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Rigidity of mapping class group actions on S^1

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The mapping class group $\mathrm{Mod}_{g,1}$ of a surface with one marked point can be identified with an index two subgroup of $\mathrm{Aut}(\pi_1\Sigma_g)$. For a surface of genus $g \geq 2$, we show that any action of $\mathrm{Mod}_{g,1}$ on the circle is either semiconjugate to its natural faithful action on the Gromov boundary of $\pi_1\Sigma_g$, or factors through a finite cyclic group. For $g \geq 3$, all finite actions are trivial. This answers a question of Farb.

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1 Introduction

Let Σ_g be an oriented surface of genus $g \geq 2$, and let Γ_g denote $\pi_1(\Sigma_g)$. The Gromov boundary of Γ_g is a topological circle, on which the group $\mathrm{Aut}(\Gamma_g)$ of automorphisms of Γ_g acts faithfully by homeomorphisms. Geometrically, this boundary action of $\mathrm{Aut}(\Gamma_g)$ can be seen as follows. By the Dehn–Nielsen–Baer theorem, the exact sequence $\mathrm{Inn}(\Gamma_g) \to \mathrm{Aut}(\Gamma_g) \to \mathrm{Out}(\Gamma_g)$ is isomorphic, term by term, to the Birman exact sequence $\pi_1(\Sigma_g) \to \mathrm{Mod}^{\pm}_{g,1} \to \mathrm{Mod}^{\pm}_g$ of the extended mapping class group of a surface with one marked point. Fixing a hyperbolic metric on Σ_g, the universal cover $\widetilde{\Sigma}_g$ can be identified with $\mathbb{H}^2$, which has a natural compactification to a closed disc. Let $x \in \mathbb{H}^2$ be a lift of the marked point on Σ_g. For $f \in \mathrm{Homeo}(\Sigma_g)$ fixing the marked point, let $\tilde{f}$ denote the unique lift of f to $\mathbb{H}^2$ that fixes x. Using the fact that quasigeodesics remain bounded distance from geodesics in negative curvature, one can show that the action of $\tilde{f}$ on $\mathbb{H}^2$ extends to a homeomorphism of the boundary circle, which depends only on the isotopy class of f. This procedure gives a well-defined homomorphism $\mathrm{Mod}^{\pm}_{g,1} \to \mathrm{Homeo}(S^1)$, which, under the identification $\mathrm{Aut}(\Gamma_g) \cong \mathrm{Mod}^{\pm}_{g,1}$, is conjugate to the boundary action described above. We call this conjugacy class of actions the *standard action* of $\mathrm{Aut}(\Gamma_g)$ on S^1.

The mapping class group $\mathrm{Mod}_{g,1}$ consisting of isotopy classes of orientation-preserving homeomorphisms is an index two subgroup of $\mathrm{Mod}^{\pm}_{g,1}$; we let $\mathrm{Aut}_+(\Gamma_g)$ denote the corresponding subgroup of $\mathrm{Aut}(\Gamma_g)$. In [5, Question 6.2], B Farb asked whether any

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faithful action of $\mathrm{Aut}_+(\Gamma_g)$ on S^1 by homeomorphisms was necessarily conjugate to the standard action described above. In fact one needs to be a little more careful with the statement: rather than conjugacy, the appropriate notion of equivalence for C^0 actions on S^1 is Ghys's *semiconjugacy*, described below, since any action of an infinite group on S^1 can be modified (for instance, using the classical Denjoy trick) to produce nonconjugate, but semiconjugate examples. Here we give a positive answer to this version of Farb's question.

Theorem 1 *Let ρ: $\mathrm{Aut}_+(\Gamma_g) \to \mathrm{Homeo}_+(S^1)$ be a homomorphism. Up to reversing the orientation of the circle, we have the following. If $g \geq 3$, then ρ is either trivial or semiconjugate to the standard Gromov boundary action. If $g = 2$, then ρ is either conjugate to a subgroup of $\mathbb{Z}/10\mathbb{Z}$ acting by rotations, or is semiconjugate to the standard action.*

Actions conjugate to finite groups of rotations do indeed arise in the genus 2 case. As shown by Mumford [19], the abelianization of $\mathrm{Aut}_+(\Sigma_2)$ is $\mathbb{Z}/10\mathbb{Z}$. (Mumford discusses Mod_2, but the same is true of $\mathrm{Mod}_{2,1}$, which is also generated by Dehn twists in simple closed curves. See Korkmaz [11].) Finite cyclic groups do act on S^1, necessarily by an action conjugate to one by rigid rotations. Our theorem simply states that any nonstandard action factors through the abelianization. Note also that the theorem immediately gives a corresponding statement for homomorphisms of the larger group $\mathrm{Aut}(\Gamma_g)$ into $\mathrm{Homeo}(S^1)$.

As asserted above, semiconjugacy is also a necessary hypothesis. Following usage of Ghys, two actions ρ_1 and ρ_2: $\Gamma \to \mathrm{Homeo}_+(S^1)$ are said to be "semiconjugate" if there exists an equivariant, cyclic order preserving bijection from some orbit of S^1 under ρ_1 to some orbit under ρ_2. This is *not* equivalent to the usual notion of semiconjugacy from topological dynamics. Instead, it is equivalent to the condition that any continuous, conjugacy-invariant, real-valued map f on $\mathrm{Hom}(\Gamma, \mathrm{Homeo}_+(S^1))$ satisfies $f(\rho_1) = f(\rho_2)$. For this reason, we prefer the term *weakly conjugate*; it is well suited to studying analogs of character varieties. See Mann and Wolff [13] for a full discussion. However, as "weakly conjugate" is not yet standard, we defer to tradition and use the term "semiconjugacy" in the remainder of the paper.

In our situation, the standard action of $\mathrm{Aut}_+(\Sigma_g)$ is *minimal*, and applying the Denjoy trick produces a nonminimal, hence nonconjugate action. Note, however, that any two minimal, semiconjugate actions are in fact conjugate.

Such nonconjugate, but semiconjugate actions naturally arise from geometric considerations. For example, consider a complete hyperbolic surface with precisely one cusp, and identify the set of geodesic rays emanating from the cusp, $\mathcal{R}$, with the circle. Then $\mathrm{Aut}_+(\Sigma_g)$ naturally acts on $\mathcal{R}$, and this action is *not* minimal; see Bowditch and Sakuma [2, Theorem 2.1] for details. However, it is semiconjugate to the standard action: this can be seen either directly, or by using Theorem 1 above.

An easy consequence of Theorem 1 is the following:

Corollary 2 *For $g \geq 3$, any action of $\mathrm{Aut}_+(\Sigma_g)$ on S^1 by C^1 diffeomorphisms is trivial.*

Parwani [20] proved this statement under the additional assumption that $g \geq 6$, and Farb and Franks [6] proved it in C^2 regularity. Both of these works also concern other groups, and some information is obtained for subgroups of finite index of mapping class groups in [6]. More recently, Baik, Kim and Koberda [1] proved that no finite-index subgroup of $\mathrm{Aut}_+(\Sigma_g)$ acts faithfully on a compact one-manifold in regularity $C^{1+\mathrm{BV}}$; see also Kim and Koberda [10]. In this regard, the techniques used in the present article are specific to $\mathrm{Aut}_+(\Sigma_g)$ as they rely strongly on torsion; on the other hand, we consider all actions, not only the faithful ones. See also Remark 14. We give the proof of Corollary 2 in Section 3.

2 Proof of Theorem 1

Let $g \geq 2$, and let $\rho\colon \mathrm{Aut}_+(\Gamma_g) \to \mathrm{Homeo}_+(S^1)$ be a representation. The strategy of the proof is to constrain the possible behavior of the restriction of ρ to the surface subgroup $\Gamma_g \cong \mathrm{Inn}(\Gamma_g) \subset \mathrm{Aut}_+(\Gamma_g)$. We will use the notation $\mathrm{Aut}_+(\Gamma_g)$ and $\mathrm{Mod}_{g,1}$ interchangeably throughout the proof, depending on whether we prefer to evoke Γ_g as an abstract group, or whether it is useful to remember the topology of the surface Σ_g.

2.1 Action of the surface subgroup

Before embarking on the proof, we briefly recall some standard material on rotation numbers and on the Euler number of a representation. The group $\mathrm{Homeo}_+(S^1)$ fits in the exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(S^1) \to 1,$$

where $\mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$ is the group of (orientation-preserving) homeomorphisms of $\mathbb{R}$ which commute with integer translations. The *translation number of* an element

$f \in \mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$ is defined by $\widetilde{\mathrm{rot}}(f) = \lim_{n\to+\infty} f^n(x)/n$. This limit exists and is independent of the choice of $x \in \mathbb{R}$. For $f \in \mathrm{Homeo}_+(S^1)$, the (Poincaré) *rotation number* of f, denoted by $\mathrm{rot}(f)$, is defined as the translation number of any of its lifts to $\mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$ modulo $\mathbb{Z}$. It is easily verified that rot is conjugacy invariant, and satisfies $\mathrm{rot}(f^{-1}) = -\mathrm{rot}(f)$.

Any representation $\varphi\colon \Gamma_g \to \mathrm{Homeo}_+(S^1)$ can be assigned an integral *Euler number* as follows: Associated to φ, there is an S^1–bundle over Σ_g given by the quotient of $\widetilde{\Sigma}_g \times S^1$ by the diagonal action of Γ_g via deck transformations on $\widetilde{\Sigma}_g$ and φ on S^1. The Euler number $\mathrm{eu}(\varphi)$ is the pairing of the Euler class of this bundle with the fundamental class of Σ_g; it is (classically) the obstruction to a section of the bundle. Following the interpretation of Milnor and Wood [18; 22] the Euler number of φ can also be seen as the obstruction to lifting φ to a representation into $\mathrm{Homeo}^{\mathbb{Z}}(\mathbb{R})$. This can be computed by understanding rotation numbers of individual elements of $\varphi(\Gamma_g)$, or, more precisely, through the *translation cocycle*, $\tau\colon \mathrm{Homeo}_+(S^1)\times\mathrm{Homeo}_+(S^1)\to\mathbb{R}$, defined by $(f,g)\mapsto \widetilde{\mathrm{rot}}(\tilde{f}\tilde{g}) - \widetilde{\mathrm{rot}}(\tilde{f}) - \widetilde{\mathrm{rot}}(\tilde{g})$. That this cocycle takes values in $[-1,1]$ is the key to the *Milnor–Wood inequality*, which states that $|\mathrm{eu}(\varphi)| \le 2g-2$ for any representation $\varphi\colon \Gamma_g \to \mathrm{Homeo}_+(S^1)$.

One way to compute the Euler number of an action φ of Γ_g using rotation numbers is to decompose Σ_g into pairs of pants, and then sum the contributions to the Euler number coming from each pant. Suppose $P \subset \Sigma_g$ is an embedded pants subsurface with fundamental group $\langle a, b, c \mid abc = 1\rangle$, where a, b and c are freely homotopic to the boundary curves of P. The surface P inherits an orientation from Σ_g, and we require a, b and c to have the induced boundary orientation. We define $\mathrm{eu}(\varphi|_P) := \tau(\varphi(a), \varphi(b))$; this is invariant under cyclic permutations of a, b and c. Then the Euler number of φ is simply the sum $\sum_{P\in\mathcal{P}} \mathrm{eu}(\varphi|_P)$, where $\mathcal{P}$ is any pants decomposition. We refer the reader to [7] for a general introduction to Euler classes. The definition given here that applies to pants subsurfaces is a contribution of Burger, Iozzi and Wienhard (see [3, (4.4) and Theorem 4.6]). The reader may also refer to [14, Section 5] for a brief explanation and proof of well-definedness that avoids the use of bounded cohomology.

While the discussion above was general, we now return to our assumption that ρ is a representation of $\mathrm{Aut}_+(\Gamma_g)$ in $\mathrm{Homeo}_+(S^1)$. Using a technical result from work of the authors in [13], we prove a first lemma.

Lemma 3 *If* $a \in \pi_1(M) \cong \mathrm{Inn}(\Gamma_g)$ *represents a nonseparating simple closed curve, then* $\mathrm{rot}(\rho(a)) = 0$.

Proof It suffices to prove this for a single nonseparating simple closed curve, since $\mathrm{Mod}_{g,1}$ acts transitively on such curves, and so every nonsimple closed curve has the same rotation number (up to multiplying by -1 to account for orientation). Let a be a nonseparating simple closed curve. Since there is an involution of Σ_g that maps a to its inverse, it follows that $\mathrm{rot}(\rho(a)) = \mathrm{rot}(\rho(a^{-1})) = -\mathrm{rot}(\rho(a))$, so $\mathrm{rot}(\rho(a))$ is either $\frac{1}{2}$ or 0.

Now suppose for contradiction that this common rotation number for nonseparating simple curves is $\frac{1}{2}$, and let a and b be standard generators of the fundamental group of a subsurface T of Σ_g homeomorphic to a one-holed torus. Proposition 5.5 from [13] gives a procedure to produce a simple closed curve c contained in T such that $0 \leq |\mathrm{rot}(\rho(c))| < \frac{1}{2}$. But this contradicts the fact that $\mathrm{rot}(\rho(a)) = \pm\mathrm{rot}(\rho(c))$. □

Remark 4 The procedure given in [13] referred to above simply tracks the image of intervals complimentary to a finite orbit of a simple closed curve a under b to find a curve of the form $b^N a$ with strictly lower rotation number. The assumption $\mathrm{rot}(a) = \frac{1}{2}$ makes the proof very quick.

An alternative proof with a more topological than dynamical flavor was proposed to us by L Chen: One may write the automorphism represented by a as a product of two Dehn twists $\tau_1\tau_2$, supported in disjoint annuli that are freely homotopic in Σ_g, but lie on opposite sides of the marked point, with orientations such that τ_1 is conjugate to τ_2^{-1} in $\mathrm{Mod}_{g,1}$. Thus, $\mathrm{rot}(\rho(\tau_1)) = -\mathrm{rot}(\rho(\tau_2))$ by conjugacy invariance of rotation number. Since τ_1 and τ_2 have disjoint support, they commute; and since rotation number is a homomorphism on abelian groups, we have $\mathrm{rot}(\rho(\tau_1\tau_2)) = 0$, as desired.

As a consequence, we have the following:

Corollary 5 *The Euler number of $\rho|_{\Gamma_g}$ is either 0 or $\pm(2g-2)$.*

Proof Let $P_1, P_2, \dots, P_{2g-2}$ be an oriented pants decomposition of Σ_g such that each boundary curve of each P_i is nonseparating. For $i = 1, \dots, 2g-2$, let a_i, b_i and c_i be generators of the fundamental group of P_i with orientation inherited from P_i. (This is a slight abuse of notation, as we base all these curves at the same point in Σ_g.)

By Lemma 3 and the Milnor–Wood inequality statement that $|\tau| \leq 1$, for each i we have that $\mathrm{eu}(\rho|_{P_i})$ is either 0 or ± 1. Since the boundary curves of P_i are nonseparating, for each i there is an orientation-preserving homeomorphism f_i of Σ_g sending P_i to P_1.

It follows that the triples $(\rho(a_i), \rho(b_i), \rho(c_i))$ are all conjugate in $\mathrm{Homeo}_+(S^1)$, with $(\rho(a_i), \rho(b_i), \rho(c_i))$ conjugate to $(\rho(a_1), \rho(b_1), \rho(c_1))$ by the image of the mapping class of f_i under ρ. Hence the contributions of all P_i to the Euler number $\mathrm{eu}(\rho|_{\Gamma_g})$ are all equal, and so their sum is either 0 or $\pm(2g-2)$. □

To treat the case where $\mathrm{eu}(\rho) = \pm(2g-2)$, we use the following theorem of Matsumoto:

Theorem 6 (Matsumoto [16]) *If Γ_g acts on S^1 with Euler number equal to $\pm(2g-2)$, then the action is semiconjugate to the boundary action given by embedding Γ_g in $\mathrm{PSL}_2(\mathbb{R})$ as a cocompact lattice.*

The proof of this uses the translation cocycle defined above; a strategy for a more elementary approach to the proof can be found in [17].

Lemma 7 *If the Euler number of the restriction of ρ to Γ_g is nonzero, then, up to reversing orientation of the circle, ρ is semiconjugate to the standard action.*

Proof Suppose that the Euler number of the restriction of ρ to Γ_g is nonzero. Then $\rho(\Gamma_g)$ does not have a finite orbit, and so there is a canonical *minimal set* for the action (see eg [7, Proposition 5.6]). This is the unique closed, Γ_g–invariant set contained in the closure of any orbit, on which Γ_g acts minimally. Let $K \subset S^1$ denote the minimal set. It is equal to S^1 if the action is minimal, and homeomorphic to a Cantor set if not.

Since Γ_g is normal in $\mathrm{Aut}_+(\Gamma_g)$, and minimal sets are characteristic, the action of $\mathrm{Aut}_+(\Gamma_g)$ on S^1 also preserves K. Thus, K is a closed, invariant set on which $\mathrm{Aut}_+(\Gamma_g)$ acts minimally, hence is the minimal set for ρ. Up to semiconjugacy, we may in fact assume that the action of ρ is minimal. Precisely, if K happens to be a Cantor set, then let $h\colon S^1 \to S^1$ be a continuous, surjective map that is injective on K and collapses each of its complementary intervals to a point. The action of Γ_g descends naturally to an action on $h(S^1)$ which is minimal and semiconjugate to the original action. Thus, going forward, we assume that $\rho(\mathrm{Aut}_+(\Gamma_g))$ acts minimally. As noted above (see also [7]) minimality implies that the action of $\rho(\Gamma_g)$ is *conjugate* to the standard boundary action. We claim that this is enough to determine the action of $\mathrm{Aut}_+(\Gamma_g)$ up to conjugacy. Indeed, minimality of the action of Γ_g implies that the set of attracting fixed points of hyperbolic elements of Γ_g represented by closed curves in Σ_g is dense in S^1, and the cyclic order of these points is determined only by the topology of the surface. If $x \in S^1$ is the attractor of some $\gamma \in \Gamma_g$ represented by a closed curve, then each $\varphi \in \mathrm{Aut}_+(\Gamma_g)$ must necessarily have $\rho(\varphi)$ map x to the (unique) attractor of $\varphi(\gamma) \in \Gamma_g$. This determines the action of $\rho(\varphi)$ on a dense set, hence completely specifies it as a homeomorphism. □

The remainder of the proof of Theorem 1 consists of showing that the Euler number of $\rho|_{\Gamma_g}$ is 0 only when ρ factors through the abelianization of $\mathrm{Aut}_+(\Gamma_g)$.

2.2 Orbifold groups and their Euler numbers

Our motivation for the remainder of the proof comes from the following observation:

Observation 8 *Fix an embedding of* Γ_g *as a lattice in* $\mathrm{PSL}_2(\mathbb{R})$. *If* Δ *is a Fuchsian group that normalizes* Γ_g, *then* Δ *embeds faithfully into* $\mathrm{Aut}_+(\Gamma_g)$.

Proof This is just the observation that the centralizer of Γ_g in $\mathrm{PSL}_2(\mathbb{R})$ is trivial, because Γ_g is nonelementary. □

Thus, to get our hands on concrete elements of $\mathrm{Aut}_+(\Gamma_g)$, we produce embeddings of Γ_g into $\mathrm{PSL}_2(\mathbb{R})$ with large normalizers. These are constructed by realizing Σ_g as a regular cover of a hyperbolic orbifold. To this end, we recall a few facts about cocompact Fuchsian groups; the reader may refer to [9] for a general introduction. Any cocompact Fuchsian group Γ has a *signature*, of the form $(g; m_1, \dots, m_r)$, corresponding to the presentation

$$\Gamma = \langle a_1, b_1, \dots, a_g, b_g, q_1, \dots, q_r \mid q_1 \cdots q_r \cdot [a_1, b_1] \cdots [a_g, b_g] = q_1^{m_1} = \cdots = q_r^{m_r} = 1 \rangle.$$

Its covolume (the volume of the quotient $\mathbb{H}^2/\Gamma$) is given by the formula $\mu(\Gamma) = 2\pi\bigl(2g - 2 + \sum_{i=1}^{r}(1 - 1/m_i)\bigr)$, and its *orbifold Euler characteristic* $\chi(\Gamma)$ is defined to be $-\frac{1}{2\pi}\mu(\Gamma)$. When $r = 0$ this is the fundamental group of a closed surface, with the usual definition of Euler characteristic and hyperbolic volume. When $r > 1$, such a group Γ should be thought of as the holonomy representation of a hyperbolic surface with r cone points, of cone angles $2\pi/m_i$. A $(g; m_1, \dots, m_r)$ *orbifold* is simply the quotient of $\mathbb{H}^2$ by such a group Γ.

The definition of the *Euler number* of a representation $\Gamma_g \to \mathrm{Homeo}_+(S^1)$ discussed above can be extended to representations of any orbifold group. As in the case of representations of surface groups, one can form the quotient of $\mathbb{H}^2 \times S^1$ by the diagonal action of Γ; the result is not generally an S^1–bundle, but rather a Seifert fibered space. Seifert fibered spaces have associated Euler numbers; analogous to the circle bundle case, the Euler number can also be thought of as the obstruction to a Γ–invariant section of the projection $\mathbb{H}^2 \times S^1 \to \mathbb{H}^2$. An equivalent definition can be obtained by thinking of the cone points as topological boundary components and using the definition from [3] of Euler number for representations of surfaces with boundary, as was used in our pants decomposition definition in Section 2.1.

Both the orbifold Euler characteristic and the Euler number are multiplicative under covers. If $\Delta \subset \Gamma$ is a finite-index subgroup of index k, then $\chi(\Delta) = k\chi(\Gamma)$ (see eg [9, Theorem 3.1.2]) and for any representation $\rho\colon \Gamma \to \mathrm{Homeo}_+(S^1)$, we also have $\mathrm{eu}(\rho|_\Delta) = k\,\mathrm{eu}(\rho)$. The following observation follows quickly from the definition (most easily from that given in [3]) and was used by Calegari in [4]:

Observation 9 *Let Γ have signature $(g; m_1, \dots, m_r)$, and let $\rho\colon \Gamma \to \mathrm{Homeo}_+(S^1)$ be a representation. Then there exists $m \in \mathbb{Z}$ such that*

$$\mathrm{eu}(\rho) = m + \sum_{i=1}^{r} \mathrm{rot}(\rho(q_i)).$$

As such, there are situations when one can easily guarantee that the Euler number of a representation is nonzero. There are two specific examples of this which we will use in the sequel.

Example 10 The $(0; 2, 2, 2, 2g)$ orbifold group has presentation

$$\langle a, b, c, d \mid a^2, b^2, c^2, d^{2g}, abcd \rangle$$

and has Euler characteristic $(1-g)/(2g)$. If this group acts on the circle by homeomorphisms, and the action of d^2 is nontrivial, then the Euler number of the action is of the form $\frac{1}{2}k + \mathrm{rot}(d)$. As d^2 acts nontrivially and d is of finite order, we conclude that $\mathrm{rot}(d) \notin \left\{\frac{1}{2}, 0\right\}$, so the Euler number of the action is nonzero.

Example 11 The $(0; 3, 3, 4)$ orbifold group has presentation

$$\langle a, b, c \mid a^3, b^3, c^4, abc \rangle$$

and has Euler characteristic $-\frac{1}{12}$. If it acts on the circle by homeomorphisms and the action of c is nontrivial, then the Euler number of the action is of the form $\frac{1}{3}k + \frac{1}{4}m$ for integers k and m with $m \neq 0 \bmod 4$, hence it is nonzero.

We conclude these preliminaries with a final (and well-known) ingredient for our proof.

Lemma 12 *Let* Γ *be an orbifold group of signature* $(g; m_1, \dots, m_r)$, *with the standard presentation given above, and let* $\varphi\colon \Gamma \to G$ *be a surjective homomorphism to a finite group. Suppose that each finite-order generator* q_i *of* Γ *is mapped to an element of* G *of order* m_i. *Then* $\ker(\varphi)$ *is the fundamental group of a compact surface of genus* g *given by the formula* $2-2g = \chi(\Gamma) \times |G|$.

Proof It is a classical standard fact that all finite-order (ie elliptic) elements of Γ are conjugate to some power of one of its finite-order standard generators (see eg [9, Theorem 3.5.2]). Hence, it follows from the assumption that $\ker(\varphi)$ is torsion-free. As Γ is cocompact and $\ker(\varphi)$ is of finite index (of index $|G|$), the group $\ker(\varphi)$ is cocompact as well, hence it is a surface group. The genus calculation follows from multiplicativity of Euler characteristic under covers. □

2.3 Finishing the proof

We now apply the framework above to our situation, finding normal genus g surface subgroups inside of the orbifold groups given in Examples 10 and 11, and use this to conclude our proof.

Consider first the group given in Example 10, which has an (equivalent) presentation $\langle a, b, c \mid a^2, b^2, c^2, (abc)^{2g} \rangle$. Define a surjective homomorphism from this group to the dihedral group $\langle r, s \mid r^{2g}, s^2, srsr \rangle$ of order $4g$ by

$$a \mapsto r^g, \quad b \mapsto sr, \quad c \mapsto sr^{2-g},$$

so $abc \mapsto r$. Since the standard, finite-order generators a, b, c and $d := (abc)^{-1}$ map to elements of their respective orders, Lemma 12 states that the kernel K of this morphism is a torsion-free subgroup of index $4g$, and the Euler characteristic of the regular cover corresponding to the kernel is $4g \cdot (1-g)/(2g) = 2-2g$. Thus, we obtain Σ_g as a regular cover of the $(0; 2, 2, 2, 2g)$ orbifold.

Recall that ρ is assumed to be an action of $\mathrm{Aut}_+(\Gamma_g)$ on S^1. By Observation 8, the $(0; 2, 2, 2, 2g)$ orbifold group embeds in $\mathrm{Aut}_+(\Gamma_g)$, with $\mathrm{Inn}_g \cong \Gamma_g$ agreeing with the kernel K of the homomorphism defined above. It follows from Example 10 and multiplicativity of the Euler number that if $\rho(d)$ has order greater than 2, then the restriction of ρ to K also has nonzero Euler number. Thus, by Lemma 3, the Euler number of the restriction of ρ to this subgroup equals $\pm(2g-2)$, so, by Lemma 7, ρ agrees with the standard action.

Thus, we have proved Theorem 1 under the additional hypothesis that $\rho(d)^2 \neq \mathrm{id}$. To remove this hypothesis, we use recent work of Lanier and Margalit on normally generating mapping class groups.

Theorem 13 (Lanier and Margalit [12]) *Let $g \geq 2$. Then every nontrivial, periodic mapping class that is not a hyperelliptic involution normally generates the commutator subgroup of* Mod_g.

Recall that the abelianization of Mod_g is trivial if $g \geq 3$, and is $\mathbb{Z}/10\mathbb{Z}$ if $g = 2$ (see eg [11]); hence these periodic mapping classes normally generate Mod_g if $g \geq 3$.

The key step of Lanier–Margalit's proof is as follows. Given any such periodic mapping class f, they find simple closed curves α and β that intersect once such that the product of Dehn twists $\tau_\alpha \tau_\beta^{-1}$ lies in the normal closure of f. (See [12, Lemma 2.3].) This step can be carried out in exactly the same way in the group $\mathrm{Mod}_{g,1}$. Since such elements $\tau_\alpha \tau_\beta^{-1}$ also generate the commutator subgroup of $\mathrm{Mod}_{g,1}$, the proof goes through verbatim and the conclusion of Theorem 13 holds in this case as well.

Using this result, we may now quickly conclude our proof in the case of genus $g \geq 3$. In this case, the element d^2 has order $g \geq 3$ in $\mathrm{Mod}_{g,1} \cong \mathrm{Aut}_+(\Gamma_g)$, so is not a hyperelliptic involution. Thus, if $\rho(d)^2$ is trivial, the normal closure of d^2 is in the kernel of ρ as well, so, by Theorem 13, ρ is trivial.

For the case of genus 2, the element d^2 *is* the hyperelliptic involution of Σ_2, so the argument above does not immediately apply. So we work instead with the group $\langle a, b \mid a^3, b^3, (ab)^4 \rangle$ from Example 11, following the same strategy. Define a homomorphism φ from this group to the finite group $\mathrm{SL}_2(\mathbb{F}_3)$ by $\varphi(a) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\varphi(b) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. This morphism is easily seen to be surjective, as $\varphi(a)$ and $\varphi(b)$ are images of standard generators of $\mathrm{SL}_2(\mathbb{Z})$ under the natural map to $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$. Lemma 12 again implies that the kernel of φ is a torsion-free subgroup, hence the fundamental group of a surface. Since the Euler characteristic of the $(0; 3, 3, 4)$ orbifold is $-\frac{1}{12}$ and $|\mathrm{SL}_2(\mathbb{F}_3)| = 24$, this surface has genus 2, so, by Observation 8, we can identify the $(0; 3, 3, 4)$ group with a subgroup of $\mathrm{Aut}_+(\Gamma_2)$.

Example 11 now implies that, if $\rho\colon \mathrm{Aut}_+(\Gamma_2) \to \mathrm{Homeo}_+(S^1)$ is such that the Euler number of the restriction to the $(0; 3, 3, 4)$ group is zero, then $\rho(ab) = \mathrm{id}$. Since ab has order 4, it is not the hyperelliptic involution, so Theorem 13 implies that kernel of ρ contains the commutator subgroup of $\mathrm{Aut}_+(\Gamma_2)$, hence ρ factors through its abelianization, $\mathbb{Z}/10\mathbb{Z}$. This completes the proof. □

3 Concluding remarks

Proof of Corollary 2 Let $\rho\colon \mathrm{Aut}_+(\Sigma_g) \to \mathrm{Diff}^1_+(S^1)$, where $g \geq 3$. By Theorem 1, ρ is either trivial or is semiconjugate to the standard action (up to reversing orientation).

Suppose for contradiction that ρ is a C^1 action that is semiconjugate to the standard action. Let $\gamma \in \Gamma_g$ be an element represented by a separating simple closed curve c on Σ_g, so that one connected component of $\Sigma \smallsetminus c$ has genus $h \geq 2$. Then the stabilizer of γ in $\mathrm{Aut}_+(\Sigma_g)$ contains a copy of Mod_h^1, the mapping class group of the genus h surface with one boundary component. Since the standard boundary action restricted to this subgroup has a global fixed point, and since the property of having a global fixed point is preserved under semiconjugacy, it follows that $\rho(\mathrm{Mod}_h^1)$ acts on S^1 with a fixed point. Let x be a point on the boundary of the fixed set of $\rho(\mathrm{Mod}_h^1)$. Since Mod_h^1 has trivial abelianization (see eg [11]), the linear representation obtained by taking derivatives at x is trivial. However, the *Thurston stability theorem* [21] states that, if G is any finitely generated, nontrivial group of germs of C^1 diffeomorphisms at a point of a manifold, with trivial linear part, then G admits a surjective morphism to $\mathbb{Z}$. It follows that Mod_h^1 (via its image in the group of germs of diffeomorphisms at x) has a nontrivial morphism to $\mathbb{Z}$, contradicting the fact that its abelianization is trivial. □

Remark 14 Our proof of Theorem 1 relied heavily on torsion elements, so does not generalize to finite-index subgroups of $\mathrm{Aut}_+(\Gamma_g)$. However, quotients of finite-index subgroups of $\mathrm{Mod}_{g,1}$ are not well understood, so one does not expect an analogous result to follow along the same lines. For example, it is a longstanding question — or perhaps conjecture — of Ivanov [8] whether all finite-index subgroups of mapping class groups have trivial abelianization. (See [15] for a discussion on the current status of the problem.) There are many nonsemiconjugate actions of $\mathbb{Z}^d$ on the circle for any $d \geq 1$; for example, one may take representations into $\mathrm{SO}(2)$; then their (semi)conjugacy classes are distinguished by rotation numbers. Thus, any subgroup $\Gamma \subset \mathrm{Aut}_+(\Gamma_g)$ with $H^1(\Gamma, \mathbb{Z}) \neq 0$ would have many nontrivial and nonsemiconjugate actions on the circle.

Given the remark above the relevant remaining question is as follows:

Question 15 Is every faithful action of a finite-index subgroup of $\mathrm{Aut}_+(\Gamma_g)$ on S^1 semiconjugate to the standard action?

We hope to address this in future work.

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