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Trees of manifolds as boundaries of spaces and groups

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We show that trees of manifolds, the topological spaces introduced by Jakobsche, appear as boundaries at infinity of various spaces and groups. In particular, they appear as Gromov boundaries of some hyperbolic groups, of arbitrary dimension, obtained by the procedure of strict hyperbolization. We also recognize these spaces as boundaries of arbitrary Coxeter groups with manifold nerves and as Gromov boundaries of the fundamental groups of singular spaces obtained from some finite-volume hyperbolic manifolds by cutting off their cusps and collapsing the resulting boundary tori to points.

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Introduction

Not many explicit topological spaces are known to be homeomorphic to Gromov boundaries of hyperbolic groups. The list consists of spheres and Sierpiński compacta of arbitrary dimension, Menger compacta in dimensions $d \in \{1, 2, 3\}$, the Pontriagin sphere and Pontriagin surfaces (which are 2–dimensional), and some trees of 3–manifolds (which are certain spaces of topological dimension 3). See Przytycki and Świątkowski [18] for more detailed comments concerning this list. One of the goals of this paper is to extend this list to trees of n–manifolds in arbitrary dimension n (for which the methods used in [18], in the case $n = 3$, are insufficient).

Trees of manifolds were formally introduced by Włodzimierz Jakobsche [14], though their idea goes back to Ancel and Siebenmann [1]. These are certain explicit homogeneous metric compacta, typically not absolute neighborhood retracts (ANRs), appearing in abundance in arbitrary finite topological dimension. We describe them in detail in Section 1, and here we only mention that each closed connected topological n–manifold M determines uniquely one such space, denoted by $\mathcal{X}(M)$, which has topological dimension n, and which we call the *tree of manifolds* M. Our first result is the following (compare Theorem 5.1, which is slightly more general).

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Theorem 1 (1) *For any natural number n, if a closed connected orientable PL n–manifold M bounds a compact orientable PL $(n+1)$–manifold, then the tree of manifolds $\mathcal{X}(M)$ is homeomorphic to the Gromov boundary of some hyperbolic group.*

(2) *For any closed connected nonorientable PL manifold N the tree of manifolds $\mathcal{X}(N)$ is homeomorphic to the Gromov boundary of some hyperbolic group.*

Appropriate hyperbolic groups as in the assertion of Theorem 1 will be constructed in Section 5, using the procedure of strict hyperbolization due to Charney and Davis [5]. Theorem 1 follows from a more general result, our main theorem, formulated in Section 2 (and proved in Sections 3 and 4), which concerns boundaries of certain CAT(κ) pseudomanifolds with $\kappa \leq 0$ called $(\mathcal{M}, \kappa)$–pseudomanifolds. Another application of this theorem, discussed in detail in Section 5, is the following correction and extension of the result of H Fischer [11] (who has studied only the case of right-angled Coxeter groups with orientable manifold nerves).

Theorem 2 *Let (W, S) be a Coxeter system* (not necessarily right-angled) *whose nerve is a PL triangulation of a closed connected manifold M. Then the boundary at infinity of (W, S)* (ie the visual boundary of the corresponding Davis–Moussong complex for (W, S)) *is homeomorphic to one of the following trees of manifolds*:

(1) *$\mathcal{X}(M \# \overline{M})$ if M is orientable, where M and $\overline{M}$ are two oppositely oriented copies of M.*

(2) *$\mathcal{X}(M)$ if M is nonorientable.*

Proofs of Theorems 1 and 2 (or even slightly more general results) are presented in Section 5. In Section 6, we present a variation (Theorem 6.2) of the main theorem, which yields the following result, which applies to a class of hyperbolic groups (having representatives in arbitrary dimension) constructed by L Mosher and M Sageev [17] (see also K Fujiwara and J Manning [13]).

Theorem 3 *Let M be a finite-volume complete noncompact hyperbolic $(n+1)$–manifold with toral cusps. Suppose that after removing some open horoball neighborhoods of all cusps we get a compact $(n+1)$–manifold M° whose euclidean toral boundary components contain no closed geodesics of length $\leq 2\pi$. Let Γ be a hyperbolic group which is the fundamental group of a pseudomanifold obtained from M° by collapsing all its boundary components to points. Then the Gromov boundary $\partial\Gamma$ is homeomorphic to the tree of tori $\mathcal{X}(T^n)$.*

The proofs of the results in this paper required some new ideas and tools. One of them is a new and more flexible characterization of trees of manifolds as limits of inverse sequences of manifolds, given in Definition 1.1 and established by the author in a separate paper [22]. Remarks after Definition 1.1 clarify the novelty of this characterization. Another new ingredient is a more careful or further-pushed (than in M Davis and T Januszkiewicz [8], Fischer [11] and Fujiwara and Manning [13]) analysis of geodesic projections between concentric spheres in $\mathrm{CAT}(\kappa)$ pseudomanifolds. This analysis allows us to approximate projections as above by more regular maps, which in turn allows us to understand the boundaries of the corresponding pseudomanifolds.

The above-mentioned analysis of geodesic projections builds upon some results of Davis and Januszkiewicz [8]. Since the proofs of these results given in [8] are incomplete, as we were informed by the referee of the present paper, we include an appendix. This appendix presents complete proofs of these results, in the form provided later by the authors of [8], but never published. The authors of [8] have approved such an arrangement.

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1 Trees of manifolds

Trees of manifolds were introduced by Jakobsche in [14]. These are some homogeneous metric compacta, typically not ANRs, appearing in abundance in any finite topological dimension. We briefly recall their description, in the original setting of Jakobsche modified and extended as in [22]. Although the description is valid for arbitrary topological manifolds, in this paper we make use only of the subclass related to PL manifolds.

1.1 Definition Let $\mathcal{M}$ be a finite family of closed connected n–dimensional topological manifolds, either all oriented or at least one of which is nonorientable. Let

$$\mathcal{J} = (\{X_i : i \geq 1\}, \{\pi_i : i \geq 1\})$$

be an inverse sequence consisting of closed connected topological n–manifolds X_i and maps $\pi_i\colon X_{i+1} \to X_i$. Assume furthermore that if all the manifolds in $\mathcal{M}$ are oriented then all the X_i are also oriented. We say that $\mathcal{J}$ is a *weak Jakobsche inverse sequence for* $\mathcal{M}$ if for each $i \geq 1$ one can choose a finite family $\mathcal{D}_i$ of collared n–disks in X_i, partitioned into subfamilies $\mathcal{D}_i^M$ for $M \in \mathcal{M}$, such that:

(1) For each $i \geq 1$ the disks in the family $\mathcal{D}_i$ are pairwise disjoint.

(2) For each $i \geq 1$ the map π_i maps the preimage $\pi_i^{-1}\big(X_i \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}_i\}\big)$ homeomorphically onto $X_i \setminus \bigcup\{\mathrm{int}(D) : D \in \mathcal{D}_i\}$.

(3a) X_1 is homeomorphic to one of the manifolds from $\mathcal{M}$, and if the manifolds in $\mathcal{M}$ are oriented, we require that this homeomorphism respect orientations.

(3b) For each $i \geq 1$, for each $M \in \mathcal{M}$, and for any $D \in \mathcal{D}_i^M$, the preimage $\pi_i^{-1}(D)$ is homeomorphic to $M \setminus \mathrm{int}(\Delta)$, where Δ is some collared n–disk in M; furthermore, if the manifolds in $\mathcal{M}$ are oriented, we require that the above homeomorphism respect the orientations induced from X_{i+1} and from M.

(4) If $i < j$, $D \in \mathcal{D}_i$, and $D' \in \mathcal{D}_j$, then $\pi_{i,j}(D') \cap \partial D = \varnothing$, where $\pi_{i,j} := \pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{j-1}$.

(5) For each $i \geq 1$ the family $\{\pi_{i,j}(D) : j \geq i, D \in \mathcal{D}_j\}$ of subsets of X_i is *null*; that is, the diameters of these subsets converge to 0. Here $\pi_{i,i}$ denotes the identity map on X_i.

(6) For any $i \geq 1$ and each $M \in \mathcal{M}$, the set $\bigcup_{j=i}^{\infty} \pi_{i,j}\big(\bigcup \mathcal{D}_j^M\big)$ is dense in X_i.

Remarks (1) It follows from conditions (1), (2), (3a) and (3b) that each X_i is the connected sum of a family of manifolds each homeomorphic to one of the manifolds in $\mathcal{M}$; moreover, if the manifolds in $\mathcal{M}$ are oriented, the above-mentioned homeomorphisms (and the involved operation of connected sum) respect the orientations.

(2) When the manifolds in $\mathcal{M}$ are oriented, conditions (1)–(5) in Definition 1.1 basically coincide with conditions (1)–(6) in [14, Section 2, page 82].

(3) Condition (6) in Definition 1.1 implies condition (7) in [14], but it is essentially weaker than the conjunction of conditions (7) and (8) of [14] (except when the family $\mathcal{M}$ consists of a single manifold M, in which case condition (6) of Definition 1.1 is equivalent to the conjunction of conditions (7) and (8) of [14], as was observed and exploited in [11] and [23]).

(4) Definition 1.1 describes exactly the class of inverse sequences naturally associated to weakly saturated tree systems of manifolds from $\mathcal{M}$, as described in [22, Section 3.E]. It represents significant, and probably close to optimal, relaxation of the initial set of conditions provided by Jakobsche in [14], for which the inverse limit of the corresponding sequence depends uniquely on $\mathcal{M}$ (see Theorem 1.2 below).

The following result is a reformulation of [22, Corollary 3.E.4].

1.2 Theorem *Given $\mathcal{M}$ as in Definition 1.1, any two weak Jakobsche inverse sequences for $\mathcal{M}$ have homeomorphic inverse limits.*

1.3 Definition Given $\mathcal{M}$ as in Definition 1.1, denote by $\mathcal{X}(\mathcal{M})$, and call the *tree of manifolds from $\mathcal{M}$*, the space homeomorphic to the inverse limit of some (and hence any) weak Jakobsche inverse sequence for $\mathcal{M}$.

Remarks (1) If $\mathcal{M} = \{M\}$, we denote the corresponding space $\mathcal{X}(\mathcal{M})$ simply by $\mathcal{X}(M)$, and call it the *tree of manifolds M*.

(2) If $\mathcal{M} = \{M_1, \dots, M_k\}$, it is known that the space $\mathcal{X}(\mathcal{M})$ is homeomorphic to the tree of manifolds $M_1 \# \cdots \# M_k$, ie to the space $\mathcal{X}(M_1 \# \cdots \# M_k)$ (see eg [22, Corollary 3.E.4]).

As was shown in [14] and [21], trees of manifolds $\mathcal{X}(\mathcal{M})$ are connected homogeneous metric compacta of topological dimension equal to the dimension of the manifolds in $\mathcal{M}$. As was observed in [18, Corollary 3.3], trees of manifolds in dimensions ≥ 2 have no local cut points, and in fact they are *Cantor manifolds*, ie no subsets of topological codimension ≥ 2 separate them.

Trees of manifolds $\mathcal{X}(\mathcal{M})$ of the same topological dimension n can be sometimes distinguished by means of their homotopical or homological invariants, or the shape-theoretic invariants. For example, it is known that if $n \geq 3$ then the shape fundamental group $\check{\pi}_1(\mathcal{X}(\mathcal{M}))$ is isomorphic to the inverse limit of the increasing free products $G_1 * G_2 * \cdots * G_k$, where the additional factors in larger products are collapsed to the identity while the common factors are mapped to each other through identities and where the groups G_i are copies of the fundamental groups $\pi_1(M)$ for $M \in \mathcal{M}$, each group appearing infinitely often (see [12]). Obviously, this shape fundamental group is sometimes sufficient to distinguish some trees of manifolds. The general question of classifying trees of manifolds up to homeomorphism remains open.

2 $(\mathcal{M},\kappa)$–pseudomanifolds and the main theorem

Let $\mathcal{M}$ be a finite collection of (pairwise distinct) closed connected PL manifolds of the same dimension n, distinct from the standard PL n–sphere, and let $\kappa \in \{0,-1\}$. An $(\mathcal{M},\kappa)$*–pseudomanifold* is a metric polyhedral complex X of piecewise constant curvature κ, with finite shapes, such that:

(1) X is $\mathrm{CAT}(\kappa)$.

(2) For each point $x \in X$ the link $\mathrm{Lk}(x,X)$ (viewed as a combinatorial polyhedral complex) is either a PL n–sphere or is PL–homeomorphic to some $M \in \mathcal{M}$; furthermore, if all $M \in \mathcal{M}$ are oriented, we assume that X is also oriented and that each homeomorphism $\mathrm{Lk}(x,X) \to M$ as above respects the orientations.

(3) For each $M \in \mathcal{M}$ the set $\Lambda_M := \{x \in X : \mathrm{Lk}(x,X) \cong M\}$ is a net in X; that is, for some $R>0$ each ball of radius R in X intersects Λ_M (if all $M \in \mathcal{M}$ are oriented, the symbol $\cong$ above denotes the relation of being PL–homeomorphic as oriented manifolds).

Note that, since X above has finite shapes, it is a geodesic metric space, so that it makes sense to speak of the $\mathrm{CAT}(\kappa)$ property for it. Moreover, by condition (2), X is automatically an $(n+1)$–dimensional pseudomanifold. The assumption that the link of every point of X is a manifold implies that only the vertices of X can have links that are not spheres. Hence, if $x \in X$ and $\mathrm{Lk}(x,X) \cong M \in \mathcal{M}$, than x must be a vertex of X. The finite shapes property implies that the vertices of X form a discrete set. Therefore, each set Λ_M as in condition (3) is discrete. Setting $\Lambda := \bigcup_{M\in\mathcal{M}} \Lambda_M$, we get that Λ, which will be called the *singular set* of X, is also discrete.

We are now ready to state the main theorem, which is the main result of the paper. Its proof occupies Sections 3 and 4. In Section 5 we show a few classes of examples to which this theorem applies; in particular, we obtain the proofs of Theorems 1 and 2 of the introduction.

Main Theorem *The visual boundary ∂X of any $(\mathcal{M},\kappa)$–pseudomanifold X is homeomorphic to the tree of manifolds $\mathcal{X}(\mathcal{M})$.*

3 Proof of the main theorem

Fix any point $x_0 \in \Lambda$. Denote the spheres and the closed and open balls of radius R in X centered at x_0 by $S_R = \{x \in X : d(x,x_0) = R\}$, $B_R = \{x \in X : d(x,x_0) \le R\}$ and $B_R^\bullet = \{x \in X : d(x,x_0) < R\}$ respectively. By discreteness of Λ, the set of numbers $\{d(x_0, p) : p \in \Lambda\}$ is also discrete. Order this set into the increasing sequence R_i for $i \geq 0$ with $R_0 = 0$. For each $i \geq 0$, let $p_{i,1}, \ldots, p_{i,k_i}$ be the points of the intersection $S_{R_i} \cap \Lambda$. (In particular, we have $k_0 = 1$ and $p_{0,1} = x_0$.)

For each $i \geq 1$, let $G_i \colon X \setminus B^\bullet_{R_i} \to S_{R_i}$ be the geodesic projection towards x_0, ie the map which to any point $x \in X \setminus B^\bullet_{R_i}$ associates the unique point x' in the intersection of the sphere S_{R_i} with the geodesic segment $[x, x_0]$. Denote also by $g_i \colon S_{R_{i+1}} \to S_{R_i}$ the restriction of G_i to $S_{R_{i+1}}$. Consider the inverse sequence $\mathcal{S} = (\{S_{R_i}\}, \{g_i\})$ and recall that

$$\partial X = \varprojlim \mathcal{S}$$

(see [3, II.8.5] or [8, the comments after Definition (2b.1)]). Before getting to the core of the proof, we need a few preparatory results.

The next result was essentially proved in [8, Section 3]. Its special case, in which X is an $(\mathcal{M}, 0)$–pseudomanifold and $\mathcal{M}$ consists of homology spheres, appears in [8, Theorem (3d.1)(i)]. The proof in [8] relies on the argument given in the paragraph starting at the bottom of page 371 and ending at page 372 of that paper; this argument is incomplete (as we were informed by the referee of the present paper), and thus we provide its complete version in the appendix. In particular, the lemma below is a special case of Lemma B of the appendix.

3.1 Lemma *Let X be an $(n+1)$–dimensional $(\mathcal{M}, \kappa)$–pseudomanifold. Then each sphere $S_R(x, X)$ in X (with positive radius R) is a closed n–manifold.*

The next result is a special case of Lemma C of the appendix. The notion of a cell-like map appearing in the statement of this result is recalled in the appendix (right after the statement of Lemma C).

3.2 Lemma *Let X be an $(\mathcal{M}, \kappa)$–pseudomanifold. Given any $R' > R > 0$ and given any $x \in X$, let $g \colon S_{R'}(x, X) \to S_R(x, X)$ be the geodesic projection towards x. If the set $B^\bullet_{R'}(x, X) \setminus B^\bullet_R(x, X)$ contains no singular point from Λ then g is a cell-like map. In particular, in the notation established at the beginning of this section, for any $0 < \epsilon < R_{i+1} - R_i$ the geodesic projection $\phi_i \colon S_{R_{i+1}} \to S_{R_i+\epsilon}$ towards x_0 is cell-like.*

Recall that, by a result of M Brown [4] (*Brown's lemma*), if we replace the maps g_i in the inverse sequence $\mathcal{S}$, successively, by sufficiently close maps $g_i' \colon S_{R_{i+1}} \to S_{R_i}$, then the limit of the resulting inverse sequence $\mathcal{S}' = (\{S_{R_i}\}, \{g_i'\})$ remains unchanged (up to homeomorphism), ie $\varprojlim \mathcal{S}' \cong \varprojlim \mathcal{S}$. The next proposition describes particularly nice approximations g_i', as above, of the maps g_i.

3.3 Proposition *Let* $\dim(X) = n+1$. *For all* $i \geq 1$ *there exist maps* $g_i'\colon S_{R_{i+1}} \to S_{R_i}$ *satisfying the following*:

(1) *Each* g_i' *is close enough to* g_i *that, according to Brown's lemma, we have*

$$\varprojlim \mathcal{S}' \cong \varprojlim \mathcal{S} \cong \partial X.$$

(2) *For each* $i \geq 1$, *setting* $S_{R_i}^{\circ} = S_{R_i} \setminus \{p_{i,1}, \dots, p_{i,k_i}\}$, *the restriction*

$$g_i'|_{(g_i')^{-1}(S_{R_i}^{\circ})}\colon (g_i')^{-1}(S_{R_i}^{\circ}) \to S_{R_i}^{\circ}$$

is a homeomorphism.

(3) *For each* $i \geq 1$ *and each* $1 \leq j \leq k_i$, *if* $D_{i,j}$ *is a sufficiently small collared* n*–disk in* S_{R_i} *such that* $p_{i,j} \in \operatorname{int}(D_{i,j})$, *then*

$$(g_i')^{-1}(D_{i,j}) \cong M \setminus \operatorname{int}(\Delta),$$

where $M = \operatorname{Lk}(p_{i,j}, X) \in \mathcal{M}$ *and where* Δ *is any collared* n*–disk in* M.

(4) *For each* $M \in \mathcal{M}$ *and for all* $i \geq 1$, *the images in* S_{R_i} *of the points from the set* $\Lambda_M \cap (X \setminus B_{R_i})$, *through appropriate compositions of the maps* g_k' *with* $k \geq i$, *form a dense subset of* S_{R_i}.

We postpone the proof of Proposition 3.3 until the next section, and in the remaining part of this section we complete the proof of the main theorem, using the proposition. We do this by showing that an inverse sequence $\mathcal{S}' = (\{S_{R_i}\}, \{g_i'\})$ resulting from Proposition 3.3 is a weak Jakobsche inverse sequence for $\mathcal{M}$. More precisely, we describe families $\mathcal{D}_i^M$ of disks as required in Definition 1.1, inductively with respect to i.

We start with checking condition (3a) of Definition 1.1, ie showing that S_{R_1} is homeomorphic to one of the manifolds from $\mathcal{M}$. Here, and later in Section 4, we will need the concept of a *cone neighborhood* of a point in a piecewise constant curvature polyhedral complex. Let Y be a metric polyhedral complex of constant curvature k equal to 1, -1 or 0, with finite shapes. Then for any point $y \in Y$, and any sufficiently small $\epsilon > 0$, the ball $B_\epsilon(y, Y)$ isometrically coincides with the ball of the same radius ϵ in the k–cone $C_k(\operatorname{Lk}(y, Y))$ centered at the cone vertex (see [3, Definition I.5.6 on page 59] or [7, page 505] for the definition of the k–cone, and see [3, Theorem I.7.16 on page 103] for justification of the above claim). A *cone neighborhood* of y in Y is any ball $B_\epsilon(y, Y)$ as above. An obvious (and useful) property is that geodesics starting at any point of Y do not bifurcate inside a cone neighborhood of this point. It follows that the natural map from the boundary sphere $S_\epsilon(y, Y)$ of any cone neighborhood to the link $\mathrm{Lk}(y, Y)$ is a homeomorphism, which we will view as the natural identification of the sphere with the link.

Coming back to checking condition (3a), choose ϵ small enough that B_ϵ is a cone neighborhood of x_0 in X. Then the sphere S_ϵ is homeomorphic to the link $\mathrm{Lk}(x_0, X)$, ie to some manifold $M_0 \in \mathcal{M}$. By Lemma 3.2, the geodesic projection $\phi_0\colon S_{R_1} \to S_\epsilon$ is a cell-like map. By the approximation theorem concerning cell-like maps (recalled in the appendix), ϕ_0 can be approximated by homeomorphisms. It follows that S_{R_1} is homeomorphic to S_ϵ, and hence also to M_0, as required.

We now turn to describing families $\mathcal{D}_i^M$ of disks as required in Definition 1.1. Fix an auxiliary sequence ϵ_i of positive real numbers, converging to 0. To start the inductive description, choose a family $\{D_{1,j} : 1 \le j \le k_1\}$ of pairwise disjoint collared disks in S_{R_1} such that for each j the following conditions hold:

(d1) $D_{1,j}$ contains the point $p_{1,j}$ in its interior.

(d2) The diameter of $D_{1,j}$ is less than ϵ_1.

(d3) The preimage $(g_1')^{-1}(D_{1,j})$ is homeomorphic to $M \setminus \mathrm{int}(\Delta)$, where $M \in \mathcal{M}$ is homeomorphic to the link $\mathrm{Lk}(p_{1,j}, X)$, and Δ is a collared n–disk in M.

(d4) Denoting by $\Lambda(1)$ the set of images in S_{R_1}, through appropriate compositions of the maps g_l' with $l \ge 1$, of all points in $\Lambda \cap (X \setminus B_{R_1})$, we have that $\Lambda(1) \cap \partial D_{1,j} = \varnothing$.

A choice as above is possible because of property (3) in Proposition 3.3 (which allows us to obtain (d3)) and because the set $\Lambda(1)$ is countable (which allows us to get (d4)). Set $\mathcal{D}_1 = \{D_{1,j} : 1 \le j \le k_1\}$, and split this set into subsets $\mathcal{D}_1^M$ according to the rule that $D_{1,j} \in \mathcal{D}_1^M$ if and only if $\mathrm{Lk}(p_{1,j}, X) \cong M$.

Now, suppose that $m \ge 2$ and that for all $1 \le i < m$ and all $M \in \mathcal{M}$, we have already described the families $\mathcal{D}_i^M$ satisfying conditions (1), (2), (3b) and (4) of Definition 1.1. Suppose also that any disk in a so-far described family $\mathcal{D}_i^M$, as well as its image in any S_{R_k} with $k < i$ through the appropriate composition of the maps g_l', has diameter less than ϵ_i. (We demand this to ensure that condition (5) of Definition 1.1 hold, after describing all families of disks.) Clearly, if $m = 2$ then these assumptions hold, which can be easily deduced from the first step of the construction given in the previous paragraph.

Choose a family $\{D_{m,j} : 1 \le j \le k_m\}$ of pairwise disjoint collared disks in S_{R_m} such that for each j the following conditions hold:

(d1′) $D_{m,j}$ contains the point $p_{m,j}$ in its interior.

(d2′) The diameter of $D_{m,j}$, as well as of its image in any S_{R_k} with $k < m$ through the appropriate composition of the maps g'_l, is less than ϵ_m.

(d3′) The preimage $(g'_m)^{-1}(D_{m,j})$ is homeomorphic to $M \setminus \mathrm{int}(\Delta)$, where $M \in \mathcal{M}$ is homeomorphic to the link $\mathrm{Lk}(p_{m,j}, X)$, and Δ is a collared n–disk in M.

(d4′) Denoting by $\Lambda(m)$ the set of images in S_{R_m}, through appropriate compositions of the maps g'_l with $l \geq m$, of all points in $\Lambda \cap (X \setminus B_{R_m})$, we have that $\Lambda(m) \cap \partial D_{m,j} = \varnothing$.

(d5′) For any $k < m$ and any $D \in \mathcal{D}_k$, the image of $D_{m,j}$ in S_{R_k}, through the appropriate composition of the maps g'_l, is disjoint from ∂D.

The fact that choosing $D_{m,j}$ sufficiently small we can fulfill (d5′) follows from having condition (d4′) satisfied, by inductive hypothesis, for m replaced with any $i < m$. (Clearly, condition (d5′) corresponds to condition (4) in Definition 1.1.)

The families of disks obtained by the above inductive description clearly satisfy conditions (1)–(5) of Definition 1.1. It remains to check condition (6) of this definition. However, this condition follows fairly directly from condition (4) in Proposition 3.3 (which holds for the maps g'_i in the inverse sequence $\mathcal{S}'$), and from the fact that each point $p_{i,j} \in \Lambda_M$ is included in the disk $D_{i,j} \in \mathcal{D}_i^M$.

This completes the proof of the main theorem.

4 Proof of Proposition 3.3

The proof of Proposition 3.3 is split into a series of auxiliary observations and partial results, and it is completed in the last part of the section. We use the notation established at the beginning of Section 3, where X is an $(\mathcal{M}, \kappa)$–pseudomanifold, Λ is its singular set, S_{R_i} are the spheres in X centered at a fixed point $x_0 \in \Lambda$, of appropriately chosen radii R_i, and $G_i \colon X \setminus B^\bullet_{R_i} \to S_{R_i}$ are the geodesic projections towards x_0.

The following observation is an easy consequence of the assumptions that X is $\mathrm{CAT}(\kappa)$ for some $\kappa \in \{0, -1\}$ and that for each $M \in \mathcal{M}$ the singular set Λ_M is a net in X (we omit the proof).

4.1 Claim *For each $M \in \mathcal{M}$ and for each $i \geq 1$ the image set $G_i(\Lambda_M \cap (X \setminus B_{R_i}))$ is dense in S_{R_i}.*

The next result follows essentially by the same arguments as those used in the proof of Lemma C in the appendix. More precisely, the same arguments show that point preimages of the restricted map ψ_i are cell-like sets. Moreover, the fact that ψ_i is a surjective map between compact spaces easily implies that its restriction of the form as in the statement below is surjective and proper. We omit further details of the justification, but we make an additional important observation that, in view of Lemma 3.1, the sets $S_{R_i} \setminus \Lambda_{R_i}$ and $\psi_i^{-1}(S_{R_i} \setminus \Lambda_{R_i})$ appearing in the statement below are open subsets in manifolds, and hence are themselves manifolds.

4.2 Lemma *Under the notation established at the beginning of Section 3, set $\Lambda_{R_i} := \Lambda \cap S_{R_i}$. Given some $\epsilon \in (0, R_{i+1} - R_i)$, denote by $\psi_i \colon S_{R_i+\epsilon} \to S_{R_i}$ the geodesic projection towards x_0, ie the appropriate restriction of the map G_i. Then the restriction of ψ_i to the map from $\psi_i^{-1}(S_{R_i} \setminus \Lambda_{R_i})$ to $S_{R_i} \setminus \Lambda_{R_i}$ is a cell-like map between manifolds.*

The next three lemmas deal with approximations of geodesic projections by maps with better properties. They prepare the ground for the construction of approximations g_i' as required in Proposition 3.3. More precisely, Lemma 4.3 is related to condition (2) of the proposition, while Lemmas 4.4 and 4.5 concern conditions (3) and (4), respectively.

4.3 Lemma *Let $\epsilon \in (0, R_{i+1} - R_i)$, and let $\psi_i \colon S_{R_i+\epsilon} \to S_{R_i}$ be the geodesic projection towards x_0 in X, ie the appropriate restriction of the map G_i. Then the ψ_i can be approximated by maps ψ_i' such that*

(1) *for each $1 \le j \le k_i$ we have $(\psi_i')^{-1}(p_{i,j}) = \psi_i^{-1}(p_{i,j})$;*

(2) *ψ_i' restricted to the preimage $(\psi_i')^{-1}(S_{R_i} \setminus \Lambda_{R_i})$ maps this set homeomorphically onto $S_{R_i} \setminus \Lambda_{R_i}$.*

Proof Let $\overline{\delta} \colon S_{R_i+\epsilon} \to [0,\infty)$ be a continuous function such that $\overline{\delta}^{-1}(0) = \psi_i^{-1}(\Lambda_{R_i})$, and let $\delta \colon S_{R_i+\epsilon} \setminus \psi_i^{-1}(\Lambda_{R_i}) \to R_+$ be the restriction of $\overline{\delta}$. Since a cell-like map between manifolds is a near-homeomorphism (see the approximation theorem in the appendix), it is a consequence of Lemma 4.2 that there is a homeomorphism $\psi' \colon S_{R_i+\epsilon} \setminus \psi_i^{-1}(\Lambda_{R_i}) \to S_{R_i} \setminus \Lambda_{R_i}$ such that $d(\psi'(y), \psi_i(y)) \le \delta(y)$ for all y in the domain of ψ'. By this estimate, and since $\delta(y)$ converges to 0 as y converges to a point in $\psi_i^{-1}(\Lambda_{R_i})$, the map ψ' can be extended to a continuous map $\psi_i' \colon S_{R_i+\epsilon} \to S_{R_i}$, by setting $\psi_i'(z) = \psi_i(z)$ for all $z \in \psi_i^{-1}(\Lambda_{R_i})$. This extension necessarily satisfies properties (1) and (2), and clearly we can get in this way a map as close to ψ_i as we wish. This finishes the proof. □

4.4 Lemma *Under the notation of Lemma 4.3, if ϵ is sufficiently small then any map ψ_i' fulfilling assertions* (1) *and* (2) *of that lemma satisfies also the following*:

($\star$) *For each $1 \le j \le k_i$ and any sufficiently small collared n–disk D in S_{R_i} containing $p_{i,j}$ in its interior, the preimage $(\psi_i')^{-1}(D)$ is homeomorphic to $M \setminus \mathrm{int}(\Delta)$, where $M = \mathrm{Lk}(p_{i,j}, X) \in \mathcal{M}$ and where Δ is a collared n–disk in M.*

Proof Let ϵ be small enough that for each $1 \le j \le k_i$ the preimage $C_j := \psi_i^{-1}(p_{i,j})$, viewed as a subset of X, is contained in the interior of some cone neighborhood $B_{r_j}(p_{i,j}, X)$. Consider the canonical identification of each sphere $L_j := S_{r_j}(p_{i,j}, X)$ with the link $L_j^* := \mathrm{Lk}(p_{i,j}, X)$. Set $\{w_j\} = [p_{i,j}, x_0] \cap L_j$, and denote by w_j^* the point in L_j^* corresponding to w_j under the identification of L_j and L_j^*. Let $A_j^* := \{y \in L_j^* : d_j(y, w_j^*) \ge \pi\}$, where d_j is the piecewise spherical metric in L_j^*, be the *infinitesimal shadow* of the point $p_{i,j}$ with respect to w_j^*, and let A_j be the corresponding subset in L_j. Note that, by the first assertion of Lemma A(2) of the appendix, $L_j^* \setminus A_j^*$ is an open n–disk, and hence the same holds true for $L_j \setminus A_j$. Set $\Omega_j = B_{R_i+\epsilon} \cap B_{r_j}(p_{i,j}, X)$ and note that it is a convex set in X containing $p_{i,j}$ in its interior. It follows that the natural conical projection $c\colon L_j \to \partial\Omega_j$ (towards $p_{i,j}$) is a homeomorphism. It is also not hard to realize that $C_j = c(A_j)$.

Now, choose $\rho \in (0, \pi)$ close enough to π that, setting $N_j^* := L_j^* \setminus B_\rho^\bullet(w_j^*, L_j^*)$ and denoting by N_j the corresponding subset in L_j, this subset N_j is disjoint with the ball $B_{R_i+\epsilon}$, and consequently its image $N_j' := c(N_j)$ falls in the part of $\partial\Omega_j$ which is contained in $S_{R_i+\epsilon}$. By Lemma A(2) of the appendix, there is a collared n–disk B^* in L_j^* such that $B_\rho(w_j^*, L_j^*) \subset B^* \subset B_\pi^\bullet(w_j^*, L_j^*)$. Obviously, the corresponding n–disk B in L_j is also collared, so we get that $L_j \setminus \mathrm{int}(B)$ is homeomorphic to $M \setminus \mathrm{int}(\Delta)$. We also have $L_j \setminus B \subset N_j$, so that the image $P_j := c(L_j \setminus B)$ is a subset of the part of $\partial\Omega_j$ which is contained in $S_{R_i+\epsilon}$. Moreover, since $L_j \setminus A_j$ is an open n–disk, it follows from the annulus theorem that $[L_j \setminus \mathrm{int}(B)] \setminus A_j$ is homeomorphic to the product $S^{n-1} \times [0,1)$. Since the conical projection c is a homeomorphism, we also get that $P_j \cong M \setminus \mathrm{int}(\Delta)$ and $P_j \setminus C_j \cong S^{n-1} \times [0,1)$. Set $D_j' := \psi_i'(P_j)$. By Lemma 4.3, D_j' is homeomorphic to the quotient space P_j / C_j, and by what was said above, this quotient is homeomorphic to the closed n–disk which contains $p_{i,j}$ in its interior. Clearly, we have $(\psi_i')^{-1}(D_j') = P_j \cong M \setminus \mathrm{int}(\Delta)$. By the annulus theorem, the same is true for any collared n–disk D contained in the interior of D_j' and containing $p_{i,j}$ in its interior, hence the lemma. □

4.5 Lemma *Let $g_i\colon S_{R_{i+1}} \to S_{R_i}$ be the geodesic projection, and let $\Lambda_{R_i} = \{p_{i,1}, \dots, p_{i,k_i}\}$ be the set of singular points in the sphere S_{R_i}, as in the proof of the main theorem. Let Z be an arbitrary finite subset of S_{R_i} disjoint with Λ_{R_i}, and for each $z \in Z$ let $y_z \in S_{R_{i+1}}$ be an arbitrary point in the preimage $(g_i)^{-1}(z)$. Then g_i can be approximated arbitrarily closely by a map $g_i^Z\colon S_{R_{i+1}} \to S_{R_i}$ satisfying conditions (2) and (3) of Proposition 3.3 (with g_i^Z substituted for g_i') and such that*

(1) $$g_i^Z(y_z) = z \quad \text{for all } z \in Z.$$

Proof Choose positive ϵ as small as required in Lemma 4.4, and consider the corresponding geodesic projections $\psi_i\colon S_{R_i+\epsilon} \to S_{R_i}$ and $\phi_i\colon S_{R_{i+1}} \to S_{R_i+\epsilon}$ towards x_0. We then clearly have $g_i = \psi_i\phi_i$. Consider a very close approximation ψ_i' of ψ_i satisfying conditions (1) and (2) of Lemma 4.3 and condition ($\star$) of Lemma 4.4. Moreover, since by Lemma 3.2 the map ϕ_i is cell-like, consider its very close approximation by a homeomorphism ϕ_i', as guaranteed by the approximation theorem (recalled in the appendix). The composition map $\psi_i'\phi_i'$ satisfies conditions (2) and (3) of Proposition 3.3, but not necessarily (1) asserted in the lemma.

Note that, by the choices of ψ_i' and ϕ_i', for each $z \in Z$ the point $\psi_i'\phi_i'(y_z)$ is very close to $\psi_i\phi_i(y_z) = g_i(y_z) = z$. Since S_{R_i} is a manifold, we can choose a correcting homeomorphism $\omega\colon S_{R_i} \to S_{R_i}$, very close to the identity, fixing all points of Λ_{R_i}, and such that $\omega(\psi_i'\phi_i'(y_z)) = z$ for all $z \in Z$. The lemma follows by taking $g_i^Z := \omega\psi_i'\phi_i'$, since clearly such a composition can be chosen to approximate the map g_i arbitrarily closely. □

We now pass to the final phase of the proof of Proposition 3.3. Observe that Lemma 4.5 guarantees existence of approximations of the maps g_i satisfying conditions (2) and (3) of the proposition. It remains to justify that appropriately chosen such approximations satisfy also the last condition (4).

Chose any sequence i_m of natural numbers such that $i_m \le m$ and each $i \ge 1$ appears in this sequence infinitely often. Choose also a sequence ϵ_m of positive reals converging to 0. Proceed inductively with respect to m, as follows. For $m = 1$ and for each $M \in \mathcal{M}$ choose a finite subset $Y_{M,1} \subset \Lambda_M \cap [X \setminus B_{R_1}]$ such that its image $Z_{M,1} = G_1(Y_{M,1})$ is an ϵ_1–net in S_{R_1} (ie any point of S_{R_1} lies at distance at most ϵ_1 from $Z_{M,1}$). Set $Y_1 = \bigcup_{M\in\mathcal{M}} Y_{M,1}$ and $Z_1 = \bigcup_{M\in\mathcal{M}} Z_{M,1}$, and assume, without loss of generality, that Z_1 is disjoint with Λ_{R_1}. (The possibility of such a choice of the above sets follows from Claim 4.1.) Set ν_1 to be the largest i such that the intersection $Y_1 \cap S_{R_i}$

is nonempty. For $1 \le i \le \nu_1 - 1$, set $Y_{1,i} := Y_1 \cap [X \setminus B_{R_i}]$ and $Z_{1,i} := G_i(Y_{1,i})$, and for each $z \in Z_{1,i}$, set $y_z = G_{i+1}(y)$, where $y \in Y_{1,i}$ is the point for which $z = G_i(y)$. Again for $1 \le i \le \nu_1 - 1$ choose successively approximations $g_i' = g_i^{Z_{1,i}}$ as in Lemma 4.5, close enough to g_i that the requirements of Brown's lemma are fulfilled. Note that, apart from satisfying conditions (2) and (3) of Proposition 3.3, the maps g_i' with $1 \le i \le \nu_1 - 1$ have the following property: for each $M \in \mathcal{M}$ the images in S_{R_1} of the points from $\bigcup_{i=1}^{\nu_1} (\Lambda_M \cap S_{R_i})$, through the appropriate compositions of the maps g_i', form an ϵ_1–net in S_{R_1}.

Now, assume that $m \ge 2$, that $\nu_{m-1} \ge m$, and that we have already defined the approximations g_i' for all $i \le \nu_{m-1} - 1$. For any $M \in \mathcal{M}$ choose a finite subset $Y_{M,m} \subset \Lambda_M \cap [X \setminus B_{R_{\nu_{m-1}}}]$ such that its image $Z_{M,m} = G_{\nu_{m-1}}(Y_{M,m})$, further projected by the composition $g_{i_m}' \circ g_{i_m+1}' \circ \cdots \circ g_{\nu_{m-1}-1}'$, is an ϵ_m–net in $S_{R_{i_m}}$. Set $Y_m = \bigcup_{M \in \mathcal{M}} Y_{M,m}$ and $Z_m = \bigcup_{M \in \mathcal{M}} Z_{M,m}$, and assume, without loss of generality, that the image of Z_m in $S_{R_{i_m}}$, through the above-mentioned composition of the maps g_i', omits the points $p_{i_m,1}, \dots, p_{i_m,k_{i_m}}$. (The possibility of such a choice of the above sets follows from Claim 4.1 and from the fact that the image of a dense subset through a surjective map is dense.) Set ν_m to be the largest i such that the intersection $Y_m \cap S_{R_i}$ is nonempty. For $\nu_{m-1} \le i \le \nu_m - 1$, set $Y_{m,i} := Y_m \cap [X \setminus B_{R_i}]$ and $Z_{m,i} := G_i(Y_{m,i})$, and for each $z \in Z_{m,i}$, set $y_z = G_{i+1}(y)$ for the $y \in Y_{m,i}$ for which $G_i(y) = z$. Then for $\nu_{m-1} \le i \le \nu_m - 1$ choose successively approximations $g_i' = g_i^{Z_{m,i}}$ as in Lemma 4.5, close enough to g_i to fulfill the requirements of Brown's lemma. Note that, apart from satisfying conditions (2) and (3) of Proposition 3.3, the maps g_i' with $1 \le i \le \nu_m - 1$ have the following property: for each $M \in \mathcal{M}$ and for each $j \in \{i_1, \dots, i_m\}$, the images in S_{R_j} of the points from $\bigcup_{i=j+1}^{\nu_m} (\Lambda_M \cap S_{R_i})$, through the appropriate compositions of the maps g_i', form an ϵ_k–net in S_{R_j}, where $k \le m$ is the largest number such that $i_k = j$.

A direct verification shows that the whole sequence of maps g_i' for $i \ge 1$ described above meets all the requirements of Proposition 3.3, which finishes the proof.

5 Applications of the main theorem

In this section we describe vast classes of examples to which the main theorem (as presented in Section 2) applies. Among others, we provide proofs of Theorems 1 and 2 from the introduction.

Hyperbolizations of $\mathcal{M}$–pseudomanifolds and the proof of Theorem 1 Given a finite family $\mathcal{M}$ of PL manifolds as in Section 2 (all of the same dimension n), a *closed $\mathcal{M}$–pseudomanifold* is a compact connected polyhedral cell complex P such that:

(1) For each point $x \in P$ the link $\mathrm{Lk}(x, P)$ either is a PL n–sphere or is PL–homeomorphic to some $M \in \mathcal{M}$; furthermore, if all $M \in \mathcal{M}$ are oriented, we assume that P is also oriented and that each homeomorphism $\mathrm{Lk}(x, P) \to M$ as above respects the orientations.

(2) For each $M \in \mathcal{M}$ the set $\Lambda_M^P := \{x \in P : \mathrm{Lk}(x, P) \cong M\}$ is nonempty (if all $M \in \mathcal{M}$ are oriented, the symbol $\cong$ denotes the relation of being PL–homeomorphic as oriented manifolds).

Recall that *hyperbolization*, as described eg in [8], is a procedure that turns an arbitrary compact simplicial complex into a compact nonpositively curved piecewise euclidean polyhedral complex with the same local PL topology. In particular, if we apply this procedure to a simplicial closed $\mathcal{M}$–pseudomanifold P, we get a nonpositively curved piecewise euclidean complex P_h which is also a closed $\mathcal{M}$–pseudomanifold. (If all manifolds in $\mathcal{M}$ are oriented, we consider the procedure which respects all requirements concerning orientations, eg the procedure described in [8, Section (4c)], for which each hyperbolized simplex is an oriented manifold, and the associated hyperbolization map is of degree 1.) Observe that the universal cover $\widetilde{P}_h$ is then an $(\mathcal{M}, 0)$–pseudomanifold.

There is also an analogous procedure, called *strict hyperbolization* and described in [5], which turns any simplicial closed $\mathcal{M}$–pseudomanifold P into a piecewise hyperbolic locally CAT(-1) closed $\mathcal{M}$–pseudomanifold P_h^s. The universal cover $\widetilde{P}_h^s$ is then an $(\mathcal{M}, -1)$–pseudomanifold.

Theorem 1 of the introduction is an easy consequence of the following.

5.1 Theorem *Let $\mathcal{M} = \{M_1, \dots, M_k\}$ be a finite family of closed connected PL manifolds of the same dimension n, either all oriented or at least one of which is nonorientable. Suppose that for some positive integers $m_1, \dots, m_k$, the disjoint union $\bigsqcup_{i=1}^k m_i M_i$ bounds a compact $(n+1)$–dimensional PL manifold W (in the oriented sense if all manifolds from $\mathcal{M}$ are oriented). Then there is a hyperbolic group G such that its Gromov boundary ∂G is homeomorphic to the tree of manifolds $\mathcal{X}(\mathcal{M})$.*

Proof Consider any PL triangulation of any manifold W as in the assumptions. For each boundary component M of W consider the simplicial cone over M and glue it to W via the identity of M. This gives a simplicial closed $\mathcal{M}$–pseudomanifold which we denote by P. Consider its strict hyperbolization P_h^s and its universal cover $\widetilde{P}_h^s$, which is a $(\mathcal{M},-1)$–pseudomanifold. The group $G=\pi_1(P_h^s)$ is then a word-hyperbolic group, and its Gromov boundary ∂G coincides with the visual boundary $\partial \widetilde{P}_h^s$. Since, by the main theorem, the latter boundary is homeomorphic to the tree of manifolds $\mathcal{X}(\mathcal{M})$, this completes the proof. □

Proof of Theorem 1 Part (1) follows from Theorem 5.1 by taking $\mathcal{M}=\{M\}$, while part (2) follows by taking $\mathcal{M}=\{N\}$ and $W=N\times[0,1]$. □

5.2 Remark Note that Theorem 1 provides new examples of Gromov boundaries already in dimension 3. More precisely, since each closed connected PL 3–manifold M bounds a compact PL 4–manifold, if $\mathcal{X}(M)$ is a tree of 3–manifolds M, then $\mathcal{X}(M)$ is homeomorphic to the Gromov boundary of some hyperbolic group. On the other hand, in the case of orientable 3–manifolds the arguments of [18] (after appropriate correction of the main result of [11] used in [18]) justify this statement only for manifolds of the form $M\#\overline{M}$, where M and $\overline{M}$ are the oppositely oriented copies of any orientable 3–manifold.

5.3 Question Theorem 5.1 leaves open the following question: *is there a hyperbolic group G whose Gromov boundary ∂G is homeomorphic to the tree of complex projective planes $\mathcal{X}(\mathbb{CP}^2)$?* Recall that neither $\mathbb{CP}^2$ nor any positive number of its copies bounds a compact oriented 5–manifold, so G cannot be obtained by referring to Theorem 5.1. On the other hand, this theorem easily implies that the space $\mathcal{X}(\{\mathbb{CP}^2,\overline{\mathbb{CP}^2}\})$ is homeomorphic to the Gromov boundary of some hyperbolic group. Since one can use properties of Čech cohomology rings to show that the spaces $\mathcal{X}(\mathbb{CP}^2)$ and $\mathcal{X}(\{\mathbb{CP}^2,\overline{\mathbb{CP}^2}\})$ are not homeomorphic, this also does not help to answer the question. My guess is that the answer is negative. Of course, a similar question can be asked for manifolds other than $\mathbb{CP}^2$ which represent the elements of infinite order in the corresponding oriented cobordism additive semigroup.

Coxeter groups with manifold nerves and the proof of Theorem 2 Our main reference concerning Coxeter groups is the book by Davis [7]. We start with explaining the terms appearing in the statement of Theorem 2, following the terminology and notation from [7].

Given a finite set S, a *Coxeter matrix* on S is a matrix $\boldsymbol{m}=(m_{st})_{s,t\in S}$ such that

(1) for each $s\in S$ we have $m_{ss}=1$;

(2) for all $s,t\in S$ such that $s\neq t$, we have that m_{st} is an integer ≥ 2 or is ∞, and $m_{st}=m_{ts}$.

Associated to a Coxeter matrix $\boldsymbol{m}$ on S, there is a group W given by the presentation

$$\langle S \mid \{(st)^{m_{st}} : s,t \in S\}\rangle,$$

where the symbol $(ab)^\infty$ denotes absence of any relation of the form $(ab)^k$ in the set of relations. W is called the *Coxeter group* associated to $\boldsymbol{m}$, and it is known that the set S canonically injects into W. The pair (W,S) is called the *Coxeter system* associated to $\boldsymbol{m}$.

For any subset $T \subset S$ the *special subgroup* $W_T < W$ is the subgroup generated by T. It is known that (W_T, T) can be canonically identified with the Coxeter system associated to the restricted matrix $\boldsymbol{m}_T$, ie the matrix $(m_{st})_{s,t\in T}$.

The *nerve* of a Coxeter system (W,S) is the simplicial complex $L = L(W,S)$ whose vertex set coincides with S and is such that $T \subset S$ spans a simplex of L if and only if the special subgroup W_T is finite.

As described in [7, Chapter 7], to any Coxeter system (W,S) there is associated a polyhedral cell complex $\Sigma = \Sigma(W,S)$, called the *Davis–Moussong complex* of (W,S), which satisfies the following properties:

(Σ1) Each vertex link of Σ is a simplicial complex isomorphic with the nerve L.

(Σ2) Σ carries a natural piecewise euclidean metric with respect to which it is a CAT(0) space; see [7, Theorem 12.3.3 on page 235].

(Σ3) The group W acts on Σ by isometries, properly discontinuously and cocompactly, so that the generators from S correspond to certain geometric reflections in Σ and so that the action is simply transitive on the vertex set of Σ.

Proof of Theorem 2 Suppose that the nerve $L(W,S)$ is a PL triangulation of a closed connected manifold M. If M is orientable, it follows from conditions (Σ1)–(Σ3) above that $\Sigma(W,S)$ is an $(\mathcal{M},0)$–pseudomanifold with $\mathcal{M} = \{M, \overline{M}\}$, where M and $\overline{M}$ are the two oppositely oriented copies of M. Similarly, if M is nonorientable, $\Sigma(W,S)$ is an $(\mathcal{M},0)$–pseudomanifold with $\mathcal{M} = \{M\}$. Thus, applying the main theorem to the Davis–Moussong complex $\Sigma(W,S)$, we immediately get Theorem 2. □

5.4 Remark It is known that, when a Coxeter group W is word-hyperbolic, its Gromov boundary ∂W coincides with the visual boundary $\partial\Sigma(W,S)$ (see eg [7, Remark I.8.5 on page 527]). In such a case, if the nerve $L(W,S)$ is a PL triangulation of a closed connected manifold M, the Gromov boundary ∂W is homeomorphic to the space $\mathcal{X}(\mathcal{M})$ as in the statement of Theorem 2.

6 Riemannian $(\mathcal{M}, 0)$–pseudomanifolds with log-injective singularities

In this section we explain how to adapt our proof of the main theorem to a class of smooth CAT(0) pseudomanifolds with Riemannian metrics on their regular part. Using this, we deduce Theorem 3. We start with describing the relevant class of pseudomanifolds. We refer the reader to [13, Section 3.2] for the definition of the *space of directions* $\Sigma_p(X)$ at a point p of a CAT(0) space X (or to [3, Definition II.3.18 on page 190], where the same object is denoted by $S_p(X)$).

6.1 Definition Given a finite collection $\mathcal{M}$ of closed connected manifolds of the same dimension n, a CAT(0) complete geodesic metric space (X, d) is a *Riemannian* $(\mathcal{M}, 0)$*–pseudomanifold with log-injective singularities* if these conditions are satisfied:

(1) There is a subset $\Lambda \subset X$, called the *singular set* of X, which is discrete and whose complement $X \setminus \Lambda$ is a smooth manifold.

(2) The set Λ is partitioned into subsets $\{\Lambda_M : M \in \mathcal{M}\}$ such that for each $M \in \mathcal{M}$ and any $p \in \Lambda_M$, the space of directions $\Sigma_p(X)$ is homeomorphic to M.

(3) Each of the subsets Λ_M is a net in X.

(4) The *regular part* $X \setminus \Lambda$ is equipped with a Riemannian metric g of nonpositive sectional curvature such that d restricted to $X \setminus \Lambda$ coincides with the path metric induced by g, and the completion of $(X \setminus \Lambda, d)$ coincides with (X, d).

(5) Each $p \in \Lambda$ has a neighborhood U such that the logarithmic map

$$\log_p \colon U \setminus \{p\} \to \Sigma_p(X) \times R_+$$

(which to each $x \in U \setminus \{p\}$ associates the pair (a, r) such that $a \in \Sigma_p(X)$ is the direction of the geodesic $[p, x]$ and $r = d(p, x)$) is injective.

(6) The space $\Sigma_p(X)$ at any singular point p has the property that every ball of radius $r \in (0, \pi)$ in it (with respect to the angle metric) is a collared n–disk in $\Sigma_p(X)$.

6.2 Theorem *Suppose that X is a Riemannian $(\mathcal{M}, 0)$–pseudomanifold with log-injective singularities. Then the visual boundary ∂X is homeomorphic to the tree of manifolds $\mathcal{X}(\mathcal{M})$.*

Theorem 6.2 follows by the same arguments as in the proof of the main theorem, slightly adapted and simplified according to the following features:

(1) Whenever in the proof of the main theorem we use cone neighborhoods of singular points, we need to use instead small balls which are log-injective neighborhoods of singular points; existence of the latter balls is justified by condition (5) in Definition 6.1.

(2) The references to Lemma A of the appendix appearing in the proof of the main theorem, when applied to links at singular points, need to be replaced by references to condition (6) of Definition 6.1; moreover, references to properties of links at nonsingular points also need to be replaced by references to the properties of the corresponding spaces of directions, which are just the standard round n–spheres of constant curvature 1.

(3) Since geodesics in Riemannian $(\mathcal{M},0)$–pseudomanifolds do not bifurcate outside the singular set, while proving Theorem 6.2 we never need to approximate (by referring to the approximation theorem for cell-like maps) various geodesic projections between the spheres (or their restrictions) by homeomorphisms, since in this setting the corresponding maps are automatically homeomorphisms.

Proof of Theorem 3 In [13], Fujiwara and Manning explain how to put a Riemannian metric on a regular part of any space appearing in the statement of Theorem 3 so that its lift to the universal cover of this space (and the induced path metric and its completion) satisfies all the requirements of Definition 6.1. In fact, the metrics constructed in [13] are even $\mathrm{CAT}(-1)$. In view of this, Theorem 3 follows fairly directly from Theorem 6.2. □

A related class of examples and an open question In [6] Coulon studies a class of examples related to those appearing in Theorem 3. Namely, he considers a negatively curved closed Riemannian manifold M and its closed totally geodesic submanifold N, and the space $Y = M \cup_N \mathrm{Cone}(N)$ obtained by attaching to M a cone of base N. He shows that, under certain conditions, the resulting space is aspherical, and its fundamental group $\Gamma = \pi_1 Y$ is hyperbolic.

Note that the same group Γ is the fundamental group of the pseudomanifold X obtained from M by simply collapsing N to a point. Equivalently, we can delete from M some open tubular neighborhood V of N, and collapse the boundary $\partial V = \partial(M \setminus V)$ to a point, which we denote by p. Clearly, X is then homotopy equivalent to Y. Moreover, it is a pseudomanifold with natural PL structure, and p is its only singular point. The link $\mathrm{Lk}(p, X)$ is PL–homeomorphic to ∂V, and thus X is a $\{\partial V\}$–pseudomanifold (as defined at the beginning of Section 5).

The arguments used by Coulon in [6] do not guarantee existence of a negatively or nonpositively curved metric on X. Thus, the following question naturally appears.

6.3 Question Is the Gromov boundary $\partial\Gamma$ of a hyperbolic group Γ as above, resulting from the construction of Coulon [6], homeomorphic to the tree of manifolds $\mathcal{X}(\partial V)$?

Appendix

In this appendix we present (slightly changed and extended variants of) some results of Davis and Januszkiewicz [8, Section 3] (see Lemmas A–C below). These results are necessary ingredients in the arguments in Sections 3 and 4 of the present paper. We provide also proofs of these results. The reasons for doing this are as follows:

(a) Some of these results are not explicitly stated in [8], though they appear implicitly in the proof of Lemma 3b.1 and Theorem 3b.2 on pages 371–372 of that paper.

(b) The statements we give are more general than those in [8], although they follow by essentially the same arguments; for our applications in Sections 3 and 4 we need the extended versions of the results.

(c) In the above-mentioned proof of [8, Lemma 3b.1 and Theorem 3b.2], there are two gaps, as was pointed out to me by the referee of the present paper; the first gap was found by Hanspeter Fischer and rectified in a 1997 letter of Mike Davis to Fischer, but the new argument has been never published; the idea how to rectify the second gap was communicated to me by the referee of the present paper; the essential part of this appendix is devoted to presentation of these two new arguments; see Remark A.2 below for more details.

(d) The proofs of (the analogues of) Lemmas A–C given in [8] form a scheme of induction (with respect to the dimension n) involving all of these results at once; even though the first above-mentioned mistake concerns the proof of Lemma C, and the second one the proof of Lemma A, it is hard to present the correction without mentioning the other results, and providing the whole of their common proof.

We thank the referee of the present paper for suggestions concerning the scope and the shape of this appendix.

The results We use the terminology as in [3]. In our notation, $B_r(x, X)$ is the *closed* metric ball, and $B_r^\bullet(x, X)$ the *open* one. The notions of a cellular set and a cell-like set and map (appearing in the statements of Lemmas A and C) are recalled in the next part of this appendix.

Lemma A *Suppose that L is a* $\mathrm{CAT}(1)$ *piecewise spherical closed PL manifold of dimension n, and let $v \in L$ be any point.*

(1) *For any $r \in \left(0, \frac{\pi}{2}\right)$ the closed ball $B_r(v, L)$ is homeomorphic to the n–disk B^n, with the sphere $S_r(v, L)$ corresponding to the boundary ∂B^n, and it is collared in L.*

(2) *For any $r \in (0, \pi]$ the open ball $B^\bullet_r(v, L)$ is homeomorphic to the open n–disk. As a consequence, for any $r \in (0, \pi)$ and any $\varepsilon \in (0, \pi - r)$, there is a collared n–disk D in L such that $B_r(v, L) \subset D \subset B^\bullet_{r+\varepsilon}(v, L) \subset B^\bullet_\pi(v, L)$.*

(3) *If, moreover, L is the standard PL n–sphere, then for any $r \in (0, \pi)$ the complement $L \setminus B^\bullet_r(v, L)$ is a cellular subset of L. In particular, $L \setminus B^\bullet_\pi(v, L)$ is cellular in L.*

Lemma B *Suppose that X is a locally compact* $\mathrm{CAT}(\kappa)$ *M_κ–polyhedral complex with* $\mathrm{Shapes}(X)$ *finite. Fix any $x_0 \in X$ and any $r > 0$ $\left(r < \frac{\pi}{2}\sqrt{\kappa} \text{ if } \kappa > 0\right)$. Suppose that for each $x \in X$ such that $d(x, x_0) = r$ the metric link* $\mathrm{Lk}(x, X)$ *(viewed as a piecewise spherical complex) is a closed PL manifold of dimension $n-1$. Then the sphere $S_r(x_0, X)$ is a closed $(n-1)$–manifold, and a neighborhood U of $S_r(x_0, X)$ in $B_r(x_0, X)$ is an n–manifold with boundary $\partial U = S_r(x_0, X)$.*

Lemma C *Suppose that X is a locally compact* $\mathrm{CAT}(\kappa)$ *M_κ–polyhedral complex with* $\mathrm{Shapes}(X)$ *finite. Let $x_0 \in X$, and for any $t > r > 0$ such that $t < \pi/\sqrt{\kappa}$ if $\kappa > 0$, denote by $c_{t,r}\colon S_t(x_0, X) \to S_r(x_0, X)$ the geodesic projection towards x_0 between the metric spheres. Suppose that for each $x \in X$ such that $r \le d(x, x_0) < t$, the metric link* $\mathrm{Lk}(x, X)$ *(viewed as a piecewise spherical complex) is PL–homeomorphic to the standard PL sphere S^{n-1}. Then $c_{t,r}$ is a cell-like map.*

Before proceeding to the proof of the above lemmas we need some preparations.

Preparations concerning cellular sets and cell-like sets and maps A nonempty compact subset C of a metric n–manifold (M, d) is *cellular* if C has arbitrarily close open neighborhoods which are homeomorphic to the open n–disk. More precisely, given any $\epsilon > 0$ there is a neighborhood of C of the relevant form mentioned above which is contained in the ϵ–neighborhood $N_\epsilon(C) = \{x \in M : d(x, C) < \epsilon\}$.

A nonempty compact metric space C is *cell-like* if it can be embedded into the Hilbert cube Q so that for any neighborhood U of C in Q the space C is null-homotopic in U. It is known (see eg [9, remark on page 114]) that a finite-dimensional compact metric space is cell-like if it can be embedded as a cellular subset in some manifold.

Obviously, each cellular subset of a manifold is cell-like, but the converse is not true (ie there are embeddings of cell-like spaces in manifolds with noncellular images). A map between metric spaces is *cell-like* if it is a proper surjection and each point preimage is cell-like.

The following result is due to Siebenmann [20] for $n \geq 5$, Quinn [19] for $n = 4$, Armentrout [2] for $n = 3$ and R L Moore [16] for $n \leq 2$. The case of manifolds with nonempty boundary is carefully addressed in [20], and the cases involving dimension 3 hold because the Poincaré conjecture is true.

Approximation Theorem *Each cell-like map between manifolds with boundary is a near-homeomorphism; that is, it can be approximated by homeomorphisms. More precisely, if $f\colon M \to N$ is cell-like and d is any metric in N (compatible with the topology), then for any continuous positive function $\delta\colon M \to R_+$ there is a homeomorphism $h\colon M \to N$ such that for all $x \in M$ we have $d(f(x), h(x)) \leq \delta(x)$.*

The proof of the next result was indicated to me by the referee of the present paper.

A.1 Lemma *If $f\colon A \to B$ is a cell-like surjection between finite-dimensional metric compacta, then A is cell-like if and only if B is cell-like.*

Proof Recall that an ENR is a locally compact finite-dimensional ANR. Embed A in a compact ENR X (eg a sphere of dimension $2\dim A + 1$). Let Y be the quotient space obtained from X by shrinking each set $f^{-1}(b)$ with $b \in B$ (viewed as a subset of X) to a point. Since Y is easily seen to be a Hausdorff space, the quotient map $F\colon X \to Y$ is closed, and hence also perfect. Since metrizability is preserved by perfect maps (see eg [10, Theorem 4.4.15]), Y is metrizable.

By definition of Y, the map F is cell-like. Since $\dim Y \leq \dim B + \dim(X \setminus A) + 1$, we get that $\dim Y$ is finite. It follows then from [15, Corollary 3.3] that Y is an ENR. Since $F^{-1}(B) = A$, Theorem 1.4 of [15] implies that A is cell-like if and only if B is cell-like, as required. □

The scheme of the inductive proof of Lemmas A–C Observe that for $n = 1$ the statements of Lemmas A–C are clearly true. Denote by (A_n), (B_n) and (C_n) respectively the statements of Lemmas A–C in dimension n. We shall prove Lemmas A–C according to the scheme

$$(A_{n-1}) \Rightarrow (B_n) \Rightarrow (C_n) \Rightarrow (A_n).$$

A.2 Remark The first gap in the paper [8], mentioned in comment (c) at the beginning of this appendix, concerns the argument for showing the analogue of our implication $[(A_{n-1})$ and $(B_n)] \Rightarrow (C_n)$. The gap appears in the paragraph beginning in the middle of page 372 of [8], and more precisely in the seventh sentence of this paragraph, which starts with "Assume that we have chosen s close enough to r ...". Examples can be constructed in which for every $s > r$ there are geodesic rays emanating from x that have two or more bifurcation points (ie points with nontrivial infinitesimal shadow) in $A_{r,s}$. The new argument provided by the authors of [8], which rectifies this gap, is presented below, in the part concerning the proof of the implication $[(A_{n-1})$ and $(B_n)] \Rightarrow (C_n)$.

The second gap in [8] concerns the proof of Lemma (3b.1) from that paper, and it appears in line -6 on page 371. In fact, the proof of the implication $(L_{n-1}) \Rightarrow (T_n)$, as presented in [8], does not apply to the implication $(L_{n-1}) \Rightarrow (L_n)$, for balls with radius $r \in \left[\frac{\pi}{2}, \pi\right)$. The reason is that balls with such radii need not be locally geodesically strictly convex in the corresponding space L (in the sense that the geodesic between any two sufficiently close points on the boundary sphere of such a ball intersects this sphere only at the endpoints). The latter property is essentially exploited in the argument in [8]; see eg the statement in line 1 on page 372 of that paper and a few sentences following. In fact, it is not clear whether [8, Lemma (3b.1)] holds true in its full generality. Thus, in our exposition we only formulate (as Lemma A(1)) its special case for $r \in \left(0, \frac{\pi}{2}\right)$. We also prove a certain result slightly weaker than [8, Lemma (3b.1)], namely Lemma A(2). We provide arguments that replace those from [8], whenever the latter arguments refer to the full statement of Lemma (3b.1).

Proof of $(A_{n-1}) \Rightarrow (B_n)$ The proof of the implication $(L_{n-1}) \Rightarrow (T_n)$ in [8], in a long paragraph beginning at the end of page 371 and ending in the middle of the next page, works without essential changes in the slightly more general setting of (B_n). More precisely, for any point $y \in S_r(x_0, X)$, denoting by $v \in \mathrm{Lk}(y, X)$ the point induced by the geodesic segment $[y, x_0]$, the above-mentioned argument from [8] shows that a small closed neighborhood of y in $B_r(x_0, X)$ is homeomorphic to the space obtained from the product $B_{\pi/2}(v, \mathrm{Lk}(y, X)) \times [0, \varepsilon]$ by collapsing to a point its subset

$$\left[B_{\pi/2}(v, \mathrm{Lk}(y, X)) \times \{0\}\right] \cup \left[S_{\pi/2}(v, \mathrm{Lk}(y, X)) \times [0, \varepsilon)\right],$$

where the latter point corresponds to y. Since, by part (2) of (A_{n-1}), the open ball $B^{\bullet}_{\pi/2}(v, \mathrm{Lk}(y, X))$ is homeomorphic to the open $(n-1)$–disk, it follows easily that the above quotient is homeomorphic to the closed n–disk B^n, with the point corresponding to y lying on its boundary ∂B^n. We omit further details.

Proof of $[(A_{n-1})$ and $(B_n)] \Rightarrow (C_n)$ The argument below essentially coincides with the one communicated by Davis in his 1997 letter to Fischer.

Suppose $g\colon [a,b] \to X$ is a geodesic segment in a metric space X, and let $a < t < b$. Recall that $g(t)$ is a *bifurcation point* of g if there is a geodesic segment $g'\colon [a,c] \to X$, with $t < c$, such that $g(s) = g'(s)$ for all $a \le s \le t$ and $g(s) \ne g'(s)$ for all $t < s \le \min(b,c)$. If X is a $\mathrm{CAT}(\kappa)$ M_κ–polyhedral complex, and if v is the point in the metric link $L = \mathrm{Lk}(g(t), X)$ corresponding to the incoming direction of g (ie the direction induced by $g|_{[a,t]}$) then $g(t)$ is a bifurcation point of g precisely when the infinitesimal shadow of $g(t)$ with respect to v (as defined in [8, page 369]) is nontrivial, ie consists of more than one point. This infinitesimal shadow, denoted by $\mathrm{Shad}_{g(t)}(v)$, is known to coincide with the set $\{y \in L : d_L(y,v) \ge \pi\}$, where d_L is the canonical piecewise spherical metric in L.

A.3 Fact *If $g\colon [a,b] \to X$ is a geodesic segment in an M_κ–polyhedral complex with* $\mathrm{Shapes}(X)$ *finite, then the set of bifurcation points of g is finite.*

Proof Since each M_κ–polyhedral complex with finite shapes can be subdivided into an isometric M_κ–simplicial complex with finite shapes [3, Proposition 7.49, page 118], we can assume X is an M_κ–simplicial complex with finite shapes. Since every geodesic segment in such a simplicial complex is the concatenation of a finite number of segments, each of which is contained in a single simplex [3, Corollary 7.29, page 110], this also holds for g. If γ is one such subsegment for g, we claim that no interior point y of γ is a bifurcation point of g. Indeed, since the interior of γ is contained in the interior of some simplex of X, if v and w denote the points of the link $\mathrm{Lk}(y, X)$ corresponding to the incoming and outgoing directions of g, then the whole metric link $\mathrm{Lk}(y, X)$ is isometric to some spherical suspension in which v and w are the suspension points. It follows that w is the only point in $\mathrm{Lk}(y, X)$ at distance π from v, so the infinitesimal shadow of y at v is trivial. Thus only the ends of the subsegments γ of g as above can be the bifurcation points of g, hence the fact. $\square$

A.4 Fact *Let X be a* $\mathrm{CAT}(\kappa)$ *M_κ–polyhedral complex with* $\mathrm{Shapes}(X)$ *finite. Fix any $x_0 \in X$, $\rho > 0$ ($\rho < \pi/\sqrt{\kappa}$ if $\kappa > 0$) and $x \in S_\rho(x_0, X)$, and let v be the point of the link* $\mathrm{Lk}(x, X)$ *corresponding to the incoming direction of the geodesic from x_0 to x. Then there is an $\epsilon > 0$ such that if $\rho < s < \rho + \epsilon$ then the preimage $c_{s,\rho}^{-1}(x)$ is homeomorphic to the infinitesimal shadow* $\mathrm{Shad}_x(v)$.

Proof There is an $\epsilon > 0$ such that the open ball $B^\bullet_\epsilon(x, X)$ is naturally isometric to the open ball of radius ϵ centered at the cone point in the κ–cone $C_\kappa(\mathrm{Lk}(x, X))$

(see [3, Theorem I.7.16 on page 103]). We may additionally assume that $\rho+\epsilon<\pi/\sqrt{\kappa}$ if $\kappa>0$. It follows that for any s with $\rho<s<\rho+\epsilon$ there is a canonical map $\Phi\colon \mathrm{Lk}(x,X)\to S_{s-\rho}(x,X)$ which is a homeomorphism. Moreover, it is not hard to observe that Φ maps the infinitesimal shadow $\mathrm{Shad}_x(v)$ to the set $c_{s,\rho}^{-1}(x)$, thus establishing a homeomorphism as required. □

We now pass to the essential part of the proof of $[(A_{n-1})$ and $(B_n)]\Rightarrow(C_n)$. Note that, by the assumption concerning links, the region $\{z\in X : r<d(z,x_0)<t\}$ has the geodesic extension property (compare [3, Proposition II.5.10, page 208]), and thus $c_{t,r}$ is a surjection. Moreover, since by our assumptions the spheres in X are compact, this map is also proper. Arguing by contradiction, suppose that $c_{t,r}$ is not cell-like. Hence there is an $x\in S_r(x_0,X)$ such that $c_{t,r}^{-1}(x)$ is not a cell-like set. We will construct a geodesic segment, starting at x_0 and passing through x, that contains an infinite sequence (y_i) of bifurcation points, thus contradicting Fact A.3.

Claim 1 *There is a point $y_1\in X$ such that*

(1) *x lies on the geodesic $[x_0,y_1]$ from x_0 to y_1, and*

(2) *every geodesic from x_0 to a point of $c_{t,r}^{-1}(x)$ passes through y_1* (ie contains $[x_0,y_1]$) *and bifurcates at y_1.*

Proof of Claim 1 Since the set $c_{t,r}^{-1}(x)$ is not cell-like, it contains more than one point. For any two distinct points of $c_{t,r}^{-1}(x)$ there is a point z where the two geodesics from these points to x_0 meet for the first time. Set $s=d(x_0,z)$, note that $r\le s<t$, and consider the infimum t_1 of all such s, for all pairs of distinct points in $c_{t,r}^{-1}(x)$. Then $r\le t_1<t$, and all geodesics from x_0 to a point of $c_{t,r}^{-1}(x)$ must coincide on the interval $[0,t_1]$. Let $y_1\in S_{t_1}(x_0,X)$ be the common point that all these geodesics pass through. Then y_1 is a bifurcation point for all of these geodesics. Indeed, if this is not true, the infinitesimal shadow at y_1 for all of these geodesics must be trivial. The existence of a cone neighborhood of y_1 in X yields then existence of a y' such that y_1 is an interior point of the geodesic $[x_0,y']$ and all geodesics from x_0 to a point of $c_{t,r}^{-1}(x)$ contain $[x_0,y']$. But this contradicts the choice of t_1, thus completing the proof. □

Claim 2 *There is an $r_1\in(t_1,t)$ and a point $x_1\in S_{r_1}(x_0,X)$ such that*

(1) *the points x and y_1 lie on the geodesic $[x_0,x_1]$, and*

(2) *the set $c_{t,r_1}^{-1}(x_1)$ is not cell-like.*

Proof of Claim 2 Consider the link $L = \mathrm{Lk}(y_1, X)$ and note that, by the assumptions in Lemma C, L is a CAT(1) piecewise spherical complex (with $\kappa = 1$) PL–homeomorphic to the standard PL sphere S^{n-1}. Let $v \in L$ be the point corresponding to the incoming direction of the geodesic $[x_0, y_1]$. By part (3) of (A_{n-1}), the infinitesimal shadow $\mathrm{Shad}_{y_1}(v) = L \setminus B^{\bullet}_{\pi}(v, L)$ is a cellular subset of L, and hence it is cell-like. Moreover, it follows from Fact A.4 that there is an $r_1 \in (t_1, t)$ such that the preimage $c^{-1}_{r_1,t_1}(y_1)$ (which coincides with the preimage $c^{-1}_{r_1,r}(x)$) is homeomorphic to the infinitesimal shadow $\mathrm{Shad}_{y_1}(v)$, and hence it is also a cell-like set. Note that for the geodesic projection c_{t,r_1} the preimage $c^{-1}_{t,r_1}(c^{-1}_{r_1,r}(x))$ coincides with the set $c^{-1}_{t,r}(x)$. Moreover, the restriction $f\colon c^{-1}_{t,r}(x) \to c^{-1}_{r_1,r}(x)$ of c_{t,r_1} is a surjection, because, by the assumption concerning links, we have the appropriate geodesic extension property. Since the set $c^{-1}_{t,r}(x)$ is not cell-like, it follows from Lemma A.1 that there is an $x_1 \in c^{-1}_{r_1,r}(x)$ such that $f^{-1}(x_1) = c^{-1}_{t,r_1}(x_1)$ is not a cell-like set, which completes the proof. □

Iterating Claims 1 and 2 we construct an infinite sequence of real numbers

$$r \le t_1 < r_1 \le t_2 < r_2 \le t_3 < \cdots < t$$

and sequences of points

$$y_i \in S_{t_i}(x_0, X) \quad \text{and} \quad x_i \in S_{r_i}(x_0, X) \quad \text{for } i \ge 1$$

such that for each $i \ge 1$ the geodesic $[x_0, x_i]$ bifurcates at y_i and is contained in $[x_0, x_{i+1}]$. By the latter property, there is a limit $x_* = \lim_{i\to\infty} x_i \in X$, and the geodesic $[x_0, x_*]$ contains the union of the geodesics $[x_0, x_i]$. In particular, all points y_i belong to $[x_0, x_*]$, and each of them is a bifurcation point for this geodesic. Since this contradicts Fact A.3, the assertion of (C_n) follows.

Proof of $[(B_n)$ and $(C_n)] \Rightarrow (A_n)$ First, assuming (B_n) and (C_n), we establish two auxiliary facts.

A.5 Fact *Under assumptions as in Lemma C, with the additional assumption that $t < \frac{\pi}{2}\sqrt{\kappa}$ if $\kappa > 0$, consider the annular region*

$$A_{r,t} = B_t(x_0, X) \setminus B^{\bullet}_r(x_0, X) = \{x \in X : r \le d(x, x_0 \le t)\}.$$

Then $A_{r,t}$ is homeomorphic to the product $S_r(x_0, X) \times [r, t]$, with $S_r(x_0, X) \times \{r\}$ and $S_r(x_0, X) \times \{t\}$ corresponding respectively to the spheres $S_r(x_0, X) \subset A_{r,t}$ and $S_t(x_0, X) \subset A_{r,t}$.

Proof We will construct a homeomorphism $A_{r,t} \to S_r(x_0, X) \times [r, t]$ which maps the spheres $S_t(x_0, X)$ and $S_r(x_0, X)$ onto the sets $S_r(x_0, X) \times \{t\}$ and $S_r(x_0, X) \times \{r\}$, respectively.

Note first that $A_{r,t} \setminus S_r(x_0, X)$ is a manifold with boundary $\partial(A_{r,t} \setminus S_r(x_0, X))$ equal to the sphere $S_t(x_0, X)$. Indeed, by the assumption on links, each point in $A_{r,t} \setminus (S_r(x_0, X) \cup S_t(x_0, X))$ has a neighborhood in $A_{r,t} \setminus S_r(x_0, X)$ homeomorphic to the open n–disk. Furthermore, by the last assertion in (B_n), each point of $S_t(x_0, X)$ has a neighborhood in $A_{r,t}$ homeomorphic to the open half-n–disk.

Consider now the map $\psi\colon A_{r,t} \setminus S_r(x_0, X) \to S_r(x_0, X) \times (r, t]$ given by $\psi(x) = (c_{s,r}(x), s)$, where $s = d(x, x_0)$. Note that, by the geodesic extension property in $A_{r,t}$, the map ψ is surjective. Because ψ extends, with the same formula, to a continuous map $\overline{\psi}\colon A_{r,t} \to S_r(x_0, X) \times [r, t]$ between compact spaces, and because $\overline{\psi}^{-1}(S_r(x_0, X) \times \{t\}) = S_t(x_0, X)$, it is easy to deduce that ψ is proper. Moreover, for each $(y, s) \in S_r(x_0, X) \times (r, t]$ the inverse image $\psi^{-1}((y, s))$ coincides with the set $c_{s,r}^{-1}(y)$, and hence (C_n) implies that this set is cell-like. Consequently, ψ is a cell-like map of manifolds. Consider the function $\delta\colon S_r(x_0, X) \times (r, t] \to R_+$ given by $\delta(y, s) = s - r$. By the approximation theorem, there is a homeomorphism $h\colon A_{r,t} \setminus S_r(x_0, X) \to S_r(x_0, X) \times (r, t]$ such that $d_X(h(y, s), \psi(y, s)) < \delta(y, s) = s - r$ for each $(y, s) \in S_r(x_0, X) \times (r, t]$. Consequently, if an argument $z \in A_{r,t} \setminus S_r(x_0, X)$ converges to some $z_0 \in S_r(x_0, X)$, then $h(z)$ converges to (z_0, r). Hence h can be extended to a continuous map $H\colon A_{r,t} \to S_r(x_0, X) \times [r, t]$ by setting $H(x) = (x, r)$ for $x \in S_r(x_0, X)$. Moreover, H is easily seen to be a bijection, and since its domain is compact, it is a homeomorphism. Finally, since homeomorphisms preserve the boundary, h maps $S_t(x_0, X)$ onto the set $S_r(x_0, X) \times \{t\}$, and the same is true for H. It is also clear that H maps $S_r(x_0, X)$ onto the set $S_r(x_0, X) \times \{r\}$, and this completes the proof. □

A.6 Fact *Under assumptions as in Lemma B, suppose additionally that for each $x \in X$ such that $d(x, x_0) = r$ the metric link* $\mathrm{Lk}(x, X)$ *(viewed as a piecewise spherical complex) is PL–homeomorphic to the standard PL sphere S^{n-1}. Then the sphere $S_r(x_0, X)$ is a closed $(n-1)$–manifold which is bicollared in X* (ie $S_r(x_0, X)$ has a neighborhood homeomorphic to the product $Y \times (-1, 1)$, with $Y \times \{0\}$ corresponding to $S_r(x_0, X)$).

Proof The assumption concerning links implies that there is an $\epsilon > 0$ such that for any $x \in X$ satisfying $d(x, X) \in (r - \epsilon, r + \epsilon)$ the link $\mathrm{Lk}(x, X)$ is PL–homeomorphic to the standard PL sphere S^{n-1}. This follows by observing that the sphere $S_r(x_0, X)$ is compact and each of its points has a cone neighborhood in X. Then, it follows from (D_n) that the annular region $A_{r-\epsilon/2,r+\epsilon/2}$ is a bicollared neighborhood of $S_r(x_0, X)$ in X, as required. □

We now prove part (1) of (A_n). By the existence of a cone neighborhood, there is an $\epsilon > 0$ such that the closed ball $B_\epsilon(v, L)$ is homeomorphic to the n–disk B_n, with the sphere $S_\epsilon(v, L)$ corresponding to the boundary. By Fact A.5, the annular region $A_{\epsilon,r} = B_r(v, L) \setminus B^\bullet_\epsilon(v, L)$ is homeomorphic to $S_\epsilon(v, L) \times [\epsilon, r] \cong S^{n-1} \times [\epsilon, r]$, with $S_\epsilon(v, L)$ coinciding with one of the boundary components of this region. Consequently, the pair $(B_r(v, L), S_r(v, L))$ is homeomorphic to the pair $(B^n, \partial B^n)$, as required. It is also an obvious consequence of Fact A.6 that $B_r(v, L)$ is collared in L. This completes the proof of part (1).

We now turn to proving part (2) of (A_n). Note that for $r \in \left(0, \frac{\pi}{2}\right)$ the first assertion of part (2) follows directly from part (1). Thus, we only need to justify this assertion for $r \in \left[\frac{\pi}{2}, \pi\right]$. Given any such r, consider any homeomorphism $\lambda\colon [0, r) \to \left[0, \frac{\pi}{4}\right)$ such that $\lambda(t) < t$ for all $t > 0$ in the domain. Define the map $f\colon B^\bullet_r(x_0, X) \to B^\bullet_{\pi/4}(x_0, X)$ by letting the restriction of f to any sphere $S_t(x_0, X)$ with $t < r$ be the geodesic projection $c_{t,\lambda(t)}$ which maps $S_t(x_0, X)$ onto $S_{\lambda(t)}(x_0, X)$. It follows then from (C_n) that f is a cell-like map between n–manifolds. Hence, by the approximation theorem, f can be approximated by homeomorphisms. In particular, the open ball $B^\bullet_r(x_0, X)$ is homeomorphic to $B^\bullet_{\pi/4}(x_0, X)$, and thus also to the open n–disk.

The second assertion of part (2) follows immediately from the first one.

To prove part (3) of (A_n), set $C = L \setminus B^\bullet_r(v, L)$. Note that, by part (2) of (A_n), the complement of C in the PL n–sphere L is an open n–disk. This property of C is known under the term *point-like*. It is also known that any point-like subset of a sphere is cellular (see eg [9, Theorem on page 114]), so the assertion follows simply by referring to this result.

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