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Recognition of being fibered for compact 3–manifolds

ANDREI JAIKIN-ZAPIRAIN

Let M be a compact orientable aspherical 3–manifold. We show that if the profinite completion of $\pi_1(M)$ is isomorphic to the profinite completion of a free-by-cyclic group or to the profinite completion of a surface-by-cyclic group, then M fibers over the circle with compact fiber.

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1 Introduction

Let Γ be a finitely generated residually finite group. It is natural to ask what properties of Γ are determined by the set of its finite quotients. Recall that two finitely generated groups have the same collection of finite quotients if and only if their profinite completions are isomorphic (see the discussion around Theorem 2.2 of A Reid [15]). Thus, our previous question can be reformulated as follows:

Question 1 *Let Γ_1 and Γ_2 be two finitely generated groups. Assume that they have isomorphic profinite completions. What group-theoretic properties are shared by Γ_1 and Γ_2?*

There are many recent developments around this question. We recommend the reader to consult the survey of Reid [15], where all these new results are described.

In this paper we consider a particular case of Question 1, where the groups Γ_1 and Γ_2 are the fundamental groups of 3–manifolds. This case has been also studied actively in recent years (for more details, see [15, Section 4]).

Throughout this paper, all manifolds are connected and aspherical. We allow 3–manifolds to have nonempty boundary, but we assume that there are no spherical components.

By a free-by-cyclic group we mean a group $F \rtimes \mathbb{Z}$, where F is a finitely generated free group, and a surface-by-cyclic group is a group $S \rtimes \mathbb{Z}$, where S is the fundamental group of a compact closed orientable surface. Our result shows that the profinite

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completion of the fundamental group of a 3–manifold determines whether the manifold fibers over the circle with compact fiber.

Theorem 1.1 *Let M be a compact orientable 3–manifold. Assume that the profinite completion of $\pi_1(M)$ is isomorphic to the profinite completion of a free-by-cyclic group or to the profinite completion of a surface-by-cyclic group. Then M fibers over the circle with compact fiber.*

Under some additional conditions, this result has been proved by Bridson and Reid [5], Boileau and Friedl [4] and Bridson, Reid and Wilton [6]. Our proof and all the previous proofs use in an essential way results of I Agol [1; 2] and P Przytycki and D Wise [14; 18] on separability of 3–manifold groups.

Corollary 1.2 *Let M and N be two compact orientable 3–manifolds such that $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$. Then M fibers over the circle if and only if N does.*

Let us explain briefly the strategy of the proof of Theorem 1.1. If Γ is a finitely generated group, then we say that a nontrivial class $\phi \in H^1(\Gamma, \mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z})$ is *fibered* if $\ker \phi$ is finitely generated. If $\Gamma = \pi_1(M)$ is the fundamental group of a compact orientable 3–manifold M, then, by a well-known result of J Stallings [17], fibered classes in $H^1(\pi_1(M), \mathbb{Z})$ are in one-to-one correspondence with fibrations of M over the circle with compact fiber.

Now, let us assume that M satisfies the conditions of Theorem 1.1. First we have to find a useful criterion for a class $\phi \in H^1(\pi_1(M), \mathbb{Z})$ to be fibered. We start with a criterion proved by S Friedl and S Vidussi (Proposition 3.3). In Section 3.1 we explain the notion of cohomological goodness introduced by J-P Serre [16]. Using that the fundamental group of a 3–manifold is cohomologically good we reformulate Proposition 3.3 in terms of the profinite completion of $\pi_1(M)$ and obtain Corollary 3.4. It tells us that $\phi \in H^1(\pi_1(M), \mathbb{Z})$ is fibered if certain homology groups of some subgroups of $\widehat{\pi_1(M)}$ are finite. The latter property can be understood in terms of certain rank functions, which we study in Section 2. In Section 4 we prove a proposition that together with Corollary 2.2 allows us to find a desired class. All this is explained again in more detail in Section 5.

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2 Rank functions

Let R be a commutative domain and let $Q(R)$ denote its field of fractions. Given a matrix A over R we denote by $\mathrm{rk}_R(A)$ the $Q(R)$–rank of A. Similarly, if $\phi\colon M_1 \to M_2$ is a homomorphism of finitely generated free left R–modules, we denote by $\mathrm{rk}_R(\phi)$ the $Q(R)$–rank of the induced map $Q(R)\otimes_R M_1 \to Q(R)\otimes_R M_2$.

We may extend the previous notation in the following way. Let S be a ring and assume that S_0 is a subring of S such that $S \cong (S_0)^k$ for some $k \geq 1$ as a left S_0–module. If $\alpha\colon S_0 \to R$ is a homomorphism of rings, then α induces a structure of right S_0–module on R. Take $A \in \mathrm{Mat}_{n\times m}(S)$ and consider the homomorphism of finitely generated free left R–modules

$$\phi_\alpha^A\colon (R\otimes_{S_0} S)^n \to (R\otimes_{S_0} S)^m, \quad (x_1,\dots,x_n)\mapsto (x_1,\dots,x_n)A.$$

If R is a commutative domain, then we put $\mathrm{rk}_\alpha(A) = \mathrm{rk}_R(\phi_\alpha^A)$.

Let us describe two concrete examples of rank functions that will appear in this paper. Let Γ be a residually finite group and H a subgroup of finite index. Put $S = \mathbb{Z}[\Gamma]$ and $S_0 = \mathbb{Z}[H]$. It is clear that S is a left free S_0–module. We fix a prime p once and for all throughout the paper.

(A) Let K be a normal subgroup of Γ such that Γ/K is abelian and torsion-free. Let $\alpha_1\colon \mathbb{Z}[H] \to \mathbb{F}_p[H/H\cap K]$ be the canonical map. We define

$$\mathrm{rk}_{H,K}(A) = \frac{\mathrm{rk}_{\alpha_1}(A)}{|\Gamma : H|}, \quad \text{where } A \text{ is a matrix over } \mathbb{Z}[\Gamma].$$

(B) For a profinite group G and a profinite ring T we denote by $T[\![G]\!]$ *the completed group algebra*

$$T[\![G]\!] = \varprojlim_{U\trianglelefteq_o G} T[\![G/U]\!].$$

Now let G be a profinite completion of Γ. We identify Γ with its canonical image in G. Let U be the closure of H in G and let N be a closed normal subgroup of G such that G/N is an abelian and torsion-free pro- p group. Let $\alpha_2\colon \mathbb{Z}[H] \to \mathbb{F}_p[\![U/U\cap N]\!]$ be the canonical map. In this context we define

$$\mathrm{rk}_{H,N}(A) = \frac{\mathrm{rk}_{\alpha_2}(A)}{|G:U|}, \quad \text{where } A \text{ is a matrix over } \mathbb{Z}[\Gamma].$$

This rank function coincides with the previous one in the case $K = \Gamma \cap N$. Indeed, for every matrix A over $\mathbb{Z}[\Gamma]$ we have the equality

$$\mathrm{rk}_{\alpha_1}(A) = \mathrm{rk}_{\alpha_2}(A)$$

because α_2 is the composition of α_1 and the canonical embedding of $\mathbb{F}_p[H/H\cap K]$ into $\mathbb{F}_p[\![U/U\cap N]\!]$. Thus, we obtain that

$$\mathrm{rk}_{H,\Gamma\cap N}(A) = \mathrm{rk}_{H,N}(A). \tag{1}$$

The rank functions $\mathrm{rk}_{H,K}$ and $\mathrm{rk}_{H,N}$ are examples of Sylvester matrix rank functions on $\mathbb{Z}[\Gamma]$. The interested reader can find more information about them in [11]. We need the following elementary lemma about rank functions:

Lemma 2.1 *Let S be a ring and assume that S_0 is a subring of S such that S is a finitely generated free left S_0–module. Let $\pi\colon R_1 \to R_2$ be a homomorphism between two commutative domains and $\alpha_1\colon S_0 \to R_1$ a homomorphism. We put $\alpha_2 = \pi\circ\alpha_1$. Then for any matrix A over S we have*

$$\mathrm{rk}_{\alpha_1}(A) \ge \mathrm{rk}_{\alpha_2}(A).$$

Proof For simplicity of exposition, let us assume that $A \in S$ (that is, A is a 1-by-1 matrix).

Let $\{v_1,\dots,v_k\}$ be a basis of S as a left S_0–module. Consider the matrix associated to ϕ_{Id}^A with respect to $\{v_1,\dots,v_k\}$. It is a k-by-k matrix $B=(b_{ij})$ over S_0 defined by means of

$$v_i A = \sum_{j=1}^k b_{ij} v_j.$$

Let $B_1 = \alpha_1(B)$ and $B_2 = \alpha_2(B)$, and observe that $B_2 = \pi(B_1)$. Thus, $\mathrm{rk}_{R_1}(B_1) \ge \mathrm{rk}_{R_2}(B_2)$. Therefore, we obtain that

$$\mathrm{rk}_{\alpha_1}(A) = \mathrm{rk}_{R_1}(B_1) \ge \mathrm{rk}_{R_2}(B_2) = \mathrm{rk}_{\alpha_2}(A). \qquad \square$$

We will apply the lemma in the following situation:

Corollary 2.2 *Let Γ be a residually finite group and let $G = \widehat{\Gamma}$. Let N be a closed normal subgroup of G such that $G/N \cong \mathbb{Z}_p$ and let K be a normal subgroup of Γ such that $\Gamma/K \cong \mathbb{Z}$. Assume that $\Gamma \cap N \leq K$. Then, for every matrix A over $\mathbb{Z}[\Gamma]$ and every subgroup H of Γ of finite index,*

$$\mathrm{rk}_{H,N}(A) \geq \mathrm{rk}_{H,K}(A).$$

Proof Using (1), we obtain that $\mathrm{rk}_{H,\Gamma\cap N}(A) = \mathrm{rk}_{H,N}(A)$. Since $\Gamma \cap N \leq K$, the previous lemma implies that $\mathrm{rk}_{H,\Gamma\cap N}(A) \geq \mathrm{rk}_{H,K}(A)$ Hence we are done. □

3 Preliminaries on 3–manifolds

3.1 Cohomologically good groups

Let Γ be a group and $\widehat{\Gamma}$ its profinite completion. The group Γ is called *cohomologically good* if the homomorphism of cohomology groups

$$i^n(M)\colon H^n(\widehat{\Gamma}, M) \to H^n(\Gamma, M)$$

induced by the natural homomorphism $i\colon \Gamma \to \widehat{\Gamma}$ is an isomorphism for every $\mathbb{Z}[\Gamma]$–module M having a finite number of elements. This notion was introduced by Serre (see [16, Section I.2.6]) and has been studied in several papers (see for example [10; 9; 13]).

It is known that free, surface, free-by-cyclic, surface-by-cyclic and 3–manifold groups are cohomologically good. The result on 3–manifold groups (see for example [7; 9, Remark 5.14; 3]) uses the recent advances in the theory of 3–manifolds, including the solution of the virtual Haken conjecture by Agol [2]. We want to point out that under the hypothesis of Theorem 1.1 that the profinite completion of $\pi_1(M)$ is isomorphic to the profinite completion of a free-by-cyclic group or to the profinite completion of a surface-by-cyclic group, the cohomological goodness of $\pi_1(M)$ can be proved directly as in [9, Corollary 5.4].

In the case where Γ is an FP_∞–group there is an alternative way to define the cohomological goodness.

Proposition 3.1 *Let Γ be an FP_∞–group and let*

$$\cdots \to \mathbb{Z}[\Gamma]^{d_i} \xrightarrow{\phi_i} \mathbb{Z}[\Gamma]^{d_{i-1}} \to \cdots \to \mathbb{Z}[\Gamma]^{d_1} \xrightarrow{\phi_1} \mathbb{Z}[\Gamma] \to \mathbb{Z} \to 0$$

be a resolution of the trivial $\mathbb{Z}[\Gamma]$*–module* $\mathbb{Z}$*. Then* Γ *is cohomologically good if and only if the induced sequence*

$$\cdots \to \widehat{\mathbb{Z}}[\![\widehat{\Gamma}]\!]^{d_i} \xrightarrow{\widehat{\phi}_i} \widehat{\mathbb{Z}}[\![\widehat{\Gamma}]\!]^{d_{i-1}} \to \cdots \to \widehat{\mathbb{Z}}[\![\widehat{\Gamma}]\!]^{d_1} \xrightarrow{\widehat{\phi}_1} \widehat{\mathbb{Z}}[\![\widehat{\Gamma}]\!] \to \widehat{\mathbb{Z}} \to 0$$

is exact.

Proof The "if" part is clear. Let us show the "only if" part.

By [16, page 15], since Γ is cohomologically good we have that for every $n \geq 1$, every prime p, every subgroup H of Γ of finite index and every $\alpha \in H^k(H, \mathbb{F}_p)$, there exists another subgroup T of finite index in H such that the restriction of α to T vanishes in $H^k(T, \mathbb{F}_p)$. Hence the direct limit of $H^k(H, \mathbb{F}_p)$ (when H runs over all subgroups of finite index of Γ) is equal to zero, and so the inverse limit of $H_k(H, \mathbb{F}_p)$ is also equal to zero. Now we can apply [12, Theorem 2.5] and conclude that $H_k(\Gamma, \mathbb{Z}_p[\![\widehat{\Gamma}]\!]) = 0$ for each p, and therefore $H_k(\Gamma, \widehat{\mathbb{Z}}[\![\widehat{\Gamma}]\!]) = 0$. This implies the exactness of the second sequence in the statement of the proposition. □

3.2 Profinite completions of 3–manifold groups

In the following proposition we show that the hypotheses of Theorem 1.1 impose several restrictions on the boundary of M.

Proposition 3.2 *Let* M *be a compact orientable* 3*–manifold and let* Γ *be a free-by-cyclic group or a surface-by-cyclic group. Assume that* $\widehat{\pi_1(M)} \cong \widehat{\Gamma}$*. Then the boundary of* M *is a union of incompressible tori or it is empty.*

Proof If Γ is a free-by-cyclic group, the proposition follows from [5, Corollary 4.3]. If Γ is a surface-by-cyclic group, then $\widehat{\pi_1(M)}$ is of cohomological dimension 3. Since $\pi_1(M)$ is cohomologically good, $\pi_1(M)$ is of cohomological dimension 3 as well, and so M does not have boundary. □

3.3 A criterion for fibering

Let $\phi\colon \Gamma \to \mathbb{Z}$ be a nontrivial homomorphism and let H be a subgroup of finite index in Γ. We denote by H_ϕ the intersection $H \cap \ker\phi$. We have the following criterion of fibering, stated in a slightly different way in a paper of Friedl and Vidussi [8]:

Proposition 3.3 *Let M be a compact orientable 3–manifold with toroidal or empty boundary and let $\phi\colon \pi_1(M)\to\mathbb{Z}$ be a nontrivial map. Then the class ϕ is fibered if and only if for every normal subgroup H of $\pi_1(M)$ of finite index the group $H_1(H_\phi,\mathbb{F}_p)$ is finite.*

Proof The "only if" part is clear, since if ϕ fibers, $\ker\phi$ is finitely generated.

Let H be a normal subgroup of $\pi_1(M)$ of finite index and assume that $H_1(H_\phi,\mathbb{F}_p)$ is finite. Hence $H_1(\pi_1(M),\mathbb{F}_p[\pi_1(M)/H_\phi])\cong H_1(H_\phi,\mathbb{F}_p)$ is also finite. Put $L=\mathbb{F}_p[\pi_1(M)/H]$. The tensor product

$$\widetilde{L}=\mathbb{F}_p[\pi_1(M)/\ker\phi]\otimes_{\mathbb{F}_p}L$$

is a finitely generated right $\pi_1(M)$–module (the elements of $\pi_1(M)$ act diagonally on L_ϕ). The i^{th} twisted Alexander polynomial $\Delta^L_{M,\phi,i}$ is the order of the $\mathbb{F}_p[\pi_1(M)/\ker\phi]$–module $H_i(\pi_1(M),\widetilde{L})$ (see [4] for details). Observe that $\widetilde{L}$ is a direct sum of a finite number of copies of $\mathbb{F}_p[\pi_1(M)/H_\phi]$. Therefore, $H_1(\pi_1(M),\widetilde{L})$ is also finite. Hence

$$\Delta^{\mathbb{F}_p[\pi_1(M)/H]}_{M,\phi,1}\neq 0.$$

Thus, we can apply [8, Theorem 1.1] and obtain the "if" part of the proposition. □

Let Γ be a finitely generated group. A map $\phi\colon\Gamma\to\mathbb{Z}$ induces a homomorphism $\phi_k\colon\widehat{\Gamma}\to\mathbb{Z}_p/p^k\mathbb{Z}_p$. We put

$$\widehat{\phi}=\lim_k\phi_k\colon\widehat{\Gamma}\to\mathbb{Z}_p$$

and say that $\widehat{\phi}$ is *associated* to ϕ.

Let G be a profinite group, U an open subgroup and $\varphi\colon G\to\mathbb{Z}_p$ a nontrivial homomorphism. We denote by U_φ the intersection $U\cap\ker\varphi$. Note that $U/U_\varphi\cong\mathbb{Z}_p$. Now we present a profinite analogue of Proposition 3.3.

Corollary 3.4 *Let M be a compact orientable 3–manifold with toroidal or empty boundary and let $\phi\colon\pi_1(M)\to\mathbb{Z}$ be a nontrivial map. Denote by $\widehat{\phi}\colon\widehat{\pi_1(M)}\to\mathbb{Z}_p$ the homomorphism associated to ϕ. Then the class ϕ is fibered if and only if for every normal open subgroup U of $\widehat{\pi_1(M)}$ the group $H_1(U_{\widehat{\phi}},\mathbb{F}_p)$ is finite.*

Proof The "only if" part is clear. Let us prove the "if" part.

We write a projective resolution of $\mathbb{Z}$ as a left $\mathbb{Z}[\pi_1(M)]$–module

$$\mathbb{Z}[\pi_1(M)]^r \xrightarrow{m_B} \mathbb{Z}[\pi_1(M)]^d \to \mathbb{Z}[\pi_1(M)] \to \mathbb{Z} \to 0,$$

where m_B is multiplication by a matrix B. Since, $\pi_1(M)$ is cohomologically good, using Proposition 3.1, we obtain an exact sequence

$$\widehat{\mathbb{Z}}[\![\widehat{\pi_1(M)}]\!]^r \xrightarrow{m_B} \widehat{\mathbb{Z}}[\![\widehat{\pi_1(M)}]\!]^d \to \widehat{\mathbb{Z}}[\![\widehat{\pi_1(M)}]\!] \to \widehat{\mathbb{Z}} \to 0.$$

Let H be a normal subgroup of $\pi_1(M)$ of finite index and let U be the closure of H in $\widehat{\pi_1(M)}$. Then U is a normal open subgroup of $\widehat{\pi_1(M)}$. Note that

$$H_1(U_{\widehat{\phi}}, \mathbb{F}_p) \cong H_1(\widehat{\pi_1(M)}, \mathbb{F}_p[\![\widehat{\pi_1(M)}/U_{\widehat{\phi}}]\!]).$$

Thus, since $H_1(U_{\widehat{\phi}}, \mathbb{F}_p)$ is finite, we obtain that $\operatorname{rk}_{H,\ker\widehat{\phi}}(B) = d-1$. Now, observe that, by (1),

$$\operatorname{rk}_{H,\ker\phi}(B) = \operatorname{rk}_{H,\ker\widehat{\phi}}(B) = d-1.$$

Therefore,

$$H_1(H_\phi, \mathbb{F}_p) \cong H_1(\pi_1(M), \mathbb{F}_p[\pi_1(M)/H_\phi])$$

is also finite. Hence we can apply Proposition 3.3 and obtain the "if" part of the corollary. □

4 Finding a fibered class

The following proposition will be a key result that will help us to find a fibered class in Theorem 1.1.

Proposition 4.1 *Let $V = \mathbb{Z}_p^k$ and let $I \cong J \cong \mathbb{Z}^k$ be two dense subgroups of V. Put $V^* = \operatorname{Hom}(V, \mathbb{Z}_p)$. For $\varphi \in V^*$ we denote by $\varphi|_I$ the restriction of φ to I. Then, for every $\psi \in V^*$ such that $\psi(I) \le \mathbb{Z}$, there exists an element $\varphi \in V^*$ such that $\varphi(J) \le \mathbb{Z}$ and $\ker\varphi|_I \le \ker\psi|_I$.*

Proof Let $\{v_1, \dots, v_k\}$ and $\{u_1, \dots, u_k\}$ be $\mathbb{Z}$–bases of I and J, respectively. Define elements $v_1^*, \dots v_k^*, u_1^*, \dots, u_k^*$ of V^* by means of

$$v_i^*(v_j) = u_i^*(u_j) = \delta_{ij}.$$

Since $\{v_1, \dots, v_k\}$ and $\{u_1, \dots, u_k\}$ are $\mathbb{Z}_p$–bases of V, $\{v_1^*, \dots, v_k^*\}$ and $\{u_1^*, \dots, u_k^*\}$ are $\mathbb{Z}_p$–bases of V^*. Hence there exists a matrix $A \in \mathrm{GL}_k(\mathbb{Z}_p)$ such that

(2) $$(u_1^*, \dots, u_k^*) = (v_1^*, \dots, v_k^*)A.$$

Write A as

$$A = \sum_{i=1}^{s} A_i z_i, \tag{3}$$

where $A_i \in \mathrm{Mat}_k(\mathbb{Z})$ and $\{z_1, \dots, z_s\} \subset \mathbb{Z}_p$ are $\mathbb{Z}$–linearly independent.

Let $R = \mathbb{Q}[t_1, \dots, t_s]$ be a polynomial ring. We put $B = \sum_{i=1}^{s} A_i t_i \in \mathrm{Mat}_k(R)$. Since $A = \sum_{i=1}^{s} A_i z_i$ is invertible over $\mathbb{Z}_p$, $\det A \neq 0$. Hence $\det B \neq 0$ as well and, as a consequence, there are $q_1, \dots, q_s \in \mathbb{Z}$ such that

$$\det\left(\sum_{i=1}^{s} A_i q_i\right) \neq 0.$$

Let us put

$$C = \sum_{i=1}^{s} A_i q_i. \tag{4}$$

We can express ψ as

$$\psi = \alpha_1 v_1^* + \dots + \alpha_k v_k^*, \quad \text{where } \alpha_i \in \mathbb{Z}. \tag{5}$$

Since C is invertible over $\mathbb{Q}$, there are $\beta_1, \dots, \beta_k \in \mathbb{Z}$ and $0 \neq n \in \mathbb{Z}$ such that

$$C \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = n \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}. \tag{6}$$

Now we set

$$\begin{aligned} \varphi = \sum_{i=1}^{k} \beta_i u_i^* &= (v_1^*, \dots, v_k^*) A \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} && \text{(by (2))} \\ &= \sum_{j=1}^{s} z_j (v_1^*, \dots, v_k^*) A_j \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} && \text{(by (3))}, \end{aligned}$$

and define

$$\varphi_j = (v_1^*, \dots, v_k^*) A_j \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \in \mathbb{Z} v_1^* + \dots + \mathbb{Z} v_k^*. \tag{7}$$

Then, since $\{z_1, \dots, z_s\}$ are $\mathbb{Z}$–linearly independent,

$$\ker \varphi|_I = \ker(z_1(\varphi_1)|_I + \dots + z_s(\varphi_s)|_I) = \bigcap_{i=1}^{s} \ker(\varphi_j)|_I.$$

Thus, we obtain

$$\begin{aligned}\ker\varphi|_I \le \ker\Big(\sum_{j=1}^{s} q_j(\varphi_j)|_I\Big) &= \ker\left((v_1^*,\dots,v_k^*)C\begin{pmatrix}\beta_1\\ \vdots\\ \beta_k\end{pmatrix}|_I\right) && \text{(by (7) and (4))}\\ &= \ker\left((v_1^*,\dots,v_k^*)\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_k\end{pmatrix}|_I\right) && \text{(by (6))}\\ &= \ker\psi|_I && \text{(by (5)).} \qquad\square\end{aligned}$$

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let $\Gamma = S\rtimes\mathbb{Z}$ be a free-by-cyclic or a surface-by-cyclic group. Then $\widehat{\Gamma}\cong\widehat{S}\rtimes\widehat{\mathbb{Z}}$. We write a projective resolution of $\mathbb{Z}$ as a left $\mathbb{Z}[\Gamma]$–module,

$$\mathbb{Z}[\Gamma]^r \xrightarrow{m_A} \mathbb{Z}[\Gamma]^d \to \mathbb{Z}[\Gamma]\to\mathbb{Z}\to 0,$$

where m_A is multiplication by a matrix A. We will need the following lemma:

Lemma 5.1 *Let $\varphi\colon\widehat{\Gamma}\to\mathbb{Z}_p$ be a nontrivial homomorphism and let H be a subgroup of Γ of finite index. Denote by U the closure of H in $\widehat{\Gamma}$. Then $H_1(U_\varphi,\mathbb{F}_p)$ is finite if and only if $\operatorname{rk}_{H,\ker\varphi}(A) = d-1$.*

Proof We may use the same argument as in the proof of Corollary 3.4, because Γ is cohomologically good. □

Let M be a compact orientable 3–manifold and assume that $\widehat{\pi_1(M)}$ and $\widehat{\Gamma}$ are isomorphic. By Proposition 3.2, M has toroidal or empty boundary.

Denote by $\alpha\colon\widehat{\pi_1(M)}\to\widehat{\Gamma}$ an isomorphism between $\widehat{\pi_1(M)}$ and $\widehat{\Gamma}$. In order to show that $\pi_1(M)$ is fibered we want to use Corollary 3.4. To do so we are going to work in $\widehat{\Gamma}$ and then transport the obtained information from $\widehat{\Gamma}$ to $\widehat{\pi_1(M)}$ using α^{-1}.

Let $f\colon\Gamma\to\mathbb{Z}$ be such that $\ker f = S$ and let $\widehat{f}\colon\widehat{\Gamma}\to\mathbb{Z}_p$ be the map associated to f. Denote by V the maximal torsion-free abelian pro-p quotient of $\widehat{\Gamma}$. Then $V\cong\mathbb{Z}_p^k$ for some $k\ge 1$. Let I and J be the images in V of Γ and $\alpha(\pi_1(M))$, respectively.

Applying Proposition 4.1 for $\psi=\widehat{f}$, we obtain that there exists $\varphi\colon\widehat{\Gamma}\to\mathbb{Z}_p$ such that

$$\varphi(\alpha(\pi_1(M)))\le\mathbb{Z} \quad\text{and}\quad \ker\varphi\cap\Gamma\le\ker f.$$

Let U be an open normal subgroup of $\widehat{\Gamma}$. Put $H = \Gamma \cap U$. Observe that U is equal to the closure of H in $\widehat{\Gamma}$. By Corollary 2.2,

$$\mathrm{rk}_{H,\ker\varphi}(A) \geq \mathrm{rk}_{H,\ker f}(A) = d - 1.$$

Hence, by Lemma 5.1, $H_1(U_\varphi, \mathbb{F}_p)$ is finite.

Now, we put

$$\phi = \varphi \circ \alpha|_{\pi_1(M)} \in H^1(\pi_1(M), \mathbb{Z}).$$

Let $\widehat{\phi}\colon \widehat{\pi_1(M)} \to \mathbb{Z}_p$ be associated to ϕ. Then we have that $\widehat{\phi} = \varphi \circ \alpha$.

Thus, we obtain that

$$H_1(U_{\widehat{\phi}}, \mathbb{F}_p) \cong H_1(\alpha(U_{\widehat{\phi}}), \mathbb{F}_p) = H_1(\alpha(U)_\varphi, \mathbb{F}_p)$$

is finite for every normal open subgroup U of $\widehat{\pi_1(M)}$. Therefore, applying Corollary 3.4, we obtain that ϕ is fibered. This finishes the proof of Theorem 1.1.

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Departamento de Matemáticas, University Autónoma de Madrid and Instituto de Ciencias Matemáticas
Madrid, Spain

andrei.jaikin@uam.es

http://matematicas.uam.es/~andrei.jaikin

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