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Trees of metric compacta and trees of manifolds

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We present a construction, called a *tree of spaces*, that allows us to produce many compact metric spaces that are good candidates for being (up to homeomorphism) Gromov boundaries of some hyperbolic groups. We develop also a technique that allows us (1) to work effectively with the spaces in this class and (2) to recognize ideal boundaries of various classes of infinite groups, up to homeomorphism, as some spaces in this class.

We illustrate the effectiveness of the presented technique by clarifying, correcting and extending various results concerning the already widely studied class of spaces called *trees of manifolds*.

In a companion paper (Geom. Topol. 24 (2020) 593–622), which builds upon results from the present paper, we show that trees of manifolds in arbitrary dimension appear as Gromov boundaries of some hyperbolic groups.

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Introduction

We present a construction, called the *limit of a tree system of spaces* (or, less formally, a *tree of spaces*). The construction is designed to produce compact metric spaces that resemble fractals out of more regular spaces, such as closed manifolds, compact polyhedra, compact Menger manifolds, etc. Such spaces are potential candidates to be homeomorphic to ideal boundaries of infinite groups.

A very special case of this construction, *trees of manifolds* (known also as *Jakobsche spaces*), has been studied in the literature; see Ancel and Siebenmann [1], Jakobsche [12; 13], Stallings [21], Fischer [9], Przytycki and Świątkowski [19] and Zawiślak [26]. We present here a different approach, much more general, and, we believe, much more convenient for establishing various basic properties of the resulting spaces in a more general setting. Already in the case of trees of manifolds, using this approach we clarify, correct and extend known results and properties; see eg Propositions A and C, and Theorem B in this introduction. The most significant (although a bit technical) new results in this direction are Theorem 3.E.2 and Corollary 3.E.4, which pave the

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way to the results of the companion paper [25], in which ideal boundaries of many groups, in arbitrary dimension, are identified as trees of manifolds (see eg Theorem D in this introduction).

For general trees of spaces, we provide a toolbox of results that allow us, for example, to

- identify the resulting spaces as limits of some associated inverse systems, which allows further study of their properties (see Proposition 1.D.1 and Theorem 2.B.4);
- estimate (in many cases sharply) topological dimension of the resulting spaces (Corollary 1.D.3.2 and Propositions 2.D.2 and 2.D.5);
- manipulate the data involved in a tree system, without affecting the resulting limit space (Theorem 3.A.1 and Proposition 3.C.8);
- use manipulations as above to study orbits in the resulting spaces under homeomorphisms.

This toolbox can be widely used in the study of boundaries of groups other than trees of manifolds, as has been exemplified by its direct application in work of Hruska and Ruane [11] and Pawlik and Zabłocki [18].

Outline In Section 1 we introduce the notions of a tree system of metric compacta (Section 1.B) and its limit (Section 1.C). We then prove that such a limit is always a compact metrizable space (Section 1.D) by showing that it is homeomorphic to the limit of some inverse system naturally associated to a tree system. In Section 1.E we introduce the notion of isomorphism of tree systems. We also present a class of natural examples — dense tree systems of closed (topological) manifolds M — and a class of spaces obtained as their limits — the trees of manifolds M, denoted by $\mathcal{X}(M)$ — depending uniquely up to homeomorphism on M. As we show in Section 2, trees of manifolds M coincide with the spaces studied earlier by Włodzimierz Jakobsche in [13] (for M oriented) and by Paul Stallings in [21] (for nonorientable M). Our exposition of the case of nonorientable manifolds M, in Example 1.E.4, concerns all topological manifolds (and not only PL ones, as in [21]), and it frees the description from certain inconvenient and unnecessary conditions present in Stallings' approach (see Remark 1.E.4.3).

In Section 2 we exhibit some other useful inverse systems associated to tree systems. In Section 2.A we introduce some additional data, called the *extended system of spaces and maps*, necessary to produce such inverse systems. In Section 2.B we show that inverse limits of these new inverse systems canonically coincide with limits of the corresponding tree systems. Section 2.C deals with a subclass of tree systems called peripherally ANR, and describes inverse systems of particularly nice form associated to such tree systems. This allows us to relate our construction of limit of a tree system, in the case of dense tree systems of manifolds, to some earlier constructions from the literature, notably the construction of Jakobsche [13]. In Section 2.D we apply associated inverse systems to provide estimates from above for the topological dimension of limits of tree systems. We indicate some general cases in which these estimates are sharp.

In Section 3 we introduce some natural and useful operations on tree systems. In Section 3.A we describe an operation of consolidation, by which the spaces appearing in the initial system are merged together into bigger spaces, constituting naturally a new tree system. We show that this operation does not affect the limit. As an application, we derive equalities (up to homeomorphism) between the trees of manifolds for some families of different manifolds. In particular, we show the following (see Corollary 3.A.2.2 and Proposition 3.A.2.3).

Proposition A *Let $N_1, \dots, N_k$ be a family of closed connected topological manifolds that are of the same dimension. For any positive integers $m_1, \dots, m_k$, the space $\mathcal{X}(m_1 N_1 \# \cdots \# m_k N_k)$ is homeomorphic to $\mathcal{X}(N_1 \# \cdots \# N_k)$, where each $m_i N_i$ is the connected sum of m_i copies of N_i.*

As another application of the described technique of consolidation of tree systems, we give the following correction to the main result of Hanspeter Fischer in [9] (see Theorem 3.A.3 and the comment after its statement).

Theorem B *Suppose W is a right-angled Coxeter group whose nerve is a flag PL triangulation of a closed oriented manifold M, and let $\overline{M}$ be the same manifold with reversed orientation. Then the* $\mathrm{CAT}(0)$ *boundary of W* (ie the boundary of the Coxeter–Davis complex for W) *is the Jakobsche space $\mathcal{X}(M \# \overline{M})$.*

In Section 3.B we show how to decompose a compact metric space X into pieces which form a tree system, so that X naturally coincides with the limit of this tree system. We also show how to use such decompositions to determine limits of certain tree systems. In Section 3.C we introduce subdivision of a tree system, an operation opposite to consolidation, which generalizes the operation of decomposition from Section 3.B. In Section 3.D we apply operations of subdivision and consolidation to study orbits under homeomorphisms in trees of manifolds. In particular, we get the following extension of a result of Stallings [21] to topological (and not only PL) nonorientable manifolds (see Proposition 3.D.1).

Proposition C *Let N be a closed connected topological nonorientable manifold. Then the tree of manifolds N, ie the Jakobsche space $\mathcal{X}(N)$* (as defined in Example 1.E.4), *is homogeneous.*

We want to emphasize that the technique of subdivision of tree systems is potentially applicable in the study of orbits under homeomorphisms for limit spaces of more general tree systems. This is exemplified by Propositions 3.D.6.1 and 3.D.6.2.

Finally, in Section 3.E we apply the operations of consolidation and subdivision to provide a much more flexible description of certain trees of manifolds than those so far present in the literature (see Theorem 3.E.2 and Corollary 3.E.4). In the companion paper [25], among other applications, we use this description to identify ideal boundaries of many groups as trees of manifolds. In particular, we show in [25] that trees of manifolds in arbitrary dimension appear as Gromov boundaries of certain hyperbolic groups. More precisely, we show in [25] the following.

Theorem D (1) *For any natural number n, if a closed connected orientable PL n–manifold M bounds a compact orientable PL $(n+1)$–manifold, then the tree of manifolds $\mathcal{X}(M)$ is homeomorphic to the Gromov boundary of some hyperbolic group.*

(2) *For any closed connected nonorientable PL manifold N the tree of manifolds $\mathcal{X}(N)$ is homeomorphic to the Gromov boundary of some hyperbolic group.*

Wider context for results Our motivation for dealing with the general construction as presented in this paper comes from an attempt to understand which topological spaces are boundaries of hyperbolic groups. This problem, stated eg by Kapovich and Kleiner [16, Question A], remains wide open. An overview of the limited knowledge concerning this problem can be found in Kapovich and Benakli [15, Section 17] or in Przytycki and Świątkowski [19, Introduction].

The present paper initiates a larger research project, investigated by the author, concerning the above problem. We briefly outline the aims and expected lines of further development in this project.

- In a paper under preparation we describe a vast class of topological spaces called *trees of polyhedra*. These spaces are compact, metrizable, have finite topological dimension, and typically they are "wild" (eg they are usually not locally contractible, and hence not ANRs). They are obtained as limits of some tree systems, and depend uniquely up to homeomorphism on certain finite data, part of which is a finite collection of compact polyhedra. Thus, the spaces are given not by a universal characterization in terms of a list of properties, but rather by a sort of "presentation" (similar in spirit to a presentation of a group in combinatorial group theory). Typically, the same space has many distinct "presentations", and clarification of the relationship between such "presentations" will be one of the challenges. Some partial progress in studying very special trees of polyhedra, other than trees of manifolds, using (and explicitly referring to) the techniques developed in the present paper, has been obtained recently by other authors; see Zawiślak [27] and Pawlik and Zabłocki [18].

- The next part of the project consists of identifying boundaries of various classes of groups as explicit trees of polyhedra. We have formulated several conjectures in this direction, based on some results from the literature (see Remark 2.C.8), and on our new partial results. One of these conjectures deals with Gromov boundaries of all groups obtained by any procedure of strict hyperbolization (eg the one described in Charney and Davis [4]). Another conjecture concerns CAT(0) boundaries of a large class of right-angled Coxeter groups (not only word-hyperbolic ones). We have already confirmed these conjectures in some cases when the boundaries are trees of manifolds in arbitrary dimension (see [25, Theorem 1] or Theorem D in this introduction), trees of graphs, or trees of n–discs which are also known as the $(n-1)$–dimensional Sierpiński compacta (our recent results in [23; 24] provide implicitly such confirmations). Also, by referring to the methods developed in the present paper, boundaries of some amalgamated free products have been recognized as trees of multidiscs (ie of joins of spheres with finite sets); see Pawlik and Zabłocki [18]. Hruska and Ruane [11] have applied the results of the present paper in their study of Bowditch boundaries of some relatively hyperbolic groups. It seems that the boundaries of numerous (maybe even most) other groups so far studied in the literature are trees of polyhedra.

- A question arises of which hyperbolic groups have Gromov boundaries that are not trees of polyhedra. The following examples seem to belong to this class:

(1) Groups whose Gromov boundaries are Menger compacta of dimension ≥ 2 (see Dymara and Osajda [7] for examples of such groups, with boundaries of dimensions 2 and 3); we suspect that Menger compacta satisfy stronger disjoint disk properties than any trees of polyhedra of the same topological dimension.

(2) The 7–systolic groups, as defined by Januszkiewicz and Świątkowski [14], with boundaries of dimension ≥ 3; by Świątkowski [22], Gromov boundaries of 7–systolic groups contain no copy of the 2–disk, which cannot happen for a tree of polyhedra in dimensions ≥ 3.

(3) Topologically rigid hyperbolic groups, examples of which have been constructed by Kapovich and Kleiner [16]; we expect that trees of polyhedra always admit homeomorphisms whose dynamics is different from that occurring for the induced action of a group on its boundary.

• Boundaries of hyperbolic groups fall in the class of spaces called Markov compacta, as defined by Dranishnikov [6]; see Pawlik [17]. We work on showing that trees of polyhedra coincide with a subclass of Markov compacta of a certain rather simple form. This suggests the possibility of introducing a certain notion of degree of complexity for Markov compacta, with the lowest degree corresponding to trees of polyhedra. A big challenge is to explore more fully the territory of hyperbolic groups with higher-degree boundaries (ie boundaries which are not trees of polyhedra). For example, one can ask for new strict hyperbolization procedures, resulting with groups of higher-degree boundaries. One can also ask for explicit description of such boundaries, perhaps starting with degree just above the lowest one.

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1 Tree systems and their limits

1.A Some terminology and notation concerning trees

Trees under our consideration will be usually countably infinite, and locally infinite. We denote by V_T the set of all vertices of a tree T, and by O_T the set of all its oriented edges. For any $e \in O_T$, we denote by $\alpha(e)$ and $\omega(e)$ the initial and the terminal vertex of e, respectively. We also denote by $\bar{e}$ the same geometric edge as e, but oppositely oriented. For any $t \in V_T$, we denote by $N_t = \{e \in O_T : \alpha(e) = t\}$ the set of all oriented edges of T with initial vertex t.

We denote the combinatorial (embedded) paths in T as sequences of consecutive vertices $[t_0, t_1, \dots, t_m]$, or as sequences $[e_1, \dots, e_m]$ of consecutive oriented edges, or shortly by $[t, s]$ if t and s are the ends of the path. An infinite combinatorial path $[t_0, t_1, \dots]$ in T is called a *ray*, and denoted usually by ρ. We denote by $\rho(0)$ the initial vertex t_0, and by $e_1(\rho)$ the initial oriented edge (t_0, t_1) of a ray ρ.

We denote by E_T the set of *ends* of T, ie the set of equivalence classes of rays in T with respect to the relation of coincidence except possibly at some finite initial parts. We denote the end determined by a ray ρ as $[\rho]$.

Let S be a subtree of T. We then distinguish the set

$$N_S = \{e \in O_T : \alpha(e) \in V_S \text{ and } \omega(e) \notin V_S\}.$$

Note that, in case when S is reduced to a single vertex t, this notation agrees with the notation introduced earlier for the set N_t. A finite subtree of T will be usually denoted by F, and we shall consider the poset $(\mathcal{F}_T, \subset)$ of all finite subtrees of T.

1.B Tree systems of spaces

Recall that a family of subsets of a compact metric space is *null* if for each $\epsilon > 0$ all but finitely many of these subsets have diameters less than ϵ.

1.B.1 Definition A *tree system of metric compacta* is a tuple

$$\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$$

such that:

(TS1) T is a countable tree.

(TS2) To each $t \in V_T$ there is associated a compact metric space K_t.

(TS3) To each $e \in O_T$ there is associated a nonempty compact subset $\Sigma_e \subset K_{\alpha(e)}$, and a homeomorphism $\phi_e \colon \Sigma_e \to \Sigma_{\bar{e}}$ such that $\phi_{\bar{e}} = \phi_e^{-1}$.

(TS4) For each $t \in V_T$ the family $\{\Sigma_e : e \in N_t\}$ is null and consists of pairwise disjoint sets.

We call T the *underlying tree*, $\{K_t : t \in V_T\}$ the *constituent spaces*, $\{\Sigma_e : e \in O_T\}$ the *peripheral subspaces*, and $\{\phi_e : e \in O_T\}$ the *connecting maps* of the tree system Θ.

Remark In future applications of tree systems of spaces we will often additionally require that for any $t \in V_T$ the family $\{\Sigma_e : e \in N_t\}$ be *dense* in the space K_t (which means that the union of this family is a dense subset). However, to establish many basic properties of tree systems of metric compacta we do not need this requirement.

1.B.2 Example (tree system of manifolds) Let T be a countable tree, and let $\{M_t : t \in V_T\}$ be a family of closed manifolds of the same dimension n. For each $t \in V_T$ and each $e \in N_t$, let Δ_e be a collared n–dimensional disk embedded in M_t, and suppose that each $\mathcal{D}_t = \{\Delta_e : e \in N_t\}$ is a null family of pairwise disjoint subsets of M_t.

For each $t \in V_T$, set

$$K_t = M_t \setminus \Big(\bigcup \{\mathrm{int}(\Delta_e) : e \in N_t\}\Big).$$

This defines a family $\{K_t\}$ in a tree system of manifolds.

For each $e \in O_T$, set $\Sigma_e = \partial \Delta_e$, and note that $\Sigma_e \subset K_{\alpha(e)}$. For each $e \in O_T$ consider also a homeomorphism $\phi_e \colon \Sigma_{\alpha(e)} \to \Sigma_{\omega(e)}$ between the corresponding $(n-1)$–spheres such that $\phi_{\bar{e}} = \phi_e^{-1}$ for each e. A *tree system of manifolds* is a tuple $\mathcal{M} = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ as described above.

Intuitively, such a system $\mathcal{M}$ may be viewed as a pattern for a connected sum operation applied at the same time to a countable (in general infinite) family of closed manifolds.

1.C Limit of a tree system of spaces

We now describe the *limit* of a tree system $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$, denoted by $\lim \Theta$, starting with its description as a set. Denote by $\#\Theta$ the quotient

$$\Big(\bigsqcup_{t \in V_T} K_t\Big) \Big/ \sim$$

of the disjoint union of the sets K_t by the equivalence relation $\sim$ induced by the equivalences $x \sim \phi_e(x)$ for any $e \in O_T$ and any $x \in \Sigma_e$. This set may be viewed as obtained from the family $\{K_t : t \in V\}$ as a result of gluings provided by the maps ϕ_e. Observe that any set K_t canonically injects into $\#\Theta$. Define $\lim \Theta$ to be the disjoint union $\#\Theta \sqcup E_T$, where E_T is the set of ends of T.

To put the appropriate topology on the set $\lim \Theta$ we need some terminology. Given a family $\mathcal{A}$ of subsets in a set X, we say that $U \subset X$ is *saturated* with respect to $\mathcal{A}$ (shortly, $\mathcal{A}$*–saturated*) if for any $A \in \mathcal{A}$ we have either $A \subset U$ or $A \cap U = \varnothing$.

For any finite subtree F of T, denote by V_F and O_F the sets of vertices and of oriented edges in F, respectively. Write

$$\Theta_F = (F, \{K_t : t \in V_F\}, \{\Sigma_e : e \in O_F\}, \{\phi_e : e \in O_F\})$$

to denote the tree system of spaces Θ *restricted* to F. The set $K_F := \#\Theta_F$, equipped with the natural quotient topology (with which it is clearly a metrizable compact space), will be called the *partial union* of the system Θ related to F. Observe that any partial union K_F is canonically a subset in $\#\Theta$, and thus also in $\lim\Theta$.

For any finite subtree F, set $\mathcal{A}_F = \{\Sigma_e : e \in N_F\}$, and view the elements of this family as subsets in the partial union K_F. Observe that the family $\mathcal{A}_F$ consists of pairwise disjoint compact sets, and that it is null. Let $U \subset K_F$ be a subset which is saturated with respect to $\mathcal{A}_F$. Set $N_U := \{e \in N_F : \Sigma_e \subset U\}$ and

$$D_U := \{t \in V_T : [\omega(e), t] \cap V_F = \varnothing \text{ for some } e \in N_U\}$$

(ie D_U is the set of those vertices $t \in V_T \setminus V_F$ for which the shortest path in T connecting F to t starts with an edge $e \in N_U$). Here we mean in particular that the vertices $\omega(e)$ with $e \in N_U$ all belong to D_U.

Again, for any finite subtree F of T, let R_F be the set of all rays ρ in T with $\rho(0)$ in V_F and all other vertices outside V_F. Note that the map $\rho \to [\rho]$ is then a bijection from R_F to the set E_T of all ends of T. Given a subset $U \subset K_F$ as above (ie saturated with respect to $\mathcal{A}_F$), set $R_U := \{\rho \in R_F : e_1(\rho) \in N_U\}$ and $E_U := \{[\rho] \in E_T : \rho \in R_U\}$.

Finally, for any subset $U \subset K_F$ as above, set

$$G(U) := U \cup \left(\bigcup_{t \in D_U} K_t\right) \cup E_U,$$

where all summands in the above union are viewed as subsets of $\lim\Theta$; clearly, $G(U)$ is then also a subset in $\lim\Theta$. We consider the topology in the set $\lim\Theta$ given by the basis $\mathcal{B}$ consisting of all sets $G(U)$, for all finite subtrees F in T, and all open subsets $U \subset K_F$ saturated with respect to $\mathcal{A}_F$. The family $\mathcal{B}$ satisfies the axioms for a topological basis because intersection of any two sets from $\mathcal{B}$ is again in $\mathcal{B}$, as can be quickly deduced from the following two easy observations:

1. If $U, U' \subset K_F$ are open and $\mathcal{A}_F$–saturated then $U \cap U'$ is also $\mathcal{A}_F$–saturated, and $G(U) \cap G(U') = G(U \cap U')$.
2. If $U \subset K_F$ is open and $\mathcal{A}_F$–saturated, then for any finite $F' \supset F$ the set $U' := G(U) \cap K_{F'}$ is $\mathcal{A}_{F'}$–saturated, and $G(U') = G(U)$.

1.C.1 Proposition *For any tree system Θ of metric compacta the limit $\lim\Theta$, with topology given by the above-described basis $\mathcal{B}$, is a metrizable compact space. Moreover, for each finite subtree F of T the canonical inclusion (as a set) of the partial union $K_F = \#\Theta_F$ in $\lim\Theta$ is a topological embedding. In particular, for any $t \in V_T$ the canonical inclusion of K_t in $\lim\Theta$ is a topological embedding. Finally, both families of subsets $\{K_t : t \in V_T\}$ and $\{\Sigma_e : e \in O_T\}$ in $\lim\Theta$ are null (with respect to any metric compatible with the topology).*

We present a proof of Proposition 1.C.1 in the next section, after we have established Proposition 1.D.1, which we need in that proof.

1.C.2 Remark If we do not assume that each of the families of subsets $\{\Sigma_e : e \in N_t\}$ in K_t, for any $t \in V_T$, is null, then the above description still defines a compact topological space $\lim\Theta$, but then the limit is in general not Hausdorff. Moreover, canonical inclusions $K_t \to \lim\Theta$ are then in general not embeddings (though they are always continuous).

1.D The standard inverse system associated to a tree system

We describe a certain inverse system of metric compacta naturally associated to a tree system Θ, and show that the inverse limit of this system is canonically homeomorphic to the limit $\lim\Theta$. We use this fact to prove Proposition 1.C.1, and to derive a few further properties of limits of tree systems.

Given a tree system $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$, we proceed using the notation of Section 1.C. For any finite subtree $F \subset T$, denote by $K_F^* = K_F/\mathcal{A}_F$ the quotient of K_F in which all the subsets from the family $\mathcal{A}_F$ are shrunk to points. More precisely, let $\mathcal{D}_F$ be the decomposition of K_F consisting of the sets in $\mathcal{A}_F$ and the singletons from the complement $K_F \setminus \bigcup \mathcal{A}_F$, and let $K_F^* = K_F/\mathcal{D}_F$. Since the family $\mathcal{A}_F$ is null, the decomposition $\mathcal{D}_F$ is upper semicontinuous, and hence the quotient K_F^* is metrizable (see [5, Proposition 3 on page 14 and Proposition 2 on page 13]). Consequently, K_F^* is a metric compactum, and we call it the *reduced partial union of* Θ related to F. We denote by $q_F\colon K_F \to K_F^*$ the quotient map resulting from the above description of K_F^*.

For any pair $F_1 \subset F_2$ of finite subtrees of T, define a map $f_{F_1F_2}\colon K_{F_2}^* \to K_{F_1}^*$ as follows. For each edge $e \in N_{F_1} \cap O_{F_2}$ denote by $\mathcal{V}_e$ the set of all vertices $s \in V_{F_2} \setminus V_{F_1}$ such that the shortest path in T connecting F_1 with s starts with e. Then the family $\{\mathcal{V}_e : e \in N_{F_1} \cap O_{F_2}\}$ is a partition of $V_{F_2} \setminus V_{F_1}$, and each $\mathcal{V}_e$ is the vertex set of a subtree of F_2, which we denote by S_e. Viewing each K_{S_e} canonically as a subset in K_{F_2}, note that it is $\mathcal{A}_{F_2}$–saturated. Thus, the corresponding subset $q_{F_2}(K_{S_e})$ in $K^*_{F_2}$ is well defined and closed. Moreover, by shrinking each of the subsets $q_{F_2}(K_{S_e})$ with $e \in N_{F_1} \cap O_{F_2}$ to a point we get a quotient of $K^*_{F_2}$ which is canonically homeomorphic to (and which we identify with) $K^*_{F_1}$. Take the corresponding quotient map as $f_{F_1F_2}$, and observe that this map is continuous and surjective.

Given any finite subtrees $F_1 \subset F_2 \subset F_3$ of T, it is not hard to see that $f_{F_1F_2} \circ f_{F_2F_3} = f_{F_1F_3}$. Consequently, the system

$$\mathcal{S}_\Theta = (\{K^*_F : F \in \mathcal{F}_T\}, \{f_{FF'} : F \subset F'\}) \tag{1.D.1}$$

is an inverse system of metric compacta over the poset $\mathcal{F}_T$ of all finite subtrees of T. We call it the *standard inverse system associated to* Θ.

We now turn to describing a natural map $\beta\colon \lim\Theta \to \varprojlim \mathcal{S}_\Theta$ (initially as a map between the sets, but then we show that it is a homeomorphism). We split the description into two parts, accordingly with the partition $\lim\Theta = \#\Theta \sqcup E_T$.

To describe β on the subset $\#\Theta \subset \lim\Theta$, for each finite subtree $F \subset T$ consider the map $f_{F\#}\colon \#\Theta \to K^*_F$ defined similarly to the maps $f_{FF'}$, as follows. For each edge $e \in N_F$ denote by $\mathcal{V}_e$ the set of all vertices $s \in V_T \setminus V_F$ such that the shortest path in T connecting F with s starts with e. Then the family $\{\mathcal{V}_e : e \in N_F\}$ is a partition of $V_T \setminus V_F$, and each $\mathcal{V}_e$ is the vertex set of a subtree of T, which we denote by S_e. For each $e \in N_F$, denote by Θ_{S_e} the tree system obtained by restricting Θ to S_e, and view $\#\Theta_{S_e}$ canonically as a subset in $\#\Theta$. Note that $\#\Theta$ splits as

$$\#\Theta = \Bigl(K_F \setminus \bigcup_{e\in N_F} \Sigma_e\Bigr) \sqcup \bigsqcup_{e\in N_F} \#\Theta_{S_e}.$$

Viewing K_F as a subset in $\#\Theta$, for any $x \in K_F \setminus \bigcup_{e\in N_F} \Sigma_e$, we set $f_{F\#}(x) := x$. For any $e \in N_F$ and any $x \in \#\Theta_{S_e}$ we set $f_{F\#}(x) := q_F(\Sigma_e) \in K^*_F$, where the latter is a single point by definition of K^*_F. A straightforward verification shows that for any finite subtrees $F \subset F'$ in T we have $f_{FF'} \circ f_{F'\#} = f_{F\#}$. Thus the family $\{f_{F\#} : F \in \mathcal{F}_T\}$ induces a well-defined map $f_\#\colon \#\Theta \to \varprojlim \mathcal{S}_\Theta$, and we set $\beta(x) := f_\#(x)$ for any $x \in \#\Theta$.

To describe β on the subset $E_T \subset \lim\Theta$, consider any $x \in E_T$. For any finite subtree $F \subset T$, let e^F_x be the first oriented edge in the "shortest" path in T connecting F to x (ie in the unique ray in T starting at a vertex of F, passing through no other vertex of F, and representing x). Set $x(F) = q_F(\Sigma_{e^F_x}) \in K^*_F$. It is not hard to observe that the tuple $(x(F))_{F\in\mathcal{F}_T}$ is a thread of the inverse system $\mathcal{S}_\Theta$, ie an element of the inverse limit $\varprojlim \mathcal{S}_\Theta$, which we denote by ξ_x. We then set $\beta(x) := \xi_x$.

1.D.1 Proposition *The above-described map $\beta\colon \lim\Theta \to \varprojlim \mathcal{S}_\Theta$ is a homeomorphism.*

Proof We will first show that β is a bijection and then that both β and β^{-1} are continuous.

***β* is injective** Consider first any two distinct points $x, y \in \#\Theta$. Let F be any finite subtree of T such that $x, y \in K_F \setminus \bigcup_{e\in N_F} \Sigma_e$ (which obviously exists). By definition of the map $f_{F\#}$, we get $f_{F\#}(x) \neq f_{F\#}(y)$, and this implies $f_\#(x) \neq f_\#(y)$, and hence also $\beta(x) \neq \beta(y)$.

Second, consider any two distinct $x, y \in E_T$. Obviously, there is a finite subtree $F \subset T$ such that the oriented edges e_x^F and e_y^F are distinct. Since then $x(F) \neq y(F)$, we get $\xi_x \neq \xi_y$, and thus $\beta(x) \neq \beta(y)$.

To finish the proof of injectivity, we need to show that for any $x \in \#\Theta$ and any $y \in E_T$, we have $\beta(x) \neq \beta(y)$. To do this, for any $F \in \mathcal{F}_T$ denote by P_F the set of all points in K_F^* of the form $q_F(\Sigma_e)$ with $e \in N_F$. Observe that $\beta(y) = (y(F))_{F\in\mathcal{F}_T}$ has the property that for each $F \in \mathcal{F}_T$ we have $y(F) \in P_F$. On the other hand, writing $\beta(x) = (x_F)_{F\in\mathcal{F}_T}$, we have $x_F = f_{F\#}(x)$ for any $F \in \mathcal{F}_T$. Taking F_0 such that $x \in K_{F_0} \setminus \bigcup_{e\in N_{F_0}} \Sigma_e$, we get $x_{F_0} \notin P_{F_0}$, and thus $\beta(x) \neq \beta(y)$, as required.

***β* is surjective** Let $x_0 = (x_F)_{F\in\mathcal{F}_T}$ be an arbitrary point of $\varprojlim \mathcal{S}_\Theta$. Suppose first that for some $F_0 \in \mathcal{F}_T$ we have $x_{F_0} \notin P_F$. We may then view x_{F_0} as a point of $K_{F_0} \setminus \bigcup_{e\in N_{F_0}} \Sigma_e$. Observe that for any finite $F \supset F_0$ there is unique $z_F \in K_F^*$ such that $f_{F_0F}(z_F) = x_{F_0}$, and in fact we have $z_F = q_F(x_{F_0}) = f_{F\#}(x_{F_0})$, under the canonical inclusions $x_{F_0} \in K_{F_0} \subset K_F \subset \#\Theta$. From this it easily follows that, viewing x_{F_0} as an element of $\#\Theta$, we have $x_0 = \beta(x_{F_0})$.

In the remaining case, for each $F \in \mathcal{F}_T$ we have $x_F \in P_F$. As we will see, in this case the tree T is necessarily unbounded. For any $F \in \mathcal{F}_T$ denote by e_F the edge in N_F corresponding to $x_F \in P_F$ (ie such that $x_F = q_F(\Sigma_{e_F})$), and set $t_F := \alpha(e_F) \in V_F$. Consider any increasing sequence $\sigma = (F_i)_{i\geq 1}$ of finite subtrees of T such that $\bigcup_{i\geq 1} F_i = T$ (we will call any such sequence an *exhausting sequence*). It is not hard to realize that the associated sequence $(t_{F_i})_{i\geq 1}$, after deleting potential repetitions of subsequent terms, is necessarily infinite, and determines a ray in T starting at t_{F_1} and passing through all vertices t_{F_i} as well as through all oriented edges e_{F_i} (and possibly through some other vertices and edges). We denote this ray by ρ_σ. Given any two exhausting sequences $\sigma = (F_i)_{i\geq 1}$ and $\sigma' = (F_i')_{i\geq 1}$ of finite subtrees in T, we obviously have the following property: *for any $i \geq 1$ there is $j \geq 1$ such that $F_i \subset F_j'$ and $F_i' \subset F_j$*. From this property one deduces that the rays ρ_σ and $\rho_{\sigma'}$ eventually coincide. It follows that in the considered case the point $x_0 = (x_F)_{F\in\mathcal{F}_T}$ uniquely determines an end $\xi_0 \in E_T$. It is also clear that $\beta(\xi_0) = x_0$, which completes the proof.

β and β^{-1} are continuous Recall that, by definition, the inverse limit $\varprojlim \mathcal{S}_\Theta$ is a subspace in the topological product $\prod_{F\in\mathcal{F}_T} K_F^*$. Given any open subset $U \subset K_{F_0}^*$, for any $F_0 \in \mathcal{F}_T$, set

$$G_{\mathcal{S}}(U) := \Bigg(U \times \prod_{F\in\mathcal{F}_T\setminus\{F_0\}} K_F^*\Bigg) \cap \varprojlim \mathcal{S}_\Theta.$$

Note that $G_{\mathcal{S}}(U)$ is an open subset of $\varprojlim \mathcal{S}_\Theta$, and that all subsets of this form constitute a subbasis of the topology of $\varprojlim \mathcal{S}_\Theta$.

Define $U' := q_{F_0}^{-1}(U)$ and note that, as U runs through all open subsets of $K_{F_0}^*$, U' runs through all open $\mathcal{A}_{F_0}$–saturated subsets of K_{F_0}. Moreover, a direct observation shows that

$$\beta^{-1}(G_{\mathcal{S}}(U)) = G(U'), \tag{1.D.2}$$

where $G(U')$ is an element of the basis $\mathcal{B}$ (of the topology in $\lim\Theta$) described in Section 1.C. Since, by what was said above, (1.D.2) implies that both β and β^{-1} are continuous, this completes the proof of Proposition 1.D.1. □

Proof of Proposition 1.C.1 We apply Proposition 1.D.1. Since $\lim\Theta$ is homeomorphic to the inverse limit of an inverse system of metric compacta, it is itself a compact metrizable space, which yields the first assertion.

To see the second assertion, for each $F_0 \in \mathcal{F}_T$ consider the subposet $\mathcal{F}_T^{F_0}$ consisting of all subtrees $F \in \mathcal{F}_T$ that contain F_0, and note that this subposet is cofinal with $\mathcal{F}_T$. Denote by $\mathcal{S}_\Theta^{F_0}$ the restriction of the inverse system $\mathcal{S}_\Theta$ to this subposet, and note that we have a canonical identifications of the limits $\varprojlim \mathcal{S}_\Theta^{F_0} = \varprojlim \mathcal{S}_\Theta$. For each $F \in \mathcal{F}_T^{F_0}$ consider the map $h_F^{F_0}\colon K_{F_0} \to K_F^*$ given as the composition of the inclusion map $K_{F_0} \to K_F$ and the map $q_F\colon K_F \to K_F^*$. Note that the family of maps $\{h_F^{F_0} : F \in \mathcal{F}_T^{F_0}\}$ gives a morphism $K_{F_0} \to \mathcal{S}_\Theta^{F_0}$, and thus induces a continuous map $h^{F_0}\colon K_{F_0} \to \varprojlim \mathcal{S}_\Theta$. Since it is not hard to observe that, under identification of $\varprojlim \mathcal{S}_\Theta$ with $\lim\Theta$ through β, the map h^{F_0} coincides with the inclusion map $K_{F_0} \to \lim\Theta$, it follows that the latter map is an embedding.

To see the last assertion, consider an auxiliary increasing sequence $(F_i)_{i\geq 1}$ of finite subtrees of T such that $T=\bigcup_{i\geq 1} F_i$. Obviously, this sequence yields a cofinal subposet of the poset $\mathcal{F}_T$. Denote by $\mathcal{S}'$ the inverse sequence obtained by restricting $\mathcal{S}_\Theta$ to this subposet, and note that there is a canonical identification $\varprojlim \mathcal{S}'=\varprojlim \mathcal{S}_\Theta$. The map β is then naturally viewed as a map $\lim\Theta\to\varprojlim \mathcal{S}'$. Call a subset of the product $\prod_{i\geq 1} K^*_{F_i}$ a *k–slice* if it is of the form $\{(p_1,\dots,p_k)\}\times\prod_{i\geq k+1} K^*_{F_i}$, where $p_i\in K^*_{F_i}$ for $1\leq i\leq k$. Consider any metric on the product $\prod_{i\geq 1} K^*_{F_i}$ compatible with the product topology. Then for each $\epsilon>0$ there is a k_0 such that each k_0–slice has diameter $<\epsilon$. Observe that for $1\leq i\leq k_0$, each of the maps $f_{F_i\#}\colon \#\Theta\to K^*_{F_i}$ squeezes each subset from the collection $\{K_t : t\in V_T\setminus V_{F_{k_0}}\}$ to a point. As a consequence, viewing $\varprojlim \mathcal{S}'=\varprojlim \mathcal{S}_\Theta$ as a subspace in $\prod_{i\geq 1} K^*_{F_i}$, we realize that for $t\in V_T\setminus V_{F_{k_0}}$ the images $\beta(K_t)$ are contained in k_0–slices, and thus they all have diameters $<\epsilon$ for the metric restricted from the product. It follows that the family $\{\beta(K_t) : t\in V_t\}$ is null in $\varprojlim \mathcal{S}'$, and hence the family $\{K_t : t\in V_T\}$ is null in $\lim\Theta$.

To see that the family $\{\Sigma_e : e\in O_T\}$ is null, we apply the following general observation: *if a family of subsets $\{A_i : i\geq 1\}$ of a metric space is null, and if for each $i\geq 1$ the family $\{B_{i,j} : j\geq 1\}$ of subsets of A_i is null, then the full family $\{B_{i,j} : i\geq 1, j\geq 1\}$ is also null.* We omit further details, thus completing the proof. □

1.D.2 Example (limit of a tree system of spheres) For arbitrary $n\geq 1$, consider a *tree system of n–spheres*, ie a tree system $\mathcal{M}$ of manifolds, as described in Example 1.B.2, for which all manifolds in $\{M_t : t\in V_T\}$ are the n–sphere S^n. The next result is an application of Proposition 1.D.1.

1.D.2.1 Lemma *The limit of any tree system of n–spheres is homeomorphic to S^n.*

To prove Lemma 1.D.2.1, we need a special case of the following well-known result (which we will also need later, in its full generality).

1.D.2.2 Lemma *Let M be an n–dimensional compact topological manifold with boundary, and let $\mathcal{D}$ be a null and dense family of pairwise disjoint collared n–disks contained in the interior of M. Let $M/\mathcal{D}$ be the quotient space obtained by collapsing all disks $D\in\mathcal{D}$ to points, ie the quotient space of the decomposition of M induced by $\mathcal{D}$. Then $M/\mathcal{D}$ is homeomorphic to M, via a homeomorphism which is the identity on ∂M.*

Proof It follows from a theorem of Bing [10, Theorem 7.2] that the decomposition of M induced by $\mathcal{D}$ is shrinkable (see [5, Section II.5] for the definition of shrinkability).

By [5, Theorem 5.3], this implies that the quotient map $M \to M/\mathcal{D}$ can be approximated by homeomorphisms, which clearly implies our assertion. □

Proof of Lemma 1.D.2.1 Let $\mathcal{M} = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be any tree system of n–spheres. Note that, by Lemma 1.D.2.2, each space K_F^* with $F \in \mathcal{F}_T$ is homeomorphic to S^n. Moreover, for each pair of finite subtrees $F \subset F'$ of T the map $f_{FF'}\colon K_{F'}^* \to K_F^*$ is the quotient map from S^n to S^n modulo a decomposition induced by a finite collection of closed collared n–disks. Since such a map is obviously a near homeomorphism, we get that the associated standard inverse system $\mathcal{S}_{\mathcal{M}}$ consists of n–spheres and near homeomorphisms.

Let $(F_i)_i \geq 1$ be an increasing sequence of finite subtrees of T such that $\bigcup_{i\geq 1} F_i = T$. Since this sequence forms a cofinal subposet of the poset $\mathcal{F}_T$, the inverse sequence $\mathcal{S}'$ obtained from $\mathcal{S}_{\mathcal{M}}$ by restriction to this sequence has the same inverse limit. Since $\mathcal{S}'$ consists of n–spheres and near homeomorphisms, it follows from a result of M Brown [2, Theorem 4] that $\varprojlim \mathcal{S}' \cong S^n$. Applying Proposition 1.D.1, we get

$$\lim \mathcal{M} \cong \varprojlim \mathcal{S}_{\mathcal{M}} = \varprojlim \mathcal{S}' \cong S^n,$$

as required. □

1.D.3 Estimates of the dimension of lim Θ In the remaining part of this section we use Proposition 1.D.1 to estimate the topological dimension of the limit of a tree system. Some further estimates (or rather exact calculations) will be provided in Section 2.D.

An obvious estimate, implied by the fact that each constituent space K_t embeds in $\lim \Theta$, is

$$\dim(\lim \Theta) \geq \sup\{\dim(K_t) : t \in V_T\}. \tag{1.D.3}$$

Below we provide an upper bound for $\dim(\lim \Theta)$. If $\sup\{\dim(K_t) : t \in V_T\} = \infty$, then, clearly, $\dim(\lim \Theta) = \infty$ as well. Thus, we restrict our attention to tree systems which have a universal finite upper bound for the dimensions of the constituent spaces K_t.

1.D.3.1 Proposition *Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta such that*

$$\sup\{\dim(K_t) : t \in V_T\} = n < \infty.$$

Then $\dim(\lim \Theta) \leq n + 1$.

Proof By the closed sum theorem in dimension theory, for any $F \in \mathcal{F}_T$ we have $\dim(K_F) \leq n$, and hence also $\dim\big(K_F \setminus \bigcup_{e \in N_t} \Sigma_e\big) \leq n$. Recall that, for each $F \in \mathcal{F}_T$, we denote by P_F the set of all points in K_F^* which are obtained by shrinking the subsets Σ_e with $e \in N_F$. Since $K_F^* \setminus P_F$ is homeomorphic to $K_F \setminus \bigcup_{e \in N_t} \Sigma_e$, we get $\dim(K_F^* \setminus P_F) \le n$. Since each P_F is countable, it has dimension ≤ 0, and by the addition theorem we get $\dim(K_F^*) \le n+1$. By Proposition 1.D.1, and by the properties of inverse limits, we get $\dim(\lim \Theta) = \dim(\varprojlim \mathcal{S}_\Theta) \le \sup_{F \in \mathcal{F}_T} \dim(K_F^*) \le n+1$, as required. □

1.D.3.2 Corollary *Let* $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ *be a tree system of metric compacta such that*

$$\sup\{\dim(K_t) : t \in V_T\} = n < \infty.$$

Then $\dim(\lim \Theta) \in \{n, n+1\}$.

In Section 2.D we show that both values for the dimension asserted in the above corollary appear, for various classes of examples.

1.E An isomorphism of tree systems of spaces

Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ and $\Theta' = (T', \{K_t'\}, \{\Sigma_e'\}, \{\phi_e'\})$ be two tree systems of spaces. An *isomorphism* $F\colon \Theta \to \Theta'$ is a tuple $F = (\lambda, \{f_t\})$ such that:

(I1) $\lambda\colon T \to T'$ is an isomorphism of trees.

(I2) For each $t \in V_T$ the map $f_t\colon K_t \to K'_{\lambda(t)}$ is a homeomorphism.

(I3) For each $e \in N_t$ we have $f_t(\Sigma_e) = \Sigma'_{\lambda(e)}$.

(I4) For each $e \in N_t$ the commutation rule $\phi'_{\lambda(e)} \circ (f_{\alpha(e)}|_{\Sigma_{\alpha(e)}}) = f_{\omega(e)} \circ \phi_e$ holds.

An easy consequence of the definition of the limit (of a tree system of metric compacta) is the following.

1.E.1 Lemma *If* Θ *and* Θ' *are isomorphic tree systems of metric compacta then their limits* $\lim \Theta$ *and* $\lim \Theta'$ *are homeomorphic.*

1.E.2 Example (Toruńczyk's lemma and dense tree systems of oriented manifolds) The following result proved by Henryk Toruńczyk (see [12]) has interesting consequences concerning existence of isomorphisms for certain natural classes of tree systems of manifolds. (Throughout this paper, by a manifold we mean a topological manifold.) Recall that a family of subsets of a topological space is *dense* if the union of this family is a dense subset.

1.E.2.1 Toruńczyk's Lemma *Let M be a compact n–dimensional topological manifold with or without boundary, and let $\mathcal{D}$ and $\mathcal{D}'$ be two families of collared n–disks in* $\operatorname{int}(M)$ *such that each family consists of pairwise disjoint sets and both families are null and dense. Then each homeomorphism $h\colon \partial M \to \partial M$ which is extendable to a homeomorphism of M admits an extension to a homeomorphism $H\colon M \to M$ which maps $\mathcal{D}$ to $\mathcal{D}'$. More precisely, this means that there is an associated bijective map $\nu\colon \mathcal{D} \to \mathcal{D}'$ such that for each $\Delta \in \mathcal{D}$ the restriction $H|_\Delta$ maps Δ homeomorphically onto the disk $\nu(\Delta) \in \mathcal{D}'$.*

The above lemma provides motivation for the following.

1.E.2.2 Definition Let $\mathcal{M}$ be a tree system of closed manifolds, with families of manifolds $\{M_t\}$ and disks $\{\Delta_e\}$ as in Example 1.B.2. We say that this system is *dense* if for each $t \in V_T$ the family $\mathcal{D}_t = \{\Delta_e : e \in N_t\}$ is dense in the manifold M_t.

We denote the constituent spaces of a dense system of manifolds M_t by M_t°. The symbol M_t° is meant to contain information both of the space itself and of the peripheral subspaces contained in this space. Given M_t, it follows from Toruńczyk's Lemma 1.E.2.1 that the corresponding M_t° is unique up to a homeomorphism preserving the peripheral subspaces. We will call any constituent space of the form M_t° (viewed again as equipped with its standard family of peripheral subspaces) a *densely punctured manifold M_t*.

A recursive application of Toruńczyk's Lemma 1.E.2.1, together with Lemma 1.E.1, immediately yield the following.

1.E.2.3 Proposition *Let M be a closed connected oriented topological manifold. Let $\mathcal{M}$ and $\mathcal{M}'$ be two dense tree systems of manifolds such that*

(1) *all manifolds in the corresponding families $\{M_t\}$ and $\{M'_{t'}\}$ are homeomorphic to M,*

(2) *all maps $\phi_e\colon \Sigma_e = \partial\Delta_e \to \Sigma_{\bar{e}} = \partial\Delta_{\bar{e}}$ respect orientations, ie reverse the induced orientations on the corresponding spheres, and the same holds for all maps $\phi'_{e'}$.*

Then the tree systems $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic, and consequently, their limits are homeomorphic.

Note that, by the above proposition, each closed connected oriented manifold M determines uniquely up to isomorphism the dense tree system of manifolds satisfying conditions (1) and (2) of the proposition. We denote this tree system by $\mathcal{M}(M)$ and call it the *dense tree system of manifolds* M or the *Jakobsche system for* M. The latter term is motivated by the fact that the tree systems $\mathcal{M}(M)$ are intimately related to some inverse sequences described by Jakobsche in [13]. We describe this relationship in Section 2, especially in Example 2.C.7.

As a consequence of Lemma 1.E.1, M as above determines, uniquely up to homeomorphism, the compact metric space $\lim \mathcal{M}(M)$, which we denote by $\mathcal{X}(M)$ and call the *tree of manifolds* M or the *Jakobsche space for* M. In Jakobsche's paper [13] this space is denoted by $X(M, \{M\})$, and it is obtained as the inverse limit of one of the inverse sequences mentioned in the previous paragraph.

Remark Jakobsche [13] considered also a more general class of spaces obtained as inverse limits and determined uniquely up to homeomorphism by a finite or infinite family $\mathcal{N}$ of closed connected oriented manifolds of the same dimension. We discuss the corresponding dense tree systems of manifolds, for finite families $\mathcal{N}$, in Example 3.A.2.

1.E.3 Example (tree of spheres is a sphere) For arbitrary $n \geq 1$, consider the dense tree system $\mathcal{M}(S^n)$ of n–spheres, as described in Example 1.E.2. Note that this tree system can be alternatively thought of as follows.

1.E.3.1 Remark (an alternative description of the tree system $\mathcal{M}(S^n)$) Recall that the unique topological space $(S^n)^\circ$ obtained from S^n by deleting interiors of n–disks D from any null and dense family $\mathcal{D}$ consisting of pairwise disjoint collared disks is called the $(n-1)$–dimensional Sierpiński compactum. (The uniqueness follows from [3] if $n \neq 4$, and in the remaining case the argument in [3] also holds true in view of the later proofs of the Annulus Theorem and the Approximation Theorem for $n = 4$, due to F Quinn [20]; in fact, this follows also from Toruńczyk's Lemma 1.E.2.1.) The space $(S^n)^\circ$ contains a family of distinguished subsets, called *peripheral spheres*, which coincides with the family of boundaries ∂D of the disks $D \in \mathcal{D}$. The tree system $\mathcal{M}(S^n)$ can be described as follows. It is the unique tree system of metric compacta in which all constituent spaces are $(n-1)$–dimensional Sierpiński compacta, and families of peripheral subsets in all of these spaces coincide with the families of peripheral spheres.

The next result is a special case of Lemma 1.D.2.1.

1.E.3.2 Corollary *The limit* $\lim \mathcal{M}(S^n)$, *ie the tree of spheres* $\mathcal{X}(S^n)$, *is homeomorphic to* S^n.

A different argument for proving Corollary 1.E.3.2 is sketched in Remark 3.B.12.2.

1.E.4 Example (the tree of nonorientable manifolds N) Recall the following rather well-known fact.

1.E.4.1 Lemma *Let N be a closed connected nonorientable topological manifold of dimension n and let D and D' be any collared n–disks in N. Then each homeomorphism $D \to D'$* (no matter how it behaves with respect to local orientations in the disks) *extends to a homeomorphism of N.*

A recursive application of Toruńczyk's Lemma 1.E.2.1 and Lemma 1.E.4.1 yields the following.

1.E.4.2 Proposition *Let N be a closed connected nonorientable topological manifold, and let $\mathcal{M}$ and $\mathcal{M}'$ be any two dense tree systems of manifolds such that all manifolds in the corresponding families $\{M_t\}$ and $\{M'_{t'}\}$ are homeomorphic to N. Then the tree systems $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic, and consequently, their limits are homeomorphic.*

The tree system as above, uniquely determined by N, will be denoted by $\mathcal{M}(N)$ and called the *dense tree system of manifolds N*, or the *Jakobsche system for N*. Its limit, denoted by $\mathcal{X}(N)$, will be called the *tree of manifolds N* or the *Jakobsche space for N*.

1.E.4.3 Remark Proposition 1.E.4.2 clarifies the picture with trees of nonorientable manifolds, as documented so far in the literature. Namely, in [21], Stallings describes these spaces in a way which could be translated to our setting as follows: the tree of manifolds N is the limit of a dense tree system as in Proposition 1.E.4.2 satisfying a certain additional technical condition for the connecting maps called the *dense orientation condition* (we do not recall this condition). Proposition 1.E.4.2 shows that this additional condition plays in fact no role. Moreover, Proposition 1.E.4.2 applies to all topological manifolds, while the methods used by Stallings in [21] allowed him to deal only with PL manifolds.

2 Extensions of tree systems and associated inverse systems

In this section we present some alternative expressions of trees of spaces as inverse limits. It turns out that these expressions (rather than those related to standard associated inverse systems described in Section 1.D) allow us to relate the spaces obtained as limits of tree systems with certain previously studied classes of topological spaces defined in terms of inverse limits (eg Jakobsche trees of manifolds, Markov compacta as defined in [6], etc). Moreover, these expressions seem to be potentially more convenient for the purpose of recognizing ideal boundaries of various spaces and groups as (homeomorphic to) some specific trees of spaces (see Remark 2.C.8 for examples of such applications occurring in the literature).

The alternative expressions of trees of spaces as inverse limits, presented in this section, turn out to be useful for exact calculations of topological dimension for certain classes of trees of spaces (see Section 2.D).

2.A Extended spaces and maps

Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta. Suppose that for each $e \in O_T$ we are given a compact metric space Δ_e, its compact subspace S_e, and a homeomorphism $\varphi_e\colon S_e \to \Sigma_e$.

For any $t \in V_T$, let $T(t)$ be a subtree of T spanned by the set $\{t\} \cup \{\omega(e) : e \in N_t\}$ (ie t and all vertices adjacent to t). We define a tree system

$$\Theta(t) = (T(t), \{K'_s\}, \{\Sigma'_e\}, \{\phi'_e\})$$

as follows. Set $K'_t = K_t$ and $K'_{\omega(e)} = \Delta_e$ for each $e \in N_t$. Set $\Sigma'_e = \Sigma_e$ and $\Sigma'_{\bar{e}} = S_e$ for each $e \in N_t$. Finally, set $\phi'_e = \varphi_e^{-1}$ and $\phi'_{\bar{e}} = \varphi_e$ for each $e \in N_t$. Denote by $\widehat{K}_t = \lim \Theta(t)$ the limit of the above tree system. Observe that K_t and the sets Δ_e with $e \in N_t$ are then canonically the subspaces of $\widehat{K}_t$ (in particular, they are compact subspaces). Moreover, the sets Σ_e and S_e, viewed in the above way as subspaces of $\widehat{K}_t$, do coincide. We call any family $\{\widehat{K}_t : t \in V_t\}$ as above a *family of extended spaces* for Θ.

2.A.1 Examples (1) For each $e \in O_T$, set $\Delta_e = S_e = \Sigma_e$ and $\varphi_e = \mathrm{id}_{\Sigma_e}$. This defines what we call the *trivial family of extended spaces* for Θ. Note that in this family we have $\widehat{K}_t = K_t$ for each $t \in V_T$.

(2) Let $\mathcal{M}$ be a tree system of manifolds as in Example 1.B.2. Let $\{\Delta_e : e \in O_T\}$ be the family of n–disks as in this example, and set $S_e = \partial\Delta_e$ and $\varphi_e = \mathrm{id}_{S_e}$ for each such e. Observe that then we have $\widehat{K}_t = M_t$ for each $t \in V_T$. Indeed, by Lemma 1.D.2.2 all reduced partial unions of the system $\Theta(t)$ corresponding to finite subtrees $F \subset T(t)$ containing t (ie all spaces in the standard associated inverse system $\mathcal{S}_{\Theta(t)}$ restricted to the cofinal subposet in $\mathcal{F}_{T(t)}$ consisting of all subtrees containing t) are then

homeomorphic to M_t. Moreover, an argument as in the proof of Lemma 1.D.2.2 shows that the maps in the above restriction of the system $\mathcal{S}_{\Theta(t)}$ are all near homeomorphisms, and thus (applying also Proposition 1.D.1) we get $\widehat{K}_t = \lim \Theta(t) \cong \varprojlim \mathcal{S}_{\Theta(t)} \cong M_t$. (A more canonical identification of $\widehat{K}_t$ with M_t is easily provided by Theorem 3.B.10.) This means that as a family of extended spaces for $\mathcal{M}$ we can take the initial family of manifolds M_t. We will call the family $\{\widehat{K}_t = M_t\}$ as above the *standard family of extended spaces for* $\mathcal{M}$.

(3) The previous example can be generalized as follows. Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be any tree system of metric compacta. For each $e \in N_T$, set $\Delta_e = \mathrm{cone}(\Sigma_e)$ and set S_e to be the base of this cone, and let φ_e be the identity on $S_e = \Sigma_e$. We call the associated family $\{\widehat{K}_t : t \in V_T\}$ the *family of conically extended spaces* for Θ.

Given a family of extended spaces for Θ determined, as above, by a family of pairs $\{(\Delta_e, S_e) : e \in O_T\}$, consider the family $\{\widehat{K}_e : e \in O_T\}$ of subspaces defined by $\widehat{K}_e = \widehat{K}_{\omega(e)} \setminus (\Delta_{\bar{e}} \setminus S_{\bar{e}})$. Suppose that we are given a family of maps $\{\delta_e : \widehat{K}_e \to \Delta_e\}$ such that $\delta_e|_{\Sigma_{\bar{e}}} = \phi_{\bar{e}}$. Any tuple $\mathcal{E} = (\{\widehat{K}_t : t \in V_T\}, \{\delta_e : e \in O_T\})$ as above will be called a *family of extended spaces and maps* for Θ.

To any family $\mathcal{E}$ of spaces and maps as above we associate an inverse system of metric compacta, denoted by $\mathcal{S}_{\mathcal{E}}$, as follows. Let $(\mathcal{F}_T, \subset)$ be the poset of all finite subtrees of T. For any $F \in \mathcal{F}_T$, let $\widehat{K}_F$ be the quotient of the disjoint union

$$\widehat{K}_F = \bigsqcup_{t \in V_F} \Bigg(\widehat{K}_t \setminus \bigcup_{e \in N_t \cap O_F} (\Delta_e \setminus S_e) \Bigg) \Big/ {\sim},$$

where $\sim$ is the equivalence relation induced by the equivalences of the form $x \sim \phi_e(x)$ for all $e \in O_F$ and all $x \in \Sigma_e$ (where we view Σ_e and $\Sigma_{\bar{e}}$ canonically as subsets in $\widehat{K}_{\alpha(e)}$ and $\widehat{K}_{\omega(e)}$, respectively). This gives us the family $\{\widehat{K}_F : F \in \mathcal{F}_T\}$ of spaces in the inverse system $\mathcal{S}_{\mathcal{E}}$.

We now turn to defining the maps $h_{F_1 F_2} \colon \widehat{K}_{F_2} \to \widehat{K}_{F_1}$, for all pairs $F_1 \subset F_2$ of finite subtrees of T. We do this in two steps, as follows. First, suppose that $V_{F_2} = V_{F_1} \cup \{t\}$ for some $t \notin V_{F_1}$. Let e be the unique oriented edge in F_2 with $\omega(e) = t$. Viewing canonically Δ_e as a subset of $\widehat{K}_{F_1}$, we set

$$h_{F_1 F_2}(x) = \begin{cases} x & \text{for } x \in \widehat{K}_{F_1} \setminus (\Delta_e \setminus S_e) \subset \widehat{K}_{F_2}, \\ \delta_e(x) & \text{for } x \in \widehat{K}_e \subset \widehat{K}_{F_2}. \end{cases}$$

In the second step, for any pair $F' \subset F$ of finite subtrees of T we consider a sequence $F_1 \subset F_2 \subset \cdots \subset F_m$ such that $F_1 = F'$, $F_m = F$, and each pair F_i and F_{i+1} is of the form as in the first step. We then set

$$h_{FF'} = h_{F_1F_2} \circ h_{F_2F_3} \circ \cdots \circ h_{F_{m-1}F_m}$$

and we observe that the resulting map does not depend on the choice of the above sequence $F_1, \dots, F_m$ (which is in general not unique). A related observation is that $h_{F_1F_3} = h_{F_1F_2} \circ h_{F_2F_3}$ whenever $F_1 \subset F_2 \subset F_3$.

2.A.2 Definition Let Θ be a tree system of metric compacta, and let $\mathcal{E}$ be an associated family of extended spaces and maps for Θ. The *associated inverse system for* Θ *induced by* $\mathcal{E}$ is the tuple

$$\mathcal{S}_{\mathcal{E}} = (\{\widehat{K}_F : F \in \mathcal{F}_T\}, \{h_{FF'} : F \subset F'\}).$$

In the next section we show (see Theorem 2.B.4) that in some cases one can use the limit of an associated inverse system $\mathcal{S}_{\mathcal{E}}$ as an alternative description of the limit of a tree system.

2.A.3 Remark Any associated inverse system $\mathcal{S}_{\mathcal{E}}$ as above has various cofinal subsequences. Since any cofinal subsystem is viewed as equivalent to the original one (for example, its inverse limit is canonically the same), this allows simpler (in a certain sense) descriptions of the system $\mathcal{S}_{\mathcal{E}}$, which might be more convenient for some purposes. More precisely, let $F_1 \subset F_2 \subset \cdots$ be an increasing sequence of finite subtrees of T. Clearly, this sequence is cofinal with $\mathcal{F}_T$ if and only if $\bigcup_{i=1}^{\infty} V_{F_i} = V_T$. For any such cofinal sequence (F_i) the tuple

$$\mathcal{S}_{\mathcal{E},(F_i)} = (\{\widehat{K}_{F_i} : i \in \mathbb{N}\}, \{h_{F_jF_i} : i < j\})$$

is an inverse sequence equivalent to the inverse system $\mathcal{S}_{\mathcal{E}}$.

2.B Relationship to the limit of a tree system

In order to relate the limit $\lim \Theta$ with the inverse limit of some associated inverse system $\mathcal{S}_{\mathcal{E}}$, we need one more condition on the associated family $\mathcal{E}$ of extended spaces and maps. We use the notation of the previous section.

Let $\gamma = (t_0, t_1, \dots, t_m)$ be any finite combinatorial path in T of length $m \geq 2$. For $i = 1, \dots, m$, let $e_i = (t_{i-1}, t_i)$ be the consecutive oriented edges in γ. We then have

$$\Delta_{e_m} \subset \widehat{K}_{e_{m-1}} \xrightarrow{\delta_{e_{m-1}}} \Delta_{e_{m-1}} \subset \widehat{K}_{e_{m-2}} \xrightarrow{\delta_{e_{m-2}}} \cdots \xrightarrow{\delta_{e_2}} \Delta_{e_2} \subset \widehat{K}_{e_1} \xrightarrow{\delta_{e_1}} \Delta_{e_1}.$$

We denote the composition map $\delta_\gamma = \delta_{e_1} \circ \cdots \circ \delta_{e_{m-2}} \circ \delta_{e_{m-1}}|_{\Delta_{e_m}}$ by $\delta_\gamma \colon \Delta_{e_m} \to \Delta_{e_1}$.

2.B.1 Definition We say that an associated family $\mathcal{E}$ of extended spaces and maps is *fine* if for each $e \in O_T$ the family of images $\delta_\gamma(\Delta_{e_m})$ (where e_m is the terminal edge in γ), for all combinatorial paths γ in T of length ≥ 2 and starting with $e_1 = e$, is a null family of subsets in Δ_e.

Remark The condition of fineness does not follow automatically from the nullity conditions in Θ (for families $\{\Sigma_e : e \in N_t\}$).

The next definition describes a subclass of fine families $\mathcal{E}$ of extended spaces and maps which occur in practical situations where we express limits of tree systems as inverse limits.

2.B.2 Definition Let $\mathcal{E} = (\{\widehat{K}_t\}, \{\delta_e\})$ be a family of extended spaces and maps related to a family of pairs $\{(\Delta_e, S_e)\}$. We say that $\mathcal{E}$ is *contracting* if there are metrics $\hat{d}_t$ on the extended spaces $\widehat{K}_t$ and a constant $0 \leq c < 1$ such that:

(1) For each $e, e' \in O_T$ such that $e' \neq \bar{e}$ and $\alpha(e') = \omega(e)$ the restricted map $\delta_e|_{\Delta_{e'}}$ is a c–contraction with respect to the metrics $\hat{d}_{\omega(e)}$ in $\Delta_{e'} \subset \widehat{K}_{\omega(e)}$ and $\hat{d}_{\alpha(e)}$ in $\Delta_e \subset \widehat{K}_{\alpha(e)}$; more precisely, for any $x, y \in \Delta_{e'}$ we have
$$\hat{d}_{\alpha(e)}(\delta_e(x), \delta_e(y)) \leq c \cdot \hat{d}_{\omega(e)}(x, y).$$

(2) If $c > 0$ then there is another constant $C > 0$ such that for each $t \in V_T$ we have $\operatorname{diam}(\widehat{K}_t, \hat{d}_t) \leq C$.

We omit the proof of the following apparent fact.

2.B.3 Fact *Any contracting family $\mathcal{E}$ of extended spaces and maps is fine.*

2.B.4 Theorem *Let Θ be a tree system of metric compacta, let $\mathcal{E}$ be an associated family of extended spaces and maps for Θ, and suppose that $\mathcal{E}$ is fine. Let $\mathcal{S}_{\mathcal{E}}$ be the associated inverse system of compact topological spaces. Then the limit $\lim \Theta$ is canonically homeomorphic to the inverse limit $\varprojlim \mathcal{S}_{\mathcal{E}}$.*

Proof The proof consists of three parts. In the first part we describe explicitly strings $(x_F)_{F \in \mathcal{F}_T}$ which represent points of $\varprojlim \mathcal{S}_{\mathcal{E}}$. In the second part, we use this description to define a natural map $\beta \colon \varprojlim \mathcal{S}_{\mathcal{E}} \to \lim \Theta$ which is a bijection. Finally, in the third part we prove that β is a homeomorphism.

Recall that, by definition of inverse limit, each element $x \in \varprojlim \mathcal{S}_{\mathcal{E}}$ is a tuple $(x_F)_{F \in \mathcal{F}_T}$, with $x_F \in \widehat{K}_F$, such that $h_{FF'}(x_F) = x_{F'}$ whenever $F' \subset F$. We will show that there are two kinds of such tuples in $\varprojlim \mathcal{S}_{\mathcal{E}}$. At first, we consider tuples (x_F) satisfying the following property:

($\star$) For any cofinal subsequence $F_1 \subset F_2 \subset \cdots$ in the poset $\mathcal{F}_T$ the sequence x_{F_n} *eventually stabilizes*; that is, there is an N such that for all $n \geq N$ we have $x_{F_n} \in K_{F_n} \subset \widehat{K}_{F_n}$, and under the natural inclusions $K_{F_n} \subset K_{F_{n+1}}$, we have the equalities $x_{F_{n+1}} = x_{F_n}$.

A large class of such tuples can be described as follows. Let $y \in K_t$ for some $t \in V_T$. This y determines the tuple (x_F^y) as follows. If F contains t, we have the canonical inclusion $K_t \subset \widehat{K}_F$, and we set $x_F^y = x$. If F does not contain t, let F' be the smallest subtree of T containing both F and t. Viewing K_t as a subset of $\widehat{K}_{F'}$ we set $x_F^y := h_{F'F}(x)$. It is easy to check that $x^y = (x_F^y)$ is then a string, ie $x^y \in \varprojlim \mathcal{S}_{\mathcal{E}}$, and it obviously satisfies condition ($\star$). Moreover, we have the following.

Claim 1 *Each string $(x_F) \in \varprojlim \mathcal{S}_{\mathcal{E}}$ which satisfies property ($\star$) has the form (x_F^y) described above.*

The claim follows by observing that if y is the element to which some sequence x_{F_n} stabilizes, then any other such sequence stabilizes to the same element, and consequently the string necessarily has the asserted form (x_F^y).

We now turn to strings (x_F) not satisfying condition ($\star$). A large class of such strings is induced by ends $z \in E_T$. Given any $z \in E_T$, we define the tuple (x_F^z) as follows. For any $F \in \mathcal{F}_T$, let ρ_F be the minimal (with respect to inclusion) ray starting at a vertex of F such that $[\rho_F] = z$. Let $\gamma_n = (e'_1, \dots, e'_n)$ be the initial path of length n in ρ_F, and let $\delta_{\gamma_n}\colon \Delta_{e'_n} \to \Delta_{e'_1}$ be the map described at the beginning of this section. Note that the sequence (Q_n) of compact subsets in $\Delta_{e'_1}$ given by $Q_n = \delta_{\gamma_n}(\Delta_{e'_n})$ is then decreasing, and by the fineness of $\mathcal{E}$, we have $\lim_{n\to\infty} \operatorname{diam}(Q_n) = 0$. Consequently, the intersection $\bigcap Q_n$ is a single point in $\Delta_{e'_1} \subset \widehat{K}_F$, and we take it as x_F^z. We make the following observations (omitting their straightforward proofs):

(1) (x_F^z) is a string; that is, it belongs to $\varprojlim \mathcal{S}_{\mathcal{E}}$.

(2) (x_F^z) does not satisfy condition ($\star$).

(3) If $z \neq z'$ then $(x_F^z) \neq (x_F^{z'})$.

Claim 2 *Each element $(x_F) \in \varprojlim \mathcal{S}_{\mathcal{E}}$ which does not satisfy condition ($\star$) has the form (x_F^z) as above, for some end $z \in E_T$. Moreover, $z \to (x_F^z)$ is a bijective correspondence between the set of ends of T and the set of elements of $\varprojlim \mathcal{S}_{\mathcal{E}}$ not satisfying ($\star$).*

To prove the claim, note that for each string (x_F) not satisfying $(\star)$ there is a cofinal sequence $F_1 \subset F_2 \subset \cdots$ for which elements x_{F_n} change infinitely often. By passing to a subsequence, we may assume that $x_{F_{n+1}} \neq x_{F_n}$ (ie it is not true that $x_{F_{n+1}} = x_{F_n} \in K_{F_n} \subset K_{F_{n+1}}$) for all n. Observe that in this situation we have $x_{F_n} \in \Delta_{e_n}$ for some unique $e_n \in N_{F_n}$, and that the edges e_n induce uniquely a ray ρ in T of the form $\rho = (e_1, \dots, e_2, \dots, e_3, \dots)$. It is not hard to realize that then $(x_F) = (x_F^{[\rho]})$, which yields the first assertion of the claim. The second assertion follows then from observation (3) stated just before Claim 2.

We are now ready to describe the natural map $\beta\colon \varprojlim \mathcal{S}_{\mathcal{E}} \to \lim \Theta$ which will be our candidate for a homeomorphism. If a string $x = (x_F)$ satisfies condition $(\star)$, there is a point y to which (x_F) stabilizes, and this y belongs to some K_t. Viewing K_t as a subset in $\lim \Theta$, we set $\beta(x) = y$. If x does not satisfy $(\star)$, it corresponds uniquely to some end z of T. Viewing ends of T as elements of $\lim \Theta$, we set $\beta(x) = z$. We skip a direct verification of the facts that β is well defined and that it is a bijection.

Since both spaces $\varprojlim \mathcal{S}_{\mathcal{E}}$ and $\lim \Theta$ are compact, to prove that β is a homeomorphism, it is sufficient to show that it is continuous. To do this, we will show that for any open set $G(U) \in \mathcal{B}$ its preimage $\beta^{-1}(G(U))$ is open in $\varprojlim \mathcal{S}_{\mathcal{E}}$. More precisely, viewing $\varprojlim \mathcal{S}_{\mathcal{E}}$ as subspace of the product $\prod_F \widehat{K}_F$, we will show that $\beta^{-1}(G(U)) = W \cap \varprojlim \mathcal{S}_{\mathcal{E}}$ for some open set W in $\prod_F \widehat{K}_F$. Suppose that U is an open subset of K_{F_0} saturated with respect to the family $\mathcal{A}_{F_0}$. Set

$$\widehat{U} = U \cup \bigcup \{\Delta_e : e \in N_{F_0} \text{ and } \Sigma_e \subset U\}$$

and note that $\widehat{U}$ is an open subset of $\widehat{K}_{F_0}$. Take $W = \prod_F W_F$, where $W_{F_0} = \widehat{U}$ and $W_F = \widehat{K}_F$ for $F \neq F_0$. Clearly, W is open in $\prod_F \widehat{K}_F$. It is also not hard to deduce from Claims 1 and 2, and from the descriptions of strings, that $\beta^{-1}(G(U)) = W \cap \varprojlim \mathcal{S}_{\mathcal{E}}$. This completes the proof. □

2.C Tree systems with ANR peripheral subspaces

This section is devoted to showing that any tree system Θ in which all peripheral subspaces Σ_e are ANRs admits a fine family $\mathcal{E} = (\{\widehat{K}_t\}, \{\delta_e\})$ of extended spaces and maps in which $\{\widehat{K}_t\}$ is the family of conically extended spaces (see Examples 2.A.1(3)) and $\{\delta_e\}$ is an associated family of 0–contracting maps. See Proposition 2.C.2 for the precise statement of this main result of the section.

Recall that a compact metric space Σ is an ANR (absolute neighborhood retract) if for any embedding of Σ in another compact metric space K there is a neighborhood of Σ

in K which retracts onto Σ. Recall also that every compact polyhedron is an ANR. Lemma 2.C.3 presents a different characterization of ANR spaces, more convenient for our purposes.

2.C.1 Definition Let $\mathcal{E} = (\{\widehat{K}_t\}, \{\delta_e\})$ be a family of extended spaces and maps for a tree system Θ.

(1) We say that Θ is *peripherally ANR* if all peripheral subspaces Σ_e with $e \in O_T$ of Θ are ANRs.

(2) We say that $\mathcal{E}$ is *conical* if the corresponding family $\{\widehat{K}_t\}$ is the family of conically extended spaces for Θ.

(3) We say that $\mathcal{E}$ is 0*–contracting* if for each $e, e' \in O_T$ such that $e' \neq \bar{e}$ and $\alpha(e') = \omega(e)$ the restricted map $\delta_e|_{\Delta_{e'}}$ is a 0–contraction; equivalently, for all e and e' as above, δ_e maps the subspace $\Delta_{e'} \subset \widehat{K}_e$ to a point.

Note that a 0–contracting family $\mathcal{E}$ is automatically fine (compare Fact 2.B.3) and thus, by Theorem 2.B.4, can be used to express the limit $\lim \Theta$ as an inverse limit. The main result of this section is the following.

2.C.2 Proposition *Each peripherally ANR tree system has a family of extended spaces and maps that is conical and* 0*–contracting.*

The proof of Proposition 2.C.2 requires various preparations. We start with a lemma which characterizes an ANR space Σ in terms of maps to the cones over Σ. Since this characterization is an obvious reformulation of the definition of an ANR space, we omit its proof.

2.C.3 Lemma *Let* Σ *be a compact metric space, and let* $\operatorname{cone}(\Sigma)$ *be the cone over* Σ, *with* Σ *canonically identified as the cone base. Then the following two conditions are equivalent:*

(1) *For any metric compactum* K *containing* Σ *as a subspace there is a continuous map* $f\colon K \to \operatorname{cone}(\Sigma)$ *such that* $f|_\Sigma = \mathrm{id}_\Sigma$.

(2) Σ *is an ANR.*

We now turn to discussing decompositions of the constituent spaces K_t induced by families of their peripheral subsets. For any $e \in O_T$, set

$$\mathcal{A}_e = \{\Sigma_{e'} : e' \neq \bar{e} \text{ and } \alpha(e') = \omega(e)\}.$$

Clearly, $\mathcal{A}_e$ is a null family of pairwise disjoint compact subsets of the space $K_{\omega(e)}$.

We will view $\mathcal{A}_e$ as a decomposition of $K_{\omega(e)}$ into closed subsets by considering all singletons $\{x\}$ with $x \notin \bigcup \mathcal{A}_e$, together with the sets from $\mathcal{A}_e$, as elements of this decomposition. Equivalently, we identify the family $\mathcal{A}_e$ with the decomposition of $K_{\omega(e)}$ in which $\mathcal{A}_e$ is the set of nondegenerate elements. By the fact that the family $\mathcal{A}_e$ is null we immediately get the following (compare [5, Proposition 3 on page 14]).

2.C.4 Fact *For each $e \in O_T$ the decomposition $\mathcal{A}_e$ of the space $K_{\omega(e)}$ is upper semicontinuous.*

Denote by $K_{\omega(e)}/\mathcal{A}_e$ the quotient space of the decomposition $\mathcal{A}_e$. Since any quotient of an upper semicontinuous decomposition of a metric space is metrizable (see [5, Proposition 2 on page 13]), we get:

2.C.5 Fact *For each $e \in O_T$ the quotient $K_{\omega(e)}/\mathcal{A}_e$ is a compact metrizable space, and the subspace $\Sigma_{\bar{e}}$ canonically topologically embeds in this quotient.*

Fact 2.C.5, together with Lemma 2.C.3, yields the following.

2.C.6 Corollary *Suppose that the peripheral subset $\Sigma_{\bar{e}}$ is an ANR. Then there is a continuous map $f_e\colon K_{\omega(e)}/\mathcal{A}_e \to \mathrm{cone}(\Sigma_{\bar{e}})$ which is the identity on $\Sigma_{\bar{e}}$. Consequently, there is an induced continuous map $f_e'\colon K_{\omega(e)} \to \mathrm{cone}(\Sigma_{\bar{e}})$ which is the identity on $\Sigma_{\bar{e}}$ and which maps each subset $\Sigma_{e'} \in \mathcal{A}_e$ to a point.*

Proof of Proposition 2.C.2 Let $\{\widehat{K}_t : t \in V_T\}$ be the associated family of conically extended spaces for Θ, as described in Examples 2.A.1(3), and let $\{\widehat{K}_e : e \in O_T\}$ be the corresponding family of subspaces. For each $e \in O_T$ define a map $\delta_e'\colon \widehat{K}_e \to \mathrm{cone}(\Sigma_e)$ by

$$\delta_e'(x) = \begin{cases} f_e'(x) & \text{if } x \in K_{\omega(e)} \subset \widehat{K}_e, \\ f_e'(\Sigma_{e'}) & \text{if } x \in \Delta_{e'} \subset \widehat{K}_e \text{ for some } e' \in N_{\omega(e)} \setminus \{\bar{e}\}, \end{cases}$$

where f_e' is a map as in Corollary 2.C.6.

After identifying $\mathrm{cone}(\Sigma_{\bar{e}})$ with $\mathrm{cone}(\Sigma_e)$ via cone of the map $\phi_{\bar{e}}\colon \Sigma_{\bar{e}} \to \Sigma_e$, and then identifying $\mathrm{cone}(\Sigma_e)$ with Δ_e, the above-described map δ_e' gives the map $\delta_e\colon \widehat{K}_e \to \Delta_e$ such that

(1) δ_e maps each subset $\Delta_{e'} \subset \widehat{K}_e$ for $e' \in N_{\omega(e)} \setminus \{\bar{e}\}$ to a point;

(2) the restriction of δ_e to $\Sigma_{\bar{e}}$ coincides with $\phi_{\bar{e}}$.

Thus, the tuple $\mathcal{E} = (\{\widehat{K}_e\}, \{\delta_e\})$ is a conical and 0–contracting family of extended spaces and maps for Θ, as required. This finishes the proof. □

2.C.7 Example (Jakobsche's inverse sequences) As we have already mentioned in Section 1.E, Jakobsche [12; 13] has introduced and studied the spaces which appear in the present paper as limits of dense tree systems of manifolds, in particular the tree systems of manifolds M, as described in Proposition 1.E.2.3 (the Jakobsche spaces $\mathcal{X}(M)$). Jakobsche has introduced those spaces as inverse limits of certain inverse sequences of manifolds, which we can describe in our terms as follows.

Given a closed oriented manifold M, let $\mathcal{M} = \mathcal{M}(M)$ be the dense tree system of manifolds M (as defined in Section 1.E, right after Proposition 1.E.2.3). Let $\mathcal{E} = (\{\widehat{K}_t\}, \{\delta_e\})$ be any fine conical family of extended spaces and maps for $\mathcal{M}$ (existence of which is justified by Proposition 2.C.2). Let $\mathcal{S}_{\mathcal{E}} = (\{\widehat{K}_F\}, \{h_{F,F'}\})$ be the associated inverse system for $\mathcal{M}$, as in Definition 2.A.2. Note that the spaces $\widehat{K}_F$ in this system are homeomorphic to iterated connected (oriented) sums of copies of M.

Jakobsche has considered cofinal inverse subsequences $\mathcal{S}_{\mathcal{E},(F_i)}$ of the system $\mathcal{S}_{\mathcal{E}}$, as described in Remark 2.A.3, in which the sequences $F_1 \subset F_2 \subset \cdots$ of finite subtrees of T satisfy the following conditions:

(j1) F_1 is a subtree of T reduced to a single vertex.

(j2) For each $i \geq 1$ the vertex set $V_{F_{i+1}} \setminus V_{F_i}$ consists of vertices adjacent to F_i (ie of the form $\omega(e)$ for some $e \in N_{F_i}$).

A choice of a sequence (F_i) in the above way yields an inverse sequence with the following properties:

(p1) Each manifold $\widehat{K}_{F_{i+1}}$ is obtained from $\widehat{K}_{F_i}$ by means of operations of connected sum, with copies of the manifold M, performed at a finite set of pairwise disjoint connecting disks in the manifold $\widehat{K}_{F_i}$.

(p2) Each map h_{F_{i+1},F_i} maps each new copy of M in $\widehat{K}_{F_{i+1}}$, attached to $\widehat{K}_{F_i}$ by means of connected sum at an appropriate connecting disk, to this disk, and it is the identity in the complement of the union of the connecting disks.

2.C.8 Remark There are at least two examples in the literature where ideal boundaries of groups have been identified as certain trees of manifolds, and this has been achieved by referring to the description of these spaces as limits of Jakobsche's inverse sequences, as described in Example 2.C.7.

First, Fischer [9] identifies the CAT(0) boundaries of right-angled Coxeter groups having manifold nerves as trees of the corresponding manifolds. A minor mistake in the statement of the main result of [9] is corrected in Theorem 3.A.3 below. The work of Fisher is complemented by [19], by Piotr Przytycki and the author, where a vast class of trees of manifolds, in dimensions ≤ 3, is realized as Gromov boundaries of Coxeter groups which are hyperbolic.

A different class of spaces, and associated hyperbolic groups, is studied by Paweł Zawiślak in [26]. He shows, among other results, that the Gromov boundary of a 7–systolic 3–dimensional orientable simplicial pseudomanifold is the tree of 2–tori (known also as the Pontriagin sphere). A related class of spaces is exhibited, for which the Gromov boundary is the tree of projective planes.

2.D More on the dimension of the limit of a tree system

For the reasons explained in Section 1.D.3, we restrict our attention to tree systems which have a universal finite upper bound for the dimensions of the constituent spaces K_t.

2.D.1 Proposition *Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta such that*

$$\sup\{\dim(K_t) : t \in V_T\} = n < \infty.$$

Suppose that for some constant $0 \leq c < 1$, for each $e \in O_T$ there is a retraction $r_e \colon K_{\alpha(e)} \to \Sigma_e$ such that for each $e' \in N_{\alpha(e)} \setminus \{e\}$ the restriction $r_e|_{\Sigma_{e'}}$ is a c–contraction. Suppose also that if the constant c in the previous assumption is positive, there exists another constant $C > 0$ such that $\operatorname{diam}(K_t) \leq C$ for each $t \in V_T$. Then $\dim(\lim \Theta) = n$.

Any family of maps $\{r_e : e \in O_T\}$ as in Proposition 2.D.1, together with the trivial family of extended spaces for Θ (see Examples 2.A.1(1)), form a contracting system $\mathcal{E} = (\{\widehat{K}_t = K_t\}, \{r_e\})$ of extended spaces and maps for Θ. Further, by Fact 2.B.3, $\mathcal{E}$ is then fine. So Proposition 2.D.1 is a special case of the following slightly more general result.

2.D.2 Proposition *Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta such that*

$$\sup\{\dim(K_t) : t \in V_T\} = n < \infty.$$

Suppose that Θ admits a fine family $\mathcal{E}$ of extended spaces and maps in which the corresponding family of extended spaces is trivial. Then $\dim(\lim \Theta) = n$.

Proof Since we have an easy estimate

$$\dim(\lim \Theta) \geq \sup\{\dim(K_t) : t \in V_T\} = n,$$

it is sufficient to show the converse inequality $\dim(\lim \Theta) \leq n$.

Let $\mathcal{S}_{\mathcal{E}} = (\{\widehat{K}_F\}, \{h_{FF'}\})$ be the inverse system associated to $\mathcal{E}$. Since the family $\mathcal{E}$ is fine, Theorem 2.B.4 implies that $\lim \Theta \cong \varprojlim \mathcal{S}_{\mathcal{E}}$. Since the family of extended spaces in $\mathcal{E}$ is trivial, for each finite subtree $F \subset T$ we have $\widehat{K}_F = \#\{K_t : t \in V_F\}$ (finite partial union), and hence $\dim(\widehat{K}_F) = \max\{\dim(K_t) : t \in V_F\} \leq n$. Moreover, by the properties of inverse limits we have $\dim(\varprojlim \mathcal{S}_{\mathcal{E}}) \leq \sup_{F \in \mathcal{F}_T} \dim(\widehat{K}_F)$. This gives the required converse inequality $\dim(\lim \Theta) \leq n$, thus proving the proposition. □

2.D.3 Example (tree of internally punctured manifolds with boundary) We show how to apply Proposition 2.D.1 to calculate dimension of the limit for the following class of tree systems $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$. Fix any $n \geq 1$, and suppose that for each $t \in V_T$,

$$K_t = M_t \setminus \bigcup \{\mathrm{int}(D) : D \in \mathcal{D}\} \quad \text{and} \quad \{\Sigma_e : e \in N_t\} = \{\partial D : D \in \mathcal{D}\},$$

where M_t is a compact n–dimensional topological manifold with nonempty boundary, and $\mathcal{D}$ is a family of pairwise disjoint collared n–disks $D \subset \mathrm{int}(M_t)$ that is null and dense in M_t. We call any Θ as above a *dense tree system of internally punctured manifolds with boundary*. We claim that for any such Θ the topological dimension satisfies $\dim(\lim \Theta) = n - 1$.

To prove the claim, fix any $t \in V_T$ and any $e \in N_t$, and suppose that $\Sigma_e = \partial D_0$, where $D_0 \in \mathcal{D}$. Observe that by collapsing all peripheral subsets $\Sigma_{e'}$ with $e' \in N_t \setminus \{e\}$ to points one gets the quotient space $K_t/{\sim_e}$ homeomorphic to $M_t \setminus \mathrm{int}(D_0)$, via a homeomorphism which sends $\partial M_t \cup \partial D_0 \subset K_t/{\sim_e}$ identically to $\partial M_t \cup \partial D_0 \subset M_t \setminus \mathrm{int}(D_0)$ (Lemma 1.D.2.2). We denote a homeomorphism $K_t/{\sim_e} \to M_t \setminus \mathrm{int}(D_0)$ as above by h_e. Observe also that, because $\partial M_t \neq \varnothing$, there exists a retraction $g_e \colon M_t \setminus \mathrm{int}(D_0) \to \partial D_0 = \Sigma_e$. (Existence of such a retraction is pretty obvious when M_t is either smooth or PL, and it requires some effort in the topological category; the latter case is carefully dealt with in [27].) Thus, setting $r_e := g_e \circ h_e$, we obtain a family $\{r_e\}$ of retractions as in the assumption of Proposition 2.D.1, with constant $c = 0$. Since we clearly have $\dim(K_t) = n - 1$ for each t, Proposition 2.D.1 directly implies that $\dim(\lim \Theta) = n - 1$.

2.D.3.1 Remark Let $\mathcal{M}$ be a dense tree system of internally punctured manifolds with boundary, as defined above. Let M be a connected topological manifold with nonempty boundary, oriented or nonorientable, and suppose that all manifolds M_t used to describe $\mathcal{M}$ are homeomorphic to M. Suppose also that, in case when M is oriented, all connecting maps ϕ_e of $\mathcal{M}$ respect orientations, ie they reverse the induced orientations on the corresponding spheres. By the arguments as before (see Examples 1.E.2 and 1.E.4) one easily shows that the system $\mathcal{M}$ depends uniquely up to isomorphism on M only. We will call such a system $\mathcal{M}$ the *dense tree system of internally punctured manifolds* M, and its limit $\lim \mathcal{M}$ the *tree of internally punctured manifolds* M, denoted by $\mathcal{X}_{\mathrm{int}}(M)$.

2.D.4 Remark The following example shows that the equality as in the assertion of Proposition 2.D.1 or 2.D.2 does not hold universally. Let $\mathcal{M}$ be a dense tree system of closed oriented $(n+1)$–dimensional manifolds $\{M_t\}$. Then the corresponding constituent spaces K_t (obtained from the manifolds M_t as in Definition 1.E.2.2) can be easily shown to have the topological dimension $\dim(K_t) = n$. In particular, we have $\sup\{\dim(K_t) : t \in V_T\} = n$. On the other hand, it is known that $\dim(\lim \mathcal{M}) = n+1$; see [13, Proposition (2.2)]. In the special case when all the M_t are $(n+1)$–spheres, this follows also from Lemma 1.D.2.1.

The next result concerns peripherally ANR tree systems Θ, as defined in Section 2.C. It exhibits another class of examples for which we have the equality $\dim(\lim \Theta) = \sup\{\dim(K_t) : t \in V_T\}$.

2.D.5 Proposition *For any peripherally ANR tree system of metric compacta* $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ *we have*

$$\dim(\lim \Theta) \le \max(\sup_{t \in V_t} \dim(K_t), \sup_{e \in O_T} \dim(\Sigma_e) + 1).$$

In particular, if Θ *is peripherally ANR, if* $\sup_t \dim(K_t) = n < \infty$ *and* $\sup_e \dim(\Sigma_e) \le n-1$, *then* $\dim(\lim \Theta) = n$.

Proof Consider any conical and 0–contracting family $\mathcal{E}$ of extended spaces and maps for Θ, the existence of which is justified by Proposition 2.C.2, and recall that it is fine. Let $\mathcal{S}_{\mathcal{E}}$ be the associated inverse system for Θ induced by $\mathcal{E}$. For each $t \in V_T$ the family $\{K_t\} \cup \{\Delta_e : e \in N_t\}$ is a countable closed covering of $\widehat{K}_t$, and hence, by the countable union theorem (see [8, Theorem 7.2.1]), we have $\dim(\widehat{K}_t) = \max(\dim(K_t), \sup_{e \in N_t} \dim(\Delta_e))$. Since the family $\mathcal{E}$ is conical, and since we have the equality $\dim(\mathrm{cone}(X)) = \dim(X) + 1$ for any compact metric space, it follows that

$$\dim(\widehat{K}_t) = \max(\dim(K_t), \sup_{e \in N_t} \dim(\Sigma_e) + 1).$$

Using this, and applying once again the countable union theorem (this time to a finite union), we get the following estimate for each finite subtree F of T:

$$\dim(\widehat{K}_F) \le (\sup_{t \in V_t} \dim(K_t), \sup_{e \in O_T} \dim(\Sigma_e) + 1).$$

Finally, since by Theorem 2.B.4 we have $\lim \Theta \cong \varprojlim \mathcal{S}_{\mathcal{E}}$, and since by the properties of inverse limits we have $\dim(\varprojlim \mathcal{S}_{\mathcal{E}}) \le \sup_{F\in\mathcal{F}_T} \dim(\widehat{K}_F)$, we get the required estimate for $\dim(\lim \Theta)$, as in the first assertion of the proposition.

The second assertion follows from the first one, and from the inequality (1.D.3). □

3 Modifications of tree systems

In this section we describe some natural operations on tree systems which do not affect their limits. We also present several applications of these operations for justifying various properties of trees of manifolds. We are convinced that these operations provide a powerful tool for the future study of more general classes of trees of spaces.

3.A Consolidation of a tree system

We describe an operation which turns one tree system of spaces into another by merging the constituent spaces of the initial system, and forming a new system out of bigger pieces (corresponding to a family of pairwise disjoint subtrees in the underlying tree of the initial system). As we show below (see Theorem 3.A.1), this operation does not affect the limit of a system.

Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta. Let Π be a *partition of a tree T into subtrees*, ie a family of subtrees $S \subset T$ such that the vertex sets V_S with $S \in \Pi$ are pairwise disjoint and cover all of V_T. We allow that some of the subtrees $S \in \Pi$ are just single vertices of T.

We define a *consolidation of Θ with respect to Π*, denoted by Θ_Π, to be the following tree system $(T_\Pi, \{K_S\}, \{\Sigma_e\}, \{\phi_e\})$. As a vertex set V_Π of the tree T_Π we take the family Π, and as the edge set O_Π the set $\{e \in O_T : e \notin \bigcup_{S\in\Pi} O_S\}$. Clearly, for each oriented edge $e \in O_\Pi$ the initial vertex $\alpha_\Pi(e)$ is the subtree $S \in \Pi$ for which $\alpha(e) \in V_S$. Similarly, $\omega_\Pi(e)$ is the subtree $S \in \Pi$ for which $\omega(e) \in V_S$.

For any subtree $S \in \Pi$ denote by Θ_S the restriction of Θ to S, and set $K_S := \lim \Theta_S$. Note that for any $e \in O_\Pi$ we have the canonical inclusion $\Sigma_e \subset K_{\alpha_\Pi(e)}$, and if we set $N_S = \{e \in O_\Pi : \alpha_\Pi(e) = S\}$, the family $\{\Sigma_e : e \in N_S\}$ of subsets of K_S consists of pairwise disjoint sets. Moreover, by Proposition 1.C.1, the subsets $\Sigma_e \subset K_{\alpha_\Pi(e)}$ are all compact and each family $\{\Sigma_e : e \in N_S\}$ is null. This justifies that the just-described tuple

$$\Theta_\Pi := (T_\Pi, \{K_S : S \in \Pi\}, \{\Sigma_e : e \in O_\Pi\}, \{\phi_e : e \in O_\Pi\})$$

is a tree system of metric compacta.

3.A.1 Theorem *For any tree system* Θ *of metric compacta and any consolidation* Θ_Π *of* Θ*, the limits* $\lim\Theta$ *and* $\lim\Theta_\Pi$ *are canonically homeomorphic.*

Proof We first describe the natural map $i_\Pi\colon \lim\Theta_\Pi \to \lim\Theta$, and then show it is a homeomorphism.

To define $i_\Pi(x)$ for all $x \in \lim\Theta_\Pi$, we consider three possible positions of x in $\lim\Theta_\Pi$. First, suppose that $x \in K_t \subset K_S \subset \lim\Theta_\Pi$, for some $t \in V_S$ and some $S \in \Pi$. View x as an element of $\lim\Theta$ via the canonical inclusion $K_t \subset \lim\Theta$, and set $i_\Pi(x) = x$. Second, suppose that $x \in E_S \subset \lim\Theta_S = K_S \subset \lim\Theta_\Pi$ is an end of some tree $S \in \Pi$. Since we have the canonical inclusion $E_S \subset E_T$, the element x is also an end of T, and hence an element of $\lim\Theta$. We set again $i_\Pi(x) = x$. Finally, in the remaining case, x is an end of the tree T_Π. Suppose that x is represented by a ray $\rho = (S_0, S_1, \dots)$ in the tree T_Π. Then ρ determines the ray ρ_T in T by

$$\rho_T = (e_1, [\omega(e_1), \alpha(e_2)], e_2, [\omega(e_2), \alpha(e_3)], e_3, \dots),$$

where e_i is the edge in T connecting S_{i-1} to S_i, and $[\omega(e_i), \alpha(e_{i+1})]$ are the paths in T (sometimes perhaps empty) connecting the corresponding vertices. Set $i_\Pi(x) = [\rho_T] \in E_T \subset \lim\Theta$. We skip the straightforward verification that i_Π is well defined, and that it is a bijection.

Since any continuous bijection between compact metric spaces is a homeomorphism, to prove that i_Π is a homeomorphism, it is sufficient to show that it is continuous. To do this, we will show that for any open set $V \subset \lim\Theta$ from the basis $\mathcal{B}$ the preimage $i_\Pi^{-1}(V)$ is open in $\lim\Theta_\Pi$. For this we need two claims.

Claim 1 *For any* $S \in \Pi$ *the restriction* $i_\Pi|_{K_S}$ *is a homeomorphism onto its image* (a topological embedding).

Since K_S is compact and i_Π is injective, to prove Claim 1 it is sufficient to show that $i_\Pi|_{K_S}$ is continuous. Let $G(U) \in \mathcal{B}$, where U is an open subset in K_F for some finite $F \subset T$, and U is $\mathcal{A}_F$–saturated. We need to show that $K_S \cap i_\Pi^{-1}(G(U))$ is open in K_S. Suppose first that $F \cap S = \varnothing$. It is not hard to realize that in this case $K_S \cap i_\Pi^{-1}(G(U))$ is either empty or coincides with K_S, and thus the assertion follows. Suppose then that $F \cap S \neq \varnothing$. Viewing $K_{F\cap S}$ as subset in K_F, set $U_S := U \cap K_{F\cap S}$, and note that U_S is open in $K_{F\cap S}$ and saturated with respect to the family $\mathcal{A}^{\Theta_S}_{F\cap S}$ of peripheral subsets in $K_{F\cap S}$ viewed as a partial union of the system Θ_S. Moreover, it is not hard to observe that $K_S \cap i_\Pi^{-1}(G(U)) = G(U_S) \in \mathcal{B}_S$, where $\mathcal{B}_S$ is the standard basis in the limit K_S of the system Θ_S (as described in Section 1.C). Thus Claim 1 follows.

Claim 2 *For any finite subtree $F_0 \subset T_\Pi$ the restriction $i_\Pi|_{K_{F_0}}$ is a homeomorphism onto its image.*

Since K_{F_0} is a finite union of its compact subsets of the form K_S, it is compact itself. Moreover, in view of Claim 1, the same fact implies that $i_\Pi|_{K_{F_0}}$ is continuous. Since this map is also injective, the assertion of Claim 2 follows.

Coming back to the proof that i_Π is continuous, let $V = G(U)$ for some open and $\mathcal{A}_F$–saturated subset $U \subset K_F$, where $F \subset T$ is some finite subtree. Let F_Π be the subtree of T_Π spanned by the vertices represented by those subtrees $S \in \Pi$ which intersect F; clearly, F_Π is finite. Note that, by Claim 2, the set $U_\Pi := K_{F_\Pi} \cap i_\Pi^{-1}(V)$ is open in K_{F_Π} (because $i_\Pi(K_{F_\Pi}) \cap V$ is open in $i_\Pi(K_{F_\Pi})$). Moreover, since U is $\mathcal{A}_F$–saturated, it is not hard to realize that U_Π is $\mathcal{A}_{F_\Pi}$–saturated, where $\mathcal{A}_{F_\Pi}$ is the appropriate family of peripheral subsets of the system Θ_Π in its partial union K_{F_Π}. Finally, observe that $i_\Pi^{-1}(V) = G(U_\Pi) \in \mathcal{B}_\Pi$ (where $\mathcal{B}_\Pi$ denotes the standard basis for the topology in $\lim \Theta_\Pi$), and hence this preimage is open in $\lim \Theta_\Pi$. This finishes the proof. $\square$

The next example illustrates how one can apply the procedure of consolidation to justify that limits of various classes of tree systems are homeomorphic.

3.A.2 Example (dense trees of finite families of manifolds) Let $\mathcal{N} = \{M_1, \dots, M_k\}$ be a finite family of closed connected oriented topological manifolds of the same dimension. Let $\mathcal{M} = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a dense tree system of manifolds from $\mathcal{N}$. For each $t \in V_T$, let $i_t \in \{1, \dots, k\}$ be the index for which the space K_t is of the form $M_{i_t}^\circ$, as described in Section 1.E, just after Definition 1.E.2.2. We say that $\mathcal{M}$ is 2*–saturated* if for each $t \in V_T$ and every $j \in \{1, \dots, k\}$ there are at least two distinct edges $e \in N_t$ such that $i_{\omega(e)} = j$.

If $\mathcal{M}$ is 2–saturated, it is not hard to construct a partition Π of T such that for each $S \in \Pi$ the vertex set V_S contains exactly one vertex s with $i_s = j$, for each $j \in \{1, \dots, k\}$ (in particular, the cardinality of each V_S is k). Consider the consolidated tree system $\mathcal{M}_\Pi$ for the partition Π. It is easy to note that the constituent spaces of $\mathcal{M}_\Pi$ are all of the form $(M_1 \# \cdots \# M_k)^\circ$, and that $\mathcal{M}_\Pi$ is (isomorphic to) the dense tree system of manifolds $M_1 \# \cdots \# M_k$. In view of Theorem 3.A.1 and Proposition 1.E.2.3 we get the following.

3.A.2.1 Proposition *For any 2–saturated dense tree system $\mathcal{M}$ of manifolds from $\mathcal{N}$ the limit $\lim \mathcal{M}$ is homeomorphic to the Jakobsche space $\mathcal{X}(M_1 \# \cdots \# M_k)$.*

The above observation can be generalized as follows. Let $\mathcal{N}$ be a family of manifolds as above, and let $\mu = (m_1, \dots, m_k)$ be a tuple of arbitrary positive integers. If $\mathcal{M}$ is a tree system as above, one can easily construct a partition Π_μ such that for each $S \in \Pi_\mu$ and each $j \in \{1, \dots, k\}$ the vertex set V_S contains exactly m_j elements s with $i_s = j$. By the argument as above, we get that $\lim \mathcal{M} = \lim \mathcal{M}_{\Pi_\mu}$ is homeomorphic to the Jakobsche space $\mathcal{X}(m_1 M_1 \# \cdots \# m_k M_k)$, where each $m_i M_i$ is the connected sum of m_i copies of M_i. As a consequence, we get the following.

3.A.2.2 Corollary *Let $M_1, \dots, M_k$ be any closed connected oriented topological manifolds of the same dimension, and let $m_1, \dots, m_k$ be any positive integers. Then the Jakobsche spaces $\mathcal{X}(M_1 \# \cdots \# M_k)$ and $\mathcal{X}(m_1 M_1 \# \cdots \# m_k M_k)$ are homeomorphic.*

Now, let $\mathcal{K} = \{N_1, \dots, N_k\}$ be a family of closed connected topological manifolds of the same dimension, at least one of which is nonorientable. We also assume that the orientable manifolds in $\mathcal{K}$ have no distinguished orientation. By a *tree system of manifolds from* $\mathcal{K}$ we mean a tree system of manifolds in which every constituent space is of the form N° for some $N \in \mathcal{K}$, and in which the connecting maps are arbitrary homeomorphisms between the corresponding spherical peripheral subspaces. Note that a connected sum $N_1 \# \cdots \# N_k$ (again, with arbitrary connecting homeomorphisms) is unique up to homeomorphism (this is a consequence of Lemma 1.E.4.1).

The arguments as above, enriched by application of Lemma 1.E.4.1, yield the following variation on Proposition 3.A.2.1 and Corollary 3.A.2.2.

3.A.2.3 Proposition *Let $\mathcal{K} = \{N_1, \dots, N_k\}$ be a family of closed connected topological manifolds of the same dimension, at least one of which is nonorientable. Let $\mathcal{M}$ be any 2–saturated dense tree system of manifolds from $\mathcal{K}$. Then the limit $\lim \mathcal{M}$ is homeomorphic to the Jakobsche space $\mathcal{X}(N_1 \# \cdots \# N_k)$* (as defined in Example 1.E.4). *Moreover, for any positive integers $m_1, \dots, m_k$ the space $\mathcal{X}(m_1 N_1 \# \cdots \# m_k N_k)$ is also homeomorphic to $\mathcal{X}(N_1 \# \cdots \# N_k)$.*

3.A.2.4 Remark Jakobsche [13] considered inverse sequences related to dense tree systems of manifolds from a family $\mathcal{N}$ and established a much weaker variant of Proposition 3.A.2.1. Namely, using our terms, he got uniqueness up to homeomorphism under the following (much stronger than 2–saturation) assumption for a dense tree system of manifolds from $\mathcal{N}$: *for each $M \in \mathcal{N}$ and for each $t \in T$ the set of collared disks $\{\Delta_e \subset M_t : M_{\omega(e)} \cong M\}$ is dense in M_t* (see [13, Theorem (4.6)]). His proof consisted of showing something equivalent to the fact that any two tree systems satisfying the above assumption are isomorphic. Observe that, under our weaker assumption of 2–saturation, tree systems as in Proposition 3.A.2.1 needn't be isomorphic, so the proof of this proposition necessarily requires some argument involving modifications of tree systems.

As another application of the technique of consolidation we present the following correction to the main result from [9] by Fischer.

3.A.3 Theorem *Suppose W is a right-angled Coxeter group whose nerve is a flag PL triangulation of a closed oriented manifold M, and let $\overline{M}$ be the same manifold with reversed orientation. Then the* $\mathrm{CAT}(0)$ *boundary of W* (ie the boundary of the Coxeter–Davis complex for W) *is the Jakobsche space $\mathcal{X}(M \,\#\, \overline{M})$.*

In the original (wrong) statement of this result in [9] instead of the space $\mathcal{X}(M \,\#\, \overline{M})$, there appears the space $\mathcal{X}(M)$, which is in general different (for example, it is not hard to show that the spaces $\mathcal{X}(\mathbb{CP}^2)$ and $\mathcal{X}(\mathbb{CP}^2 \,\#\, \overline{\mathbb{CP}^2})$ are not homeomorphic, by referring to the properties of their Čech cohomology rings).

Proof of Theorem 3.A.3 We indicate a necessary minor modification of the argument provided in [9]. The author argues by showing that ∂W is homeomorphic to the inverse limit of some inverse sequence of manifolds of the form as in Example 2.C.7. A part of his argument which requires correction is this (see the beginning of the proof of [9, Theorem 3.7]). Let X_W be the Coxeter–Davis complex for W, obtained as the union $X_W = \bigcup\{gQ : g \in W\}$, where Q is the Davis cell for W (topologically equal to the cone over the manifold M). Let $|g|$ be the word norm for elements $g \in W$ with respect to the standard generating set. Let $X_k = \bigcup\{gQ : |g| \leq k\}$ and let $M_k = \partial X_k$. The author claims that M_k is the connected sum of the appropriate number of copies of M, but it is clear from the way X_k is formed out of copies of Q that in fact M_k is the connected sum of copies of both M and $\overline{M}$. Moreover, since any two adjacent copies of Q in X_W are obtained from one another by reflection, M_{k+1} is obtained from M_k by connected sum with copies of M if k is odd, and with copies of $\overline{M}$ if k is even (in particular, $M_0 = \partial Q = M$). From this, using the remaining arguments of Fischer, one deduces that ∂W is homeomorphic to the limit of a 2–saturated dense tree system of manifolds M and $\overline{M}$. By Proposition 3.A.2.1, this yields the assertion. □

3.B Tree decomposition of a compact metric space

In Sections 3.B and 3.C we describe an operation on a tree system that is inverse to that of consolidation (as described in Section 3.A). In this section, we start with an elementary case when the initial tree system is trivial, ie when the underlying tree T

is reduced to a single vertex. The operation is then called *tree decomposition* of a compact metric space.

We start with introducing terminology related to the concept of tree decomposition. The main result of the section, which relates this concept to that of a tree system and its limit, is Theorem 3.B.10.

At the end of the section we show how to use tree decompositions to prove that limits of some tree systems are homeomorphic to some explicit spaces (Example 3.B.12).

3.B.1 Definition An *elementary splitting* of a compact metric space K is a triple $(A, \{Y, Z\})$ of compact subspaces of K such that $Y \cup Z = K$, $Y \cap Z = A$, and A is a nonempty proper subset both in Y and in Z. The set A is called the *separator* of the splitting, and the sets Y and Z are the *half-spaces*. Moreover, the sets $Y \setminus A$ and $Z \setminus A$ will be called the *open half-spaces of the splitting*. If H is any half-space of the splitting above (ie $H = Y$ or $H = Z$), we denote by $\dot{H}$ the corresponding open half-space, and by H^c the *complementary* (or *opposite*) half-space (equal to $K \setminus \dot{H}$).

Note that for any splitting as above we have the following:

(1) The set $K \setminus A$ is disconnected and the open half-spaces are the unions of connected components in this set.
(2) K is canonically homeomorphic to the space $Y \cup_A Z$ obtained from the disjoint union of Y and Z by gluing through the identity on A; equivalently, K is the limit of the tree system whose underlying tree T is a single edge, the constituent spaces K_t are Y and Z, the peripheral subspaces Σ_e coincide with A, and the connecting maps ϕ_e are the identities on A.

3.B.2 Definition Given two elementary splittings $(A_i, \{Y_i, Z_i\})$ for $i = 1, 2$ of a compact metric space K, we say they *do not cross* if for at least one pair of half-spaces H_1 and H_2 selected from those splittings we have $H_1 \cap H_2 = \varnothing$.

3.B.3 Remark The noncrossing condition $H_1 \cap H_2 = \varnothing$ has these consequences:

(1) $A_1 \cap A_2 = \varnothing$.
(2) $H_1 \subset \dot{H}_2^c$ and $H_2 \subset \dot{H}_1^c$.

3.B.4 Definition Given three pairwise noncrossing splittings $(A_i, \{Y_i, Z_i\})$ for $i = 1, 2, 3$ of a compact metric space K, we say that A_2 separates A_1 from A_3 if for an appropriately chosen half-space H for A_2 we have $A_1 \subset \dot{H}$ and $A_3 \subset \dot{H}^c$.

We will be interested in countable (usually infinite) families of pairwise noncrossing splittings satisfying some additional properties, which we now describe.

3.B.5 Definition Let $\mathcal{C} = (A_\lambda, \{Y_\lambda, Z_\lambda\})_{\lambda\in\Lambda}$ be a family of pairwise noncrossing splittings of a compact metric space K. We say $\mathcal{C}$ is *discrete* if for any two separators A and A' from $\mathcal{C}$ there is only finitely many separators in $\mathcal{C}$ that separate A from A'.

Any discrete family $\mathcal{C}$ of pairwise noncrossing splittings of K determines the associated family of *domains obtained by splitting*, and the *dual tree* $T_{\mathcal{C}}$. We start with describing the domains.

Consider any separator A from $\mathcal{C}$ and any half-space H related to A. The pair (A, H) determines a domain $\Omega_{A,H}$ described as follows. Let $A = A_{\lambda_0}$, and set $\Lambda_0 = \Lambda \setminus \{\lambda_0\}$. For any separator A_λ with $\lambda \in \Lambda_0$ consider the half-space H_λ which contains A. We have to consider the following two cases. First, suppose that for each $\lambda \in \Lambda_0$ we have $H \subset H_\lambda$. We then set $\Omega_{A,H} := H$ and note that this domain is disjoint with all separators from $\mathcal{C}$ other than A. In the second case, there is some $\lambda \in \Lambda_0$ with $H \not\subset H_\lambda$. By discreteness, there is also such λ for which no separator from $\mathcal{C}$ separates A from A_λ. Denote by $\Lambda_{A,H}$ the set of all $\lambda \in \Lambda_0$ for which $H \not\subset H_\lambda$ and no separator from $\mathcal{C}$ separates A from A_λ. By what was said above, this set is nonempty. Set

$$\Omega_{A,H} := H \cap \bigcap_{\lambda\in\Lambda_{A,H}} H_\lambda,$$

and note that this set satisfies the following properties:

(1) $\Omega_{A,H}$ is compact, contains the sets A and A_λ with $\lambda \in \Lambda_{A,H}$, and it is disjoint with all other separators from $\mathcal{C}$.

(2) For each $\lambda \in \Lambda_{A,H}$ we have $\Omega_{A_\lambda,H_\lambda} = \Omega_{A,H}$.

A *domain* in K induced by $\mathcal{C}$ is any subset $\Omega = \Omega_{A,H}$ as above. A domain Ω is called *adjacent* to a separator A of $\mathcal{C}$ if it contains A or, equivalently, if $\Omega = \Omega_{A,H}$ for some half-space H related to A. For each separator A of $\mathcal{C}$ there are exactly two domains induced by $\mathcal{C}$ and adjacent to A.

We are now ready to describe the dual tree $T_{\mathcal{C}}$ of a discrete family of pairwise noncrossing splittings $\mathcal{C}$. As a vertex set $V_{\mathcal{C}}$ we take the set of all domains for $\mathcal{C}$, as described above. As the set $O_{\mathcal{C}}$ of oriented edges we take the set of all pairs (A, H) as above, and we set $\omega_{\mathcal{C}}(A, H) = \Omega_{A,H}$ and $\overline{(A, H)} = (A, H^c)$. It is an easy exercise to show that the graph obtained in the above way is connected and contains no loops, and thus is a tree. We denote this tree by $T_{\mathcal{C}}$ and call the *dual tree of* $\mathcal{C}$.

Remark Note that for any discrete family $\mathcal{C}$ of pairwise noncrossing splittings of a compact metric space K the corresponding dual tree $T_{\mathcal{C}}$ is locally countable (and hence countable). More precisely, given a vertex Ω of $T_{\mathcal{C}}$ (ie a domain for $\mathcal{C}$), consider the set N_Ω of all edges (A, H) in $T_{\mathcal{C}}$ satisfying $\alpha_{\mathcal{C}}(A, H) = \Omega$. Note that the corresponding family $\{\dot{H} : (A, H) \in N_\Omega\}$ of open half-spaces in K consists of pairwise disjoint nonempty open subsets of K. Since any compact metric space is separable, it follows that the set N_Ω is countable, thus justifying the remark.

The above construction of the dual tree motivates the first part of the following.

3.B.6 Definition (1) A discrete family $\mathcal{C}$ of pairwise noncrossing splittings of a compact metric space K will be called a *tree decomposition* of K.

(2) A tree decomposition $\mathcal{C} = \{(A_\lambda, \{Y_\lambda, Z_\lambda\})\}_{\lambda\in\Lambda}$ of K is *fine* if for each $\epsilon > 0$ the set $\{\lambda \in \Lambda : \min[\operatorname{diam}(Y_\lambda), \operatorname{diam}(Z_\lambda)] > \epsilon\}$ is finite.

3.B.7 Example Let $K = \lim\Theta$ be the limit of a tree system Θ of metric compacta, and suppose that it is *essential*, in the sense that for any $e \in O_T$ the set Σ_e is a proper subset in $K_{\alpha(e)}$. For any edge $e \in O_T$, let T_e denote the maximal subtree of $T \setminus \operatorname{int}(|e|)$ that contains $\omega(e)$. Set H_e to be the limit of the restricted system Θ_{T_e}. Note that for each $e \in O_T$ the triple $(\Sigma_e, \{H_e, H_{\bar{e}}\})$, viewed as consisting of subsets of K, is an elementary splitting of K. Note also that the family of all splittings of K having this form yields a tree decomposition of K. We denote this decomposition by $\mathcal{C}(\Theta)$. Moreover, this decomposition is fine, which can be deduced by the same methods as in the proof of the last assertions in Proposition 1.C.1 (as given in Section 1.D).

One of the consequences of fineness of a tree decomposition is the following.

3.B.8 Fact *Let $\mathcal{C}$ be a fine tree decomposition of a compact metric space K. Then for any domain Ω for $\mathcal{C}$ the family $\mathcal{A}_\Omega$ of all separators from $\mathcal{C}$ adjacent to Ω is null.*

Proof Note that $\mathcal{A}_\Omega$ coincides with the family of those separators A from $\mathcal{C}$ for which there is a half-space H (related to A) such that $\alpha_{\mathcal{C}}(A, H) = \Omega$. Since any opposite half-space H^c to a half-space H as in the previous sentence contains Ω, and since for such H^c we have $\operatorname{diam}(H^c) \geq \operatorname{diam}(\Omega) > 0$, it follows from fineness of $\mathcal{C}$ that the family of half-spaces $\{H : \alpha_{\mathcal{C}}(A, H) = \Omega\}$ is null. Consequently, since we have inclusions $A \subset H$, the family $\mathcal{A}_\Omega$ is also null. □

3.B.9 Definition Given a fine tree decomposition $\mathcal{C}$ of a compact metric space K, the *tree system $\Theta_{\mathcal{C}}$ associated to $\mathcal{C}$* is described as follows. The underlying tree for $\Theta_{\mathcal{C}}$ is the dual tree $T_{\mathcal{C}}$. For each vertex $t \in V_{\mathcal{C}}$ represented by some domain Ω we set $K_t = \Omega$. For each oriented edge $e = (A, H) \in O_{\mathcal{C}}$ we set $\Sigma_e = A$ and $\phi_e = \mathrm{id}_A$. In view of Fact 3.B.8, this well defines a tree system of metric compacta (which is moreover essential).

Remark It is not hard to realize that if $\mathcal{C}(\Theta)$ is the tree decomposition of $\lim \Theta$ described in Example 3.B.7 then the associated tree system $\Theta_{\mathcal{C}(\Theta)}$ is canonically isomorphic with Θ.

The main result of this section is the following.

3.B.10 Theorem *For any fine tree decomposition $\mathcal{C}$ of a compact metric space K the limit $\lim \Theta_{\mathcal{C}}$ of the associated tree system is canonically homeomorphic to K.*

To prove Theorem 3.B.10, we need the following result, which exhibits consequences of fineness.

3.B.11 Fact *Let $\mathcal{C}$ be a fine tree decomposition of a compact metric space K.*

(1) *For each domain Ω for $\mathcal{C}$ the family of half-spaces from $\mathcal{C}$ given by $\mathcal{H}_\Omega := \{H : \alpha_{\mathcal{C}}(A, H) = \Omega\}$ is null.*

(2) *For each ray $\rho = (e_1, e_2, \dots)$ in the dual tree $T_{\mathcal{C}}$ with $e_i = (A_i, H_i)$, the corresponding family $\{H_i\}$ of half-spaces is null in K.*

Proof The proof of part (1) has already been given in the proof of Fact 3.B.8. To prove part (2), set $\Omega = \alpha_{\mathcal{C}}(e_1)$ and note that for each i we have $\Omega \subset H_i^c$. Consequently, we have $\operatorname{diam}(H_i^c) \geq \operatorname{diam}(\Omega) > 0$. It follows then from fineness of $\mathcal{C}$ that $\lim_{i\to\infty} \operatorname{diam}(H_i) = 0$, which completes the proof. $\square$

Proof of Theorem 3.B.10 We use the identification of $\lim \Theta_{\mathcal{C}}$ with the inverse limit $\varprojlim \mathcal{S}_{\Theta_{\mathcal{C}}}$, provided by Proposition 1.D.1.

We describe a natural map $\mu\colon K \to \lim \Theta_{\mathcal{C}}$. Writing $\Theta_{\mathcal{C}} = (T_{\mathcal{C}}, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$, for each finite subtree F of $T_{\mathcal{C}}$ consider a map $\mu_F\colon K \to K_F^*$ (where K_F^* is the corresponding reduced partial union of the system $\Theta_{\mathcal{C}}$) defined as follows. Writing $e = (A_e, H_e)$ for all $e \in O_{T_{\mathcal{C}}}$, we have that

$$K_F = \bigcap_{e \in N_F} H_e^c \quad \text{and} \quad K_F^* = K_F/\{A_e : e \in N_F\} = K/\{H_e : e \in N_F\}.$$

The latter equality is a homeomorphism by nullness of the family $\{H_e : e \in N_F\}$ (which follows from Fact 3.B.11(1) by observing that $\{H_e : e \in N_F\}$ is a subfamily of the union of finitely many families as in the fact). Thus, we take as μ_F the corresponding quotient map $K \to K/\{H_e : e \in N_F\} = K_F^*$. Since the maps μ_F commute with the maps in the inverse system $\mathcal{S}_{\Theta_\mathcal{C}}$, they induce a continuous map μ: $K \to \varprojlim \mathcal{S}_{\Theta_\mathcal{C}} = \lim \Theta_\mathcal{C}$. Using the arguments as in the proof of Proposition 1.D.1, we verify that μ is a bijection. (In the proof of injectivity one needs to use Fact 3.B.11(2).) Since both spaces K and $\lim \Theta_\mathcal{C}$ are compact, μ is a homeomorphism, which completes the proof. □

3.B.12 Example (boundary tree of disks is a disk) For any $n \geq 2$, let D^n be the n–disk, and let $\mathcal{D}$ be a dense and null family of pairwise disjoint collared $(n-1)$–disks in the boundary sphere ∂D^n. Note that, by Toruńczyk's Lemma 1.E.2.1 (followed by Alexander's trick), the tuple $(D^n, \mathcal{D})$ is unique up to homeomorphism. Consider the unique tree system in which all constituent spaces are homeomorphic to D^n, and all families of peripheral subspaces coincide, up to ambient homeomorphism, with $\mathcal{D}$. We denote this tree system by $\mathcal{M}_\partial(D^n)$ and call it the *dense boundary tree system of n–disks*.

3.B.12.1 Lemma *The limit* $\lim \mathcal{M}_\partial(D^n)$ *is the n–disk.*

Proof We will describe a fine tree decomposition $\mathcal{C}$ of the n–disk D^n such that the associated tree system $\Theta_\mathcal{C}$ is isomorphic to $\mathcal{M}_\partial(D^n)$. By applying Theorem 3.B.10, and in view of Lemma 1.E.1, this gives the assertion.

To construct $\mathcal{C}$, view D^n as the standard n–disk in R^n, and consider the group Möb(D^n) of all Möbius transformations of D^n, ie those Möbius transformations of R^n which preserve D^n. Viewing int(D^n) as the Poincaré disk model for hyperbolic n–space, we will think of Möb(D^n) as the group of all hyperbolic isometries in its action on the completion of hyperbolic n–space by its ideal boundary.

Let $\mathcal{D}$ be any null and dense family of pairwise disjoint round $(n-1)$–disks in ∂D^n. For each $D \in \mathcal{D}$, let H_D be the hyperbolic half-space in D^n such that $H_D \cap \partial D^n = D$, and let H_D^c be the opposite half-space. Denote by A_D the hyperplane in D^n bounding H_D, and by s_D the hyperbolic reflection with respect to A_D, which clearly belongs to Möb(D^n).

Let Γ be the subgroup of Möb(D^n) generated by all elements s_D with $D \in \mathcal{D}$. Clearly, Γ is then an infinitely generated free reflection group with fundamental domain

$$\Omega_0 := \bigcap_{D \in \mathcal{D}} H_D^c.$$

Algebraically, Γ is the free product of its order 2 subgroups $\langle s_D \rangle$ with $D \in \mathcal{D}$.

Let $\mathcal{A}$ be the family of reflection hyperplanes in D^n for all reflections from Γ. In other words, $\mathcal{A}$ is the family of all images through elements of Γ of the hyperplanes A_D with $D \in \mathcal{D}$. Each $A \in \mathcal{A}$ splits D^n into two components. Denote the closures of these components in D^n by Y and Z, and observe that $(A, \{Y, Z\})$ is an elementary splitting of D^n. Denote by $\mathcal{C}$ the set of elementary splittings $(A, \{Y, Z\})$ as above, for all $A \in \mathcal{A}$. It is fairly clear that $\mathcal{C}$ is then a discrete family of pairwise noncrossing splittings of D^n, ie a tree decomposition of D^n, and that it is fine.

It remains to show that the tree system $\Theta_{\mathcal{C}}$ associated to $\mathcal{C}$ is isomorphic to $\mathcal{M}_\partial(D^n)$. To see this, note that for each $D \in \mathcal{D}$ the domain Ω_{A_D, H_D^c} coincides with the fundamental domain Ω_0 for Γ. It is not hard too see that this domain is homeomorphic to the n–disk. The separators of $\mathcal{C}$ contained in Ω_0 are exactly the elements A_D with $D \in \mathcal{D}$, and they clearly form a null and dense family of collared and pairwise disjoint $(n-1)$–disks in the boundary $\partial\Omega_0$. Consequently, the constituent space Ω_0 of $\Theta_{\mathcal{C}}$, together with the family $\{A_D : D \in \mathcal{D}\}$ of all its peripheral subspaces, is homeomorphic to $(D^n, \mathcal{D})$. To see that the same is true for all other constituent spaces of $\Theta_{\mathcal{C}}$, note that each such space is of the form $\gamma\Omega_0$ for some $\gamma \in \Gamma$, and the corresponding family of peripheral subspaces is of the form $\{\gamma A_D : D \in \mathcal{D}\}$. This shows that $\Theta_{\mathcal{C}}$ is isomorphic to $\mathcal{M}_\partial(D^n)$, thus completing the proof. □

3.B.12.2 Remark Let $\mathcal{C}$ be a family of elementary splittings of D^n, as constructed in the proof of Lemma 3.B.12.1. Restricting $\mathcal{C}$ to the boundary ∂D^n, we obviously get a fine tree decomposition of the $(n-1)$–sphere, and we denote it by $\mathcal{C}|_{\partial D^n}$. Moreover, the domains of this new decomposition are the intersections of the domains of $\mathcal{C}$ with ∂D^n, and it is not hard to verify that they are the $(n-2)$–dimensional Sierpiński compacta $(S^{n-1})^\circ$. Thus, these Sierpiński compacta are the constituent spaces of the associated tree system $\Theta_{\mathcal{C}|_{\partial D^n}}$, and the peripheral subspaces correspond exactly to the peripheral spheres in these Sierpiński compacta. It follows that the associated tree system $\Theta_{\mathcal{C}|_{\partial D^n}}$ is isomorphic to the tree system $\mathcal{M}(S^{n-1})$. Since, by Theorem 3.B.10, we have

$$\lim \Theta_{\mathcal{C}|_{\partial D^n}} = \partial D^n = S^{n-1},$$

it follows that $\lim \mathcal{M}(S^{n-1}) = S^{n-1}$. Thus, we get an alternative (and more elementary) proof of Corollary 1.E.3.2.

3.B.13 Remark/Example/Exercise Using a similar argument as in the proof of Lemma 3.B.12.1 one can identify, up to homeomorphism, limits of various other tree systems. For example, one can show that the limit of any dense tree system of internally punctured n–disks (see Example 2.D.3) is homeomorphic to the $(n-1)$–dimensional Sierpiński compactum $(S^n)^\circ$. Once this is known, one can use a consolidation procedure from Section 3.A to show that the limit of any dense tree system of internally punctured connected planar surfaces is homeomorphic to the Sierpiński curve.

3.C Subdivision of a tree system

Generalizing the concepts from the previous section, we describe in this section the operation of subdivision of a tree system, opposite to the operation of consolidation described in Section 3.A.

Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system, let $t \in V_T$ be any vertex, and suppose that $\mathcal{C}$ is a tree decomposition of the space K_t. We say that $\mathcal{C}$ *does not cross* Θ if for each separator A of $\mathcal{C}$ and each $e \in N_t$ there is a half-space H for A such that $\Sigma_e \subset \dot{H}$. This means in particular that $A \cap \Sigma_e = \varnothing$.

Furthermore, given a tree decomposition $\mathcal{C}$ of K_t not crossing Θ, and any edge $e \in N_t$, we say that $\mathcal{C}$ is *discrete at* e if for some (and hence any) separator A of $\mathcal{C}$ there are only finitely many separators in $\mathcal{C}$ that separate A from Σ_e (ie finitely many separators A' from $\mathcal{C}$ such that for some half-space H related to A' we have $A \subset \dot{H}$ and $\Sigma_e \subset \dot{H}^c$). It is not hard to realize that $\mathcal{C}$ is discrete at e if and only if $\Sigma_e \subset \Omega$ for some domain $\Omega \subset K_t$ for $\mathcal{C}$. We also have the following sufficient condition for discreteness at e.

3.C.1 Lemma *Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system, let $\mathcal{C}$ be a tree decomposition not crossing Θ of some constituent space K_t, and let $e \in N_t$. If $\mathcal{C}$ is fine and if* $\operatorname{diam}(\Sigma_e) > 0$, *then $\mathcal{C}$ is discrete at e.*

Proof Let A be any separator from $\mathcal{C}$, and let H_A be the half-space for A which does not contain Σ_e. Let A' be any separator from $\mathcal{C}$ that separates A from Σ_e, with $A \subset \dot{H}$ and $\Sigma_e \subset \dot{H}^c$ for the corresponding half-spaces H and H^c for A'. Note that $H_A \subset \dot{H}$, and consequently

$$\operatorname{diam}(H) \geq \operatorname{diam}(H_A) > 0 \quad \text{and} \quad \operatorname{diam}(H^c) \geq \operatorname{diam}(\Sigma_e) > 0.$$

In view of fineness of $\mathcal{C}$, this implies that there are only finitely many separators A' as above, which completes the proof. □

3.C.2 Definition A tree decomposition $\mathcal{C}$ of a space K_t from a tree system Θ is *compatible with* Θ if it does not cross Θ and if it is discrete at e for each $e \in N_t$.

3.C.3 Example Given a tree system $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ and a vertex $t \in V_T$, we present a criterion for a family of elementary splittings of the space K_t to be a tree decomposition of K_t compatible with Θ. We will use this criterion in Section 3.D.

Let $x_0 \in K_t \setminus \bigcup\{\Sigma_e : e \in N_t\}$. Suppose that a family $\mathcal{C}_t$ of elementary splittings of K_t is of the form

$$\mathcal{C}_t = \{(A_k, \{H_k, H_k^c\}) : k \in \mathbb{N}\},$$

where the following three conditions hold:

(1) Splittings from $\mathcal{C}_t$ do not cross Θ; ie each separator A_k is disjoint with each of the sets from the family $\mathcal{A}_t = \{\Sigma_e : e \in N_t\}$ and each half-space H_k is $\mathcal{A}_t$–saturated.

(2) $H_{k+1} \subset \dot{H}_k$ for each natural number k.

(3) $\bigcap_{k=1}^{\infty} H_k = \{x_0\}$ $\big($equivalently, $x_0 \in \bigcap_{k=1}^{\infty} H_k$ and $\lim_{k\to\infty} \operatorname{diam}(H_k) = 0\big)$.

We then get the following.

3.C.3.1 Lemma *If a family $\mathcal{C}_t$ satisfies conditions* (1)–(3) *above then $\mathcal{C}_t$ is fine, discrete, and discrete at all $e \in N_t$. In particular, it is a tree decomposition of K_t, and it is compatible with Θ.*

Proof By condition (2), splittings in $\mathcal{C}_t$ are pairwise noncrossing, and $\mathcal{C}_t$ is discrete. By condition (3), $\mathcal{C}_t$ is fine. Thus, in view of condition (1), $\mathcal{C}_t$ is a tree decomposition of K_t. Moreover, by condition (3), for each $e \in N_t$ there is a k such that $\Sigma_e \cap H_k = \varnothing$. It follows that for each $e \in N_t$ the family $\mathcal{C}_t$ is discrete at e. This completes the proof. □

3.C.4 Definition A *tree decomposition of a tree system* Θ is a family $\mathcal{C} = \{\mathcal{C}_t : t \in V_T\}$ of tree decompositions $\mathcal{C}_t$ of the spaces K_t which are all compatible with Θ.

We now describe some elementary splittings of the limit space $\lim \Theta$ induced by any elementary splittings from any tree decomposition $\mathcal{C} = \{\mathcal{C}_t\}$ of Θ. Let t be an arbitrary vertex in the underlying tree T of Θ, and let $(A, \{H, H^c\})$ be an elementary splitting of the space K_t belonging to $\mathcal{C}_t$. Note that both subsets H and H^c of K_t are saturated with respect to the family $\mathcal{A}_t = \{\Sigma_e : e \in N_t\}$. In particular, it makes sense to speak of the subsets $G(H)$ and $G(H^c)$ in $\lim \Theta$ of the form described in Section 1.C (just before Proposition 1.C.1). Moreover, when we view A as a subset of $\lim \Theta$, we clearly have $G(H) \cap G(H^c) = A$, where all three subsets in this expression are compact subspaces of $\lim \Theta$. Thus, the triple $\bigl(A, \{G(H), G(H^c)\}\bigr)$ is an elementary splitting of the limit space $\lim \Theta$. For each $t \in V_T$ we set

$$\mathcal{C}_t^{\lim} = \bigl\{\bigl(A, \{G(H), G(H^c)\}\bigr) : A \text{ is a separator in } \mathcal{C}_t\bigr\}$$

3.C.5 Lemma *Let Θ be a tree system of metric compacta, and let $\mathcal{C}(\Theta)$ be the associated tree decomposition of the limit $\lim \Theta$. Then for any tree decomposition $\mathcal{C}$ of Θ the family*

$$\mathcal{C}_{\lim} = \Bigl(\bigcup_{t \in V_T} \mathcal{C}_t^{\lim}\Bigr) \cup \mathcal{C}(\Theta)$$

is a tree decomposition of $\lim \Theta$.

Proof The fact that the elementary splittings from $\mathcal{C}_{\lim}$ are pairwise noncrossing follows directly from the fact that the families $\mathcal{C}_t$ do not cross Θ. The discreteness of $\mathcal{C}_{\lim}$ follows fairly directly from discreteness of each $\mathcal{C}_t$ at each $e \in N_t$. We omit further details. □

3.C.6 Proposition *Under the notation of Lemma 3.C.5, the tree decomposition $\mathcal{C}_{\lim}$ is fine if and only if each tree decomposition $\mathcal{C}_t$ from $\mathcal{C}$ is fine.*

Proof One implication, namely that fineness of $\mathcal{C}_{\lim}$ implies fineness of every $\mathcal{C}_t$, is immediate just by observing that half-spaces for $\mathcal{C}_t$ are simply restrictions to K_t of the appropriate half-spaces for $\mathcal{C}_{\lim}$. The converse implication requires much more effort and some preparatory claims. We start with a claim which gives a useful characterization of fineness of a tree decomposition.

Claim 1 *A tree decomposition $\mathcal{C}$ of a compact K is fine if and only if for any $\epsilon > 0$ there is a finite collection $\Omega_1, \dots, \Omega_m$ of domains for $\mathcal{C}$ such that if H is any half-space from $\mathcal{C}$ not containing any of the above domains then $\operatorname{diam}(H) < \epsilon$.*

To prove Claim 1, suppose first that $\mathcal{C}$ is fine. Given $\epsilon > 0$, let $(A_i, \{Y_i, Z_i\})$ for $i = 1, \dots, q$ be all elementary splittings in $\mathcal{C}$ for which $\operatorname{diam}(Y_i) \ge \epsilon$ and $\operatorname{diam}(Z_i) \ge \epsilon$. Let $\Omega_1, \dots, \Omega_m$ be the set of all domains for $\mathcal{C}$ adjacent to some of the separators $A_1, \dots, A_q$. Since each separator has exactly two adjacent domains, the above set of domains is finite. Let H be a half-space from $\mathcal{C}$ not containing any of the above domains. It is not hard to realize that then for any $1 \le i \le q$ either Y_i or Z_i is contained in H^c, and hence $\operatorname{diam}(H^c) \ge \epsilon$. On the other hand, the separator A corresponding to H is clearly distinct from each of the separators $A_1, \dots, A_q$, and hence $\operatorname{diam}(H) < \epsilon$, as required.

To prove the converse implication in Claim 1, fix $\epsilon > 0$ and let $\Omega_1, \dots, \Omega_m$ be some domains associated to ϵ, as in the assumption of the implication. Note that, by the assumption, for any splitting $(A, \{Y, Z\}) \in \mathcal{C}$ with at least one half-space not containing any of the above domains we have $\min(\operatorname{diam}(Y), \operatorname{diam}(Z)) < \epsilon$. Thus, it is sufficient to show that the number of splittings with both half-spaces containing some of the domains $\Omega_1, \dots, \Omega_m$ is finite. Translating this to the language of the dual tree, we need to know that for any finite set V_0 of vertices in $T_{\mathcal{C}}$ the set of edges in $T_{\mathcal{C}}$ which separate some two of the vertices of V_0 is finite. Since this set of edges clearly coincides with the set of edges in the subtree of $T_{\mathcal{C}}$ spanned by V_0, this completes the proof of Claim 1.

We come back to the tree decomposition $\mathcal{C}_{\mathrm{lim}}$. For each $t \in V_T$ and any $e \in N_t$, let H_e be the half-space from $\mathcal{C}(\Theta)$ associated to the separator Σ_e which does not contain K_t. For each half-space H from $\mathcal{C}_t$, set $\mathcal{H}_H := \{H_e : e \in N_t \text{ and } \Sigma_e \subset H\}$. The next preparatory claim provides some estimate for the diameter of a half-space from $\mathcal{C}_{\mathrm{lim}}$ in terms of diameters of appropriate half-spaces from $\mathcal{C}$ and $\mathcal{C}(\Theta)$.

Claim 2 *Let t be any vertex of T and let H be any half-space from $\mathcal{C}_t$. Then the induced half-space $G(H)$ from $\mathcal{C}_t^{\mathrm{lim}}$ has form $G(H) = H \cup \big(\bigcup \mathcal{H}_H\big)$ and its diameter is estimated by*

$$\operatorname{diam}(G(H)) \le \operatorname{diam}(H) + 2 \cdot \max\{\operatorname{diam}(H') : H' \in \mathcal{H}_H\}.$$

The first assertion of Claim 2, ie that $G(H) = H \cup \big(\bigcup \mathcal{H}_H\big)$, follows directly from the definition of $G(H)$. To prove the second assertion, we estimate the distance of any two points of $G(H)$. Suppose that x and y are some points of $G(H)$ not contained in H. Then, by the first assertion, there exist $e, e' \in N_t$ such that $x \in H_e$ and $y \in H_{e'}$. Let $x' \in \Sigma_e$ and $y' \in \Sigma_{e'}$ be arbitrary points. Since $\Sigma_e = H \cap H_e$ and $\Sigma_{e'} = H \cap H_{e'}$, we get the estimate

$$d(x, y) \le d(x, x') + d(x', y') + d(y', y) \le \operatorname{diam}(H_e) + \operatorname{diam}(H) + \operatorname{diam}(H_{e'}),$$

which can be further estimated from above by the right-hand side term in the desired inequality. When one or both of the points x and y belong to H, a similar (even simpler) estimate can be obtained in the same way. This completes the proof of Claim 2.

We come back to the proof of the following implication: if each tree decomposition $\mathcal{C}_t$ from $\mathcal{C}$ is fine then $\mathcal{C}_{\mathrm{lim}}$ is fine. We use the characterization of fineness given in Claim 1. Fix any $\epsilon > 0$. Since $\mathcal{C}(\Theta)$ is fine (see the last comment in Example 3.B.7), we choose a finite subtree $S \subset T$ such that any half-space H' from $\mathcal{C}(\Theta)$ which does not contain any of the spaces K_t with $t \in V_S$ satisfies $\operatorname{diam}(H') < \frac{1}{3}\epsilon$. Note that this implies also that for any $s \in V_T \setminus V_S$ we have $\mathrm{diam}(K_s) < \frac{1}{3}\epsilon$. For each $t \in V_S$, using the fact that $\mathcal{C}_t$ is fine, choose some domains $\Omega^t_1, \dots, \Omega^t_{m_t}$, with $m_t \geq 1$, such that any half-space H from $\mathcal{C}_t$ which does not contain any of these domains satisfies $\mathrm{diam}(H) < \frac{1}{3}\epsilon$. Set $\mathcal{Z} := \bigcup_{t \in V_S} \{\Omega^t_1, \dots, \Omega^t_{m_t}\}$ and note that $\mathcal{Z}$ is a finite family of domains for the tree decomposition $\mathcal{C}_{\mathrm{lim}}$. We claim that any half-space H_{lim} from $\mathcal{C}_{\mathrm{lim}}$ which does not contain any of the domains from $\mathcal{Z}$ satisfies $\mathrm{diam}(H_{\mathrm{lim}}) < \epsilon$.

To prove the above claim, suppose first that $H_{\mathrm{lim}} = H$ for some half-space H from $\mathcal{C}(\Theta)$. Since $H = H_{\mathrm{lim}}$ does not contain any of the domains from $\mathcal{Z}$, it also does not contain any of the spaces K_t with $t \in V_S$. By the choice of S, this implies that $\mathrm{diam}(H_{\mathrm{lim}}) < \frac{1}{3}\epsilon < \epsilon$, as required.

In the remaining case we have $H_{\mathrm{lim}} = G(H)$, where H is a half-space from $\mathcal{C}_t$ in the space K_t for some $t \in V_T$. We will estimate the diameter of $H_{\mathrm{lim}} = G(H)$ using Claim 2. To do this, we claim that $\mathrm{diam}(H) < \frac{1}{3}\epsilon$. Indeed, if $H \subset K_t$ for some $t \in V_S$, this estimate follows from our choice of the domains $\Omega^t_1, \dots, \Omega^t_{m_t}$, in view of the fact that H does not contain any of them. If $H \subset K_s$ for some $s \in V_T \setminus V_S$, we get $\mathrm{diam}(H) \leq \mathrm{diam}(K_s) < \frac{1}{3}\epsilon$, where the last inequality follows from the choice of S.

We now estimate diameters of the half-spaces $H' \in \mathcal{H}_H$. Since any such H' (being a half-space from $\mathcal{C}(\Theta)$) does not contain any of the spaces K_t with $t \in V_S$ (because it does not contain any of the domains from $\mathcal{Z}$), it follows from the choice of S that $\mathrm{diam}(H') < \frac{1}{3}\epsilon$.

In view of Claim 2, the estimates from the two previous paragraphs yield the inequality $\mathrm{diam}(G(H)) < \epsilon$, as required. By Claim 1, the tree decomposition $\mathcal{C}_{\mathrm{lim}}$ is then fine, which completes the proof. □

3.C.7 Definition Let Θ be a tree system of metric compacta. A *subdivision* of Θ is any tree system of the form $\Theta_{\mathcal{C}_{\mathrm{lim}}}$, for any fine tree decomposition $\mathcal{C}_{\mathrm{lim}}$ as in Lemma 3.C.5.

3.C.8 Proposition *Let Ξ be any subdivision of Θ. Then Θ can be canonically obtained from Ξ by means of a consolidation. Moreover, the limits* $\lim \Xi$ *and* $\lim \Theta$ *are canonically homeomorphic.*

Proof Let $\Xi = \Theta_{\mathcal{C}_{\mathrm{lim}}}$. Note that the constituent spaces of the tree system Ξ are precisely the constituent spaces of the systems $\Theta_{\mathcal{C}_t}$ for all $t \in V_T$. Since each $\mathcal{C}_t$ is a fine tree decomposition of K_t, it follows from Theorem 3.B.10 that $\lim \Theta_{\mathcal{C}_t} = K_t$.

This shows that Θ is a consolidation of Ξ, for the canonical partition of the dual tree $T_{\mathcal{C}_{\lim}}$ into subtrees $T_{\mathcal{C}_t}$. The second assertion follows either from Theorem 3.B.10 and Lemma 3.C.5 or, in view of the first assertion, from Theorem 3.A.1. $\square$

3.D Homogeneity of trees of manifolds

In this section we prove the following.

3.D.1 Proposition *Let M be a closed connected topological manifold* (*either oriented or nonorientable*). *Then the tree of manifolds M, ie the Jakobsche space $\mathcal{X}(M)$* (as defined in Section 1.E), *is homogeneous.*

This has been proved before in [13] (for oriented M) and in [21] (for nonorientable M which are PL). The proof we present here uses the technique of subdivisions and consolidations of tree systems. It is inspired by the corresponding proof in [13, Sections 7 and 8]. Our argument, and the technique used in it, has a potential for extensions. It can be applied to various other classes of tree systems (eg to the tree systems of polyhedra mentioned in the introduction) to study orbits of the group of homeomorphisms of the corresponding limit space. One easy instance of such an extension is presented below, as Propositions 3.D.6.1 and 3.D.6.2.

We start with few technical preparatory results.

3.D.2 Lemma *Let M be a closed connected topological manifold, oriented or nonorientable, and let $\mathcal{M}$ be the dense tree system of manifolds M with underlying tree T. Then the points of the limit $\mathcal{X}(M) = \lim \mathcal{M}$ corresponding to the set E_T of the ends of T are all in the same orbit of the group of homeomorphisms of $\mathcal{X}(M)$.*

Proof Let $z_1, z_2 \in E_T \subset \lim \mathcal{M}$. Using Toruńczyk's Lemma 1.E.2.1 (together with Lemma 1.E.4.1 in case when M is nonorientable) it is not hard to get an automorphism of the tree system $\mathcal{M}$ for which the corresponding automorphism of the underlying tree T maps z_1 to z_2. Clearly, this automorphism of $\mathcal{M}$ induces a homeomorphism of $\lim \mathcal{M}$ which maps z_1 to z_2. This finishes the proof. $\square$

The next result is an extension of [12, Lemma 5], and it appears implicitly inside the proof of [13, Lemma 7.1].

We will need the following corollary to Lemma 1.D.2.2.

3.D.3 Corollary *Let M and $\mathcal{D}$ be as in Lemma 1.D.2.2, and let x_0 be an interior point of M not contained in any $D \in \mathcal{D}$. Then there is a sequence of collared n–disks Q_k contained in* $\operatorname{int}(M)$ *such that*

(1) *for each k the boundary ∂Q_k is disjoint with the union of $\mathcal{D}$,*

(2) *for each k we have $Q_{k+1} \subset \operatorname{int}(Q_k)$,*

(3) $\{x_0\} = \bigcap_k Q_k$.

Proof By Lemma 1.D.2.2, the quotient $M/\mathcal{D}$ is homeomorphic to M. Let $A_{\mathcal{D}}$ be the set of points in the quotient $M/\mathcal{D}$ corresponding to the collapsed elements of $\mathcal{D}$. Clearly, $A_{\mathcal{D}}$ is then countably infinite, and $x_0 \in (M/\mathcal{D}) \setminus A_{\mathcal{D}}$. Obviously, there exists a sequence of collared n–disks P_k in $M/\mathcal{D}$ such that

(1) for each k the boundary ∂P_k is disjoint with $A_{\mathcal{D}}$,

(2) for each k we have $P_{k+1} \subset \operatorname{int}(P_k)$,

(3) $x_0 = \bigcap_k P_k$.

For each k, set $Q_k = q^{-1}(P_k)$, where $q\colon M \to M/\mathcal{D}$ is the quotient map. To see that the sequence Q_k is as required, it obviously suffices to show that each Q_k is a collared n–disk in M. To do this, consider the decomposition of Q_k induced by the family $\mathcal{D}_k = \{D \in \mathcal{D} : D \subset Q_k\}$. Then we clearly have $P_k = Q_k/\mathcal{D}_k$, and we denote the quotient map by q_k. Arguing as in the proof of Lemma 1.D.2.2, we get that the above decomposition is shrinkable, and hence q_k can be approximated by homeomorphisms. This shows Q_k is an n–disk. By applying the same argument to the complement $M \setminus \operatorname{int}(Q_k)$ (instead of Q_k), we get that Q_k is collared, which finishes the proof. □

A crucial ingredient in the proof of Proposition 3.D.1 is the following result, in the proof of which we use the technique of modifications of tree systems, developed in Sections 3.A–3.C. This proof is inspired by Jakobsche's idea from his proof of [13, Lemma (7.1)].

3.D.4 Lemma *Let M be a closed connected manifold, oriented or nonorientable, and let $\mathcal{M} = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be the dense tree system of manifolds M. Let*

$$x \in K_{t_*} \setminus \bigcup \{\Sigma_e : e \in N_{t_*}\} \subset \lim \mathcal{M}$$

for some $t_* \in V_T$, and let $y \in E_T \subset \lim \mathcal{M}$. Then there is a homeomorphism of $\lim \mathcal{M}$ which maps x to y.

Proof Writing $n = \dim M$, recall that for all $t \in V_T$ we have

$$K_t = M_t^\circ = M_t \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}\},$$

where M_t is a homeomorphic copy of M and where $\mathcal{D}$ is a null and dense family of pairwise disjoint collared n–disks in M_t. View x as a point of M_{t_*} and consider a sequence Q_k of n–disks as in Corollary 3.D.3 for $x_0 = x$. For each k consider the elementary splitting $(A_k, \{H_k, H_k^c\})$ of K_{t_*} given by $A_k := \partial Q_k$ and $H_k := Q_k \cap K_{t_*}$, and denote by $\mathcal{C}_{t_*}$ the set of these elementary splittings for all natural numbers k. By Lemma 3.C.3.1, the family $\mathcal{C}_{t_*}$ is a tree decomposition of K_{t_*} compatible with $\mathcal{M}$.

Let $\mathcal{C}^x = \{\mathcal{C}_t : t \in V_T\}$ be a tree decomposition of $\mathcal{M}$ such that $\mathcal{C}_{t_*}$ is as above and $\mathcal{C}_t = \varnothing$ for $t \neq t_*$. Let $\mathcal{M}^x = (T^x, \{K_t^x\}, \{\Sigma_e^x\}, \{\phi_e^x\})$ be the subdivision of $\mathcal{M}$ induced by $\mathcal{C}^x$. The underlying tree T^x may be viewed as obtained from T by expanding the vertex t_* into an infinite polygonal ray $\rho = (t_0, t_1, \dots)$. We may identify all other vertices of T^x bijectively with the vertices in $V_T \setminus \{t_*\}$. The set of the edges of T adjacent to t_* canonically splits into subsets which may be identified with the sets of edges of T^x adjacent to the vertices t_i (for $i = 0, 1, \dots$). The edges of T^x disjoint from the ray ρ are in the natural bijective correspondence with the edges of T not adjacent to t_*. Accordingly, the constituent spaces K_t with $t \neq t_*$ are still the constituent spaces in $\mathcal{M}^x$ at the corresponding vertices, with the families of peripheral subspaces unchanged. In particular, they are still the densely punctured manifolds M. The constituent space K_{t_*} of $\mathcal{M}$ splits into the constituent spaces $K_{t_i}^x$ for $i \geq 0$ of $\mathcal{M}^x$, which are described as follows. We have $K_{t_0}^x := H_1^c \cap K_{t_*} = K_{t_*} \setminus \mathrm{int}(Q_1)$, and for $i \geq 1$ we have $K_{t_i}^x := (H_{i+1}^c \setminus \dot{H}_i) \cap K_{t_*} = K_{t_*} \cap (Q_i \setminus \mathrm{int}(Q_{i+1}))$. It is not hard to see that $K_{t_0}^x$ still has the form of the densely punctured manifold M, while each of $K_{t_i}^x$ for $i \geq 1$ is the densely punctured sphere S^n.

By Proposition 3.C.8, we have the canonical identification of the limits $\lim \mathcal{M}$ and $\lim \mathcal{M}^x$. The point x, viewed as an element of $\lim \mathcal{M}^x$, clearly corresponds to the end of the tree T^x induced by the ray ρ, ie $x = [\rho] \in E_{T^x}$. On the other hand, the point y viewed as an element of $\lim \mathcal{M}^x$ still corresponds to an end of the underlying tree, ie $y \in E_{T^x} \subset \lim \mathcal{M}^x$.

We now apply to the system $\mathcal{M}^x$ an operation of consolidation, as described in Section 3.A. More precisely, for each $i \geq 1$ choose any vertex s_i in T^x adjacent to t_i and distinct from both t_{i-1} and t_{i+1}, and denote by Π the partition of T^x consisting of the subtrees S_i for $i \geq 1$ spanned by the pairs t_i and s_i (these subtrees are just edges) and of subtrees reduced to vertices for all remaining vertices of T^x.

Let $\mathcal{M}^x_\Pi$ be the tree system obtained from $\mathcal{M}^x$ by consolidation with respect to Π. By Proposition 3.C.8, the limit $\lim \mathcal{M}^x_\Pi$ is canonically homeomorphic to $\lim \mathcal{M}^x$, and hence also to $\lim \mathcal{M}$.

From what was said above about the system $\mathcal{M}^x$ it is not hard to deduce that $\mathcal{M}^x_\Pi$ is a dense tree system of manifolds M. Moreover, the point x (viewed now as a point of $\lim \mathcal{M}^x_\Pi$) clearly corresponds to the end of the tree T^x_Π induced by the ray $\rho_\Pi = (S_1, S_2, \dots)$. On the other hand, the point y still corresponds to an end of the underlying tree, ie $y \in E_{T^x_\Pi} \subset \lim \mathcal{M}^x_\Pi$. Thus, by Lemma 3.D.2, there is a homeomorphism of $\lim \mathcal{M}^x_\Pi$ which maps x to y, which finishes the proof. □

The next result is a consequence of Lemma 3.D.4.

3.D.5 Corollary *Let M and $\mathcal{M}$ be as in Lemma 3.D.4, let $x \in \Sigma_{e_*} \subset \lim \mathcal{M}$ for some $e_* \in O_T$, and let $y \in E_T \subset \lim \mathcal{M}$. Then there is a homeomorphism of $\lim \mathcal{M}$ which maps x to y.*

Proof Writing $n = \dim M$, view the constituent space $K_{\alpha(e_*)}$ of $\mathcal{M}$ as obtained from $M_{\alpha(e_*)} = M$ by deleting interiors of disks D from a null and dense family $\mathcal{D}$ of pairwise disjoint collared n–disks. Let $D_* \in \mathcal{D}$ be the disk for which $\partial D_* = \Sigma_{e_*}$. Let Q be a collared n–disk in $M_{\alpha(e_*)}$ with $D_* \subset Q$ and with ∂Q disjoint from the union of $\mathcal{D}$ (existence of such Q follows by arguments as in the proof of Corollary 3.D.3).

Consider the elementary splitting $(A, \{H, H^c\})$ of $K_{\alpha(e_*)}$ given by $A = \partial Q$ and $H = Q \cap K_{\alpha(e_*)}$. Let $\mathcal{M}^Q$ be the subdivision of $\mathcal{M}$ induced by this single splitting. Let t_H be the vertex of the underlying tree T^Q of $\mathcal{M}^Q$ corresponding to the constituent space H. Viewing $\omega(e_*)$ naturally as a vertex in T^Q, note that t_H and $\omega(e_*)$ are adjacent, and denote by S_Q the subtree of T^Q consisting of these two vertices and the edge which connects them. Consider the consolidation $\mathcal{M}^Q_\Pi$ of $\mathcal{M}^Q$ for the partition Π consisting of the subtree S_Q and the singleton subtrees for all other vertices. Clearly, the limits $\lim \mathcal{M}^Q_\Pi$ and $\lim \mathcal{M}$ are canonically homeomorphic.

Note that $\mathcal{M}^Q_\Pi$ is again a dense tree of manifolds M and that x, naturally viewed as element of $\lim \mathcal{M}^Q_\Pi$, belongs to the constituent space $K_{S_Q} \subset \lim \mathcal{M}^Q_\Pi$ and lies outside all its peripheral subspaces. Since the point y, viewed as element of $\lim \mathcal{M}^Q_\Pi$, still corresponds to an end of the underlying tree, the corollary follows by applying Lemma 3.D.4. □

Proof of Proposition 3.D.1 The proposition is a direct consequence of Lemmas 3.D.2 and 3.D.4 and Corollary 3.D.5. □

3.D.6 Remark The argument of this section yields also the following two results.

3.D.6.1 Proposition *Let M be a connected compact topological manifold with boundary, either oriented or nonorientable, and let $\mathcal{M}$ be the dense tree system of internally punctured manifolds M, as defined in Remark 2.D.3.1. Then, viewing the boundaries ∂M_t as subsets of the constituent spaces $K_t = M_t^\circ$ of $\mathcal{M}$, all points of $K_t \setminus \partial M_t \subset \lim \mathcal{M}$ (for all t) and all points of $E_T \subset \lim \mathcal{M}$ are in the same orbit of the group of homeomorphisms of $\lim \mathcal{M} = \mathcal{X}_{\mathrm{int}}(M)$.*

Recall that, given any natural number m, a topological space X is *m–homogeneous* if the group of its homeomorphisms acts transitively on the set of all m–tuples of pairwise distinct points of X. A straightforward extension of the arguments of this section allows us to prove [13, Theorem 8.1], ie the fact that the Jakobsche space $\mathcal{X}(M)$ is m–homogeneous for arbitrary m. It also allows us to prove the following variant of this result, for trees of manifolds with boundary.

3.D.6.2 Proposition *Let M and $\mathcal{M}$ be as in Proposition 3.D.6.1. Consider the subspace*

$$U = E_T \cup \Big(\bigcup_{t \in V_T} K_t \setminus \partial M_t \Big) \subset \lim \mathcal{M}.$$

Then for any natural number m the group of homeomorphisms of $\lim \mathcal{M}$ acts transitively on the set of m–tuples of pairwise distinct points of U.

3.E Weakly saturated tree systems of manifolds

In this section we present another application of the operations of consolidation and subdivision. It concerns dense trees of finite families of manifolds, as in Example 3.A.2, and provides a significant strengthening of Propositions 3.A.2.1 and 3.A.2.3. More precisely, we will show that under a much weaker assumption than 2–saturation, the limit of a dense tree of a finite family of manifolds $\{M_1, \dots, M_k\}$ is still homeomorphic to the space $\mathcal{X}(M_1 \# \cdots \# M_k)$. The weaker condition will be called *weak saturation*. The results of this section are used by the author in [25] to show that various trees of manifolds, in arbitrary dimension, appear as Gromov boundaries of some hyperbolic groups.

Let $\mathcal{N} = \{M_1, \dots, M_k\}$ be a finite family of closed connected topological manifolds of the same dimension, either all oriented or at least one of which is nonorientable. Let $\mathcal{M} = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a dense tree system of manifolds from $\mathcal{N}$. As in Example 3.A.2, for each $t \in V_T$, let $i_t \in \{1, \dots, k\}$ be the index for which K_t is of the form $M_{i_t}^\circ$. Recall also that a *half-tree* in a tree is any maximal subtree of it obtained by deleting the interior of any edge.

3.E.1 Definition We say that a dense tree system $\mathcal{M}$ of manifolds from $\mathcal{N}$ is *weakly saturated* if for each $j \in \{1, \dots, k\}$ any half-tree in the underlying tree T contains a vertex t with $i_t = j$ (equivalently, for each $j \in \{1, \dots, k\}$ the set $V_T^j = \{t \in V_T : i_t = j\}$ spans T).

The main result of this section is the following.

3.E.2 Theorem *Let $\mathcal{M}$ be any weakly saturated dense tree system of manifolds from a finite family $\mathcal{N} = \{M_1, \dots, M_k\}$ (where the M_i are closed connected topological manifolds of the same dimension, either all oriented or at least one of which is nonorientable). Then the limit $\lim \mathcal{M}$ is homeomorphic to the Jakobsche space $\mathcal{X}(M_1 \# \cdots \# M_k)$.*

Proof The proof consists of showing that, by applying a certain consolidation followed by some subdivision, the tree system $\mathcal{M}$ can be transformed into a 2–saturated dense tree system of manifolds from $\mathcal{N}$. Because of invariance of the limit under operations as above, and in view of Propositions 3.A.2.1 and 3.A.2.3, this will give the assertion.

We use the following notation. Given an oriented edge $e \in O_T$ (where T is the underlying tree of the system $\mathcal{M}$), denote by T_e the half-tree obtained by deleting from T the interior of e, which contains the terminal vertex $\omega(e)$.

We describe the appropriate consolidation and subdivision simultaneously, according to the following scheme. We successively choose finite subtrees S of a partition Π of the tree T (which will induce a desired consolidation $\mathcal{M}_\Pi$ of $\mathcal{M}$), and finite tree decompositions $\mathcal{C}_S$ of the corresponding constituent spaces K_S in $\mathcal{M}_\Pi$ (which will give the tree decomposition $\mathcal{C} = \{\mathcal{C}_S : S \in \Pi\}$ inducing a desired subdivision of $\mathcal{M}_\Pi$). The choices of subtrees S and decompositions $\mathcal{C}_S$ are made inductively, using an auxiliary ordering of the vertices of T into a sequence $(u_n)_{n \geq 1}$, as follows.

Step 1 Set $S_1 = \{u_1\}$ (ie the subtree consisting of a single vertex u_1), and set $\Pi_1 = \{S_1\}$. Furthermore, let $\mathcal{C}_{S_1}$ be the empty tree decomposition of the corresponding space $K_{S_1} = K_{u_1}$.

Step 2 Having constructed a finite family Π_n of finite subtrees, and a corresponding family of decompositions $\{\mathcal{C}_S : S \in \Pi_n\}$, we keep as a part of inductive assumption the following properties (which clearly hold true for $n = 1$):

(i0) The subtrees in Π_n are pairwise disjoint.

(i1) $u_n \in \bigcup\{V_S : S \in \Pi_n\}$.

(i2) The union $\bigcup\{V_S : S \in \Pi_n\}$ is the vertex set of a finite subtree of T, which we denote by T_n.

(i3) For each $S \in \Pi_n$, for each separator A of the decomposition $\mathcal{C}_S$, and for any edge $e \in N_S$, there is a half-space H for A such that $\Sigma_e \subset \dot{H}$.

For each subtree $S \in \Pi_n \setminus \Pi_{n-1}$ and for each domain $\Omega \subset K_S$ corresponding to $\mathcal{C}_S$, choose arbitrary pairwise distinct oriented edges $e_1, \dots, e_k, e'_1, \dots, e'_k$ from N_S, not belonging to T_n, such that for $1 \le j \le k$ the corresponding peripheral subsets Σ_{e_j} and $\Sigma_{e'_j}$ are contained in Ω. This is possible because, by the finiteness of $\mathcal{C}_S$, Ω has nonempty interior in K_S, and by the denseness of $\mathcal{M}$, it thus contains infinitely many peripheral subsets Σ_e with e from N_S and not belonging to T_n. For each $1 \le j \le k$ choose a vertex t_j in T_{e_j} and t'_j in $T_{e'_j}$ such that $i_{t_j} = i_{t'_j} = j$. This is possible since $\mathcal{M}$ is weakly saturated. Furthermore, for each j choose a finite subtree $S_j(\Omega)$ in T_{e_j} containing the vertices $\omega(e_j)$ and t_j, and a finite subtree $S'_j(\Omega)$ in $T_{e'_j}$ containing the vertices $\omega(e'_j)$ and t'_j. We also require that, setting

$$\Pi_{n+1} = \Pi_n \cup \bigcup_{\Omega} \{S_1(\Omega), \dots, S_k(\Omega), S'_1(\Omega), \dots, S'_k(\Omega)\}$$

(where Ω runs through all domains in all spaces K_S with $S \in \Pi_n \setminus \Pi_{n-1}$), we have $u_{n+1} \in \bigcup\{V_S : S \in \Pi_{n+1}\}$. This clearly holds true if $u_{n+1} \in \bigcup\{V_S : S \in \Pi_n\}$; otherwise, this can be assured as follows. Let s be the vertex in T_n which is closest to u_{n+1} in T, and let e be the first oriented edge on the path from s to u_{n+1}. Let $S \in \Pi_n$ be the subtree for which $s \in V_S$, and let $\Omega \subset K_S$ be the domain corresponding to $\mathcal{C}_S$ for which $\Sigma_e \subset \Omega$. We then choose e_1 as above so that additionally we have $u_{n+1} \in T_{e_1}$ (ie we set $e_1 = e$), and then choose $S_1(\Omega)$ so that it contains u_{n+1}. As a consequence of all our choices above, the family Π_{n+1} satisfies conditions (i0)–(i2). For example, (i0) holds because if S and S' are distinct elements of $\Pi_{n+1} \setminus \Pi_n$ then $S \subset T_e$ and $S' \subset T_{e'}$ where e and e' are distinct elements of N_{T_n}.

Now, for each $S = S_j(\Omega)$ or $S = S'_j(\Omega)$ as above, we choose an appropriate tree decomposition $\mathcal{C}_S$ of the space K_S. To describe it, note that K_S (together with its peripheral subspaces of the system $\mathcal{M}_\Pi$) is homeomorphic to the densely punctured manifold, denoted by M_S, which is a connected sum of the manifolds M_t with $t \in V_S$. We denote by Δ_e the disks in M_S corresponding to the peripheral subspaces Σ_e

of K_S. We also denote by Δ_j and K_j the spaces $\Delta_{\bar{e}_j}$ and K_{t_j} if $S = S_j(\Omega)$, and the spaces $\Delta_{\bar{e}'_j}$ and $K_{t'_j}$ if $S = S'_j(\Omega)$.

Choose any $\Delta_{e_0} \subset M_S$ such that $\Sigma_{e_0} \subset K_j$ and note that, by applying Toruńczyk's Lemma 1.E.2.1 to the manifolds $M_S \setminus \mathrm{int}(\Delta_{e_0})$ and $M_S \setminus \mathrm{int}(\Delta_j)$, we get a homeomorphism $h\colon M_S \to M_S$ (preserving the orientation if all manifolds in $\mathcal{N}$ are oriented) which maps Δ_{e_0} onto Δ_j and which preserves the family of all disks Δ_e in M_S. We denote by $h^\circ\colon K_S \to K_S$ the restricted homeomorphism of the densely punctured manifold. Now, we consider the finite tree decomposition $\mathcal{C}_S$ of K_S which is induced by pushing through h° the original tree decomposition of K_S into constituent spaces K_t with $t \in V_S$ (of the system $\mathcal{M}$ restricted to S). Obviously, $\mathcal{C}_S$ satisfies property (i3) above, and it also has the following property:

$(*)$ All domains $\Omega \subset K_S$ for $\mathcal{C}_S$ are densely punctured manifolds from $\mathcal{N}$, and the domain Ω which contains the peripheral subspace $\Sigma_j = \partial\Delta_j$ is homeomorphic to K_j, ie to the densely punctured manifold M_j.

We now set $\Pi = \bigcup_{i=1}^{\infty} \Pi_n$ and note that, by conditions (i0) and (i1), Π describes a decomposition of the tree T into finite subtrees. We thus consider the induced consolidation $\mathcal{M}_\Pi$. By condition (i3), and by finiteness of the decompositions $\mathcal{C}_S$, the family $\mathcal{C} = \{\mathcal{C}_S : S \in \Pi\}$ is a tree decomposition of the system $\mathcal{M}_\Pi$. Denoting by $\mathcal{M}'$ the tree system obtained from $\mathcal{M}_\Pi$ by the subdivision induced by $\mathcal{C}$ (ie setting $\mathcal{M}' = (\mathcal{M}_\Pi)_{\mathcal{C}_{\lim}}$), we get from the construction, and in particular from the property $(*)$ above, that $\mathcal{M}'$ is a 2–saturated dense tree system of manifolds from $\mathcal{N}$. By the comment in the first paragraph of the proof, Theorem 3.E.2 follows. □

As an application of Theorem 3.E.2, we now describe a class of inverse sequences of manifolds whose limits are the Jakobsche spaces $\mathcal{X}(M_1 \# \cdots \# M_k)$. This class of inverse sequences is much more flexible, and much more convenient to deal with, than the corresponding class considered by Jakobsche in [13] (compare Remark 3.A.2.4). For this reason, it can be more efficiently used to identify boundaries of some spaces and groups as appropriate trees of manifolds; see [25].

3.E.3 Definition Let $\mathcal{N}$ be a finite family of closed connected n–dimensional topological manifolds, either all oriented or at least one of which is nonorientable. Let

$$\mathcal{J} = (\{X_i : i \geq 1\}, \{\pi_i : i \geq 1\})$$

be an inverse sequence consisting of closed connected topological n–manifolds X_i and maps $\pi_i\colon X_{i+1} \to X_i$. Assume furthermore that if the manifolds in $\mathcal{N}$ are oriented then all the X_i are also oriented. We say that $\mathcal{J}$ is a *weak Jakobsche inverse sequence for* $\mathcal{N}$ if for each $i \ge 1$ one can choose a finite family $\mathcal{D}_i$ of collared n–disks in X_i, partitioned into subfamilies $\mathcal{D}_i^M$ for $M \in \mathcal{N}$, such that:

(1) For each $i \ge 1$ the disks in the family $\mathcal{D}_i$ are pairwise disjoint.

(2) For each $i \ge 1$ the map π_i maps the preimage $\pi_i^{-1}\big(X_i \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}_i\}\big)$ homeomorphically onto $X_i \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}_i\}$.

(3a) X_1 is homeomorphic to one of the manifolds from $\mathcal{N}$, and if the manifolds in $\mathcal{N}$ are oriented, we require that this homeomorphism respect orientations.

(3b) For each $i \ge 1$, for each $M \in \mathcal{N}$, and for any $D \in \mathcal{D}_i^M$, the preimage $\pi_i^{-1}(D)$ is homeomorphic to $M \setminus \mathrm{int}(\Delta)$, where Δ is some collared n–disk in M; furthermore, if the manifolds in $\mathcal{N}$ are oriented, we require that the above homeomorphism respect the orientations induced from X_{i+1} and from M.

(4) If $i < j$, $D \in \mathcal{D}_i$, and $D' \in \mathcal{D}_j$, then $\pi_{i,j}(D') \cap \partial D = \varnothing$, where $\pi_{i,j} := \pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{j-1}$.

(5) For each $i \ge 1$ the family $\{\pi_{i,j}(D) : j \ge i, D \in \mathcal{D}_j\}$ of subsets of X_i is *null*; that is, the diameters of these subsets converge to 0. Here $\pi_{i,i}$ denotes the identity map on X_i.

(6) For any $i \ge 1$ and each $M \in \mathcal{N}$, the set $\bigcup_{j=i}^{\infty} \pi_{i,j}\big(\bigcup \mathcal{D}_j^M\big)$ is dense in X_i.

Remarks (1) It follows from conditions (1), (2), (3a) and (3b) that each X_i is the connected sum of a family of manifolds each homeomorphic to one of the manifolds in $\mathcal{N}$; moreover, if the manifolds in $\mathcal{N}$ are oriented, the above-mentioned homeomorphisms and the connected sum respect the orientations.

(2) When the manifolds in $\mathcal{N}$ are oriented, conditions (1)–(5) in Definition 3.E.3 coincide with conditions (1)–(6) in [13, Section 2, page 82].

(3) Condition (6) in Definition 3.E.3 implies condition (7) in [13], but it is essentially weaker than the conjunction of conditions (7) and (8) of [13] (except when the family $\mathcal{N}$ consists of a single manifold M, in which case (6) is equivalent to the conjunction of (7) and (8), as was observed and exploited in [9] and [26]).

3.E.4 Corollary *Given $\mathcal{N} = \{M_1, \dots, M_k\}$ as in Definition 3.E.3, the limit $\varprojlim \mathcal{J}$ of any weak Jakobsche inverse sequence $\mathcal{J}$ for $\mathcal{N}$ is homeomorphic to the Jakobsche space $\mathcal{X}(M_1 \# \cdots \# M_k)$.*

Proof First, observe that by conditions (1)–(5) of Definition 3.E.3, there is a tree system $\mathcal{M}$ of manifolds from $\mathcal{N}$ such that $\mathcal{J}$ has the form of an inverse sequence associated to $\mathcal{M}$, as in Example 2.C.7, for an appropriate choice of a conical family of extended spaces and maps. More precisely, the constituent spaces of $\mathcal{M}$ coincide with the spaces Y of the following two kinds:

(1) For any $i \geq 1$, set

$$\mathcal{D}_i' = \{D \in \mathcal{D}_i : \text{there is no } j < i \text{ with } \pi_{i,j}(D) \subset D' \text{ for some } D' \in \mathcal{D}_j\},$$

and set

$$Y = X_1 \setminus \bigcup_{i=1}^{\infty} \bigcup_{D \in \mathcal{D}_i'} \pi_{1,i}(\mathrm{int}(D)).$$

(2) For any $m \geq 1$, any $\Delta \in \mathcal{D}_m$, and any $i \geq m+1$, set $\mathcal{D}_{\Delta,i}$ to be the family of all $D \in \mathcal{D}_i$ such that $\pi_{m,i}(D) \subset \Delta$ and there is no $m+1 \leq j < i$ with $\pi_{i,j}(D) \subset D'$ for some $D' \in \mathcal{D}_j$, and set

$$Y = \pi_m^{-1}(\Delta) \setminus \bigcup_{i=m+1}^{\infty} \bigcup_{D \in \mathcal{D}_{\Delta,i}} \pi_{m+1,i}(\mathrm{int}(D)).$$

We skip further explanations and justifications concerning this first observation, and we note that, by Theorem 2.B.4, we have $\varprojlim \mathcal{J} = \lim \mathcal{M}$.

Next, it follows fairly directly from condition (6) of Definition 3.E.3 that the tree system $\mathcal{M}$ of manifolds from $\mathcal{N}$, as above, is dense and weakly saturated. The assertion follows then directly from Theorem 3.E.2. $\square$

We finish by briefly presenting a concept that is more restrictive, but slightly less technical than that of weak Jakobsche inverse sequence, the concept of a *special Jakobsche inverse sequence*.

3.E.5 Definition Let $\mathcal{N}$ be a finite family of closed connected n–dimensional topological manifolds, either all oriented or at least one of which is nonorientable. Let

$$\mathcal{G} = (\{X_i : i \geq 1\}, \{\pi_i : i \geq 1\})$$

be an inverse sequence consisting of closed connected topological n–manifolds X_i and maps $\pi_i \colon X_{i+1} \to X_i$. Assume furthermore that if the manifolds in $\mathcal{N}$ are oriented then all the X_i are also oriented. We say that $\mathcal{G}$ is a *special Jakobsche inverse sequence for* $\mathcal{N}$ if for each $i \geq 1$ one can choose a finite set $\mathcal{Q}_i$ in X_i, partitioned into subsets $\mathcal{Q}_i^M$ for $M \in \mathcal{N}$, such that:

(1) For each $i \geq 1$ the map π_i maps the preimage $\pi_i^{-1}(X_i \setminus \mathcal{Q}_i)$ homeomorphically onto $X_i \setminus \mathcal{Q}_i$.

(2a) X_1 is homeomorphic to one of the manifolds from $\mathcal{N}$, and if the manifolds in $\mathcal{N}$ are oriented, we require that this homeomorphism respect orientations.

(2b) For each $i \geq 1$, for each $M \in \mathcal{N}$, and for any $q \in \mathcal{Q}_i^M$, the preimage $\pi_i^{-1}(q)$ is a submanifold of X_{i+1} homeomorphic to $M \setminus \operatorname{int}(\Delta)$, where Δ is some collared n–disk in M; furthermore, if the manifolds in $\mathcal{N}$ are oriented, we require that the above homeomorphism respect the orientations induced from X_{i+1} and from M.

(3) If $i < j$ and $q \in \mathcal{Q}_i$, then $\mathcal{Q}_j \cap \partial[\pi_{ij}^{-1}(q)] = \varnothing$.

(4) For any $i \geq 1$ and each $M \in \mathcal{N}$ the set $\bigcup_{j=i}^{\infty} \pi_{i,j}(\mathcal{Q}_j^M)$ is dense in X_i.

Remark Condition (3) in the above definition requires a comment. Note that, by condition (2b), if $q \in \mathcal{Q}_i$ and if $j = i+1$ then $\pi_{ij}^{-1}(q)$ is an n–submanifold with boundary in X_j. Moreover, if for some $j > i$ we have that $\pi_{ij}^{-1}(q)$ is an n–submanifold with boundary in X_j, and if $\mathcal{Q}_j \cap \partial[\pi_{ij}^{-1}(q)] = \varnothing$, then it follows from conditions (1) and (2b) that $\pi_{i,j+1}^{-1}(q)$ is an n–submanifold with boundary in X_{j+1}. Thus, by induction, all the preimages $\pi_{ij}^{-1}(q)$ occurring in the statement of condition (3) are n–submanifolds with boundary in the corresponding manifolds X_j, and hence it makes sense to speak of their boundaries $\partial[\pi_{ij}^{-1}(q)]$.

An argument similar to that in the proof of Corollary 3.E.4 shows that a special Jakobsche inverse sequence $\mathcal{G}$ (for $\mathcal{N}$) has the form of the standard inverse sequence associated to a weakly saturated tree system of manifolds from $\mathcal{N}$. In this argument the reference to Theorem 2.B.4 has to be replaced with the corresponding reference to Proposition 1.D.1. As a consequence, we get the following.

3.E.6 Corollary *Given $\mathcal{N} = \{M_1, \dots, M_k\}$ as in Definition 3.E.5, the limit $\varprojlim \mathcal{G}$ of any special Jakobsche inverse sequence $\mathcal{G}$ for $\mathcal{N}$ is homeomorphic to the Jakobsche space $\mathcal{X}(M_1 \# \cdots \# M_k)$.*

Remark It is not hard to see that any special Jakobsche inverse sequence for $\mathcal{N}$ is also a weak Jakobsche inverse sequence for $\mathcal{N}$, but we omit the details. (Obviously, this can be used as another justification of Corollary 3.E.6.) The converse is not true.

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