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Pluripotential Kähler–Ricci flows

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We develop a parabolic pluripotential theory on compact Kähler manifolds, defining and studying weak solutions to degenerate parabolic complex Monge–Ampère equations. We provide a parabolic analogue of the celebrated Bedford–Taylor theory and apply it to the study of the Kähler–Ricci flow on varieties with log terminal singularities.

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Introduction

The Ricci flow, first introduced by Hamilton [24], is the equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

evolving a Riemannian metric by its Ricci curvature. If the Ricci flow starts from a Kähler metric — the underlying Riemannian manifold being complex Kähler — the evolving metrics remain Kähler and the resulting PDE is called the Kähler–Ricci flow.

After the spectacular use of the Ricci flow by Perelman to settle the Poincaré and geometrization conjectures, it is expected that the Kähler–Ricci flow can be used similarly to give a geometric classification of complex algebraic and Kähler manifolds, and produce canonical metrics at the same time.

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Understanding the existence of canonical Kähler metrics on compact Kähler manifolds has been a central question in the last forty years, following Yau's solution to the Calabi conjecture [40]. The Kähler–Ricci flow provides a canonical deformation process towards such metrics, as shown by the works of many authors (see eg Cao [5], Phong, Song, Sturm and Weinkove [29; 30], Song and Tian [32], Song and Weinkove [34], Song, Székelyhidi and Weinkove [31], Collins and Tosatti [8], Tosatti and Zhang [38], Collins and Székelyhidi [7] and Berman, Boucksom, Eyssidieux, Guedj and Zeriahi [3]).

Writing locally $g_{ij} = \psi_{ij} = \partial_i \overline{\partial}_j \psi$, it is classical that the Kähler–Ricci flow can be reduced to a nonlinear parabolic scalar equation in ψ, of the form

$$\det(\psi_{ij}) = e^{\dot{\psi}_t + H(t,x) + \lambda \psi_t},$$

where H is a smooth density and $\lambda \in \mathbb{R}$ depends on $c_1(X)$.

The classification of complex algebraic manifolds requires to work on singular varieties, as advocated by the minimal model program. Defining the Kähler–Ricci flow on mildly singular projective varieties was undertaken by Song and Tian [33] and requires a theory of weak solutions for degenerate parabolic complex Monge–Ampère equations, where ψ is no longer smooth and H can blow up.

Eyssidieux, Guedj and Zeriahi [14] have developed a parabolic viscosity approach. It applies to the Kähler context, but requires the densities to be continuous. This enabled one to study the behavior of the Kähler–Ricci flow on minimal models with positive Kodaira dimension and canonical singularities; see Eyssidieux, Guedj and Zeriahi [15].

While both the approach of Song and Tian and the viscosity one permit a good understanding of the first singular situations encountered in the minimal model program, one needs to extend these theories in order to treat the fundamental case of Kähler pairs with Kawamata log terminal (klt) singularities. *This is the main objective of the present work.*

From an analytic point of view, klt singularities lead one to deal with densities that may blow up, though belonging to L^p for some exponent $p > 1$ whose size is related to the algebraic nature of the singularities.

We develop in this article a parabolic pluripotential approach to the complex Monge–Ampère flows

$$\text{(CMAF)} \qquad (\omega_t + i\partial\overline{\partial}\varphi_t)^n = e^{\dot{\varphi}_t + F(t,x,\varphi)} g(x)\, dV(x)$$

in $X_T :=]0, T[\times X$, where $T \in]0, +\infty]$ and

- X is a compact Kähler n–dimensional manifold;

- $t \mapsto \omega(t,x)$ is a $\mathcal{C}^2$–family of closed semipositive $(1,1)$–forms such that $\theta(x) \le \omega_t(x)$, where θ is a closed semipositive big form with
$$-A\omega_t \le \dot{\omega}_t \le \frac{A}{t}\omega_t \quad \text{and} \quad \ddot{\omega}_t \le A\omega_t$$
for some fixed constant $A > 0$;
- $(t,x,r) \mapsto F(t,x,r)$ is continuous in $[0,T[\times X \times \mathbb{R}$, quasi-increasing in r, locally uniformly Lipschitz and semiconvex in (t,r);
- $g \in L^p(X,dV)$, $p > 1$, with $g > 0$ almost everywhere;
- $\varphi\colon [0,T[\times X \to \mathbb{R}$ is the unknown function, with $\varphi_t := \varphi(t,\cdot)$.

Here dV is a fixed normalized volume form on X.

We introduce a notion of pluripotential solutions to such equations, a parabolic analogue of the theory developed by Bedford and Taylor in their celebrated articles [1; 2].

We interpret the above parabolic equation on X as a second-order PDE on the $(2n+1)$–dimensional manifold X_T:

- the left-hand side becomes a positive Radon measure $(\omega_t + dd^c\varphi_t)^n \wedge dt$, which is well defined for paths $t \mapsto \varphi_t$ of bounded ω_t–psh functions [2];
- the right-hand side $e^{\dot{\varphi}_t + F(t,x,\varphi)} g(x)\, dV(x) \wedge dt$ is a well-defined Radon measure if $t \mapsto \varphi_t(x)$ is (locally) uniformly Lipschitz.

It is useful in practice to allow the Lipschitz constant to blow up as t approaches zero, so we introduce the corresponding class $\mathcal{P}(X_T,\omega)$ of *parabolic potentials* (see Definition 1.1).

We develop the local side of this theory in [19] by a direct approach, taking advantage of the Euclidean structure of $\mathbb{C}^n$. We approximate here (CMAF) by smooth complex Monge–Ampère flows and establish various a priori estimates to prove our first main result:

Theorem A *Let φ_0 be a bounded ω_0–psh function. There exists a parabolic potential $\varphi \in \mathcal{P}(X_T,\omega)$ such that*

- $(t,x) \mapsto \varphi(t,x)$ *is locally bounded in* $[0,T[\times X$;
- $(t,x) \mapsto \varphi(t,x)$ *is continuous in* $]0,T[\times \mathrm{Amp}(\theta)$;
- $t \mapsto \varphi_t$ *is locally uniformly semiconcave in* $]0,T[\times X$;

- *φ is a pluripotential solution to (CMAF);*
- *$\varphi_t \to \varphi_0$ as $t \to 0^+$ in $L^1(X)$ and pointwise.*

Here $\mathrm{Amp}(\theta)$ denotes the ample locus of θ, ie the largest Zariski open subset of X where the cohomology class of θ behaves like a Kähler class.

It turns out that $t \mapsto \varphi_t(x) - n(t \log t - t) + Ct$ is increasing for some fixed $C > 0$. The convergence at time zero is therefore rather strong (it is eg uniform if φ_0 is continuous).

The semiconcavity information of the solution φ constructed in Theorem A is a crucial tool for approximation purpose (see Theorem 1.14). We show that it is the unique pluripotential solution with such time regularity, by establishing the following comparison principle:

Theorem B *If $\varphi \in \mathcal{P}(X_T, \omega)$ is a bounded pluripotential subsolution to (CMAF) and $\psi \in \mathcal{P}(X_T, \omega)$ is a bounded pluripotential supersolution which is locally uniformly semiconcave in t, then*

$$\varphi_0 \le \psi_0 \implies \varphi \le \psi.$$

In particular, there is a unique bounded pluripotential solution $\Phi(g, F, \omega_t, \varphi_0)$ to (CMAF) which is locally uniformly semiconcave in t.

This comparison principle also allows us to establish the following stability result, which generalizes our [16, Theorem B]:

Theorem C *Assume*

- *(g_j) are densities which converge to g in L^p,*
- *F_j converges to F with uniform constants,*
- *$\omega_{t,j}$ are smooth semipositive forms smoothly converging to ω_t,*
- *$\varphi_{0,j}$ are bounded $\omega_{0,j}$–psh functions converging in $L^1(X, dV)$ to φ_0.*

Then $\Phi(g_j, F_j, \omega_{t,j}, \varphi_{0,j})$ locally uniformly converges to $\Phi(g, F, \omega_t, \varphi_0)$.

It is delicate to compare pluripotential and viscosity concepts in general. We refer the interested reader to [17], where we prove, when g is continuous, that the viscosity solution constructed in [14] coincides with the pluripotential solution $\Phi(g, F, \omega_t, \varphi_0)$.

The present pluripotential approach allows us to deal with noncontinuous data. We can, in particular, define a good notion of weak Kähler–Ricci flow on varieties with terminal singularities (and more generally on klt pairs), as we explain in Section 5, where we prove the following:

Theorem D *Let (Y,ω_0) be a compact n–dimensional Kähler variety with log terminal singularities and trivial first Chern class ($\mathbb{Q}$–Calabi–Yau variety).*

Fix S_0 a positive closed current with bounded potentials, whose cohomology class is Kähler. The Kähler–Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

exists for all times $t>0$, and deforms S_0 towards the unique Ricci flat Kähler–Einstein current ω_{KE} cohomologous to S_0 as $t \to +\infty$.

This extends previous results of Cao [5], Tsuji [39] and Tian and Zhang [35], avoiding any projectivity assumption on X [33], nor any restriction on the type of singularities [14; 15]. We refer the reader to Section 5 for much more general and precise results.

Assumptions on the data and notation

Assumptions on the manifold In the whole article we let X be a compact Kähler n–dimensional manifold. We fix $T \in]0,+\infty]$. Except for Section 5 we are mainly concerned with finite time intervals, ie $T<+\infty$, and we implicitly assume that our data are possibly defined in a slightly larger time interval, ie on $]0,T+\varepsilon[$ for some $\varepsilon>0$.

We let X_T denote the $(2n+1)$–dimensional manifold $X_T =]0,T[\times X$ with parabolic boundary

$$\partial X_T := \{0\}\times X.$$

We fix θ a smooth closed semipositive $(1,1)$–form whose cohomology class is big, ie contains a (singular) positive closed current of bidegree $(1,1)$ which dominates a Kähler form. We let Ω denote the ample locus of θ,

$$\Omega := \mathrm{Amp}(\theta),$$

which is a nonempty Zariski open subset of X.

Assumptions on the forms We assume throughout the article that $(\omega_t)_{t\in[0,T[}$ is a $\mathcal{C}^2$–smooth family of closed semipositive $(1,1)$–forms on X satisfying

$$\theta \le \omega_t$$

for all $t\in[0,T[$. For finite times we can also assume without loss of generality that $\omega_t \le \Theta$ for some Kähler form Θ.

By the end of Section 2 we need to assume that $t \mapsto \omega_t$ moreover satisfies

$$\ddot{\omega}_t \le A\omega_t$$

and

$$-A\omega_t \le \dot{\omega}_t \le A\omega_t \tag{0-1}$$

for some constant $A > 0$. The lower bound in (0-1) is equivalent to the fact that $t \mapsto e^{At}\omega_t$ is increasing. In particular,

$$\omega_{t+s} \ge e^{-As}\omega_t \ge (1 - As)\omega_t, \quad s > 0.$$

The latter will be used on several occasions in the sequel.

Assumptions on the densities We assume throughout the article that

- dV is a fixed volume form on X;
- $0 \le g \in L^p(X, dV)$ for some $p > 1$, and $\mathrm{Vol}(\{g = 0\}) = 0$;
- $(t, x, r) \mapsto F(t, x, r)$ is a continuous function on $[0, T[\times X \times \mathbb{R}$;
- $r \mapsto F(\cdot, \cdot, r)$ is uniformly quasi-increasing, ie there exists a constant $\lambda_F \ge 0$ such that for every $(t, x) \in [0, T[\times X$, the function

$$r \mapsto F(t, x, r) + \lambda_F r \quad \text{is increasing in } \mathbb{R}; \tag{0-2}$$

- $(t, r) \mapsto F(t, \cdot, r)$ is locally uniformly Lipschitz, ie for all $J \Subset [0, T[\times \mathbb{R}$ there is $\kappa_J > 0$ such that for every $x \in X$, (t, r), $(t', r') \in J$,

$$|F(t, x, r) - F(t', x, r')| \le \kappa_J(|t - t'| + |r - r'|); \tag{0-3}$$

- $(t, r) \mapsto F(t, x, r)$ is locally uniformly semiconvex, ie for every compact $J \Subset [0, T[\times \mathbb{R}$ there exists $C_J > 0$ such that for every $x \in X$,

$$(t, r) \mapsto F(t, x, r) + C_J(t^2 + r^2) \quad \text{is convex in } J. \tag{0-4}$$

Note that if F is $\mathcal{C}^2$–smooth then the local conditions (0-3) and (0-4) are automatically satisfied, while (0-2) is a global assumption.

Invariance properties of the set of assumptions We check in Section 5.1.2 that the above conditions are satisfied for the parabolic equations that describe the evolution of the normalized (as well as the nonnormalized) Kähler–Ricci flow on a mildly singular Kähler variety.

The family of parabolic complex Monge–Ampère equations we consider enjoys several useful invariance properties. We refer the reader to Section 3.3 for more details.

Organization of the paper

We describe the class of potentials we are using in Section 1.1 and define parabolic complex Monge–Ampère operators in Section 1.2. We establish fundamental a priori estimates in Section 2, which are then used to prove Theorem A in Section 3. We study uniqueness and stability of pluripotential solutions in Section 4, establishing Theorems B and C. In Section 5 we use these tools to study the long-term behavior of the normalized Kähler–Ricci flow on varieties with log terminal singularities and nonnegative Kodaira dimension, proving Theorem D and several other convergence results.

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1 Parabolic potentials and Monge–Ampère operators

1.1 Families of quasiplurisubharmonic functions

1.1.1 Compactness properties Recall that a function $u\colon X \to [-\infty,+\infty[$ is ω_t–plurisubharmonic (ω_t–psh for short) if it is locally given as the sum of a smooth and a plurisubharmonic function and the current

$$\omega_t + dd^c u \geq 0$$

is positive on X. Here $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$ are both real operators.

Definition 1.1 The set of parabolic potentials $\mathcal{P}(X_T,\omega)$ is the set of functions $\varphi\colon\,]0,T[\, \times X \to [-\infty,+\infty[$ such that

- $x \mapsto \varphi(t,x)$ is ω_t–plurisubharmonic on X for all $t \in\,]0,T[$,
- φ is locally uniformly Lipschitz in $]0,T[$.

The last condition means that for any compact subset $J \subset\,]0,T[$ there exists $\kappa = \kappa_J(\varphi) > 0$ such that

$$\varphi(t,x) \leq \varphi(s,x) + \kappa|t-s| \quad \text{for all } s,t \in J \text{ and } x \in X. \tag{1-1}$$

We say that a family $\Phi \subset \mathcal{P}(X_T, \omega)$ is locally uniformly Lipschitz in $]0, T[$ if the inequality (1-1) is satisfied for all $\varphi \in \Phi$ with a uniform constant $\kappa = \kappa(J, \Phi) > 0$ which only depends on J and Φ.

A parabolic potential $\varphi \in \mathcal{P}(X_T, \omega)$ can be extended as an upper semicontinuous function on $[0, T[\times X$ with ω_t–psh slices.

Proposition 1.2 *Assume φ_0 is ω_0–psh and $\varphi \in \mathcal{P}(X_T, \omega)$ satisfies $\varphi_t \to \varphi_0$ in L^1 as $t \to 0$. Then the extension $\varphi\colon [0, T[\times X \to [-\infty, +\infty[$ is upper semicontinuous.*

Proof It is classical that for all $x \in X$, $\varphi_0(x) = \limsup_{y \to x} \limsup_{t \to 0} \varphi_t(y)$. It therefore suffices to prove the following more general result: Assume that $u \in \mathcal{P}(X_T, \omega)$ is bounded from above near $t = 0$ and define

$$u_0(x) := \limsup_{y \to x} \Bigl(\limsup_{t \to 0} u_t(y)\Bigr).$$

Then the extension $u\colon [0, T[\times X \to [-\infty, +\infty[$ is upper semicontinuous.

The upper semicontinuity inside X_T follows from the semicontinuity in space and Lipschitz regularity in time. Assume that (t_j, x_j) is a sequence in X_T converging to $(0, x_0)$ with $x_0 \in X$. We want to prove that

$$\limsup_j u(t_j, x_j) \le u_0(x_0).$$

Since the problem is local, we can assume that the functions u_{t_j} are psh and negative in a neighborhood $B \subset \mathbb{C}^n$ of x_0. Fix $r > 0$ such that $B(x_0, 2r) \subset B$. Fix $\delta \in]0, r[$. For j large enough, $x_j \in B(x_0, \delta)$, hence $B(x_0, r) \subset B(x_j, r + \delta)$ and, since $u_{t_j} \le 0$ in B, we have

$$\begin{aligned} u(t_j, x_j) &\le \frac{1}{\mathrm{Vol}(B(x_j, r+\delta))} \int_{B(x_j, r+\delta)} u(t_j, x)\, dV(x) \\ &\le \frac{1}{\mathrm{Vol}(B(x_j, r+\delta))} \int_{B(x_0, r)} u(t_j, x)\, dV(x) \\ &= \frac{\mathrm{Vol}(B(x_0, r))}{\mathrm{Vol}(B(x_j, r+\delta))} \frac{1}{\mathrm{Vol}(B(x_0, r))} \int_{B(x_0, r)} u(t_j, x)\, dV(x). \end{aligned}$$

Since $\limsup_j u_{t_j}(x) \le u_0(x)$ for all $x \in X$, letting $j \to +\infty$ we obtain

$$\limsup_j u(t_j, x_j) \le \frac{1}{\mathrm{Vol}(B(x_0, r+\delta))} \int_{B(x_0, r)} u_0(x)\, dV(x).$$

Now, we first let $\delta \to 0$ and then $r \to 0$ to obtain the result since u_0 is psh. □

We next prove a compactness result for this class of functions.

Theorem 1.3 *Let $(\varphi_j) \subset \mathcal{P}(X_T, \omega)$ be a sequence which*

- *is locally uniformly bounded from above in X_T;*
- *is locally uniformly Lipschitz in $]0, T[$;*
- *does not converge locally uniformly to $-\infty$ in X_T.*

Then (φ_j) is bounded in $L^1_{\mathrm{loc}}(X_T)$ and there exists a subsequence which converges to some function $\varphi \in \mathcal{P}(X_T)$ in the $L^1_{\mathrm{loc}}(X_T)$–topology.

If (φ_j) converges weakly (in the sense of distributions) to φ in X_T, then it converges in $L^p_{\mathrm{loc}}(X_T)$ for all $p \geq 1$.

The classes L^p are here defined with respect to the $(2n+1)$–dimensional Lebesgue measure associated to a fixed volume form $dt \wedge dV$. For convenience we normalize dV so that $\int_X dV = 1$.

Proof The proof of this result is local in nature and follows closely the classical proof of the analogous result for quasiplurisubharmonic functions, once we have a substitute for the submean value inequality.

We can thus assume here that $X = \Omega \subset \mathbb{C}^n$ is a bounded strictly pseudoconvex domain. The Poincaré lemma ensures that $\omega_t = dd^c \rho_t$ for a family of plurisubharmonic functions ρ_t which is Lipschitz in t. Changing φ_t in $\varphi_t + \rho_t$, we reduce further to the case when $\omega_t = 0$. The corresponding compactness and convergence properties have then been obtained in [19]. □

Corollary 1.4 *The class $\mathcal{P}(X_T, \omega)$ is a subset of $L^p_{\mathrm{loc}}(X_T)$ for all $1 \leq p$, and the inclusions $\mathcal{P}(X_T, \omega) \hookrightarrow L^p_{\mathrm{loc}}(X_T)$ are continuous.*

The topologies induced by the classes L^p are thus all equivalent when restricted to the class $\mathcal{P}(X_T, \omega)$.

1.1.2 Slices and time derivatives

We now estimate the L^1–norm on slices.

Lemma 1.5 *Fix $u, v \in \mathcal{P}(X_T, \omega)$ and $0 < T_0 < T_1 < T$. Then*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(X)} \leq 2M \max\{\|u-v\|^{1/2}_{L^1(X_T)}, \|u-v\|_{L^1(X_T)}\}$$

for all $T_0 \leq t \leq T_1$, where $M := \max\{\sqrt{\kappa}, (T-T_1)^{-1}\}$ and κ is the uniform Lipschitz constant of $u - v$ in $[T_0, T]$.

This lemma expresses in a quantitative way the following fact: for functions in $\mathcal{P}(X_T,\omega)$, the convergence in $L^1(X_T)$ implies the local uniform convergence of their slices in $L^1(X)$: if $(u_j)\subset\mathcal{P}(X_T,\omega)$ converges to u in $L^1(X_T)$ and is locally uniformly Lipschitz in $]0,T[$, then $u_j(t,\cdot)$ converges to $u(t,\cdot)$ in $L^1(X)$ for each slice t.

Proof The proof is identical to the corresponding one in the local context; we refer the reader to [19]. □

Fix μ a (finite) Borel measure on X, and let ℓ denote the Lebesgue measure on $\mathbb{R}^+$.

Lemma 1.6 *Fix $\varphi\in\mathcal{P}(X_T,\omega)$. Then $\partial_t\varphi(t,x)$ exists for all $(t,x)\notin E$, where $E\subset X_T$ is $\ell\otimes\mu$–negligible.*

In particular, $\partial_t\varphi\in L^\infty_{\mathrm{loc}}(X_T)$ and $h(\partial_t\varphi)\ell\otimes\mu$ is a well-defined Borel measure on X_T for any continuous function $h\in\mathcal{C}^0(\mathbb{R},\mathbb{R})$.

Proof The proof is identical to the corresponding one in the local context; we refer the reader to [19]. □

When φ is semiconvex or semiconcave in t, we can improve this result.

Definition 1.7 We say that $\varphi\colon X_T\to\mathbb{R}$ is uniformly semiconcave in $]0,T[$ if for any compact $J\Subset\,]0,T[$, there exists $\kappa=\kappa(J,\varphi)>0$ such that for all $x\in X$, the function $t\mapsto\varphi(t,x)-\kappa t^2$ is concave in J.

The definition of uniformly semiconvex functions is analogous. Note that such functions are automatically locally uniformly Lipschitz.

Lemma 1.8 *Let $\varphi\colon X_T\to\mathbb{R}$ be a continuous function which is uniformly semiconvex in $]0,T[$. Then*

$$\partial_t^+\varphi(t,x)=\lim_{s\to 0^+}\frac{\varphi(t+s,x)-\varphi(t,x)}{s}$$

is upper semicontinuous in X_T, while

$$\partial_t^-\varphi(t,x):=\lim_{s\to 0^-}\frac{\varphi(t+s,x)-\varphi(t,x)}{s}$$

is lower semicontinuous in X_T. In particular, $\partial_t^+\varphi$ and $\partial_t^-\varphi$ coincide and are continuous $\ell\otimes\mu$–almost everywhere in X_T.

Proof The proof is identical to the corresponding one in the local context; we refer the reader to [19]. □

1.1.3 Topology on $\mathcal{P}(X_T,\omega)$ We introduce a natural complete metrizable topology on the convex set $\mathcal{P}(X_T,\omega)$.

We first consider a partial Sobolev space $W^{(1,0),\infty}_{\mathrm{loc}}(X_T)$: this is the set of functions $u \in L^1_{\mathrm{loc}}(X_T)$ whose partial time derivative (in the sense of distribution) satisfies $\dot{u} = \partial_t u \in L^\infty_{\mathrm{loc}}(X_T)$. It follows from Lemma 1.6 that

$$\mathcal{P}(X_T,\omega) \subset W^{(1,0),\infty}_{\mathrm{loc}}(X_T).$$

The local uniform Lipschitz constant of $\varphi \in \mathcal{P}(X_T,\omega)$ on a compact subset $J \Subset]0,T[$ is given by

$$\sup_{t,s\in J,\, s\neq t}\ \sup_{x\in X}{}^{*} \frac{|\varphi(s,x)-\varphi(t,x)|}{|s-t|} = \|\dot{\varphi}\|_{L^\infty(J\times X)},$$

where $\sup^*$ is the essential sup with respect to a volume form dV on X.

We can therefore consider the following seminorms on $W^{(1,0),\infty}_{\mathrm{loc}}(X_T)$: given a compact subset $J \Subset]0,T[$ and $u \in W^{(1,0),\infty}_{\mathrm{loc}}(X_T)$, we set

$$\rho_J(u) := \|\dot{u}\|_{L^\infty(J\times X)} + \int_J \int_X |u(t,x)|\, dV(x)\, dt.$$

1.2 Parabolic complex Monge–Ampère operators

As explained in the introduction, we assume in this section (without loss of generality) that $\theta \le \omega_t \le \Theta$, where θ is a semipositive and big $(1,1)$–form and Θ is a Kähler form.

1.2.1 Parabolic Chern–Levine–Nirenberg inequalities We assume here that $\varphi \in \mathcal{P}(X_T,\omega) \cap L^\infty_{\mathrm{loc}}(X_T)$. For all $t \in]0,T[$, the function

$$X \to \mathbb{R} \quad x \mapsto \varphi_t(x) = \varphi(t,x),$$

is ω_t–psh and bounded, hence $(\omega_t + dd^c\varphi_t)^n$ is well defined as a positive Borel measure on X, as follows from the works of Bedford and Taylor [1; 2].

Since $0 \le \omega_t \le \Theta$ for $0 \le t \le T$, the positive Borel measures $(\omega_t + dd^c\varphi_t)^n$ have uniformly bounded masses on X,

$$\int_X (\omega_t + dd^c\varphi_t)^n \le \int_X (\Theta + dd^c\varphi_t)^n \le \int_X \Theta^n.$$

These can be considered, alternatively, as a family of currents of degree $2n$ on the real $(2n+1)$–dimensional manifold $X_T =]0,T[\times X$. It follows from Bedford and Taylor's convergence theorem [1; 2] that $t \mapsto (\omega_t + dd^c\varphi_t)^n$ is continuous as a map from $]0,T[$ to the space $\mathcal{M}(X)$ of positive Radon measures on X endowed with the weak*–topology. More generally we have:

Lemma 1.9 *Fix $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\mathrm{loc}}(X_T)$ and χ a continuous test function in X_T. The function $t \mapsto \int_X \chi(t,\cdot)(\omega_t + dd^c\varphi_t)^n$ is continuous in $]0,T[$ and bounded, with*

$$\sup_{0<t<T} \left| \int_X \chi(t,\cdot)(\omega_t + dd^c\varphi_t)^n \right| \le \big(\max_{X_T} |\chi|\big) \int_X \Theta^n.$$

More generally, if $\chi \in L^\infty(X_T)$ is upper semicontinuous (resp. lower semicontinuous) on X_T with compact support, then so is the function $t \mapsto \int_X \chi(t,\cdot)(\omega_t + dd^c\varphi_t)^n$.

Proof Fix a continuous test function χ on X_T and fix a compact interval $J \Subset]0,T[$ such that $J \times X$ contains the support of χ.

Fix $t_0 \in]0,T[$. The Lipschitz property of φ ensures that φ_t uniformly converges on X to φ_{t_0} as $t \to t_0$. The continuity of $t \mapsto \omega_t$ and Bedford and Taylor's continuity theorem then ensure that $(\omega_t + dd^c\varphi_t)^n$ converges to $(\omega_{t_0} + dd^c\varphi_{t_0})^n$ as $t \to t_0$. Since χ_t uniformly converges on X to χ_{t_0}, the first statement follows. The second statement follows from the fact that $\int_X (\omega_t + dd^c\varphi_t)^n \le \int_X \Theta^n$ for all $t \in]0,T[$.

If χ is merely upper/lower semicontinuous with compact support, there is a sequence of continuous test functions χ_j which decreases (resp. increases) to χ. By the monotone convergence theorem we have that, for all $t \in]0,T[$,

$$\lim_j \int_X \chi_j(t,\cdot)(\omega_t + dd^c\varphi_t)^n = \int_X \chi(t,\cdot)(\omega_t + dd^c\varphi_t)^n.$$

By the previous result the functions $t \mapsto \int_X \chi_j(t,\cdot)(\omega_t + dd^c\varphi_t)^n$ are continuous in $]0,T[$. Therefore, the limit is upper semicontinuous (resp. lower semicontinuous) in $]0,T[$. □

Definition 1.10 Let $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\mathrm{loc}}(X_T)$. The map

$$\chi \mapsto \int_{X_T} \chi \, dt \wedge (\omega_t + dd^c\varphi_t)^n := \int_0^T dt \left(\int_X \chi(t,\cdot)(\omega_t + dd^c\varphi_t)^n \right) \tag{1-2}$$

defines a $(2n+1)$–current on X_T, denoted by $dt \wedge (\omega_t + dd^c\varphi_t)^n$, which can be identified with a positive Radon measure on X_T.

That (1-2) is well defined for continuous test (or Borel) functions χ follows from Lemma 1.9. The operator can also be defined by approximation in the spirit of Bedford and Taylor's convergence results [1; 2]:

Proposition 1.11 *Fix $\varphi \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\mathrm{loc}}(X_T)$ and let φ^j be a monotone sequence of functions (φ^j) in $\mathcal{P}(X_T, \omega) \cap L^\infty_{\mathrm{loc}}(X_T)$ converging to φ almost everywhere in X_T.*

Then

$$dt \wedge (\omega_t + dd^c \varphi_t^j)^n \to dt \wedge (\omega_t + dd^c \varphi_t)^n,$$

in the sense of measures on X_T*.*

Proof Let χ be a continuous test function in X_T. By definition, for all j, we have

$$\int_{X_T} \chi\, dt \wedge \mathrm{MA}(\varphi^j) := \int_0^T dt \left(\int_X \chi(t,\cdot)\, \mathrm{MA}(\varphi_t^j) \right).$$

We can apply Bedford and Taylor's convergence theorems [2] to conclude that, for all $t \in]0,T[$,

$$\int_X \chi(t,\cdot)(\omega_t + dd^c \varphi_t^j)^n \to \int_X \chi(t,\cdot)(\omega_t + dd^c \varphi_t)^n.$$

Since $\int_X \chi(t,\cdot)(\omega_t + dd^c \varphi_t^j)^n$ is uniformly bounded (Lemma 1.9), the conclusion follows from the Lebesgue convergence theorem. □

It is classical that one can then define similarly mixed parabolic Monge–Ampère operators

$$dt \wedge (\omega_t + dd^c \varphi_t^1) \wedge \cdots \wedge (\omega_t + dd^c \varphi_t^n)$$

whenever $\varphi^1, \dots, \varphi^n \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\mathrm{loc}}(X_T)$. We note, for later use, the following stronger version of the Chern–Levine–Nirenberg inequalities:

Proposition 1.12 *Assume* $\varphi^1, \dots, \varphi^n \in \mathcal{P}(X_T, \omega) \cap L^\infty_{\mathrm{loc}}(X_T)$ *and* $\psi \in \mathcal{P}(X_T, \omega)$*. Then, for all* $J \Subset]0,T[$,

$$\int_{J \times X} |\psi|\, dt \wedge (\omega_t + dd^c \varphi_t^1) \wedge \cdots \wedge (\omega_t + dd^c \varphi_t^n) \le \mathrm{Vol}(\Theta) \int_J \left(|\sup_X \psi_t| + \sum_{j=1}^n \mathrm{osc}(\varphi_t^j) \right) dt + \int_{J \times X} |\psi|\, dt \wedge \Theta^n.$$

In particular, $\psi \in L^1_{\mathrm{loc}}(X_T, dt \wedge (\omega_t + dd^c \varphi_t^1) \wedge \cdots \wedge (\omega_t + dd^c \varphi_t^n))$*.*

Proof Fix $\psi \in \mathcal{P}(X_T, \omega)$ and $J \Subset]0,T[$. Setting $\psi_t = \tilde{\psi}_t + \sup_X \psi_t$ and using the triangle inequality, we can write

$$\int_{J \times X} |\psi|\, dt \wedge (\omega_t + dd^c \varphi_t^1) \wedge \cdots \wedge (\omega_t + dd^c \varphi_t^n) \le \int_{J \times X} |\tilde{\psi}|\, dt \wedge (\omega_t + dd^c \varphi_t^1) \wedge \cdots \wedge (\omega_t + dd^c \varphi_t^n) + \mathrm{Vol}(\Theta) \int_J |\sup_X \psi_t|\, dt.$$

We can thus assume that $\sup_X \psi_t = 0$ for all $t \in J$. A series of integration by parts as in [20, Corollary 3.3] yields

$$\int_X |\psi_t|(\omega_t + dd^c\varphi_t^1) \wedge \cdots \wedge (\omega_t + dd^c\varphi_t^n) \le \sum_{j=1}^n \operatorname{osc}_X(\varphi_t^j)\operatorname{Vol}(\omega_t) + \int_X |\psi_t|\omega_t^n$$
$$\le \operatorname{Vol}(\Theta)\sum_{j=1}^n \operatorname{osc}_X(\varphi_t^j) + \int_X |\psi_t|\Theta^n.$$

Integrating on J yields the desired estimate. □

1.2.2 Convergence results

Definition 1.13 A family $\Phi \subset \mathcal{P}(X_T,\omega)$ is uniformly semiconcave in $]0,T[$ if, for any compact subset $J \Subset]0,T[$, there exists a constant $\kappa = \kappa(J,\Phi) > 0$ such that any $\varphi \in \Phi$ is uniformly κ–concave in J.

Fix μ a Borel measure on X, and let ℓ denote the Lebesgue measure on $\mathbb{R}$.

Theorem 1.14 *Let (f_j) be a sequence of positive functions which converge to f in $L^1(X_T,\ell\otimes\mu)$. Let (φ^j) be a sequence of functions in $\mathcal{P}(X_T,\omega)$ which*

- *converge $\ell\otimes\mu$–almost everywhere in X_T to a function $\varphi \in \mathcal{P}(X_T,\omega)$;*
- *is uniformly semiconcave in $]0,T[$.*

Then $\lim_{j\to+\infty}\dot\varphi^j(t,x) = \dot\varphi(t,x)$ for $\ell\otimes\mu$–almost any $(t,x) \in X_T$, and

$$h(\dot\varphi^j) f_j \ell\otimes\mu \to h(\dot\varphi) f \ell\otimes\mu,$$

in the weak sense of Radon measures on X_T, for all $h \in \mathcal{C}^0(\mathbb{R},\mathbb{R})$.

Proof The proof is identical to the corresponding one in the local context; see [19, Proposition 2.9]. For the reader's convenience we sketch the argument here. Fixing compact subintervals $J \Subset J' \Subset]0,T[$, there exists a constant C such that the functions $u^j(t,x) := \varphi_j(t,x) - Ct^2$ are concave in J' for all $x \in X$ fixed. The t–concavity of u^j on J' ensures that $\dot u^j(t,x)$ is uniformly bounded in $J\times X$ and $\dot\varphi^j(t,x)$ converges pointwise a.e. to $\dot\varphi(t,x)$ in $J\times X$.

To conclude, it suffices to prove that $h(\dot\varphi_j)f_j$ converges to $h(\dot\varphi)f$ in $L^1(J\times X,\ell\otimes\mu)$. Assume by contradiction that there exist a subsequence of (f_j), still denoted by (f_j), and a constant $a > 0$ such that

$$\|h(\dot\varphi_j)f_j - h(\dot\varphi)f\|_{L^1(J\times X,\ell\otimes\mu)} \ge a, \quad j \ge 1. \tag{1-3}$$

Extracting (f_j) again we can assume that $\|f_j - f_{j+1}\|_{L^1(J\times X,\ell\otimes\mu)} \le 2^{-j}$. Setting $g := f + f_1 + \sum_{j=1}^{+\infty} |f_{j+1} - f_j|$, we have that

$$\max(f_j, f) \le g \in L^1(J \times X, \ell \otimes \mu), \quad j \ge 1.$$

It therefore follows from Lebesgue's theorem that $h(\dot{\varphi}_j) f_j$ converges to $h(\dot{\varphi}) f$ in $L^1(J \times X, \ell \otimes \mu)$, contradicting (1-3). This completes the proof. □

2 A priori estimates

In this section we assume that $\varphi(t,x) = \varphi_t(x)$ is a smooth ω_t–psh solution to (CMAF), where $t \mapsto \omega_t$ is a smooth family of Kähler forms, and F and g are smooth with g positive. Our aim is to establish various a priori estimates that will allow us to construct weak solutions to the corresponding degenerate equations.

For convenience we will also assume that θ is Kähler, and φ_0 is smooth and strictly ω_0–psh. It follows however from [12; 23; 36] that all the a priori bounds below remain valid when θ is semipositive and big, and φ_0 is merely ω_0–psh and bounded.

We will make various extra assumptions, depending on the a priori estimates that we are interested in.

2.1 Controlling the oscillation of φ_t

Recall that $\theta(x) \le \omega_t(x) \le \Theta(x)$, where θ and Θ are Kähler forms. We let V_1 (resp. V_2) denote the volume of $\{\theta\}$ (resp. $\{\Theta\}$),

$$V_1 = \int_X \theta^n \quad \text{and} \quad V_2 = \int_X \Theta^n.$$

We fix $c_1, c_2 \in \mathbb{R}$ normalizing constants such that $V_i = e^{c_i} \mu(X)$, where $\mu = g\,dV$. It follows from [27; 12] that there exists ρ_1 a bounded θ–psh function (respectively ρ_2 a bounded Θ–psh function) such that

$$(\theta + dd^c \rho_1)^n = e^{c_1} \mu \quad \text{and} \quad (\Theta + dd^c \rho_2)^n = e^{c_2} \mu.$$

The functions ρ_1 and ρ_2 are moreover unique once normalized by

$$\sup_X \rho_1 = \inf_X \rho_2 = 0.$$

Note that in proving existence of solutions the form θ will no longer be Kähler but merely semipositive and big. But the L^∞ bound on ρ_1 remains uniform thanks to [12, Proposition 2.6].

Proposition 2.1 *The following uniform a priori bound on φ_t holds:*

$$|\varphi_t(x)| \le C_0 := C\left(e^{\lambda_F T} + \frac{e^{\lambda_F T}-1}{\lambda_F}\right),$$

where C is the uniform constant

$$C = \sup_{X_T} |F(t,x,0)| + (\lambda_F+1)\sup_X(|\rho_1|+|\rho_2|) + \sup_X |\varphi_0| + \max(-c_1, c_2).$$

Recall that $\lambda_F \ge 0$ is a constant such that, for all $(t,x) \in X_T$, the function $r \mapsto F(t,x,r) + \lambda_F r$ is increasing on $\mathbb{R}$.

Proof Set, for $t \in \mathbb{R}$,

$$\gamma(t) := \sup_X |\varphi_0| e^{\lambda_F t} + \frac{C(e^{\lambda_F t}-1)}{\lambda_F},$$

where C is as in the statement of the proposition. A direct computation shows that $\gamma(0) = \sup_X |\varphi_0|$ and $\gamma'(t) - \lambda_F\gamma(t) = C$.

Set $u(t,x) := \rho_1(x) - \gamma(t)$ for $(t,x) \in X_T$. Observe that u_t is θ–psh, hence ω_t–psh, that $u_0 \le \varphi_0$ and that

$$(\omega_t + dd^c u_t)^n \ge (\theta + dd^c\rho_1)^n = e^{c_1}\mu \ge e^{\dot{u}_t + F(t,x,u_t)}\mu.$$

The last inequality follows from our choice of C: since $r \mapsto F(\cdot,\cdot,r) + \lambda_F r$ is increasing, and $\rho_1 \le 0$ and $u_t \le 0$, we obtain

$$\begin{aligned} F(t,x,u_t(x)) + \dot{u}_t &= F(t,x,u_t(x)) - \gamma'(t) \\ &\le F(t,x,0) - \lambda_F(\rho_1 - \gamma(t)) - \gamma'(t) \\ &\le F(t,x,0) + \lambda_F|\rho_1| - C \\ &\le c_1. \end{aligned}$$

It thus follows from the maximum principle that $\varphi \ge u$ on X_T.

Set now $v(t,x) = \rho_2(x) + \gamma(t)$, $(t,x) \in X_T$. We let the reader check similarly that v_t is Θ–psh, it satisfies

$$(\Theta + dd^c v_t)^n \le e^{\partial_t v(t,x) + F(t,x,v_t)} g\, dV,$$

and $v_0 \ge \varphi_0$. Now φ_t is a subsolution to this new parabolic equation since $\omega_t \le \Theta$. It follows therefore from the maximum principle that $\varphi \le v$ on X_T, and the desired estimates follow. □

The following construction of the subbarrier will be useful in showing that the pluripotential solution to (CMAF) has the right value at $t=0$:

Proposition 2.2 *For all* $0 \le t \le 1$,

$$\varphi_t \ge (1-t)e^{-At}\varphi_0 + t\rho_1 + n(t\log t - t) - C\frac{e^{\lambda_F t}-1}{\lambda_F},$$

where C *is the uniform constant*

$$C := \sup_{X_T} F(t,x,0) + (A+\lambda_F+1)\big(\sup_X|\varphi_0| + \sup_X|\rho_1| + n\big) - c_1.$$

Proof Recall that A denotes a positive constant such that $\dot{\omega}_t \ge -A\omega_t$ for all $t \in]0,T[$. In particular, $\omega_t \ge e^{-At}\omega_0$ and $\omega_t \ge \theta$. Recall also that $\lambda_F \ge 0$ is a constant such that $r \mapsto F(t,x,r) + \lambda_F r$ is increasing in $\mathbb{R}$ for all $(t,x) \in X_T$.

Consider the function

$$u_t(x) := (1-t)e^{-At}\varphi_0 + t\rho_1 + n(t\log t - t) - C\frac{e^{\lambda_F t}-1}{\lambda_F},$$

where C is the uniform constant defined in the proposition.

Using that $\omega_t \ge e^{-At}\omega_0$ and $\omega_t \ge \theta$, we have

$$\begin{aligned}(\omega_t + dd^c u_t)^n &= ((1-t)\omega_t + (1-t)e^{-At}dd^c\varphi_0 + t(\omega_t + dd^c\rho_1))^n \\ &\ge t^n(\omega_t + dd^c\rho_1)^n \\ &\ge t^n e^{c_1} g\,dV.\end{aligned}$$

Since $u_t \le 0$ and $r \mapsto F(t,x,r) + \lambda_F r$ is increasing, a direct computation yields

$$\begin{aligned}&\dot{u}_t + F(t,x,u_t) \\ &= n\log t + \rho_1 + e^{-At}(A(1-t)+1)(-\varphi_0) - Ce^{\lambda_F t} + F(t,x,u_t) + \lambda_F u_t - \lambda_F u_t \\ &\le n\log t + (A+1)\sup_X|\varphi_0| + \sup_{X_T} F(t,x,0) + \lambda_F\big(\sup_X|\varphi_0| + \sup_X|\rho_1| + n\big) - C \\ &\le n\log t + c_1.\end{aligned}$$

It thus follows that u_t is a subsolution to (CMAF) with $u_0 \le \varphi_0$. The desired estimate follows from the classical maximum principle. □

2.2 Controlling the average

We establish the following control on the average of φ_t, which will be useful in proving convergence at zero.

Proposition 2.3 *Set $\mu = g\,dV$. The following bound holds*:

$$\int_X \varphi_t \, d\mu \le \int_X \varphi_0 \, d\mu + Ct,$$

where C_3 is the uniform constant

$$C := -\mu(X)\log(\mu(X)/V_2) - \inf_{X_T\times[-C_0,C_0]} F(t,x,r)\mu(X)$$

and C_0 is the uniform constant defined in Proposition 2.1.

Proof Set $-C' := \inf_{X_T\times[-C_0,C_0]} F(t,x,r) > -\infty$. It follows from the flow equation that

$$\int_X e^{\dot\varphi_t - C'} \, d\mu \le \int_X \omega_t^n \le V_2.$$

On the other hand, it follows from Jensen's inequality that

$$\int_X e^{\dot\varphi_t} \frac{d\mu}{\mu(X)} \ge \exp\biggl(\int_X \dot\varphi_t \frac{d\mu}{\mu(X)}\biggr).$$

Combining these two estimates we arrive at

$$\int_X \dot\varphi_t g \, dV \le C := C'\mu(X) + \mu(X)\log V_2 - \mu(X)\log\mu(X).$$

The function $t \mapsto \int_X \varphi_t \, d\mu - Ct$ is therefore nonincreasing, hence

$$\int_X \varphi_t \, d\mu \le \int_X \varphi_0 \, d\mu + Ct. \qquad \square$$

2.3 Lipschitz control in time

We now establish an a priori bound which will allow us to show that the solutions φ_t to degenerate complex Monge–Ampère flows are locally uniformly Lipschitz in time, away from zero.

For the convenience of the reader we first state and prove our theorem in the simpler case when $t \mapsto \omega_t$ is affine and $r \mapsto F(t,x,r)$ is increasing. A more technical statement follows, together with its proof.

Recall that we assume here our data are smooth ($g > 0$ is smooth, F is smooth, θ is Kähler, and φ_0 is smooth and strictly ω_0–psh). We will explain in Theorem 3.4 below how to reduce to this case.

2.3.1 Affine dependence on time

Theorem 2.4 *Assume $t \mapsto \omega_t = \omega_0 + t\chi$ is affine and $r \mapsto F(\cdot,\cdot,r)$ is increasing. Then, for all $(t,x) \in X_T$,*

$$n\log t - C \le \dot{\varphi}_t(x) \le \frac{C}{t},$$

where C depends explicitly on T, $\|\partial F/\partial r\|_{L^\infty}$, $\|\partial F/\partial t\|_{L^\infty}$, $\|g\|_p$ and C_0.

Here and below C_0 denotes the constant from Proposition 2.1, and the Lipschitz constants $\|\partial F/\partial r\|_{L^\infty}, \|\partial F/\partial t\|_{L^\infty}$ are computed on $X_T \times [-C_0, C_0]$.

Proof For notational convenience we set $\mu = g\,dV$. We first establish the bound from above. Consider

$$H(t,x) = t\dot{\varphi}_t(x) - (\varphi_t - \varphi_0) - Bt,$$

where

$$B = n + 1 - \inf_{X_T\times[-C_0,C_0]}\Big[t\frac{\partial F}{\partial t}(t,x,r)\Big].$$

Set $S_t := \omega_t + dd^c\varphi_t$ and observe that

$$\frac{\partial H}{\partial t} = t\ddot{\varphi}_t - B,$$

with

$$\dot{\varphi}_t = \log(S_t^n/\mu) - F(t,x,\varphi_t),$$

hence

$$\ddot{\varphi}_t = \Delta_{S_t}(\dot{\varphi}_t) + \mathrm{Tr}_{S_t}(\dot{\omega}_t) - \frac{\partial F}{\partial t}(x,t,\varphi_t) - \dot{\varphi}_t\frac{\partial F}{\partial r}(x,t,\varphi_t),$$

where

$$\Delta_{S_t} f := n\frac{dd^c f \wedge S_t^{n-1}}{S_t^n} \quad\text{and}\quad \mathrm{Tr}_{S_t}(\eta) := n\frac{\eta\wedge S_t^{n-1}}{S_t^n}.$$

On the other hand,

$$\Delta_{S_t}(H) = t\Delta_{S_t}(\dot{\varphi}_t) - n + \mathrm{Tr}_{S_t}(\omega_t + dd^c\varphi_0),$$

therefore

$$\Big(\frac{\partial}{\partial t} - \Delta_{S_t}\Big)(H) = \Big\{-t\frac{\partial F}{\partial t} - t\dot{\varphi}_t\frac{\partial F}{\partial r} + n - B\Big\} - \mathrm{Tr}_{S_t}(S_0) + \mathrm{Tr}_{S_t}(\omega_0 + t\dot{\omega}_t - \omega_t).$$

The assumption that $t \mapsto \omega_t$ is affine ensures $\mathrm{Tr}_{S_t}(\omega_0 + t\dot{\omega}_t - \omega_t) = 0$, while our choice of B yields

$$\Big(\frac{\partial}{\partial t} - \Delta_{S_t}\Big)(H) \le -1 - t\dot{\varphi}_t\frac{\partial F}{\partial r}(t,x,\varphi_t).$$

If H realizes its maximum $H_{\max}$ along $(t=0)$, we obtain

$$H(t,x) \le H_{\max} = \sup_{x\in X} H(0,x) = 0,$$

which yields the desired upper bound for $\dot{\varphi}_t$.

If H realizes its maximum $H_{\max}$ at some point (t_0,x_0) with $t_0>0$, then

$$0 \le \Big(\frac{\partial}{\partial t} - \Delta_{S_t}\Big)(H)(t_0,x_0),$$

hence

$$t_0\dot{\varphi}_{t_0}(x_0)\frac{\partial F}{\partial r}(t_0,x_0,\varphi_{t_0}(x_0)) \le -1 < 0.$$

Since $\partial F/\partial r \ge 0$ and $t_0>0$, we infer $\dot{\varphi}_{t_0}(x_0)<0$, hence

$$H_{\max} \le -(\varphi_{t_0}-\varphi_0)(x_0) \le C,$$

where the last inequality follows from Proposition 2.1. This yields again the desired upper bound.

We now take care of the lower bound. We first deal with the particular case when

$$\mu(x) = h(x)\theta^n(x),$$

where $h\ge 0$ is a bounded density. Fix $D \gg 1$ so large that all our quantities are well defined and under control on $[0,T+1/D]\times X$. Observe that

$$\chi + D\omega_t = D\omega_{t+1/D} \ge D\theta.$$

We set

$$G(t,x) = \dot{\varphi}_t(x) + D\varphi_t(x) - n\log t,$$

and compute

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta_{S_t}\Big)(G) &= \mathrm{Tr}_{S_t}(\chi + D\omega_t) - Dn - \frac{\partial F}{\partial t} + \Big[D - \frac{\partial F}{\partial r}\Big]\dot{\varphi}_t - \frac{n}{t}\\ &\ge \frac{f_t^{-1/n}}{C_1} - C_2 - \frac{n}{t},\end{aligned}$$

where $f_t = e^{\dot{\varphi}_t}$. We have used here

$$\begin{aligned}\mathrm{Tr}_{S_t}(\chi + D\omega_t) \ge D\,\mathrm{Tr}_{S_t}(\theta) \ge nD\Big(\frac{\theta^n}{S_t^n}\Big)^{1/n} &= nDf_t^{-1/n}\Big(\frac{\theta^n(x)}{e^{F(t,x,\varphi_t)}\mu(x)}\Big)^{1/n}\\ &\ge \frac{f_t^{-1/n}}{C_1'}\end{aligned}$$

(the last inequality uses our assumption that $\mu = h\theta^n$ with h bounded), the fact that $\partial F/\partial t \le c$ and $\partial F/\partial r \le c'$, and the inequality

$$\dot{\varphi}_t = \log f_t \ge -n\varepsilon f_t^{-1/n} - nC_\varepsilon,$$

valid for $\varepsilon > 0$ arbitrarily small (since $\varepsilon x > \log x - C_\varepsilon$ for $x > 0$).

The function G attains its minimum on $]0,T] \times X$ at a point (t_0, x_0) with $t_0 > 0$. At this point we therefore have a control on the density f_t, namely

$$f_{t_0}^{-1/n}(x_0) \le C_1 C_2 + \frac{nC_2}{t_0},$$

hence

$$\dot{\varphi}_{t_0}(x_0) = \log f_{t_0}(x_0) \ge -n \log\left[C_1 C_2 + \frac{nC_2}{t_0}\right].$$

We infer

$$G(t_0, x_0) \ge D\varphi_{t_0}(x_0) - n \log[C_1 C_2 t_0 + nC_2] \ge -C_3,$$

using Proposition 2.1. The desired lower bound follows.

We now get rid of the extra assumption made on μ. We fix as earlier ρ_1 a smooth θ–psh function such that

$$(\theta + dd^c \rho_1)^n = e^{c_1}\mu$$

and $\sup_X \rho_1 = 0$. We set

$$\widetilde{\omega}_t = \omega_t + dd^c \rho_1, \quad \widetilde{\varphi}_t := \varphi_t - \rho_1 \in \mathrm{PSH}(X, \widetilde{\omega}_t)$$

and

$$\widetilde{F}(t,x,r) = F(t,x,r+\rho_1(x)).$$

Observe that $\dot{\widetilde{\varphi}}_t = \dot{\varphi}_t$, $\partial_t \widetilde{\omega}_t = \partial_t \omega_t$ and

$$\widetilde{\omega}_t \ge \widetilde{\theta} = \theta + dd^c \rho_1.$$

Moreover, $\widetilde{F}$ has the same Lipschitz constant (in t and r) as that of F and

$$(\widetilde{\omega}_t + dd^c \widetilde{\varphi}_t)^n = e^{\dot{\widetilde{\varphi}}_t + \widetilde{F}(t,x,\widetilde{\varphi}_t)}\mu(x)$$

with $\widetilde{\theta}^n = e^{c_1}\mu$, hence $\mu = \widetilde{h}\widetilde{\theta}^n$ with bounded density $\widetilde{h} = e^{-c_1}$. We can thus use the same reasoning as above to conclude. □

2.3.2 Refining the hypotheses We now establish similar uniform bounds on $\dot{\varphi}_t$ under less restrictive assumptions on $t \mapsto \omega_t$ and $r \mapsto F(\cdot,\cdot,r)$. Recall that $r \mapsto F(t,x,r)+\lambda_F r$ is increasing on $\mathbb{R}$.

Theorem 2.5 *Assume $\dot{\omega}_t \geq -A\omega_t$. Then, for all $(t,x) \in X_T$,*

$$n \log t - C \leq \dot{\varphi}_t(x).$$

If there exists $A \geq 0$ with $\dot{\omega}_t \leq A\omega_t$, then, for all $(t,x) \in X_T$,

$$\dot{\varphi}_t(x) \leq \frac{C}{t}.$$

Here again C depends explicitly on T, $\|\partial F/\partial r\|_{L^\infty}$, $\|\partial F/\partial t\|_{L^\infty}$, $\|g\|_{L^p}$ and C_0 (defined in Proposition 2.1). The norms $\|\partial F/\partial r\|_{L^\infty}$ and $\|\partial F/\partial t\|_{L^\infty}$ are computed on $[0,T[\times X \times [-C_0,C_0]$.

Proof Consider

$$H(t,x) = t\dot{\varphi}_t(x) - C\varphi_t,$$

where $C := (A+\lambda_F)T+2$. We let the reader check that

$$\Big(\frac{\partial}{\partial t} - \Delta_{S_t}\Big)(H) \leq C' - [t\partial_r F + C - 1]\dot{\varphi}_t.$$

The upper bound then follows just as in the proof of Theorem 2.4.

We now establish the lower bound. Consider, for $(t,x) \in\,]0,T] \times X$,

$$G(t,x) := \dot{\varphi}_t + A(2\varphi_t - \rho_1) - n\log t,$$

where $\rho_1 \in \mathrm{PSH}(X,\theta)$ is the unique normalized solution to

$$(\theta + dd^c\rho_1)^n = e^{c_1} g\, dV.$$

Using the same notation as in the proof of Theorem 2.4 we obtain

$$\begin{aligned}
\Delta_{S_t} G &= \Delta_{S_t}\dot{\varphi}_t + A\big(\mathrm{Tr}_{S_t}(2\omega_t + 2dd^c\varphi_t - \omega_t - (\omega_t + dd^c\rho_1))\big)\\
&\leq \Delta_{S_t}\dot{\varphi}_t + 2nA - A\,\mathrm{Tr}_{S_t}(\omega_t) - \mathrm{Tr}_{S_t}(\theta + dd^c\rho_1)\\
&\leq \Delta_{S_t}\dot{\varphi}_t + 2nA + \mathrm{Tr}_{S_t}(\dot{\omega}_t) - n e^{(c_1 - \dot{\varphi}_t - F(t,\cdot,\varphi_t))/n}\\
&\leq \mathrm{Tr}_{S_t}(\dot{\omega}_t + dd^c\dot{\varphi}_t) + 2nA - \frac{f_t^{-1/n}}{C_1}.
\end{aligned}$$

It thus follows that

$$\left(\frac{\partial}{\partial t}-\Delta_{S_t}\right)(G)=\ddot{\varphi}_t-2A\dot{\varphi}_t-\frac{n}{t}-\Delta_{S_t}G\geq-\frac{\partial F}{\partial t}-\left(2A+\frac{\partial F}{\partial r}\right)\dot{\varphi}_t+\frac{f_t^{-1/n}}{C_1}-\frac{n}{t}-2nA.$$

We can then conclude as in the proof of Theorem 2.4. □

Remark 2.6 The lower bound for $\dot{\varphi}_t$ ensures that

$$\varphi_t \geq \varphi_0 + n(t\log t - t) - Ct,$$

which is a simpler lower bound than the one provided by Proposition 2.2.

2.4 Semiconcavity in time

Our goal in this section is to establish that ω_t–psh solutions to (CMAF) are κ–concave in time away from zero, with a uniform a priori constant κ.

2.4.1 A particular case

Theorem 2.7 *Assume that $t \mapsto \omega_t$ is affine and $r \mapsto F(\cdot,\cdot,r)$ is convex and increasing. Let φ_t be a smooth solution to (CMAF). Then there exists $C > 0$ such that*

$$\ddot{\varphi}_t(x) \leq \frac{C}{t} \quad \textit{for all } (t,x) \in X_T,$$

where C depends explicitly on T, the L^∞–norms of $\partial F/\partial r$, $\partial F/\partial t$, $\partial^2 F/\partial r\partial t$ and $\partial^2 F/\partial t^2$, $\|g\|_p$ and C_0.

Recall that C_0 is an upper bound for $|\varphi_t|$ established in Proposition 2.1, and the norms on the partial derivatives of F are computed on $X_T \times [-C_0, C_0]$.

We will establish a similar (though less precise) control under less restrictive assumptions on $t \mapsto \omega_t$ and F. We postpone this to the next subsection, as the a priori estimates are already quite involved.

Proof Set $\omega_t = \omega + t\chi$, so that $\dot{\omega}_t = \chi$. Writing

$$\dot{\varphi}_t = \log[(\omega_t + dd^c\varphi_t)^n / g(x)\, dV(x)] - F(t,x,\varphi),$$

we differentiate in time to obtain

$$\ddot{\varphi}_t = \Delta_{S_t}(\dot{\varphi}_t) + \mathrm{Tr}_{S_t}(\dot{\omega}_t) - \frac{\partial F}{\partial t}(t,x,\varphi_t) - \dot{\varphi}_t\frac{\partial F}{\partial r}(t,x,\varphi_t).$$

It follows from the Lipschitz a priori estimate (Theorem 2.4) that

$$-C \le t\dot{\varphi}_t \frac{\partial F}{\partial r}(t,x,\varphi_t) \le C$$

is uniformly bounded on X_T, hence

$$t\ddot{\varphi}_t = t\,\mathrm{Tr}_{S_t}(\chi + dd^c\dot{\varphi}_t) + O(1).$$

Differentiating again yields

$$\dddot{\varphi}_t = \Delta_{S_t}(\ddot{\varphi}_t) - n^2\left(\frac{(\chi+dd^c\dot{\varphi}_t)\wedge S_t^{n-1}}{S_t^n}\right)^2 + n(n-1)\frac{(\chi+dd^c\dot{\varphi}_t)^2\wedge S_t^{n-2}}{S_t^n}$$
$$-\frac{\partial^2 F}{\partial t^2}(x,t,\varphi_t) - 2\dot{\varphi}_t\frac{\partial^2 F}{\partial r\partial t}(x,t,\varphi_t) - \ddot{\varphi}_t\frac{\partial F}{\partial r}(x,t,\varphi_t) - (\dot{\varphi}_t)^2\frac{\partial^2 F}{\partial r^2}(x,t,\varphi_t).$$

Set $H(t,x) = t\ddot{\varphi}_t - Bt$ with $B>0$. It follows from the Lipschitz control $t|\dot{\varphi}_t| \le C$ and Lemma 2.8 below that

$$\left(\frac{\partial}{\partial t} - \Delta_{S_t}\right)H \le \left[1 - t\frac{\partial F}{\partial r}\right]\ddot{\varphi}_t - nt\left(\frac{(\chi+dd^c\dot{\varphi}_t)\wedge S_t^{n-1}}{S_t^n}\right)^2$$

if we choose $B>0$ so large that

$$t\frac{\partial^2 F}{\partial t^2}(t,x,\varphi_t) + 2t\dot{\varphi}_t\frac{\partial^2 F}{\partial r\partial t}(t,x,\varphi_t) + B \ge 0.$$

We use here the simplifying assumption that $r \mapsto F(\cdot,\cdot,r)$ is convex, so that

$$-(\dot{\varphi}_t)^2\frac{\partial^2 F}{\partial r^2}(t,x,\varphi_t) \le 0.$$

We will remove this assumption in the next subsection.

Let $(t_0,x_0) \in X_T$ be a point at which H realizes its maximum. If $t_0 = 0$ then $H \le 0$, hence $\ddot{\varphi}_t \le B$ and we are done. If $t_0 > 0$, then $0 \le (\partial/\partial t - \Delta_{S_t})H$ at the point (t_0,x_0); thus, for $(t,x) = (t_0,x_0)$,

$$\frac{1}{n}(t\,\mathrm{Tr}_{S_t}(\chi+dd^c\dot{\varphi}_t))^2 \le \left[1 - t\frac{\partial F}{\partial r}\right]t\ddot{\varphi}_t$$

with

$$\begin{aligned} t\ddot{\varphi}_t &= t\,\mathrm{Tr}_{S_t}(\chi+dd^c\dot{\varphi}_t) - t\frac{\partial F}{\partial t}(t,x,\varphi_t) - t\dot{\varphi}_t\frac{\partial F}{\partial r}(t,x,\varphi_t) \\ &= t\,\mathrm{Tr}_{S_t}(\chi+dd^c\dot{\varphi}_t) + O(1). \end{aligned}$$

It follows that $t_0\ddot{\varphi}_{t_0}(x_0)$ is uniformly bounded from above, hence so is $H \le C$. Thus $t\ddot{\varphi}_t \le Bt + C \le C'$ on X_T. □

We have used the following differential inequality, which is probably well known. We include a proof for the reader's convenience.

Lemma 2.8 *Assume $n \geq 2$. Let ω be a Kähler form and let η be a closed $(1,1)$–differential form. Then*

$$\frac{\eta^2 \wedge \omega^{n-2}}{\omega^n} \leq \left(\frac{\eta \wedge \omega^{n-1}}{\omega^n}\right)^2.$$

Proof This is a pointwise inequality, hence it reduces to linear algebra. Since ω is a Kähler form, we can assume that $\omega(x)$ is the Euclidean Kähler metric. Perturbing $\eta(x)$ if necessary, we can also make a change of local coordinates so that $\eta(x)$ is given by a diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. We infer

$$\frac{\eta^2 \wedge \omega^{n-2}}{\omega^n}(x) = \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta,$$

while

$$\left(\frac{\eta \wedge \omega^{n-1}}{\omega^n}\right)^2(x) = \left(\frac{1}{n}\sum_\alpha \lambda_\alpha\right)^2.$$

The desired inequality follows from the elementary computation

$$\left(\frac{1}{n}\sum_{\alpha=1}^n \lambda_\alpha\right)^2 - \frac{1}{n(n-1)} \sum_{\alpha \neq \beta} \lambda_\alpha \lambda_\beta = \frac{1}{n^2(n-1)} \sum_{\alpha < \beta} (\lambda_\alpha - \lambda_\beta)^2 \geq 0. \qquad \square$$

2.4.2 More general bounds We assume in this subsection that there exists a constant $A > 0$ such that, for all $t \in]0, T[$,

$$-A\omega_t \leq \dot{\omega}_t \leq +A\omega_t \quad \text{and} \quad \ddot{\omega}_t \leq A\omega_t. \tag{2-1}$$

We also assume that $(t, r) \mapsto F(t, x, r)$ is uniformly semiconvex, ie there exists a constant $C_F > 0$ such that for every $x \in X$, the function

$$(t, r) \mapsto F(t, x, r) + C_F(t^2 + r^2) \quad \text{is convex in } [0, T[\times [-C_0, C_0]. \tag{2-2}$$

Theorem 2.9 *Assume that ω_t and $F(\cdot, \cdot, r)$ are as above. Let φ_t be a solution of the above parabolic Monge–Ampère equation. Then there exists $C > 0$ such that*

$$\ddot{\varphi}_t(x) \leq \frac{C}{t^2} \quad \textit{for all } (t, x) \in X_T,$$

where C depends explicitly on A, T, C_F, λ_F, $\|\partial F/\partial r\|_{L^\infty}, \|\partial F/\partial t\|_{L^\infty}$, $\|g\|_p$ and C_0.

Here C_0 is the constant given in Proposition 2.1 and the norms $\|\partial F/\partial r\|_{L^\infty}$ and $\|\partial F/\partial t\|_{L^\infty}$ are computed on $X_T \times [-C_0, C_0]$.

Proof In the proof below we use C to denote various constants under control. Set $\alpha_t := \dot\omega_t + dd^c\dot\varphi_t$ and $S_t := \omega_t + dd^c\varphi_t$, and, for $h \in \mathcal{C}^\infty(X,\mathbb{R})$,

$$\Delta_t h := \mathrm{Tr}_t(dd^c h) := \mathrm{Tr}_{S_t}(dd^c h) = n\frac{dd^c h \wedge S_t^{n-1}}{S_t^n}.$$

Writing

$$\dot\varphi_t = \log[(\omega_t + dd^c\varphi_t)^n / g(x)\,dV(x)] - F(t,x,\varphi),$$

we differentiate twice in time to obtain, as in the proof of Theorem 2.7, that

$$t\ddot\varphi_t = t\,\mathrm{Tr}_t\alpha_t - t\partial_t F - t\dot\varphi_t\partial_r F = t\,\mathrm{Tr}_t\alpha_t + O(1), \tag{2-3}$$

where we use the uniform bound $t|\dot\varphi_t| \le C$ (thanks to Theorem 2.5), and

$$\begin{aligned}\dddot\varphi_t &= \mathrm{Tr}_t(\dot\alpha_t) + n(n-1)\frac{\alpha_t^2 \wedge S_t^{n-2}}{S_t^n} - n^2\left(\frac{\alpha_t \wedge S_t^{n-1}}{S_t^n}\right)^2 - \ddot\varphi_t\frac{\partial F}{\partial r}(t,x,\varphi_t)\\ &\qquad - \frac{\partial^2 F}{\partial t^2}(t,x,\varphi_t) - 2\dot\varphi_t\frac{\partial^2 F}{\partial r\partial t}(t,x,\varphi_t) - (\dot\varphi_t)^2\frac{\partial^2 F}{\partial r^2}(t,x,\varphi_t)\\ &\le \mathrm{Tr}_t(\dot\alpha_t) - \frac{1}{n}(\mathrm{Tr}_t\alpha_t)^2 - \ddot\varphi_t\frac{\partial F}{\partial r}(t,x,\varphi_t) + C_F((\dot\varphi_t)^2 + 1),\end{aligned}$$

using the convexity condition (2-2) and Lemma 2.8. The Lipschitz control $t|\dot\varphi_t| \le C$ (provided by Theorem 2.5) yields

$$t^2\dddot\varphi_t \le t^2\mathrm{Tr}_t\dot\alpha_t - t^2 n^{-1}[\mathrm{Tr}_t(\alpha_t)]^2 - t^2\ddot\varphi_t\frac{\partial F}{\partial r}(t,x,\varphi_t) + C. \tag{2-4}$$

Set $H(t,x) = t^2\ddot\varphi_t - ATt\varphi_t$. It follows from (2-1) and a direct computation that

$$\Delta_t H \ge t^2\mathrm{Tr}_t\dot\alpha_t + (ATt - At^2)\mathrm{Tr}_t(\omega_t) - AnTt \ge t^2\mathrm{Tr}_t\dot\alpha_t - C. \tag{2-5}$$

It follows therefore from (2-4) and (2-5) that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)H \le t\ddot\varphi_t(2 - t\partial_r F) - t^2 n^{-1}(\mathrm{Tr}_t\alpha_t)^2 + C.$$

Let $(t_0, x_0) \in X_T$ be a point at which the function H realizes its maximum. If $t_0 = 0$ then $H \le 0$, hence $t^2\ddot\varphi_t \le C$ and we are done. If $t_0 > 0$, then $0 \le (\partial/\partial t - \Delta_{S_t})H$ at the point (t_0, x_0); thus, for $(t,x) = (t_0,x_0)$,

$$t\ddot\varphi_t(t\partial_r F - 2) + t^2 n^{-1}(\mathrm{Tr}_t\alpha_t)^2 \le C.$$

Using (2-3) we conclude that $t^2\ddot\varphi_t \le C$ on X_T, finishing the proof. □

2.5 Conclusion

2.5.1 The estimates We summarize here the a priori estimates we have obtained so far. We assume that the forms and densities are *smooth* and satisfy the uniform bounds listed in the introduction, involving the constants A, p, λ_F and C_F.

Theorem 2.10 *There exist $C_0, C_1, C_2 > 0$ such that, for all $(t,x) \in X_T$,*

(1) $-C_0 \le \varphi_t(x) \le C_0$;

(2) $n \log t - C_1 \le \dot{\varphi}_t(x) \le C_1/t$;

(3) $\ddot{\varphi}_t(x) \le C_2/t^2$;

where the C_j depend on A, B, p, C_F and λ_F, and

- *C_0 explicitly depends on T, θ, Θ, $\inf_X \varphi_0$, $\sup_X \varphi_0$ and $\sup_{X_T} |F(t,x,0)|$;*
- *C_1 explicitly depends on C_0, T, $\|\partial F/\partial r\|_{L^\infty}$, $\|\partial F/\partial t\|_{L^\infty}$ and $\|g\|_{L^p}$;*
- *C_2 explicitly depends on C_0, C_1 and T.*

The norms $\|\partial F/\partial r\|_{L^\infty}$ and $\|\partial F/\partial t\|_{L^\infty}$ are computed on $X_T \times [-C_0, C_0]$.

2.5.2 Convergence of semiconcave functions It is useful to know when a sequence of ω_t–psh functions is uniformly semiconcave. It allows one to obtain the convergence of the associated parabolic Monge–Ampère operators, as the following result shows:

Theorem 2.11 *Let $g_j(t,x)$ be a family of $L^1(X_T)$–densities such that $g_j \to g$ in $L^1(X_T)$. Let $F_j(t,x,r)$ be continuous densities which uniformly converge towards F. Let $\varphi_j(t,x)$ be a family of ω_t–psh functions such that*

- *(φ_j) is uniformly bounded;*
- *$\ddot{\varphi}_j \le C/t^2$ for some uniform constant $C > 0$.*

Then there exists a bounded function $\varphi \in \mathcal{P}(X_T, \omega)$ such that, up to extracting and relabeling, $\varphi_j \to \varphi$ in $L^1_{\mathrm{loc}}(X_T)$ and

$$e^{\dot{\varphi}_j + F_j(t,x,\varphi_j(t,x))} g_j(t,x)\, dt \wedge dV(x) \to e^{\dot{\varphi} + F(t,x,\varphi(t,x))} g(t,x)\, dt \wedge dV(x),$$

in the weak sense of Radon measures on X_T.

Proof Since (φ_j) is bounded in $L^2(X_T)$, it is weakly compact. Extracting and relabeling, we assume that (φ_j) weakly converges to $\varphi \in L^2(X_T)$.

Fix a compact subinterval $J \Subset]0, T[$. There exists a constant $C = C_J > 0$ such that the functions $t \mapsto \varphi_j(t, x) - Ct^2$ are concave in J for all $x \in X$ fixed. The same property holds for the limiting function $\varphi(t, x)$ by letting $j \to +\infty$. For t fixed, the functions $x \mapsto \varphi_j(t, x)$ are ω_t–psh and uniformly bounded, hence $x \mapsto \varphi(t, x)$ is ω_t–psh and uniformly bounded in X_T.

It follows from Theorem 1.3 that $\varphi_j \to \varphi$ in $L^1_{\mathrm{loc}}(X_T)$ and $\varphi_j(t, x) \to \varphi(t, x)$ almost everywhere in X_T with respect to the Lebesgue measure. The conclusion follows by applying Theorem 1.14. □

3 Existence and properties of sub/super/solutions

From now on we assume that $t \mapsto \omega_t$ and the densities g and F satisfy the conditions listed in the introduction.

For bounded parabolic potentials $\varphi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$, the equation (CMAF) should be understood in the sense of measures on $]0, T[\times X$:

$$(\omega_t + dd^c \varphi_t)^n \wedge dt = e^{\dot{\varphi}_t + F(t,x,\varphi_t)} g(x)\, dV(x) \wedge dt. \tag{CMAF}$$

It follows from Definition 1.10 that the left-hand side is a well-defined Radon measure, while Lemma 1.6 ensures that so is the right-hand side.

3.1 Stability estimates

We establish in this section uniform L^∞–L^1 stability estimates needed in the proof of the existence theorem.

Proposition 3.1 *Fix $0 \le g_1, g_2 \in L^p(X)$ with $p > 1$, and $0 < T_0 < T_1 < T$. Assume $\varphi^1, \varphi^2 \in \mathcal{P}(X_T, \omega) \cap \mathcal{C}^\infty(X_T)$ both satisfy*

$$dt \wedge (\omega_t + dd^c \varphi^i)^n = e^{\dot{\varphi}^i(t,\cdot) + F_i(t,\cdot,\varphi^i)} g_i\, dt \wedge dV.$$

Then, for all $(t, x) \in [T_0, T_1] \times X$,

$$|\varphi^1(t, x) - \varphi^2(t, x)| \le B \|\varphi^1 - \varphi^2\|^\alpha_{L^1(X_T)}, \tag{3-1}$$

where $0 < \alpha = \alpha(n, p)$ while $0 < B$ depends on T_0, T_1, θ, Θ, and upper bounds for $\|g^i\|_{L^p(X)}$, $\|F_i\|_{L^\infty(X_T)}$, $\|\partial_t F_i\|_{L^\infty(X_T)}$ and $\|\partial_r F_i\|_{L^\infty(X_T)}$.

Proof We are going to use the stability results of [12; 21]. These rely on important estimates which we recall for the convenience of the reader. The uniform bounds $\theta \le \omega_t \le \Theta$ and [11, Lemma 2.2] show that there exists a uniform constant $A_1 > 0$ such that

$$\mathrm{Vol}(K) \le A_1 \, \mathrm{Cap}_{\omega_t}(K)^2$$

for all $t \in [0, T]$ and all compact sets $K \subset X$, where

$$\mathrm{Cap}_{\omega_t}(K) := \sup\left\{ \int_K (\omega_t + dd^c u)^n : u \in \mathrm{PSH}(X, \omega_t) \text{ with } 0 \le u \le 1 \right\}$$

is the Monge–Ampère capacity associated to the form ω_t.

Fix $T_0 < T_1 < T$ and consider the densities

$$f_t^i := e^{\dot{\varphi}^i(t,\cdot) + F_i(t,x,\varphi^i)} g_i(x), \quad i = 1, 2.$$

It follows from Theorem 2.5 that $t\dot{\varphi}^i(t,x)$ is uniformly bounded by C_1, while Proposition 2.1 ensures that the φ^i are uniformly bounded. The L^p–norms of the densities f_t^i are thus uniformly bounded from above by

$$A_2 := e^{C_1/T_0 + C_2}(\|g_0\|_{L^p(X)} + \|g_1\|_{L^p(X)})$$

when $t \in [T_0, T_1]$. It follows therefore from [12, Proposition 3.3] that

$$\max_X |\varphi^1(t,\cdot) - \varphi^2(t,\cdot)| \le C \|\varphi^1(t,\cdot) - \varphi^2(t,\cdot)\|_{L^1(X)}^{\gamma}$$

for all $t \in [T_0, T_1]$, where $\gamma \in \,]0, 1[$ only depends on p and n. Lemma 1.5 yields

$$\|\varphi^1(t,\cdot) - \varphi^2(t,\cdot)\|_{L^1(X)} \le A \max\{\|\varphi^1 - \varphi^2\|_{L^1(X_T)}, \|\varphi^1 - \varphi^2\|_{L^1(X_T)}^{1/2}\},$$

where $A := 2\max\{\sqrt{\kappa}, (T - T_1)^{-1}\}$.

The proof is completed by combining the last two inequalities. □

In practice, this proposition yields the following useful information:

Corollary 3.2 *Assume $\|g_j\|_{L^p}$, $\|F_j\|_{L^\infty}$, $\|\partial_t F_j\|_{L^\infty}$ and $\|\partial_r F_j\|_{L^\infty}$ are uniformly bounded. If a sequence (φ^j) of solutions to (CMAF)$_{F_j,g_j}$ converges in $L^1(X_T)$ to φ, then it uniformly converges on compact subsets of $]0, T[\, \times X$.*

3.2 The Cauchy problem

We are now in position to prove Theorem A of the introduction.

Definition 3.3 A parabolic potential $\varphi \in \mathcal{P}(X_T, \omega)$ is a pluripotential solution (respectively sub/supersolution) to (CMAF) with initial values $\varphi_0 \in \mathrm{PSH}(X, \omega_0) \cap L^\infty(X)$ if φ satisfies (CMAF) (or the inequality $\geq$ or $\leq$, respectively) in the sense of measures on X_T and $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$.

Theorem 3.4 *Assume that φ_0 is a bounded ω_0–psh function in X, and (ω, F, g) is as in the introduction. There exists $\varphi \in \mathcal{P}(X_T, \omega)$ solving (CMAF) such that, for all $0 < T' < T$,*

- $(t,x) \mapsto \varphi(t,x)$ *is uniformly bounded in* $]0, T'] \times X$;
- $(t,x) \mapsto \varphi(t,x)$ *is continuous in* $]0, T[\times \Omega$;
- $t \mapsto \varphi(t,x) - n(t \log t - t) + C_1 t$ *is increasing on* $]0, T'[$ *for some* $C_1 > 0$;
- $t \mapsto \varphi(t,x) + C_2 \log t$ *is concave on* $]0, T'[$ *for some* $C_2 > 0$;
- $\varphi_t \to \varphi_0$ *as* $t \to 0^+$ *in* $L^1(X)$ *and pointwise.*

Recall that Ω is the ample locus of θ. The solution we provide is in particular locally uniformly semiconcave in $t \in]0, T[$. We will study the uniqueness issue in the next section.

Proof Fix $0 < T' < T$. We prove the existence of a solution on $]0, T'[\times X$. The uniqueness result (Corollary 4.5) then ensures that a solution exists in $]0, T[\times X$.

We approximate

- g by smooth densities $g_j > 0$ in $L^p(X)$;
- F by smooth densities F_j with uniform constants κ_{F_j}, C_{F_j} and λ_{F_j};
- φ_0 from above on X by smooth $(\omega_0 + 2^{-j}\Theta)$–psh functions $\varphi_{0,j}$;
- ω_t by smooth (in t) ω_t^j such that $\omega_t^j \geq \omega_t + 2^{-j}\Theta$ and ω_t^j satisfies the assumptions in the introduction.

It is well known (see eg [37]) that there exists a unique smooth solution $\varphi^j \in \mathcal{P}(X_T, \omega^j)$ to $\text{(CMAF)}_{F_j, g_j}$, ie

$$dt \wedge (\omega_t^j + dd^c \varphi_t^j)^n = e^{\dot{\varphi}^j(t,x) + F_j(t,x,\varphi^j(t,x))} g_j \, dt \wedge dV(x). \tag{3-2}$$

It follows from Theorem 2.10 that the φ^j are uniformly bounded and the derivatives $\ddot{\varphi}^j$ are locally uniformly bounded from above in X_T. Extracting a subsequence of (φ^j) and relabeling it, Theorem 2.11 ensures that there exists $\varphi \in \mathcal{P}(X_T) \cap L^\infty_{\mathrm{loc}}(X_T)$ such that $\varphi^j \to \varphi$ in $L^1(X_T)$ and

$$e^{\dot{\varphi}^j(t,x)+F_j(t,x,\varphi^j(t,x))} g_j(t,x)\,dt \wedge dV(x) \to e^{\dot{\varphi}(t,x)+F(t,x,\varphi(t,x))} g(x)\,dt \wedge dV(x),$$

in the sense of currents on X_T.

We claim that $\varphi^j \to \varphi$ locally uniformly in X_T. This follows indeed from the stability estimates established in Proposition 3.1 above. Fix $0 < T_0 < T_1 < T$. Since the densities g_j have uniform L^p–norms, Theorem 2.5 ensures that the sequence $(\dot{\varphi}^j)$ is uniformly bounded in $[T_0, T] \times X$. By (3-1), for all j and k large enough, $t \in [T_0, T_1]$ and $x \in X$, we have

$$|\varphi^j(t,x) - \varphi^k(t,x)| \le C \|\varphi^j - \varphi^k\|^\alpha_{L^1(X_T)},$$

where $C > 0$ and $0 < \alpha < 1$ are uniform constants which do not depend on j, k and $t \in [T_0, T_1]$. This proves our claim.

Therefore $dt \wedge (\omega_t^j + dd^c\varphi_t^j)^n \to dt \wedge (\omega_t + dd^c\varphi_t)^n$ in the sense of measures on X_T, hence φ solves (CMAF).

One shows similarly that φ is uniformly semiconcave in $]0, T[$: the densities g_j in (3-2) are uniformly bounded in $L^p(X)$, hence Theorem 2.9 ensures the existence of a uniform constant $C > 0$ such that

$$\ddot{\varphi}^j(t,x) \le C/t^2$$

for all $j \in \mathbb{N}$, $(t,x) \in X_T$. Thus, for each compact subinterval $J \Subset]0, T[$ there exists a constant $C_J > 0$ such that the functions $t \mapsto \varphi^j(t) - C_J t^2$ are concave in J, and the same property holds for φ by letting $j \to \infty$.

The continuity of φ on $]0, T[\times \Omega$ follows from the elliptic theory, as will be shown in Proposition 3.12 below. The lower bound $\dot{\varphi}_t \ge n \log t - C_1$, provided by Theorem 2.5, ensures that $t \mapsto \varphi_t - n(t \log t - t) + C_1 t$ is increasing, hence any cluster point (in L^1–topology) of φ_t (as $t \to 0^+$) is greater than φ_0. On the other hand, it follows from Proposition 2.3 and Lemma 1.5 that

$$\int_X \varphi_t g\,dV \le \int_X \varphi_0 g\,dV + Ct$$

for a uniform constant $C > 0$. Let u_0 be any cluster point of (φ_t) as $t \to 0^+$. Then, as explained above, $u_0 \geq \varphi_0$. On the other hand, the average control above ensures that

$$\int_X u_0 g \, dV \leq \int_X \varphi_0 g \, dV.$$

Since the set $\{g = 0\}$ has Lebesgue measure zero, we infer $u_0 = \varphi_0$ almost everywhere, hence everywhere. □

Remark 3.5 Proposition 1.2 ensures that the pluripotential solution constructed above is upper semicontinuous on $[0, T[\times X$. The functions φ_t quasidecrease to φ_0 as $t \searrow 0$. The convergence at time zero is thus quite strong: If φ_0 is continuous, it follows for instance that the convergence is uniform. For noncontinuous initial φ_0, there is convergence in capacity: a sequence of functions u_j converges in capacity to u if for all $\varepsilon > 0$ we have

$$\lim_{j \to +\infty} \mathrm{Cap}_\theta\big(\{x \in X : |u_j(x) - u(x)| > \varepsilon\}\big) = 0.$$

Remark 3.6 The way the density is allowed to vanish is crucial. Theorem A does not hold for an arbitrary density $g \geq 0$: If g vanishes in a nonempty open set $D \subset X$ then (CMAF) has no solution with initial value φ_0 unless φ_0 is a maximal ω_0–psh function in D. Indeed, the complex Monge–Ampère operator is continuous for the convergence in capacity, so $(\omega_t + dd^c \varphi_t)^n = 0$ would converge to $(\omega_0 + dd^c \varphi_0)^n = 0$ in D.

3.3 Invariance properties of the set of assumptions

The family of parabolic complex Monge–Ampère equations we consider,

$$(\omega_t + dd^c \varphi_t)^n = e^{\dot{\varphi}_t + F(t,x,\varphi)} g(x) \, dV(x),$$

has several invariance properties, as we now briefly explain.

3.3.1 Translations We can replace $\varphi_t(x)$ by $\psi_t(x) = \varphi_t(x) + C(t)$ without changing the Monge–Ampère term, while the density F is modified into

$$\widetilde{F}(t,x,r) = F(t,x,r - C(t)) - C'(t).$$

We let the reader check that $\widetilde{F}$ satisfies the same set of assumptions as F.

More generally we can replace ω_t by $\eta_t = \omega_t - dd^c \rho_t$, changing $\varphi_t(x)$ in $\varphi_t(x) + \rho_t(x)$. The density g remains unchanged while the new density F is

$$\widetilde{F}(t,x,r) = F(t,x,r - \rho(t,x)) - \partial_t \rho(t,x).$$

3.3.2 Scaling A more involved transformation consists in scaling in space and renormalizing in time, so that the equation keeps the same shape. Namely we replace ω_t by $\gamma(s)\omega_{t(s)}$ as well as $\varphi_t(x)$ by $\psi_s(x)=\gamma(s)\varphi_{t(s)}(x)$, where $s\mapsto\gamma(s)>0$ is smooth and positive, $t(0)=0$ and $t'(s)=1/\gamma(s)$, so that

$$\partial_s\psi_s=\frac{\gamma'(s)}{\gamma(s)}\psi_s+\partial_t\varphi_{t(s)}.$$

The density g remains unchanged while the density F is transformed into

$$\widetilde{F}(s,x,R)=F(t(s),x,r(s,R))+n\log\gamma(s)-\frac{\gamma'(s)}{\gamma(s)}R,$$

where $r(s,R)=R/\gamma(s)$.

A classical example of such a transformation is when $\gamma(s)=e^s$ and $t(s)=1-e^{-s}$, allowing one to pass from the Kähler–Ricci flow to the *normalized* Kähler–Ricci flow.

We let the reader check that $\widetilde{F}$ remains quasi-increasing in R and locally uniformly Lipschitz in (s,R). It is slightly more involved to keep track of the semiconvexity property:

Lemma 3.7 *The function $(s,R)\mapsto\widetilde{F}(s,x,R)$ is locally uniformly semiconvex in (s,R).*

Proof Fix $0<S_0<S$, $T_0=t(S_0)$ and a compact interval $J\Subset\mathbb{R}$. We want to prove that $(s,R)\mapsto\widetilde{F}(s,R)$ is semiconvex in $[0,S_0]\times J$. We omit in the sequel the dependence on x as it is not affected by the transformation.

We can assume that F is smooth and proceed by approximation. The goal is to prove that the Hessian matrix $H(s,R)$ of $(s,R)\mapsto\widetilde{F}(s,R)$ satisfies

$$H(s,R)+CI_2\geq 0,$$

where I_2 is the identity matrix in $M_2(\mathbb{R})$, and the constant C is under control. Increasing C, we can also assume that F is convex in $[0,S_0]\times J$. Recall that F is Lipschitz on $[0,T_0]\times J$ and $s\mapsto\gamma(s)>0$ is smooth. Using this we can write

$$\begin{aligned}
\frac{\partial^2\widetilde{F}}{\partial s^2}&=\frac{\partial^2F}{\partial t^2}\Big(\frac{\partial t}{\partial s}\Big)^2+2\frac{\partial^2F}{\partial t\,\partial r}\frac{\partial r}{\partial s}\frac{\partial t}{\partial s}+\frac{\partial^2F}{\partial r^2}\Big(\frac{\partial r}{\partial s}\Big)^2+O(1),\\
\frac{\partial^2\widetilde{F}}{\partial R^2}&=\frac{\partial^2F}{\partial r^2}\Big(\frac{\partial r}{\partial R}\Big)^2+O(1),\\
\frac{\partial^2\widetilde{F}}{\partial R\,\partial s}&=\frac{\partial^2F}{\partial r^2}\frac{\partial r}{\partial R}\frac{\partial r}{\partial s}+\frac{\partial^2F}{\partial r\,\partial t}\frac{\partial r}{\partial R}\frac{\partial t}{\partial s}+O(1).
\end{aligned}$$

It remains to check that

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \geq 0,$$

where

$$\begin{aligned} a &= \frac{\partial^2 F}{\partial t^2}\Big(\frac{\partial t}{\partial s}\Big)^2 + 2\frac{\partial^2 F}{\partial t \partial r}\frac{\partial r}{\partial s}\frac{\partial t}{\partial s} + \frac{\partial^2 F}{\partial r^2}\Big(\frac{\partial r}{\partial s}\Big)^2, \\ c &= \frac{\partial^2 F}{\partial r^2}\Big(\frac{\partial r}{\partial R}\Big)^2, \\ b &= \frac{\partial^2 F}{\partial r^2}\frac{\partial r}{\partial R}\frac{\partial r}{\partial s} + \frac{\partial^2 F}{\partial r \partial t}\frac{\partial r}{\partial R}\frac{\partial t}{\partial s}. \end{aligned}$$

The convexity of F and a direct computation ensure that $a, c \geq 0$ and $ac - b^2 \geq 0$. □

We note, for later use, that such a transformation allows one to reduce to the case when $r \mapsto F(\cdot,\cdot,r)$ is increasing:

Lemma 3.8 *Assume that $r \mapsto F(\cdot, r)$ is quasi-increasing with $\lambda_F > 0$. Consider $\gamma\colon s \in [0, S[\mapsto 1 - \lambda_F s \in [0, T[$, where $S < \lambda_F^{-1}$ is defined by $\int_0^S (1-\lambda_F r)^{-1}\,dr = T$. The function $R \mapsto \widetilde{F}(s, R)$ is increasing for all $s \in [0, S[$.*

Proof The function $\widetilde{F}$ is given, for $(s, R) \in [0, S[\times \mathbb{R}$, by

$$\widetilde{F}(s, R) = F(t(s), R/\gamma(s)) + n \log \gamma(s) + \frac{\lambda R}{\gamma(s)}.$$

Using that $r \mapsto F(t, x, r) + \lambda r$ is increasing, it is straightforward to check that $\widetilde{F}$ is increasing in R. □

3.4 Pluripotential sub/supersolutions

3.4.1 Definitions Our plan is to establish a pluripotential parabolic comparison principle. The latter is easier to obtain under an extra regularity assumption in the time variable, so we introduce the following terminology for convenience:

Definition 3.9 A parabolic potential $u \in \mathcal{P}(X_T, \omega)$ is called of class $\mathcal{C}^{1/0}$ if for every $t \in]0, T[$ fixed, $\partial_t u(t, x)$ exists and is continuous in Ω.

Definition 3.10 A parabolic potential $\varphi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ is called a pluripotential subsolution of (CMAF) if

$$(\omega_t + dd^c \varphi_t)^n \wedge dt \geq e^{\dot{\varphi}_t + F(t,x,\varphi)} g(x)\, dV(x) \wedge dt$$

holds in the sense of measures in $]0, T[\times X$.

Similarly a parabolic potential $\varphi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ is called a pluripotential supersolution of (CMAF) if

$$(\omega_t + dd^c \varphi_t)^n \wedge dt \le e^{\dot{\varphi}_t + F(t,x,\varphi)} g(x)\, dV(x) \wedge dt$$

holds in the sense of measures in $]0,T[\times X$.

In many cases one can interpret these notions by considering a family of inequalities on slices:

Lemma 3.11 *Fix $u \in \mathcal{P}(X_T) \cap L^\infty(X_T)$.*

(1) *If u is a pluripotential subsolution of (CMAF) such that $\partial_t^+ u$ exists and is lower semicontinuous in $t \in]0,T[$, then, for all $t \in]0,T[$,*

$$(\omega_t + dd^c u_t)^n \ge e^{\partial_t^+ u + F} g\, dV \quad \text{in the sense of measures in } X.$$

(2) *If u is a pluripotential supersolution of (CMAF) such that $\partial_t^- u$ exists and is upper semicontinuous in $t \in]0,T[$, then, for all $t \in]0,T[$,*

$$(\omega_t + dd^c u_t)^n \le e^{\partial_t^- u + F} g\, dV \quad \text{in the sense of measures in } X.$$

Proof We will prove the result for subsolutions. The corresponding result for supersolutions follows similarly. Assume the right derivative $\partial_t^+ u$ exists for all $(t,x) \in X_T$ and is lower semicontinuous in t for x fixed. It follows from [19, Proposition 3.2] that for almost every $t \in]0,T[$,

$$(\omega_t + dd^c u_t)^n \ge e^{\partial_t^+ u + F} g\, dV,$$

in the sense of measures on X. Any $t \in]0,T[$ can be approximated by a sequence $(t_j)_{j\in\mathbb{N}}$ for which the inequality above holds. The limiting inequality follows from the lower semicontinuity of $\partial_t^+ u(t,x)$ in t and Fatou's lemma. □

3.4.2 Properties of supersolutions We use some properties of solutions to complex Monge–Ampère equations to show that parabolic supersolutions automatically have continuity properties.

Proposition 3.12 *Assume that $\psi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ is a supersolution to (CMAF). Then ψ is continuous in $]0,T[\times\Omega$.*

Proof Fix $0 < T_0 < T_1 < T$. For almost every $t \in]0, T[$ we have

$$(\omega_t + dd^c \psi_t)^n \leq e^{\dot{\psi}(t,\cdot)+F(t,\cdot,\psi_t)} g\, dV$$

in the weak sense on X.

Since ψ is locally uniformly Lipschitz in t and F is bounded, there exists $M > 0$ such that $\dot{\psi}(t,\cdot) + F(t,\cdot,\psi_t) \leq M$ for almost any $t \in [T_0, T_1]$. Thus,

$$(\omega_t + dd^c \psi_t)^n \leq e^M g\, dV,$$

for almost every $t \in [T_0, T_1]$. By (weak) continuity (in t) of the left-hand side, it follows that this inequality actually holds for any $t \in [T_0, T_1]$.

The elliptic theory (see eg [22, Theorem 12.23]) implies that ψ_t is continuous in Ω for any $t \in [T_0, T_1]$. Since ψ is uniformly Lipschitz in $[T_0, T_1]$, it follows that ψ is continuous in $[T_0, T_1] \times \Omega$. Indeed, let κ be the uniform Lipschitz constant of ψ on $[T_0, T_1]$. Then, for any $s, t \in [T_0, T_1]$ and $x, y \in \Omega$, we have

$$\begin{aligned}|\psi(s,x) - \psi(t,y)| &\leq |\psi(s,x) - \psi(t,x)| + |\psi(t,x) - \psi(t,y)| \\ &\leq \kappa|s-t| + |\psi(t,x) - \psi(t,y)|,\end{aligned}$$

which implies the continuity of ψ in $[T_0, T_1] \times \Omega$. □

Supersolutions admit uniform bounds from below:

Proposition 3.13 *Assume that $\psi \in \mathcal{P}(X_T) \cap L^\infty_{\text{loc}}(X_T)$ is a pluripotential supersolution to (CMAF) which is locally uniformly semiconcave in t. There exist $C > 0$ and $t_0 > 0$ such that, for all $(t, x) \in]0, t_0] \times X$,*

$$\psi(t,x) \geq (1-t)e^{-At}\psi_0(x) + C(t \log t - t).$$

Here A is a positive constant such that

$$-A\omega_t \leq \dot{\omega}_t \leq A\omega_t \quad \text{for all } t \in]0, T[.$$

This implies in particular that $\omega_{t+s} \geq e^{-At}\omega_s$ for all $t, s > 0$ with $t + s < T$.

Proof Set $M := M_\psi := \sup_{X_{T/2}} |\psi|$. The Lipschitz condition on F ensures that there exists a constant $\kappa = \kappa_F$ such that, for all $t, t' \in \left[0, \frac{1}{2}T\right]$, $x \in X$ and $r \in [-M, M]$,

$$|F(t,x,r) - F(t',x,r)| \leq \kappa|t - t'|.$$

Set $t_0 := \min\big(1, \frac{1}{4}T, \frac{1}{2}\lambda\big)$. As observed above, $\omega_{t+s} \geq e^{-At}\omega_s$ for all $s \in \big]0, \frac{1}{2}t_0\big]$ and $t \in]0, t_0[$. Fix $s \in]0, t_0]$ and consider, for $t \in]0, t_0]$,

$$u_s(t,x) := (1-t)e^{-At}\psi_s(x) + t\rho + C(t\log t - t),$$
$$v_s(t,x) := \psi(t+s,x) + 2\kappa ts,$$

where ρ is a θ–psh function on X, normalized by $\sup_X \rho = 0$, which solves

$$(\theta + dd^c\rho)^n = e^{c_1} g\, dV$$

with a normalization constant c_1, and C is a positive constant to be specified later.

The existence and boundedness of ρ follows from [12]. Observe that u_s is of class $\mathcal{C}^{1/0}$ in t and, for each $t \in]0, t_0]$ fixed, $u_s(t,\cdot)$ is continuous in Ω (see Proposition 3.12).

A direct computation shows that, for $t \in]0, t_0]$,

$$\begin{aligned}(\omega_{t+s} + dd^c u_s)^n &= \big((1-t)\omega_{t+s} + t\omega_{t+s} + dd^c((1-t)e^{-At}\psi_s) + t\,dd^c\rho\big)^n \\ &\geq \big((1-t)e^{-At}(\omega_s + dd^c\psi_s) + t(\theta + dd^c\rho)\big)^n \\ &\geq e^{c_1} t^n g\, dV.\end{aligned}$$

In the second line above we have used $\omega_{t+s} \geq e^{-At}\omega_s$ while in the last line we have used $\omega_s + dd^c\psi_s \geq 0$. Thus, since ψ_s is uniformly bounded, by choosing $C > 0$ large enough (depending on M_F) we obtain

$$(\omega_{t+s} + dd^c u_s)^n \geq e^{\partial_t u_s(t,\cdot) + F(t,\cdot,u_s(t,\cdot))} g\, dV.$$

It is also clear from the definition that $u_s(t,\cdot)$ converges in $L^1(X, dV)$ to $u_s(0,\cdot) = \psi_s$ as $t \to 0^+$.

On the other hand, since ψ is a supersolution to (CMAF), by Lemma 3.11 we have

$$\begin{aligned}(\omega_{t+s} + dd^c v_s)^n &\leq e^{\partial_t^- \psi(t+s,\cdot) + F(t+s,\cdot,\psi(t+s,\cdot))} g\, dV \\ &\leq e^{\partial_t^- v_s(t,\cdot) - 2\kappa s + F(t+s,\cdot,\psi(t+s,\cdot))} g\, dV.\end{aligned}$$

The Lipschitz condition on F ensures that, for all $t, s \in [0, t_0]$ and $x \in X$,

$$F(t+s, x, \psi(t+s,x)) \leq F(t, x, \psi(t+s,x)) + \kappa s.$$

The quasi-increasing property of F ensures that

$$F(t, x, \psi(t+s,x)) = F(t, x, v_s(t,x) - 2\kappa ts) \leq F(t, x, v_s(t,x)) + 2\kappa\lambda ts.$$

Thus, for $t \le t_0 \le \frac{1}{2}\lambda$,

$$(\omega_{t+s} + dd^c v_s)^n \le e^{\partial_t v_s(t,\cdot)+F(t,\cdot,v_s(t,\cdot))} g\, dV.$$

It follows from Proposition 3.12 that v_s is continuous on $[0,t_0]\times\Omega$ and $v_s(0,\cdot)=\psi_s$. We can thus apply Proposition 4.2 below and obtain $u_s \le v_s$ on $]0,t_0[\times X$. Letting $s\to 0$ we obtain that, for all $(t,x)\in]0,t_0]\times X$,

$$(1-t)e^{-At}\psi_0(x) + t\rho + C(t\log t - t) \le \psi(t,x).$$

The result follows since ρ is bounded. □

3.4.3 Regularization of subsolutions We introduce a regularization process for subsolutions. Fix $0<T'<T$ and $\varepsilon_0>0$ such that $(1+\varepsilon_0)T'<T$. It follows from (0-1) that there exists $A_1>0$ such that, for all $t\in]0,T'[$ and $s\in[1-\varepsilon_0,1+\varepsilon_0]$,

$$\omega_t \ge (1-A_1|s-1|)\omega_{ts}, \tag{3-3}$$

where $\varepsilon_0>0$ is a fixed small constant. For $|s-1|<\varepsilon_0$ we set

$$\lambda_s := \frac{|1-s|}{s}, \quad \alpha_s := s(1-\lambda_s)(1-A_1|s-1|) \in]0,1[.$$

Up to shrinking ε_0 we can also assume that, for all $|s-1|\le\varepsilon_0$,

$$\gamma_s := \frac{\lambda_s}{1-\alpha_s} \ge \varepsilon_1,$$

where $\varepsilon_1 = (5+A_1)^{-1} > 0$.

We let $\rho\in \mathrm{PSH}(X,\theta)$ with $\sup_X \rho = 0$ be the unique bounded solution to

$$(\varepsilon_1\theta + dd^c\rho)^n = e^{c_1} g\, dV$$

for some normalization constant $c_1\in\mathbb{R}$ (see [12]).

Lemma 3.14 *Assume that $u\in\mathcal{P}(X_T)$ is a bounded pluripotential subsolution of (CMAF). Then there exists a uniform constant $C>0$, depending on $M_u := \sup_{X_T}|u|$ and the data, such that, for every $s\in[1-\varepsilon_0,1+\varepsilon_0]$,*

$$(t,z)\mapsto v_s(t,z) := \frac{\alpha_s}{s}u(ts,x) + (1-\alpha_s)\rho(x) - C|s-1|t$$

is a pluripotential subsolution of (CMAF) in $X_{T'}$.

Proof For notational convenience we set

$$\beta_s := \frac{1-\lambda_s}{\alpha_s} = 1 + O(|s-1|).$$

Since u is a pluripotential subsolution of (CMAF), using (3-3) we can write

$$\begin{aligned}(\beta_s\omega_t + s^{-1}dd^c u(st,\cdot))^n &\geq s^{-n}(\omega_{ts} + dd^c u(st,\cdot))^n \\ &\geq e^{-n\log s + \partial_\tau u(st,x) + F(ts,x,u(st,x))} g(x)\,dV.\end{aligned}$$

By the choice of ρ we also have

$$(\gamma_s\omega_t + dd^c\rho)^n \geq (\varepsilon_1\theta + dd^c\rho)^n = e^{c_1} g\,dV.$$

Combining these with Lemma 3.15 below, we arrive at

$$\begin{aligned}(\omega_t + dd^c v_s(t,\cdot))^n &= [(1-\lambda_s)\omega_t + s^{-1}\alpha_s dd^c u(st,\cdot) + \lambda_s\omega_t + (1-\alpha_s)dd^c\rho]^n \\ &= [\alpha_s(\beta_s\omega_t + s^{-1}dd^c u(st,\cdot)) + (1-\alpha_s)(\gamma_s\omega_t + dd^c\rho)]^n \\ &\geq e^{\alpha_s\partial_\tau u(st,x) + \alpha_s F(t,x,u(st,x)) + (1-\alpha_s)c_1 - n\alpha_s\log s} g(x)\,dV.\end{aligned}$$

Since $F(t,x,r)$ is uniformly bounded on $[0,T[\times X\times[-M_u,M_u]$ and $\alpha_s - 1 = O(|s-1|)$, up to enlarging C we infer

$$(\omega_t + dd^c v_s(t,\cdot))^n \geq e^{\partial_t v_s(t,x) + F(t,x,v_s(t,x))} g(x)\,dV.$$

This concludes the proof. □

We have used the following mixed inequalities:

Lemma 3.15 *Let θ_1 and θ_2 be two closed smooth semipositive $(1,1)$–forms on X. Let $u_1 \in \mathrm{PSH}(X,\theta_1)$ and $u_2 \in \mathrm{PSH}(X,\theta_2)$ be bounded and such that*

$$(\theta_1 + dd^c u_1)^n \geq e^{f_1}\mu \quad \text{and} \quad (\theta_2 + dd^c u_2)^n \geq e^{f_2}\mu,$$

where f_1 and f_2 are bounded measurable functions and $\mu = h\,dV \geq 0$. Then, for every $\alpha \in\,]0,1[$,

$$\big(\alpha(\theta_1 + dd^c u_1) + (1-\alpha)(\theta_2 + dd^c u_2)\big)^n \geq e^{\alpha f_1 + (1-\alpha)f_2}\mu.$$

Proof The proof is identical to that of [19, Lemma 5.9] using the convexity of the exponential together with the mixed Monge–Ampère inequalities due to S Kołodziej [28] (see also [10]). □

Let $\chi\colon \mathbb{R} \to [0, +\infty[$ be a smooth function with compact support in $[-1, 1]$ such that $\int_{\mathbb{R}} \chi(s)\, ds = 1$. For $\varepsilon > 0$ we set $\chi_\varepsilon(s) := \varepsilon^{-1}\chi(s/\varepsilon)$.

Proposition 3.16 *Assume that $u \in \mathcal{P}(X_T)$ is a bounded pluripotential subsolution of (CMAF). Let v_s be defined as in Lemma 3.14.*

If $r \mapsto F(\cdot,\cdot,r)$ is convex then there exists a uniform constant $B > 0$ such that, for $\varepsilon > 0$ small enough, the function

$$u^\varepsilon(t,x) := \int_{\mathbb{R}} v_s(t,x)\chi_\varepsilon(s-1)\, ds - B\varepsilon(t+1)$$

is a pluripotential subsolution of (CMAF) which is $\mathcal{C}^{1/0}$ in t and such that

$$\sup_X [u^\varepsilon(0,x) - u_0(x)] \xrightarrow{\varepsilon\to 0} 0.$$

When $r \mapsto F(\cdot,\cdot,r)$ is merely semiconvex, the same conclusion holds if we further assume that

$$|\partial_t u(t,x)| \le C/t \quad \text{for all } (t,x) \in]0,T'] \times X. \tag{3-4}$$

Proof We first assume that $r \mapsto F(\cdot,\cdot,r)$ is uniformly convex and increasing. In this case we do not need the assumption (3-4). Fix ε_0 as in Lemma 3.14. For each s such that $|s-1| \le \varepsilon_0$ the function $v_s(t,z)$ is a pluripotential subsolution to (CMAF).

As in [19, Theorem 5.7, Step 3], we use [18, Main Theorem] and Jensen's inequality to show that, for any $t \in]0,T[$,

$$(\omega_t + dd^c u^\varepsilon)^n \ge \exp\left(\partial_t u^\varepsilon(t,x) + B\varepsilon + \int_{\mathbb{R}} F(t,x,v_s(t,x))\chi_\varepsilon(s-1)\, ds\right) g\, dV, \tag{3-5}$$

in the weak sense on X.

If $F(t,x,\cdot)$ is convex for any (t,x), then

$$\int_{\mathbb{R}} F(t,x,v_s(t,x))\chi_\varepsilon(s-1)\, ds \ge F(t,x,u^\varepsilon + B\varepsilon) \ge F(t,x,u^\varepsilon),$$

since F is nondecreasing in r. Plugging this inequality in (3-5) we conclude that, for any $B \ge 0$, u^ε is a subsolution to the equation (CMAF).

If F is merely semiconvex, the function $r \mapsto F(t,x,r) + \lambda r^2$ is convex for any $(t,x) \in X_T$ for some constant $\lambda > 0$. Thus,

$$\int_{\mathbb{R}} F(t,x,v_s(t,x))\chi_\varepsilon(s-1)\, ds \ge F(t,x,u^\varepsilon(t,x))\chi_\varepsilon(s-1)\, ds + \lambda Q_\varepsilon(t,x), \tag{3-6}$$

where

$$Q_\varepsilon(t,x) := \int_{\mathbb{R}} v_s(t,x)^2 \chi_\varepsilon(s-1)\,ds - \left(\int_{\mathbb{R}} v_s(t,x)\chi_\varepsilon(s-1)\,ds\right)^2.$$

We claim that there is $C_1 > 0$ such that $|Q_\varepsilon(t,x)| \le C_1\varepsilon$ for all $(t,x) \in [0,T]\times X$. Indeed the family of function $s \mapsto v_s$ is uniformly bounded in $[0,T]\times X$ by a constant $M > 0$. Hence, for any $(t,x) \in [0,T]\times X$ and $\varepsilon > 0$ small enough, we have

$$|Q_\varepsilon(t,x)| \le 2M \int_{\mathbb{R}} |v_s(t,x) - v_s^\varepsilon(t,x)|\chi_\varepsilon(s-1)\,ds,$$

where $v_s^\varepsilon(t,x) := \int_{\mathbb{R}} v_s(t,x)\chi_\varepsilon(s-1)\,ds$.

Recall that $v_s(t,x) := (\alpha_s/s)u(s\cdot t,x) + (1-\alpha_s)\rho(x) - C|s-1|t$. The condition (3-4) ensures that the function $\partial_s v_s$ is uniformly bounded in $s \in [1-\varepsilon_0, 1+\varepsilon_0]$ and $(t,x) \in X_T$. Thus the family $s \mapsto v_s$ is uniformly L–Lipschitz in $s \in [1-\varepsilon_0, 1+\varepsilon_0]$, which proves our claim with $C_1 := 2ML$.

By (3-6) this implies that, for any $(t,x) \in X_T$,

$$\int_{\mathbb{R}} F(t,x,v_s(t,x))\chi_\varepsilon(s-1)\,ds \ge F(t,x,u^\varepsilon(t,x))\chi_\varepsilon(s-1)\,ds - C_1\varepsilon.$$

Pluging this inequality in (3-5) and taking $B \ge C_1$ we see that u^ε is a subsolution to the equation (CMAF). Taking B large enough we obtain furthermore that $u^\varepsilon(0,x) \le u_0(x)$ for all $x \in X$.

We let the reader adapt these arguments to the situation when $r \mapsto F(\cdot,\cdot,r)$ is merely quasi-increasing. □

4 Uniqueness

We have shown in the previous section that the Cauchy problem for (CMAF) with bounded initial data $\varphi_0 \in \mathrm{PSH}(X,\omega_0)$ admits a pluripotential solution which is locally uniformly semiconcave in t. We now prove that there is only one such solution.

4.1 Comparison principle, I

Our goal in this section is to establish the following comparison principle:

Theorem 4.1 *Fix $\varphi, \psi \in \mathcal{P}(X_T,\omega) \cap L^\infty(X_T)$ and assume that*

(a) *φ is a pluripotential subsolution to (CMAF);*

(b) ψ *is a pluripotential supersolution to* (CMAF);

(c) $x \mapsto \varphi(\cdot, x)$ *is continuous in* Ω *and* $|\partial_t \varphi(t,x)| \le C/t$ *for all* $(t,x) \in X_T$;

(d) ψ *is locally uniformly semiconcave in* $t \in]0,T[$;

(e) $\varphi_t \to \varphi_0$ *and* $\psi_t \to \psi_0$ *in* L^1 *as* $t \to 0$.

If $\varphi_0 \le \psi_0$ *then* $\varphi \le \psi$.

We first establish this result under extra assumptions:

Proposition 4.2 *Fix* $\varphi, \psi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$. *Assume that* φ (*resp.* ψ) *is a pluripotential subsolution* (*resp. supersolution*) *to* (CMAF) *such that*

(a) φ *is* $\mathcal{C}^{1/0}$ *in* t *and, for any* $t > 0$, $\varphi(t,\cdot)$ *is continuous on* Ω;

(b) ψ *is locally uniformly semiconcave in* t;

(c) $\varphi_t \to \varphi_0$ *and* $\psi_t \to \psi_0$ *in* L^1 *as* $t \to 0$;

(d) *the function* $(t,x) \mapsto \psi(t,x)$ *is continuous on* $[0,T[\times \Omega$.

Then

$$\varphi_0 \le \psi_0 \implies \varphi \le \psi \quad \text{on } X_T.$$

A particular case of this result was established in [16, Theorem 3.1].

Proof We fix $T' \in]0,T[$ and we prove that $\varphi \le \psi$ on $[0,T'] \times X$. The result then follows by letting $T' \to T$. Using the invariance properties of the family of equations (see Section 3.3), we can assume without loss of generality that $r \mapsto F(\cdot,\cdot,r)$ is increasing. We proceed in several steps.

Construction of auxiliary functions We first introduce two auxiliary functions. Let $\phi_1 \in \mathrm{PSH}\big(X, \frac{1}{2}\theta\big)$ be a $\frac{1}{2}\theta$–psh function with analytic singularities (in particular ϕ_1 is smooth in Ω) such that $\phi_1 = -\infty$ on $\partial\Omega$. We need to use this function in order to apply the classical maximum principle in Ω. Note that the ample locus of θ coincides with that of $\frac{1}{2}\theta$.

The standard strategy is to replace φ by $(1-\delta)\varphi_t + \delta\phi_1$. However, the time derivative $\dot{\varphi}_t$ may blow up as $t \to 0$, so we need to use a second auxiliary function. Let $\phi_2 \in \mathrm{PSH}\big(X, \frac{1}{2}\theta\big)$ be the unique solution to

$$\big(\tfrac{1}{2}\theta + dd^c\phi_2\big)^n = e^{c_1} g \, dV,$$

normalized by $\sup_X \phi_2 = 0$, where $c_1 \in \mathbb{R}$ is a normalization constant. Then $\phi := \phi_1 + \phi_2$ is a θ–psh function which is continuous in Ω and tends to $-\infty$ on $\partial\Omega$.

Adaptation of the arguments in [16] Fix $\varepsilon, \delta > 0$ small enough and set

$$w(t,x) := (1-\delta)\varphi(t,x) + \delta\phi(x) - \psi(t,x) - 3\varepsilon t, \quad t \in [0,T'],\ x \in \Omega.$$

This function is upper semicontinuous on $[0,T'] \times \Omega$ (see Proposition 1.2) and tends to $-\infty$ on $\partial\Omega$, hence attains a maximum at some $(t_0,x_0) \in [0,T'] \times \Omega$.

We claim that $w(t_0,x_0) \le 0$. Assume by contradiction that $w(t_0,x_0) > 0$. Then $t_0 > 0$ and we set

$$K := \{x \in \Omega : w(t_0,x) = w(t_0,x_0)\}.$$

Since w tends to $-\infty$ on $\partial\Omega$ it follows from upper semicontinuity of φ that K is a compact subset of Ω. Since $\varphi(\cdot\,,x_0)$ is differentiable in $]0,T[$, the classical maximum principle ensures that, for all $x \in K$,

$$(1-\delta)\partial_t\varphi(t_0,x) \ge \partial_t^-\psi(t_0,x) + 3\varepsilon.$$

By assumption, the partial derivative $\partial_t\varphi(t,x)$ is continuous on Ω. Moreover, by Proposition 3.12, ψ_t is continuous on Ω for all $t \in]0,T[$. By local semiconcavity of $t \mapsto \psi_t$ it then follows that, for $t \in]0,T[$ fixed, $\partial_t^-\psi(t,x)$ is upper semicontinuous in Ω. We thus can find $\eta > 0$ small enough that, by introducing the open set containing K,

$$D := \{x \in \Omega : w(t_0,x) > w(t_0,x_0) - \eta\} \Subset \Omega,$$

we have

$$(1-\delta)\partial_t\varphi(t_0,x) > \partial_t^-\psi(t_0,x) + 2\varepsilon \quad \text{for all } x \in D. \tag{4-1}$$

We note here that D is open because the function $x \mapsto w(t_0,x)$ is continuous in Ω by assumptions (a) and (d) and by the continuity of ϕ in Ω.

Set $u := (1-\delta)\varphi(t_0,\cdot) + \delta\phi$ and $v := \psi(t_0,\cdot)$. Since φ is a subsolution to (CMAF), using Lemma 3.15 we infer

$$\begin{aligned}(\omega_{t_0} + dd^c u)^n &\ge \big[(1-\delta)(\omega_{t_0} + dd^c\varphi_{t_0}) + \delta\big(\tfrac{1}{2}\theta + dd^c\phi_2\big)\big]^n \\ &\ge e^{(1-\delta)\left(\partial_t\varphi(t_0,\cdot) + F(t_0,x,\varphi(t_0,\cdot))\right) + \delta c_1} g\, dV.\end{aligned}$$

Since F is bounded on $[0,T'] \times X \times [-M,M]$ for each $M > 0$ and φ is bounded on $[0,T'] \times X$, there exists a constant $C > 0$ under control such that

$$(\omega_{t_0} + dd^c u)^n \ge e^{(1-\delta)\partial_t\varphi(t_0,\cdot) + F(t_0,x,\varphi(t_0,\cdot)) - \delta C} g\, dV,$$

in the weak sense of measures in Ω. Using (4-1) and choosing $\delta < C^{-1}\varepsilon$, we then have

$$(\omega_{t_0} + dd^c u)^n \geq e^{\partial_t^- \psi(t_0,\cdot)+F(t_0,x,\varphi(t_0,\cdot))+\varepsilon} g\, dV,$$

in the weak sense of measures in D. Since ψ is a pluripotential supersolution, Lemma 3.11 ensures

$$(\omega_{t_0} + dd^c \psi_{t_0})^n \leq e^{\partial_t^- \psi(t_0,\cdot)+F(t_0,x,\psi(t_0,\cdot))} g\, dV,$$

in the weak sense of measures in D. The last two estimates yield

$$(\omega_{t_0} + dd^c u)^n \geq e^{F(t_0,\cdot,u(\cdot))-F(t_0,\cdot,v(\cdot))+\varepsilon} (\omega_{t_0} + dd^c v)^n.$$

Recall that $u(x) > v(x) + \varepsilon t_0$ for any $x \in K$. Shrinking D if necessary, we can thus assume that $u(x) > v(x)$ for all $x \in D$.

Since $r \mapsto F(t,x,r)$ is increasing, we thus get

$$(\omega_{t_0} + dd^c u)^n \geq e^{\varepsilon} (\omega_{t_0} + dd^c v)^n,$$

in the sense of positive measures in D.

Consider now $\tilde{u} := u + \min_{\partial D}(v-u)$. Since $v \geq \tilde{u}$ on ∂D, the comparison principle Proposition 4.3, below, yields

$$\int_{\{v<\tilde{u}\}\cap D} e^{\varepsilon} (\omega_{t_0} + dd^c v)^n \leq \int_{\{v<\tilde{u}\}\cap D} (\omega_{t_0} + dd^c u)^n \leq \int_{\{v<\tilde{u}\}\cap D} (\omega_{t_0} + dd^c v)^n.$$

It then follows that $\tilde{u} \leq v$ almost everywhere in D with respect to the measure $(\omega_{t_0} + dd^c v)^n$, hence everywhere in D by the domination principle (see Proposition 4.3).

In particular,

$$u(x_0) - v(x_0) + \min_{\partial D}(v-u) = \tilde{u}(x) - v(x) \leq 0. \tag{4-2}$$

Since $K \cap \partial D = \varnothing$, we infer $w(t_0,x) < w(t_0,x_0)$ for all $x \in \partial D$, hence

$$u(x) - v(x) < u(x_0) - v(x_0) \quad \text{for all } x \in \partial D,$$

contradicting (4-2). Altogether this shows that $t_0 = 0$, thus

$$(1-\delta)\varphi + \delta\phi - \psi - 3\varepsilon t \leq \delta \sup_X |\varphi_0|$$

in $[0,T'] \times \Omega$. Letting $\delta \to 0$ and then $\varepsilon \to 0$ we arrive at $\varphi \leq \psi$ in $[0,T'] \times \Omega$ hence in $[0,T'] \times X$. □

We have used the following semilocal version of the *domination principle*:

Proposition 4.3 *Fix a nonempty open subset $D \subset X$ and let u and v be θ–psh functions on X, which are bounded in a neighborhood of D, such that*

$$\liminf_{D \ni z \to \partial D} (u-v)(z) \geq 0.$$

Then

$$\int_{\{u<v\}\cap D} \theta_v^n \leq \int_{\{u<v\}\cap D} \theta_u^n.$$

Moreover, if $\mathrm{MA}_\theta(u)(\{u<v\}\cap D)=0$ then $u \geq v$ in D.

The proof is classical (see [6; 16]) but we include it for the convenience of the reader.

Proof Up to replacing v with $\max(v,-C)$, and $\tilde{u}$ with $\max(\tilde{u},-C)$ for a large constant C, we can assume that u and v are bounded on X.

Fix $\varepsilon>0$. By the assumption on the boundary values of u and v we can find a compact subset $K \Subset D$ such that $\max(u, v-\varepsilon)=u$ on $D \setminus K$. It follows from the Stokes theorem that, for any smooth test function $\chi \in C^\infty(D)$ with compact support and $\chi=1$ in a neighborhood of K,

$$\int_D \chi(\theta + dd^c \max(u, v-\varepsilon))^n = \int_D \chi(\theta + dd^c u)^n.$$

Letting χ increase to $\mathbf{1}_D$, we arrive at

$$\int_D (\theta + dd^c \max(u, v-\varepsilon))^n = \int_D (\theta + dd^c u)^n.$$

Since the Monge–Ampère operator is local with respect to the plurifine topology, we have that

$$\mathbf{1}_{\{u>v-\varepsilon\}}(\theta + dd^c \max(u, v-\varepsilon))^n = \mathbf{1}_{\{u>v-\varepsilon\}}(\theta + dd^c u)^n$$

and

$$\mathbf{1}_{\{u<v-\varepsilon\}}(\theta + dd^c \max(u, v-\varepsilon))^n = \mathbf{1}_{\{u<v-\varepsilon\}}(\theta + dd^c v)^n.$$

Combining all these, we obtain

$$\int_D (\theta + dd^c u)^n \geq \int_{\{u>v-\varepsilon\}\cap D} (\theta + dd^c u)^n + \int_{\{u<v-\varepsilon\}\cap D} (\theta + dd^c v)^n,$$

hence

$$\int_{\{u<v-\varepsilon\}\cap D} (\theta + dd^c v)^n \leq \int_{\{u \leq v-\varepsilon\}\cap D} (\theta + dd^c u)^n.$$

Letting $\varepsilon \to 0$, we prove the first statement.

Now, assume moreover that $\mathrm{MA}_\theta(u)(\{u < v\} \cap D) = 0$. Adding a constant to both u and v, we can assume that $v \geq 0$. We can then apply the comparison principle above to u and λv for $\lambda \in]0,1[$ to obtain that $(\theta + dd^c \lambda v)^n$ vanishes in $\{u < \lambda v\} \cap D$. Since $\theta > 0$ in Ω, it follows that $u \geq \lambda v$ a.e. in D, hence everywhere in D because these are quasi-psh functions. Letting $\lambda \to 1$, we arrive at the conclusion. □

We now remove assumption (d) in Proposition 4.2:

Proposition 4.4 *Fix $\varphi, \psi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ such that φ (respectively ψ) is a pluripotential subsolution (respectively supersolution) to (CMAF). If the assumptions* (a), (b) *and* (c) *of Proposition 4.2 are satisfied then*

$$\varphi_0 \leq \psi_0 \implies \varphi \leq \psi \quad \text{on } X_T.$$

Proof We assume without loss of generality that $r \mapsto F(\cdot,\cdot,r)$ is increasing and use an argument similar to that of Lemma 3.14. Fix $s > 0$ small enough and consider

$$v_s(t,x) := \psi(t+s,x) + Cst + Cs - Cs\log s$$

and

$$u_s(t,x) := \alpha_s\varphi(t,x) + (1-\alpha_s)\rho - Cst - Cs,$$

where $\alpha_s := (1-s)(1-As) \in]0,1[$, $A > 0$ is defined in (0-1), ρ is a quasi-psh function and C is a positive constant to be specified later. The goal is to show that for $C > 0$ large enough (under control), v_s is a supersolution while u_s is a subsolution to a parabolic equation and $u_s(0,\cdot) \leq v_s(0,\cdot)$. We can then invoke Proposition 4.2 and let $s \to 0$ to obtain the result.

By considering s small enough we can assume that

$$\beta_s := \frac{(1-As)s}{1-\alpha_s} \geq \varepsilon_1 > 0$$

for some uniform constant ε_1. We let ρ be the unique $\varepsilon_1\theta$–psh function on X such that $\sup_X \rho = 0$ and $(\varepsilon_1\theta + dd^c\rho)^n = e^{c_1} g\, dV$. By definition one has that $\alpha_s + (1-\alpha_s)\beta_s = 1 - As$. Then one can show that

$$\begin{aligned}(\omega_{t+s} + dd^c u_s)^n &\geq [(1-As)\omega_t + \alpha_s dd^c\varphi_t + (1-\alpha_s)dd^c\rho]^n \\ &= [\alpha_s(\omega_t + dd^c\varphi_t) + (1-\alpha_s)(\beta_s\omega_t + dd^c\rho)]^n \\ &\geq e^{\alpha_s(\partial_t\varphi_t + F(t,\cdot,\varphi(t,\cdot))) + (1-\alpha_s)c_1} g\, dV,\end{aligned}$$

where in the last line we use Lemma 3.15. Since $\alpha_s = 1 + O(s)$ and F is bounded, by choosing $C > 0$ large enough (depending on M_F, c_1) we have

$$(\omega_{t+s} + dd^c u_s)^n \geq e^{\partial_t u_s + F(t,\cdot,u_s(t,\cdot))} g\, dV.$$

On the other hand, for $C > 0$ large enough (which depends on κ_F) we have

$$(\omega_{t+s} + dd^c v_s)^n \leq e^{\partial_t v_s - Cs + F(t+s,\cdot,v_s(t,\cdot))} g\, dV \leq e^{\partial_t v_s + F(t,\cdot,v_s(t,\cdot))} g\, dV.$$

Up to increasing C it follows from Proposition 3.13 that $v_s(0,x) \geq \psi_0(x)$. Since $u_s(0,x) \leq v_s(0,x)$ it then follows from Proposition 4.2 that

$$u_s(t,x) \leq v_s(t,x) \quad \text{for all } (t,x) \in X_T.$$

Letting $s \to 0$, we arrive at the conclusion, finishing the proof. □

Proof of Theorem 4.1 Fix $T' < T$. We regularize the subsolution φ by applying Proposition 3.16. The family of subsolutions obtained this way is denoted by u_ε. Then u_ε is a pluripotential subsolution to (CMAF) and, up to enlarging the constant $B > 0$ in Proposition 3.16, we can also assume that $u_\varepsilon(t,x)$ converges to φ_0 in $L^1(X, dV)$ as $t \to 0^+$.

It follows moreover from Proposition 3.12 that for $t \in]0,T[$ fixed, ψ_t is continuous in Ω. We can thus apply Proposition 4.2 and obtain $u_\varepsilon \leq \psi$ on $]0,T'] \times X$. Letting $\varepsilon \to 0$ and then $T' \to T$, we arrive at the conclusion. □

Corollary 4.5 *Fix φ_0 a bounded ω_0–psh function. There exists a unique function $\varphi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ such that*

- *$x \mapsto \varphi(t,x)$ is continuous on Ω for each $t \in]0,T[$;*
- *$|\partial_t \varphi(t,x)| \leq C/t$ for all $(t,x) \in]0,T[\times X$;*
- *$t \mapsto \varphi(t,\cdot)$ is locally uniformly semiconcave in $]0,T[$;*
- *$\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0$;*
- *φ solves (CMAF) in X_T.*

In particular, any smooth approximants converge towards this solution, hence the latter is independent of the approximants.

Definition 4.6 Given data $(F, g, \omega, \varphi_0)$ we let $\Phi(F, g, \omega, \varphi_0)$ denotes the unique pluripotential solution to (CMAF), as in Corollary 4.5.

4.2 Uniqueness and stability

We now establish a more general uniqueness result. The proof relies on a delicate comparison principle and also yields stability results.

4.2.1 Stability, I

Proposition 4.7 *Assume that (g, F, ω_t) and $(g_j, F_j, \omega_{t,j})$ satisfy the assumptions in the introduction with uniform constants independent of j, and*

- *g_j and F_j are smooth;*
- *$0 < g_j$ converge in $L^p(X)$ to $g \in L^p(X)$;*
- *$\varphi_{0,j}$ are uniformly bounded ω_0–psh functions which converge in $L^1(X)$ towards $\varphi_0 \in L^\infty \cap \mathrm{PSH}(X, \omega_0)$;*
- *F_j uniformly converge to F on $[0, T[\times X \times J$ for each $J \Subset \mathbb{R}$;*
- *$\omega_{t,j}$ uniformly converges to ω_t.*

Let φ_j be the unique smooth solutions to (CMAF) with data $(g_j, F_j, \omega_{t,j}, \varphi_{0,j})$. Then (φ_j) converges in $L^1_{\mathrm{loc}}(X_T)$ to φ, where φ is the unique solution of (CMAF) with data $(g, F, \omega_t, \varphi_0)$ provided by Corollary 4.5.

Proof The sequence (φ_j) satisfies the conditions of Theorem 2.10. Hence, Theorem 2.11 ensures that a subsequence of (φ_j), still denoted by (φ_j), converges in $L^1_{\mathrm{loc}}(X_T)$ to a function $\varphi \in \mathcal{P}(X_T, \omega)$ which

- is a pluripotential solution to (CMAF);
- is locally uniformly semiconcave in $t \in]0, T[$;
- satisfies $|\partial_t \varphi| \le C/t$ on $]0, T[\times X$.

It follows from Proposition 3.13 that there exist uniform constants $C > 0$ and $t_0 > 0$ such that, for all $(t, x) \in]0, t_0] \times X$,

$$\varphi_j(t, x) \ge (1-t)e^{-Ct}\varphi_{0,j}(x) + C(t \log t - t).$$

Letting $j \to +\infty$ we obtain, for all $(t, x) \in]0, t_0] \times X$,

$$\varphi(t, x) \ge (1-t)e^{-Ct}\varphi_0(x) + C(t \log t - t).$$

This lower bound and Proposition 2.3 ensure that $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$. It finally follows from Corollary 4.5 that $\varphi \in \mathcal{P}(X_T, \omega)$ is uniquely determined. □

4.2.2 Comparison principle, II We now extend the comparison principle (Theorem 4.1), avoiding the space continuity assumption on the subsolution φ, as well as the Lipschitz-type control at the origin:

Theorem 4.8 *Assume that $\varphi, \psi \in \mathcal{P}(X_T) \cap L^\infty(X_T)$ are such that*

- *φ is a pluripotential subsolution to* (CMAF);
- *ψ is a pluripotential supersolution to* (CMAF);
- *ψ is locally uniformly semiconcave in t;*
- *$\varphi_t \to \varphi_0$ and $\psi_t \to \psi_0$ in L^1 as $t \to 0$.*

If $\varphi_0 \le \psi_0$ then $\varphi \le \psi$.

Proof We fix $T' < T$ and prove that $\varphi \le \psi$ on $[0, T'] \times X$. The result follows then by letting $T' \to T$. Using the invariance properties of our family of equations, we can assume that $r \mapsto F(\cdot, \cdot, r)$ is increasing.

If $\widetilde{\psi}$ is the unique pluripotential solution constructed by approximation (see Corollary 4.5), Theorem 4.1 ensures that $\widetilde{\psi} \le \psi$ on $X_{T'}$. We can thus assume without loss of generality that $\psi = \widetilde{\psi}$. We proceed in several steps.

Step 1 Assume that the data (ω_t, g, F, ψ_0) is smooth, $g > 0$, θ is Kähler and the time derivative $\partial_t \varphi(t, x)$ is uniformly bounded in $]0, T'] \times X$.

Then ψ is smooth since there exists a unique smooth solution (as follows from [23; 9; 36] and Corollary 4.5).

If $x \mapsto \varphi(\cdot, x)$ were known to be continuous, we could invoke Theorem 4.1 to conclude. In absence of this extra assumption, we take a little detour inspired by viscosity techniques.

Step 1.1 Assume the following conditions:

(1) For all $x \in X$, $\dot{\varphi}_t(x)$ exists and $t \mapsto \dot{\varphi}_t(x)$ is continuous in $t \in]0, T']$.

(2) φ_t and $\dot{\varphi}_t$ are uniformly quasicontinuous on X. This means, for any $\varepsilon > 0$, there exists an open subset U with $\mathrm{Vol}(U) < \varepsilon$ such that on the compact set $X \setminus U$ the functions $x \mapsto \varphi_t(x)$ and $x \mapsto \dot{\varphi}_t(x)$ are continuous for all $t \in]0, T']$.

The first condition ensures that the inequality

$$(\omega_t + dd^c \varphi_t)^n \ge e^{\dot{\varphi}_t + F(t, \cdot, \varphi_t(\cdot))} g \, dV$$

holds in the pluripotential sense on X for all $t \in]0, T']$. As will be shown later, the regularizing family $\varphi_\varepsilon(t, x)$ as constructed in Proposition 3.16 satisfies these two conditions.

We introduce the constants $M := 2 + M_1 + M_2$, where

$$M_1 := \sup_{[0,T']\times X} |\varphi(t, x)|$$

and

$$M_2 := \sup_{[0,T']\times X} \big(|\dot{\varphi}_t(x)| + |F(t, x, \varphi_t(x))| + |F(t, x, \psi_t(x))|\big).$$

Fix $\varepsilon > 0$ small enough. By uniform quasicontinuity of φ_t and $\dot{\varphi}_t$, there exists an open set U such that

- $\mathrm{Vol}(U) < e^{-4M/\varepsilon}$;
- for all $t \in]0, T']$, the functions φ_t and $\dot{\varphi}_t$ are continuous on $X \setminus U$.

Let ρ_ε be the unique bounded θ–psh function on X such that

$$(\theta + dd^c \rho_\varepsilon)^n = e^{2M/\varepsilon} \mathbf{1}_U g\, dV + a_\varepsilon g\, dV, \quad \sup_X \rho_\varepsilon = 0.$$

Here $0 \le a_\varepsilon$ is a normalization constant. The boundedness of g yields

$$\int_U e^{2M/\varepsilon} g\, dV \le e^{2M/\varepsilon} \sup_X g,$$

hence, for $\varepsilon > 0$ small enough, $a_\varepsilon \ge \frac{1}{2}$. Also, the L^2–norm of the density of $(\theta + dd^c \rho_\varepsilon)^n$ is uniformly bounded, hence, by [12, Proposition 2.6], ρ_ε is uniformly bounded.

Set, for $(t, x) \in X_{T'}$,

$$u_\varepsilon(t, x) := (1 - \varepsilon)\varphi_t + \varepsilon\rho_\varepsilon.$$

We prove that $u_\varepsilon - \psi - C\varepsilon t \le 0$ in $[0, T'] \times X$, where $C := 2M + M_1\kappa_F$. By contradiction assume that it were not the case. Since the function is upper semicontinuous on the compact set $[0, T'] \times X$, its maximum is attained at some $(t_0, x_0) \in]0, T'] \times X$. We then have

$$u_\varepsilon(t_0, x_0) - \psi(t_0, x_0) \ge C\varepsilon t_0 > 0,$$

hence

$$\varphi(t_0, x_0) \ge \psi(t_0, x_0) - \varepsilon M_1. \tag{4-3}$$

By the classical maximum principle we have

$$(1-\varepsilon)\dot{\varphi}_{t_0}(x_0) \geq C\varepsilon + \dot{\psi}_{t_0}(x_0). \tag{4-4}$$

By assumption (1), $t \mapsto \dot{\varphi}_t(x)$ is continuous on $]0, T'[$ for all $x \in X$. Since φ is a subsolution to (CMAF), Lemma 3.11 then ensures that

$$(\omega_{t_0} + dd^c \varphi_{t_0})^n \geq e^{\dot{\varphi}_{t_0} + F(t_0, \cdot, \varphi_{t_0}(\cdot))} g\, dV$$

holds in the sense of measures on X. By construction of ρ_ε, we also have

$$(\omega_{t_0} + dd^c \rho_\varepsilon)^n \geq (\theta + dd^c \rho_\varepsilon)^n \geq e^{f_\varepsilon} g\, dV,$$

where

$$f_\varepsilon := \begin{cases} 2M/\varepsilon & \text{if } x \in U, \\ -\log 2 & \text{if } x \in X \setminus U. \end{cases}$$

It then follows from Lemma 3.15 that

$$(\omega_{t_0} + dd^c u_\varepsilon(t_0, \cdot))^n \geq e^{h_\varepsilon} g\, dV, \tag{4-5}$$

where

$$h_\varepsilon := \begin{cases} M & \text{if } x \in U, \\ (1-\varepsilon)\dot{\varphi}_{t_0} + (1-\varepsilon)F(t_0, x, \varphi_{t_0}(x)) - \varepsilon \log 2 & \text{if } x \in X \setminus U. \end{cases}$$

The assumption (2) ensures that h_ε is lower semicontinuous on $X \setminus U$. The choice of M then shows that h_ε is lower semicontinuous on X and

$$h_\varepsilon(x) \geq (1-\varepsilon)\dot{\varphi}_{t_0}(x) + (1-\varepsilon)F(t_0, x, \varphi_{t_0}(x)) - \varepsilon \log 2 \quad \text{for all } x \in X. \tag{4-6}$$

It then follows from Lemma 4.9 that (4-5) holds in the viscosity sense. The function $x \mapsto \psi_{t_0}(x) - \psi_{t_0}(x_0) + u_\varepsilon(t_0, x_0)$ is a smooth upper test for $u_\varepsilon(t_0, \cdot)$ at x_0, hence

$$(\omega_{t_0} + dd^c \psi_{t_0})^n \geq e^{h_\varepsilon} g\, dV$$

holds in the classical sense at x_0. Now, from (4-4) and (4-6) we have

$$(\omega_{t_0} + dd^c \psi_{t_0})^n(x_0) \geq e^{\dot{\psi}_{t_0}(x_0) + (C - \log 2)\varepsilon + (1-\varepsilon)F(t_0, x_0, \varphi_{t_0}(x_0))} g(x_0)\, dV. \tag{4-7}$$

It follows from (4-3) and the monotonicity of $r \mapsto F(t, x, r)$ that

$$F(t_0, x_0, \varphi_{t_0}(x_0)) \geq F(t_0, x_0, \psi_{t_0}(x_0) - M_1\varepsilon) \geq F(t_0, x_0, \psi_{t_0}(x_0)) - \kappa_F M_1 \varepsilon.$$

Hence,

$$\begin{aligned} (1-\varepsilon)F(t_0, x_0, \varphi_{t_0}(x_0)) &\geq F(t_0, x_0, \psi_{t_0}(x_0)) - \kappa_F M_1 \varepsilon - M\varepsilon \\ &\geq F(t_0, x_0, \psi_{t_0}(x_0)) - (C - M)\varepsilon. \end{aligned}$$

This together with (4-7) gives a contradiction since ψ is a solution to (CMAF).

We thus have that $u_\varepsilon \le \psi + C\varepsilon t$ on $[0,T'] \times X$. Letting $\varepsilon \to 0$ we arrive at the conclusion.

Step 1.2 We next remove the assumptions on φ in Step 1.1.

Using Proposition 3.16 we can find φ^ε which are pluripotential subsolutions to (CMAF) and which satisfy the assumptions in Step 1.1. Indeed, it suffices to check that φ_t is uniformly quasicontinuous on X. But this holds by quasicontinuity of φ_t and by the Lipschitz condition. More precisely, for fixed $\varepsilon > 0$, there exists an open subset U with $\mathrm{Vol}(U) < \varepsilon$ such that φ_t is continuous on $X \setminus U$ for all $t \in]0,T[\cap \mathbb{Q}$. The continuity of φ_t on $X \setminus U$ for irrational points t follows from the Lipschitz property of the family φ_t.

Thus the previous step applies and yields $\varphi^\varepsilon \le \psi + O(\varepsilon)$. Letting $\varepsilon \to 0$, we arrive at $\varphi \le \psi$.

Step 2 We finally remove the smoothness assumption on the data and the Lipschitz condition on φ by using the stability result above together with an argument from [16].

Let (g_j, F_j) be smooth approximants of (g, F). Fix $\varepsilon > 0$ small enough and consider

$$\varphi_{\varepsilon,j}(t,x) := (1-\delta_j)\varphi(t+\varepsilon,x) + \delta_j\rho_j + n\log(1-\delta_j) - (B\delta_j + C\varepsilon + \eta_j)t,$$

where $\delta_j \in \left]0, \frac{1}{2}\right[$ and $\eta_j \ge 0$ will be specified later, and $\rho_j \in \mathrm{PSH}(X,\theta)$ is the unique solution to

$$(\theta + dd^c\rho_j)^n = \left(a_j + \frac{|g_j - g|}{\|g_j - g\|_p}\right) dV,$$

normalized by $\sup_X \rho_j = 0$ for a normalization constant $a_j \ge 0$.

We are going to prove that, for suitable choices of B, C, δ_j and η_j, the function $\varphi_{\varepsilon,j}$ is a pluripotential subsolution to (CMAF) with data $(\omega_{\varepsilon,j}, g_j, F_j)$, where

- $\omega_{\varepsilon,j}(t)$ is a smooth family of Kähler forms such that $\omega_{\varepsilon,j}$ satisfies the assumptions in the introduction, $\omega_{\varepsilon,j}(t) \ge \omega(t+\varepsilon)$, and $\omega_{\varepsilon,j}(t)$ converges to $\omega(t+\varepsilon)$ as $j \to +\infty$;
- $0 < g_j$ is smooth and $\|g_j - g\|_p \to 0$;
- F_j is smooth in $]0,T[\times X \times \mathbb{R}$ with the same Lipschitz and semiconvexity constants as F, and F_j locally uniformly converge to F.

Let $\psi_{\varepsilon,j}$ be the unique smooth solution to (CMAF) with the above data $(\omega_{\varepsilon,j}, g_j, F_j)$ such that $\psi_{\varepsilon,j}(0,\cdot) \ge (1-\delta_j)\varphi(\varepsilon,\cdot)$ and

$$\int_X \psi_{\varepsilon,j}(0,x)\,dV(x) \le \int_X (1-\delta_j)\varphi(\varepsilon,x)\,dV(x) + 2^{-j}.$$

It follows from Proposition 4.7 that the $\psi_{\varepsilon,j}$ converge, as $j \to +\infty$ and $\varepsilon \to 0$, to some $\widetilde{\psi} \in \mathcal{P}(X_{T'})$ which is a pluripotential solution to (CMAF) with data (ω_t, g, F) and initial value φ_0. Moreover, $\widetilde{\psi}$ and ψ satisfy the assumptions of Theorem 4.1, hence $\widetilde{\psi} \le \psi$.

We now prove that $\varphi \le \widetilde{\psi}$ by showing that $\varphi_{\varepsilon,j}$ is a subsolution to an approximate (CMAF). Since φ is locally uniformly Lipschitz, there exists a constant C_1 (depending also on ε) such that

$$\sup_{[\varepsilon,T']\times X} |\dot{\varphi}| \le C_1,$$

hence

$$\partial_t\varphi(t+\varepsilon,x) \ge (1-\delta_j)\partial_t\varphi(t+\varepsilon,x) - C_1\delta_j = \partial_t\varphi_{\varepsilon,j}(t,x) + (B-C_1)\delta_j + C\varepsilon + \eta_j.$$

A direct computation yields

$$(\omega_{\varepsilon,j} + dd^c\varphi_{\varepsilon,j})^n \ge e^{n\log(1-\delta_j)+\partial_t\varphi(t+\varepsilon,x)+F(t+\varepsilon,x,\varphi(t+\varepsilon,x))} g\,dV + \delta_j^n \frac{|g_j-g|}{\|g_j-g\|_p}\,dV.$$

Set $M_\varphi := \sup_{X_T}|\varphi|$ and $J := [-M_\varphi, M_\varphi]$ and

$$\eta_j := \sup\{|F(t,x,r)-F_j(t,x,r)| : (t,x,r) \in [0,T'] \times X \times J\}.$$

Then $\eta_j \to 0$ as $j \to +\infty$. Setting

$$\delta_j^n := e^{C_1+M_F}\|g_j-g\|_p,$$

where

$$M_F := \sup\{|F(t,x,r)| : (t,x,r) \in [0,T[\times J\},$$

and considering j large enough $\bigl($so that $\delta_j \le \frac{1}{2}\bigr)$, we obtain

$$(\omega_{\varepsilon,j} + dd^c\varphi_{\varepsilon,j})^n \ge e^{n\log(1-\delta_j)+\partial_t\varphi(t+\varepsilon,x)+F(t+\varepsilon,x,\varphi(t+\varepsilon,x))} g_j\,dV.$$

The Lipschitz condition on F ensures that

$$F(t+\varepsilon,x,\varphi(t+\varepsilon,x)) - F(t,x,(1-\delta_j)\varphi(t+\varepsilon,x)) \ge -C_2(\delta_j+\varepsilon)$$

for a uniform constant $C_2 > 0$. Choosing positive B and C large enough and using $\log(1-\delta_j) \geq -\delta_j$, we conclude that

$$\begin{aligned}(\omega_{\varepsilon,j}(t,x) + dd^c\varphi_{\varepsilon,j}(t,x))^n &\geq e^{\partial_t\varphi_{\varepsilon,j} + F(t,x,\varphi_{\varepsilon,j}) + \eta_j} g_j\, dV \\ &\geq e^{\partial_t\varphi_{\varepsilon,j} + F_j(t,x,\varphi_{\varepsilon,j})} g_j\, dV.\end{aligned}$$

Thus $\varphi_{\varepsilon,j}$ is a subsolution to (CMAF) for the data $(g_j, F_j, \omega_{\varepsilon,j})$.

We can now apply Step 1 to obtain $\varphi_{\varepsilon,j} \leq \psi_{\varepsilon,j}$. Letting $j \to +\infty$ and then $\varepsilon \to 0$ finally shows that $\varphi \leq \psi$. □

We have used the following straightforward extension of [13, Proposition 1.5]:

Lemma 4.9 *Assume that u is a psh function in an open set $U \subset \mathbb{C}^n$ and $(dd^c u)^n \geq e^f\, dV$ in the pluripotential sense, where f is lower semicontinuous in U. Then the inequality holds in the viscosity sense.*

Corollary 4.10 *There exists a unique solution to the Cauchy problem for (CMAF) which is locally uniformly semiconcave in t. It is the envelope of pluripotential subsolutions.*

4.2.3 Stability, II We are now ready to prove Theorem C of the introduction. We assume here that

- $F, G\colon \hat{X}_T := [0,T[\times X \times \mathbb{R} \to \mathbb{R}$ are continuous;
- F and G are increasing in the last variable;
- F and G are uniformly Lipschitz in r with Lipschitz constants L_F and L_G.
- $0 \leq f, g \in L^p(X)$ with $p > 1$.

The Lipschitz assumption on F means that, for all $(t,x) \in X_T$,

$$|F(t,x,r) - F(t,x,r')| \leq L_F |r - r'|.$$

Theorem 4.11 *Assume that $\varphi \in \mathcal{P}(X_T,\omega)$ is a solution to the parabolic equation (CMAF) with admissible data (F, f) and $\psi \in \mathcal{P}(X_T,\omega)$ is a bounded solution to (CMAF) with admissible data (G, g).*

There exists $\alpha \in]0,1[$ and for any $\varepsilon > 0$ there exists $A(\varepsilon) > 0$ such that

$$\sup_{[\varepsilon,T[\times X} |\varphi - \psi| \leq A(\varepsilon)\|\varphi_\varepsilon - \psi_\varepsilon\|^{\alpha}_{L^1(X)} + T \sup_{\hat{X}_T} |F - G| + A(\varepsilon)\|g - f\|_p^{1/n}.$$

In particular, if

- *(g_j) are densities which converge to g in $L^p(X)$,*

- *F_j converges to F locally uniformly,*
- *$\varphi_{0,j}$ are bounded ω_0–psh functions converging in $L^1(X)$ to φ_0,*

then $\Phi(F_j, g_j, \varphi_{0,j})$ locally uniformly converges to $\Phi(F, g, \varphi_0)$.

Here we denote by $\Phi(F, g, \varphi_0)$ the solution to the Cauchy problem for the admissible data (F, g, φ_0).

Proof Set $\Phi^j = \Phi(F_j, g_j, \varphi_{0,j})$ and $\Phi = \Phi(F, g, \varphi_0)$. The quantitative estimate is a simple consequence of Proposition 4.12 below. The norm $\|\Phi_\varepsilon^j - \Phi_\varepsilon\|_{L^1(X)}$ is controlled by $\|\Phi^j - \Phi\|_{L^1([\varepsilon,T]\times X)}$ as follows from Lemma 1.5. By Proposition 3.1, Φ^j converges in $L^1_{\mathrm{loc}}(X_T)$ to Φ, hence the last statement of the theorem follows. □

The stability result is a consequence of the following quantitative version of the comparison principle:

Proposition 4.12 *Assume $\varphi \in \mathcal{P}(X_T, \omega)$ is a subsolution to (CMAF) with data (F, f) and $\psi \in \mathcal{P}(X_T, \omega)$ is a supersolution to (CMAF) with data (G, g).*

Fix $\varepsilon > 0$. There exists $\alpha, A, B > 0$ such that, for all $(t, x) \in [\varepsilon, T[\times X$,

$$\varphi(t,x) - \psi(t,x) \le B\|(\varphi_\varepsilon - \psi_\varepsilon)_+\|^\alpha_{L^1(X)} + T\sup_{\hat{X}_T}(G-F)_+ + A\|(g-f)_+\|_p^{1/n},$$

where $A, B > 0$ depend on X, θ, n, p, a uniform bound on $\dot{\varphi}$, $\dot{\psi}$, φ and ψ on the set $[\varepsilon, T[\times X$, $\sup_{X_T} G(t, x, \sup_{X_T}\varphi)$, and L_G.

Proof We use a perturbation argument as in [16] which goes back to the work of Kołodziej [26]. For convenience we normalize θ so that $\int_X dV = \int_X \theta^n = 1$. Set

$$m_0 := \inf_{X_T}\varphi, \quad m_1(\varepsilon) := \inf_{[\varepsilon,T[\times X}\dot{\varphi} \quad \text{and} \quad M := \sup_{\hat{X}_T}(G-F)_+.$$

We first assume that $\|(g-f)_+\|_p > 0$. It follows from [12] (see also [27] in the Kähler case) that there exists $\rho \in \mathrm{PSH}(X, \theta) \cap L^\infty(X)$, normalized by $\max_X \rho = 0$, such that

$$(\theta + dd^c\rho)^n = \left(a + \frac{(g-f)_+}{\|(g-f)_+\|_p}\right) dV, \tag{4-8}$$

where $a \ge 0$ is a normalizing constant given by

$$a := 1 - \frac{\|(g-f)_+\|_1}{\|(g-f)_+\|_p} \in [0, 1].$$

We moreover have a uniform bound on ρ which only depends on the L^p–norm of the density of $(\theta+dd^c\rho)^n$, which is here bounded from above by 2,

$$\|\rho\|_\infty \le C_0(a+1) \le 2C_0, \tag{4-9}$$

where $C_0>0$ is a uniform constant depending only on (X,θ,p).

For $0<\delta<1$ and $(t,x)\in X_T$, we set

$$\varphi_\delta(t,x) := (1-\delta)\varphi(t,x)+\delta\rho+n\log(1-\delta)-B\delta t-Mt.$$

The plan is to choose $B>0$ in such a way that φ_δ is a subsolution to (CMAF) on $[\varepsilon,T[$ with data (G,g). The conclusion will then follow from the comparison principle (Theorem 4.8).

Observe that for almost all $t\in[\varepsilon,T[$ fixed, $\varphi_\delta(t,\cdot)$ is ω_t–plurisubharmonic on X and

$$(\omega_t+dd^c\varphi_\delta(t,\cdot))^n \ge (1-\delta)^n(\omega_t+dd^c\varphi_t)^n+\delta^n(\theta+dd^c\rho)^n.$$

Using that φ is a subsolution to (CMAF) with data (F,f), we infer

$$(\omega_t+dd^c\varphi_\delta(t,\cdot))^n \ge e^{\dot\varphi_t+F(t,\cdot,\varphi_t)+n\log(1-\delta)}f\,dV+\delta^n\frac{(g-f)_+}{\|(g-f)_+\|_p}\,dV. \tag{4-10}$$

Noting that $\varphi\ge\varphi_\delta+\delta\varphi$ and recalling that G is increasing in the last variable, we obtain

$$\begin{aligned}
&\dot\varphi(t,x)+F(t,x,\varphi(t,x))+n\log(1-\delta)\\
&\qquad\ge \dot\varphi_\delta(t,x)+\delta\dot\varphi(t,x)+G(t,x,\varphi(t,x))-M+n\log(1-\delta)+B\delta+M\\
&\qquad\ge \dot\varphi_\delta(t,x)+\delta\dot\varphi(t,x)+G(t,x,\varphi_\delta(t,x)+\delta\varphi(t,x))+n\log(1-\delta)+B\delta\\
&\qquad\ge \dot\varphi_\delta(t,x)+G(t,x,\varphi_\delta(t,x)+\delta m_0)+\delta m_1(\varepsilon)+n\log(1-\delta)+B\delta.
\end{aligned}$$

The Lipschitz condition on G yields, writing $L=L_G$,

$$\begin{aligned}
&\dot\varphi(t,x)+F(t,x,\varphi(t,x))+n\log(1-\delta)\\
&\qquad\ge \dot\varphi_\delta(t,x)+G(t,x,\varphi_\delta(t,x))+B\delta-L\delta|m_0|+\delta m_1(\varepsilon)+n\log(1-\delta).
\end{aligned}$$

Using the elementary inequality $\log(1-\delta)\ge -2(\log 2)\delta$ for $0<\delta\le\frac12$, it follows that, for $0<\delta\le\frac12$,

$$B\delta-L\delta|m_0|+m_1(\varepsilon)\delta+n\log(1-\delta)\ge(B-L|m_0|+m_1(\varepsilon)-2n\log 2)\delta.$$

We now choose $B:=L|m_0|+2n\log 2-m_1(\varepsilon)$, so that

$$\dot\varphi(t,x)+F(t,x,\varphi(t,x))+n\log(1-\delta)\ge\dot\varphi_\delta(t,x)+G(t,x,\varphi_\delta(t,x)),$$

which, together with (4-10), yields

$$(\omega_t + dd^c\varphi_\delta(t,\cdot))^n \ge e^{\dot\varphi_\delta(t,\cdot)+G(t,\cdot,\varphi_\delta(t,\cdot))} f\, dV + \delta^n \frac{(g-f)_+}{\|(g-f)_+\|_p}. \tag{4-11}$$

On the other hand, if we set

$$M_1(\varepsilon) := \sup_{[\varepsilon,T[\times X} \dot\varphi, \quad M_0 := \sup_{X_T} \varphi \quad \text{and} \quad M_2 := \sup_{X_T} G(t,x,M_0),$$

then the Lipschitz property of G ensures, for $(t,x) \in [\varepsilon, T[\times X$,

$$\begin{aligned}\dot\varphi_\delta(t,x) + G(t,x,\varphi_\delta(t,x)) &\le (1-\delta) \sup_{[\varepsilon,T[\times X} \dot\varphi + \sup_{X_T} G(t,x,(1-\delta)\varphi(t,x)) \\ &\le (1-\delta)M_1(\varepsilon) + \sup_{X_T} G(t,x,(1-\delta)M_0) \\ &\le (1-\delta)M_1(\varepsilon) + M_0 L\delta + M_2 \\ &\le M_2 + \max\{LM_0, M_1(\varepsilon)\}.\end{aligned}$$

Using (4-11), we conclude that, for $0<\delta<\frac{1}{2}$, $x \in X$ and almost all $t \in [\varepsilon, T[$,

$$\begin{aligned}&(\omega_t + dd^c\varphi_\delta(t,\cdot))^n \\ &\qquad \ge e^{\dot\varphi_\delta(t,\cdot)+G(t,\cdot,\varphi_\delta(t,\cdot))}\left(f + \delta^n e^{-M_3(\varepsilon)} \frac{(g-f)_+}{\|(g-f)_+\|_p}\right) dV,\end{aligned} \tag{4-12}$$

where $M_3(\varepsilon) := M_2 + \max\{LM_0, M_1(\varepsilon)\}$.

To conclude that φ_δ is a subsolution, we finally set

$$\delta := \|(g-f)_+\|_p^{1/n} e^{M_3(\varepsilon)/n}. \tag{4-13}$$

Assume first that $\|(g-f)_+\|_p \le 2^{-n} e^{-M_3(\varepsilon)}$, so that $\delta \le \frac{1}{2}$. It follows from (4-12) that, for almost all $t \in [\varepsilon, T[$,

$$\begin{aligned}(\omega_t + dd^c\varphi_\delta(t,\cdot))^n &\ge e^{\dot\varphi_\delta(t,\cdot)+G(t,\cdot,\varphi_\delta(t,\cdot))}(f + (g-f)_+)\, dV \\ &\ge e^{\dot\varphi_\delta(t,\cdot)+G(t,\cdot,\varphi_\delta(t,\cdot))} g\, dV,\end{aligned}$$

hence φ_δ is a subsolution to (CMAF) for the data (G,g) on $[\varepsilon,T[$. The comparison principle ensures that, for all $(t,x)\in[\varepsilon,T[\times X$,

$$\varphi_\delta(t,x) - \psi(t,x) \le \max_X (\varphi_\delta(\varepsilon,\cdot) - \psi(\varepsilon,\cdot))_+.$$

Together with (4-9) and (4-13) we obtain, for $(t,x)\in[\varepsilon,T[\times X$,

$$\varphi(t,x) - \psi(t,x) \le \max_X(\varphi(\varepsilon,\cdot)-\psi(\varepsilon,\cdot))_+ + TM + A_1(\varepsilon)\|(g-f)_+\|_p^{1/n},$$

where

$$A_1(\varepsilon) := (M_0 + 2n\log 2 + BT)e^{M_3(\varepsilon)/n}.$$

When $\|(g-f)_+\|_p > 2^{-n}e^{-M_3(\varepsilon)}$, we choose $A_2(\varepsilon) > 0$ so that

$$\sup_{X_T}(\varphi(t,x) - \psi(t,x)) \le \max_X(\varphi_\varepsilon - \psi_\varepsilon)_+ + A_2(\varepsilon)2^{-n}e^{-M_3(\varepsilon)}.$$

We lastly set $A(\varepsilon) = \max\{A_1(\varepsilon), A_2(\varepsilon)\}$.

Assume finally that $\|(g-f)_+\|_p = 0$, which means that $g \le f$ almost everywhere in X. In this case, the function $(t,x) \mapsto \varphi(t,x) - Mt$ is a subsolution to (CMAF) with data (G,g) and the conclusion follows from the comparison principle.

Now observe that ψ_ε is a supersolution to the degenerate elliptic equation

$$(\omega_\varepsilon + dd^c\psi_\varepsilon)^n \le e^{D(\varepsilon)} g\,dV,$$

where $D(\varepsilon)$ is an upper bound of $\dot{\psi}(t,x) + G(t,x,\psi(t,x))$ on $[\varepsilon, T[\times X$.

By the L^∞–L^1 semistability theorem of [12, Proposition 3.3], it follows that there exists $\alpha \in]0,1[$ and a constant $C(\varepsilon) > 0$ depending on $D(\varepsilon)$, p, θ and $\|g\|_p$ such that

$$\max_X(\varphi_\varepsilon - \psi_\varepsilon)_+ \le C(\varepsilon)\|(\varphi_\varepsilon - \psi_\varepsilon)_+\|^\alpha_{L^1(X)},$$

concluding the proof. □

5 Geometric applications

In this section we show that our hypotheses are satisfied when studying the Kähler–Ricci flow on a compact Kähler variety with log terminal singularities. We prove the existence and study the long-term behavior of the normalized Kähler–Ricci flow (NKRF for short) on such varieties starting from an arbitrary closed positive current with bounded potential.

The definition and study of the Kähler–Ricci flow on mildly singular projective varieties has been undertaken by Song and Tian in [32; 33]. A different viscosity approach has been developed by Eyssidieux, Guedj and Zeriahi in [14; 15].

Our approach allows one to avoid any projectivity assumption on the varieties, to deal with more general singularities, to avoid any continuity assumption on the data, and also provide more general uniqueness and stability results. The whole discussion extends to the case of Kawamata log terminal pairs but we leave this discussion for later work.

5.1 Analytic approach to the minimal model program

5.1.1 Log terminal singularities Let Y be an irreducible compact Kähler normal complex analytic space with only terminal singularities. Let $\pi\colon X\to Y$ be a log-resolution, ie X is a compact Kähler manifold, π is a bimeromorphic projective morphism and $\mathrm{Exc}(\pi)$ is a divisor with simple normal crossings. Here $\mathrm{Exc}(\pi):=\pi^{-1}(Y\setminus Y_{\mathrm{reg}})$ is the exceptional divisor. Denote by $\{E\}_{E\in\mathcal{E}}$ the family of the irreducible components of $\mathrm{Exc}(\pi)$. With this notation, one has furthermore

$$K_X\equiv\pi^*K_Y+\sum_E a_E E,$$

where $-1<a_E\in\mathbb{Q}$, K_Y denotes the canonical $\mathbb{Q}$–line bundle on Y, whose restriction to the smooth locus is the line bundle whose sections are holomorphic top-dimensional forms (canonical forms), K_X the canonical class of X and E also denotes, with a slight abuse of language, the cohomology class of E (we refer to [25] for more details).

The log terminal condition $a_E>-1$ means that for every nonvanishing locally defined multivalued canonical form η defined over Y, the holomorphic multivalued canonical form $\pi^*\eta$ on X has poles or zeroes of order a_E along E, so that the corresponding volume form decomposes as

$$\pi^*(c_n\eta\wedge\overline{\eta})=e^{w^+-w^-}\,dV(x),$$

where $w^+=\sum_{a_E>0}a_E\log|s_E|_{h_E}$ and $w^-=\sum_{0>a_E>-1}a_E\log|s_E|_{h_E}$ are quasi-plurisubharmonic with e^{w^+} continuous and $e^{-w^-}\in L^p$ for some $p>1$ whose precise value depends on $\min_E(a_E+1)$.

5.1.2 The (normalized) Kähler–Ricci flow The Kähler–Ricci flow is the evolution equation of Kähler forms on Y

$$\frac{\partial\theta_t}{\partial t}=-\mathrm{Ric}(\theta_t),$$

starting from an initial Kähler form θ_0. These can be written as

$$\theta_t=\chi_t+dd^c\phi_t\quad\text{with }\chi_t=\theta_0+t\chi,$$

where $\chi\in c_1(K_Y)$.

One can pull back these forms via a log-resolution of singularities and consider the corresponding forms $\omega_t=\pi^*\chi_t$, which are big and semipositive (they vanish along $\mathrm{Exc}(\pi)$). The latter satisfy our main assumptions:

Lemma 5.1 *Assume that $\pi\colon X \to Y$ is a proper holomorphic map onto a compact normal Kähler space Y, with $\omega_t = \pi^* \chi_t$ and $\theta = \pi^* \omega_Y$, where ω_Y is a Kähler form on Y. Then there exists $A > 0$ such that*

$$\theta/A \le \omega_t, \quad -A\omega_t \le \dot{\omega}_t \le A\omega_t \quad \text{and} \quad \ddot{\omega}_t \le A\omega_t.$$

Proof The corresponding inequalities are valid on Y since ω_Y and θ_0 are Kähler and $t \mapsto \chi_t$ is smooth, as long as we work on a finite time interval (which is implicit). One can then transpose the inequalities from Y to X by the holomorphic mapping π. □

One can similarly consider the normalized Kähler–Ricci flow on Y

$$\frac{\partial \theta_t}{\partial t} = -\mathrm{Ric}(\theta_t) - \lambda \theta_t,$$

starting from an initial data $\theta_0 = \chi_0 + dd^c \phi_0$ with ϕ_0 being a bounded potential which is plurisubharmonic with respect to the given Kähler form χ_0 on Y, and where $\lambda \in \mathbb{R}$.

By rescaling, one can reduce to the cases $\lambda = 1, 0, -1$. To simplify the discussion we restrict to the case $\lambda = 1$. At the cohomological level, this yields a first-order ODE showing that the cohomology class of θ_t evolves as

$$\{\theta_t\} = e^{-t}\{\theta_0\} + (1 - e^{-t}) K_Y.$$

We thus define by

$$T_{\max} := \sup\{t > 0 : e^{-t}\{\theta_0\} + (1 - e^{-t}) K_Y \in \mathcal{K}(Y)\}$$

the maximal (cohomological) time of existence of the flow.

Denote by $\mathcal{K}(Y) \subset H^1(Y, \mathcal{PH}_Y)$ the open convex cone of Kähler classes and let χ_0 be a smooth Kähler representative of the Kähler class $\{\theta_0\}$. Here $\mathcal{PH}_Y$ is the sheaf of real-valued pluriharmonic functions on Y (see [12, Section 5.2] for more details). Assume h is a smooth Hermitian metric on the holomorphic $\mathbb{Q}$–line bundle K_Y. Then $\chi := -dd^c \log h$ is a smooth representative of $K_Y \in H^1(Y, \mathcal{PH}_Y)$ and we set

$$\chi_t = e^{-t} \chi_0 + (1 - e^{-t}) \chi.$$

The solution to the normalized Kähler–Ricci flow can be written as $\theta_t = \chi_t + dd^c \phi_t$, with $\pi^* \phi \in \mathcal{P}(X_T)$. We now define

$$\mu_{\mathrm{NKRF}} = c_n \frac{\pi^* \eta \wedge \overline{\pi^* \eta}}{\pi^* \|\eta\|_h^2} \in C^0(X, \Omega_X^{n,n}),$$

which we view as a continuous element of $C^0(X_T, \Omega^{n,n}_{X_T/[0,T[})$, where c_n is the unique complex number of modulus 1 such that the expression is positive. As the notation suggests, μ_{NKRF} is independent of the auxiliary multivalued holomorphic form η but depends on h.

In local coordinates μ_{NKRF} has density of the form

$$v_{\mathrm{NKRF}} = \prod_E |f_E|^{2a_E} v,$$

where $v > 0$ is smooth and f_E is an equation of E in these local coordinates.

Theorem 5.2 *The Cauchy problem with initial data $S_0 := \chi_0 + dd^c\phi_0$ for the normalized Kähler–Ricci flow on Y admits a unique pluripotential solution defined on $[0, T_{\max}[\times Y$.*

Proof Fix $T < T_{\max}$. Since $e^{-t}\{\omega_0\} + (1-e^{-t})K_Y \in \mathcal{K}(Y)$ for any $t \in [0,T]$, there exists a smooth family of Kähler forms $(\chi_t)_{0\le t\le T} \in \mathcal{K}(Y)$ such that for any $t \in [0,T]$, $\{\chi_t\} = \{\omega_t\}$.

We can write $\theta_t = \chi_t + dd^c\phi_t$, where ϕ is a solution to the corresponding Monge–Ampère flow at the level of potentials,

$$(\chi_t + dd^c\phi_t)^n = e^{\partial_t\phi+\phi_t} v_Y,$$

on Y_T for some admissible volume form v_Y on Y, or, equivalently,

$$(\omega_t + dd^c\varphi_t)^n = e^{\partial_t\varphi+\varphi_t}\mu_{\mathrm{NKRF}} = e^{\partial_t\varphi+F(t,x,\varphi_t)} g\, dV_X, \tag{5-1}$$

on a log-resolution $\pi\colon X \to Y$, where μ_{NKRF} is a volume form on X which can be locally written as

$$\mu_{\mathrm{NKRF}} = \prod_E |f_E|^{2a_E}\, dV_X = g\, dV_X,$$

where $g = \prod_E |f_E|^{2a_E} \in L^p$ for some $p > 1$, since $-1 < a_E$ for all E, and $g > 0$ almost everywhere.

We write here $\varphi := \pi^*\phi$ and $\omega_t := \pi^*\chi_t$. Since $(\chi_t)_{0\le t\le T}$ is a smooth family of Kähler forms on Y, it follows that the family of semipositive forms $[0,T[\ni t \mapsto \theta_t$ satisfies all our requirements.

Theorem 3.4 can thus be applied (with $F(t,x,r) \equiv r$) and guarantees the existence of a unique pluripotential solution to the Monge–Ampère flow on X_T for any fixed

$T < T_{\max}$ starting at φ_0. By uniqueness, all these solutions glue into a unique solution of the Monge–Ampère flow on $[0, T_{\max}[\times X$ starting at φ_0, denoted by φ_t. Pushing this solution down to Y, we obtain a solution to the NKRF starting at S_0, denoted by ϕ_t.

Now, assume that ψ_t is another solution to the flow on Y. Then $\pi^*\psi_t$ is a solution to the flow (5-1) on $\pi^{-1}(Y_{\mathrm{reg}})$. Since $X \setminus \pi^{-1}(Y_{\mathrm{reg}})$ is pluripolar, the equation (5-1) extends trivially to the whole of X. Corollary 4.10 thus yields that $\pi^*\psi_t = \varphi_t$ on X, hence $\phi_t = \psi_t$ on Y. □

5.1.3 Song–Tian program A natural and difficult problem is to understand the asymptotic behavior of ω_t as $t \to T_{\max}$. Song and Tian have proposed in [33] an ambitious program, combining the minimal model program and Hamilton–Perelman approach to the Poincaré conjecture.

We focus here on the case when X has nonnegative Kodaira dimension. One would ideally like to proceed as follows:

Step 1 Show that (Y, ω_t) converges to a mildly singular Kähler variety (Y_1, S_1) equipped with a singular *Kähler current* S_1, as $t \to T_{1,\max}$.

Step 2 Restart the NKRF on Y_1 with initial data S_1.

Step 3 Repeat finitely many times to reach a minimal model Y_r (K_{Y_r} is *nef*).

Step 4 Study the long-term behavior of the NKRF and show that (Y_r, ω_t) converges to a canonical model $(Y_{\mathrm{can}}, \omega_{\mathrm{can}})$, as $t \to +\infty$.

This program is more or less complete in dimension ≤ 2 (see the lecture notes by Song and Weinkove in [34] or Tosatti [37]). It is largely open in dimension ≥ 3, but for Step 2, which has been completed in [33; 23; 9; 14; 36].

In the sequel we focus on the final Step 4, ie we assume that $T_{\max} = +\infty$, so that Y is a minimal model with log terminal singularities. The normalized Kähler–Ricci flow is then well defined for all times $t > 0$, and our goal is to understand its asymptotic behavior as $t \to +\infty$.

5.2 Convergence of the NKRF

5.2.1 Convergence of the NKRF on log terminal varieties of general type Let Y be a compact Kähler variety with terminal singularities and assume K_Y is big and nef. It has been shown in [12] that there exists a unique positive closed current ω_{KE} on Y

such that

- $\omega_{\mathrm{KE}} \in C_1(K_Y)$ and it has bounded potentials;
- ω_{KE} is smooth in $\mathrm{Amp}(K_Y)$, where it satisfies $\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}$.

The current ω_{KE} is called the *singular Kähler–Einstein current.*

Theorem 5.3 *Fix S_0 a positive closed current with bounded potentials, whose cohomology class is Kähler. The normalized Kähler–Ricci flow continuously deforms S_0 towards ω_{KE} as $t \to +\infty$ at an exponential speed.*

Proof It is classical that the problem boils down to solving and studying the long-term behavior of the parabolic scalar equation

$$(\chi_t + dd^c\varphi_t)^n = e^{\partial_t\varphi_t+\varphi_t} v_Y, \tag{5-2}$$

with initial data φ_0, where $S_0 = \chi_0 + dd^c\varphi_0$ and $\chi_t = e^{-t}\chi_0 + (1-e^{-t})\chi$. Here χ is a Kähler form representing $c_1(K_Y)$.

The existence of the unique maximal solution φ_t has been explained in Theorem 5.2, so the problem is to show that $\varphi_t \to \varphi_{\mathrm{KE}}$ as $t \to +\infty$, where $\omega_{\mathrm{KE}} = \chi + dd^c\varphi_{\mathrm{KE}}$. We let the reader check that

$$u(t,x) = e^{-t}\varphi_0 + (1-e^{-t})\varphi_{\mathrm{KE}} + h(t)e^{-t}$$

is a subsolution to (5-2), where

$$h(t) = n(e^t-1)\log(e^t-1) - ne^t\log e^t = O(t).$$

The computations are the same as that of [14, Theorem 4.3, Step 1]. The comparison principle (Theorem 4.8) yields

$$\varphi_{\mathrm{KE}}(x) - C(t+1)e^{-t} \le u(t,x) \le \varphi(t,x)$$

for some uniform constant $C > 0$.

The proof for the upper bound is similar. Since χ is Kähler, we can fix $B > 0$ such that $\omega_0 \le (1+B)\chi$, thus $\chi_t \le (1+Be^{-t})\chi$ for all t. We set

$$v_t(x) := (1+Be^{-t})\varphi_{\mathrm{KE}} + Ce^{-t},$$

where C is chosen so that $v_0 \ge \varphi_0$. The function v is a supersolution to the Cauchy problem for the parabolic equation

$$([1+Be^{-t}]\chi + dd^c v_t)^n = e^{\partial_t v_t + v_t + n\log[1+Be^{-t}]} v_Y \le e^{\partial_t v_t + v_t + nBe^{-t}} v_Y$$

with initial data φ_0, while $w(t,x) = \varphi(t,x) - nBte^{-t}$ is a subsolution to this equation since

$$([1+Be^{-t}]\chi + dd^c w)^n \geq (\chi_t + dd^c \varphi_t)^n = e^{\partial_t \varphi_t + \varphi_t} v_Y = e^{\partial_t w_t + w_t + nBe^{-t}} v_Y.$$

The comparison principle thus yields

$$\varphi(t,x) \leq \varphi_{\mathrm{KE}}(x) + C'(t+1)e^{-t}.$$

The conclusion follows. □

5.2.2 Convergence of the KRF on log terminal $\mathbb{Q}$–Calabi–Yau varieties In this section we study the Kähler–Ricci flow on a log terminal $\mathbb{Q}$–Calabi–Yau variety Y (ie a Gorenstein Kähler space of finite index with trivial first Chern class and log terminal singularities), and prove Theorem D of the introduction.

Theorem 5.4 *Fix S_0 a positive closed current with bounded potentials, whose cohomology class is Kähler. The weak Kähler–Ricci flow*

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t)$$

exists for all times $t > 0$, and deforms S_0 towards the unique Ricci flat Kähler–Einstein current ω_{KE} cohomologous to S_0 as $t \to +\infty$.

The existence of the singular Ricci flat Kähler–Einstein current ω_{KE} has been shown in [12], generalizing Yau's celebrated solution to the Calabi conjecture [40].

Proof It is classical that the problem boils down to solving and studying the long-term behavior of the parabolic scalar equation

$$(\theta_0 + dd^c \varphi_t)^n = e^{\partial_t \varphi_t} v_Y, \tag{5-3}$$

with initial data φ_0, where $S_0 = \theta_0 + dd^c \varphi_0$.

The existence of the unique semiconcave solution φ_t has been explained in Theorem 5.2. We are going to show that φ_t uniformly converges to φ_{KE} as $t \to +\infty$, where $\omega_{\mathrm{KE}} = \theta_0 + dd^c \varphi_{\mathrm{KE}}$ with

$$(\theta_0 + dd^c \varphi_{\mathrm{KE}})^n = v_Y,$$

and the bounded θ_0-plurisubharmonic function φ_{KE} is properly normalized. We proceed in several steps.

Step 1 ($\mathcal{C}^0$–bounds and normalization) It follows from the comparison principle that (φ_t) remains uniformly bounded: indeed $\varphi_{\mathrm{KE}} - C$ (resp. $\varphi_{\mathrm{KE}} + C$) provides a static subsolution (resp. supersolution) to the Cauchy problem if $C > 0$ is so large that $\varphi_{\mathrm{KE}} - C \le \varphi_0$ (resp. $\varphi_{\mathrm{KE}} + C \ge \varphi_0$).

We assume without loss of generality that v_Y and θ_0 are normalized by

$$\int_Y \theta_0^n = v_Y(Y) = 1.$$

The concavity of the logarithm ensures that

$$\int_Y \partial_t \varphi_t \, v_Y = \int_Y \log[(\theta_0 + dd^c \varphi_t)^n / v_Y] v_Y \le \log\left[\int_Y (\theta_0 + dd^c \varphi_t)^n\right] = 0,$$

hence $t \mapsto \int_Y \varphi_t \, v_Y$ is decreasing. We therefore impose the normalization

$$\int_Y \varphi_{\mathrm{KE}} \, v_Y = \lim_{t \to +\infty} \int_Y \varphi_t \, v_Y.$$

Step 2 (monotonicity of the Monge–Ampère energy along the flow) We now observe that $t \mapsto E(\varphi_t)$ is increasing, where

$$E(\varphi_t) := \frac{1}{n+1} \sum_{j=0}^{n} \int_Y \varphi_t (\theta_0 + dd^c \varphi_t)^j \wedge \theta_0^{n-j}.$$

More precisely:

Lemma 5.5 *The function $t \mapsto E(\varphi_t)$ is differentiable almost everywhere with*

$$\frac{d}{dt} E(\varphi_t) = \int_Y \dot{\varphi}_t (\theta_0 + dd^c \varphi_t)^n \ge 0,$$

for almost every $t \in]0, +\infty[$.

Proof It is straightforward to check that $t \mapsto E(\varphi_t)$ is locally Lipschitz, its differentiability almost everywhere thus follows from Rademacher theorem. Our goal is now to compute its derivative.

By the Lipschitz property of $t \mapsto \varphi_t$ we can find a subset $I \subset]0, T[$ with $]0, +\infty[\setminus I$ having measure zero such that, for every $t_0 \in I$ fixed, the function $t \mapsto \varphi(t, x)$ is differentiable at t_0 for almost every $x \in Y$. By the first observation we can also assume that $t \mapsto E(\varphi_t)$ is differentiable at every $t \in I$. The semiconcavity property of φ_t in t moreover ensures that, for every $t \in I$ and almost every $x \in Y$,

$$\dot{\varphi}_t^+(x) = \dot{\varphi}_t^-(x).$$

The semiconcavity property of $t \mapsto \varphi_t$ ensures that, for $x \in Y$ fixed, the function $t \mapsto \dot{\varphi}_t^+(x)$ is lower semicontinuous, while $t \mapsto \dot{\varphi}_t^-(x)$ is upper semicontinuous in $]0, +\infty[$. In particular, for $t_0 \in I$ fixed,

$$\liminf_{t \to t_0} \dot{\varphi}_t{}^+(x) \geq \dot{\varphi}_{t_0}^+(x) = \dot{\varphi}_{t_0}^-(x) \geq \limsup_{t \to t_0} \dot{\varphi}_t^-(x),$$

for almost every $x \in Y$.

Fix $t_0 \in I$ and $t \in I$ with $t > t_0$. By concavity of the Monge–Ampère energy (see [4, Proposition 2.1]) we obtain

$$\int_Y \frac{\varphi_t - \varphi_{t_0}}{t - t_0} (\theta_0 + dd^c \varphi_t)^n \leq \frac{E(\varphi_t) - E(\varphi_{t_0})}{t - t_0} \leq \int_Y \frac{\varphi_t - \varphi_{t_0}}{t - t_0} (\theta_0 + dd^c \varphi_{t_0})^n.$$

Using that φ_t is a solution to (5-3) and $\dot{\varphi}_t^+(x) = \dot{\varphi}_t^-(x)$ a.e., we get

$$\int_Y \frac{\varphi_t - \varphi_{t_0}}{t - t_0} e^{\dot{\varphi}_t} g \, dV \leq \frac{E(\varphi_t) - E(\varphi_{t_0})}{t - t_0} \leq \int_Y \frac{\varphi_t - \varphi_{t_0}}{t - t_0} e^{\dot{\varphi}_{t_0}} g \, dV.$$

Letting $I \ni t \to t_0$ and using the Lebesgue dominated convergence theorem, we arrive at the desired formula for the derivative of $t \mapsto E(\varphi_t)$ at t_0.

It remains to check that $\frac{d}{dt} E(\varphi_t) \geq 0$. This follows from Jensen's inequality,

$$\frac{d}{dt} E(\varphi_t) = \int_Y \dot{\varphi}_t (\theta_0 + dd^c \varphi_t)^n \geq -\log \int_Y v_Y = 0. \qquad \square$$

Step 3 (asymptotic behavior of $\dot{\varphi}_t(x)$) We claim that there exists a constant $C > 0$ such that, for all $t \geq 1$ and $x \in Y$,

$$|\dot{\varphi}_t(x)| \leq C.$$

Indeed, since $t \mapsto \varphi_t(x)$ is locally uniformly Lipschitz (away from $t = 0$), there is $C > 0$ such that $|\varphi_{s+1} - \varphi_1| \leq Cs$ for every $s \in [0, 1]$. Fix such s and consider, for $t > 0$ and $x \in Y$,

$$u_t(x) := \varphi(s + t + 1, x) - Cs.$$

Observe that $u_0 \leq \varphi_1$ and

$$(\theta_0 + dd^c u_t)^n = e^{\dot{u}_t} v_Y.$$

Since the function $(t, x) \mapsto \varphi(t + 1, x)$ solves the above equation, it follows from Theorem 4.8 that $u_t \leq \varphi_{t+1}$ for all $t > 0$. Thus

$$\varphi_{s+t+1} \leq \varphi_{t+1} + Cs,$$

and letting $s \to 0$ yields a uniform upper bound for $\dot{\varphi}_t$. The lower bound follows similarly.

We now claim that there is a sequence of times $t_j \to +\infty$ such that $\dot{\varphi}_{t_j}(x) \to 0$ for almost every $x \in Y$. Indeed, observe that the functional $t \mapsto \mathcal{F}(\varphi_t) := E(\varphi_t) - \int_Y \varphi_t \, v_Y$ is increasing along the flow: for a.e. $t \geq 1$,

$$\frac{d}{dt}\mathcal{F}(\varphi_t) = \int_Y \dot{\varphi}_t (e^{\dot{\varphi}_t} - 1) \, dv_Y \geq C^{-1} \int_Y |\dot{\varphi}_t|^2 \, dv_Y \geq 0.$$

Since $\mathcal{F}$ is uniformly bounded along the flow, there is $t_j \to +\infty$ such that

$$\int_Y |\dot{\varphi}_{t_j}|^2 \, dv_Y \to 0.$$

Since the time derivative $\dot{\varphi}_t$ is uniformly bounded for $t \geq 1$, it follows that

$$e^{\dot{\varphi}_{t_j}} \to 1$$

in $L^q(Y, dv_Y)$ for all $1 < q < 2$, and $\dot{\varphi}_{t_j}(x) \to 0$ for almost every $x \in Y$ (up to extracting and relabeling). It follows from the elliptic L^1–L^∞ stability [21, Theorem C] that φ_{t_j} uniformly converges to some ψ which satisfies

$$(\theta_0 + dd^c \psi)^n = v_Y$$

and $\int_Y \psi \, dv_Y = \int_Y \varphi_{\mathrm{KE}} \, dv_Y$, since $\int_X \varphi_t \, dv_Y$ decreases to $\int_Y \varphi_{\mathrm{KE}} \, dv_Y$. The uniqueness of the normalized Kähler–Einstein potential [12] now ensures that $\psi = \varphi_{\mathrm{KE}}$, ie φ_{t_j} uniformly converges to φ_{KE}.

Step 4 (the semigroup property) The conclusion follows now from the fact that our equation is invariant under translations in time: Observe that for all $s > 0$, the function $(t, x) \mapsto \psi(t, x) = \varphi(t + s, x)$ is again a bounded parabolic potential solution to the equation

$$(\theta_0 + dd^c \psi_t)^n = e^{\partial_t \psi_t} dv_Y.$$

Fix $\varepsilon > 0$ and j large enough that

$$\sup_X |\varphi_{t_j} - \varphi_{\mathrm{KE}}| < \varepsilon.$$

The function $\psi(t, x) = \varphi_{\mathrm{KE}}(x) - \varepsilon$ is a subsolution to the Cauchy problem for the above equation with initial data φ_{t_j}. Similarly, $\varphi_{\mathrm{KE}}(x) + \varepsilon$ is a supersolution to the same Cauchy problem. The comparison principle (Theorem 4.8) therefore yields, for all $t \geq 0$ and $x \in X$,

$$\varphi_{\mathrm{KE}}(x) - \varepsilon \leq \varphi(t + t_j, x) \leq \varphi_{\mathrm{KE}}(x) + \varepsilon.$$

Letting $t \to +\infty$ and then $\varepsilon \to 0$ yields the conclusion. □

5.2.3 Minimal models of intermediate Kodaira dimension We finally just say a few words about the more delicate volume-collapsing case. We assume here Y is an abundant minimal model of Kodaira dimension $1 < \kappa = \mathrm{kod}(Y) < n$, ie K_Y is a semiample $\mathbb{Q}$–line bundle with $K_Y = f^* K_{Y_{\mathrm{can}}}$, where $f\colon Y \to Y_{\mathrm{can}}$ is the Iitaka fibration, with $K_{Y_{\mathrm{can}}}$ ample.

A generic fiber $X_y = f^{-1}(y)$ is a $\mathbb{Q}$–Calabi–Yau variety. We fix h_A a positive Hermitian metric of $A = K_{Y_{\mathrm{can}}}$ with curvature form θ_A, and η local (multivalued) nonvanishing holomorphic section of K_Y. Then

$$v(h_A) = c_n \frac{\eta \wedge \overline{\eta}}{\|\eta\|^2_{f^* h_A}}$$

is a globally well-defined volume form on Y such that the measure $f_* v(h_A)$ has density in $L^{1+\varepsilon}$ with respect to θ_A^κ.

Generalizing [32; 33], it has been shown in [15] that there exists a unique bounded θ_A–psh function φ_{can} on Y_{can} such that

- $(\theta_A + dd^c \varphi_{\mathrm{can}})^\kappa = e^{\varphi_{\mathrm{can}}} f_*(v(h_A))$;
- the current $\omega_{\mathrm{can}} = \theta_A + dd^c \varphi_{\mathrm{can}}$ is independent of h_A;
- it is smooth in $Y_{\mathrm{can}}^{\mathrm{reg}} \setminus$ critical values of f;
- it satisfies $\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}}$ in $Y_{\mathrm{can}}^{\mathrm{reg}} \setminus$ critical values.

The Weil–Petersson term ω_{WP} is a semipositive $(1,1)$–form which measures the change of complex structures in the fibers of the Iitaka fibration. The current $T_{\mathrm{can}} = f^* \omega_{\mathrm{can}}$ is an important birational invariant such that

$$T_{\mathrm{can}}^\kappa \wedge \omega_{\mathrm{SF}}^{n-\kappa} = e^{\varphi_{\mathrm{can}} \circ f} v(h_A).$$

Here $\omega_{\mathrm{SF}} = \omega_0 + dd^c \rho$ denotes the fiberwise family of Ricci flat KE metrics,

$$\omega_{\mathrm{SF}}|_{X_y} = \text{unique Ricci flat metric in } \{\omega_0\}|_{X_y},$$

whose existence has been obtained in [12].

Extending the main result of [15], the tools developed in this article allow one to establish the following:

Theorem 5.6 *If* $\dim_{\mathbb{C}} Y \le 3$ *then the normalized Kähler–Ricci flow deforms* ω_0 *towards the canonical current* T_{can} *as* $t \to +\infty$*.*

Proof For a suitable choice of the normalizing constants, the normalized Kähler–Ricci flow is equivalent to the parabolic complex Monge–Ampère flow of potentials

$$\frac{(\omega_t + dd^c\varphi_t)^n}{C_n^\kappa e^{-(n-\kappa)t}} = e^{\partial_t\varphi + \varphi_t} v(h),$$

starting from an initial bounded potential $\varphi_0 \in \mathrm{PSH}(X, \omega_0)$. We have normalized here both sides so that the volume of the left-hand side converges to 1 as $t \to +\infty$. Here C_n^k denotes the binomial coefficient $C_n^k = \binom{n}{k}$. It follows from Theorem 5.2 that this flow admits a unique bounded pluripotential solution.

Once the objects are well defined, the proof is then identical to that in [15, Theorem D]. The restriction on $\dim_{\mathbb{C}} Y$ is related to a regularity issue for some families of Ricci flat metrics. □

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