\gamma}$.

7.1 The contact structure $\xi_{\alpha,\beta,\gamma}$ on $Z_{\alpha,\beta,\gamma}$

The sutured manifold $(Z_{\alpha,\beta,\gamma}, \kappa_{\alpha,\beta,\gamma})$ has a natural contact structure $\xi_{\alpha,\beta,\gamma}$ that we define by decomposing $Z_{\alpha,\beta,\gamma}$ along convex surfaces. The construction generalizes to the case of sutured multidiagrams $(\Sigma, \eta_1, \dots, \eta_n)$ for arbitrary $n \geq 1$, though in this paper, we will only need it for $n \in \{1,2,3\}$.

Let P_n be an n–gon, viewed as the complex unit disk with boundary divided into n arcs labeled $e_{\eta_1}, \dots, e_{\eta_n}$ clockwise. We write v_{η_{i-1},η_i} for the terminal endpoint of e_{η_i} for $i \in \{1,\dots,n\}$, where $\eta_0 := \eta_n$. Let

$$Z_{\eta_1,\dots,\eta_n} = (\Delta \times \partial\Sigma) \cup (e_{\eta_1} \times R_{\eta_1}) \cup \dots \cup (e_{\eta_n} \times R_{\eta_n}) \tag{7-1}$$

with sutures

$$\kappa_{\eta_1,\dots,\eta_n} = \{v_{\eta_1,\eta_2}, \dots, v_{\eta_n,\eta_1}\} \times \partial\Sigma = \gamma_{\eta_1,\eta_2} \cup \dots \cup \gamma_{\eta_n,\eta_1}$$

and

$$R_+(\kappa_{\eta_1,\dots,\eta_n}) = (\{v_{\eta_1,\eta_2}\} \times R_{\eta_2}) \cup \dots \cup (\{v_{\eta_n,\eta_1}\} \times R_{\eta_1}).$$

We now define the contact structure $\xi_{\eta_1,\dots,\eta_n}$ on $(Z_{\eta_1,\dots,\eta_n}, \kappa_{\eta_1,\dots,\eta_n})$. Let us write Z_0 for $P_n \times \partial\Sigma$, which is a union of $|\partial\Sigma|$ solid tori. Let γ_0 consist of two parallel longitudinal sutures on each component of ∂Z_0 such that their projections to P_n wind positively around ∂P_n with respect to the orientation of P_n (ie they wind counterclockwise around $\partial P_n \subseteq \mathbb{C}$). Since (Z_0, γ_0) is product disc decomposable, it admits a unique tight contact structure ξ_0, up to equivalence, which has ∂Z_0 as a convex surface with dividing set γ_0.

For $\tau \in \{\eta_1, \dots, \eta_n\}$, we write (Z_τ, γ_τ) to denote the product sutured manifold $(e_\tau \times R_\tau, \{m_\tau\} \times \partial R_\tau)$, where m_τ is the midpoint of e_τ. The sutured manifold (Z_τ, γ_τ) is product disc decomposable, and hence admits a unique tight contact structure ξ_τ with dividing set γ_τ, up to equivalence. Let s_τ denote a small translation of the suture γ_τ for $\tau \in \{\eta_1, \dots, \eta_n\}$ such that each component of s_τ intersects the corresponding component of $\gamma_{\boldsymbol{\tau}}$ transversely at exactly two points. Let $N(s_{\boldsymbol{\tau}}) \subseteq \partial Z_{\boldsymbol{\tau}}$ denote a small regular neighborhood of $s_{\boldsymbol{\tau}}$ that intersects $\gamma_{\boldsymbol{\tau}}$ along two arcs. Using Giroux's Legendrian realization principle, we may assume that each $\partial N(s_{\boldsymbol{\tau}})$ is Legendrian.

We now describe how the subsurfaces $N(s_{\boldsymbol{\tau}}) \subseteq Z_{\boldsymbol{\tau}}$ are glued to Z_0. By picking γ_0 appropriately, we may assume that each component of $\{m_{\boldsymbol{\tau}}\} \times \partial\Sigma$ intersects γ_0 transversely at two points. For each $\boldsymbol{\tau} \in \{\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_n\}$, we pick a small neighborhood $N(\{m_{\boldsymbol{\tau}}\} \times \partial\Sigma) \subseteq \partial Z_0$ such that each component of $N(\{m_{\boldsymbol{\tau}}\} \times \partial\Sigma)$ intersects γ_0 along two arcs. Using Legendrian realization, we may assume that $\partial N(\{m_{\boldsymbol{\tau}}\} \times \partial\Sigma)$ is Legendrian.

We glue $(Z_{\boldsymbol{\tau}}, \xi_{\boldsymbol{\tau}})$ for every $\boldsymbol{\tau} \in \{\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_n\}$ to (Z_0, ξ_0) by identifying $N(s_{\boldsymbol{\tau}})$ and $N(\{m_{\boldsymbol{\tau}}\} \times \partial\Sigma)$. We let $\xi_{\boldsymbol{\eta}_1,\dots,\boldsymbol{\eta}_n}$ be the resulting contact structure. After rounding the Legendrian corners, the contact structure $\xi_{\boldsymbol{\eta}_1,\dots,\boldsymbol{\eta}_n}$ has dividing set isotopic to $\kappa_{\boldsymbol{\eta}_1,\dots,\boldsymbol{\eta}_n}$. This is shown in Figure 30 for $n = 3$ and $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3) = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. As $N(s_{\boldsymbol{\tau}})$ is unique up to isotopy, $\xi_{\boldsymbol{\eta}_1,\dots,\boldsymbol{\eta}_n}$ is well defined up to equivalence.

The following will be useful throughout the paper:

Lemma 7.2 *If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a sutured Heegaard diagram, then the sutured manifold $(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \kappa_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ is diffeomorphic to*

$$\big(I \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}, (-\{0\} \times \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \cup (\{1\} \times \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}})\big),$$

and the contact structure $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is isotopic to the I–invariant contact structure on $I \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ with dividing set $\{t\} \times \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ on $\{t\} \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for every $t \in I$.

Proof By construction, $Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is obtained by gluing $e_{\boldsymbol{\alpha}} \times R_{\boldsymbol{\alpha}}$ and $e_{\boldsymbol{\beta}} \times R_{\boldsymbol{\beta}}$ to $P_2 \times \partial\Sigma$. This is diffeomorphic to gluing $I \times R_{\boldsymbol{\alpha}}$ to $I \times R_{\boldsymbol{\beta}}$ by identifying $(t, x) \in I \times \partial\Sigma$ with $(1-t, x) \in I \times \partial\Sigma$, where $\partial R_{\boldsymbol{\alpha}} = \partial R_{\boldsymbol{\beta}} = \partial\Sigma$. Hence $Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is diffeomorphic to $I \times (\bar{R}_{\boldsymbol{\alpha}} \cup R_{\boldsymbol{\beta}}) = I \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Under this diffeomorphism, the sutures $\kappa_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup \gamma_{\boldsymbol{\beta},\boldsymbol{\alpha}}$ are mapped to $(-\{0\} \times \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \cup (\{1\} \times \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}})$, since $\gamma_{\boldsymbol{\beta},\boldsymbol{\alpha}} = -\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. In particular, $R_+(\kappa_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ is identified with $(\{0\} \times R_{\boldsymbol{\alpha}}) \cup (\{1\} \times R_{\boldsymbol{\beta}})$.

To see that the contact structure $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is isotopic to the I–invariant contact structure in the statement, we will construct a decomposition of the latter along convex annuli that cuts $I \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ into the disjoint union of the three contact manifolds $(Z_{\boldsymbol{\alpha}}, \xi_{\boldsymbol{\alpha}})$, $(Z_{\boldsymbol{\beta}}, \xi_{\boldsymbol{\beta}})$, and (Z_0, ξ_0). Let $s_1, s_2 \subseteq R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be two disjoint curves that are both small translates of the dividing set $\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \partial\Sigma$. We assume that s_1 and s_2 each intersect $\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ transversely at two points. Using Legendrian realization, we can assume that both s_1 and s_2 are Legendrian.

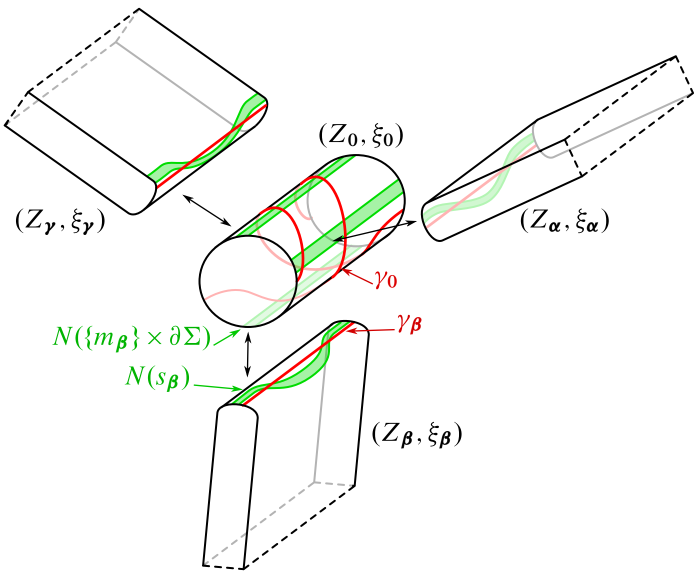

Figure 30: Constructing the contact structure $\xi_{\alpha,\beta,\gamma}$ on $Z_{\alpha,\beta,\gamma}$ by gluing along convex surfaces. We glue $(Z_{\alpha}, \xi_{\alpha})$, (Z_{β}, ξ_{β}), and $(Z_{\gamma}, \xi_{\gamma})$ to (Z_0, ξ_0) along the green strips, which have Legendrian boundary. The faces on the front side are identified with the faces on the back.

We will cut $I \times R_{\alpha,\beta}$ along the two annuli $I \times s_1$ and $I \times s_2$. We note that since s_1 and s_2 are Legendrians, the characteristic foliations on $I \times s_1$ and $I \times s_2$ are simple to describe. They consist of the horizontal leaves $\{t\} \times s_i$ for all $t \in I$, as well as two vertical lines of singularities along $I \times \{p\}$ for $p \in s_i \cap \gamma_{\alpha,\beta}$. This is shown in Figure 31. We note that the characteristic foliation satisfies the Poincaré–Bendixson property (the limit set of a flow line consists of a singular point, a periodic orbit, or a finite union of singular points and connecting orbits) and has no closed orbits or retrograde saddle connections. So, by the work of Giroux [7], the surfaces $I \times s_i$ are

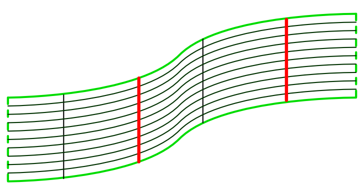

Figure 31: The characteristic foliation (thin black) and the dividing set (thick red) on the annulus $I \times s_i \subseteq I \times R_{\alpha,\beta}$. The vertical black lines consist of singularities. The left- and right-hand sides of the surface are identified to form an annulus.

convex. Furthermore, the dividing set on $I \times s_i$ consists of two curves of the form $I \times \{q\}$, where $q \in s_i \setminus \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. See also [3, Example 2.24].

After rounding the Legendrian corners that appear when we cut along $I \times s_i$, we obtain the disjoint union of the sutured manifolds $(Z_{\boldsymbol{\alpha}}, \xi_{\boldsymbol{\alpha}})$, $(Z_{\boldsymbol{\beta}}, \xi_{\boldsymbol{\beta}})$, and (Z_0, γ_0). Furthermore, the contact structures obtained on the three pieces are isotopic to $\xi_{\boldsymbol{\alpha}}$, $\xi_{\boldsymbol{\beta}}$, and ξ_0, since they are tight by Giroux's criterion [10, Theorem 3.5], and $\xi_{\boldsymbol{\alpha}}$, $\xi_{\boldsymbol{\beta}}$, and ξ_0 are the unique tight contact structures, by definition. A picture of the convex decomposition of $I \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ along $I \times s_1$ and $I \times s_2$ is shown in Figure 32. □

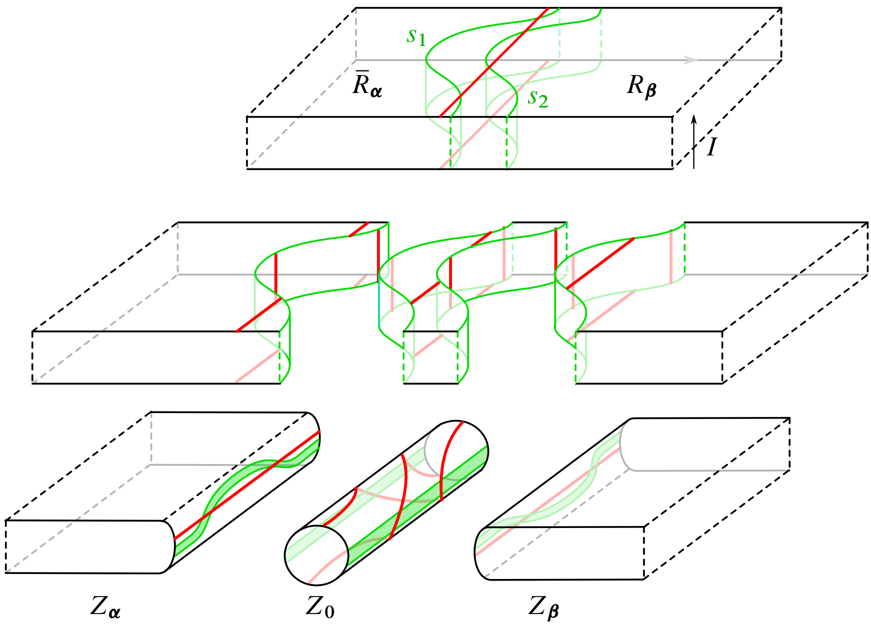

Figure 32: Decomposing $I \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ (top) along two convex annuli $I \times s_1$ and $I \times s_2$, we obtain the middle picture. After rounding the Legendrian corners, we obtain the bottom picture, which is the disjoint union of $Z_{\boldsymbol{\alpha}}$, $Z_{\boldsymbol{\beta}}$, and Z_0. The front and back sides of each picture are identified.

Finally, we need one additional description of the contact structure $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ on $Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$, in terms of gluing $(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ and $(Z_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \xi_{\boldsymbol{\beta},\boldsymbol{\gamma}})$ together. By Lemma 7.2, the contact manifolds $(Z_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \xi_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ and $(Z_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \xi_{\boldsymbol{\beta},\boldsymbol{\gamma}})$ are contactomorphic to $I \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $I \times R_{\boldsymbol{\beta},\boldsymbol{\gamma}}$, respectively, with I–invariant contact structures. We pick automorphisms of each of the surfaces $R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $R_{\boldsymbol{\beta},\boldsymbol{\gamma}}$, supported in neighborhoods of $\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}}$, that perturb the dividing sets $\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ slightly. Write $s_{\boldsymbol{\alpha},\boldsymbol{\beta}} \subseteq R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $s_{\boldsymbol{\beta},\boldsymbol{\gamma}} \subseteq R_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ for the images of $\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}}$, respectively. We assume that $s_{\boldsymbol{\alpha},\boldsymbol{\beta}}$

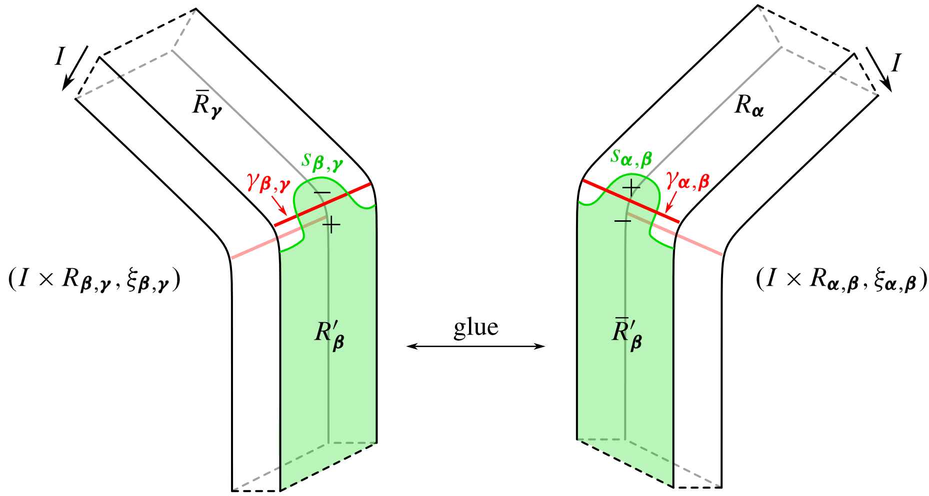

Figure 33: Gluing $(I \times R_{\alpha,\beta}, \xi_{\alpha,\beta})$ and $(I \times R_{\beta,\gamma}, \xi_{\beta,\gamma})$ together to obtain $(Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})$. We glue along the green shaded subsurfaces labeled R'_{β} and $\overline{R}'_{\beta}$. The dividing sets $\gamma_{\alpha,\beta}$ and $\gamma_{\beta,\gamma}$ are shown in red. The curves $s_{\alpha,\beta}$ and $s_{\beta,\gamma}$ are small perturbations of the dividing sets and are Legendrian.

and $\gamma_{\alpha,\beta}$ intersect transversely at two points, and similarly for $s_{\beta,\gamma}$ and $\gamma_{\beta,\gamma}$. Let $\overline{R}'_{\beta} \subseteq \{0\} \times R_{\alpha,\beta}$, oriented as the boundary of $I \times R_{\alpha,\beta}$, denote the image of $\{0\} \times \overline{R}_{\beta} \subseteq \{0\} \times R_{\alpha,\beta}$, and let $R'_{\beta} \subseteq \{0\} \times R_{\beta,\gamma}$ denote the image of $\{0\} \times R_{\beta} \subseteq \{0\} \times R_{\beta,\gamma}$. Using the Legendrian realization principle, we may assume that $s_{\alpha,\beta}$ and $s_{\beta,\gamma}$ are Legendrian. The following description of $(Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})$ will be useful for our purposes:

Lemma 7.3 *The contact structure* $(Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})$ *is equivalent to the one obtained by gluing* $(I \times R_{\alpha,\beta}, \xi_{\alpha,\beta})$ *and* $(I \times R_{\beta,\gamma}, \xi_{\beta,\gamma})$ *together along the surfaces* R'_{β} *and* $\overline{R}'_{\beta}$ (*which are convex with Legendrian boundary*) *described above; see Figure 33.*

Proof As in Lemma 7.2, we work backwards, and provide a convex decomposition of $(Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})$ into the disjoint union of $(I \times R_{\alpha,\beta}, \xi_{\alpha,\beta})$ and $(I \times R_{\beta,\gamma}, \xi_{\beta,\gamma})$. By definition, $Z_{\alpha,\beta,\gamma}$ is the union of Z_{α}, Z_{β}, Z_{γ}, and Z_0. Let us view Z_{β} as $I \times R_{\beta}$ with I–invariant contact structure, and Legendrian corners along $\{0, 1\} \times \partial R_{\beta}$, as on the right side of the middle level of Figure 32. We start with the surface $S_{\beta} := \{\frac{1}{2}\} \times R_{\beta}$, which is convex with Legendrian boundary. We can view ∂S_{β} as a Legendrian arc on ∂Z_0 that intersects the dividing set on ∂Z_0 at two points. There is an annulus $S_0 \subseteq Z_0$ with one boundary component on ∂S_{β}, and another boundary component on ∂Z_0 between Z_{γ} and Z_{α}. Furthermore, after perturbing the surface S_0 to be convex, it cuts (Z_0, ξ_0) into two copies of (Z_0, ξ_0). To see this, we consider (Z_0, ξ_0) as the I–invariant contact manifold (with Legendrian corners) in the center of the middle of Figure 32. In this picture, an example of an annulus cutting (Z_0, ξ_0) into two copies of (Z_0, ξ_0) would be the intersection of Z_0 with the slice $\{\frac{1}{2}\} \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. We let our decomposing surface S be the union $S_{\boldsymbol{\beta}} \cup S_0$. This is shown schematically in Figure 34. When we cut along S, we get two components, one of which is $Z_{\boldsymbol{\alpha}} \cup Z_0 \cup Z_{\boldsymbol{\beta}}$, and one of which is $Z_{\boldsymbol{\beta}} \cup Z_0 \cup Z_{\boldsymbol{\gamma}}$. Lemma 7.2 identifies these latter two contact manifolds with $I \times R_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $I \times R_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ with I–invariant contact structures. □

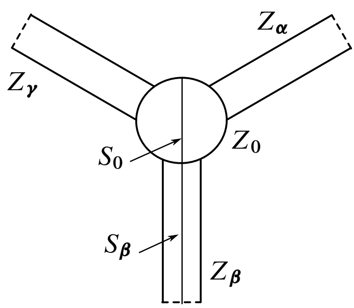

Figure 34: Decomposing $Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ along the convex surface $S = S_{\boldsymbol{\beta}} \cup S_0$ into $Z_{\boldsymbol{\alpha}} \cup Z_0 \cup Z_{\boldsymbol{\beta}}$ and $Z_{\boldsymbol{\beta}} \cup Z_0 \cup Z_{\boldsymbol{\gamma}}$.

7.2 A handle decomposition of $(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s$

Recall that $\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} = (W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}, Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}, \xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})$ is a sutured manifold cobordism from $(M_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \sqcup (M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}})$ to $(M_{\boldsymbol{\alpha},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$. The boundary cobordism $(\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^b$ corresponds to gluing $(Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}, -\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})$ to

$$(-M_{\boldsymbol{\alpha},\boldsymbol{\beta}}, -\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \sqcup (-M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, -\gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}}).$$

In light of Lemma 7.3, we can topologically view this as gluing $\bar{R}_{\boldsymbol{\beta}} \subseteq -\partial M_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ to $R_{\boldsymbol{\beta}} \subseteq -\partial M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$. Hence, the special cobordism $(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s$ goes from $(M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$ to $(M_{\boldsymbol{\alpha},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$. In this section, we give a topological description of the special cobordism $(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s$ in terms of 4–dimensional handle attachments.

A handle decomposition of $(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s$ can be constructed from a sutured Morse function on $U_{\boldsymbol{\beta}}$, as we now describe. Let $f_{\boldsymbol{\beta}}\colon U_{\boldsymbol{\beta}} \to I$ be a Morse function induced by the diagram $(\Sigma, \boldsymbol{\beta})$; ie we view $U_{\boldsymbol{\beta}}$ as a collection of 2–handles glued to $\Sigma \times I$ along $\boldsymbol{\beta} \times \{0\} \subseteq \Sigma \times \{0\}$. Furthermore, we pick $f_{\boldsymbol{\beta}}$ such that $f_{\boldsymbol{\beta}}^{-1}(1) = \Sigma \times \{1\}$, $f_{\boldsymbol{\beta}}^{-1}(0) = R_{\boldsymbol{\beta}}$, and $f_{\boldsymbol{\beta}}(y, t) = t$ for $y \in \partial\Sigma$. We also assume that $f_{\boldsymbol{\beta}}$ has $|\boldsymbol{\beta}|$ index 1 critical points, whose ascending manifolds intersect $\Sigma \times \{1\}$ along the $\boldsymbol{\beta}$ curves, and that $f_{\boldsymbol{\beta}}$ has no other critical points. For a curve $\beta_i \in \boldsymbol{\beta}$, let $\lambda_i \subseteq U_{\boldsymbol{\beta}}$ denote the descending manifold of the critical point of $f_{\boldsymbol{\beta}}$ corresponding to β_i. We have the following topological description of $W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$:

Lemma 7.4 *The special cobordism $(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s$ from $(M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$ to $(M_{\boldsymbol{\alpha},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$ consists of $|\boldsymbol{\beta}|$ 2–handle attachments. The 2–handles are attached along the link formed by concatenating the arcs $\lambda_i \subseteq U_{\boldsymbol{\beta}}$ (which have boundary on $\bar{R}_{\boldsymbol{\beta}}$) with their reflections in $-U_{\boldsymbol{\beta}}$. The framing on this link is determined by picking an arbitrary framing on λ_i and concatenating it with the mirrored framing on the image of λ_i in $-U_{\boldsymbol{\beta}}$.*

Proof This can be proven similarly to Lemma 6.7, by using a Morse function built from $f_{\boldsymbol{\beta}}$ as well as another auxiliary function. We leave the details to the reader. □

7.3 Arc slides and bases of arcs

In this section, we describe some basic topological facts about arc decompositions of surfaces with boundary.

Definition 7.5 Suppose that Σ is a compact, orientable surface with no closed components and $\mathcal{I} \subseteq \partial\Sigma$ is a collection of subarcs of $\partial\Sigma$ such that each component of $\partial\Sigma$ contains at least one component of $\mathcal{I}$.

(1) We say that $\boldsymbol{a} = \{a_1, \dots, a_n\}$ is an *arc basis* for $(\Sigma, \mathcal{I})$ if $\boldsymbol{a}$ is a set of pairwise disjoint, properly embedded arcs with boundary on $\mathcal{I}$ that form a basis of $H_1(\Sigma, \mathcal{I})$ (or, equivalently, $\boldsymbol{a}$ cuts Σ into a collection of closed disks, each containing exactly one component of $\partial\Sigma \setminus \mathcal{I}$).

(2) We say an arc a_i' is formed by an *allowable arc slide* of a_i across a_j if there is an embedded hexagon $P_6 \subseteq \Sigma$ whose boundary consists of a_i, a_j, a_i', as well as three subarcs of $\mathcal{I}$, and is otherwise disjoint from the arcs $a_1, \dots, a_n$; see Figure 35.

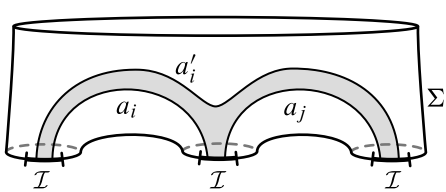

Figure 35: An allowable arc slide of a_i across a_j.

The reader may compare the following to [13, Lemma 3.3]:

Lemma 7.6 *Suppose Σ is a compact, orientable surface with boundary and no closed components and $\mathcal{I}$ is a collection of pairwise disjoint, closed subintervals of $\partial\Sigma$ such that each boundary component of $\partial\Sigma$ contains at least one component of $\mathcal{I}$. If $\boldsymbol{a}$ and $\boldsymbol{a}'$ are two arc bases of $(\Sigma,\mathcal{I})$, then $\boldsymbol{a}$ can be obtained from $\boldsymbol{a}'$ by a sequence of allowable arc slides.*

Proof Pick another collection of closed, pairwise disjoint subarcs $\mathcal{I}' \subseteq \operatorname{int}(\partial\Sigma \setminus \mathcal{I})$ such that each component of $\partial\Sigma \setminus \mathcal{I}$ contains exactly one component of $\mathcal{I}'$. We view Σ as a cobordism from $\mathcal{I}$ to $\mathcal{I}'$, with horizontal boundary $\mathcal{I} \cup \mathcal{I}'$, and vertical boundary $\partial\Sigma \setminus \operatorname{int}(\mathcal{I} \cup \mathcal{I}')$. We can view each basis of arcs $\boldsymbol{a}$ as corresponding to a Morse function f and gradient-like vector field v on Σ such that f has only index 1 critical points with stable manifolds $\boldsymbol{a}$. Furthermore, f is minimized along $\mathcal{I}$, maximized along $\mathcal{I}'$, and the components of $\partial\Sigma \setminus \operatorname{int}(\mathcal{I} \cup \mathcal{I}')$ are flow lines of v.

More generally, we can consider pairs (f, v) of Morse functions f with gradient-like vector fields v such that f is minimized on $\mathcal{I}$, maximized on $\mathcal{I}'$, has no critical points along $\partial\Sigma$, but is allowed to have critical points of index 0, 1, and 2. Assuming (f, v) is also Morse–Smale, we can construct a graph $\Gamma \subseteq \Sigma$ whose vertices are the index 0 critical points of f and the points of $\mathcal{I}$ that flow to index 1 critical points along v and whose edges are the stable manifolds of the index 1 critical points. The graph Γ intersects $\partial\Sigma$ along a collection of valence 1 vertices in $\mathcal{I}$.

With this in mind, we say a graph $\Gamma \subseteq \Sigma$ is a *decomposing graph* of $(\Sigma,\mathcal{I})$ if Γ intersects $\partial\Sigma$ in a collection of valence 1 vertices in $\mathcal{I}$, and the interior of each component of $\Sigma \setminus \Gamma$ is homeomorphic to an open 2–ball that contains at most one component of $\partial\Sigma \setminus \mathcal{I}$. Note that, for a decomposing graph of an orientable surface, Definition 7.5 modifies easily to describe an arc slide of two edges of Γ that meet at a vertex in $\operatorname{int}(\Sigma)$ and are consecutive with respect to the cyclic ordering of the edges adjacent to the vertex.

By interpreting decomposing graphs in terms of Morse functions and gradient-like vector fields, and considering the bifurcations occurring in generic 1–parameter families of smooth functions, it follows that any two decomposing graphs can be connected by a sequence of the following moves and their inverses:

(G-1) **Index 0/1 births** The addition of a vertex $v \in \operatorname{int}(\Sigma) \setminus \Gamma$ as well as a new edge e connecting v to an existing vertex of $\Gamma \setminus \partial\Sigma$ or to a point in $\mathcal{I} \setminus \Gamma$.

(G-2) **Index 1/2 births** The addition of a new edge to Γ whose interior is contained in $\Sigma \setminus \Gamma$ and which has both endpoints on the same vertex of $\Gamma \setminus \partial\Sigma$.

(G-3) **Arc slides** An allowable arc slide of two adjacent edges along $\mathcal{I}$ or an arc slide of a pair of edges that meet at a vertex of $\Gamma \cap \operatorname{int}(\Sigma)$ and are consecutive with respect to the cyclic ordering of the edges adjacent to the vertex.

Note that if an index $1/2$ birth occurs in a generic 1–parameter family of Morse functions and gradient-like vector fields, then the corresponding decomposing graph Γ changes by adding an edge e to a component B of $\Sigma \setminus \Gamma$ with both endpoints on $\partial B \setminus (\partial\Sigma \setminus \mathcal{I})$. By assumption, B is an open 2–ball and $\partial B \cap (\partial\Sigma \setminus \mathcal{I})$ is either empty or a single arc. We can add e using moves (G-1)–(G-3). Indeed, using move (G-1), we add an edge e' to Γ with one endpoint along $\partial B \cap \mathcal{I}$ on the subarc of $\partial B \setminus (\partial\Sigma \setminus \mathcal{I})$ between the endpoints of e, and one endpoint at a new interior vertex $v \in B$. We then add a loop edge e'' at v using move (G-2). We slide the endpoints of e'' along the two sides of e' to $\mathcal{I}$, and finally slide them to the endpoints of e using move (G-3).

If $\boldsymbol{\Gamma} = (\Gamma_1, \dots, \Gamma_n)$ is a sequence of decomposing graphs such that consecutive terms differ up to isotopy by an instance of moves (G-1)–(G-3) and their inverses, then we say that $\boldsymbol{\Gamma}$ is a *Cerf sequence*. We define $n_1(\boldsymbol{\Gamma})$ to be the number of moves (G-1), and $n_2(\boldsymbol{\Gamma})$ to be the number of moves (G-2) that occur in $\boldsymbol{\Gamma}$. Note that if Γ_1 and Γ_n are arc bases, then $n_1(\boldsymbol{\Gamma}) = n_2(\boldsymbol{\Gamma}) = 0$ if an only if $\boldsymbol{\Gamma}$ consists only of allowable arc slides.

We will now show that if $\boldsymbol{\Gamma} = (\Gamma_1, \dots, \Gamma_n)$ is a Cerf sequence of decomposing graphs connecting two arc bases $\boldsymbol{a}$ and $\boldsymbol{a}'$, and $n_1(\boldsymbol{\Gamma}) > 0$, then there is a Cerf sequence $\boldsymbol{\Gamma}'$ connecting $\boldsymbol{a}$ and $\boldsymbol{a}'$ with

$$n_1(\boldsymbol{\Gamma}') = n_1(\boldsymbol{\Gamma}) - 1 \quad \text{and} \quad n_2(\boldsymbol{\Gamma}') = n_2(\boldsymbol{\Gamma}). \tag{7-2}$$

Such a sequence $\boldsymbol{\Gamma}'$ will be constructed by contacting certain edges of the Γ_i.

We say that an edge e of Γ_i is *special* if ∂e consists of two distinct points, at least one of which is contained in $\operatorname{int}(\Sigma)$. It is easy to see that if Γ is a decomposing graph of $(\Sigma, \mathcal{I})$ with at least one vertex in $\operatorname{int}(\Sigma)$, then Γ has at least one special edge. If e is a special edge of Γ, we construct a new decomposing graph $C_e(\Gamma)$ of Σ, which we call the *contraction* of Γ along e. Outside a small neighborhood of e, we define the graph $C_e(\Gamma)$ to be Γ. If e has two vertices in $\operatorname{int}(\Sigma)$, then $C_e(\Gamma)$ is formed by collapsing e and its two endpoints into a single vertex. If instead e has one endpoint on $\partial\Sigma$, then $C_e(\Gamma)$ is formed by deleting e and performing an isotopy of the edges previously incident to e until they hit $\partial\Sigma$. The contraction operation is shown Figure 36.

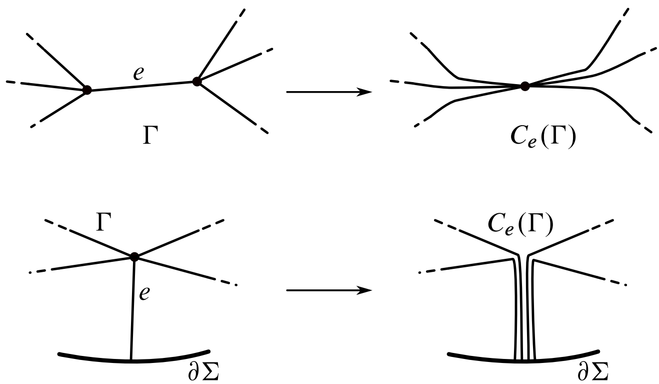

Figure 36: Contracting a decomposing graph along a special edge. On the top is when e has boundary equal to two vertices in $\operatorname{int}(\Sigma)$. On the bottom is the case when ∂e has a vertex on $\partial\Sigma$.

Now suppose $\mathbf{\Gamma}$ is a Cerf sequence connecting $\boldsymbol{a}$ and $\boldsymbol{a}'$ with $n_1(\Gamma) > 0$. Let k be the first index where Γ_k is obtained from Γ_{k-1} by move (G-1) (an index $0/1$ birth). We construct a sequence of special edges $e_k, \dots, e_l$ in $\Gamma_k, \dots, \Gamma_l$, respectively, for some $l \le n$. We define the first special edge, e_k, to be the edge which is added to Γ_{k-1} to form Γ_k. We construct the remaining special edges e_i recursively. Supposing e_i has already been chosen, we define e_{i+1} as follows, depending on which move is used to form Γ_{i+1} from Γ_i:

(1) **Γ_{i+1} is obtained by move (G-1) or its inverse** If e_i is deleted by this move, we declare e_i to be the final special edge in our sequence and set $l = i$. If e_i is unchanged, we set $e_{i+1} = e_i$.

(2) **Γ_{i+1} is obtained by move (G-2) or its inverse** Since e_i is special, it cannot be deleted by this move. Noting this, we set $e_{i+1} = e_i$.

(3) **Γ_{i+1} is obtained by move (G-3)** There are two subcases:

   (a) **An edge e is slid over another edge, and $e \neq e_i$** Noting that e_i remains special, we set $e_{i+1} = e_i$.

   (b) **The edge e_i is slid over another edge e** Write e_i' for the edge obtained by arc sliding e_i over e. If e_i' is not special, then its easy to check that e must be special, and we set $e_{i+1} = e$. If e_i' is special, we set $e_{i+1} = e_i'$.

We now demonstrate that consecutive terms in the sequence

$$\mathbf{\Gamma}' := (\Gamma_1, \dots, \Gamma_{k-1}, C_{e_k}(\Gamma_k), \dots, C_{e_l}(\Gamma_l), \Gamma_{l+1}, \dots, \Gamma_n)$$

are related by either a single instance of moves (G-1) or (G-2) or their inverses, by

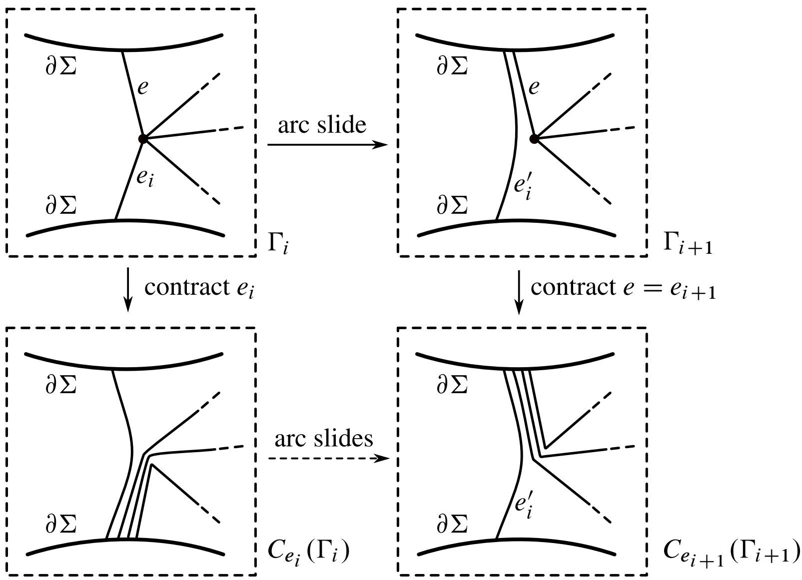

Figure 37: An illustration of the fact that $C_{e_{i+1}}(\Gamma_{i+1})$ is obtained from $C_{e_i}(\Gamma_i)$ by a sequence of arc slides when Γ_{i+1} is obtained from Γ_i by an arc slide of the special edge e_i across another edge e, and the resulting arc e_i' is no longer special.

multiple instances of move (G-3), or by no moves. Furthermore, we will show that $\mathbf{\Gamma}'$ satisfies (7-2).

We first consider the case when Γ_{i+1} is obtained from Γ_i by adding or removing an edge e via an instance of moves (G-1) or (G-2) or their inverses. If $e = e_i$ is removed, then, by definition, e_i is the last special edge (ie $i = l$), and clearly $C_{e_i}(\Gamma_i) = \Gamma_{i+1}$. If $e \neq e_i$, then $C_{e_{i+1}}(\Gamma_{i+1})$ is also obtained from $C_{e_i}(\Gamma_i)$ by adding or removing the edge e via a move of the same type as the one relating Γ_i and Γ_{i+1}.

We now consider the case when Γ_{i+1} is obtained from Γ_i by an instance of move (G-3). As indicated above, we break the argument into two cases. Case (1) occurs when we slide an edge e over another edge e', and $e \neq e_i$. Since e_i is unchanged, it remains special. If $e_i \neq e'$, then it is straightforward to see that $C_{e_{i+1}}(\Gamma_{i+1})$ is also obtained from $C_{e_i}(\Gamma_i)$ by an arc slide. If $e_i = e'$, then, in fact, $C_{e_{i+1}}(\Gamma_{i+1})$ and $C_{e_i}(\Gamma_i)$ are isotopic.

We finally consider the most complicated case, Case (2), in which Γ_{i+1} is obtained by sliding e_i over another edge e. We write e_i' for the edge resulting from arc sliding e_i over e. There are two further subcases, depending on whether e_i' is special or not.

Suppose first that e_i' is not special, so $e_{i+1}=e$. By definition, either e_i' has both endpoints on $\partial\Sigma$, or e_i' has both endpoints on the same vertex. We claim that in both cases $C_{e_{i+1}}(\Gamma_{i+1})$ differs from $C_{e_i}(\Gamma_i)$ by a sequence of arc slides. See Figure 37 for an illustration when e_i' has both endpoints on $\partial\Sigma$. A similar argument holds when e_i' has both endpoints on the same vertex of Γ_{i+1}.

The other subcase of (2) occurs when e_i' is special. Write $\{v_1,v_2\}=\partial e_i$ and $\{v_2,v_3\}=\partial e$. Since e_i and e_i' are special, it follows that both $v_1\neq v_2$ and $v_1\neq v_3$. The argument can be further divided into the cases that $v_2=v_3$ and $v_2\neq v_3$. In both cases, a manipulation similar to the one shown in Figure 37 shows that $C_{e_{i+1}}(\Gamma_{i+1})$ is obtained from $C_{e_i}(\Gamma_i)$ by a sequence of allowable arc slides.

By the above, for $i\in\{k,\dots,l-1\}$, the graph $C_{e_{i+1}}(\Gamma_{i+1})$ is obtained from $C_{e_i}(\Gamma_i)$ using the same type of moves as the one relating Γ_{i+1} and Γ_i, and exactly one move is used in case of a move of type (G-1). Noting that $\Gamma_{k-1}=C_{e_k}(\Gamma_k)$ and $C_{e_l}(\Gamma_l)=\Gamma_{l+1}$, Equation (7-2) is immediate.

Repeating this procedure, starting with a fixed $\mathbf{\Gamma}$, we can construct a Cerf sequence $\mathbf{\Gamma}''$ such that

$$n_1(\mathbf{\Gamma}'')=0 \quad\text{and}\quad n_2(\mathbf{\Gamma}'')=n_2(\mathbf{\Gamma}).$$

By turning Morse functions corresponding to each decomposing graph upside down (switching the roles of the index 0 and 2 critical points) and repeating the above procedure to the induced dual decomposing graphs, we obtain a sequence of decomposing graphs from $\boldsymbol{a}$ to $\boldsymbol{a}'$ which differ only by a sequence of allowable arc slides, completing the proof. □

7.4 Doubled diagrams

Suppose that $\mathcal{H}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is a sutured Heegaard diagram for the balanced sutured manifold (M,γ). There is a special way of constructing a new diagram of (M,γ) from $\mathcal{H}$ which will be important for computing the cobordism map $F_{\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}}$. Let us first pick two collections of subintervals $\mathcal{I}_0$ and $\mathcal{I}_1$ of $\partial\Sigma$ such that each component of $\partial\Sigma$ contains exactly one subinterval from $\mathcal{I}_0$ and one subinterval from $\mathcal{I}_1$ that are disjoint.

We now define the *Heegaard surface that is doubled along* $R_{\boldsymbol{\beta}}$ to be

$$D_{\boldsymbol{\beta}}(\Sigma):=\Sigma\natural_{\mathcal{I}_0}\overline{\Sigma}\natural_{\mathcal{I}_1}R_{\boldsymbol{\beta}},$$

where $\natural$ denotes the boundary connected sum operation. Here $\overline{\Sigma}$ denotes a push-off of Σ into $U_{\boldsymbol{\beta}}$, with the opposite orientation.

Before we define the attaching curves, let us first describe how $D_{\boldsymbol{\beta}}(\Sigma)$ is embedded in $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. A schematic is shown in Figure 38. Strictly speaking, we have changed the sutures. An isotopy supported in a neighborhood of the original sutures moves the new sutures to the original sutures.

We now describe compressing curves on $D_{\boldsymbol{\beta}}(\Sigma)$. First, pick a collection of pairwise disjoint arcs on Σ with boundary on $\mathcal{I}_0$ that form a basis of $H_1(\Sigma, \mathcal{I}_0)$. One then doubles these across $\mathcal{I}_0$ and obtains a collection of curves $\boldsymbol{\Delta}_\Sigma \subseteq \Sigma \natural_{\mathcal{I}_0} \overline{\Sigma} \subseteq D_{\boldsymbol{\beta}}(\Sigma)$. Similarly, one picks a collection of pairwise disjoint arcs on $R_{\boldsymbol{\beta}}$ (or equivalently, a collection of arcs on Σ that avoid the $\boldsymbol{\beta}$ curves, up to handle slides across $\boldsymbol{\beta}$) with boundary on $\mathcal{I}_1$ that form a basis of $H_1(R_{\boldsymbol{\beta}}, \mathcal{I}_1)$. Doubling these curves across $\mathcal{I}_1$ yields closed curves $\boldsymbol{\delta}_{\boldsymbol{\beta}} \subseteq \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}} \subseteq D_{\boldsymbol{\beta}}(\Sigma)$.

Write $\overline{\boldsymbol{\beta}}$ for the images of the $\boldsymbol{\beta}$ curves on $\overline{\Sigma}$. Note that the $\boldsymbol{\alpha}$, $\overline{\boldsymbol{\beta}}$, and $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ curves are all disjoint. Since (M, γ) is balanced, $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, and an easy computation shows that

$$|\boldsymbol{\alpha}| + |\boldsymbol{\beta}| + |\boldsymbol{\delta}_{\boldsymbol{\beta}}| = |\boldsymbol{\Delta}_\Sigma|.$$

The doubled Heegaard diagram is now defined as

$$D_{\boldsymbol{\beta}}(\mathcal{H}) = (D_{\boldsymbol{\beta}}(\Sigma), D_{\boldsymbol{\beta}}(\boldsymbol{\alpha}), \boldsymbol{\Delta}_\Sigma) = (\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}}, \boldsymbol{\alpha} \cup \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}, \boldsymbol{\Delta}_\Sigma).$$

Remark 7.7 There is some asymmetry in the above construction, since we took Σ and connected it along $\mathcal{I}_0$ to a surface $\overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}}$ that was in the $U_{\boldsymbol{\beta}}$ handlebody. We could instead connect Σ along $\mathcal{I}_0$ to the surface $R_{\boldsymbol{\alpha}} \natural_{\mathcal{I}_1} \overline{\Sigma}$ in the $U_{\boldsymbol{\alpha}}$ handlebody and construct analogous attaching curves $\boldsymbol{\Delta}_\Sigma$ and $\boldsymbol{\delta}_{\boldsymbol{\alpha}} \cup \overline{\boldsymbol{\alpha}} \cup \boldsymbol{\beta}$ for the Heegaard surface $D_{\boldsymbol{\alpha}}(\Sigma) := R_{\boldsymbol{\alpha}} \natural_{\mathcal{I}_1} \overline{\Sigma} \natural_{\mathcal{I}_0} \Sigma$. We will write

$$D_{\boldsymbol{\alpha}}(\mathcal{H}) = (D_{\boldsymbol{\alpha}}(\Sigma),\ \boldsymbol{\Delta}_\Sigma,\ \boldsymbol{\delta}_{\boldsymbol{\alpha}} \cup \overline{\boldsymbol{\alpha}} \cup \boldsymbol{\beta})$$

for this Heegaard diagram of (M, γ). If there is any ambiguity, we will call the Heegaard diagram $D_{\boldsymbol{\beta}}(\mathcal{H})$ the *β–double* and $D_{\boldsymbol{\alpha}}(\mathcal{H})$ the *α–double*.

7.5 Weakly conjugated diagrams

Given a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ of a based 3–manifold, we can consider the conjugate diagram $(\overline{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\alpha}, w)$ that represents the same based 3–manifold. This was described by Ozsváth and Szabó [24] and was explored further by Hendricks and Manolescu [9]. Given a sutured diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for (M, γ), one can consider the sutured diagram $(\overline{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\alpha})$; however, this is now a diagram for $(M, -\gamma)$. So, unlike in

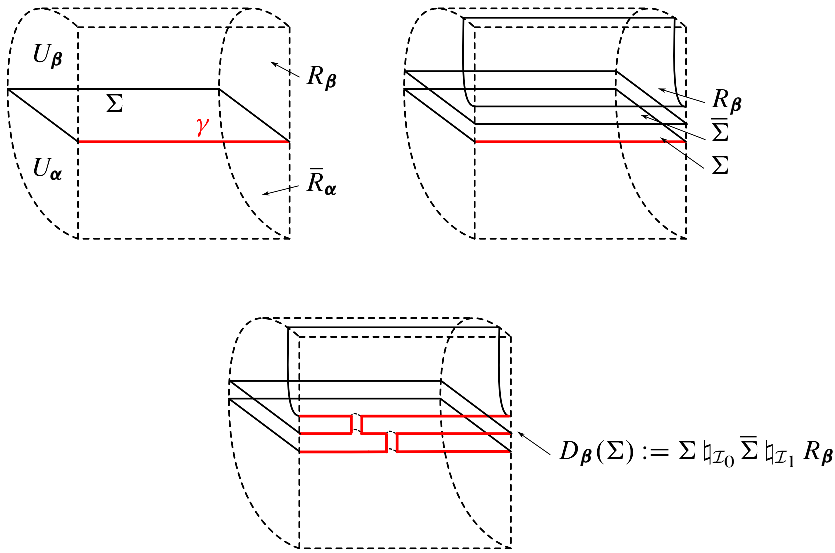

Figure 38: The doubled Heegaard surface $D_{\boldsymbol\beta}(\Sigma)$. A neighborhood of a portion of the sutures γ in M is shown. The sutures are drawn in red. Strictly speaking, the new sutures $\partial D_{\boldsymbol\beta}(\Sigma)$ are different from γ; however, an isotopy supported in a neighborhood of γ moves $\partial D_{\boldsymbol\beta}(\Sigma)$ back to γ.

the case of closed 3–manifolds, this operation does not induce a conjugation action on $\mathrm{SFH}(M,\gamma)$.

Nonetheless, a similar diagrammatic manipulation appears when we compute the cobordism map for $\mathcal{W}_{\boldsymbol\alpha,\boldsymbol\beta,\boldsymbol\gamma}$. In analogy to the terminology for the conjugation action

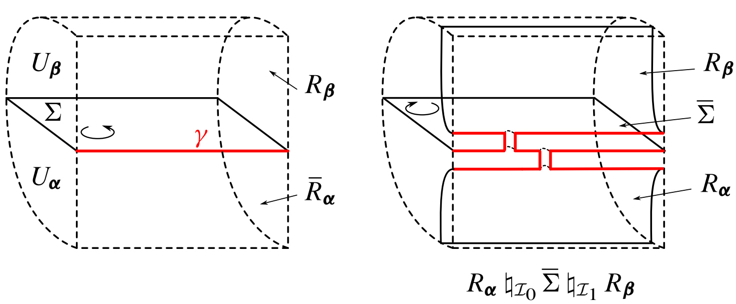

Figure 39: The weakly conjugated Heegaard surface $C(\Sigma)$. A neighborhood of a portion of the sutures is shown. The sutures are drawn in red. An isotopy supported in a neighborhood of the sutures moves the boundary of the new Heegaard surface, as we have drawn it, to the position of $\partial\Sigma$, the boundary of the old Heegaard surface.

on Heegaard diagrams for closed 3–manifolds, we will say that the diagrams that appear are *weakly conjugated.* We describe the construction of weakly conjugated diagrams in this section.

As with the doubled diagrams, we pick collections of subintervals $\mathcal{I}_0$ and $\mathcal{I}_1$ in $\partial\Sigma$ such that each component of $\partial\Sigma$ contains exactly one subinterval from $\mathcal{I}_0$ and from $\mathcal{I}_1$ that are disjoint. We can then form the weakly conjugated Heegaard surface

$$C(\Sigma) := R_{\boldsymbol{\alpha}} \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}}.$$

This is shown in Figure 39. As described, $\partial C(\Sigma)$ is different from $\partial\Sigma$, but an isotopy supported in a neighborhood of $\partial\Sigma$ moves $\partial C(\Sigma)$ to $\partial\Sigma$.

We now describe compressing curves on $C(\Sigma)$. Note that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ still bound compressing disks on Σ. As curves on $\overline{\Sigma}$, we denote them by $\overline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\beta}}$. However, these are not complete collections of compressing curves, as we have increased the genus of the Heegaard surface by attaching $R_{\boldsymbol{\alpha}}$ and $R_{\boldsymbol{\beta}}$. Hence, we pick a collection of pairwise disjoint arcs on $R_{\boldsymbol{\alpha}}$ that form a basis of $H_1(R_{\boldsymbol{\alpha}}, \mathcal{I}_0)$ and double them across $\mathcal{I}_0$ to get a collection of curves $\boldsymbol{\delta}_{\boldsymbol{\alpha}}$ on $R_{\boldsymbol{\alpha}} \natural_{\mathcal{I}_0} \overline{\Sigma}$. Similarly, we pick a collection of pairwise disjoint arcs on $R_{\boldsymbol{\beta}}$ that form a basis of $H_1(R_{\boldsymbol{\beta}}, \mathcal{I}_1)$ and double them across $\mathcal{I}_1$ to get a collection of curves $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ on $\overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}}$. We define the weakly conjugated Heegaard diagram of $\mathcal{H}$ to be

$$C(\mathcal{H}) = (C(\Sigma), C(\boldsymbol{\beta}), C(\boldsymbol{\alpha})) := (R_{\boldsymbol{\alpha}} \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}, \overline{\boldsymbol{\alpha}} \cup \boldsymbol{\delta}_{\boldsymbol{\alpha}}).$$

Then $C(\mathcal{H})$ is also a diagram of (M, γ).

7.6 The change of diagrams map from $D_{\boldsymbol{\beta}}(\mathcal{H})$ to $\mathcal{H}$

In this section, we prove a relatively simple formula for the change of diagrams map from the β–double of a diagram $D_{\boldsymbol{\beta}}(\mathcal{H})$ to the original diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Recall that

$$D_{\boldsymbol{\beta}}(\mathcal{H}) = (\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}}, \boldsymbol{\alpha} \cup \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}, \boldsymbol{\Delta}_{\Sigma}).$$

We will write $\boldsymbol{\beta}' := \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}$. Note that $(D_{\boldsymbol{\beta}}(\Sigma), \boldsymbol{\Delta}_{\Sigma}, \boldsymbol{\beta} \cup \boldsymbol{\beta}')$ is the α–double of the diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta})$ (see Remark 7.7), which represents

$$(M_{\boldsymbol{\beta},\boldsymbol{\beta}}, \gamma_{\boldsymbol{\beta},\boldsymbol{\beta}}),$$

which we note is the sutured manifold obtained by gluing two copies of the sutured handlebody $U_{\boldsymbol{\beta}}$ together along Σ.

Lemma 7.8 *There is a relatively graded isomorphism*

$$\mathrm{SFH}(M_{\boldsymbol{\beta},\boldsymbol{\beta}}, \gamma_{\boldsymbol{\beta},\boldsymbol{\beta}}) \cong \bigotimes_{i=1}^{|\boldsymbol{\beta}|}((\mathbb{F}_2)_{\frac{1}{2}} \oplus (\mathbb{F}_2)_{-\frac{1}{2}}).$$

In particular, there is a top-graded element $[\Theta_{\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}'}] \in \mathrm{SFH}(D_{\boldsymbol{\beta}}(\mathcal{H}))$.

Proof Let the $\boldsymbol{\gamma}$ curves be small Hamiltonian translates of the $\boldsymbol{\beta}$ curves such that $|\beta_i \cap \gamma_j| = 2\delta_{ij}$. Then $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$ is an admissible diagram for $(M_{\boldsymbol{\beta},\boldsymbol{\beta}}, \gamma_{\boldsymbol{\beta},\boldsymbol{\beta}})$ whose sutured Floer complex contains $2^{|\boldsymbol{\beta}|}$ generators. Furthermore, since every component of $\Sigma\setminus\boldsymbol{\beta}$ contains a component of $\partial\Sigma$, it is straightforward to see that the only homology classes of disks on $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$ have domains which are supported in the small bigons between β_i and γ_i. From these considerations, the only index 1 holomorphic disks are the classes whose domain consists of a bigon connecting the higher graded point of $\beta_i \cap \gamma_i$ to the lower graded point. Hence, modulo 2, the differential on $\mathrm{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$ vanishes, and as a relatively graded group, we have

$$\mathrm{SFH}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \mathrm{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}) \cong \bigotimes_{i=1}^{|\boldsymbol{\beta}|}((\mathbb{F}_2)_{\frac{1}{2}} \oplus (\mathbb{F}_2)_{-\frac{1}{2}}),$$

completing the proof. □

Using the holomorphic triangle map and the class $[\Theta_{\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}'}]$, we construct a map

$$F_2\colon \mathrm{SFH}(D_{\boldsymbol{\beta}}(\Sigma), \boldsymbol{\alpha}\cup\boldsymbol{\beta}', \boldsymbol{\Delta}_\Sigma)) \to \mathrm{SFH}(D_{\boldsymbol{\beta}}(\Sigma), \boldsymbol{\alpha}\cup\boldsymbol{\beta}', \boldsymbol{\beta}\cup\boldsymbol{\beta}')$$

using the formula

$$F_2 := F_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}'}(-, \Theta_{\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}'}).$$

We can also define a 3–handle map

$$F_3 := F_3^{\boldsymbol{\beta}',\boldsymbol{\beta}'}\colon \mathrm{SFH}(D_{\boldsymbol{\beta}}(\Sigma), \boldsymbol{\alpha}\cup\boldsymbol{\beta}', \boldsymbol{\beta}\cup\boldsymbol{\beta}') \to \mathrm{SFH}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

(here, the second copy of $\boldsymbol{\beta}'$ is a small Hamiltonian translate of $\boldsymbol{\beta}'$, though we omit this from the notation).

Lemma 7.9 *The composition $F_3 \circ F_2$ is chain homotopic to the change of diagrams map from* $\mathrm{CF}(D_{\boldsymbol{\beta}}(\mathcal{H}))$ *to* $\mathrm{CF}(\mathcal{H})$.

Proof The idea is simple: The map F_2 can be interpreted as a composition of 2–handle maps, and the map F_3 can be interpreted as a composition of 3–handle maps, for a collection of 4–dimensional 2–handles and 3–handles that topologically cancel. Hence

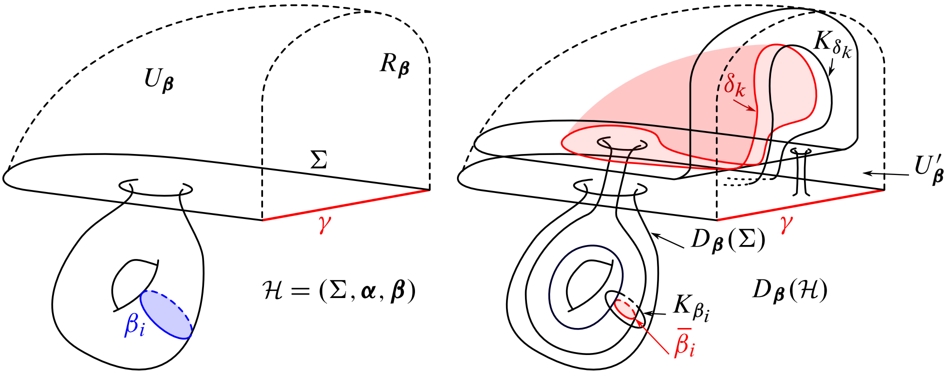

Figure 40: Constructing the link $\mathbb{L}\subseteq U_{\boldsymbol{\beta}}$. The left shows a part of the handlebody $U_{\boldsymbol{\beta}}$ for a diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$, together with a curve $\beta_i\in\boldsymbol{\beta}$. The right depicts the β–double $D_{\boldsymbol{\beta}}(\mathcal{H})$ of $\mathcal{H}$. The corresponding curve $\overline{\beta}_i\in\overline{\boldsymbol{\beta}}$ and a curve $\delta_k\in\boldsymbol{\delta}_{\boldsymbol{\beta}}$ are shown, as well as the corresponding components K_{β_i} and a portion of K_{δ_k} of $\mathbb{L}$. The shaded regions denote compressing disks bounded by the curves β_i, $\overline{\beta}_i$, and δ_k.

their composition represents the transition map from naturality, by the well-definedness of the sutured cobordism maps [15].

We now explain the technical details. Let us first describe the framed link along which surgery induces F_2. Let $U'_{\boldsymbol{\beta}}$ denote the β–handlebody of the diagram $D_{\boldsymbol{\beta}}(\mathcal{H})$ (ie $\boldsymbol{\Delta}_\Sigma$ bounds compressing disks in $U'_{\boldsymbol{\beta}}$). For each curve in $\boldsymbol{\beta}$ and each curve in $\boldsymbol{\delta}_{\boldsymbol{\beta}}$, we will construct a component of $\mathbb{L}$. For a curve $\beta_i\in\boldsymbol{\beta}$, there is a knot $K_{\beta_i}\subseteq U'_{\boldsymbol{\beta}}$, obtained by pushing β_i into $U'_{\boldsymbol{\beta}}$ slightly. We choose the framing of K_{β_i} to be parallel to Σ. Similarly, given $\delta_k\in\boldsymbol{\delta}_{\boldsymbol{\beta}}$, we can construct a framed knot K_{δ_k} by pushing δ_k into $U'_{\boldsymbol{\beta}}$, and take the framing induced by the tangent space of the surface $D_{\boldsymbol{\beta}}(\Sigma)$. The construction is illustrated in Figure 40. We define the framed link

$$\mathbb{L}:=\bigcup_{\tau\in\boldsymbol{\beta}\cup\boldsymbol{\delta}_{\boldsymbol{\beta}}}K_\tau.$$

The map F_3 is clearly the 3–handle map for a collection of $|\boldsymbol{\beta}'|$ 2–spheres in $M(\mathbb{L})$ that topologically cancel the link $\mathbb{L}$. In Lemma 7.10, we will show that the triple $(D_{\boldsymbol{\beta}}(\Sigma),\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}')$ can be related to a triple subordinate to a bouquet for $\mathbb{L}$ by a sequence of handle slides and isotopies of the $\boldsymbol{\Delta}_\Sigma$ and $\boldsymbol{\beta}\cup\boldsymbol{\beta}'$ curves. Assuming this topological fact for the moment, we now show that this implies that F_2 is the 2–handle map for surgery on $\mathbb{L}$. Suppose $(D_{\boldsymbol{\beta}}(\Sigma),\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\xi},\boldsymbol{\zeta})$ is a Heegaard triple which is subordinate to a bouquet for $\mathbb{L}$, and $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are related to $\boldsymbol{\Delta}_\Sigma$ and $\boldsymbol{\beta}\cup\boldsymbol{\beta}'$, respectively, by a sequence of handle slides and isotopies. To show that F_2 is the 2–handle map, it is sufficient to show that

$$F_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}'}(-,\Theta_{\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}'})\simeq\Psi^{\boldsymbol{\zeta}\to\boldsymbol{\beta}\cup\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}'}\circ F_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\xi},\boldsymbol{\zeta}}(\Psi^{\boldsymbol{\Delta}_\Sigma\to\boldsymbol{\xi}}_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}'}(-),\Theta_{\boldsymbol{\xi},\boldsymbol{\zeta}}),\tag{7-3}$$

the left-hand side being F_2, and the right-hand side being the definition of the 2–handle map precomposed with the transition map for changing $\boldsymbol{\Delta}_\Sigma$ to $\boldsymbol{\xi}$, and postcomposed with the transition map for changing $\boldsymbol{\zeta}$ to $\boldsymbol{\beta}\cup\boldsymbol{\beta}'$. We compute

$$\begin{aligned}\Psi^{\boldsymbol{\zeta}\to\boldsymbol{\beta}\cup\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}'}&\circ F_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\xi},\boldsymbol{\zeta}}(\Psi^{\boldsymbol{\Delta}_\Sigma\to\boldsymbol{\xi}}_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}'}(-),\Theta_{\boldsymbol{\xi},\boldsymbol{\zeta}})\\ &\simeq\Psi^{\boldsymbol{\zeta}\to\boldsymbol{\beta}\cup\boldsymbol{\beta}'}_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}'}\circ F_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\Delta}_\Sigma,\boldsymbol{\zeta}}(-,\Psi^{\boldsymbol{\zeta}}_{\boldsymbol{\xi}\to\boldsymbol{\Delta}_\Sigma}(\Theta_{\boldsymbol{\xi},\boldsymbol{\zeta}}))\\ &\simeq F_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\Delta}_\Sigma,\boldsymbol{\zeta}}\big(-,\Psi^{\boldsymbol{\zeta}\to\boldsymbol{\beta}\cup\boldsymbol{\beta}'}_{\boldsymbol{\Delta}_\Sigma}(\Psi^{\boldsymbol{\zeta}}_{\boldsymbol{\xi}\to\boldsymbol{\Delta}_\Sigma}(\Theta_{\boldsymbol{\xi},\boldsymbol{\zeta}}))\big).\end{aligned}\tag{7-4}$$

The first chain homotopy follows from associativity for the 4–tuple $(\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\Delta}_\Sigma,\boldsymbol{\xi},\boldsymbol{\zeta})$, and the second follows from associativity applied to the 4–tuple $(\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\Delta}_\Sigma,\boldsymbol{\zeta},\boldsymbol{\beta}\cup\boldsymbol{\beta}')$. Using the fact that the top-graded generator is well defined on the level of homology, as in Lemma 7.8, it follows that $\Psi^{\boldsymbol{\zeta}\to\boldsymbol{\beta}\cup\boldsymbol{\beta}'}_{\boldsymbol{\Delta}_\Sigma}(\Psi^{\boldsymbol{\zeta}}_{\boldsymbol{\xi}\to\boldsymbol{\Delta}_\Sigma}(\Theta_{\boldsymbol{\xi},\boldsymbol{\zeta}}))$ and $\Theta_{\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}'}$ differ by a boundary, so the last line of (7-4) is chain homotopic to F_2, establishing (7-3).

Using the fact that F_2 is chain homotopic to the 2–handle map for $\mathbb{L}$, it follows that the composition $F_3\circ F_2$ is chain homotopic to the map from naturality. □

To finish the proof of Lemma 7.9, it is sufficient to show the following:

Lemma 7.10 *The Heegaard triple $(D_{\boldsymbol{\beta}}(\Sigma),\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\Delta}_\Sigma,\boldsymbol{\beta}\cup\boldsymbol{\beta}')$ is related to a triple subordinate to the framed link $\mathbb{L}$ by a sequence of handle slides and isotopies of the $\boldsymbol{\Delta}_\Sigma$ and $\boldsymbol{\beta}\cup\boldsymbol{\beta}'$ curves.*

Proof We recall that the $\boldsymbol{\Delta}_\Sigma$ curves were constructed by picking a set of arcs $\boldsymbol{s}=\{s_1,\dots,s_n\}$ forming a basis of $H_1(\Sigma,\mathcal{I}_0)$ and then doubling the arcs of $\boldsymbol{s}$ across $\mathcal{I}_0$ onto $\overline{\Sigma}$ to obtain a collection of $n:=2(g(\Sigma)+|\partial\Sigma|-|\Sigma|)$ closed curves. By Lemma 7.6, any two such bases $\boldsymbol{s}$ of $H_1(\Sigma,\mathcal{I}_0)$ can be connected by a sequence of allowable arc slides. An allowable arc slide of two arcs in $\boldsymbol{s}$ induces a handle slide of the corresponding doubled curves in $\boldsymbol{\Delta}_\Sigma$. Consequently, we can assume that $\boldsymbol{\Delta}_\Sigma$ is constructed using any convenient basis of $H_1(\Sigma,\mathcal{I}_0)$.

The curves $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ are obtained by picking a set of

$$k:=\operatorname{rk}H_1(R_{\boldsymbol{\beta}},\mathcal{I}_1)=2(g(R_{\boldsymbol{\beta}})+|\partial\Sigma|-|R_{\boldsymbol{\beta}}|)$$

arcs $d_1,\dots,d_k$ which form a basis of $H_1(R_{\boldsymbol{\beta}},\mathcal{I}_1)$. Noting that we view $R_{\boldsymbol{\beta}}$ as the result of surgering Σ on $\boldsymbol{\beta}$, we can assume that $\boldsymbol{\Delta}_\Sigma$ and $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ are constructed using arcs $\boldsymbol{s}$ and $\boldsymbol{d}$ satisfying the following:

(1) For each $\beta \in \boldsymbol{\beta}$, there is a pair of arcs $s, s' \in \boldsymbol{s}$ such that $|s \cap \beta| = 1$, and s is disjoint from all the other $\boldsymbol{\beta}$ and $\boldsymbol{d}$ curves. The curve s' is obtained by taking β and isotoping a neighborhood of the point $\beta \cap s$ along s until it intersects $\partial\Sigma$. In particular, s' is disjoint from $\boldsymbol{d} \cup \boldsymbol{\beta}$. Furthermore, we can handle slide β across its image $\bar{\beta}$ on $\bar{\Sigma}$ along the arc s to obtain a curve isotopic to $s' \cup \bar{s}'$, where $\bar{s}'$ is the image of s' on $\bar{\Sigma}$.

(2) For each $d \in \boldsymbol{d}$, there is a corresponding arc $s \in \boldsymbol{s}$ satisfying $|s \cap d| = 1$. Furthermore, s is disjoint from $(\boldsymbol{d} \setminus \{d\}) \cup \boldsymbol{\beta}$.

See Figure 41 for an example of the arcs $\boldsymbol{s}$ and $\boldsymbol{d}$, and the resulting attaching curves $\boldsymbol{\Delta}_\Sigma$ and $\boldsymbol{\delta}_{\boldsymbol{\beta}}$.

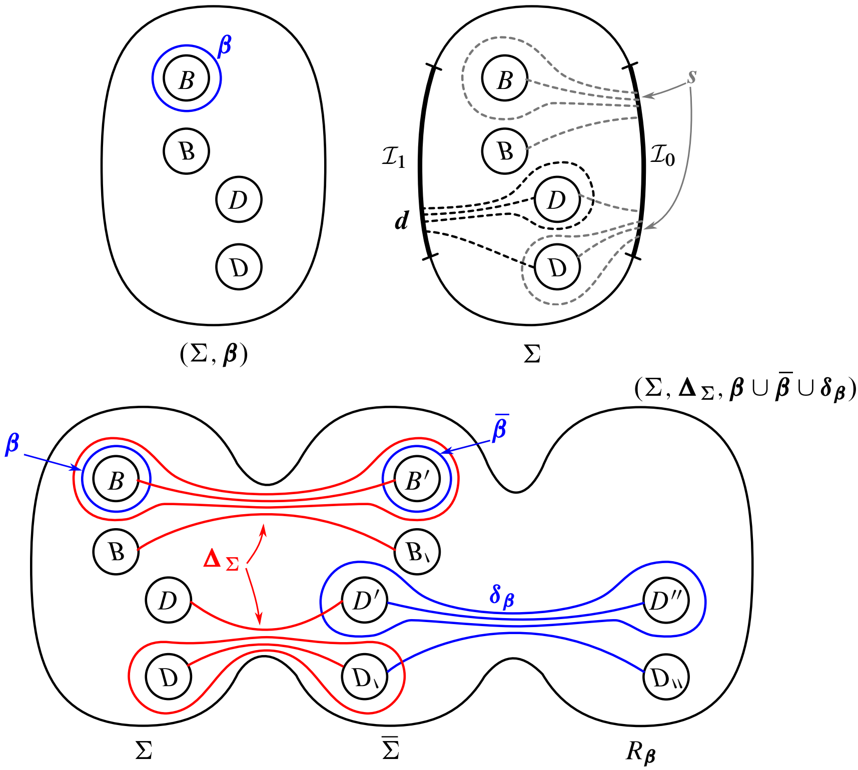

Figure 41: On the top left is an example of a monodiagram $(\Sigma, \boldsymbol{\beta})$ with $|\boldsymbol{\beta}| = 1$, $g(\Sigma) = 2$, and $|\partial\Sigma| = 1$. On the top right are the arcs $\boldsymbol{s}$ used to form $\boldsymbol{\Delta}_\Sigma$, as well as the arcs $\boldsymbol{d}$ used to form $\boldsymbol{\delta}_{\boldsymbol{\beta}}$. On the bottom is the diagram $(D_{\boldsymbol{\beta}}(\Sigma), \boldsymbol{\Delta}_\Sigma, \boldsymbol{\beta} \cup \boldsymbol{\beta}')$.

We handle slide each $\beta \in \boldsymbol{\beta}$ across the corresponding $\overline{\beta} \in \overline{\boldsymbol{\beta}}$ along the arcs $\boldsymbol{s}$ to form the curve β^H. Let us call the resulting set of curves $\boldsymbol{\beta}^H$. We note that each curve in $\overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}$ is a longitude of a component of $\mathbb{L}$, and each curve in $\boldsymbol{\Delta}_\Sigma$ is either a meridian or a small Hamiltonian isotopy of a curve in $\boldsymbol{\beta}^H$. It follows that the triple

$$(D_{\boldsymbol{\beta}}(\Sigma), \boldsymbol{\alpha} \cup \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}, \boldsymbol{\Delta}_\Sigma, \boldsymbol{\beta}^H \cup \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}})$$

is subordinate to a bouquet for $\mathbb{L}$. □

7.7 The change of diagrams map from $C(\mathcal{H})$ to $D_{\boldsymbol{\beta}}(\mathcal{H})$

Analogously to the formula in Lemma 7.9, we can describe the change of diagrams map from $C(\mathcal{H})$ to $D_{\boldsymbol{\beta}}(\mathcal{H})$ in concrete terms.

Pick a collection of pairwise disjoint arcs on $R_{\boldsymbol{\alpha}}$ with endpoints in $\mathcal{I}_0$ that form a basis of $H_1(R_{\boldsymbol{\alpha}}, \mathcal{I}_0)$. These induce arcs on Σ which are disjoint from $\boldsymbol{\alpha}$ and well defined up to handle slides across $\boldsymbol{\alpha}$; let $\boldsymbol{\Delta}_{\boldsymbol{\alpha}} \subseteq \Sigma \natural_{\mathcal{I}_0} \overline{\Sigma}$ be the curves formed by doubling them across $\mathcal{I}_0$. By construction, the curves in $\boldsymbol{\Delta}_{\boldsymbol{\alpha}}$ are disjoint from $\boldsymbol{\alpha}$ and $\overline{\boldsymbol{\alpha}}$. Also, let $\boldsymbol{\delta}_{\boldsymbol{\alpha}}$ denote the images of the curves $\boldsymbol{\Delta}_{\boldsymbol{\alpha}}$ on $R_{\boldsymbol{\alpha}} \natural_{\mathcal{I}_0} \overline{\Sigma}$. Let $\boldsymbol{\alpha}' := \overline{\boldsymbol{\alpha}} \cup \boldsymbol{\Delta}_{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}' := \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}$.

Recall the surface $D_{\boldsymbol{\beta}}(\Sigma)$ is $\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}}$ and $C(\Sigma)$ is defined as $R_{\boldsymbol{\alpha}} \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\beta}}$. Since surgering the surface Σ along the $\boldsymbol{\alpha}$ curves yields $R_{\boldsymbol{\alpha}}$, we see that there is a 1–handle map

$$G_1 := F_1^{\boldsymbol{\alpha},\boldsymbol{\alpha}} \colon \mathrm{CF}(C(\mathcal{H})) = \mathrm{CF}(C(\Sigma),\ \boldsymbol{\beta}',\ \overline{\boldsymbol{\alpha}} \cup \boldsymbol{\delta}_{\boldsymbol{\alpha}}) \to \mathrm{CF}(D_{\boldsymbol{\beta}}(\Sigma),\ \boldsymbol{\alpha} \cup \boldsymbol{\beta}',\ \boldsymbol{\alpha} \cup \boldsymbol{\alpha}').$$

Lemma 7.11 *The diagram $(D_{\boldsymbol{\beta}}(\Sigma), \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \boldsymbol{\Delta}_\Sigma)$ represents the sutured manifold*

$$(S^1 \times S^2)^{\# n}(|\partial\Sigma|) \natural_{\mathcal{I}_1} (R_{\boldsymbol{\beta}} \times I, \partial R_{\boldsymbol{\beta}} \times I)$$

for some n, where $(S^1 \times S^2)^{\# n}(|\partial\Sigma|)$ denotes the sutured manifold obtained by removing $|\partial\Sigma|$ balls from $(S^1 \times S^2)^{\# n}$ and adding a connected suture to each boundary component. Furthermore, $\natural_{\mathcal{I}_1}$ denotes the boundary connected sum taken along $\mathcal{I}_1$ by adding $|\mathcal{I}_1|$ product 1–handles.

Proof All the attaching curves are disjoint from $R_{\boldsymbol{\beta}}$. If we cut $R_{\boldsymbol{\beta}}$ off of $D_{\boldsymbol{\beta}}(\Sigma)$, we are left with the diagram $(\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma}, \boldsymbol{\alpha} \cup \boldsymbol{\alpha}', \boldsymbol{\Delta}_\Sigma)$. Recall that $\boldsymbol{\alpha}' = \overline{\boldsymbol{\alpha}} \cup \boldsymbol{\Delta}_{\boldsymbol{\alpha}}$, where $\boldsymbol{\Delta}_{\boldsymbol{\alpha}}$ is obtained by choosing a basis of arcs for $H_1(R_{\boldsymbol{\alpha}}, \mathcal{I}_0)$ and doubling the induced curves on Σ onto $\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma}$. We can assume that $\boldsymbol{\Delta}_\Sigma$ is constructed by starting with small Hamiltonian isotopes of the curves in $\boldsymbol{\Delta}_{\boldsymbol{\alpha}}$ and then adjoining an additional $2|\boldsymbol{\alpha}|$ curves that are disjoint from $\boldsymbol{\Delta}_{\boldsymbol{\alpha}}$. Then the curves in $\boldsymbol{\Delta}_{\boldsymbol{\alpha}}$ together with their isotopes in $\boldsymbol{\Delta}_\Sigma$ determine $|\boldsymbol{\Delta_\alpha}|$ embedded 2–spheres, which may or may not be separating. After surgering these out, we are left with the double of a diagram for a disjoint union of some number of copies of $(S^1\times S^2)^{\# l}(m)$ for various l and m. It follows that $(D_{\boldsymbol{\beta}}(\Sigma),\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\Delta}_\Sigma)$ represents

$$(S^1\times S^2)^{\# n}(|\partial\Sigma|)\natural_{\mathcal{I}_1}(R_{\boldsymbol{\beta}}\times I,\partial R_{\boldsymbol{\beta}}\times I)$$

for some n. □

As a consequence of Lemma 7.11, there is a top-graded generator

$$\Theta_{\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\Delta}_\Sigma}\in \mathrm{SFH}(D_{\boldsymbol{\beta}}(\Sigma),\ \boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\ \boldsymbol{\Delta}_\Sigma),$$

and hence we can also define a triangle map

$$G_2:=F_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\Delta}_\Sigma}(-,\Theta_{\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\Delta}_\Sigma}).$$

Lemma 7.12 *The composition $G_2\circ G_1$ is chain homotopic to the change of diagrams map from $C(\mathcal{H})$ to $D_{\boldsymbol{\beta}}(\mathcal{H})$.*

Proof The proof is similar to the proof of Lemma 7.9. We can interpret the composition $G_2\circ G_1$ as the cobordism map for a canceling collection of $|\boldsymbol{\alpha}|$ pairs of 4–dimensional 1– and 2–handles. We now describe the attaching spheres of the 1–handles and 2–handles (see also [29, Section 7.2] for a detailed account of a similar topological manipulation in the setting of closed 3–manifolds). Let $D_1,\dots,D_{|\boldsymbol{\alpha}|}$ denote compressing disks attached along the curves $\boldsymbol{\alpha}\subseteq\Sigma$, where $\Sigma\subseteq M$ denotes the original Heegaard surface.

The two feet of a 1–handle are obtained by pushing off the center point of the disk D_i in both normal directions. The canceling 2–handle for this 1–handle is attached along the core of the 1–handle concatenated with an arc connecting the two feet that intersects the center of D_i transversely. By adapting the proof of Lemma 7.10, it is not hard to see that, after a sequence of handle slides, the triple $(D_{\boldsymbol{\beta}}(\Sigma),\boldsymbol{\alpha}\cup\boldsymbol{\beta}',\boldsymbol{\alpha}\cup\boldsymbol{\alpha}',\boldsymbol{\Delta}_\Sigma)$ becomes subordinate to the link described above. □

7.8 A diagram for $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}\cup_{R_{\boldsymbol{\beta}}}M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ and a formula for the special cobordism map $(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s$

Let $\mathcal{T}=(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ be an admissible sutured triple diagram. We now describe a diagram $A(\mathcal{T})$ for the sutured manifold $(M_{\boldsymbol{\alpha},\boldsymbol{\beta}}\cup_{R_{\boldsymbol{\beta}}}M_{\boldsymbol{\beta},\boldsymbol{\gamma}},\gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$, which has sutures along $\partial R_{\boldsymbol{\beta}}$, where the two manifolds are glued together.

Analogous to the doubled diagram, we let $\mathcal{I}_0, \mathcal{I}_1 \subseteq \partial\Sigma$ be disjoint collections of subintervals such that each component of $\partial\Sigma$ contains exactly one subinterval from $\mathcal{I}_0$ and $\mathcal{I}_1$. We form the diagram

$$A(\mathcal{T}) = (D_{\boldsymbol{\gamma}}(\Sigma), D_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}), A(\boldsymbol{\beta})) = (\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\gamma}},\ \boldsymbol{\alpha} \cup \overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}},\ \boldsymbol{\beta} \cup \overline{\boldsymbol{\beta}} \cup \boldsymbol{\Delta}_{\boldsymbol{\beta}}),$$

where the component Σ of $D_{\boldsymbol{\gamma}}(\Sigma)$ is embedded in $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and the component $\overline{\Sigma}$ in $M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$. We call $A(\mathcal{T})$ the *amalgamation of* $\mathcal{T}$ *along* $R_{\boldsymbol{\beta}}$. Here $\boldsymbol{\delta}_{\boldsymbol{\gamma}} \subseteq \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\gamma}}$ is obtained by doubling a collection of arcs forming a basis of $H_1(R_{\boldsymbol{\gamma}}, \mathcal{I}_1)$. Similarly, $\boldsymbol{\Delta}_{\boldsymbol{\beta}} \subseteq \Sigma \natural_{\mathcal{I}_0} \overline{\Sigma}$ is obtained by choosing a basis of arcs in $H_1(R_{\boldsymbol{\beta}}, \mathcal{I}_0)$ and doubling a lift of them to Σ across $\mathcal{I}_0$.

Note that the doubling construction can be viewed as an instance of amalgamation, in the sense that

$$D_{\boldsymbol{\beta}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) = A(\Sigma, \boldsymbol{\alpha}, \varnothing, \boldsymbol{\beta}).$$

Here we interpret $R_{\varnothing}$ as Σ, so that $\boldsymbol{\Delta}_{\boldsymbol{\beta}}$ is the collection $\boldsymbol{\Delta}_{\Sigma}$ defined in the construction of a doubled diagram.

Lemma 7.13 *The diagram* $A(\mathcal{T})$ *is a diagram for* $(M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$.

Proof The diagram $A(\mathcal{T})$ is obtained by first replacing $R_{\boldsymbol{\beta}}$ with Σ in

$$C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}) = (R_{\boldsymbol{\beta}} \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\gamma}},\ \overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}},\ \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}),$$

after which $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ becomes $\boldsymbol{\Delta}_{\boldsymbol{\beta}}$. As compressing Σ along $\boldsymbol{\beta}$ gives $R_{\boldsymbol{\beta}}$, if we add $\boldsymbol{\beta}$, the resulting diagram

$$(\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\gamma}},\ \overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}},\ \boldsymbol{\beta} \cup \overline{\boldsymbol{\beta}} \cup \boldsymbol{\Delta}_{\boldsymbol{\beta}})$$

represents $(-U_{\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} U_{\boldsymbol{\beta}} \cup_{\Sigma} -U_{\boldsymbol{\gamma}}, \partial\Sigma)$. Finally, we add $\boldsymbol{\alpha}$, which amounts to gluing $U_{\boldsymbol{\alpha}}$ to $-U_{\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} U_{\boldsymbol{\beta}} \cup_{\Sigma} -U_{\boldsymbol{\gamma}}$ by identifying $\Sigma \subseteq U_{\boldsymbol{\alpha}}$ with $\overline{\Sigma} \subseteq \partial(-U_{\boldsymbol{\beta}})$. □

By Lemma 7.4, the special cobordism

$$(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s \colon (M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}},\ \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}}) \to (M_{\boldsymbol{\alpha},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$$

is a 2–handle cobordism, for surgery on a framed link $\mathbb{L} \subseteq M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$. We recall the description of the framed link $\mathbb{L}$. One takes a Morse function $f_{\boldsymbol{\beta}}$ on $U_{\boldsymbol{\beta}}$ that is 0 on $\overline{R}_{\boldsymbol{\beta}}$ and 1 on Σ and has $\boldsymbol{\beta}$ as the intersection of the ascending manifolds of the critical points of $f_{\boldsymbol{\beta}}$ with Σ. Then the descending manifolds of the critical points of $f_{\boldsymbol{\beta}}$ determine a collection of $|\boldsymbol{\beta}|$ properly embedded arcs $\lambda_i \subseteq U_{\boldsymbol{\beta}}$ that have both ends on $R_{\boldsymbol{\beta}}$. The link $\mathbb{L}$ is obtained by taking the union of the arcs $\lambda_i \subseteq U_{\boldsymbol{\beta}} \subseteq M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$, together with their images in $-U_{\boldsymbol{\beta}} \subseteq M_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

Let $\mathbf{\Delta}_\Sigma \subseteq \Sigma \natural_{\mathcal{I}_0} \overline{\Sigma}$ be a collection of curves obtained by picking arcs forming a basis of $H_1(\Sigma, \mathcal{I}_0)$ and then doubling them across $\mathcal{I}_0$. One can assume that the chosen basis of $H_1(\Sigma, \mathcal{I}_0)$ extends the lift of the basis of $H_1(R_{\boldsymbol{\beta}}, \mathcal{I}_0)$ to Σ we chose in the construction of the $\mathbf{\Delta}_{\boldsymbol{\beta}}$ curves, so that $\mathbf{\Delta}_{\boldsymbol{\beta}} \subseteq \mathbf{\Delta}_\Sigma$, though this is not essential.

We note that the diagram

$$(D_{\boldsymbol{\gamma}}(\Sigma), A(\boldsymbol{\beta}), \mathbf{\Delta}_\Sigma) = (\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\gamma}},\ \boldsymbol{\beta} \cup \overline{\boldsymbol{\beta}} \cup \mathbf{\Delta}_{\boldsymbol{\beta}},\ \mathbf{\Delta}_\Sigma)$$

represents

$$(S^1 \times S^2)^{\#n}(|\partial\Sigma|) \natural_{\mathcal{I}_1} (I \times R_{\boldsymbol{\gamma}}, I \times \partial R_{\boldsymbol{\gamma}})$$

for some n by Lemma 7.11. Consequently, there is a top-graded generator

$$\Theta_{A(\boldsymbol{\beta}),\mathbf{\Delta}_\Sigma} \in \mathrm{SFH}(D_{\boldsymbol{\gamma}}(\Sigma), A(\boldsymbol{\beta}), \mathbf{\Delta}_\Sigma) = \mathrm{SFH}(\Sigma \natural_{\mathcal{I}_0} \overline{\Sigma} \natural_{\mathcal{I}_1} R_{\boldsymbol{\gamma}},\ \boldsymbol{\beta} \cup \overline{\boldsymbol{\beta}} \cup \mathbf{\Delta}_{\boldsymbol{\beta}},\ \mathbf{\Delta}_\Sigma).$$

Note that $(D_{\boldsymbol{\gamma}}(\Sigma), D_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}), \mathbf{\Delta}_\Sigma)$ is a doubled diagram for $(M_{\boldsymbol{\alpha},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$. We have the following:

Lemma 7.14 *The special cobordism map*

$$F_{(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s}\colon \mathrm{CF}(M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}},\ \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}}) \to \mathrm{CF}(M_{\boldsymbol{\alpha},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$$

is chain homotopic to the triangle map

$$F_{D_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}),A(\boldsymbol{\beta}),\mathbf{\Delta}_\Sigma}(-, \Theta_{A(\boldsymbol{\beta}),\mathbf{\Delta}_\Sigma}).$$

Proof By Lemma 7.4, the special cobordism $(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s$ is a 2–handle cobordism, for a framed link $\mathbb{L} \subseteq -U_{\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} U_{\boldsymbol{\beta}}$ described above. By adapting the proof of Lemma 7.10, we see that the triple

$$(D_{\boldsymbol{\gamma}}(\Sigma), D_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}), A(\boldsymbol{\beta}), \mathbf{\Delta}_\Sigma)$$

can be related by a sequence of handle slides and isotopies to a triple that is subordinate to a bouquet for $\mathbb{L}$. Thus, $F_{(W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})^s}$ is chain homotopic to the triangle map above. □

7.9 A holomorphic triangle description of the gluing map

Let $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ be a sutured Heegaard triple. In this section, we will present a natural candidate for the map for gluing

$$(Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}, -\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})$$

to

$$(-M_{\boldsymbol{\alpha},\boldsymbol{\beta}}, -\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \sqcup (-M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, -\gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}})$$

and prove that it is indeed the gluing map.

Let

$$C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}) = (R_{\boldsymbol{\beta}} \natural \overline{\Sigma} \natural R_{\boldsymbol{\gamma}},\ \overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}},\ \overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}})$$

be a weak conjugate of $\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$, as described in Section 7.5. We now construct our candidate

$$\Phi\colon \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \otimes \mathrm{CF}(C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})) \to \mathrm{CF}(A(\mathcal{T}))$$

for the gluing map. Note that the domain of Φ is $\mathrm{CF}(M_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \otimes \mathrm{CF}(M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}})$, and its range is $\mathrm{CF}(M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$.

The definition of the map Φ is formally similar to the map for connected sums due to Ozsváth and Szabó [23]. We will call Φ the *amalgamation map*, since its image is in the Floer homology of the amalgamated diagram from the previous section. We also remark that the Ozsváth–Szabó maps that inspire the construction of Φ were called the *intertwining maps* in [29], where they played a similar role as in our present context.

Note that there is a 1–handle map

$$F_1^{\overline{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\overline{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}}}\colon \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \to \mathrm{CF}(D_{\boldsymbol{\gamma}}(\Sigma), D_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}), D_{\boldsymbol{\gamma}}(\boldsymbol{\beta})),$$

where $D_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \cup \overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}}$ and $D_{\boldsymbol{\gamma}}(\boldsymbol{\beta}) = \boldsymbol{\beta} \cup \overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}}$.

Recall that the underlying Heegaard surface of $C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})$ is $R_{\boldsymbol{\beta}} \natural \overline{\Sigma} \natural R_{\boldsymbol{\gamma}}$, and the Heegaard surface $D_{\boldsymbol{\gamma}}(\Sigma)$ is $\Sigma \natural \overline{\Sigma} \natural R_{\boldsymbol{\gamma}}$. Because the surface $R_{\boldsymbol{\beta}}$ is the result of surgering Σ along the $\boldsymbol{\beta}$ curves, it follows that there is a 1–handle map

$$F_1^{\boldsymbol{\beta},\boldsymbol{\beta}}\colon \mathrm{CF}(C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})) \to \mathrm{CF}(D_{\boldsymbol{\gamma}}(\Sigma), D_{\boldsymbol{\gamma}}(\boldsymbol{\beta}), A(\boldsymbol{\beta})),$$

obtained by adding the $\boldsymbol{\beta}$ curves back into $R_{\boldsymbol{\beta}} \subseteq C(\Sigma)$. Note that, after we add these 1–handles, turning $R_{\boldsymbol{\beta}}$ into Σ, the curves $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ become $\boldsymbol{\Delta}_{\boldsymbol{\beta}}$.

Finally, we define

$$\Phi := F_{D_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}),D_{\boldsymbol{\gamma}}(\boldsymbol{\beta}),A(\boldsymbol{\beta})} \circ (F_1^{\overline{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\overline{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}}} \otimes F_1^{\boldsymbol{\beta},\boldsymbol{\beta}}).$$

A key ingredient in our analysis of the sutured cobordism $\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ is the following:

Proposition 7.15 *The amalgamation map*

$$\Phi\colon \mathrm{SFH}(M_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \otimes \mathrm{SFH}(M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}}) \to \mathrm{SFH}(M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$$

defined above is chain homotopic to the contact gluing map for gluing $(Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}, -\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}})$ *to*

$$(-M_{\boldsymbol{\alpha},\boldsymbol{\beta}}, -\gamma_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \sqcup (-M_{\boldsymbol{\beta},\boldsymbol{\gamma}}, -\gamma_{\boldsymbol{\beta},\boldsymbol{\gamma}}).$$

Proof First, we describe a contact handle decomposition of $(Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})$, relative to $R_{\alpha,\beta} \sqcup R_{\beta,\gamma}$. We pick a collection of subintervals $\mathcal{I}_0 \subseteq \partial R_{\beta}$ such that each component of ∂R_{β} contains exactly one subinterval. We pick an arc basis $\lambda_1, \dots, \lambda_n$ for $(R_{\beta}, \mathcal{I}_0)$. We claim that a contact handle decomposition of $(Z_{\alpha,\beta,\gamma}, \xi_{\alpha,\beta,\gamma})$, relative to $R_{\alpha,\beta} \sqcup R_{\beta,\gamma}$, can be constructed as follows:

(1) The contact 1–handles are the components of $N(I \times \mathcal{I}_0)$ (in particular, we have one 1–handle for each component of ∂R_{β}).

(2) The contact 2–handles are $N(I \times \lambda_i)$ for $i \in \{1, \dots, n\}$.

The 1–handles are simply added with feet along the corresponding subintervals of $\mathcal{I}_0 \subseteq R_{\beta} \subseteq \partial M_{\alpha,\beta}$ and $\bar{\mathcal{I}}_0 \subseteq \bar{R}_{\beta} \subseteq \partial M_{\beta,\gamma}$. The 2–handles are attached along curves obtained by concatenating an arc λ_i on R_{β} with its mirror on $\bar{R}_{\beta}$.

To show that the above description determines a contact handle decomposition, we use Lemma 7.3, which allows us to decompose $Z_{\alpha,\beta,\gamma}$ along the convex surface obtained by extending $\{\frac{1}{2}\} \times R_{\beta} \subseteq Z_{\beta}$ across the solid tori Z_0 to get a decomposing surface which is diffeomorphic to R_{β}. See Figures 33 and 34 for schematics of the decomposing surface. Phrased another way, we can write $(Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})$ as $(I \times R_{\alpha,\beta}, -\xi_{\alpha,\beta})$ and $(I \times R_{\beta,\gamma}, -\xi_{\beta,\gamma})$ glued together along two subsurfaces $\bar{R}'_{\beta} \subseteq \{0\} \times R_{\alpha,\beta}$ and $R'_{\beta} \subseteq \{0\} \times R_{\beta,\gamma}$ that are small perturbations of $\bar{R}_{\beta} \subseteq \{0\} \times \bar{R}_{\alpha,\beta}$ and $R_{\beta} \subseteq \{0\} \times R_{\beta,\gamma}$, respectively. We pick R'_{β} and $\bar{R}'_{\beta}$ such that $\mathcal{I}_0 \subseteq R'_{\beta}$ and $\bar{\mathcal{I}}_0 \subseteq \bar{R}'_{\beta}$. We identify R'_{β} and $\bar{R}'_{\beta}$ with their images on $\partial M_{\alpha,\beta}$ and $\partial M_{\beta,\gamma}$, respectively.

The above description implies that the cores of the 1–handles in our contact handle decomposition are Legendrian, and the attaching circles of the 2–handles cross the dividing set exactly twice. Using Legendrian realization after attaching the contact 1–handles, we can assume the attaching curves of the 2–handles are Legendrian, and hence have $\mathrm{tb} = -1$. It follows that the above description determines a contact handle decomposition of $Z_{\alpha,\beta,\gamma}$, starting at $R_{\alpha,\beta} \sqcup R_{\beta,\gamma}$ and ending at $R_{\alpha,\gamma}$. Using the above contact handle decomposition of $(Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})$, we can now give a description of the gluing map associated to $(Z_{\alpha,\beta,\gamma}, -\xi_{\alpha,\beta,\gamma})$ on the level of Heegaard diagrams.

The contact 1–handle maps have a simple description in terms of diagrams; one simply adds a band with ends at the feet of the 1–handle. There are two dividing arcs on the boundary of the band, and they are distinguished: One intersects $I \times R_{\beta} \subseteq Z_{\alpha,\beta,\gamma}$ nontrivially, while the other is disjoint from $I \times R_{\beta}$; see Figure 42. Let us call the edge that intersects $I \times R_{\beta}$ the "interior" dividing arc of the 1–handle. The other edge

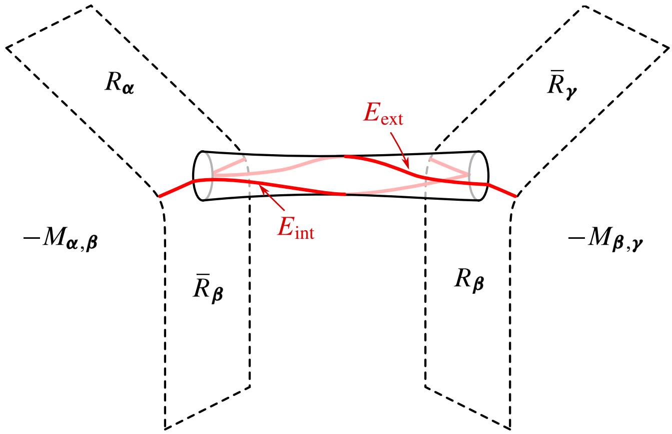

Figure 42: A contact 1–handle added to $-M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \sqcup -M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$. On the 1–handle, the interior edge of the dividing set (in E_{int}) is contained in $I \times R_{\boldsymbol{\beta}}$, and the exterior edge (in E_{ext}) is disjoint from $I \times R_{\boldsymbol{\beta}}$.

we call the "exterior" dividing arc. We will write E_{int} for the interior arcs, and E_{ext} for the exterior arcs.

Note that the attaching circles of the 2–handles only intersect the dividing set along the contact 1–handles, and only along the interior arcs E_{int}. Hence, on the level of diagrams, the contact 2–handle map involves adding a band to E_{int} and also adding an α–curve and a β–curve.

We will write Σ_0 for the new portion of the Heegaard surface which is added by the contact 1–handles and 2–handles. We will shortly see that Σ_0 can be identified with $\bar{R}_{\boldsymbol{\beta}}$; see (7-5). We call the collection of β–curves that we add with the contact 2–handles $\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ (the notation will be justified below; see (7-5)), and we call the α–curves that we add $\boldsymbol{\epsilon}$.

By construction, the diagram for the domain of the amalgamation map is the disjoint union of $\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and a weak conjugate

$$C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}) = (R_{\boldsymbol{\beta}} \natural \bar{\Sigma} \natural R_{\boldsymbol{\gamma}},\ \bar{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}},\ \bar{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}})$$

of $\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}} = (\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$. After adding all the 1–handles and 2–handles, we obtain the Heegaard diagram

$$G(\mathcal{T}) = (\Sigma(\mathcal{T}), \boldsymbol{\alpha}(\mathcal{T}), \boldsymbol{\beta}(\mathcal{T})) := (\Sigma \natural \Sigma_0 \natural R_{\boldsymbol{\beta}} \natural \bar{\Sigma} \natural R_{\boldsymbol{\gamma}},\ \boldsymbol{\alpha} \cup \boldsymbol{\epsilon} \cup \bar{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}},\ \boldsymbol{\beta} \cup \bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}} \cup \bar{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}})$$

for $M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$. Note that the diagram $G(\mathcal{T})$ is similar to, but not equal to, the Heegaard diagram

$$A(\mathcal{T}) = (\Sigma \natural \bar{\Sigma} \natural R_{\boldsymbol{\gamma}}, \boldsymbol{\alpha} \cup \bar{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}}, \boldsymbol{\beta} \cup \bar{\boldsymbol{\beta}} \cup \boldsymbol{\Delta}_{\boldsymbol{\beta}})$$

of $M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ that appears in the target of the map in Proposition 7.15. We now describe the curves $\boldsymbol{\delta}_{\boldsymbol{\beta}}$, $\boldsymbol{\epsilon}$, and $\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ and the surface Σ_0 more explicitly. For each contact 1–handle, there is a corresponding 0–handle of the surface Σ_0. For each contact 2–handle, we attach a 1–handle to Σ_0, along the interior edges E_{int} of the portion of Σ_0 built when attaching the contact 1–handles. For each contact 2–handle, we also add a curve in $\boldsymbol{\epsilon}$ and a curve in $\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$.

Let $c_i \subseteq \Sigma_0$ be the core of the band of the contact 2–handle associated to the arc λ_i on $R_{\boldsymbol{\beta}}$. We extend the curves c_i across the bands of the 1–handles so that they have both ends on E_{ext}; see the top left and top right of Figure 43. If we isotope each c_i near E_{ext} so that its ends are in $\mathcal{I}_0$ and then double it across $\mathcal{I}_0$ onto $R_{\boldsymbol{\beta}} \subseteq -M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$, in a sense specified in (7-6), then we get the $\boldsymbol{\epsilon}$ curves. If we isotope each c_i near E_{ext} in the opposite direction until its ends are in $\bar{\mathcal{I}}_0$, and then we double it across $\bar{\mathcal{I}}_0$ onto $\Sigma \subseteq -M_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, in a sense specified in (7-7), then we get the $\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ curves. Examples of the curves $\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ and $\boldsymbol{\epsilon}$ on $\bar{R}_{\boldsymbol{\beta}}$ are shown in Figure 43.

Let $\mathcal{I}_1$ denote the collection of subarcs of $\partial R_{\boldsymbol{\beta}}$ that are used to connect $R_{\boldsymbol{\beta}}$ to $\bar{\Sigma}$ in the weakly conjugated diagram $C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})$. Since the curves $\boldsymbol{\epsilon}$ were added by the contact 2–handles, and the 2–handles have attaching circles constructed using the basis of arcs $\lambda_1, \dots, \lambda_n$ for $(R_{\boldsymbol{\beta}}, \mathcal{I}_0)$, the arcs $\boldsymbol{\epsilon} \cap R_{\boldsymbol{\beta}}$ cut $R_{\boldsymbol{\beta}}$ into a collection of disks, each of which contains a single component of $\mathcal{I}_1$. Similarly, since $\boldsymbol{\epsilon} \cap \Sigma_0$ are the cores of the 1–handles used to build Σ_0, the arcs $\boldsymbol{\epsilon} \cap \Sigma_0$ cut Σ_0 into a collection of disks, each of which contains exactly one arc of $\bar{\mathcal{I}}_0$ along its boundary. Consequently, there is an orientation-reversing diffeomorphism

$$\phi\colon \Sigma_0 \to R_{\boldsymbol{\beta}}, \tag{7-5}$$

specified up to isotopy by the property that

$$\phi(\boldsymbol{\epsilon} \cap \Sigma_0) = \boldsymbol{\epsilon} \cap R_{\boldsymbol{\beta}} \tag{7-6}$$

and also that $\mathcal{I}_0 \subseteq \Sigma_0$ is mapped to $\mathcal{I}_0 \subseteq R_{\boldsymbol{\beta}}$, and $\bar{\mathcal{I}}_0 \subseteq \Sigma_0$ is mapped to $\mathcal{I}_1 \subseteq R_{\boldsymbol{\beta}}$.

Since we have freedom to assume that the curves $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ are constructed with any convenient basis of $H_1(R_{\boldsymbol{\beta}}, \mathcal{I}_1)$, we can assume that they are constructed using the images of the

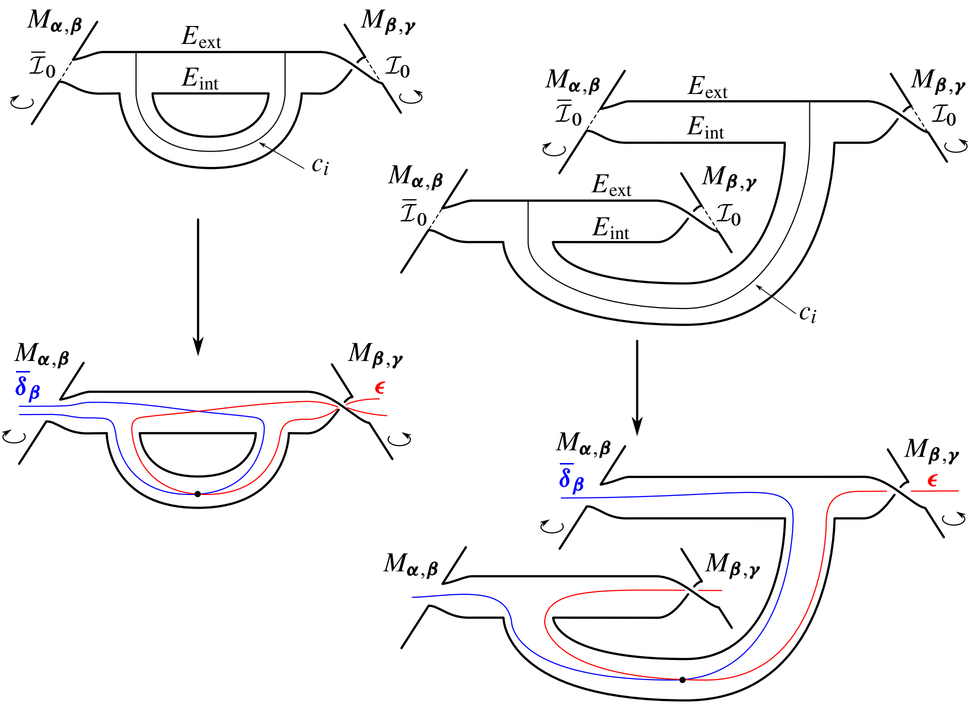

Figure 43: The effect on the Heegaard diagram of attaching a single contact 2–handle in our decomposition of $Z_{\alpha,\beta,\gamma}$. The regions shown consist of a band for a contact 2–handle with both ends attached to a single contact 1–handle (left) or a pair of contact 1–handles (right). Viewing the band from the contact 2–handle as a 1–handle added to the Heegaard surface, the core c_i is shown in the top row. Isotoping the ends of c_i in a neighborhood of $E_{\text{ext}} \cup \mathcal{I}_0$ so they lie in $\mathcal{I}_0$, and then doubling across $\mathcal{I}_0$, we get ϵ. Isotoping the ends of c_i in a neighborhood of $E_{\text{ext}} \cup \overline{\mathcal{I}}_0$ so they lie in $\overline{\mathcal{I}}_0$, we get $\overline{\delta}_\beta$. The orientations of the Heegaard surfaces for $M_{\alpha,\beta}$ and $M_{\beta,\gamma}$ are shown.

arcs $\overline{\delta}_\beta \cap \Sigma_0$ under ϕ. Consequently, we assume

$$\phi(\overline{\delta}_\beta \cap \Sigma_0) = \delta_\beta \cap R_\beta. \tag{7-7}$$

In light of (7-5), we will henceforth write $\overline{R}_\beta$ for Σ_0.

The next step to understanding the gluing map is to destabilize the region $\overline{R}_\beta \natural R_\beta$. Unfortunately, the curves $\overline{\delta}_\beta$ and ϵ are not suitable for this (even if we use the compound destabilization operation from Section 2.2). In order to present the curves in a reasonable manner, we need to do some handle slides amongst the δ_β, $\overline{\delta}_\beta$, and ϵ curves, as we now describe.

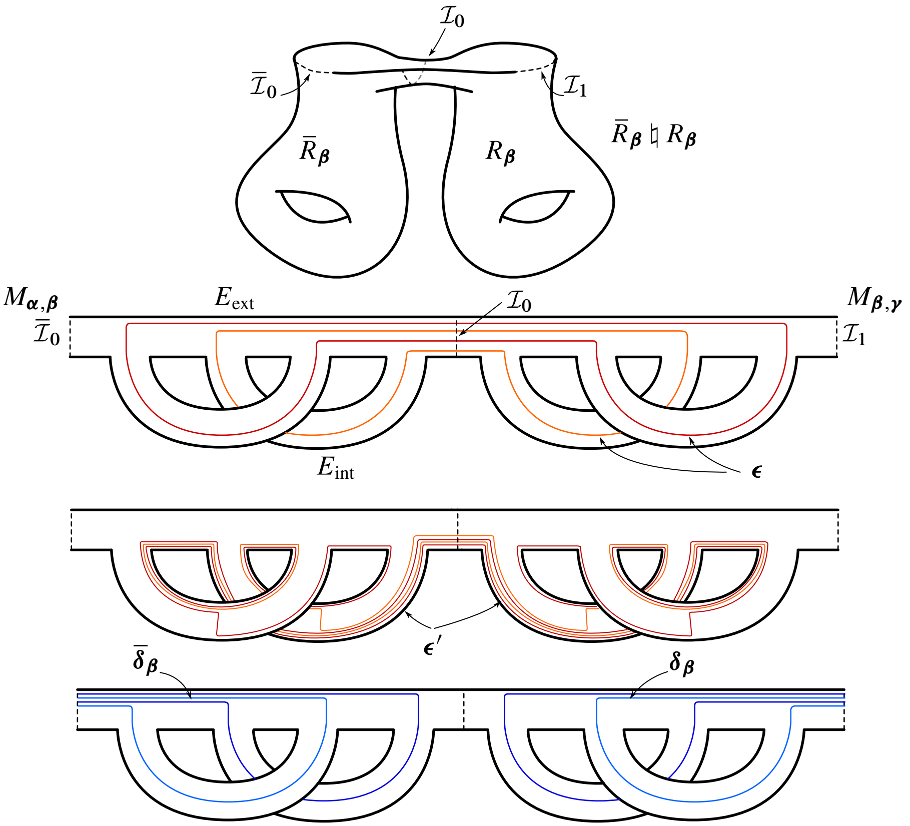

Figure 44: The portion of the Heegaard surface identified with $\overline{R}_{\beta} \natural R_{\beta}$. In this case, R_{β} is a genus 1 surface with one boundary component. The curves ϵ are shown in the second row. The curves ϵ' are shown in the third. The curves δ_{β} and $\overline{\delta}_{\beta}$ are shown in the last row.

We perform two moves. First, we modify the ϵ curves, as follows. Recall that we obtained $\overline{\delta}_{\beta}$ and ϵ by isotoping the cores of the 2–dimensional 1–handles near the E_{ext} boundary arcs of the contact 1–handles. Let us write c_i^* for the cocore of the handle with core c_i. By construction, $|c_i^* \cap c_j| = \delta_{ij}$.

Since the bands associated to the contact 2–handles are all attached to E_{int}, we can subdivide each component of $\partial \overline{R}_{\beta}$ into four subarcs, which we label as $\mathcal{I}_0$, E_{ext}, $\overline{\mathcal{I}}_0$, or E_{int}, in a way which is compatible with the analogous designation of the boundaries of the contact 1–handle bands.

We now isotope each c_i^* in a neighborhood of $E_{\text{int}} \cup \mathcal{I}_0$ such that its ends lie in $\mathcal{I}_0$. Let ϵ' denote the closed curves obtained by doubling the resulting curves across $\mathcal{I}_0$

onto $R_{\boldsymbol{\beta}}$. Note that we perform this isotopy after *all* the 2–handle bands have been added (not after just the corresponding 2–handle has been attached). An example is shown in Figure 44.

Since

$$|(\boldsymbol{\delta}_{\boldsymbol{\beta}})_i \cap \boldsymbol{\epsilon}'_j| = |(\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}})_i \cap \boldsymbol{\epsilon}'_j| = \delta_{ij},$$

we can handle slide $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ over $\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ along $\boldsymbol{\epsilon}'$ in such a way that the resulting curves $\widehat{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ do not intersect the $\boldsymbol{\epsilon}'$ curves. With this configuration, we note that $\boldsymbol{\epsilon}'$ intersects only $\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$, and, furthermore, the two sets of curves come in pairs. A sequence of destabilizations can then be used to surger out the $\boldsymbol{\epsilon}'$ curves, while removing the $\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ curves. Once we do this, we are left with the diagram $A(\mathcal{T})$ of $M_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cup_{R_{\boldsymbol{\beta}}} M_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ described in Section 7.8.

Note that $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are both obtained by picking a set of arcs which form a basis of $H_1(\overline{R}_{\boldsymbol{\beta}}, \mathcal{I}_0)$ and then doubling them onto $\overline{R}_{\boldsymbol{\beta}} \natural R_{\boldsymbol{\beta}}$. By Lemma 7.6, two such collections of arcs can be connected by a sequence of allowable arc slides (Definition 7.5). An allowable arc slide with respect to $\mathcal{I}_0$ induces a handle slide of the corresponding curves obtained by doubling across $\mathcal{I}_0$. Consequently, it follows that $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ can be connected by a sequence of handle slides on $\overline{R}_{\boldsymbol{\beta}} \natural R_{\boldsymbol{\beta}}$, though the particular sequence of handle slides is not of importance for us.

We now describe the effect of these Heegaard moves on the sutured Floer complexes and relate them to the desired triangle map formula. We write

$$C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}} \colon \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \otimes \mathrm{CF}(C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})) \to \mathrm{CF}(G(\mathcal{T}))$$

for the contact gluing map, obtained by adding the surface $\overline{R}_{\boldsymbol{\beta}}$ to $\Sigma \sqcup (R_{\boldsymbol{\beta}} \natural \overline{\Sigma} \natural R_{\boldsymbol{\gamma}})$ to get $\Sigma(\mathcal{T}) = \Sigma \natural \overline{R}_{\boldsymbol{\beta}} \natural R_{\boldsymbol{\beta}} \natural \overline{\Sigma} \natural R_{\boldsymbol{\gamma}}$ and then adding in the curves $\boldsymbol{\epsilon}$ and $\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$. If $\boldsymbol{x} \in \mathrm{CF}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ and $\boldsymbol{y} \in \mathrm{CF}(C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}))$, the map $C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}}$ is defined by the formula $\boldsymbol{x} \times \boldsymbol{y} \mapsto \boldsymbol{x} \times \boldsymbol{c} \times \boldsymbol{y}$, where $\boldsymbol{c}$ is the distinguished intersection point of $\mathbb{T}_{\boldsymbol{\epsilon}} \cap \mathbb{T}_{\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}}$.

On the other hand, we can precompose the map $C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}}$ with a transition map for a small isotopy of the curves $\boldsymbol{\beta}$ on $\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and of the curves $\overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}}$ on $C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})$. By a small abuse of notation, we also write $\boldsymbol{\beta}$ and $\overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}}$ for the translates. Hence

$$\Phi_{-\xi}(\boldsymbol{x} \times \boldsymbol{y}) := C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}} \circ (\Psi_{\boldsymbol{\alpha}}^{\boldsymbol{\beta} \to \boldsymbol{\beta}}(\boldsymbol{x}) \otimes \Psi^{\overline{\boldsymbol{\beta}} \cup \boldsymbol{\delta}_{\boldsymbol{\beta}}}_{\overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}} \to \overline{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}}}(\boldsymbol{y})). \tag{7-8}$$

The transition maps in (7-8) can be computed using holomorphic triangle maps. We note that, even though we are doing a move of the curves $\boldsymbol{\beta}$ on $\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and a move of the curves $\bar{\boldsymbol{\gamma}} \cup \boldsymbol{\delta}_{\boldsymbol{\gamma}}$ on $C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})$, since the two diagrams are disjoint, we can compute the holomorphic triangles simultaneously to arrive at the formula

$$\text{(7-9)}\quad \Phi_{-\xi}(\mathbf{x}\times\mathbf{y}) = C^{\boldsymbol{\epsilon},\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}} \circ F_{\boldsymbol{\alpha}\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\boldsymbol{\beta}\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\boldsymbol{\beta}\cup\bar{\boldsymbol{\beta}}\cup\boldsymbol{\delta}_{\boldsymbol{\beta}}}(\mathbf{x}\times\Theta_{\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}}}, \mathbf{y}\times\Theta_{\boldsymbol{\beta},\boldsymbol{\beta}}).$$

Using the local computation shown in Figure 19, we can commute the map $C^{\boldsymbol{\epsilon},\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}}$ to the right of the triangle map. When we do this, the top-graded generators $\Theta_{\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}}}$ and $\Theta_{\boldsymbol{\beta},\boldsymbol{\beta}}$ can be rewritten as the images of sequences of 1–handle maps. Hence, we can write the right-hand side of (7-9) as

$$\text{(7-10)}\quad F_{\boldsymbol{\alpha}\cup\boldsymbol{\epsilon}\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\boldsymbol{\beta}\cup\boldsymbol{\epsilon}\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\boldsymbol{\beta}\cup\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}\cup\bar{\boldsymbol{\beta}}\cup\boldsymbol{\delta}_{\boldsymbol{\beta}}}\big(F_1^{\boldsymbol{\epsilon}\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}},\boldsymbol{\epsilon}\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}}}(\mathbf{x}), F_1^{\boldsymbol{\beta},\boldsymbol{\beta}}(C^{\boldsymbol{\epsilon},\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}}(\mathbf{y}))\big).$$

Replacing the right-hand side of (7-9) with (7-10), we streamline the notation by writing

$$\text{(7-11)}\qquad \Phi_{-\xi} = F_{\boldsymbol{\alpha}(\mathcal{T}),\boldsymbol{\beta}\cup\boldsymbol{\gamma}',\boldsymbol{\beta}(\mathcal{T})} \circ (F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'} \otimes (F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})),$$

where

$$\boldsymbol{\alpha}(\mathcal{T}) := \boldsymbol{\alpha}\cup\boldsymbol{\epsilon}\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}}, \quad \boldsymbol{\beta}(\mathcal{T}) := \boldsymbol{\beta}\cup\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}\cup\bar{\boldsymbol{\beta}}\cup\boldsymbol{\delta}_{\boldsymbol{\beta}}, \quad \text{and} \quad \boldsymbol{\gamma}' := \boldsymbol{\epsilon}\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}}.$$

In the above expression, the contact handle map $C^{\boldsymbol{\epsilon},\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}}$ is defined by adding the surface $R_{\boldsymbol{\beta}} \natural \bar{R}_{\boldsymbol{\beta}}$ to $R_{\boldsymbol{\beta}} \natural \bar{\Sigma} \natural R_{\boldsymbol{\gamma}}$ to get $R_{\boldsymbol{\beta}} \natural \bar{R}_{\boldsymbol{\beta}} \natural R_{\boldsymbol{\beta}} \natural \bar{\Sigma} \natural R_{\boldsymbol{\gamma}}$ and then adding in the curves $\boldsymbol{\epsilon}$ and $\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$. On the level of complexes, it is defined by the formula $\mathbf{y} \mapsto \mathbf{c}\times\mathbf{y}$, where $\mathbf{c}$ is the distinguished intersection point of $\mathbb{T}_{\boldsymbol{\epsilon}} \cap \mathbb{T}_{\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}}$. The copy of $R_{\boldsymbol{\beta}}$ that is added should be thought of as Σ surgered along the $\boldsymbol{\beta}$ curves. Note that $C^{\boldsymbol{\epsilon},\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}}$ can be described by attaching contact 0–handles and 1–handles to add $R_{\boldsymbol{\beta}}$ and then attaching contact 2–handles to add $\bar{R}_{\boldsymbol{\beta}}$ and the curves $\boldsymbol{\epsilon}$ and $\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$.

We now change $\boldsymbol{\epsilon}$ to $\boldsymbol{\epsilon}'$. Define

$$\boldsymbol{\gamma}'' := \boldsymbol{\epsilon}'\cup\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_{\boldsymbol{\gamma}}.$$

Applying associativity to the 4–tuple $(\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'', \boldsymbol{\alpha}(\mathcal{T}), \boldsymbol{\beta}\cup\boldsymbol{\gamma}', \boldsymbol{\beta}(\mathcal{T}))$, we conclude from (7-11) that

$$\begin{aligned}\text{(7-12)}\quad \Psi^{\boldsymbol{\beta}(\mathcal{T})}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ \Phi_{-\xi} &\simeq \Psi^{\boldsymbol{\beta}(\mathcal{T})}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ F_{\boldsymbol{\alpha}(\mathcal{T}),\boldsymbol{\beta}\cup\boldsymbol{\gamma}',\boldsymbol{\beta}(\mathcal{T})} \circ (F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'} \otimes (F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})) \\ &\simeq F_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}\cup\boldsymbol{\gamma}',\boldsymbol{\beta}(\mathcal{T})} \circ ((\Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'}) \otimes (F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\bar{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})).\end{aligned}$$

Applying naturality of sutured Floer homology as well as associativity for the 4–tuple $(\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'', \boldsymbol{\beta}\cup\boldsymbol{\gamma}', \boldsymbol{\beta}\cup\boldsymbol{\gamma}'', \boldsymbol{\beta}(\mathcal{T}))$ to (7-12), we conclude that

$$
\begin{aligned}
(7\text{-}13)\quad & F_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}\cup\boldsymbol{\gamma}',\boldsymbol{\beta}(\mathcal{T})} \circ ((\Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'}) \otimes (F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})) \\
&\simeq F_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}\cup\boldsymbol{\gamma}',\boldsymbol{\beta}(\mathcal{T})} \circ ((\mathrm{id}\circ\Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'}) \otimes (F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})) \\
&\simeq F_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}\cup\boldsymbol{\gamma}',\boldsymbol{\beta}(\mathcal{T})} \\
&\qquad\qquad \circ \Big(((\Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}''\to\boldsymbol{\beta}\cup\boldsymbol{\gamma}'}_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ \Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'\to\boldsymbol{\beta}\cup\boldsymbol{\gamma}''}_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''}) \circ \Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'}) \\
&\qquad\qquad\qquad \otimes (F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})\Big) \\
&\simeq F_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}(\mathcal{T})} \\
&\qquad\qquad \circ ((\Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'\to\boldsymbol{\beta}\cup\boldsymbol{\gamma}''}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'}) \otimes (\Psi^{\boldsymbol{\beta}(\mathcal{T})}_{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'\to\boldsymbol{\beta}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})).
\end{aligned}
$$

Combining (7-11), (7-12), and (7-13), we conclude

$$
\begin{aligned}
&\Psi^{\boldsymbol{\beta}(\mathcal{T})}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ \Phi_{-\xi} \\
&\quad \simeq F_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}(\mathcal{T})} \circ ((\Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'\to\boldsymbol{\beta}\cup\boldsymbol{\gamma}''}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'}) \otimes (\Psi^{\boldsymbol{\beta}(\mathcal{T})}_{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'\to\boldsymbol{\beta}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})).
\end{aligned}
$$

We claim that

$$
(7\text{-}14)\qquad \Psi^{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'\to\boldsymbol{\beta}\cup\boldsymbol{\gamma}''}_{\boldsymbol{\alpha}(\mathcal{T})\to\boldsymbol{\alpha}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'} \simeq F_1^{\boldsymbol{\gamma}'',\boldsymbol{\gamma}''}.
$$

To establish (7-14), first note that the domain of both maps is $\mathrm{SFH}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$. The 1–handle maps $F_1^{\boldsymbol{\gamma}',\boldsymbol{\gamma}'}$ and $F_1^{\boldsymbol{\gamma}'',\boldsymbol{\gamma}''}$ are defined by taking the boundary connected sum of Σ with $\Sigma' := \overline{R}_{\boldsymbol{\beta}} \natural R_{\boldsymbol{\beta}} \natural \overline{\Sigma} \natural R_{\boldsymbol{\gamma}}$. The boundary connected sum operation yields $|\partial\Sigma|$ arcs on $\Sigma(\mathcal{T}) = \Sigma \natural \Sigma'$ that separate Σ from Σ' and intersect none of the attaching curves in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}'$, or $\boldsymbol{\gamma}''$. Hence, the change of diagrams map appearing on the left-hand side of (7-14) involves only counting holomorphic triangles with image on the disjoint union of Σ and Σ'. Since there is a unique top-graded generator of $\mathrm{SFH}(\Sigma',\boldsymbol{\gamma}',\boldsymbol{\gamma}')$, and $\boldsymbol{\gamma}''$ is obtained from $\boldsymbol{\gamma}'$ by a sequence of handle slides, that generator will be preserved by the change of diagrams map from $\mathrm{SFH}(\Sigma',\boldsymbol{\gamma}',\boldsymbol{\gamma}')$ to $\mathrm{SFH}(\Sigma',\boldsymbol{\gamma}'',\boldsymbol{\gamma}'')$. Hence, the 1–handle map will be preserved. Equation (7-14) now follows.

We obtain that the gluing map is chain homotopic to

$$
F_{\boldsymbol{\alpha}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}\cup\boldsymbol{\gamma}'',\boldsymbol{\beta}(\mathcal{T})} \circ (F_1^{\boldsymbol{\gamma}'',\boldsymbol{\gamma}''} \otimes (\Psi^{\boldsymbol{\beta}(\mathcal{T})}_{\boldsymbol{\beta}\cup\boldsymbol{\gamma}'\to\boldsymbol{\beta}\cup\boldsymbol{\gamma}''} \circ F_1^{\boldsymbol{\beta},\boldsymbol{\beta}} \circ C^{\boldsymbol{\epsilon},\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}})).
$$

Note that, on the diagram $(\Sigma(\mathcal{T}), \boldsymbol{\beta}\cup\boldsymbol{\gamma}'', \boldsymbol{\beta}(\mathcal{T}))$, the $\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ curves each have only one intersection point, which occurs with an $\boldsymbol{\epsilon}'$ curve. The $\boldsymbol{\epsilon}'$ curves still intersect both $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ and $\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$. Further, each $\boldsymbol{\epsilon}'$ curve intersects exactly one $\overline{\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ curve. Hence, we can consider the compound stabilization map $\sigma^{\epsilon',\bar{\delta}_\beta}$ defined in Section 2.2. It agrees with the map from naturality by Proposition 2.2.

We claim that

$$\Psi^{\beta(\mathcal{T})}_{\beta\cup\gamma'\to\beta\cup\gamma''}\circ F_1^{\beta,\beta}\circ C^{\epsilon,\bar{\delta}_\beta} \simeq F_1^{\beta,\beta}\circ\sigma^{\epsilon',\bar{\delta}_\beta}\colon \mathrm{SFH}(C(\mathcal{H}_{\beta,\gamma}))\to\mathrm{SFH}(\Sigma(\mathcal{T}),\beta\cup\gamma'',\beta(\mathcal{T})).$$

By Proposition 2.1, it suffices to show the claim with the β curves surgered out and with no 1–handle maps (note that it is easy to show that on Σ, there is a path from each β curve to $\partial\Sigma$ that avoids $\bar{\delta}_\beta\cap\Sigma$, so the hypotheses of the previously mentioned proposition are satisfied). Therefore, it suffices to show that

$$\sigma^{\epsilon',\bar{\delta}_\beta}\simeq\Psi^{\bar{\delta}_\beta\cup\bar{\beta}\cup\delta_\beta}_{\gamma'\to\gamma''}\circ C^{\epsilon,\bar{\delta}_\beta},$$

or, equivalently, that

$$\mathrm{id}\simeq(\sigma^{\epsilon',\bar{\delta}_\beta})^{-1}\circ\Psi^{\bar{\delta}_\beta\cup\bar{\beta}\cup\delta_\beta}_{\gamma'\to\gamma''}\circ C^{\epsilon,\bar{\delta}_\beta}.$$

However, this holds because of the functoriality of the gluing map, ie because the right-hand side represents the map induced by gluing a trivial a copy of $I\times R'_\beta$ to $-M_{\beta,\gamma}$.

Hence, we conclude that the gluing map $\Phi_{-\xi}$ is chain homotopic to

$$F_{\alpha\cup\gamma'',\beta\cup\gamma'',\beta(\mathcal{T})}\circ(F_1^{\gamma'',\gamma''}\otimes(F_1^{\beta,\beta}\circ\sigma^{\epsilon',\bar{\delta}_\beta})).$$

We still need to handle slide the δ_β over $\bar{\delta}_\beta$ to become $\hat{\delta}_\beta$ and then compound destabilize. That is, the gluing map is chain homotopic to

$$(\sigma^{\epsilon',\bar{\delta}_\beta})^{-1}\circ\Psi^{\beta(\mathcal{T})\to\beta(\mathcal{T})'}_{\alpha\cup\gamma''}\circ F_{\alpha\cup\gamma'',\beta\cup\gamma'',\beta(\mathcal{T})}\circ(F_1^{\gamma'',\gamma''}\otimes(F_1^{\beta,\beta}\circ\sigma^{\epsilon',\bar{\delta}_\beta})),$$

where $\beta(\mathcal{T})'$ is equal to $\beta\cup\bar{\delta}_\beta\cup\bar{\beta}\cup\hat{\delta}_\beta$. By associativity applied to the 4–tuple $(\alpha\cup\gamma'',\beta\cup\gamma'',\beta(\mathcal{T}),\beta(\mathcal{T})')$, this is chain homotopic to

$$(\sigma^{\epsilon',\bar{\delta}_\beta})^{-1}\circ\big(F_{\alpha\cup\gamma'',\beta\cup\gamma'',\beta(\mathcal{T})'}\circ(F_1^{\gamma'',\gamma''}\otimes(\Psi^{\beta(\mathcal{T})\to\beta(\mathcal{T})'}_{\beta\cup\gamma''}\circ F_1^{\beta,\beta}\circ\sigma^{\epsilon',\bar{\delta}_\beta}))\big).$$

By a simple generalization of Lemma 2.3 to deal with multiple compound stabilizations of the Heegaard triple simultaneously, this now becomes

$$F_{D_\gamma(\alpha),D_\gamma(\beta),A(\beta)}\circ\big(F_1^{C(\gamma),C(\gamma)}\otimes((\sigma^{\epsilon',\bar{\delta}_\beta})^{-1}\circ\Psi^{\beta(\mathcal{T})\to\beta(\mathcal{T})'}_{\beta\cup\gamma''}\circ F_1^{\beta,\beta}\circ\sigma^{\epsilon',\bar{\delta}_\beta})\big),$$

where $C(\gamma)=\bar{\gamma}\cup\delta_\gamma$. Applying the triangle counts from Proposition 2.1 to the map $F_1^{\beta,\beta}$ for 1–handles added near the boundary and also commuting the map $F_1^{\beta,\beta}$

with the destabilization map, this becomes

$$F_{D_{\gamma}(\alpha),D_{\gamma}(\beta),A(\beta)} \circ \big(F_1^{C(\gamma),C(\gamma)} \otimes (F_1^{\beta,\beta} \circ (\sigma^{\epsilon',\bar{\delta}_\beta})^{-1} \circ \Psi_{\gamma''}^{\beta(\mathcal{T})\setminus\beta\to\beta(\mathcal{T})'\setminus\beta} \circ \sigma^{\epsilon',\bar{\delta}_\beta})\big).$$

We claim that

$$(\sigma^{\epsilon',\bar{\delta}_\beta})^{-1} \circ \Psi_{\gamma''}^{\beta(\mathcal{T})\setminus\beta\to\beta(\mathcal{T})'\setminus\beta} \circ \sigma^{\epsilon',\bar{\delta}_\beta} \simeq \mathrm{id},$$

which follows simply from naturality, as it is a loop in the space of Heegaard diagrams. We have now arrived at our desired formula, concluding the proof of Proposition 7.15. □

7.10 Computation of the triangle cobordism map

We now prove that the sutured cobordism map for $\mathcal{W}_{\alpha,\beta,\gamma}$ is chain homotopic to the map that counts holomorphic triangles on a single Heegaard triple, by using the formula from the previous section for the contact gluing map.

Theorem 7.16 *If $(\Sigma,\alpha,\beta,\gamma)$ is a sutured Heegaard triple, then the sutured cobordism map*

$$F_{\mathcal{W}_{\alpha,\beta,\gamma}} \colon \mathrm{CF}(\Sigma,\alpha,\beta) \otimes \mathrm{CF}(\Sigma,\beta,\gamma) \to \mathrm{CF}(\Sigma,\alpha,\gamma)$$

is chain homotopic to the map that counts holomorphic triangles on the triple $(\Sigma,\alpha,\beta,\gamma)$.

Proof We first compose with the change of diagrams map $\mathrm{id} \otimes \Psi_{\mathcal{H}_{\beta,\gamma}\to C(\mathcal{H}_{\beta,\gamma})}$. The next step is to use the gluing map to glue the two copies of R_β together. Then we performs surgery on a $|\beta|$–component framed link. See Section 7.8 for a description of the triple diagram

$$(D_\gamma(\Sigma), D_\gamma(\alpha), A(\beta), \mathbf{\Delta}_\Sigma)$$

used for computing the 2–handle cobordism map. This yields (omitting writing the first change of diagrams map)

$$F_{D_\gamma(\alpha),A(\beta),\mathbf{\Delta}_\Sigma} \circ \big((F_{D_\gamma(\alpha),D_\gamma(\beta),A(\beta)} \circ (F_1^{\bar{\gamma}\cup\delta_\gamma,\bar{\gamma}\cup\delta_\gamma} \otimes F_1^{\beta,\beta})) \otimes \Theta_{A(\beta),\mathbf{\Delta}_\Sigma}\big).$$

We now use associativity applied to the 4–tuple $(D_\gamma(\alpha), D_\gamma(\beta), A(\beta), \Delta_\Sigma)$ to see that this is chain homotopic to

$$F_{D_\gamma(\alpha),D_\gamma(\beta),\mathbf{\Delta}_\Sigma} \circ \big(F_1^{\bar{\gamma}\cup\delta_\gamma,\bar{\gamma}\cup\delta_\gamma} \otimes (F_{D_\gamma(\beta),A(\beta),\mathbf{\Delta}_\Sigma} \circ (F_1^{\beta,\beta} \otimes \Theta_{A(\beta),\mathbf{\Delta}_\Sigma}))\big).$$

Note that

$$\Psi := F_{D_\gamma(\beta),A(\beta),\mathbf{\Delta}_\Sigma} \circ (F_1^{\beta,\beta} \otimes \Theta_{A(\beta),\mathbf{\Delta}_\Sigma})$$

is the change of diagrams map $\Psi_{C(\mathcal{H}_{\beta,\gamma})\to D_\gamma(\mathcal{H}_{\beta,\gamma})}$, from a weakly conjugated diagram to a doubled diagram, by Lemma 7.12. Thus, the cobordism map is

$$F_{D_{\gamma}(\boldsymbol{\alpha}),D_{\gamma}(\boldsymbol{\beta}),\boldsymbol{\Delta}_\Sigma}\circ(F_1^{\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma,\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma}\otimes\Psi). \tag{7-15}$$

The range of this map is not $\mathrm{SFH}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$ but rather a double of this diagram along $R_{\boldsymbol{\gamma}}$, so we must compose with the change of diagrams map from $D_{\boldsymbol{\gamma}}(\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\gamma}})$ to $\mathcal{H}_{\boldsymbol{\alpha},\boldsymbol{\gamma}}$, which is

$$F_3^{\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma,\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma}\circ(F_{D_\gamma(\boldsymbol{\alpha}),\boldsymbol{\Delta}_\Sigma,D_\gamma(\boldsymbol{\gamma})}\circ(-\otimes\Theta_{\boldsymbol{\Delta}_\Sigma,D_\gamma(\boldsymbol{\gamma})})),$$

by Lemma 7.9. By postcomposing (7-15) with this expression, and applying associativity of the 4–tuple $(D_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}), D_{\boldsymbol{\gamma}}(\boldsymbol{\beta}), \Delta_\Sigma, D_{\boldsymbol{\gamma}}(\boldsymbol{\gamma}))$, we conclude that the composition is chain homotopic to

$$F_3^{\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma,\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma} \circ\big(F_{D_\gamma(\boldsymbol{\alpha}),D_\gamma(\boldsymbol{\beta}),D_\gamma(\boldsymbol{\gamma})}\circ\big(F_1^{\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma,\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma}\otimes\big(F_{D_\gamma(\boldsymbol{\beta}),\boldsymbol{\Delta}_\Sigma,D_\gamma(\boldsymbol{\gamma})}\circ(\Psi\otimes\Theta_{\boldsymbol{\Delta}_\Sigma,D_\gamma(\boldsymbol{\gamma})})\big)\big)\big).$$

Using the 3–handle and triangle map computation from Proposition 2.1 (note that it is an easy exercise to verify that the hypotheses of that proposition are satisfied), this is equal to

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}\circ\big(-\otimes F_3^{\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma,\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma}\circ\big(F_{D_\gamma(\boldsymbol{\beta}),\boldsymbol{\Delta}_\Sigma,D_\gamma(\boldsymbol{\gamma})}\circ(\Psi\otimes\Theta_{\boldsymbol{\Delta}_\Sigma,D_\gamma(\boldsymbol{\gamma})})\big)\big).$$

We note that

$$F_3^{\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma,\bar{\boldsymbol{\gamma}}\cup\boldsymbol{\delta}_\gamma}\circ\big(F_{D_\gamma(\boldsymbol{\beta}),\boldsymbol{\Delta}_\Sigma,D_\gamma(\boldsymbol{\gamma})}\circ(\Psi\otimes\Theta_{\boldsymbol{\Delta}_\Sigma,D_\gamma(\boldsymbol{\gamma})})\big)$$

is just a change of diagrams map. Writing Ψ' for the compositions of all the three change of diagrams maps (the initial one from $\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ to a weakly conjugated diagram $C(\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}})$, which we have been omitting writing, and then the last two, going back to $\mathcal{H}_{\boldsymbol{\beta},\boldsymbol{\gamma}}$ through a doubled diagram), the composition is thus equal to

$$F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}(-\otimes\Psi')\simeq F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}(-,-),$$

since $\Psi'\simeq\mathrm{id}_{(\Sigma,\boldsymbol{\beta},\boldsymbol{\gamma})}$ by naturality. □

8 Trace and cotrace cobordisms

Let $\xi_{I\times\partial M}$ denote the I–invariant contact structure on $-I\times\partial M$ that induces the dividing set γ on ∂M. In this section, we consider the trace cobordism

$$\Lambda_{(M,\gamma)}=(I\times M,\ -I\times\partial M,\ [\xi_{I\times\partial M}])$$

from $(M,\gamma)\sqcup(-M,\gamma)$ to $\varnothing$, and the cotrace cobordism

$$V_{(M,\gamma)} = (I\times M,\ -I\times\partial M,\ [\xi_{I\times\partial M}])$$

from $\varnothing$ to $(-M,\gamma)\sqcup(M,\gamma)$. The main result of this section is the following:

Theorem 8.1 *The trace cobordism $\Lambda_{(M,\gamma)}\colon (M,\gamma)\sqcup(-M,\gamma)\to\varnothing$ induces the canonical trace map*

$$\mathrm{tr}\colon \mathrm{SFH}(M,\gamma)\otimes\mathrm{SFH}(-M,\gamma)\to\mathbb{F}_2.$$

The cotrace cobordism $V_{(M,\gamma)}\colon \varnothing\to(-M,\gamma)\sqcup(M,\gamma)$ induces the canonical cotrace map

$$\mathrm{cotr}\colon \mathbb{F}_2\to\mathrm{SFH}(-M,\gamma)\otimes\mathrm{SFH}(M,\gamma).$$

To prove Theorem 8.1, we first need a convenient topological description of the trace cobordism. Suppose that $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ is a diagram for (M,γ). We will write $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}},\gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}})$ for the sutured manifold constructed from the diagram $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\alpha})$, and we note that $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}},\gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}})$ can be obtained by surgering $(I\times R_{\boldsymbol{\alpha}}, I\times\partial R_{\boldsymbol{\alpha}})$ along k 0–spheres. We consider the triangular sutured manifold cobordism

$$\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}} = (W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}, Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}, [\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}])$$

from $(M,\gamma)\sqcup(-M,\gamma)$ to $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}},\gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}})$, defined in Section 7.

There is also a cobordism $\mathcal{W}_{\boldsymbol{\alpha}} = (W_{\boldsymbol{\alpha}}, Z_{\boldsymbol{\alpha}}, [\xi_{\boldsymbol{\alpha}}])$ from $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}},\gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}})$ to $\varnothing$. The 4–manifold $W_{\boldsymbol{\alpha}}$ is defined as

$$W_{\boldsymbol{\alpha}} = (P_1\times\Sigma)\cup_{e_{\boldsymbol{\alpha}}\times\Sigma}(e_{\boldsymbol{\alpha}}\times U_{\boldsymbol{\alpha}}).$$

Here P_1 denotes a monogon, viewed as having a single boundary edge $e_{\boldsymbol{\alpha}}$.

Lemma 8.2 *The cobordism $\Lambda_{(M,\gamma)}$ is equivalent to the composition $\mathcal{W}_{\boldsymbol{\alpha}}\circ\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}$.*

Proof It follows from [15, Proposition 6.6] that $W_{\boldsymbol{\alpha}}\cup W_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}$ is diffeomorphic to $I\times M_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and, furthermore, that $Z_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}\cup Z_{\boldsymbol{\alpha}}$ is diffeomorphic to $-I\times\partial M$. The fact that $\xi_{\boldsymbol{\alpha}}$ and $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}$ glue up to give $\xi_{I\times\partial M}$ can be proven by adapting Lemma 7.3. Schematically, the decomposition of $\Lambda_{(M,\gamma)}$ into $\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}$ and $\mathcal{W}_{\boldsymbol{\alpha}}$ is shown in Figure 45. □

Let $\boldsymbol{\alpha}'$ denote be a small Hamiltonian translate of $\boldsymbol{\alpha}$. Note that, of course, $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}})$ and $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}'})$ are homeomorphic. It follows from Theorem 7.1 and the composition law that the map

$$F_{\Lambda_{(M,\gamma)}}\colon \mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}) \otimes \mathrm{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha}) \to \mathbb{F}_2$$

is equal to the composition

$$F_{\mathcal{W}_{\boldsymbol{\alpha}}} \circ F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}'} \circ (\mathrm{id}_{\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})} \otimes \Psi_{(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})\to(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha}')}), \tag{8-1}$$

where $F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}'}$ is the map that counts holomorphic triangles on $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}')$.

It remains to compute the cobordism map for $\mathcal{W}_{\boldsymbol{\alpha}}$. Note that, on $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\alpha}')$, there is a canonical bottom-graded intersection point $\Theta^-_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$.

Proposition 8.3 *The cobordism $\mathcal{W}_{\boldsymbol{\alpha}}$ from $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}'})$ to $\varnothing$ induces the map*

$$\boldsymbol{x} \mapsto \begin{cases} 1 & \text{if } \boldsymbol{x} = \Theta^-_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}, \\ 0 & \text{otherwise.} \end{cases}$$

Before we prove the above result, we need to find a convenient Morse function for the cobordism $\mathcal{W}_{\boldsymbol{\alpha}}$. As a first step, we prove the following:

Lemma 8.4 *Let $U_{\boldsymbol{\alpha}}$ be the sutured compression body that is formed by attaching 3–dimensional 2–handles to $I \times \Sigma$ along the curves $\{0\} \times \boldsymbol{\alpha}$. After rounding corners, we can view $U_{\boldsymbol{\alpha}}$ as a (nonsutured) handlebody of genus $|\boldsymbol{\alpha}| - \chi(R_{\boldsymbol{\alpha}}) + 1$ and boundary*

$$\partial U_{\boldsymbol{\alpha}} = (\{1\} \times \Sigma) \cup \bar{R}_{\boldsymbol{\alpha}},$$

where $R_{\boldsymbol{\alpha}}$ is the surface obtained by surgering Σ along the $\boldsymbol{\alpha}$ curves. Furthermore, a (possibly overcomplete) set of compressing disks for $U_{\boldsymbol{\alpha}}$ can be obtained by taking $|\boldsymbol{\alpha}|$ compressing disks D_α with boundary $\{1\} \times \alpha$ for $\alpha \in \boldsymbol{\alpha}$, as well as disks of the form $D_{c_i^*} := I \times c_i^*$ for pairwise disjoint, embedded arcs $c_1^*, \dots, c^*_{b_1(R_{\boldsymbol{\alpha}})}$ in Σ that avoid the $\boldsymbol{\alpha}$ curves and form a basis of $H_1(R_{\boldsymbol{\alpha}}, \partial R_{\boldsymbol{\alpha}})$. These cut $U_{\boldsymbol{\alpha}}$ into $b_0(R_{\boldsymbol{\alpha}})$ 3–balls.

Remark 8.5 A basis of arcs $c_1^*, \dots, c^*_{b_1(R_{\boldsymbol{\alpha}})}$ can be obtained by picking a Morse function $g\colon R_{\boldsymbol{\alpha}} \to I$ (viewing $R_{\boldsymbol{\alpha}}$ as a cobordism from $\varnothing$ to $\partial R_{\boldsymbol{\alpha}}$) that has $b_0(R_{\boldsymbol{\alpha}})$ local minima and $b_1(R_{\boldsymbol{\alpha}})$ index 1 critical points. The Morse function g determines a handle decomposition of $R_{\boldsymbol{\alpha}}$ with $b_0(R_{\boldsymbol{\alpha}})$ 0–handles (ie disks) and $b_1(R_{\boldsymbol{\alpha}})$ 1–handles. The arcs $c_1^*, \dots, c^*_{b_1(R_{\boldsymbol{\alpha}})}$ can be taken as the cocores of the 1–handles in this decomposition.

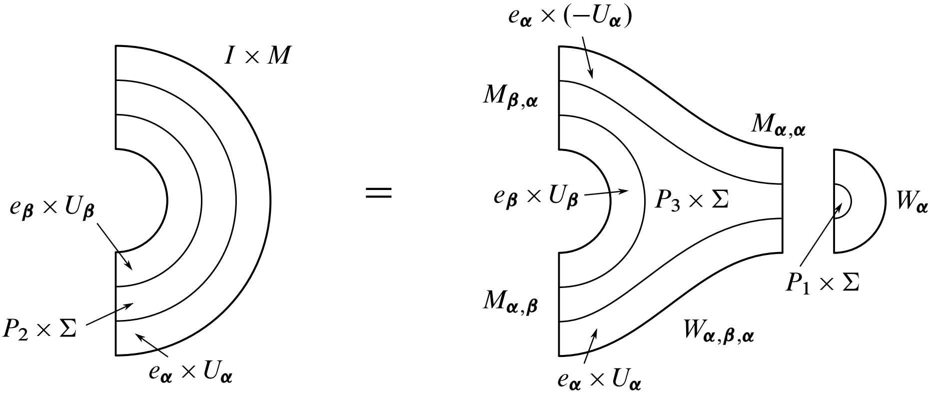

Figure 45: The decomposition $I \times M \cong W_{\boldsymbol\alpha,\boldsymbol\beta,\boldsymbol\alpha} \cup_{M_{\boldsymbol\alpha,\boldsymbol\alpha}} W_{\boldsymbol\alpha}$. It is convenient to view the polygons P_1, P_2, and P_3 as having fattened vertices.

Proof of Lemma 8.4 View $U_{\boldsymbol\alpha}$ as a cobordism from $\overline{\partial U_{\boldsymbol\alpha}} = (\{1\} \times \overline{\Sigma}) \cup R_{\boldsymbol\alpha}$ to the empty set. The $\boldsymbol\alpha$ curves determine 3–dimensional 2–handles in $U_{\boldsymbol\alpha}$. After attaching these 2–handles, the remaining cobordism is homeomorphic to $I \times R_{\boldsymbol\alpha}$ (with corners smoothed), viewed as a cobordism from $R_{\boldsymbol\alpha} \cup_{\partial R_{\boldsymbol\alpha}} \overline{R}_{\boldsymbol\alpha}$ to $\varnothing$. In such a way, we reduce the argument to the case when there are no $\boldsymbol\alpha$ curves, where it is straightforward. □

We need an additional Morse theory argument:

Lemma 8.6 *Suppose that $U_{\boldsymbol\alpha}$ is the sutured compression body induced by the sutured monodiagram $(\Sigma, \boldsymbol\alpha)$. Then $I \times U_{\boldsymbol\alpha}$ can be viewed, after smoothing corners, as a (nonsutured) cobordism from the closed manifold $U_{\boldsymbol\alpha} \cup_{\partial U_{\boldsymbol\alpha}} -U_{\boldsymbol\alpha}$ to $\varnothing$. Furthermore, there is a Morse function $F\colon I \times U_{\boldsymbol\alpha} \to I$ such that*

- $F^{-1}(0) = U_{\boldsymbol\alpha} \cup_{\partial U_{\boldsymbol\alpha}} -U_{\boldsymbol\alpha}$,
- *F has no index 0, 1, or 2 critical points,*
- *F has $|\boldsymbol\alpha| + b_1(R_{\boldsymbol\alpha})$ index 3 critical points, and*
- *F has $b_0(R_{\boldsymbol\alpha})$ index 4 critical points.*

The attaching spheres of the 3–handles are obtained taking the union of the disks D_{α_i} and $D_{c_i^*}$ (defined in Lemma 8.4) *in $U_{\boldsymbol\alpha}$, together with their images in $-U_{\boldsymbol\alpha}$.*

Proof A model for the 4–manifold obtained by rounding the corners of $I \times U_{\boldsymbol\alpha}$ can be taken to be $X := I \times U_{\boldsymbol\alpha}/\sim$, where $(t, x) \sim (t', x)$ if $x \in \partial U_{\boldsymbol\alpha}$. Using Lemma 8.4, we can construct a Morse function $f\colon U_{\boldsymbol\alpha} \to I$ with $f^{-1}(0) = \partial U_{\boldsymbol\alpha}$ such that f has no index 0 or 1 critical points, $|\boldsymbol{\alpha}| + b_1(R_{\boldsymbol{\alpha}})$ index 2 critical points, and $b_0(R_{\boldsymbol{\alpha}})$ index 3 critical points. Furthermore, the descending manifolds of the index 2 critical points in $U_{\boldsymbol{\alpha}}$ are the disks D_{α_i} and $D_{c_i^*}$.

To construct a Morse function on X with the stated critical points, the argument is now a modification of Lemma 6.7. More precisely, we will construct an auxiliary function

$$G\colon (I \times I)/(I \times \{0\}) \to I.$$

We view $(I \times I)/(I \times \{0\})$ as having the same smooth structure at the point $I \times \{0\}$ as the upper half-plane; see Figure 46. Furthermore, we assume that

- $G|_{I\times\{0\}} \equiv G|_{\{0\}\times I} \equiv G|_{\{1\}\times I} \equiv 0$,
- $\partial G/\partial s > 0$ in $(0,1) \times I$,
- $(\partial G/\partial t)(t,s) = 0$ if and only if $(t,s) \in \left\{\frac{1}{2}\right\} \times I$.

An example of such a function G is shown in Figure 46. We then consider the function $F\colon X \to I$ defined by

$$F(t,y) = G(t, f(y)).$$

It is straightforward to verify that F on $I \times U_{\boldsymbol{\alpha}}$ with its corners rounded is Morse and has critical points with attaching spheres as stated. □

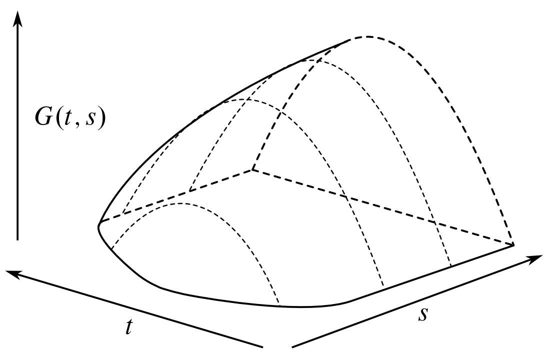

Figure 46: An example of an auxiliary function $G\colon (I \times I)/(I \times \{0\}) \to I$ in Lemma 8.6.

We are now ready to prove Proposition 8.3:

Proof of Proposition 8.3 The sutured cobordism

$$\mathcal{W}_{\boldsymbol{\alpha}} = (W_{\boldsymbol{\alpha}}, Z_{\boldsymbol{\alpha}}, [\xi_{\boldsymbol{\alpha}}])$$

from $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}'})$ to $\varnothing$ is the composition of the boundary cobordism $\mathcal{W}^b_{\boldsymbol{\alpha}}$ obtained by gluing $Z_{\boldsymbol{\alpha}} := I \times R_{\boldsymbol{\alpha}}$ to $\partial M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} \cong R_{\boldsymbol{\alpha}} \cup_{\partial R_{\boldsymbol{\alpha}}} \bar{R}_{\boldsymbol{\alpha}}$ and the special cobordism $\mathcal{W}^s_{\boldsymbol{\alpha}}$ (between closed 3–manifolds) diffeomorphic to $I \times U_{\boldsymbol{\alpha}}$ from $U_{\boldsymbol{\alpha}} \cup -U_{\boldsymbol{\alpha}}$ to $\varnothing$. We use Lemma 8.6 to give a handle decomposition of the cobordism $\mathcal{W}^s_{\boldsymbol{\alpha}}$ consisting of $|\boldsymbol{\alpha}| - \chi(R_{\boldsymbol{\alpha}}) + 1$ 3–handles and $b_0(R_{\boldsymbol{\alpha}})$ 4–handles.

Note that this description of the cobordism $\mathcal{W}_{\boldsymbol{\alpha}}$ does not quite allow us to compute the cobordism map $F_{\mathcal{W}_{\boldsymbol{\alpha}}}$, because when we glue $Z_{\boldsymbol{\alpha}} = I \times R_{\boldsymbol{\alpha}}$ to $M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$, we obtain a closed (ie nonsutured manifold). The necessary modification is to instead remove $b_0(R_{\boldsymbol{\alpha}})$ 3–balls from $Z_{\boldsymbol{\alpha}}$, to obtain a sutured manifold cobordism from $(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}, \gamma_{\boldsymbol{\alpha},\boldsymbol{\alpha}'})$ to $\bigsqcup_{i=1}^{b_0(R_\alpha)} (B^3, \gamma_0)$ (where $\gamma_0 \subseteq \partial B^3$ is a simple closed curve), and then compose with the natural isomorphism

$$\bigotimes_{i=1}^{b_0(R_\alpha)} \mathrm{SFH}(B^3, \gamma_0) \cong \mathbb{F}_2.$$

Let us write $Z'_{\boldsymbol{\alpha}}$ for $Z_{\boldsymbol{\alpha}}$ with $b_0(R_{\boldsymbol{\alpha}})$ 3–balls removed and $\mathcal{W}'$ for the induced special cobordism from $\bigl(M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} \cup Z'_{\boldsymbol{\alpha}}, \bigsqcup_{i=1}^{b_0(R_\alpha)} \gamma_0\bigr)$ to $\bigsqcup_{i=1}^{b_0(R_\alpha)} (B^3, \gamma_0)$. Note that $\mathcal{W}'$ can be given a handle decomposition that is the same as the handle decomposition for $\mathcal{W}^s_{\boldsymbol{\alpha}}$ with the 4–handles removed.

We write $S_\alpha \subseteq M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} \cup Z'_{\boldsymbol{\alpha}}$ for the 2–spheres obtained by doubling the compressing disk D_α for $\alpha \in \boldsymbol{\alpha}$ and $S_{c_i^*} \subseteq M_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} \cup Z'_{\boldsymbol{\alpha}}$ for the 2–spheres obtained by doubling the compressing disk $D_{c_i^*} \subseteq U_{\boldsymbol{\alpha}}$. We recall that the curves c_i^* were obtained by picking a Morse function on $R_{\boldsymbol{\alpha}}$ that had $b_0(R_{\boldsymbol{\alpha}})$ index 0 critical points and $b_1(R_{\boldsymbol{\alpha}})$ index 1 critical points. The index 1 critical points each determine a 2–dimensional 1–handle, added to the handles of index 0. The curves c_i^* are then the cocores of these handles.

Note that, by the composition law for sutured cobordisms, we can commute the 3–handle maps for the spheres S_α with the contact gluing map for gluing in $Z'_{\boldsymbol{\alpha}}$. As the composition of the 3–handle maps for the spheres S_α has the same formula as the one in the statement of the proposition, under the identification

$$\mathrm{SFH}(I \times R_{\boldsymbol{\alpha}}, I \times \partial R_{\boldsymbol{\alpha}}) \cong \mathbb{F}_2,$$

we thus reduce to the case when there are no $\boldsymbol{\alpha}$ curves, ie when $(\Sigma, \boldsymbol{\alpha})$ is the diagram $(R_{\boldsymbol{\alpha}}, \varnothing)$.

To see the claim when there are no $\boldsymbol{\alpha}$ curves, we note that the cobordism map is obtained by first gluing $Z'_{\boldsymbol{\alpha}}$ to the product sutured manifold $(I \times R_{\boldsymbol{\alpha}}, I \times \partial R_{\boldsymbol{\alpha}})$ and then attaching $b_1(R_{\boldsymbol{\alpha}})$ 4–dimensional 3–handles. The contact manifold $Z'_{\boldsymbol{\alpha}}$ is obtained by attaching $b_1(R_{\boldsymbol{\alpha}})$ contact 2–handles. If c_i^* denotes the arc on $R_{\boldsymbol{\alpha}}$ from Lemma 8.4, obtained as the cocore of a handle decomposition of $R_{\boldsymbol{\alpha}}$, as above, then the attaching 1–spheres s_i for the contact 2–handles forming $Z'_{\boldsymbol{\alpha}}$ are given by

$$s_i = (\{0,1\}\times c_i)\cup(I\times\partial c_i).$$

Note that s_i bounds the disk $D_{c_i^*}$, described in Lemma 8.4, which is the descending manifold of the Morse function f, constructed on $U_{\boldsymbol{\alpha}}$ in the proof of Lemma 8.6. The 2–sphere $S_{c_i^*}$ along which we attach the $b_1(R_{\boldsymbol{\alpha}})$ 3–handles are then equal to the union of $D_{c_i^*}$, together with the core of the corresponding contact 2–handle. Now an easy model computation shows that the composition of these $b_1(R_{\boldsymbol{\alpha}})$ contact 2–handle maps and the $b_1(R_{\boldsymbol{\alpha}})$ 3–handle maps sends $1\in \mathrm{SFH}(I\times R_{\boldsymbol{\alpha}}, I\times\partial R_{\boldsymbol{\alpha}})$ to $1\in\bigotimes_{i=1}^{b_0(R_\alpha)}\mathrm{SFH}(B^3,\gamma_0)$. This model computation is shown in Figures 47 and 48.

Using the composition law for sutured cobordisms, the formula for the cobordism map $F_{\mathcal{W}_{\boldsymbol{\alpha}}}$ now follows. □

We can now prove the main theorem of this section:

Proof of Theorem 8.1 Let us consider the trace cobordism map. Recall from (8-1) that $\Lambda_{(M,\gamma)}$ can be written as the composition of $\mathcal{W}_{\boldsymbol{\alpha}}$ with $\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}$. Noting that $\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}}$ is equivalent to $\mathcal{W}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}'}$ (where $\boldsymbol{\alpha}'$ is a small Hamiltonian translate of $\boldsymbol{\alpha}$) and using the formula for $F_{\mathcal{W}_{\boldsymbol{\alpha}}}$ from Proposition 8.3, we know that

$$F_{\Lambda_{(M,\gamma)}}\colon \mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})\otimes\mathrm{CF}(\Sigma,\boldsymbol{\beta},\boldsymbol{\alpha})\to\mathbb{F}_2$$

takes the form

$$F_{\Lambda_{(M,\gamma)}}(\boldsymbol{x}\otimes\boldsymbol{y}) = \bigl\langle F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}'}(\boldsymbol{x},\Psi_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}(\boldsymbol{y})),\Theta^-_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\bigr\rangle,$$

where

$$\langle \boldsymbol{z},\boldsymbol{z}'\rangle := \begin{cases} 1 & \text{if } \boldsymbol{z}=\boldsymbol{z}',\\ 0 & \text{otherwise.}\end{cases}$$

Note, however, that the triangle map $F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}'}$ counts the same triangles as the triangle map $F_{\boldsymbol{\alpha}',\boldsymbol{\alpha},\boldsymbol{\beta}}$ and that $\Theta^-_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\in\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\alpha}')$ is the same intersection point as $\Theta^+_{\boldsymbol{\alpha}',\boldsymbol{\alpha}}\in\mathrm{CF}(\Sigma,\boldsymbol{\alpha}',\boldsymbol{\alpha})$. Hence

$$\bigl\langle F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}'}(\boldsymbol{x},\Psi_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}(\boldsymbol{y})),\Theta^-_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\bigr\rangle = \bigl\langle F_{\boldsymbol{\alpha}',\boldsymbol{\alpha},\boldsymbol{\beta}}(\Theta^+_{\boldsymbol{\alpha}',\boldsymbol{\alpha}},\boldsymbol{x}),\Psi_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}(\boldsymbol{y})\bigr\rangle.$$

However, $F_{\boldsymbol{\alpha}',\boldsymbol{\alpha},\boldsymbol{\beta}}(\Theta^+_{\boldsymbol{\alpha}',\boldsymbol{\alpha}},-)$ is the transition map $\Psi^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}(-)$, from naturality. Hence the cobordism map becomes simply the composition

$$\langle\Psi^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}(\boldsymbol{x}),\Psi_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}(\boldsymbol{y})\rangle,$$

which is easily seen to be $\langle\Psi^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}'\to\boldsymbol{\alpha}}\circ\Psi^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}(\boldsymbol{x}),\boldsymbol{y}\rangle = \langle\boldsymbol{x},\boldsymbol{y}\rangle$. The formula for the cotrace cobordism map $F_{V_{(M,\gamma)}}$ follows from the above formula for the trace cobordism map $F_{\Lambda_{(M,\gamma)}}$, combined with Theorem 6.6. □

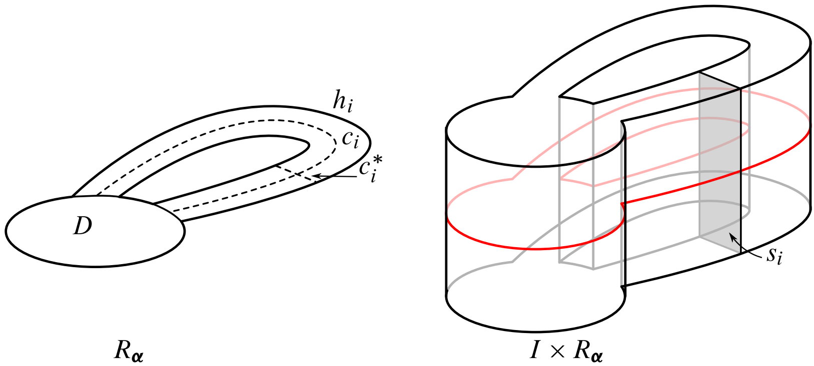

Figure 47: $R_{\boldsymbol{\alpha}}$ and $I \times R_{\boldsymbol{\alpha}}$. On the left, a 1–handle h_i from a handle decomposition of $R_{\boldsymbol{\alpha}}$ is shown, together with the core c_i and the cocore c_i^*. On the right is the product manifold $(I \times R_{\boldsymbol{\alpha}}, I \times \partial R_{\boldsymbol{\alpha}})$, together with the closed curve s_i, along which we attach a contact 2–handle. The red lines on the right indicate the sutures of $I \times R_{\boldsymbol{\alpha}}$.

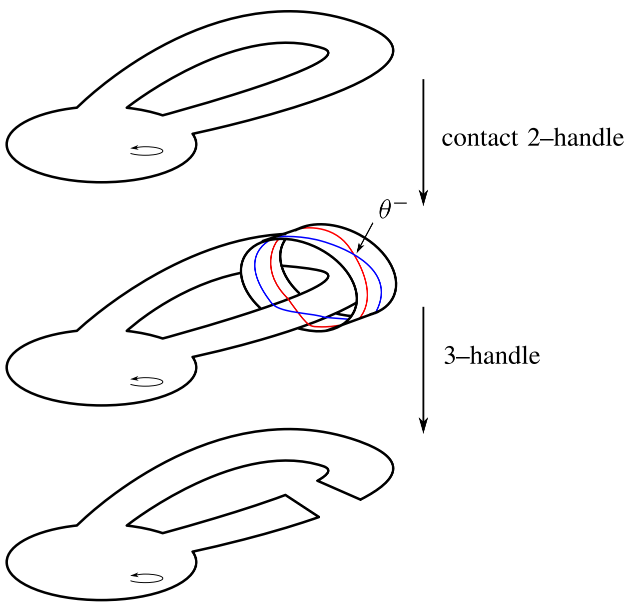

Figure 48: The model computation from Proposition 8.3. The contact 2–handle map takes the form $1 \mapsto \theta^-$, and the 3–handle map sends θ^- to 1.

9 Equivalence of two link cobordism map constructions

In this section, we describe an application of the techniques of this paper to link Floer homology.

9.1 Background on link Floer homology

Knot Floer homology is an invariant of knots embedded in 3–manifolds constructed by Ozsváth and Szabó [22] and independently by Rasmussen [27]. Link Floer homology is a generalization to links, constructed by Ozsváth and Szabó [26].

Definition 9.1 A *multibased link* $\mathbb{L} = (L, \boldsymbol{w}, \boldsymbol{z})$ in a 3–manifold Y is an oriented link $L \subseteq Y$, together with two disjoint collections of basepoints $\boldsymbol{w}, \boldsymbol{z} \subseteq L$, such that

(1) each component of L has at least two basepoints,

(2) the basepoints along a link component of L alternate between $\boldsymbol{w}$ and $\boldsymbol{z}$ as one traverses the link.

To a multibased link $\mathbb{L}$ in Y, link Floer homology associates an $\mathbb{F}_2$–module

$$\widehat{\mathrm{HFL}}(Y, \mathbb{L}).$$

To construct the modules, one picks a Heegaard diagram for the pair $(Y, \mathbb{L})$, in the following sense:

Definition 9.2 A *Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ *for an oriented multibased link* $(Y, (L, \boldsymbol{w}, \boldsymbol{z}))$ is a tuple satisfying the following:

(1) $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram for Y such that $Y \setminus \Sigma$ is the union of two handlebodies $U_{\boldsymbol{\alpha}}$ and $U_{\boldsymbol{\beta}}$ that meet along Σ.

(2) $\Sigma \cap L = \boldsymbol{w} \cup \boldsymbol{z}$.

(3) Each component of $\Sigma \setminus \boldsymbol{\alpha}$ and $\Sigma \setminus \boldsymbol{\beta}$ contains exactly one $\boldsymbol{w}$ basepoint and one $\boldsymbol{z}$ basepoint.

(4) $L \cap U_{\boldsymbol{\alpha}}$ is isotopic in $U_{\boldsymbol{\alpha}}$, relative to $\partial(L \cap U_{\boldsymbol{\alpha}})$, to a collection of arcs in $\Sigma \setminus \boldsymbol{\alpha}$. Similarly, $L \cap U_{\boldsymbol{\beta}}$ is isotopic in $U_{\boldsymbol{\beta}}$, relative to $\partial(L \cap U_{\boldsymbol{\beta}})$, to a collection of arcs in $\Sigma \setminus \boldsymbol{\beta}$.

(5) The link L intersects Σ positively at the $\boldsymbol{z}$ basepoints and negatively at the $\boldsymbol{w}$ basepoints.

A somewhat more concise way of defining a Heegaard diagram for a multibased link is as a diagram for the sutured manifold $Y(\mathbb{L})$, obtained by removing a neighborhood of the link L from Y and adding sutures to $\partial(Y \setminus N(L))$ that are positively oriented meridians of L over the $\boldsymbol{w}$ basepoints and negatively oriented meridians of L over the $\boldsymbol{z}$ basepoints. Link Floer homology, as defined by Ozsváth and Szabó [26], is easily seen to satisfy the isomorphism

$$\widehat{\mathrm{HFL}}(Y, \mathbb{L}) \cong \mathrm{SFH}(Y(\mathbb{L})).$$

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ is a diagram for $(Y, \mathbb{L})$, then the link Floer complex

$$\widehat{\mathrm{CFL}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$$

is generated over $\mathbb{F}_2$ by intersection points $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, and the differential counts Maslov index 1 pseudoholomorphic discs that go over none of the $\boldsymbol{w}$ or $\boldsymbol{z}$ basepoints.

For links in S^3 or null-homologous links in $(S^1 \times S^2)^{\# n}$, there is some additional structure on $\widehat{\mathrm{HFL}}(S^3, \mathbb{L})$. If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ is a diagram for a link in S^3, one can define three relative gradings, $\mathrm{gr}_{\boldsymbol{w}}$, $\mathrm{gr}_{\boldsymbol{z}}$, and A, on $\widehat{\mathrm{CFL}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two intersection points, then the gradings $\mathrm{gr}_{\boldsymbol{w}}$ and $\mathrm{gr}_{\boldsymbol{z}}$ are defined by picking a homology class $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ and setting

$$\mathrm{gr}_{\boldsymbol{w}}(\boldsymbol{x}, \boldsymbol{y}) := \mu(\phi) - 2n_{\boldsymbol{w}}(\phi) \quad \text{and} \quad \mathrm{gr}_{\boldsymbol{z}}(\boldsymbol{x}, \boldsymbol{y}) := \mu(\phi) - 2n_{\boldsymbol{z}}(\phi),$$

where $n_{\boldsymbol{w}}(\phi)$ and $n_{\boldsymbol{z}}(\phi)$ denote the sum of the multiplicities of ϕ over the $\boldsymbol{w}$ or $\boldsymbol{z}$ basepoints, respectively. It is easy to see that the formulas for $\mathrm{gr}_{\boldsymbol{w}}$ and $\mathrm{gr}_{\boldsymbol{z}}$ are independent of the choice of homology class ϕ. An absolute lift of the relative grading $\mathrm{gr}_{\boldsymbol{w}}$ can be fixed by requiring that $\widehat{\mathrm{HF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$, which is isomorphic as a relatively graded group to $\bigotimes^{|\boldsymbol{w}|-1}((\mathbb{F}_2)_{-\frac{1}{2}} \oplus (\mathbb{F}_2)_{\frac{1}{2}})$, have top-graded generator in grading $\frac{1}{2}(|\boldsymbol{w}| - 1)$. An absolute lift of the grading $\mathrm{gr}_{\boldsymbol{z}}$ can be specified similarly. Finally, the Alexander grading A can be defined as

$$A := \tfrac{1}{2}(\mathrm{gr}_{\boldsymbol{w}} - \mathrm{gr}_{\boldsymbol{z}}).$$

9.2 The link Floer homology TQFT

In [15], the first author provided a construction of cobordism maps for decorated link cobordisms. The construction used the following notion of cobordism between multibased links:

Definition 9.3 Let Y_1 and Y_2 be 3–manifolds containing multibased links $\mathbb{L}_1 = (L_1, \boldsymbol{w}_1, \boldsymbol{z}_1)$ and $\mathbb{L}_2 = (L_2, \boldsymbol{w}_2, \boldsymbol{z}_2)$, respectively. A *decorated link cobordism* from $(Y_1, \mathbb{L}_1)$ to $(Y_2, \mathbb{L}_2)$ is a triple $(X, S, \mathcal{A})$, where

(1) X is an oriented cobordism from Y_1 to Y_2,

(2) S is a properly embedded oriented surface in X with $\partial S = -L_1 \cup L_2$, and

(3) $\mathcal{A}$ is a properly embedded 1–manifold in S that divides S into two subsurfaces $S_{\boldsymbol{w}}$ and $S_{\boldsymbol{z}}$ that meet along $\mathcal{A}$ such that $\boldsymbol{w}_1, \boldsymbol{w}_2 \subseteq S_{\boldsymbol{w}}$ and $\boldsymbol{z}_1, \boldsymbol{z}_2 \subseteq S_{\boldsymbol{z}}$.

Note that this definition is slightly different from [15, Definition 4.5] and follows [30]. The equivalence of the two definitions is explained in [17, Section 2.3].

If $\mathcal{X} = (X, S, \mathcal{A})$ from $(Y_1, \mathbb{L}_1)$ to $(Y_2, \mathbb{L}_2)$ is a decorated link cobordism, then there is a well-defined cobordism $\mathcal{W}(\mathcal{X}) = (W, Z, [\xi])$ of sutured manifolds from $Y(\mathbb{L}_1)$ to $Y(\mathbb{L}_2)$, as we now describe. The 4–manifold W is defined by the formula

$$W := X \setminus N(S),$$

where $N(S)$ denotes a regular neighborhood of S, viewed as the unit normal disk bundle of S. The set Z is defined as the unit normal circle bundle of S, oriented as a submanifold of ∂W. Since S is an oriented surface, Z is a principal S^1–bundle over S. According to Lutz [20] and Honda [11], the dividing set $\mathcal{A}$ uniquely determines an S^1–invariant contact structure on Z with dividing set $\mathcal{A}$ on S, up to isotopy. The contact structure ξ is defined to be this S^1–invariant contact structure. The link cobordism map

$$F^J_{\mathcal{X}} \colon \widehat{\mathrm{HFL}}(Y_1, \mathbb{L}_1) \to \widehat{\mathrm{HFL}}(Y_2, \mathbb{L}_2)$$

is defined to be the sutured cobordism map

$$F_{\mathcal{W}(\mathcal{X})} \colon \mathrm{SFH}(Y_1(\mathbb{L}_1)) = \widehat{\mathrm{HFL}}(Y_1, \mathbb{L}_1) \to \mathrm{SFH}(Y_2(\mathbb{L}_2)) = \widehat{\mathrm{HFL}}(Y_2, \mathbb{L}_2).$$

The second author [30] constructed another link cobordism map $F^Z_{\mathcal{X},\mathfrak{s}}$, where $\mathfrak{s} \in \mathrm{Spin}^c(X)$ is a Spin^c structure on X, that did not use the Honda–Kazez–Matić gluing map. It instead was defined by writing a link cobordism as a composition of elementary link cobordisms. The map $F^Z_{\mathcal{X},\mathfrak{s}}$ is defined on a more general version of link Floer homology than $F^J_{\mathcal{X}}$, though it induces a map $\hat{F}^Z_{\mathcal{X},\mathfrak{s}}$ on the hat version. Let

$$\hat{F}^Z_{\mathcal{X}} := \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} \hat{F}^Z_{\mathcal{X},\mathfrak{s}}.$$

It is not obvious that the maps $F^J_{\mathcal{X}}$ and $\hat{F}^Z_{\mathcal{X}}$ agree.

The first author and Marengon [17] made some steps towards computing the maps $F_{\mathcal{X}}^J$ when $\mathcal{X}$ is an elementary link cobordism, though most of the computational results from [17] are still in terms of the Honda–Kazez–Matić gluing map, and hence it is challenging to directly compare the maps $F_{\mathcal{X}}^J$ and $\hat{F}_{\mathcal{X}}^Z$. Nonetheless, combining several results from [17] with the results of this paper, we are able to prove the following:

Theorem 9.4 *Given a decorated link cobordism $\mathcal{X}$, we have $F_{\mathcal{X}}^J = \hat{F}_{\mathcal{X}}^Z$.*

9.3 Elementary link cobordisms

In this section, we provide the following definition:

Definition 9.5 A decorated link cobordism $\mathcal{X} = (X, S, \mathcal{A})\colon (Y_1, \mathbb{L}_1) \to (Y_2, \mathbb{L}_2)$ is an *elementary link cobordism* if one of the following is satisfied:

(1) **Identity cobordism**

$$(Y_1, \mathbb{L}_1) = (Y_2, \mathbb{L}_2) = (Y, \mathbb{L}) \quad \text{and} \quad (X, S, \mathcal{A}) = (I \times Y, I \times L, I \times \boldsymbol{p}),$$

where $\boldsymbol{p} \subseteq L$ consists of exactly one point in each component of $L \setminus (\boldsymbol{w} \cup \boldsymbol{z})$.

(2) **1–, 2–, or 3–handle attachment** The cobordism $(X, S, \mathcal{A})$ is obtained by attaching a 4–dimensional 1–, 2–, or 3–handle, with framed attaching sphere disjoint from $\mathbb{L}_1$.

(3) **0– or 4–handle attachment** The cobordism $\mathcal{X}$ is obtained by attaching a 4–dimensional 0–handle or 4–handle, viewed as a smooth 4–ball, that contains a standard disk intersecting the boundary of the 4–ball in an unknot, with dividing set consisting of a single arc on the disk.

(4) **Saddle cobordism** The cobordism $\mathcal{X}$ has underlying 4–manifold $X = I \times Y$ and a surface S such that projection to I induces a Morse function that has a single index 1 critical point that occurs in a $\boldsymbol{w}$–region or a $\boldsymbol{z}$–region and such that the dividing arcs all travel from $\mathbb{L}_1$ to $\mathbb{L}_2$.

(5) **Stabilization cobordism** The cobordism $\mathcal{X}$ has underlying 4–manifold $X = I \times Y$ and surface $S = I \times L$. Furthermore, exactly one arc of $\mathcal{A}$ goes from $\mathbb{L}_1$ to $\mathbb{L}_1$ or from $\mathbb{L}_2$ to $\mathbb{L}_2$. All other arcs are of the form $I \times \{p\}$ for various $p \in L$. A stabilization cobordism is *positive* if it adds two basepoints and is *negative* if it removes two basepoints.

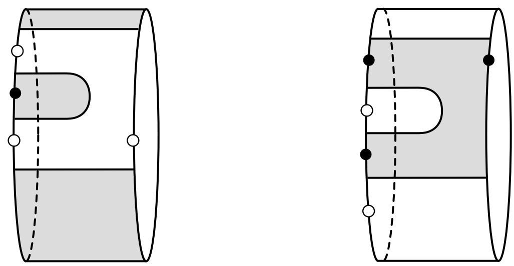

Figure 49: Two examples of stabilization cobordisms. The two surfaces are each of the form $I \times L$ and sit inside $I \times Y$. The shaded regions are the $\boldsymbol{w}$ regions, and the unshaded regions are the $\boldsymbol{z}$ regions.

A schematic of a saddle cobordism can be found in Figure 55. Examples of stabilization cobordisms are shown in Figure 49.

It is straightforward to see that an arbitrary link cobordism $\mathcal{X}$ such that $\pi_0(S) \to \pi_0(X)$ is a surjection can be decomposed into a sequence of link cobordisms that are each diffeomorphic to one of the elementary link cobordisms in the above list. We remark that the above list is overcomplete, in the following sense:

Remark 9.6 An elementary positive stabilization cobordism can be written as a composition of a 0–handle cobordism (adding an unknot with two basepoints) followed by a 1–handle, followed by a saddle cobordism. Similarly an elementary negative stabilization cobordism can be written as a composition of a saddle cobordism, followed by a 3–handle and a 4–handle.

9.4 Link triple diagrams and contact structures

Definition 9.7 A *link triple diagram*

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{w}, \boldsymbol{z})$$

is a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ with $2(n - g(\Sigma) + 1)$ basepoints $\boldsymbol{w} \cup \boldsymbol{z} \subseteq \Sigma$, where $n = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = |\boldsymbol{\gamma}|$, such that each component of $\Sigma \setminus \boldsymbol{\tau}$ for $\boldsymbol{\tau} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ is planar and contains exactly one $\boldsymbol{w}$ basepoint and exactly one $\boldsymbol{z}$ basepoint.

Given a link triple diagram $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{w}, \boldsymbol{z})$, we can construct a decorated link cobordism

$$\mathcal{X}_{\mathcal{T}} = (X_{\mathcal{T}}, S_{\mathcal{T}}, \mathcal{A}_{\mathcal{T}}),$$

as follows. The 4–manifold $X_{\mathcal{T}}$ is constructed as in [25], by the formula

$$X_{\mathcal{T}} := (\Delta \times \Sigma) \cup (e_{\boldsymbol{\alpha}} \times U_{\boldsymbol{\alpha}}) \cup (e_{\boldsymbol{\beta}} \times U_{\boldsymbol{\beta}}) \cup (e_{\boldsymbol{\gamma}} \times U_{\boldsymbol{\gamma}})/\sim.$$

Here, the handlebody $U_{\boldsymbol{\tau}}$ for $\boldsymbol{\tau} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ is obtained by attaching 3–dimensional 2–handles to $\Sigma \times I$ along $\boldsymbol{\tau} \times \{0\}$ and filling in the resulting sphere boundary components with 3–dimensional 3–handles.

We obtain the surface $S_{\mathcal{T}}$ as follows. Pick Morse functions $f_{\boldsymbol{\alpha}}$, $f_{\boldsymbol{\beta}}$, and $f_{\boldsymbol{\gamma}}$ on $U_{\boldsymbol{\alpha}}$, $U_{\boldsymbol{\beta}}$, and $U_{\boldsymbol{\gamma}}$ that induce the attaching curves $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, respectively. By concatenating the ascending flow lines passing through the basepoints in $\boldsymbol{w}$ and $\boldsymbol{z}$, we get a collection of $|\boldsymbol{w}|$ arcs $K_{\boldsymbol{\alpha}}$, $K_{\boldsymbol{\beta}}$, and $K_{\boldsymbol{\gamma}}$ in each of $U_{\boldsymbol{\alpha}}$, $U_{\boldsymbol{\beta}}$, and $U_{\boldsymbol{\gamma}}$, respectively. Each arc has exactly one endpoint in $\boldsymbol{w}$ and one endpoint in $\boldsymbol{z}$, and we orient it from $\boldsymbol{w}$ to $\boldsymbol{z}$. Then the surface $S_{\mathcal{T}}$ is defined as

$$S_{\mathcal{T}} := (\Delta \times (-\boldsymbol{w} \cup \boldsymbol{z})) \cup (e_{\boldsymbol{\alpha}} \times K_{\boldsymbol{\alpha}}) \cup (e_{\boldsymbol{\beta}} \times K_{\boldsymbol{\beta}}) \cup (e_{\boldsymbol{\gamma}} \times K_{\boldsymbol{\gamma}}).$$

Finally, we describe the dividing set $\mathcal{A}_{\mathcal{T}}$ on $S_{\mathcal{T}}$. For $\boldsymbol{\tau} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, let $p_{\boldsymbol{\tau}} \subseteq K_{\boldsymbol{\tau}}$ denote a collection of points obtained by picking a single point in each arc of $K_{\boldsymbol{\tau}}$. Then

$$\mathcal{A}_{\mathcal{T}} := (e_{\boldsymbol{\alpha}} \times p_{\boldsymbol{\alpha}}) \cup (e_{\boldsymbol{\beta}} \times p_{\boldsymbol{\beta}}) \cup (e_{\boldsymbol{\gamma}} \times p_{\boldsymbol{\gamma}})$$

is a dividing set on $S_{\mathcal{T}}$.

Given a link triple diagram $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{w}, \boldsymbol{z})$, we naturally obtain a sutured Heegaard triple $\mathcal{T}_0 = (\Sigma_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$, where $\Sigma_0 = \Sigma \setminus N(\boldsymbol{w} \cup \boldsymbol{z})$. Let $\mathcal{W}_{\mathcal{T}_0} = (W_{\mathcal{T}_0}, Z_{\mathcal{T}_0}, [\xi_{\mathcal{T}_0}])$ denote the associated sutured cobordism.

Proposition 9.8 *Let $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{w}, \boldsymbol{z})$ be a link triple diagram, and $\mathcal{T}_0 = (\Sigma_0, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ the corresponding sutured triple diagram. Then $Z_{\mathcal{T}_0}$ is the unit normal circle bundle of $S_{\mathcal{T}}$, and there is a projection map $\pi\colon Z_{\mathcal{T}_0} \to S_{\mathcal{T}}$ with fiber S^1. The contact structure $\xi_{\mathcal{T}_0}$ on $Z_{\mathcal{T}_0}$ described in Section 7 is equivalent to the S^1–invariant contact structure on $Z_{\mathcal{T}_0}$ with respect to the dividing set $\mathcal{A}_{\mathcal{T}}$ and the projection map π.*

Proof Let us write ξ_{S^1} for the S^1–invariant contact structure on the unit normal circle bundle $\mathrm{SN}(S_{\mathcal{T}})$ of $S_{\mathcal{T}}$. The proof of the proposition will be to describe a convex decomposition of $(\mathrm{SN}(S_{\mathcal{T}}), \xi_{S^1})$ into the disjoint union of the contact manifolds (Z_0, ξ_0), $(Z_{\boldsymbol{\alpha}}, \xi_{\boldsymbol{\alpha}})$, $(Z_{\boldsymbol{\beta}}, \xi_{\boldsymbol{\beta}})$, and $(Z_{\boldsymbol{\gamma}}, \xi_{\boldsymbol{\gamma}})$, whose union is $(Z_{\mathcal{T}_0}, \xi_{\mathcal{T}_0})$ by definition. Note that $S_{\mathcal{T}}$ has no closed components, so $\mathrm{SN}(S_{\mathcal{T}})$ is diffeomorphic to $S_{\mathcal{T}} \times S^1$, with the map π given by projection onto the first factor.

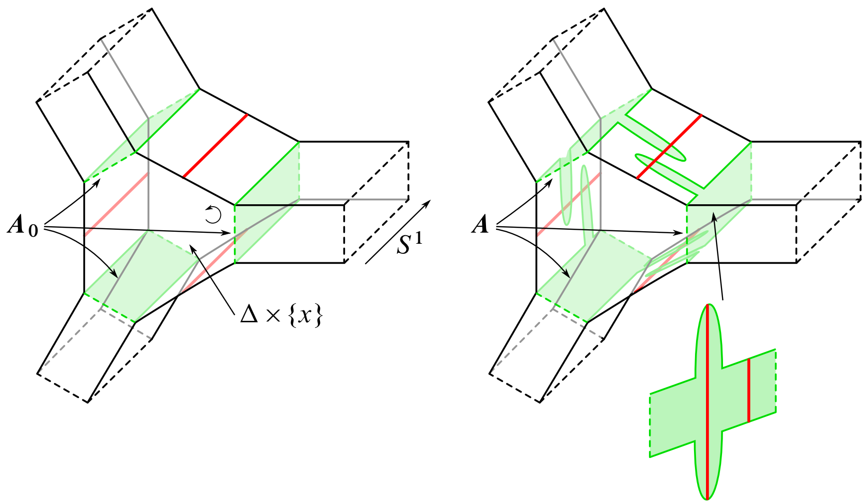

Figure 50: The annuli A_0 in $S_{\mathcal{T}} \times S^1$ (left) and the annuli A obtained by performing finger moves along the boundaries toward the dividing set on $\partial S_{\mathcal{T}} \times S^1$ (right). We prove that the dividing set on each annulus in A is as shown in the bottom. The hexagonal region in the middle corresponds to a component of $\Delta \times \{x\}$ for a basepoint $x \in \boldsymbol{w} \cup \boldsymbol{z}$. The orientation of $\Delta \times \{x\}$ is shown, and we take the product orientation on $S_{\mathcal{T}} \times S^1$ in the picture.

We will decompose $\mathrm{SN}(S_{\mathcal{T}}) \approx S_{\mathcal{T}} \times S^1$ along $3|\partial \Sigma_0|$ convex annuli. To construct the annuli, it is convenient to view Δ as a smooth 2–disk and view the edges $e_{\boldsymbol{\alpha}}$, $e_{\boldsymbol{\beta}}$, and $e_{\boldsymbol{\gamma}}$ as being closed, disjoint subintervals of $\partial \Delta$. We let A_0 denote the union of annuli of the form $e_{\boldsymbol{\tau}} \times \{w\} \times S^1$ and $e_{\boldsymbol{\tau}} \times \{z\} \times S^1$ inside $S_{\mathcal{T}} \times S^1$ for $\boldsymbol{\tau} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$. Note that we cannot decompose along the annuli in A_0, since their boundaries are disjoint from the dividing set, and hence we cannot use Legendrian realization to ensure that ∂A_0 is Legendrian and A_0 is convex. Instead, we perform a finger move along each boundary component of each annulus in A_0 until it intersects the dividing set. Let A denote the resulting collection of annuli. The configuration of the annuli A_0 and A is shown in Figure 50.

We can perturb the annuli in A so that they are convex with Legendrian boundary (note this requires using Legendrian realization along $\partial S_{\mathcal{T}} \times S^1$, and hence involves replacing ξ_{S^1} with an equivalent contact structure that is no longer S^1–invariant near the boundary).

We now claim that the dividing sets on the annuli in A are as in the bottom of Figure 50, consisting of two arcs that go from one boundary component of the annulus to the

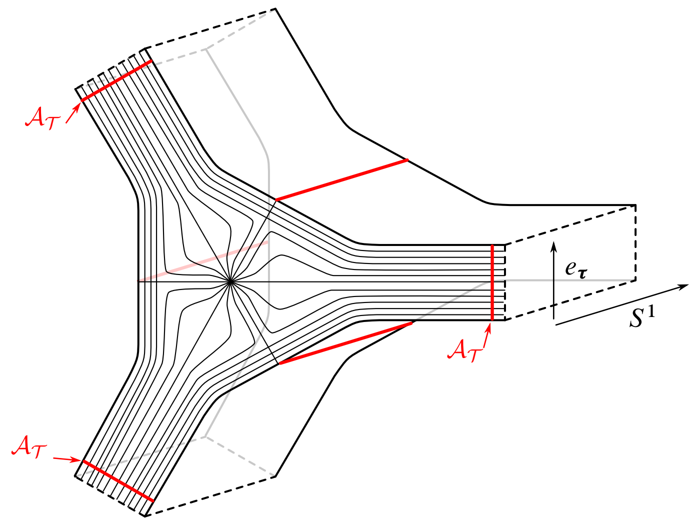

Figure 51: The characteristic foliation of $S_\mathcal{T}\times\{p\}$ of an S^1–invariant contact 1–form on $S_\mathcal{T}\times S^1$ with dividing set $\mathcal{A}_\mathcal{T}$.

other and which do not wind around the annulus. To see this, we first claim that, for an appropriately chosen S^1–invariant contact 1–form, we can embed a neighborhood of each annulus inside an $e_{\boldsymbol{\tau}}$–invariant contact structure on $e_{\boldsymbol{\tau}}\times I\times S^1$ that has dividing set on $(\partial e_{\boldsymbol{\tau}})\times I\times S^1$ equal to $(\partial e_{\boldsymbol{\tau}})\times\{\frac{1}{2}\}\times S^1$. To do this, we recall that an S^1–invariant contact 1–form on $S_\mathcal{T}\times S^1$ can be defined as $\beta+f\cdot d\theta$, where β is a 1–form on $S_\mathcal{T}$ and $f\colon S_\mathcal{T}\to\mathbb{R}$ is a function that is zero exactly on $\mathcal{A}_\mathcal{T}$. It is straightforward to write down conditions for such a 1–form to be a contact form on $S_\mathcal{T}\times S^1$. Since the contact form $\beta+f\cdot d\theta$ is essentially determined by the characteristic foliation $\ker\beta$ on $S_\mathcal{T}\times\{p\}$, we will focus on constructing a singular foliation with the appropriate dividing set that is invariant under translation along a nonvanishing vector field (thought

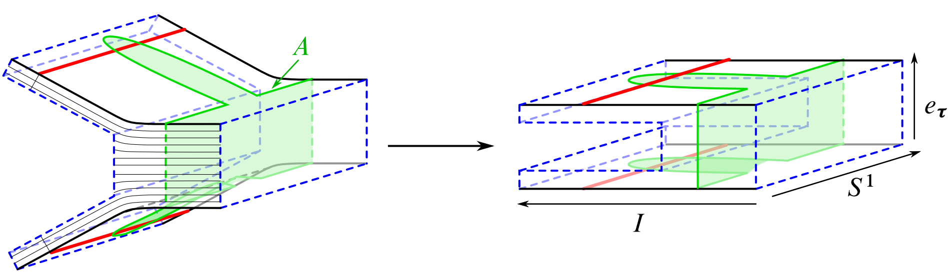

Figure 52: A neighborhood of an annulus $A\in\boldsymbol{A}$ in $S_\mathcal{T}\times S^1$ which embeds into an $e_{\boldsymbol{\tau}}$–invariant contact structure on $e_{\boldsymbol{\tau}}\times I\times S^1$.

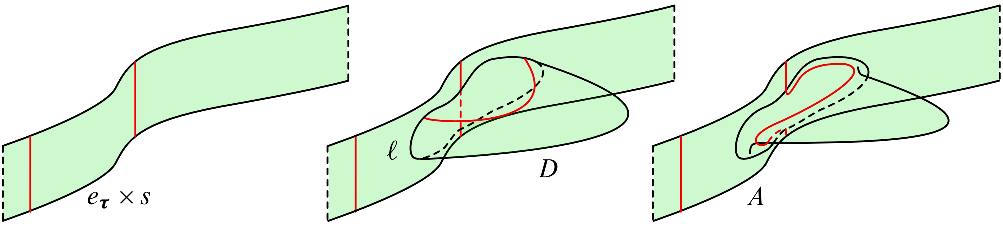

Figure 53: Attaching a convex disk to move the annulus $e_{\boldsymbol{\tau}} \times s \subseteq e_{\boldsymbol{\tau}} \times I \times S^1$ to the position of an annulus $A \in \boldsymbol{A}$. On the left is (a C^0–small perturbation of) the annulus $e_{\boldsymbol{\tau}} \times s$. In the middle is a convex disk with Legendrian boundary ℓ with $\operatorname{tb}(\ell) = -1$. On the right is a convex annulus that we can take to be A, obtained by edge rounding along ℓ.

of as $\partial/\partial e_{\boldsymbol{\tau}}$) in a neighborhood of each annulus in $\boldsymbol{A}$. It is an elementary, though somewhat tedious, exercise to explicitly construct an appropriate contact 1–form by picking β and f appropriately, so we will leave that step to the reader. An example of an appropriately chosen characteristic foliation on $S_{\mathcal{T}} \times \{p\}$ is shown in Figure 51.

We can choose a neighborhood of an annulus in $\boldsymbol{A}$ that embeds into an $e_{\boldsymbol{\tau}}$–invariant contact structure on $e_{\boldsymbol{\tau}} \times I \times S^1$ as in Figure 52. Note that, inside $e_{\boldsymbol{\tau}} \times I \times S^1$, the annulus A pictured on the right of Figure 52 is isotopic, relative to its boundary, to a surface of the form $e_{\boldsymbol{\tau}} \times s$, for a Legendrian s in $(\partial e_{\boldsymbol{\tau}}) \times I \times S^1$. The characteristic

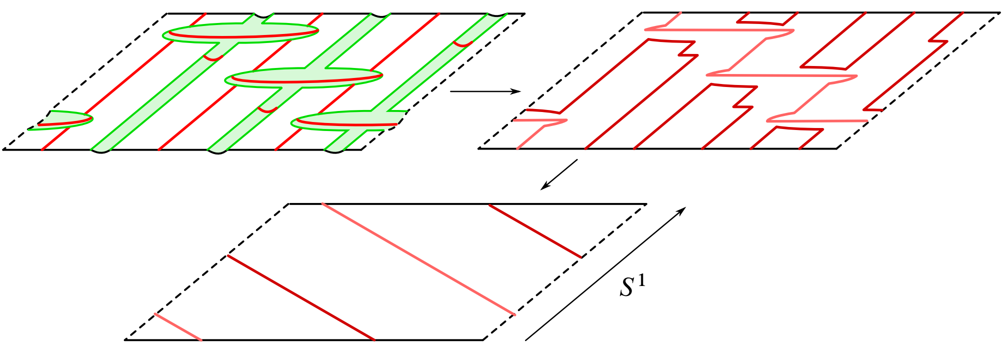

Figure 54: Rounding corners after cutting $S_{\mathcal{T}} \times S^1$ along the annuli in $\boldsymbol{A}$. Shown is the dividing set on the boundary of a solid torus component of $(S_{\mathcal{T}} \times S^1) \setminus \boldsymbol{A}$ corresponding to $\Delta \times \{x\} \times S^1$ for a basepoint $x \in \boldsymbol{w} \cup \boldsymbol{z}$. On the top left, we show the dividing before rounding the Legendrian corners. On the top right, we show the result of rounding corners. On the bottom, we show the result of isotoping the dividing set. We view Z_0 as being "below" the surface shown.

foliation and dividing set on $e_{\boldsymbol{\tau}} \times s$ are the same as on the annulus shown in Figure 31. Namely, the characteristic foliation consists of horizontal leaves lying in $\{t\} \times s$, as well as two vertical singular sets of the form $e_{\boldsymbol{\tau}} \times \{p\}$, for two points p in s. The dividing set on $e_{\boldsymbol{\tau}} \times s$ consists of two vertical arcs, as well.

Note that if we could show that A and $e_{\boldsymbol{\tau}} \times s$ were isotopic through convex surfaces, we would be done, since the dividing set on A would be isotopic to the one on $e_{\boldsymbol{\tau}} \times s$, which is what we are trying to show. This is somewhat geometrically hard to prove, so we argue as follows. Perturb $e_{\boldsymbol{\tau}} \times s$ slightly so that there is a Legendrian loop ℓ intersecting one of the dividing arcs twice. This does not change the isotopy type of the dividing set. Let $D_0 \subseteq e_{\boldsymbol{\tau}} \times s$ be the disk bounded by ℓ. Now take a convex disk D in $e_{\boldsymbol{\tau}} \times I \times S^1$ with $D \cap (e_{\boldsymbol{\tau}} \times s) = \ell$. The dividing set on D consists of a single arc, since we can assume D lies in a tight contact ball. We can replace $e_{\boldsymbol{\tau}} \times s$ by $((e_{\boldsymbol{\tau}} \times s) \setminus D_0) \cup D$ and then round along the Legendrian corner ℓ. When we do this, we do not change the isotopy type of the dividing set, and we can move $e_{\boldsymbol{\tau}} \times s$ to A via a sequence of such moves. The move is shown in Figure 53. This establishes that the annulus A has a dividing set isotopic to the one shown in Figure 50.

Having determined the dividing sets along the convex annuli in $\boldsymbol{A}$, we can cut along the annuli in $\boldsymbol{A}$, then round the Legendrian corners. The dividing set on the sutured manifold corresponding to $\Delta \times \partial\Sigma$ is shown in Figure 54. As a sutured manifold, this is the same as (Z_0, γ_0). Similarly, it is easy to see that rounding the Legendrian corners on the other 3 pieces yields the sutured manifolds $(Z_{\boldsymbol{\alpha}}, \gamma_{\boldsymbol{\alpha}})$, $(Z_{\boldsymbol{\beta}}, \gamma_{\boldsymbol{\beta}})$, and $(Z_{\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\gamma}})$. It follows that $\mathrm{SN}(S_{\boldsymbol{\tau}}) \approx Z_{\mathcal{T}_0}$. On the other hand, we note that ξ_{S^1} is tight, since the dividing set on $S_{\mathcal{T}}$ contains no contractible components. It follows that ξ_{S^1} restricts to tight contact structures on (Z_0, γ_0), $(Z_{\boldsymbol{\alpha}}, \gamma_{\boldsymbol{\alpha}})$, $(Z_{\boldsymbol{\beta}}, \gamma_{\boldsymbol{\beta}})$, and $(Z_{\boldsymbol{\gamma}}, \gamma_{\boldsymbol{\gamma}})$. However, each of these four sutured manifolds are product disk decomposable, so, up to equivalence, there is a unique tight contact structure on each one. Hence the restrictions of ξ_{S^1} are equivalent to ξ_0, $\xi_{\boldsymbol{\alpha}}$, $\xi_{\boldsymbol{\beta}}$, and $\xi_{\boldsymbol{\gamma}}$, respectively. Since $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ is constructed by gluing together ξ_0, $\xi_{\boldsymbol{\alpha}}$, $\xi_{\boldsymbol{\beta}}$, and $\xi_{\boldsymbol{\gamma}}$, it follows that ξ_{S^1} and $\xi_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$ are equivalent on $Z_{\mathcal{T}_0}$. □

9.5 Saddle cobordisms and link triple diagrams

In this section, we review the construction of saddle cobordism maps [30, Section 6]. As an important step towards proving Theorem 9.4, we show that the maps from [30] and [15] agree for such cobordisms.

Definition 9.9 Suppose that Y is a 3–manifold containing an oriented multibased link $\mathbb{L}=(L,\boldsymbol{w},\boldsymbol{z})$. We say that an oriented square $B\subseteq Y$ is a *β–band* for a multibased link $\mathbb{L}$ in Y if B is smoothly embedded in Y, it is identified with $[-1,1]\times[-1,1]$, and

(1) $B\cap L=[-1,1]\times\{-1,1\}$,

(2) the boundary orientation of B agrees with the orientation of $-L$,

(3) $B\cap(\boldsymbol{w}\cup\boldsymbol{z})=\varnothing$, and both ends of B are in regions of $L\setminus(\boldsymbol{w}\cup\boldsymbol{z})$ that go from $\boldsymbol{z}$ to $\boldsymbol{w}$.

Note that if B is a β–band for the link $\mathbb{L}$ in Y, then there is a well-defined multibased link

$$\mathbb{L}(B)=(L(B),\boldsymbol{w},\boldsymbol{z})$$

obtained by band surgery on B. The following is [30, Definition 6.4]:

Definition 9.10 The link triple diagram

$$(\Sigma,\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,\beta'_1,\dots,\beta'_n,\boldsymbol{w},\boldsymbol{z})$$

is *subordinate to the β–band B* if the following hold:

(1) $(\Sigma_0,\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_{n-1})$, where $\Sigma_0=\Sigma\setminus N(\boldsymbol{w}\cup\boldsymbol{z})$, is a diagram for the sutured manifold $Y(\mathbb{L})\setminus N(B)$.

(2) $(\Sigma,\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,\boldsymbol{w},\boldsymbol{z})$ is a diagram for $(Y,\mathbb{L})$.

(3) $\beta'_1,\dots,\beta'_{n-1}$ are small Hamiltonian translates of the curves $\beta_1,\dots,\beta_{n-1}$.

(4) The curve β'_n is induced by the band B, and $(\Sigma,\alpha_1,\dots,\alpha_n,\beta'_1,\dots,\beta'_n,\boldsymbol{w},\boldsymbol{z})$ is a diagram for $(Y,\mathbb{L}(B))$.

Notice that, if we ignore the basepoints $\boldsymbol{w}$ and $\boldsymbol{z}$, then the curve β'_n is related to β_n by a sequence of handle slides and isotopies. It follows that the 4–manifold $X_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}$ induced by a Heegaard triple subordinate to a β–band is diffeomorphic to $I\times Y$ with a neighborhood of $\{\frac{1}{2}\}\times U_{\boldsymbol{\beta}}$ removed.

The band B induces a saddle cobordism $S(B)\subseteq I\times Y$ from L to $L(B)$. The surface $S(B)$ is obtained by rounding the corners of the surface

$$S_0(B):=\left(\left[0,\tfrac{1}{2}\right]\times L\right)\cup\left(\left\{\tfrac{1}{2}\right\}\times B\right)\cup\left(\left[\tfrac{1}{2},1\right]\times L(B)\right).$$

We note that $S(B)$ has two natural choices of dividing sets. To construct them, pick points $\boldsymbol{p}\subseteq L\setminus(\boldsymbol{w}\cup\boldsymbol{z})$ and $\boldsymbol{q}\subseteq L(B)\setminus(\boldsymbol{w}\cup\boldsymbol{z})$ such that the following hold:

(1) Each component of $L \setminus (\boldsymbol{w} \cup \boldsymbol{z})$ contains exactly one point of $\boldsymbol{p}$, and each component of $L(B) \setminus (\boldsymbol{w} \cup \boldsymbol{z})$ contains exactly one point of $\boldsymbol{q}$.

(2) If $C \subseteq L \setminus (\boldsymbol{w} \cup \boldsymbol{z})$ is a component disjoint from B, then $\boldsymbol{p} \cap C = \boldsymbol{q} \cap C$.

(3) If C is a component of $L \setminus (\boldsymbol{w} \cup \boldsymbol{z})$ that intersects B, and $\{p\} = C \cap \boldsymbol{p}$, then $p \in B$. Similarly, if C' is a component of $L(B) \setminus (\boldsymbol{w} \cup \boldsymbol{z})$ that intersects B, and $\{q\} = C' \cap \boldsymbol{q}$, then $q \in B$.

A dividing $\mathcal{A}_0$ on $S(B) \setminus \left(\left\{\tfrac{1}{2}\right\} \times B\right)$ is then specified by the equation

$$\mathcal{A}_0 := \left(\left[0, \tfrac{1}{2}\right] \times \boldsymbol{p}\right) \cup \left(\left[\tfrac{1}{2}, 1\right] \times \boldsymbol{q}\right).$$

There are two natural ways to extend the dividing set $\mathcal{A}_0$ to B. We write $\mathcal{A}_{\boldsymbol{w}}$ and $\mathcal{A}_{\boldsymbol{z}}$ for the two possible extensions, as shown in Figure 55.

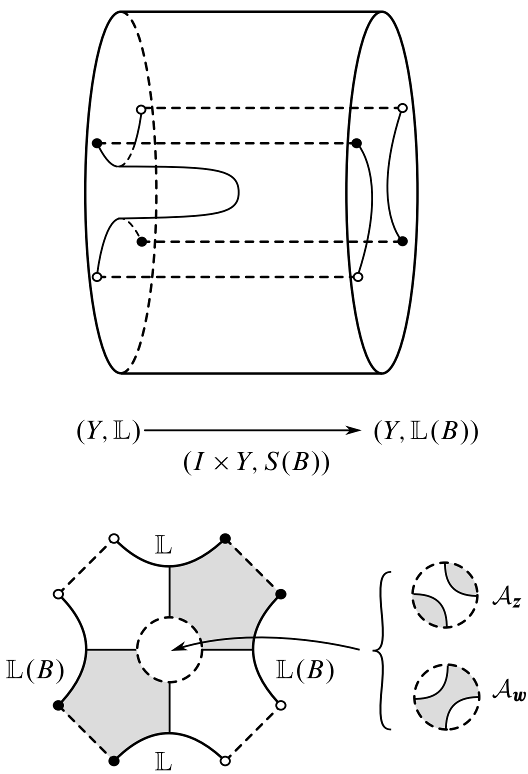

Figure 55: A portion of the surface $S(B) \subseteq I \times Y$, as well as the two dividing sets $\mathcal{A}_{\boldsymbol{w}}$ and $\mathcal{A}_{\boldsymbol{z}}$ on $S(B)$. The $\boldsymbol{w}$ basepoints are shown as solid dots, while the $\boldsymbol{z}$ basepoints are open dots. The $\boldsymbol{w}$ regions are shown as shaded, the $\boldsymbol{z}$ regions are unshaded.

Lemma 9.11 *If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$ is a triple subordinate to a β–band B, then*

$$(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$$

represents an unlink $\mathbb{U}$ in $(S^1 \times S^2)^{\# g(\Sigma)}$, where all components of $\mathbb{U}$ have two basepoints, except for one component that has four basepoints. With respect to each of the Maslov gradings $\mathrm{gr}_{\boldsymbol{w}}$ and $\mathrm{gr}_{\boldsymbol{z}}$, there is a top-graded generator of $\widehat{\mathrm{HFL}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$, for which we write $\Theta^{\boldsymbol{w}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}$ and $\Theta^{\boldsymbol{z}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}$.

Proof For $i \in \{1, \dots, n-1\}$, the curves β_i and β_i' are Hamiltonian translates of each other and hence determine a 2–sphere in the manifold $U_{\boldsymbol{\beta}} \cup_\Sigma U_{\boldsymbol{\beta}'}$. After surgering all of these out, we are left with $|\boldsymbol{w}|-2$ copies of S^3, each containing a doubly based unknot, as well as one copy of S^3 with an unknot containing four basepoints. After surgering all of these 2–spheres out, it is easy to see $\widehat{\mathrm{HFL}}$ is generated by two elements, one of which is in $(\mathrm{gr}_{\boldsymbol{w}}, \mathrm{gr}_{\boldsymbol{z}})$–grading $\left(+\frac{1}{2}, -\frac{1}{2}\right)$ and one of which is in grading $\left(-\frac{1}{2}, +\frac{1}{2}\right)$. The effect of undoing the surgeries we did on the 2–spheres corresponds to adding back in a collection of 1–handles, which clearly preserves the property of having a top $\mathrm{gr}_{\boldsymbol{w}}$–graded element and a top $\mathrm{gr}_{\boldsymbol{z}}$–graded element. □

Lemma 9.12 *Consider the link cobordism $\mathcal{X} = (X, S, \mathcal{A})$ from the empty set to an unlink in $(S^1 \times S^2)^{\# k}$ constructed by setting X to be a 4–dimensional genus k handlebody, S to be a collection of n standardly embedded disks in X intersecting $\partial X \cong (S^1 \times S^2)^{\# k}$ in n disjoint unknots, and letting $\mathcal{A}$ consist of a single arc on each component of S except for one component of S where $\mathcal{A}$ consists of two arcs. Then*

$$F^J_{\mathcal{X}}(1) = \hat{F}^Z_{\mathcal{X}}(1) = \begin{cases} \Theta^{\boldsymbol{w}} & \text{if } \chi(S_{\boldsymbol{w}}) = \chi(S_{\boldsymbol{z}}) + 1, \\ \Theta^{\boldsymbol{z}} & \text{if } \chi(S_{\boldsymbol{w}}) = \chi(S_{\boldsymbol{z}}) - 1, \end{cases}$$

where 1 denotes the generator of $\widehat{\mathrm{HFL}}(\varnothing, \varnothing) \cong \mathbb{F}_2$.

Proof We decompose the cobordism $\mathcal{X}$ into a composition of elementary link cobordisms (Definition 9.5). Write $\mathcal{X} = \mathcal{X}_4 \circ \mathcal{X}_3 \circ \mathcal{X}_2 \circ \mathcal{X}_1$, where

- $\mathcal{X}_1$ is a 0–handle cobordism which adds a doubly based unknot in S^3;
- $\mathcal{X}_2$ is a stabilization cobordism which adds two basepoints to the unknot in S^3;
- $\mathcal{X}_3$ consists of $(n-1)$ 0–handles, each adding a doubly based unknot;
- $\mathcal{X}_4$ consists of $(k+n-1)$ 1–handles.

If $\mathbb{U} \subseteq S^3$ is a doubly based unknot, then $\widehat{\mathrm{HFL}}(S^3, \mathbb{U}) \cong \mathbb{F}_2$. Noting that the 0–handle maps are nonzero, since they can be canceled topologically, using multiplicitivity of link Floer homology under disjoint unions, it follows that

$$F^J_{\mathcal{X}_i} = \hat{F}^Z_{\mathcal{X}_i}$$

for $i = 1, 3$.

If $\mathbb{U}'$ is an unknot in S^3 with four basepoints, then, by Lemma 9.11, we have $\widehat{\mathrm{HFL}}(S^3, \mathbb{U}') \cong \mathbb{F}_2 \oplus \mathbb{F}_2$. Furthermore, $\widehat{\mathrm{HFL}}(S^3, \mathbb{U}')$ is generated by two elements, $\Theta^{\boldsymbol{w}}$ and $\Theta^{\boldsymbol{z}}$, which are distinguished by grading. The element $\Theta^{\boldsymbol{w}}$ has $(\mathrm{gr}_{\boldsymbol{w}}, \mathrm{gr}_{\boldsymbol{z}})$ bigrading $\left(+\frac{1}{2}, -\frac{1}{2}\right)$, while $\Theta^{\boldsymbol{z}}$ has bigrading $\left(-\frac{1}{2}, +\frac{1}{2}\right)$.

It follows immediately from the definition of the maps in [30] that

$$\hat{F}^Z_{\mathcal{X}_2}(1) = \begin{cases} \Theta^{\boldsymbol{w}} & \text{if } \chi(S_{\boldsymbol{w}}) = \chi(S_{\boldsymbol{z}}) + 1, \\ \Theta^{\boldsymbol{z}} & \text{if } \chi(S_{\boldsymbol{w}}) = \chi(S_{\boldsymbol{z}}) - 1, \end{cases}$$

since the cobordism map for a stabilization cobordism is defined using the quasistabilization map [30, Section 3.2]. On the other hand, a straightforward functoriality argument shows that $F^J_{\mathcal{X}_2}$ must be nonzero. By [17, Theorem 5.18], it follows that $F^J_{\mathcal{X}_2}(1)$ has the same $\mathrm{gr}_{\boldsymbol{w}}$ and $\mathrm{gr}_{\boldsymbol{z}}$ grading as $\hat{F}^Z_{\mathcal{X}_2}(1)$. Since $\widehat{\mathrm{HFL}}(S^3, \mathbb{U}')$ is a 2–dimensional vector space over $\mathbb{F}_2$, it follows that $F^J_{\mathcal{X}_2} = \hat{F}^Z_{\mathcal{X}_2}$.

Finally, the two cobordism map constructions use the same 4–dimensional handle attachment maps, so

$$F^J_{\mathcal{X}_4} = \hat{F}^Z_{\mathcal{X}_4}.$$

Furthermore, as in Lemma 9.11, the 1–handle maps preserve the top-graded elements $\Theta^{\boldsymbol{w}}$ and $\Theta^{\boldsymbol{z}}$. Composing all the maps, the claim now follows. □

In [30], the link cobordism maps for

$$\mathcal{X}_{\boldsymbol{w}} = (I \times Y,\ S(B),\ \mathcal{A}_{\boldsymbol{w}}) \quad \text{and} \quad \mathcal{X}_{\boldsymbol{z}} = (I \times Y,\ S(B),\ \mathcal{A}_{\boldsymbol{z}})$$

are defined to be

$$\hat{F}^Z_{\mathcal{X}_{\boldsymbol{w}}}(-) := F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}(- \otimes \Theta^{\boldsymbol{z}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}) \quad \text{and} \quad \hat{F}^Z_{\mathcal{X}_{\boldsymbol{z}}}(-) := F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}(- \otimes \Theta^{\boldsymbol{w}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}). \tag{9-1}$$

Lemma 9.13 *For the decorated saddle cobordisms $\mathcal{X}_{\boldsymbol{w}}$ and $\mathcal{X}_{\boldsymbol{z}}$, defined above, we have $F^J_{\mathcal{X}_{\boldsymbol{w}}} = \hat{F}^Z_{\mathcal{X}_{\boldsymbol{w}}}$ and $F^J_{\mathcal{X}_{\boldsymbol{z}}} = \hat{F}^Z_{\mathcal{X}_{\boldsymbol{z}}}$.*

Proof The key observation is that if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$ is a Heegaard triple subordinate to a β–band, then the 4–manifold $X_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}$ is equal to $I \times Y$ with a neighborhood of $\{\frac{1}{2}\} \times U_{\boldsymbol{\beta}}$ removed, and the surface with divides

$$(S_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}, \mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}) = \big(S(B) \setminus \big(\{\tfrac{1}{2}\} \times B\big), \mathcal{A}_0\big).$$

There is a cobordism $X_{\boldsymbol{\beta}}\colon \varnothing \to Y_{\boldsymbol{\beta},\boldsymbol{\beta}'}$ consisting of 0–handles and 1–handles. Inside $X_{\boldsymbol{\beta}}$, there is a surface S_0 that consists of $|\boldsymbol{w}|-1$ disks. We note that $|\boldsymbol{w}|-2$ of the disks have boundary equal to a doubly based unknot in $Y_{\boldsymbol{\beta},\boldsymbol{\beta}'}$, but one disk has boundary equal to an unknot with four basepoints. It is clear that if we fill in the $Y_{\boldsymbol{\beta},\boldsymbol{\beta}'}$ boundary of the link cobordism $(X_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}, S_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'})$ with $(X_{\boldsymbol{\beta}}, S_0)$, then we obtain the (undecorated) link cobordism $(I \times Y, S(B))$. On the other hand, as described above, there are two natural dividing sets on S_0, shown in Figure 55. Define

$$\mathcal{A}_{\boldsymbol{w},0} = S_0 \cap \mathcal{A}_{\boldsymbol{w}} \quad \text{and} \quad \mathcal{A}_{\boldsymbol{z},0} = S_0 \cap \mathcal{A}_{\boldsymbol{z}}.$$

Using Lemma 9.12, we have that

$$F^J_{X_{\boldsymbol{\beta}},S_0,\mathcal{A}_{\boldsymbol{w},0}}(1) = \Theta^{\boldsymbol{z}}_{\boldsymbol{\beta},\boldsymbol{\beta}'} \quad \text{and} \quad F^J_{X_{\boldsymbol{\beta}},S_0,\mathcal{A}_{\boldsymbol{z},0}}(1) = \Theta^{\boldsymbol{w}}_{\boldsymbol{\beta},\boldsymbol{\beta}'},$$

as maps from $\widehat{\mathrm{HFL}}(\varnothing,\varnothing) \cong \mathbb{F}_2$ to $\widehat{\mathrm{HFL}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\beta}', \boldsymbol{w}, \boldsymbol{z})$.

Using Theorem 7.1 and Proposition 9.8, we know that the sutured link cobordism map $F^J_{\mathcal{X}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}}$ for the decorated link cobordism

$$\mathcal{X}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'} = (X_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}, S_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}, \mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'})$$

is the map $F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}$ that counts holomorphic triangles on the Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$. It follows from the composition law that

$$F^J_{\mathcal{X}_z}(-) = F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}(- \otimes F_{X_{\boldsymbol{\beta}},S_0,\mathcal{A}_{z,0}}(1)) = F_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\beta}'}(- \otimes \Theta^{\boldsymbol{w}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}) = \hat{F}^Z_{\mathcal{X}_z}(-).$$

The result for $F^J_{\mathcal{X}_{\boldsymbol{w}}}$ follows similarly. □

9.6 Proof of Theorem 9.4

We can now prove that the link cobordism map constructions from [15] and [30] agree:

Proof of Theorem 9.4 Given an arbitrary decorated link cobordism $\mathcal{X} = (X, S, \mathcal{A})$, it is easy to see that one can decompose $\mathcal{X}$ into a sequence of link cobordisms which are each diffeomorphic to an elementary link cobordism (Definition 9.5). Using the composition law, it remains to verify the claim for each type of elementary link cobordism. The maps obviously agree for identity cobordisms, and 4–dimensional 0–, 1–, 2–, 3–, or 4–handle cobordisms. By Lemma 9.13, they agree for decorated saddle cobordisms. Finally, as in Remark 9.6, a stabilization cobordism can be decomposed into a composition of the other elementary cobordisms, and hence, having established the claim for the other types of elementary cobordisms, it follows as well for stabilization cobordisms. □

References

[1] **S Akbulut**, *Variations on Fintushel–Stern knot surgery on 4–manifolds*, Turkish J. Math. 26 (2002) 81–92 MR

[2] **T Etgü**, **B Özbağcı**, *Partial open book decompositions and the contact class in sutured Floer homology*, Turkish J. Math. 33 (2009) 295–312 MR

[3] **J Etnyre**, *Convex surfaces in contact geometry: class notes*, course notes (2004) Available at http://people.math.gatech.edu/~etnyre/preprints/papers/surfaces.pdf

[4] **S Friedl**, **A Juhász**, **J Rasmussen**, *The decategorification of sutured Floer homology*, J. Topol. 4 (2011) 431–478 MR

[5] **D Gabai**, *Foliations and the topology of 3–manifolds*, J. Differential Geom. 18 (1983) 445–503 MR

[6] **E Giroux**, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991) 637–677 MR

[7] **E Giroux**, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. 141 (2000) 615–689 MR

[8] **E Giroux**, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, from "Proceedings of the International Congress of Mathematicians, II" (T Li, editor), Higher Ed., Beijing (2002) 405–414 MR

[9] **K Hendricks**, **C Manolescu**, *Involutive Heegaard Floer homology*, Duke Math. J. 166 (2017) 1211–1299 MR

[10] **K Honda**, *On the classification of tight contact structures, I*, Geom. Topol. 4 (2000) 309–368 MR

[11] **K Honda**, *On the classification of tight contact structures, II*, J. Differential Geom. 55 (2000) 83–143 MR

[12] **K Honda**, **W H Kazez**, **G Matić**, *Contact structures, sutured Floer homology and TQFT*, preprint (2008) arXiv

[13] **K Honda**, **W H Kazez**, **G Matić**, *The contact invariant in sutured Floer homology*, Invent. Math. 176 (2009) 637–676 MR

[14] **A Juhász**, *Holomorphic discs and sutured manifolds*, Algebr. Geom. Topol. 6 (2006) 1429–1457 MR

[15] **A Juhász**, *Cobordisms of sutured manifolds and the functoriality of link Floer homology*, Adv. Math. 299 (2016) 940–1038 MR

[16] **A Juhász**, **M Marengon**, *Concordance maps in knot Floer homology*, Geom. Topol. 20 (2016) 3623–3673 MR

[17] **A Juhász**, **M Marengon**, *Computing cobordism maps in link Floer homology and the reduced Khovanov TQFT*, Selecta Math. 24 (2018) 1315–1390 MR

[18] **A Juhász**, **D P Thurston**, **I Zemke**, *Naturality and mapping class groups in Heegaard Floer homology*, preprint (2012) arXiv To appear under Mem. Amer. Math. Soc.

[19] **R A Litherland**, *Deforming twist-spun knots*, Trans. Amer. Math. Soc. 250 (1979) 311–331 MR

[20] **R Lutz**, *Structures de contact sur les fibrés principaux en cercles de dimension trois*, Ann. Inst. Fourier (Grenoble) 27 (1977) 1–15 MR

[21] **B Ozbagci**, *Contact handle decompositions*, Topology Appl. 158 (2011) 718–727 MR

[22] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR

[23] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR

[24] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR

[25] **P Ozsváth**, **Z Szabó**, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math. 202 (2006) 326–400 MR

[26] **P Ozsváth**, **Z Szabó**, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol. 8 (2008) 615–692 MR

[27] **J A Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University (2003) MR arXiv

[28] **R Zarev**, *Joining and gluing sutured Floer homology*, preprint (2010) arXiv

[29] **I Zemke**, *Duality and mapping tori in Heegaard Floer homology*, preprint (2018) arXiv

[30] **I Zemke**, *Link cobordisms and functoriality in link Floer homology*, J. Topol. 12 (2019) 94–220 MR

Mathematical Institute, University of Oxford
Oxford, United Kingdom

Department of Mathematics, Princeton University
Princeton, NJ, United States

juhasza@maths.ox.ac.uk, izemke@math.princeton.edu

Proposed: András I Stipsicz — Received: 7 April 2018
Seconded: Ciprian Manolescu, Peter Ozsváth — Revised: 30 March 2019