

A gluing formula for families Seiberg–Witten invariants

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We prove a gluing formula for the families Seiberg–Witten invariants of families of 4–manifolds obtained by fibrewise connected sum. Our formula expresses the families Seiberg–Witten invariants of such a connected sum family in terms of the ordinary Seiberg–Witten invariants of one of the summands, under certain assumptions on the families. We construct some variants of the families Seiberg–Witten invariants and prove the gluing formula also for these variants. One variant incorporates a twist of the families moduli space using the charge conjugation symmetry of the Seiberg–Witten equations. The other variant is an equivariant Seiberg–Witten invariant of smooth group actions. We consider several applications of the gluing formula, including obstructions to smooth isotopy of diffeomorphisms, computation of the mod 2 Seiberg–Witten invariants of spin structures, and relations between mod 2 Seiberg–Witten invariants of 4–manifolds and obstructions to the existence of invariant metrics of positive scalar curvature for smooth group actions on 4–manifolds.

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1 Introduction

The Seiberg–Witten invariants of smooth 4–manifolds admit an extension to the parametrised setting. This means that instead of a single 4–manifold, we consider the moduli space of solutions to the Seiberg–Witten equations for a *family* of smooth 4–manifolds. Here a family of smooth 4–manifolds means a locally trivial fibre bundle over a base space whose fibres are a fixed compact smooth 4–manifold X equipped with a Spin^c –structure, and the transition functions for the bundle are valued in the group of diffeomorphisms of X preserving the orientation and Spin^c –structure. The idea of studying the Seiberg–Witten equations for such a family was proposed by Donaldson in [5] and later pursued by various authors, including Ruberman [19; 21], Liu [13], Li and Liu [12] and Nakamura [15].

One expects the families Seiberg–Witten invariants to have many deep applications to the study of families of smooth 4–manifolds, just as the ordinary Seiberg–Witten invariants have produced many profound results in the study of individual smooth 4–manifolds. For example, Ruberman used a certain 1–parameter families Seiberg–Witten invariant to show that the space of positive scalar metrics on certain simply connected 4–manifolds has infinitely many connected components [21]. In order to realise such applications, one needs practical tools for computing the families invariants, just as one has for the ordinary Seiberg–Witten invariants. To date only a limited set of tools are available, such as the families wall-crossing formula [12] and families blowup formula [13]. Both of these express one set of families invariants in terms of another set, so it is difficult to get off the ground with just these. In [21], Ruberman devised a technique for computing the families Seiberg–Witten invariant for a special class of 1–dimensional families, where X is a connected sum of the form $X = M \# \mathbb{C}P^2 \# \#^2 \overline{\mathbb{C}P^2}$ and the family is the mapping cylinder of a certain diffeomorphism of $\mathbb{C}P^2 \# \#^2 \overline{\mathbb{C}P^2}$.

Here we will prove a far-reaching generalisation of Ruberman’s 1–parameter formula. Our formula will relate the parametrised Seiberg–Witten invariant for a connected sum of families with fibres $X = M \# N$ to the ordinary Seiberg–Witten invariants on M . In this way, tools for computing the Seiberg–Witten invariants of M become tools for computing the families Seiberg–Witten invariants of X . Quite surprisingly, our formula for families Seiberg–Witten invariants has nontrivial consequences for the ordinary Seiberg–Witten invariants as well.

Let us briefly outline the definition of the families Seiberg–Witten invariants. Their construction is described in more detail in [Section 2](#). Our approach follows [12] but

is more general; in particular [12] assumes the existence of a Spin^c -structure on the vertical tangent bundle while our definition does not always assume this. In addition to this, we define some further variants of the families Seiberg–Witten invariants. One is a variant which incorporates twisting the families moduli space by the so-called charge conjugation symmetry, as described in Section 2. The other is a variant of the families Seiberg–Witten invariant for actions of a group G on a 4-manifold by diffeomorphisms. This invariant takes values in the group cohomology of G and is described in Section 3.

Let X be a compact smooth oriented 4-manifold and let B be a compact smooth manifold. Suppose we have a smooth fibrewise oriented fibre bundle $\pi_X: E_X \rightarrow B$ whose fibres are diffeomorphic to X . Let \mathfrak{s}_X be a Spin^c -structure on X . We will assume that \mathfrak{s}_X can be extended to a continuously varying family of Spin^c -structures on the fibres of E_X . Equivalently, the transition functions for E_X can be chosen to be diffeomorphisms of X preserving the isomorphism classes of \mathfrak{s}_X . In general this is a weaker condition than requiring that the vertical tangent bundle of E_X admit a Spin^c -structure extending \mathfrak{s}_X .

For simplicity, we concern ourselves in this introduction with just the \mathbb{Z}_2 -valued families Seiberg–Witten invariants (see Sections 2–4 for details of the \mathbb{Z} -valued invariants). Let

$$d(X, \mathfrak{s}_X) = \frac{1}{4}(c_1(\mathfrak{s}_X)^2 - \sigma(X)) - 1 - b^+(X) + b_1(X)$$

be the virtual dimension of the unparametrised Seiberg–Witten moduli space of X . Assume that

$$b^+(X) > \dim B + 1.$$

This assumption is necessary to have a well-defined families Seiberg–Witten invariant. Under this assumption, for any family of fibrewise metrics g_X on E_X , there exists a families perturbation η_X on E_X for which the families moduli space $\mathcal{M} = \mathcal{M}(E_X, g_X, \eta_X, \mathfrak{s}_X)$ is either empty or a smooth manifold of dimension

$$\dim \mathcal{M} = d(X, \mathfrak{s}_X) + \dim B.$$

If \mathfrak{s}_X extends to a Spin^c -structure on the vertical tangent bundle of E_X , the definition of the families moduli space \mathcal{M} is the standard one, as discussed in [12]. However, one can still define the families moduli space \mathcal{M} under the weaker condition that we can equip E_X with a continuously varying family of Spin^c -structures, in the sense described above.

The simplest version of the families Seiberg–Witten invariant is defined to be the homomorphism

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \cdot): H^{\dim \mathcal{M}}(B, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2,$$

given by

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) = \langle \pi^* \theta, [\mathcal{M}] \rangle,$$

where $[\mathcal{M}]$ is the fundamental class of \mathcal{M} with \mathbb{Z}_2 -coefficients.

A more general families Seiberg–Witten invariant is obtained by incorporating a certain line bundle $\mathcal{L} \rightarrow \mathcal{M}$ on the families moduli space. Recall that for the unparametrised Seiberg–Witten moduli space, one obtains \mathcal{L} as the line bundle associated to the principal circle bundle $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$, where $\tilde{\mathcal{M}}$ is the Seiberg–Witten moduli space modulo the reduced gauge group of gauge transformations which are the identity at some fixed basepoint of X . The circle bundle $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ corresponds to the residual action of constant gauge transformations. In the families setting, the definition of the line bundle \mathcal{L} is similar but requires additional care. We construct \mathcal{L} in Section 2 under the assumptions that $b_1(X) = 0$ and that \mathfrak{s}_X extends to a Spin^c -structure on the vertical tangent bundle, which we denote by $\tilde{\mathfrak{s}}_X$. Then, for any $m \geq 0$, we define a families Seiberg–Witten invariant,

$$\begin{aligned} \text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X): H^{\dim \mathcal{M} - 2m}(B, \mathbb{Z}_2) &\rightarrow \mathbb{Z}_2, \\ \text{FSW}_m^{\mathbb{Z}_2}(E_X, \tilde{\mathfrak{s}}_X, \theta) &= \langle c_1(\mathcal{L})^m \cup \pi^* \theta, [\mathcal{M}] \rangle \in \mathbb{Z}_2. \end{aligned}$$

We sometimes abuse notation and write \mathfrak{s}_X instead of $\tilde{\mathfrak{s}}_X$ in $\text{FSW}_m^{\mathbb{Z}_2}(E_X, \tilde{\mathfrak{s}}_X, \theta)$, noting however that $\text{FSW}_m^{\mathbb{Z}_2}(E_X, \tilde{\mathfrak{s}}_X, \theta)$ in general depends on the choice of extension $\tilde{\mathfrak{s}}_X$.

We now describe the setup for the gluing formula. More details can be found in Section 4. Our gluing result is for families obtained by a families connected sum. Let M and N be compact, smooth, oriented 4-manifolds and set $X = M \# N$. In this construction M and N will play specific roles and the assumptions on M and N will not be symmetric. In particular we assume that $b_1(N) = 0$ but make no assumption about $b_1(M)$ in general.

Let B be a compact smooth manifold and suppose we have smooth fibrewise oriented families $\pi_M: E_M \rightarrow B$, $\pi_N: E_N \rightarrow B$ whose fibres are M and N , respectively. Suppose we are given sections $\iota_M: B \rightarrow E_M$ and $\iota_N: B \rightarrow E_N$ whose normal bundles are orientation-reversing isomorphic. Then, by removing tubular neighbourhoods of the sections ι_M and ι_N from E_M and E_N and identifying their boundaries using the orientation-reversing isomorphism, we obtain a family E_X over B whose fibres are

diffeomorphic to the connected sum $X = M \# N$. In general E_X will depend on the isotopy classes of the sections ι_M and ι_N and the isomorphism of their normal bundles. Let \mathfrak{s}_M and \mathfrak{s}_N be Spin^c –structures on M and N and let \mathfrak{s}_X be the Spin^c –structure on X obtained as the connected sum of \mathfrak{s}_M and \mathfrak{s}_N . We will assume that \mathfrak{s}_M and \mathfrak{s}_N can be extended to continuously varying families of Spin^c –structures on the fibres of E_M and E_N . It follows that \mathfrak{s}_X similarly extends to a continuous family of Spin^c –structures on the fibres of E_X . We will assume that $d(M, \mathfrak{s}_M) = 2m \geq 0$ is nonnegative and even.

We make the following assumptions about N . Assume that

$$c_1(\mathfrak{s}_N)^2 = \sigma(N),$$

so that the index of the Spin^c –Dirac operator on N is zero. Assume also that

$$0 < b^+(N) \leq \dim B.$$

Let $\mathcal{M} = \mathcal{M}(E_X, g_X, \eta_X, \mathfrak{s}_X)$ be the Seiberg–Witten moduli space for the family E_X with respect to a choice of fibrewise metric g_X and fibrewise perturbation η_X . If $b^+(M) + b^+(N) > \dim B + 1$, then \mathcal{M} is a compact smooth manifold for a generic choice of families perturbation. Assume this is the case and that η_X has been chosen generically. Then \mathcal{M} is either empty or a compact smooth manifold of dimension

$$\dim \mathcal{M} = d(M, \mathfrak{s}_M) + d(N, \mathfrak{s}_N) + 1 + \dim B = 2m + \dim B - b^+(N).$$

The main gluing theorem of this paper is as follows:

Theorem 1.1 (\mathbb{Z}_2 –valued gluing formula) *Suppose that either*

- (i) $d(M, \mathfrak{s}_M) = 2m = 0$, or
- (ii) $b_1(M) = b_1(X) = 0$ and \mathfrak{s}_X extends to a Spin^c –structure on $T(E_X/B)$.

Then:

- (1) For any $\theta \in H^{\dim B - b^+(N)}(B, \mathbb{Z}_2)$, we have

$$\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) = \text{SW}(M, \mathfrak{s}_M) \cdot \langle \theta \cup w_{b^+(N)}(H^+(N)), [B] \rangle.$$

- (2) For any integer $k > 0$ and any $\theta \in H^{\dim B - b^+(N) + 2k}(B, \mathbb{Z}_2)$, we have

$$\text{FSW}_{m-k}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) = 0.$$

We also prove such a gluing formula for the families Seiberg–Witten invariants incorporating twists by charge conjugation (Theorem 4.3), for the \mathbb{Z} –valued invariants (Theorem 4.4) and for the Seiberg–Witten invariants of group actions (Theorem 9.20).

In Section 9 we consider various applications of the families gluing formula. Below we summarise some of the main results obtained.

Theorem 1.2 *Let M and M' be compact simply connected smooth 4–manifolds with indefinite intersection forms. Suppose that M and M' are homeomorphic and fix a homeomorphism $\phi: M \rightarrow M'$. Suppose that $b^+(M) > 1$ and that \mathfrak{s}_M is a Spin^c –structure on M with $d(M, \mathfrak{s}_M) = 0$ and that $\text{SW}(M, \mathfrak{s}_M) \not\equiv \text{SW}(M', \phi(\mathfrak{s}_M)) \pmod{2}$. Lastly, suppose that $X = M \# (S^2 \times S^2)$ is diffeomorphic to $M' \# (S^2 \times S^2)$. Then there exists a diffeomorphism on X which is continuously isotopic to the identity but not smoothly isotopic.*

Corollary 1.3 *The following 4–manifolds admit diffeomorphisms which are continuously isotopic to the identity, but are not smoothly isotopic to the identity:*

- (i) $X = \#^n(S^2 \times S^2) \# \#^n(K3)$ for any $n \geq 2$.
- (ii) $X = \#^{2n} \mathbb{C}P^2 \# \#^m \overline{\mathbb{C}P}^2$ for any $n \geq 2$ and any $m \geq 10n + 1$.

In Section 9.4 we use the gluing formula to give a simple new proof of a theorem of Morgan and Szabó:

Theorem 1.4 [14] *Let M be smooth compact spin 4–manifold with $b_1(M) = 0$, $b^+(M) = 4n - 1$ and $b^-(M) = 20n - 1$, where $n \geq 1$. Let \mathfrak{s}_M be a Spin^c –structure on M which comes from a spin structure. Then $\text{SW}(M, \mathfrak{s}_M)$ is odd if $n = 1$ and is even otherwise.*

In Section 9.5 we show how the gluing formula can be used to compute the (un-parametrised) mod 2 Seiberg–Witten invariants of 4–manifolds under certain conditions. Amongst such results we obtain the following:

Theorem 1.5 *Let M be a compact smooth 4–manifold with*

$$b_1(M) = 0 \quad \text{and} \quad b^+(M) \equiv 3 \pmod{4}.$$

Suppose that $f: M \rightarrow M$ is an involutive diffeomorphism of M whose fixed-point set contains an isolated point. Suppose \mathfrak{s}_M is a Spin^c –structure such that $f(\mathfrak{s}_M) = -\mathfrak{s}_M$ and $c_1(\mathfrak{s}_M)^2 - \sigma(M) \equiv 0 \pmod{16}$.

- (i) *If $b^+(M) = 3$ and $c_1(\mathfrak{s}_M)^2 - \sigma(M) \geq 16$, then $\text{SW}(M, \mathfrak{s}_M) \equiv 1 \pmod{2}$ if f acts trivially on $H^+(M)$ and $\text{SW}(M, \mathfrak{s}_M) \equiv 0 \pmod{2}$ otherwise.*
- (ii) *If $b^+(M) > 3$, then $\text{SW}(M, \mathfrak{s}_M) \equiv 0 \pmod{2}$.*

In [Section 9.6](#), we prove the gluing formula for the Seiberg–Witten invariants of group actions as a consequence of the gluing formula for families. The nonvanishing of this invariant gives an obstruction to the existence of an invariant metric with positive scalar curvature. In [Section 9.7](#) we apply this formula to the simplest case, where the group is \mathbb{Z}_2 . As a corollary, we find:

Corollary 1.6 *For any $n \geq 4$ and $m \geq n + 18$, the 4–manifold $X = \#^n \mathbb{C}P^2 \# \#^m \overline{\mathbb{C}P}^2$ admits a metric of positive scalar curvature and a smooth involution $f: X \rightarrow X$ such that there does not exist an f –invariant metric of positive scalar curvature on X .*

The following is a brief outline of the sections of the paper. In [Section 2](#) we give the definition of the families Seiberg–Witten invariants including several generalisations. In [Section 3](#) we define an equivariant Seiberg–Witten invariant for smooth group actions and relate it to the families invariants. In [Section 4](#) we describe in detail the setup required to formulate the families gluing formula and state the main result. Sections [5–8](#) give the proof of the gluing formula in several steps. [Section 6](#) contains a review of the relevant aspects of gluing theory for unparametrised Seiberg–Witten theory, mostly following Nicolaescu’s textbook [\[16\]](#). In particular, [Section 6.2](#) is concerned with the local Kuranishi model for glued configurations in Seiberg–Witten theory. [Section 7](#) is concerned with adapting the arguments of the previous sections to the families setting. This turns out to be not as straightforward as one might expect, in part because of the fact that certain obstruction spaces in the local Kuranishi model are nonzero. Additionally one has to choose the families perturbation very carefully in order for the gluing arguments to work well. In [Section 9](#) we consider various applications of the gluing formula: a vanishing theorem ([Section 9.2](#)), obstructions to smooth isotopy ([Section 9.3](#)), the computation of the mod 2 Seiberg–Witten invariants of spin structures ([Section 9.4](#)), relations between mod 2 Seiberg–Witten invariants of different 4–manifolds, the gluing formula for the Seiberg–Witten invariants of group actions ([Section 9.6](#)) and actions on 4–manifolds by involutions ([Section 9.7](#)).

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2 The families Seiberg–Witten invariant

In this section we describe the formulation of the families invariant which we consider in this paper. Although this type of invariant has been defined by Li and Liu [12], we exhibit here some of its generalisations. In Section 3 we formulate a version of this invariant for group actions.

Let X be a compact smooth oriented 4–manifold. Let B be a compact smooth manifold of dimension d and suppose we have a smooth fibrewise oriented fibre bundle $\pi_X: E_X \rightarrow B$ whose fibres are diffeomorphic to X . Let $T(E_X/B) = \text{Ker}(\pi_{X*})$ be the vertical tangent bundle.

Let \mathfrak{s}_X be a Spin^c –structure on X . We will assume that \mathfrak{s}_X is monodromy-invariant under the monodromy action of $\pi_1(B)$ on the set of Spin^c –structures on X induced by the family E_X . That is, we assume \mathfrak{s}_X extends to give a continuously varying family of Spin^c –structures on the fibres of E_X . Note that this is in general a weaker condition than requiring the existence of a Spin^c –structure on $T(E_X/B)$ which restricts to \mathfrak{s}_X on each fibre. See Proposition 2.1 in [2] on this point.

Let us first consider the \mathbb{Z}_2 –valued families Seiberg–Witten invariant, so we do not have to consider orientations on the moduli space at this stage. The necessary modifications for the \mathbb{Z} –valued families Seiberg–Witten invariant are straightforward and will be dealt with later. Let

$$d(X, \mathfrak{s}_X) = \frac{1}{4}(c_1(\mathfrak{s}_X)^2 - \sigma(X)) - 1 - b^+(X) + b_1(X)$$

be the virtual dimension of the ordinary Seiberg–Witten moduli space of X . Assume that

$$b^+(X) > d + 1.$$

This assumption is necessary to have a well-defined families Seiberg–Witten invariant. Under this assumption, for any family of fibrewise metrics g_X on E_X , there exists a families perturbation η_X on E_X for which the families moduli space

$\mathcal{M} = \mathcal{M}(E_X, g_X, \eta_X, \mathfrak{s}_X)$ is a smooth manifold of dimension

$$\dim \mathcal{M} = d(X, \mathfrak{s}_X) + d$$

(\mathcal{M} is empty if this number is negative). Here, if \mathfrak{s}_X extends to a Spin^c -structure on $T(E_X/B)$, the definition of the families moduli space \mathcal{M} is the standard one, such as discussed in [12]. However, in fact, even when \mathfrak{s}_X does not extend, one can define the families moduli space. See Section 2.2 in [2] on this point.

Suppose that $m \geq 0$ is a nonnegative integer and set $s = \dim \mathcal{M} - 2m$. Now we define the families invariant of interest. The invariant will be defined in the following two cases. Assume that either

- (i) $m = 0$, or
- (ii) $b_1(X) = 0$ and \mathfrak{s}_X extends to a Spin^c -structure on $T(E_X/B)$, which we denote by $\tilde{\mathfrak{s}}_X$.

In case (ii), we obtain a line bundle $\mathcal{L} \rightarrow \mathcal{M}$ over \mathcal{M} as follows. Let $\mathcal{G} = \text{Map}(X, S^1)$ be the gauge group on X . Consider the subgroup \mathcal{G}_0 of \mathcal{G} given by

$$\mathcal{G}_0 = \left\{ g \in \mathcal{G} \mid g = e^{if} \text{ for some } f \text{ with } \int_X f \, d\text{vol}_X = 0 \right\}.$$

Since $b_1(X) = 0$, we see that there is a short exact sequence $1 \rightarrow \mathcal{G}_0 \rightarrow \mathcal{G} \rightarrow S^1 \rightarrow 1$. If we consider the families moduli space of solutions to the Seiberg–Witten equations taken modulo \mathcal{G}_0 , we obtain a principal circle bundle over \mathcal{M} , the families moduli space of solutions modulo \mathcal{G} , hence an associated line bundle $\mathcal{L} \rightarrow \mathcal{M}$. Note that this line bundle depends on the choice of extension of $\tilde{\mathfrak{s}}_X$ of \mathfrak{s}_X (see again Section 2.2 in [2] on this point). In fact, since we assume $b_1(X) = 0$, any two extensions differ by the action of the pullback of a line bundle $L_B \rightarrow B$. Then it is easy to see that \mathcal{L} changes to $\mathcal{L} \otimes \pi^*(L_B)$, where $\pi: \mathcal{M} \rightarrow B$ is the natural map to B .

Definition 2.1 (families \mathbb{Z}_2 -invariant) Suppose that $\theta \in H^s(B, \mathbb{Z}_2)$. Now we define the families invariant as follows:

- (i) If $m = 0$, then $\dim \mathcal{M} = s$ and we set

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) = \langle \pi^* \theta, [\mathcal{M}] \rangle \in \mathbb{Z}_2,$$

where $[\mathcal{M}]$ is the fundamental class of \mathcal{M} with \mathbb{Z}_2 -coefficients.

- (ii) If $m > 0$, then choose an extension $\tilde{\mathfrak{s}}_X$ of \mathfrak{s}_X to a Spin^c -structure on $T(E_X/B)$. In turn this determines a line bundle $\mathcal{L} \rightarrow \mathcal{M}$. We then set

$$\text{FSW}_m^{\mathbb{Z}_2}(E_X, \tilde{\mathfrak{s}}_X, \theta) = \langle c_1(\mathcal{L})^m \cup \pi^*\theta, [\mathcal{M}] \rangle \in \mathbb{Z}_2.$$

The condition $b^+(X) > d + 1$ ensures that $\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta)$ does not depend on the choice of metric or perturbation and so gives an invariant of the family. In case (ii), the invariant will in general depend on the choice of lift $\tilde{\mathfrak{s}}_X$ of \mathfrak{s}_X . However, to unify notation we will often write the invariant in case (ii) simply as $\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta)$, suppressing the dependence on the choice of lift $\tilde{\mathfrak{s}}_X$. This will be justified later when we see that under certain conditions the invariant is actually independent of the choice of lift.

Remark 2.2 Variant (ii) of Definition 2.1 can be defined also for $m = 0$, in which case it clearly coincides with variant (i): $\text{FSW}_0^{\mathbb{Z}_2}(E_X, \tilde{\mathfrak{s}}_X, \theta) = \text{FSW}_0^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta)$.

Next, we generalise the setup slightly to incorporate the so-called charge conjugation symmetry j . This construction was considered by Ruberman in the case of 1-dimensional families in [19]. Recall that for any Spin^c -structure \mathfrak{s}_X on X there is a “dual” or “charge conjugate” Spin^c -structure, which we will denote by $-\mathfrak{s}_X$. We have that $c_1(-\mathfrak{s}_X) = -c_1(\mathfrak{s}_X)$ and that \mathfrak{s}_X comes from a spin structure precisely when $\mathfrak{s}_X = -\mathfrak{s}_X$. Let $S^\pm(\mathfrak{s}_X)$ denote the spinor bundles associated to \mathfrak{s}_X . Then charge conjugation determines an antilinear isomorphism $j: S^\pm(\mathfrak{s}_X) \rightarrow S^\pm(-\mathfrak{s}_X)$ with the property that $j^2 = -1$. Note that if \mathfrak{s}_X comes from a spin structure, then j gives a quaternionic structure on $S^\pm(\mathfrak{s}_X)$.

Let $\pi_X: E_X \rightarrow B$ be a smooth family of 4-manifolds with fibre X . Recall that $\pi_1(B)$ acts on the set of Spin^c -structures by monodromy. We will consider families where \mathfrak{s}_X is preserved up to charge conjugation. Thus we assume there exists a homomorphism $\rho: \pi_1(B) \rightarrow \mathbb{Z}_2$ such that the monodromy action of $\pi_1(B)$ preserves \mathfrak{s}_X up to a sign factor given by ρ . Note that if \mathfrak{s}_X does not come from a spin structure, then ρ is uniquely determined by the monodromy. On the other hand, if \mathfrak{s}_X comes from a spin structure, then ρ may be chosen arbitrarily, and this choice gives rise to different families Seiberg–Witten invariants.

Define $d(X, \mathfrak{s}_X)$ as before and assume that $b^+(X) > \dim B + 1$. View ρ as a class in $H^1(B, \mathbb{Z}_2)$. Represent ρ as a \mathbb{Z}_2 -valued cocycle $\rho_{\alpha\beta}$ with respect to some open cover U_α of B . We construct local families moduli spaces \mathcal{M}_α over each U_α exactly

as before. On the overlap $U_\alpha \cap U_\beta$, we patch together \mathcal{M}_α and \mathcal{M}_β using the charge conjugation symmetry $(a, \psi) \mapsto (-a, j(\psi))$. Note that this gives us a globally well-defined moduli space \mathcal{M}^ρ because j squares to a gauge transformation. Note that if η_α are the locally defined perturbation 2-forms, then for these to patch together we require $\eta_\alpha|_{U_{\alpha\beta}} = (-1)^{\rho_{\alpha\beta}} \eta_\beta|_{U_{\alpha\beta}}$. Let $\mathbb{R}_\rho \rightarrow B$ be the real line bundle on B determined by $\rho \in H^1(B, \mathbb{Z}_2)$. Then the above patching condition on the perturbations $\{\eta_\alpha\}$ means that they patch together to give a global section η of $H^+(X) \otimes \mathbb{R}_\rho$.

Definition 2.3 (ρ -twisted families \mathbb{Z}_2 -invariant) Suppose that $\theta \in H^s(B, \mathbb{Z}_2)$, where $s = d(X, \mathfrak{s}_X) + \dim B$. Now we define the ρ -twisted families invariant as

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta, \rho) = \langle \pi^* \theta, [\mathcal{M}^\rho] \rangle \in \mathbb{Z}_2,$$

where $[\mathcal{M}^\rho]$ is the fundamental class of \mathcal{M}^ρ with \mathbb{Z}_2 -coefficients.

Next we formulate a \mathbb{Z} -valued version of the families Seiberg–Witten invariant. In addition to all assumptions supposed when we defined the \mathbb{Z}_2 -valued invariant (Definition 2.1), let us assume that B is oriented and fix a homology orientation of X . Further, let $\det^+(E_X)$ denote the real line bundle on B whose fibre over $t \in B$ is $\det(H^1(X_t, \mathbb{R}) \oplus H^+(X_t))$, where X_t denote the fibre of E_X over t and $H^+(X_t)$ denotes the space of harmonic self-dual 2-forms on X_t (with respect to some smoothly varying family of fibrewise metrics). Suppose that $\det^+(E_X)$ is orientable and fix an orientation on $\det^+(E_X)$. As in [12], this gives rise to an orientation of the families moduli space \mathcal{M} for a generic families perturbation, and thus we get a fundamental class $[\mathcal{M}]$ in \mathbb{Z} -valued cohomology. By repeating Definition 2.1, we have:

Definition 2.4 (families \mathbb{Z} -invariant) Suppose that $\theta \in H^s(B, \mathbb{Z})$. Now we define the families invariant as follows:

- (i) If $m = 0$, then $\dim \mathcal{M} = s$ and we set

$$\text{FSW}^{\mathbb{Z}}(E_X, \mathfrak{s}_X, \theta) = \langle \pi^* \theta, [\mathcal{M}] \rangle \in \mathbb{Z},$$

where $[\mathcal{M}]$ is the fundamental class of \mathcal{M} with \mathbb{Z} -coefficients.

- (ii) If $m > 0$, then choose an extension $\tilde{\mathfrak{s}}_X$ of \mathfrak{s}_X to a Spin^c -structure on $T(E_X/B)$. In turn this determines a line bundle $\mathcal{L} \rightarrow \mathcal{M}$. We then set

$$\text{FSW}_m^{\mathbb{Z}}(E_X, \tilde{\mathfrak{s}}_X, \theta) = \langle c_1(\mathcal{L})^m \cup \pi^* \theta, [\mathcal{M}] \rangle \in \mathbb{Z}.$$

Remark 2.5 If B or $\det^+(E_X)$ is nonorientable, we can still define \mathbb{Z} -valued families Seiberg–Witten invariants provided we work with local coefficients. We can also consider a \mathbb{Z} -valued ρ -twisted families Seiberg–Witten invariant. For simplicity we omit the details of these generalisations.

We here give a remark on a cohomological description of the families Seiberg–Witten invariants defined above. The argument here is parallel for both the \mathbb{Z}_2 and \mathbb{Z} cases, and therefore we drop the coefficient from our notation of (co)homology and from the families Seiberg–Witten invariant. First, recall that we can define the wrong-way map

$$\pi_*: H^*(\mathcal{M}) \rightarrow H^{*+\dim B-\dim \mathcal{M}}(B)$$

by composing the usual pullback π^* on cohomology and Poincaré duality for both \mathcal{M} and B . Next, in the setting of Definition 2.1 or Definition 2.4, let us write again the dimension of the families moduli space as $\dim \mathcal{M} = 2m + s$ for some nonnegative integer m and s , and assume again that either

- (i) $m = 0$, or
- (ii) $b_1(X) = 0$ and \mathfrak{s}_X extends to a Spin^c -structure on $T(E_X/B)$, which we denote by $\tilde{\mathfrak{s}}_X$.

In case (i), we set

$$\text{FSW}(E_X, \mathfrak{s}_X) = \pi_*(1) \in H^{-d(X, \mathfrak{s}_X)}(B) = H^{d-s}(B),$$

and in case (ii), we set

$$\text{FSW}_m(E_X, \mathfrak{s}_X) = \pi_*(c_1(\mathcal{L})^m) \in H^{2m-d(X, \mathfrak{s}_X)}(B) = H^{d-s}(B).$$

Then we get the equality

$$(2-1) \quad \text{FSW}_m(E_X, \mathfrak{s}_X, \theta) = \langle \text{FSW}_m(E_X, \mathfrak{s}_X) \cup \theta, [B] \rangle$$

for any $\theta \in H^s(B, \mathbb{Z})$ because of the so-called projection formula. In particular, if we consider the \mathbb{Z}_2 -coefficient case, then (2-1) can be regarded as a defining property of the cohomology class $\text{FSW}(E_X, \mathfrak{s}_X)$. Thus we will sometime view the families Seiberg–Witten invariant as the cohomology class $\text{FSW}(E_X, \mathfrak{s}_X)$. As well as in the untwisted case, note that we can regard the ρ -twisted families invariant as a cohomology class

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho) \in H^{-d(X, \mathfrak{s}_X)}(B, \mathbb{Z}_2).$$

3 The Seiberg–Witten invariants of group actions

In this section we define a variant of the families Seiberg–Witten invariant which is defined for smooth group actions.

Let X be a smooth compact oriented 4–manifold and let G be a group acting on X by orientation-preserving diffeomorphisms. Suppose that G preserves the isomorphism class of a Spin^c –structure \mathfrak{s}_X . Suppose that $d(X, \mathfrak{s}_X) \leq 0$ and that $b^+(X) > -d(X, \mathfrak{s}_X) + 1$. Let B be any smooth compact manifold with $\dim B < b^+(X) - 1$. Let $E \rightarrow B$ be a principal G –bundle over B (where G is given the discrete topology). We obtain an associated family $E_X = E \times_G X \rightarrow B$. Associated to this family and Spin^c –structure \mathfrak{s}_X is the families Seiberg–Witten invariant

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X) \in H^{-d(X, \mathfrak{s}_X)}(B, \mathbb{Z}_2).$$

If we consider (X, \mathfrak{s}_X) and the G –action on X as fixed data, then we may think of $\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X)$ as a map which assigns to each principal G –bundle $E \rightarrow B$ over a base B with $\dim B < b^+(X) - 1$ a cohomology class in $H^{-d(X, \mathfrak{s}_X)}(B, \mathbb{Z}_2)$. It turns out that this assignment is a characteristic class, that is, there exists a cohomology class

$$\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X) \in H^{-d(X, \mathfrak{s}_X)}(BG, \mathbb{Z}_2)$$

such that if $f: B \rightarrow BG$ is the classifying map of $E \rightarrow B$ then

$$(3-1) \quad \text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X) = f^*(\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X)).$$

This follows easily from the existence of the characteristic classes $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X) \in H^{-d(X, \mathfrak{s}_X)}(B\text{Diff}(X, \mathfrak{s}_X), \mathbb{Z}_2)$ constructed by Konno [10]. Recall that if $E_X \rightarrow B$ is any family of 4–manifolds with structure group $\text{Diff}(X, \mathfrak{s}_X)$ (the group of diffeomorphisms of X preserving the isomorphism class of \mathfrak{s}_X), the families Seiberg–Witten invariant is given by

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X) = g^*(\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X)),$$

where $g: B \rightarrow B\text{Diff}(X, \mathfrak{s}_X)$ is the classifying map of $E_X \rightarrow B$. In the case that E_X is of the form $E_X = E \times_G X$, we have that g factors as

$$B \xrightarrow{f} BG \xrightarrow{\psi} B\text{Diff}(X, \mathfrak{s}_X),$$

where ψ is the map induced from the homomorphism $G \rightarrow \text{Diff}(X, \mathfrak{s}_X)$ determined by the action of G on X . Then (3-1) follows immediately if we define $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X)$

to be

$$(3-2) \quad \text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X) = \psi^*(\text{SW}^{\mathbb{Z}_2}(X, \mathfrak{s}_X)).$$

Definition 3.1 We define the G -equivariant Seiberg–Witten invariant of (X, \mathfrak{s}_X) with \mathbb{Z}_2 -coefficients to be the class $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X) \in H^{-d(X, \mathfrak{s}_X)}(BG, \mathbb{Z}_2)$ defined by (3-2).

We can also consider a twisted version of the invariant $\text{SW}_G^{\mathbb{Z}_2}$ as follows. Let X be a smooth compact oriented 4-manifold and let G be a group acting on X by orientation-preserving diffeomorphisms. Let $\rho: G \rightarrow \mathbb{Z}_2$ be a homomorphism and suppose that G preserves a Spin^c -structure \mathfrak{s}_X up to a sign factor determined by ρ :

$$g^*(\mathfrak{s}_X) = (-1)^{\rho(g)}\mathfrak{s}_X \quad \text{for all } g \in G.$$

Suppose as before that $d(X, \mathfrak{s}_X) \leq 0$ and that $b^+(X) > -d(X, \mathfrak{s}_X) + 1$. Then we obtain a ρ -twisted G -equivariant Seiberg–Witten invariant,

$$\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) \in H^{-d(X, \mathfrak{s}_X)}(BG, \mathbb{Z}_2).$$

One can also define the corresponding \mathbb{Z} -valued invariants under suitable orientability assumptions. In all applications of these invariants that we have considered so far, the \mathbb{Z}_2 -invariant suffices. Hence we omit the details of the construction of the G -equivariant \mathbb{Z} -valued Seiberg–Witten invariant.

The following proposition illustrates one of the possible applications of the G -equivariant Seiberg–Witten invariants:

Proposition 3.2 *If X admits a G -invariant metric of positive scalar curvature then $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) = 0$, whenever it is defined.*

Proof Assume that X admits such a metric. To show that $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) = 0$ it suffices to show that its pairing with any class $\beta \in H_{-d(X, \mathfrak{s}_X)}(BG, \mathbb{Z}_2)$ is zero. By the solution to the Steenrod problem with \mathbb{Z}_2 -coefficients, there exists a compact smooth manifold B of dimension $-d(X, \mathfrak{s}_X)$ and a continuous map $f: B \rightarrow BG$ such that $f_*[B] = \beta$. Let $E = f^*(EG)$ be the pullback of the universal bundle $EG \rightarrow BG$ and let $E_X = E \times_G X$ be the family associated to E . Then

$$\langle \text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho), \beta \rangle = \langle f^*(\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X), \rho), [B] \rangle = \langle \text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho), [B] \rangle.$$

So it suffices to show that $\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho) = 0$. But a G -invariant metric g on X with positive scalar curvature determines a family $\{g_b\}_{b \in B}$ of fibrewise metrics on E_X

with positive scalar curvature. Since $b^+(X) > \dim B + 1 = -d(X, \mathfrak{s}_X) + 1$ (by assumption) we can choose a families perturbation $\{\eta_b\}_{b \in B}$ which avoids the wall of reducibles and we can in addition take the C^0 norm of the perturbations $\{\eta_b\}_{b \in B}$ to be sufficiently small that there are no irreducible solutions of the Seiberg–Witten equations for (g_b, η_b) for every $b \in B$. Thus the families Seiberg–Witten moduli space is empty and hence $\text{FSW}^{\mathbb{Z}^2}(E_X, \mathfrak{s}_X, \rho) = 0$. \square

4 Setup for the general gluing construction

In this section we describe the formulation of a gluing result used to calculate the families invariant. This is a generalisation of Ruberman’s argument given in [21]. (See also [19; 20] for the Yang–Mills version.)

Our gluing result can be proved for families that are obtained by a families connected sum. Let M and N be compact, smooth, oriented 4–manifolds and set $X = M \# N$. We assume throughout that $b_1(N) = 0$. Let B be a compact smooth manifold of dimension d and suppose we have smooth fibrewise oriented families $\pi_M: E_M \rightarrow B$ and $\pi_N: E_N \rightarrow B$ whose fibres are M and N , respectively. Let $T(E_M/B) = \text{Ker}(\pi_{M*})$ and $T(E_N/B) = \text{Ker}(\pi_{N*})$ be the vertical tangent bundles. We wish to form a connected sum family. For this, suppose we are given the following additional data:

- Sections $\iota_M: B \rightarrow E_M$ and $\iota_N: B \rightarrow E_N$.
- An orientation-reversing vector bundle isomorphism $\phi: \iota_M^*(T(E_M/B)) \rightarrow \iota_N^*(T(E_N/B))$.

Let $V = \iota_M^*(T(E_M/B))$ be the normal bundle of ι_M . Then V is a real rank-4 vector bundle on B . Note that V is oriented by our assumption that $E_M \rightarrow B$ is fibrewise oriented. Fix once and for all a metric on V so that V becomes an $\text{SO}(4)$ –vector bundle. By the isomorphism ϕ we can identify $\iota_N^*(T(E_N/B))$ with V but with the opposite orientation. Let $D(V) \rightarrow B$ be the unit open disc bundle of V and $S(V) \rightarrow B$ the unit sphere bundle. Let $U_M \subset E_M$ and $U_N \subset E_N$ be tubular neighbourhoods of ι_M and ι_N equipped with diffeomorphisms $e_M: D(V) \rightarrow U_M$ and $e_N: D(V) \rightarrow U_N$, where e_M is orientation-preserving and e_N is orientation-reversing.

Let M' denote M with a small open ball removed and similarly define N' . Then M' and N' are compact 4–manifolds with boundary $\partial M' = \partial N' = S^3$. We may regard $E_{M'} = E_M \setminus U_M$ as a family of 4–manifolds diffeomorphic to M' over B and similarly regard $E_{N'} = E_N \setminus U_N$ as a family of 4–manifolds diffeomorphic to N' .

The boundary of $E_{M'}$ is the S^3 -bundle $S(V) \rightarrow B$. Similarly, the boundary of $E_{N'}$ is also $S(V) \rightarrow B$, except the fibres of $S(V) \rightarrow B$ inherit the opposite orientation from the family $E_{N'}$.

Let \widehat{M} and \widehat{N} be the cylindrical end 4-manifolds obtained from M' and N' by attaching the half-infinite cylinders $[0, \infty) \times S^3$ to M' and $(-\infty, 0] \times S^3$ to N' . We obtain a family $E_{\widehat{M}} \rightarrow B$ of cylindrical end 4-manifolds by attaching to $E_{M'}$ the family of cylinders $[0, \infty) \times S(V)$ and we obtain a family $E_{\widehat{N}} \rightarrow B$ similarly.

The fixed metric on V determines families of fibrewise metrics on $D(V)$ and $S(V)$ that in turn define fibrewise metrics on the cylinders $[0, \infty) \times S(V)$ and $(-\infty, 0] \times S(V)$ by taking the product with the standard metric $(dt)^2$ on the intervals $[0, \infty)$, $(-\infty, 0]$. Choose a collar neighbourhood $(-\epsilon, 0] \times S(V)$ of $\partial E_{M'}$ and on this neighbourhood choose the fibrewise metric $(dt)^2 + g_{S(V)}$ and extend this to a fibrewise metric on all of $E_{M'}$. Construct a fibrewise metric similarly for $E_{N'}$. Clearly the fibrewise metrics on $E_{M'}$ and $E_{N'}$ can be extended to a family of cylindrical end metrics on the families $E_{\widehat{M}}$ and $E_{\widehat{N}}$, which we denote by $\{g_{\widehat{M},b}\}_{b \in B}$ and $\{g_{\widehat{N},b}\}_{b \in B}$, or more simply just as $g_{\widehat{M}}$ and $g_{\widehat{N}}$.

For each $r > 0$, let $\widehat{M}(r) = \widehat{M} \setminus (r+2, \infty) \times S^3$ and let $\widehat{N}(r) = \widehat{N} \setminus (\infty, -r-2) \times S^3$. Let $\widehat{X}(r)$ be the 4-manifold obtained by identifying the end portions of the necks of $\widehat{M}(r)$ and $\widehat{N}(r)$ via the map $[r, r+2] \times S^3 \rightarrow [-r-2, -r] \times S^3$ given by $t \mapsto t-2r-2$. For each $r > 0$, $\widehat{X}(r)$ is diffeomorphic to $X = M \# N$, but the family of metrics $g_{\widehat{X}(r)} = g_{\widehat{M}} \#_r g_{\widehat{N}}$ depends on r . As $r \rightarrow \infty$, the length of the neck joining M and N is stretched out. We can carry out this construction in families, giving a family $E_{\widehat{X}(r)} \rightarrow B$ which depends on r .

The family $E_{\widehat{X}(r)}$ that we have just constructed depends on numerous choices (for example on the choice of tubular neighbourhoods U_M and U_N). However it is not hard to see that, up to isomorphism of families, the only data on which $E_{\widehat{X}(r)}$ depends is the tuple $(E_M, E_N, \iota_M, \iota_N, \phi)$, where two families are considered isomorphic if there is a diffeomorphism of their total spaces covering the identity on B . We will usually just write the family as E_X , hiding the dependence of the various choices involved in its construction.

Let \mathfrak{s}_M and \mathfrak{s}_N be Spin^c -structures on M and N and let \mathfrak{s}_X be the Spin^c -structure on X obtained as the connected sum of \mathfrak{s}_M and \mathfrak{s}_N . We will assume that \mathfrak{s}_M and \mathfrak{s}_N are monodromy-invariant under the monodromy action of $\pi_1(B)$ on the set of Spin^c -structures on M and N induced by the families E_M and E_N . Put another way, we

assume \mathfrak{s}_M and \mathfrak{s}_N extend to give continuously varying families of Spin^c –structures on the fibres of E_M and E_N . Then, likewise, \mathfrak{s}_X extends to a continuous family of Spin^c –structures on the fibres of $E_{\widehat{X}(r)}$.

Let us first consider the \mathbb{Z}_2 –valued families Seiberg–Witten invariant. We will assume that

$$d(M, \mathfrak{s}_M) = \frac{1}{4}(c_1(\mathfrak{s}_M)^2 - \sigma(M)) - 1 - b^+(M) + b_1(M)$$

is even and nonnegative, and define $m \geq 0$ by $d(M, \mathfrak{s}_M) = 2m$. Next, we make the following assumptions about N . Assume that

$$c_1(\mathfrak{s}_N)^2 = \sigma(N),$$

so that the index of the Spin^c –Dirac operator on N is zero. Assume in addition that

$$0 < b^+(N) \leq \dim B = d.$$

Recall also that we are assuming $b_1(N) = 0$. It follows that

$$d(N, \mathfrak{s}_N) = \frac{1}{4}(c_1(\mathfrak{s}_N)^2 - \sigma(N)) - 1 - b^+(N) + b_1(N) = -1 - b^+(N) < 0.$$

Let $\mathcal{M} = \mathcal{M}(E_{\widehat{X}(r)}, g_{\widehat{X}(r)}, \eta_{\widehat{X}(r)}, \mathfrak{s}_X)$ be the Seiberg–Witten moduli space for the family $E_{\widehat{X}(r)}$ and a choice of generic perturbation $\eta_{\widehat{X}(r)}$ (for now $\eta_{\widehat{X}(r)}$ is an arbitrary generic perturbation. Later, we will choose $\eta_{\widehat{X}(r)}$ much more carefully). Then

$$\dim \mathcal{M} = d(M, \mathfrak{s}_M) + d(N, \mathfrak{s}_N) + 1 + \dim B = 2m + \dim B - b^+(N).$$

Note that for a well-defined families Seiberg–Witten invariant attached to the family E_X we need to assume that

$$b^+(M) > (\dim B - b^+(N)) + 1.$$

Let us set

$$s = \dim B - b^+(N) \geq 0.$$

So $\dim \mathcal{M} = 2m + s$ and we require $b^+(M) > s + 1$.

Let $\theta \in H^s(B, \mathbb{Z}_2)$. Recall that the families invariant $\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta)$ is defined in the cases

- (i) $d(M, \mathfrak{s}_M) = 2m = 0$, or
- (ii) $b_1(M) = 0$ (hence also $b_1(X) = 0$) and \mathfrak{s}_X extends to a Spin^c –structure on $T(E_X/B)$, which we denote by $\tilde{\mathfrak{s}}_X$.

The main gluing theorem of this paper is as follows:

Theorem 4.1 (\mathbb{Z}_2 -valued gluing formula) *Suppose that either*

- (i) $d(M, \mathfrak{s}_M) = 2m = 0$, or
- (ii) $b_1(M) = b_1(X) = 0$ and \mathfrak{s}_X extends to a Spin^c -structure on $T(E_X/B)$.

Then we have:

- (1) For any $\theta \in H^s(B, \mathbb{Z}_2)$, we have

$$\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) = \text{SW}(M, \mathfrak{s}_M) \cdot \langle \theta \cup w_{b+(N)}(H^+(N)), [B] \rangle$$

in \mathbb{Z}_2 . Here $\text{SW}(M, \mathfrak{s}_M)$ denotes the mod 2 Seiberg–Witten invariant. In particular, in case (ii), $\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta)$ does not depend on the choice of extension $\tilde{\mathfrak{s}}_X$ of \mathfrak{s}_X to a Spin^c -structure on $T(E_X/B)$.

- (2) For any integer $k > 0$ and any $\theta \in H^{s+2k}(B, \mathbb{Z}_2)$, we have

$$\text{FSW}_{m-k}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) = 0.$$

Remark 4.2 In case (ii), the fact that $\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta)$ does not depend on the choice of $\tilde{\mathfrak{s}}_X$ for any $\theta \in H^s(B, \mathbb{Z}_2)$ can also be deduced from the fact that

$$\text{FSW}_{m-k}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) = 0$$

for any $\theta \in H^{s+2k}(B, \mathbb{Z}_2)$, where $k > 0$.

We will prove parts (1) and (2) of [Theorem 4.1](#) separately.

Similarly, we can also get ρ -twisted version of [Theorem 4.1](#). Instead of assuming that \mathfrak{s}_M and \mathfrak{s}_N are monodromy-invariant under the monodromy action of $\pi_1(B)$, we here assume that the monodromy preserves \mathfrak{s}_M , \mathfrak{s}_N and \mathfrak{s}_X up to a common sign $\rho: \pi_1(B) \rightarrow \mathbb{Z}_2$. Let us keep all other assumptions supposed before the statement of [Theorem 4.1](#). Then we have the following:

Theorem 4.3 (ρ -twisted gluing formula) *Assume that the monodromy preserves \mathfrak{s}_M , \mathfrak{s}_N and \mathfrak{s}_X up to a common sign $\rho: \pi_1(B) \rightarrow \mathbb{Z}_2$. Assume also that $d(M, \mathfrak{s}_M) = 2m = 0$, and suppose that we are given $\theta \in H^s(B, \mathbb{Z}_2)$. Then we have*

$$\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta, \rho) = \text{SW}(M, \mathfrak{s}_M) \cdot \langle \theta \cup w_{b+(N)}(H^+(N) \otimes \mathbb{R}_\rho) \rangle$$

in \mathbb{Z}_2 , where $\mathbb{R}_\rho \rightarrow B$ is the real line bundle classified by $\rho \in H^1(B, \mathbb{Z}_2)$.

As well as the \mathbb{Z}_2 case, we can show a similar gluing result also for the \mathbb{Z} -valued invariant. Besides all assumptions supposed before the statement of [Theorem 4.1](#), let

us assume that B , $\det^+(E_M)$ and $\det^+(E_N)$ are oriented and homology orientations of M and N are fixed. Note that orientations of $X = M \# N$ and $\det^+(E_X)$ are automatically determined.

Theorem 4.4 (\mathbb{Z} -valued gluing formula) *Suppose that either*

- (i) $d(M, \mathfrak{s}_M) = 2m = 0$, or
- (ii) $b_1(M) = b_1(X) = 0$ and \mathfrak{s}_X extends to a Spin^c -structure on $T(E_X/B)$.

Then we have:

- (1) For any $\theta \in H^s(B, \mathbb{Z})$, we have

$$\text{FSW}_m^{\mathbb{Z}}(E_X, \mathfrak{s}_X, \theta) = \text{SW}(M, \mathfrak{s}_M) \cdot \langle \theta \cup e(H^+(N)), [B] \rangle$$

in \mathbb{Z} . Here $\text{SW}(M, \mathfrak{s}_M)$ denotes the Seiberg–Witten invariant and $e(H^+(N))$ the Euler class of $H^+(N)$. In particular, in case (ii), $\text{FSW}_m^{\mathbb{Z}}(E_X, \mathfrak{s}_X, \theta)$ does not depend on the choice of extension $\tilde{\mathfrak{s}}_X$ of \mathfrak{s}_X to a Spin^c -structure on $T(E_X/B)$.

- (2) For any integer $k > 0$ and any $\theta \in H^{s+2k}(B, \mathbb{Z})$, we have

$$\text{FSW}_{m-k}^{\mathbb{Z}}(E_X, \mathfrak{s}_X, \theta) = 0.$$

Remark 4.5 One of the differences between Ruberman’s original gluing theorem [19; 20; 21] and ours is that we do not assume that our families of $M \# N$ consist of trivial families of M and families of N . In [19; 20; 21], Ruberman considered diffeomorphisms which are supported on N , strictly speaking on N' , and therefore the corresponding families obtained by mapping tori can be regarded as the families connected sum of trivial families of M and (typically nontrivial) families of N . On the other hand, we shall also consider diffeomorphisms on $M \# N$ acting nontrivially not only on N but also on M . In the subsequent sections, this generalisation is effectively used when we shall give examples of 4-manifolds which admit positive scalar curvature metrics but do not admit \mathbb{Z}_2 -invariant positive scalar curvature metrics for suitable \mathbb{Z}_2 -actions on them.

5 Proof of gluing formula, first step

We first concern ourselves with [Theorem 4.1](#)(1). In this section we will show that it suffices to prove [Theorem 4.1](#)(1) in the case that $s = 0$, that is, in the case $\dim B = b^+(N)$. Suppose the theorem holds in the case $s = 0$. Now we will prove the general theorem. Let $\theta \in H^s(B, \mathbb{Z}_2)$. By the solution to the Steenrod problem, θ is Poincaré dual to a

homology class of the form $f_*[S] \in H_{b^+(N)}(B, \mathbb{Z}_2)$, where S is a compact smooth manifold of dimension $\dim S = b^+(N)$ and $f: S \rightarrow B$ is a smooth map. Then

$$\begin{aligned} & \text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) \\ &= \langle c_1(\mathcal{L})^m \cup \pi^*\theta, [\mathcal{M}] \rangle \\ &= \langle \pi_*(c_1(\mathcal{L})^m \cup \pi^*(\theta)), [B] \rangle \quad (\text{by definition of the wrong-way map } \pi_*) \\ &= \langle \pi_*(c_1(\mathcal{L})^m) \cup \theta, [B] \rangle \\ &= \langle \pi_*(c_1(\mathcal{L})^m), [S] \rangle \quad (\theta \text{ is Poincaré dual to } [S]) \\ &= \langle c_1(\mathcal{L})^m, [\mathcal{M}|_S] \rangle \quad (\text{by definition of the wrong-way map } \pi_*) \\ &= \text{FSW}_m^{\mathbb{Z}_2}(E_X|_S, \mathfrak{s}_X, 1). \end{aligned}$$

Now we consider the pullbacks $f^*(E_M)$, $f^*(E_N)$ of the families to S . By assumption, [Theorem 4.1\(1\)](#) holds on S , since $\dim S = b^+(N)$, hence

$$\begin{aligned} & \text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) \\ &= \text{FSW}_m^{\mathbb{Z}_2}(E_X|_S, \mathfrak{s}_X, 1) \\ &= \text{SW}(M, \mathfrak{s}_M) \cdot \langle w_{b^+(N)}(H^+(N)), [S] \rangle \quad (\text{by } \text{Theorem 4.1(1)} \text{ on } S) \\ &= \text{SW}(M, \mathfrak{s}_M) \cdot \langle \theta \cup w_{b^+(N)}(H^+(N)), [B] \rangle \quad (\theta \text{ is Poincaré dual to } [S]). \end{aligned}$$

The last equality is exactly [Theorem 4.1\(1\)](#) on B , which is what we wanted to prove.

Remark 5.1 Almost the same argument works in the case of \mathbb{Z} -coefficients. The main difference is that the Steenrod problem can only be solved up to a multiple. That is, given $\theta \in H^s(B, \mathbb{Z})$, there is some positive integer N such that we can represent $N\theta$ as $f_*[S]$ for some $f: S \rightarrow B$. But if the \mathbb{Z} -valued version of [Theorem 4.1](#) holds for $N\theta$, then, by cancelling out the factor of N on both sides, we get that the theorem also holds for θ itself.

So, for the remainder of the proof of [Theorem 4.1\(1\)](#), we may assume that

$$\dim B = d = b^+(N).$$

6 Gluing theory for the Seiberg–Witten equations

6.1 Review of gluing theory

To proceed further with the proof of [Theorem 4.1](#), we need to employ a parametrised version of the gluing construction for the Seiberg–Witten equations on connected

sums. Our main reference will be Nicolaescu’s book [16], and we will refer to [16] as “Nicolaescu’s book” or just simply “Nicolaescu”. An alternative and quite thorough reference for gluing theory is the monograph of Frøyshov [7]. In this section, let us first briefly review the ordinary case of this construction, ie the unparametrised case.

We have 4–manifolds M and N and their connected sum $X = M \# N$. Recall that we defined the cylindrical end 4–manifolds \widehat{M} and \widehat{N} and we attached them to form $\widehat{X}(r)$, where r is a positive real number. Each $\widehat{X}(r)$ is diffeomorphic to X , but the length of the neck of $\widehat{X}(r)$ grows proportionally with r . To make the analysis work well on cylindrical end manifolds, one considers weighted Sobolev spaces $L_{\mu}^{k,p}$, where $\mu \in \mathbb{R}$ is the weight, and the extended Sobolev spaces $L_{\mu,ex}^{k,p}$, following Atiyah, Patodi and Singer [1], called the asymptotically cylindrical sections in Nicolaescu’s book (page 299 in Section 4.1.4), to deal with asymptotic values.

Here $\eta_{\widehat{M}}$ is the self-dual 2–form perturbation, which is assumed to be supported away from the neck and $\mathfrak{s}_{\widehat{M}}$ is an asymptotically cylindrical Spin^c –structure on \widehat{M} . Isomorphism classes of such Spin^c –structures are in bijection with Spin^c –structures on M (this follows from the fact that our 3–manifold used for gluing, denoted by $\partial_{\infty}\widehat{M}$ in Nicolaescu’s book, is just S^3 and $H^1(S^3, \mathbb{Z}) = H^2(S^3, \mathbb{Z}) = 0$; see Nicolaescu for more details on asymptotically cylindrical Spin^c –structures). Therefore we will simply identify $\mathfrak{s}_{\widehat{M}}$ with \mathfrak{s}_M without further mention. We have isomorphisms

$$H_{L^2}^2(\widehat{M}) \cong H^2(\widehat{M}, [0, \infty) \times S^3, \mathbb{R}) \cong H^2(\widehat{M}, \mathbb{R}) \cong H^2(M, \mathbb{R}),$$

where $H_{L^2}^2(\widehat{M})$ is the space of L^2 –integrable harmonic 2–forms on \widehat{M} . The first isomorphism is shown by Atiyah, Patodi and Singer [1], the second isomorphism is just the long exact sequence for cohomology of the pair $(\widehat{M}, [0, \infty) \times S^3)$ and the third isomorphism can be obtained from the Mayer–Vietoris sequence. Thus the L^2 –second Betti number of \widehat{M} coincides with the ordinary second Betti number of M . Note also that this isomorphism depends only on the choice of metric $g_{\widehat{M}}$.

Next, note that $H_{L^2}^2(\widehat{M})$ is equipped with a natural bilinear form, the L^2 –intersection form

$$(\alpha, \beta) = \int_{\widehat{M}} \alpha \wedge \beta.$$

On the other hand, Poincaré–Lefschetz duality implies that the cup product determines a nondegenerate bilinear form on the image of $H^2(\widehat{M}, [0, \infty) \times S^3, \mathbb{R})$ in $H^2(\widehat{M}, \mathbb{R})$. In our case, the image is all of $H^2(\widehat{M}, \mathbb{R})$ and, under the isomorphism with $H^2(M, \mathbb{R})$, coincides with the usual intersection form on M . The isomorphism given by Atiyah,

Patodi and Singer between $H^2_{L^2}(\widehat{M})$ and the image of $H^2(\widehat{M}, [0, \infty) \times S^3, \mathbb{R})$ in $H^2(\widehat{M}, \mathbb{R})$ can be shown to respect the intersection forms. Therefore, in our case we have an isomorphism $H^2_{L^2}(\widehat{M}) \cong H^2(M, \mathbb{R})$ such that the L^2 -intersection form on $H^2_{L^2}(\widehat{M})$ coincides with the topological intersection form on $H^2(M, \mathbb{R})$. Let $H^+_{L^2}(\widehat{M})$ denote the subspace of $H^2_{L^2}(\widehat{M})$ consisting of self-dual L^2 -harmonic 2-forms. It follows immediately that the dimension of $H^+_{L^2}(\widehat{M})$ equals $b^+(M)$.

Remark 6.1 In the families setting, given the family $E_{\widehat{M}} \rightarrow B$ and family $\{g_{\widehat{M}}\}$ of fibrewise metrics, we have that $H^+_{L^2}(\widehat{M})$ and $H^2(M, \mathbb{R})$ can be thought of as vector bundles on B , equipped with bilinear forms. The isomorphism $H^+_{L^2}(\widehat{M}) \cong H^2(M, \mathbb{R})$ depends only on the choice of metric $g_{\widehat{M}}$ and so, in the families setting, we get that $H^+_{L^2}(\widehat{M}) \cong H^2(M, \mathbb{R})$ is an isomorphism of vector bundles with bilinear forms.

It is shown in Nicolaescu that if $b^+(M) > 0$, then the perturbation $\eta_{\widehat{M}}$ can be chosen so as to avoid any reducible solutions of the Seiberg–Witten equations, and for generic such $\eta_{\widehat{M}}$, the moduli space $\mathcal{M}(\widehat{M}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_{\widehat{M}}, \mu)$ is a smooth compact manifold of dimension $d(M, \mathfrak{s})$ (the same dimension as the Seiberg–Witten moduli space for (M, \mathfrak{s}_M)); see also Frøyshov [7, Theorem 1.3.1 and Sections 3.4 and 8.2]. Compactness depends crucially on the assumption that $\partial_{\infty}\widehat{M} = S^3$. More precisely, within the space of 2-form perturbations on \widehat{M} supported away from the neck, the ones for which a reducible solution exists form a closed subspace of codimension $b^+(M)$. We will discuss this in more detail in Section 7.

Given a solution

$$\widehat{C} = (\widehat{A}, \widehat{\psi}) \in \mathcal{M}(\widehat{M}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_{\widehat{M}}, \mu)$$

of the Seiberg–Witten equations on \widehat{M} , one can consider the deformation theory of \widehat{C} . One obtains a three-term deformation complex, denoted by $\widehat{\mathcal{K}}_{\widehat{C}}$ in Nicolaescu’s book, which controls the deformation theory. For $i = 0, 1, 2$, let $H^i_{\widehat{C}}$ denote the i^{th} cohomology of $\widehat{\mathcal{K}}_{\widehat{C}}$. These groups have the usual interpretations: $H^0_{\widehat{C}}$ is the Lie algebra of infinitesimal automorphisms, hence $H^0_{\widehat{C}} = 0$ if \widehat{C} is irreducible and $H^0_{\widehat{C}} = \mathbb{R}$ if \widehat{C} is reducible; $H^1_{\widehat{C}}$ is the virtual tangent space and $H^2_{\widehat{C}}$ is the obstruction space. The local structure of the moduli space around \widehat{C} is given by a Kuranishi model $H^1_{\widehat{C}} \supseteq U \xrightarrow{Q} H^2_{\widehat{C}}$ for some obstruction map Q .

As in the compact case, an application of Sard–Smale shows that for generic perturbations $\eta_{\widehat{M}}$, we have $H^2_{\widehat{C}} = 0$ for every irreducible solution of the $\eta_{\widehat{M}}$ -perturbed Seiberg–Witten equations on \widehat{M} .

Now we turn to gluing (see Nicolaescu’s book, Section 4.5). The idea is roughly as follows: Let \widehat{C}_M be a solution of the Seiberg–Witten equations on \widehat{M} and \widehat{C}_N be a solution of the Seiberg–Witten equations on \widehat{N} which agree asymptotically. If \widehat{C}_M and \widehat{C}_N were cylindrical, then we could identify them on the necks of $\widehat{M}(r)$ and $\widehat{N}(r)$ to obtain a genuine solution of the Seiberg–Witten equations on $\widehat{X}(r)$. In general, \widehat{C}_M and \widehat{C}_N are only asymptotically cylindrical. We can still glue them together on $\widehat{X}(r)$ with the aid of cutoff functions to form a glued configuration $\widehat{C}_r = \widehat{C}_M \#_r \widehat{C}_N$ on $\widehat{X}(r)$. But the cutoff functions introduce error terms so that \widehat{C}_r is only an approximate solution of the Seiberg–Witten equations on $\widehat{X}(r)$. If r is very large, then the error is very small and \widehat{C}_r is very close to being a solution. If $\eta_{\widehat{M}}$ and $\eta_{\widehat{N}}$ are perturbations on \widehat{M} and \widehat{N} supported away from the necks, then they can be glued together to form a perturbation $\eta_{\widehat{X}(r)} = \eta_{\widehat{M}} \#_r \eta_{\widehat{N}}$ on $\widehat{X}(r)$ (the gluing is straightforward because $\eta_{\widehat{M}}$ and $\eta_{\widehat{N}}$ both vanish along the necks). Using a variant of the Kuranishi model, one obtains a local description on the moduli space of genuine solutions which are sufficiently close to \widehat{C}_r . Since this step is of crucial importance, we will carefully review the construction in the next section.

Remark 6.2 Henceforth the notation $\#_r$ means the gluing by a specified cutoff function according to Nicolaescu’s book. In some works the operation $\#_r$ may be called a *pregluing* rather than gluing. (In such a case the word “gluing” is used only for a genuine solution near an approximating configuration obtained from $\#_r$.) However, in this paper, we often call the operation $\#_r$ simply a gluing.

6.2 Gluing monopoles: local theory

In Section 7, we shall describe a gluing argument for families. Roughly speaking, our argument is to estimate some errors occurring from the base space direction compared with the unparametrised case. As a preliminary of this forthcoming argument, in this section we shall recall some estimates in the local theory on gluing in the unparametrised case. We will mostly adopt the notation of Section 4 in Nicolaescu’s book (see also Frøyshov [7, Part II] for a thorough treatment of gluing).

Choose strongly cylindrical connections (see Nicolaescu’s book, Example 4.1.2) $\widehat{A}_{0,M}$ and $\widehat{A}_{0,N}$ on $\det(\mathfrak{s}_M)$ and $\det(\mathfrak{s}_N)$, respectively, which can be assumed compatible (recall that *compatible* means their asymptotic values agree), and set

$$\widehat{A}_0 = \widehat{A}_0(r) = \widehat{A}_{0,M} \#_r \widehat{A}_{0,N}.$$

Let $\widehat{\mathcal{C}}_{\mu,ex}(\widehat{M}), \widehat{\mathcal{C}}_{\mu,ex}(\widehat{N})$ be the configuration spaces consisting of asymptotically cylindrical data, defined on page 369 in Section 4.3.1 in Nicolaescu’s book. For $\widehat{C} \in \widehat{\mathcal{C}}_{\mu,ex}(\widehat{M})$ or $\widehat{\mathcal{C}}_{\mu,ex}(\widehat{N})$, we will denote by $\partial_\infty \widehat{C}$ the asymptotic value of \widehat{C} . If $\widehat{C}_M \in \widehat{\mathcal{C}}_{\mu,ex}(\widehat{M})$ and $\widehat{C}_N \in \widehat{\mathcal{C}}_{\mu,ex}(\widehat{N})$ are two smooth monopoles such that

$$\partial_\infty \widehat{C}_M = \partial_\infty \widehat{C}_N,$$

then we can form

$$\widehat{C}_r = (\widehat{\psi}_r, \widehat{A}_r) = \widehat{C}_M \#_r \widehat{C}_N = (\widehat{\psi}_M \#_r \widehat{\psi}_N, \widehat{A}_M \#_r \widehat{A}_N)$$

on $\widehat{X}(r)$, where $\widehat{X}(r)$ is the closed 4–manifold obtained from \widehat{M} and \widehat{N} having a neck of length r . Of course, the resulting configuration \widehat{C}_r is usually not a solution of the Seiberg–Witten equations since we used cutoff functions to form \widehat{C}_r as we explained in Section 6.1. We would like to show there exist genuine monopoles near \widehat{C}_r . For this we consider a variant of the Kuranishi model for a nonlinear Fredholm map $f: V \rightarrow W$ between Banach spaces. Normally one assumes $f(0) = 0$ and studies the structure of the zero set $f^{-1}(0)$ near 0. In our case, we are still interested in the zero set $f^{-1}(0)$ but we can have $f(0) \neq 0$.

Let $\widehat{C}_r \in \widehat{\mathcal{C}}(\widehat{X}(r))$ be the configuration obtained by gluing two monopoles using cutoff functions, as above. Let us denote by S^+ and S^- the positive and negative spinor bundles, respectively. We also use the notation \wedge^2_+ to indicate the self-dual part. As in Section 4.5.2 in Nicolaescu’s book, form the nonlinear map

$$\mathcal{N}: L^{2,2}(\widehat{X}(r), S^+ \oplus iT^*\widehat{X}(r)) \rightarrow L^{1,2}(\widehat{X}(r), S^- \oplus i \wedge^2_+ T^*\widehat{X}(r) \oplus i\mathbb{R})$$

given by

$$\mathcal{N}(\widehat{C}) = \widehat{SW}(\widehat{C} + \widehat{C}_r) \oplus \mathcal{L}^*_{\widehat{C}_r}(\widehat{C}).$$

Here the term $\widehat{SW}(\widehat{C} + \widehat{C}_r)$ denotes the Seiberg–Witten equations and $\mathcal{L}^*_{\widehat{C}_r}(\widehat{C})$ denotes the gauge fixing condition relative to \widehat{C}_r . Namely, when we write \widehat{C} and \widehat{C}_r as $\widehat{C} = (\widehat{a}, \widehat{\psi})$ and $\widehat{C}_r = (\widehat{a}_r, \widehat{\psi}_r)$, respectively, \widehat{SW} and $\mathcal{L}^*_{\widehat{C}_r}$ are defined as

$$\widehat{SW}(\widehat{C} + \widehat{C}_r) = (F^+_{\widehat{A}_0 + \widehat{a} + \widehat{a}_r} - q(\widehat{\psi} + \widehat{\psi}_r, \widehat{\psi} + \widehat{\psi}_r), D_{\widehat{A}_0 + \widehat{a} + \widehat{a}_r}(\widehat{\psi} + \widehat{\psi}_r))$$

and

$$\mathcal{L}^*_{\widehat{C}_r}(\widehat{C}) = -2d^*\widehat{a} - i \operatorname{Im}\langle \widehat{\psi}_r, \widehat{\psi} \rangle.$$

(On this gauge fixing condition, see Nicolaescu, Section 2.2.2.) Although here we just consider the unperturbed Seiberg–Witten equations, the necessary modification to deal with perturbed Seiberg–Witten equations will be done in Section 7, where we describe

families gluing with suitable perturbations. Note that \mathcal{N} is a well-defined smooth map because of the Sobolev multiplication theorem for $L^{2,2} \times L^{2,2} \rightarrow L^{1,2}$. We are of course interested in the zero set $\mathcal{N}^{-1}(0)$.

Denote by $\widehat{T}_r = \widehat{T}_{\widehat{C}_r}$ the linearisation of \mathcal{N} at 0, which is given by the map

$$\widehat{T}_r(\widehat{C}) = D_{\widehat{C}_r} \widehat{SW}(\widehat{C}) \oplus \mathcal{L}_{\widehat{C}_r}^*(\widehat{C})$$

whose domain and codomain are same as those of \mathcal{N} . Here the notation $D_{\widehat{C}_r}$ denotes the differential at \widehat{C}_r . Observe that, from $\widehat{C}_r = \widehat{C}_1 \#_r \widehat{C}_2$, we find that

$$\widehat{T}_r = \widehat{T}_{\widehat{C}_1} \#_r \widehat{T}_{\widehat{C}_2}.$$

Here the notation $\#_r$ means the gluing operation given by using the cutoff functions which we used to define \widehat{C}_r . Now write \mathcal{N} in the form

$$\mathcal{N}(\widehat{C}) = \mathcal{N}(0) + \widehat{T}_r(\widehat{C}) + R(\widehat{C}),$$

so R is the remainder term in the linear approximation of \mathcal{N} at 0.

Lemma 6.3 (Lemma 4.5.6 of Nicolaescu’s book) *There exists a constant $C > 0$ which depends only on the geometry of \widehat{M} and \widehat{N} such that*

$$\|R(\widehat{C})\|_{L^{1,2}} \leq Cr^{3/2} \|\widehat{C}\|_{L^{2,2}}^2$$

and

$$\|R(\widehat{C}) - R(\widehat{C}')\|_{L^{1,2}} \leq Cr^{3/2} (\|\widehat{C}\|_{L^{2,2}} + \|\widehat{C}'\|_{L^{2,2}}) \|\widehat{C} - \widehat{C}'\|_{L^{2,2}}$$

hold for any $\widehat{C}, \widehat{C}' \in L^{2,2}(\widehat{X}(r), S^+ \oplus iT^*\widehat{X}(r))$.

Remark 6.4 To state this kind of inequality between Sobolev spaces correctly, we choose connections of bundles for whose section we consider Sobolev norms such that, on the neck part, the connections are the pullbacks of some fixed connections on S^3 , and define Sobolev norms using the covariant derivatives corresponding to such connections. Then we can compare Sobolev spaces for varying r , and the statement of [Lemma 6.3](#), involving r , makes sense.

Remark 6.5 The proof of Lemma 4.5.6 is a consequence of the inequality (4.5.1) on page 430 of Nicolaescu’s book. The inequality (4.5.1) is asserted to hold for p in the range $1 < p \leq 6$, however we were only able to prove the inequality for $1 < p \leq 4$ because of the limitation coming from the Sobolev embedding. Fortunately the proof of Nicolaescu’s Lemma 4.5.6 only needs the $p = 4$ case of (4.5.1), so this is not a problem for [Lemma 6.3](#).

Definition 6.6 We introduce the notation

$$X_+^k = L^{k,2}(\widehat{X}(r), S^+ \oplus iT^*\widehat{X}(r)), \quad X_-^k = L^{k,2}(\widehat{X}(r), S^- \oplus i \wedge_+^2 T^*\widehat{X}(r) \oplus i\mathbb{R}),$$

and also set

$$X^k = X_+^k \oplus X_-^k.$$

Form the closed, densely defined operator

$$\widehat{L}_r: X^0 \rightarrow X^0$$

with block decomposition

$$\widehat{L}_r = \begin{bmatrix} 0 & \widehat{T}_r^* \\ \widehat{T}_r & 0 \end{bmatrix}.$$

It may help to think of \widehat{L}_r as being a “Dirac operator”. It is self-adjoint and induces bounded Fredholm operators

$$X^{k+1} \rightarrow X^k.$$

This is obvious because $\widehat{X}(r)$ is a closed manifold, so we can use ordinary Hodge theory.

Definition 6.7 Denote by H_r the subspace of X^0 spanned by the eigenvectors of \widehat{L}_r corresponding to eigenvalues in the interval $(-r^{-2}, r^{-2})$. The decomposition $X^0 = X_+^0 \oplus X_-^0$ induces a decomposition

$$H_r = H_r^+ \oplus H_r^-.$$

Note that H_r consists entirely of smooth sections.

Remark 6.8 In the ordinary Kuranishi model, instead of H_r^+ , we would normally just use $\text{Ker}(\widehat{T}_r)$ and instead of H_r^- , we would normally use $\text{Coker}(\widehat{T}_r) = \text{Ker}(\widehat{T}_r^*)$. However, in this situation it is more convenient to use H_r^\pm because we can use linear gluing theory, which gives us means of computing the dimensions of these spaces.

Definition 6.9 Let $Y^0(r)$ be the L^2 -orthogonal complement of H_r in X^0 . Define $Y^k(r) = Y^0(r) \cap X^k$. Then $Y^k(r)$ decomposes as

$$Y^k(r) = Y_+^k(r) \oplus Y_-^k(r).$$

We also have decompositions

$$X_+^k = H_r^+ \oplus Y_+^k(r), \quad X_-^k = H_r^- \oplus Y_-^k(r).$$

We shall sometimes simply write Y^k and Y_\pm^k instead of $Y^k(r)$ and $Y_\pm^k(r)$.

Remark 6.10 \widehat{T}_r sends H_r^+ to H_r^- and $Y_+^0(r)$ to $Y_-^0(r)$. Similarly, \widehat{T}_r^* sends H_r^- to H_r^+ and $Y_-^0(r)$ to $Y_+^0(r)$. To see this, note that since L_r is self-adjoint, then the eigenvalues of \widehat{L}_r^2 are all nonnegative real numbers. Moreover, the λ^2 -eigenspace of \widehat{L}_r^2 is easily seen to be the sum of the λ - and $-\lambda$ -eigenspaces of \widehat{L}_r . It follows that H_r can alternatively be characterised as the span of the eigenspaces of \widehat{L}_r^2 with eigenvalues in the range $[0, r^{-4})$. But

$$\widehat{L}_r^2 = \begin{bmatrix} \widehat{T}_r^* \widehat{T}_r & 0 \\ 0 & \widehat{T}_r \widehat{T}_r^* \end{bmatrix}.$$

This verifies the earlier-made claim that H_r decomposes as $H_r = H_r^+ \oplus H_r^-$. Moreover, it characterises H_r^+ as the span of the eigenspaces of $\widehat{T}_r^* \widehat{T}_r$ with eigenvalues in the range $[0, r^{-4})$ and similarly H_r^- as the span of the eigenspaces of $\widehat{T}_r \widehat{T}_r^*$ with eigenvalues in the range $[0, r^{-4})$. Now, to prove the claim made in the beginning of this remark, it suffices to show that \widehat{T}_r sends the λ^2 -eigenspace of $\widehat{T}_r^* \widehat{T}_r$ to the λ^2 -eigenspace of $\widehat{T}_r \widehat{T}_r^*$ (and similarly for \widehat{T}_r^*). Now if $\widehat{T}_r^* \widehat{T}_r x = \lambda^2 x$, then $\widehat{T}_r \widehat{T}_r^* (\widehat{T}_r x) = \widehat{T}_r (\widehat{T}_r^* \widehat{T}_r x) = \widehat{T}_r (\lambda^2 x) = \lambda^2 (\widehat{T}_r x)$ (and a similar computation holds for \widehat{T}_r^*).

Definition 6.11 Denote by P_\pm^r the L^2 -orthogonal projections $X_\pm^0 \rightarrow H_r^\pm$ and let $Q_\pm^r = 1 - P_\pm^r$ be the projections to $Y_\pm^0(r)$.

We often write P_\pm^r and Q_\pm^r simply as P_\pm and Q_\pm . Observe that $Q_\pm(X_\pm^k) = Y_\pm^k(r)$.

For each $\widehat{C} \in X_+^0$, we decompose it as $\widehat{C} = \widehat{C}_0 + \widehat{C}_1$, where

$$\widehat{C}_0 = P_+ \widehat{C}, \quad \widehat{C}_1 = Q_+ \widehat{C}.$$

By Remark 6.10, one can observe that

$$P_- \widehat{T}_r \widehat{C} = \widehat{T}_r P_+ \widehat{C} = \widehat{T}_r \widehat{C}_0, \quad Q_- \widehat{T}_r \widehat{C} = \widehat{T}_r Q_+ \widehat{C} = \widehat{T}_r \widehat{C}_1.$$

For every $k > 0$, \widehat{T}_r induces (by restriction) a bounded operator

$$\widehat{T}_r: Y_+^{k+1} \rightarrow Y_-^k.$$

Clearly this operator is invertible with bounded inverse and we denote the inverse as $S: Y_-^k \rightarrow Y_+^{k+1}$. We note that the following estimate (in Nicolaescu, this is the inequality (4.5.5), but there is a typo: it uses $\|Su\|_{L^{k+2,1}}$ instead of $\|Su\|_{L^{k+1,2}}$):

Lemma 6.12 *We have an estimate*

$$\|Su\|_{L^{k+1,2}} \leq C_k r^2 \|u\|_{L^{k,2}}$$

for some $C_k > 0$ which is independent of r and for all $u \in Y_-^k$

Proof To obtain this estimate simply note that the eigenvalues of $\widehat{T}_r^* \widehat{T}_r$ restricted to Y_+ lie in the range $[r^{-4}, \infty)$. □

The equation $\mathcal{N}(\widehat{C}) = 0$ is equivalent to the pair of equations

$$P_- \mathcal{N}(\widehat{C}) = 0, \quad Q_- \mathcal{N}(\widehat{C}) = 0.$$

Expanding \mathcal{N} and \widehat{C} , these equations become

$$Q_- \mathcal{N}(0) + \widehat{T}_r \widehat{C}_1 + Q_- R(\widehat{C}_0 + \widehat{C}_1) = 0, \quad P_- \mathcal{N}(0) + \widehat{T}_r \widehat{C}_0 + P_- R(\widehat{C}_0 + \widehat{C}_1) = 0.$$

Let us define $U \in Y_+^0$ to be

$$U = -SQ_- \mathcal{N}(0).$$

Applying S to the first of the two equations above, we end up with

$$\widehat{C}_1 = U - SQ_- R(\widehat{C}_0 + \widehat{C}_1).$$

For the moment, fix some $\widehat{C}_0 \in H_r^+$. Then define $\mathcal{F}: Y_+^2 \rightarrow Y_+^2$ as

$$\mathcal{F}(\widehat{C}_1) = U - SQ_- R(\widehat{C}_0 + \widehat{C}_1).$$

Note that \mathcal{F} is well-defined as a map from Y_+^2 to itself. First of all, it is easy to see that $U \in Y_+^2$, because $\mathcal{N}(0) \in X_-^1$, Q_- sends X_-^1 to Y_-^1 and S sends Y_-^1 to Y_+^2 . Similarly, $SQ_- R(\widehat{C}_0 + \widehat{C}_1) \in Y_+^2$ because $R(\widehat{C}_0 + \widehat{C}_1) \in X_-^1$.

Let us define

$$B_1(r^{-4}) = \{\widehat{C}_1 \in Y_+^2 \mid \|\widehat{C}_1\|_{L^{2,2}} \leq r^{-4}\} \subset Y_+^2(r)$$

and, similarly,

$$B_0(r^{-4}) = \{\widehat{C}_0 \in H_r^+ \mid \|\widehat{C}_0\|_{L^{2,2}} \leq r^{-4}\} \subset H_r^+.$$

Remark 6.13 Nicolaescu uses balls of radius r^{-3} instead of r^{-4} . Unfortunately r^{-3} does not seem sufficient to prove that \mathcal{F} is a contraction. The mistake seems to be in equation (4.5.8), page 438, which has a factor $r^{5/2}$, but the correct factor should be $r^{3/2}r^2 = r^{7/2}$.

Lemma 6.14 For all sufficiently large r and for all $\widehat{C}_0 \in B_0(r^{-4})$, the map \mathcal{F} sends $B_1(r^{-4})$ to itself and is a contraction mapping.

Proof First we check that \mathcal{F} sends $B_1(r^{-4})$ to itself, provided $\widehat{C}_0 \in B_0(r^{-4})$. Throughout we use C to indicate a constant which may depend on the geometry of the \widehat{N}_i , but is independent of r . The value of C may increase from line to line. If there is no risk of confusion, we drop the notation indicating the functional spaces from our norms.

We first have

$$\begin{aligned} \|\mathcal{F}(\widehat{C}_1)\| &\leq \|SQ_-\mathcal{N}(0)\| + \|SQ_-\mathcal{R}(\widehat{C}_0 + \widehat{C}_1)\| \\ &\leq Cr^2(\|Q_-\mathcal{N}(0)\| + \|Q_-\mathcal{R}(\widehat{C}_0 + \widehat{C}_1)\|) \\ &\leq Cr^2(\|\mathcal{N}(0)\| + \|\mathcal{R}(\widehat{C}_0 + \widehat{C}_1)\|), \end{aligned}$$

since Q_- is a projection operator.

We also have (see Nicolaescu, Lemma 4.5.5)

$$\|\mathcal{N}(0)\| \leq Ce^{-\mu r}$$

and, from Lemma 6.3,

$$\|\mathcal{R}(\widehat{C}_0 + \widehat{C}_1)\| \leq Cr^{3/2}(r^{-4})^2 = Cr^{-13/2}.$$

Hence,

$$\|\mathcal{F}(\widehat{C}_1)\| \leq C(r^2e^{-\mu r} + r^{-13/2}).$$

The right-hand side is less than or equal to r^{-4} for all sufficiently large r , hence \mathcal{F} sends $B_1(r^{-4})$ to itself.

Now we show \mathcal{F} is a contraction. Using Lemma 6.3 again, we have

$$\begin{aligned} \|\mathcal{F}(\widehat{C}_1) - \mathcal{F}(\widehat{C}'_1)\| &\leq \|SQ_-(\mathcal{R}(\widehat{C}_0 + \widehat{C}_1) - \mathcal{R}(\widehat{C}_0 + \widehat{C}'_1))\| \\ &\leq Cr^2\|\mathcal{R}(\widehat{C}_0 + \widehat{C}_1) - \mathcal{R}(\widehat{C}_0 + \widehat{C}'_1)\| \\ &\leq Cr^2r^{3/2}r^{-4}\|\widehat{C}_1 - \widehat{C}'_1\| \\ &\leq Cr^{-1/2}\|\widehat{C}_1 - \widehat{C}'_1\|. \end{aligned}$$

This shows that for all sufficiently large r , \mathcal{F} is indeed a contraction. □

Thus, for each $\widehat{C}_0 \in B_0(r^{-4})$, there is a uniquely determined fixed point of \mathcal{F} , which will be denoted as $\widehat{C}_1 = \Phi(\widehat{C}_0) \in B_1(r^{-4})$. It can be shown that $\Phi(\widehat{C}_0)$ depends differentiably on \widehat{C}_0 (by the implicit function theorem).

Now consider again the pair of equations

$$P_{-\mathcal{N}}(\widehat{C}_0 + \widehat{C}_1) = 0, \quad Q_{-\mathcal{N}}(\widehat{C}_0 + \widehat{C}_1) = 0.$$

The second of these equations is solved by the substitution $\widehat{C}_1 = \Phi(\widehat{C}_0)$ and we are left with just the single equation for \widehat{C}_0

$$P_{-\mathcal{N}}(\widehat{C}_0 + \Phi(\widehat{C}_0)) = 0.$$

This defines a map

$$\kappa_r: B_0(r^{-4}) \rightarrow H_r^-, \quad \widehat{C}_0 \mapsto P_{-\mathcal{N}}(\widehat{C}_0 + \Phi(\widehat{C}_0)),$$

which we call the *Kuranishi map*. We call H_r^- the *obstruction space*.

We have just proven the following: the set of all monopoles on $\widehat{X}(r)$ located in the slice at \widehat{C}_r at an $L^{2,2}$ -distance of $\leq r^{-4}$ from \widehat{C}_r is given by the graph of the function Φ over $\kappa_r^{-1}(0)$.

If $\kappa_r = 0$ (for instance, if the obstruction space is zero), then the moduli space of monopoles on $\widehat{X}(r)$ is locally given as the graph of a function Φ defined in a neighbourhood of $0 \in H_r^+$. In this case we can identify H_r^+ with the tangent space of the moduli space at the point $\widehat{C}_r + \Phi(0)$.

In the unobstructed case $H_r^- = 0$, we can consider the map Φ as \widehat{C}_r varies within the space of configurations obtained by gluing. The map $\widehat{C}_r \mapsto \widehat{C}_r + \Phi(0)$ sends a naively glued configuration by cutoff functions to a genuine solution of the Seiberg–Witten equations and for all sufficiently large r this mapping will actually yield a diffeomorphism between glued configurations and the Seiberg–Witten moduli space on $\widehat{X}(r)$. The fact that this map is a local diffeomorphism is not so hard to show from the Kuranishi model. The fact that it is one-to-one and onto requires further analysis, carried out in Nicolaescu, Section 4.5.4 (analogous results in Frøyshov’s monograph can be found in [7, Sections 10.2 and 10.3]).

Using the grid of 3×3 asymptotic exact sequences (Nicolaescu, pages 440–441) one finds the following:

Proposition 6.15 Suppose that $\widehat{C}_{\widehat{M}}$ is irreducible and $\widehat{C}_{\widehat{N}}$ is reducible. Then, for any sufficiently large r , there are isomorphisms

$$H_r^+ \cong H_{\widehat{C}_{\widehat{M}}}^1 \oplus H_{\widehat{C}_{\widehat{N}}}^1, \quad H_r^- \cong H_{\widehat{C}_{\widehat{M}}}^2 \oplus H_{\widehat{C}_{\widehat{N}}}^2.$$

Proof Let us use the notation appearing in pages 440–441 of Nicolaescu’s book. From our assumption on irreducibility and reducibility, one can check that

$$(6-1) \quad \mathfrak{C}_1^+ \cong \mathbb{R}, \quad \mathfrak{C}_1^- = 0, \quad \mathfrak{C}_2^+ = 0, \quad \mathfrak{C}_2^- \cong \mathbb{R},$$

and

$$T\widehat{G}_1 = 0, \quad T\widehat{G}_2 \cong \mathbb{R}, \quad T\widehat{G}_\infty \cong \mathbb{R},$$

and thus we also have

$$(6-2) \quad L_1^+ = L_1^- = L_2^+ = L_2^- = 0.$$

Using the calculations (6-1)–(6-2), the two 3×3 asymptotic exact sequences in Nicolaescu, page 440–441, and Proposition 4.1.20 in Nicolaescu, we can deduce the existence of isomorphisms stated in the proposition. \square

Example 6.16 Suppose $N = S^4$ is the 4–sphere. Then $X = M \# N \cong M$. Suppose also that $b^+(M) > 0$. Now take a generic perturbation $\eta_{\widehat{M}}$ which avoids reducibles and for which the moduli space $\mathcal{M}(\widehat{M}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_{\widehat{M}}, \mu)$ is smooth and take $\eta_{\widehat{N}} = 0$. Note that \widehat{N} can be given a cylindrical end metric which has everywhere-positive scalar curvature. Since \widehat{N} is simply connected, the Seiberg–Witten equations on \widehat{N} admit a unique solution $\widehat{C}_{\widehat{N}}$ which is reducible and $H_{\widehat{C}_{\widehat{N}}}^i = 0$ for $i = 1, 2$. Now let

$$\widehat{C}_{\widehat{M}} \in \mathcal{M}(\widehat{M}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_{\widehat{M}}, \mu),$$

so we can assume $\widehat{C}_{\widehat{M}}$ is irreducible and $H_{\widehat{C}_{\widehat{M}}}^2 = 0$. Then, by Proposition 6.15, we get

$$H_r^+ \cong H_{\widehat{C}_{\widehat{M}}}^1, \quad H_r^- \cong 0.$$

Using this, one can show that the gluing map gives a diffeomorphism (for all sufficiently large r)

$$\mathcal{M}(\widehat{M}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_{\widehat{M}}, \mu) \rightarrow \mathcal{M}(\widehat{X}(r), g_{\widehat{X}(r)}, \eta_{\widehat{X}(r)}, \mathfrak{s}_{\widehat{X}(r)}).$$

But now we observe that $\widehat{X}(r) = M$. Therefore, if $b^+(M) > 0$, we deduce that the Seiberg–Witten moduli spaces on M and \widehat{M} are diffeomorphic, for suitably chosen metrics and perturbations (a similar argument can be found in Frøyshov [7, Chapter 14]).

Unfortunately, in the families setting, we cannot expect the spaces H_r^- (defined fibrewise) to vanish. Because of this, adapting the gluing construction to the families setting requires some additional work, as we will explain in the following sections.

7 Gluing in families

Now we turn to gluing in the families setting. Suppose that we have our families $E_M \rightarrow B$ and $E_N \rightarrow B$ satisfying all the assumptions stated in Section 4. Now we carry out the gluing construction in families. First of all, we note that it is possible to define families moduli spaces for the cylindrical end families $E_{\widehat{M}}$ and $E_{\widehat{N}}$. Let us denote these moduli spaces as $\mathcal{M}(E_{\widehat{M}}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_M, \mu)$ and $\mathcal{M}(E_{\widehat{N}}, g_{\widehat{N}}, \eta_{\widehat{N}}, \mathfrak{s}_N, \mu)$. Some comments are in order:

- As usual we make no distinction between Spin^c -structures on M and \widehat{M} and simply write either one as \mathfrak{s}_M . Similarly for \mathfrak{s}_N .
- The weight μ is chosen to be positive, but sufficiently small that Lockhart–McOwen theory, summarised in Section 4.1.4 of Nicolaescu’s book, can be applied to the deformation complexes of all configurations in the families moduli spaces. More precisely, we take μ so that Lockhart–McOwen theory works for all operators appearing in the deformation complexes of regularity $L_\mu^{k,2} \rightarrow L_\mu^{k-1,2}$ for $k = 1, 2, 3$. Since the base B is compact, it is not hard to see that such a μ actually exists.
- $g_{\widehat{M}}$ and $g_{\widehat{N}}$ are families of cylindrical end metrics, which may be constructed as in Section 4.
- $\eta_{\widehat{M}}$ and $\eta_{\widehat{N}}$ are families of 2-form perturbations which vanish on the necks. These glue together to form for a family $\eta_{\widehat{X}(r)} = \eta_{\widehat{M}} \#_r \eta_{\widehat{N}}$ of 2-form perturbations for the family $E_{\widehat{X}(r)}$ for each $r > 0$.
- To keep notation simple, we will describe families gluing only in the case that ρ is trivial. The more general case is a straightforward extension.

Recall the isomorphism

$$H_{L^2}^2(\widehat{N}) \cong H^2(N) \cong H^2(\widehat{N}, [0, \infty) \times S^3).$$

The first Chern class $c_1(\mathfrak{s}_N) \in H^2(N)$ of the determinant line on N can then be identified with a class in $H_{L^2}^2(\widehat{N})$. Analytically, one can find a connection A on the determinant line of class $L_\mu^{2,2}$ such that F_A is a harmonic 2-form. This follows by noting that the curvature of any connection is a closed 2-form and considering the

Hodge decomposition of the curvature of a given connection. Then $\frac{i}{2\pi}F_A$ represents the Chern class in L^2 -cohomology. Let $\mathcal{W}_{\widehat{N}} \in H_{L^2}^+(\widehat{N})$ be 2π times the projection of this class to its self-dual part; thus, as harmonic forms,

$$\mathcal{W}_{\widehat{N}} = iF_A^+.$$

Next, let us observe that the map

$$d^+ : L_{\mu}^{k,2}(\widehat{N}, T^*\widehat{N}) \rightarrow L_{\mu}^{k-1,2}(\widehat{N}, \wedge_+^2 T^*\widehat{N})$$

has closed image for $1 \leq k \leq 2$ by Lockhart–McOwen theory. (Although this holds for any $k \geq 1$, we need only the cases of lower regularities.) The cokernel of this map is therefore isomorphic to $H_{L^2}^+(\widehat{N})$ as well as the case of closed 4-manifolds. Namely, we have the L_{μ}^2 -orthogonal decomposition

$$(7-1) \quad L_{\mu}^{k-1,2}(\widehat{N}, \wedge_+^2 T^*\widehat{N}) = H_{L^2}^+(\widehat{N}) \oplus d^+(L_{\mu}^{k,2}(\widehat{N}, T^*\widehat{N})).$$

Let $\eta_{\widehat{N}}$ be a self-dual 2-form perturbation of class $L_{\mu}^{1,2}$ which vanishes on the neck. Then there exists a reducible solution to the $\eta_{\widehat{N}}$ -perturbed Seiberg–Witten equations on \widehat{N} of class $L_{\mu}^{2,2}$ if and only if

$$(7-2) \quad [\eta_{\widehat{N}}] = \mathcal{W}_{\widehat{N}},$$

where $[\eta_{\widehat{N}}]$ means the class of the projection of $\eta_{\widehat{N}} \in L_{\mu}^{1,2}(\widehat{N}, \wedge_+^2 T^*\widehat{N})$ into $H_{L^2}^+(\widehat{N})$ along the decomposition (7-1). This is because if the equality (7-2) holds for $\eta_{\widehat{N}}$, then $\eta_{\widehat{N}} = iF_A^+ + d^+a = iF_{A+a}^+$ for some imaginary valued 1-form a , and vice versa.

We call $\mathcal{W}_{\widehat{N}}$ the wall and we say $\eta_{\widehat{N}}$ lies on the wall if $[\eta_{\widehat{N}}] = \mathcal{W}_{\widehat{N}}$. Then from what we have just discussed, the moduli space of $\eta_{\widehat{N}}$ -perturbed Seiberg–Witten equations on \widehat{N} admits reducible solutions if and only if $\eta_{\widehat{N}}$ lies on the wall. Moreover, if $b_1(N) = 0$, then up to gauge equivalence there is a unique such reducible solution whenever $\eta_{\widehat{N}}$ lies on the wall.

Now consider the connected sum $\widehat{X}(r) \cong M \# N$. From the Mayer–Vietoris sequence, we get an isomorphism

$$H^2(\widehat{X}(r)) \cong H^2(\widehat{X}(r), \mathbb{R}) \cong H^2(M, \mathbb{R}) \oplus H^2(N, \mathbb{R}) \cong H_{L^2}^2(\widehat{M}) \oplus H_{L^2}^2(\widehat{N}).$$

This isomorphism is obtained by composing a series of well-defined isomorphisms depending only on r and the cylindrical end metrics $g_{\widehat{M}}$ and $g_{\widehat{N}}$ on \widehat{M} and \widehat{N} . Let

$$\varphi_r : H^2(\widehat{X}(r)) \rightarrow H_{L^2}^2(\widehat{M}) \oplus H_{L^2}^2(\widehat{N})$$

denote this isomorphism.

Inside $H_{L^2}^2(\widehat{M})$, $H_{L^2}^2(\widehat{N})$ and $H^2(\widehat{X}(r))$ we have the subspaces $H_{L^2}^+(\widehat{M})$, $H_{L^2}^+(\widehat{N})$ and $H^+(\widehat{X}(r))$ of harmonic self-dual 2-forms. Unfortunately, we cannot expect $H_{L^2}^+(\widehat{M}) \oplus H_{L^2}^+(\widehat{N})$ to coincide with $\varphi_r(H^+(\widehat{X}(r)))$. The problem is that we cannot directly apply the Mayer–Vietoris sequence to harmonic forms. However, the linear gluing theory, such as Cappell–Lee–Miller theory referred in Nicolaescu (see also Section 3 of Donaldson’s book [6]), implies that $\varphi_r(H^+(\widehat{X}(r)))$ and $H_{L^2}^+(\widehat{M}) \oplus H_{L^2}^+(\widehat{N})$ are *asymptotically isomorphic* in the terminology of Nicolaescu (see Nicolaescu, Section 4.1.5 for this type of linear gluing theory and page 304 for the definition of an asymptotic isomorphism). Let us set

$$\mathcal{V} = H_{L^2}^2(\widehat{M}) \oplus H_{L^2}^2(\widehat{N}), \quad \mathcal{V}_1 = H_{L^2}^+(\widehat{M}) \oplus H_{L^2}^+(\widehat{N}), \quad \mathcal{V}_2(r) = \varphi_r(H^+(\widehat{X}(r))).$$

Note that $\mathcal{V}_2(r)$ depends on r while \mathcal{V}_1 does not. Equip \mathcal{V} with the natural L^2 -inner product and let $\pi_1: \mathcal{V} \rightarrow \mathcal{V}_1$ and $\pi_2(r): \mathcal{V} \rightarrow \mathcal{V}_2(r)$ be the orthogonal projections to \mathcal{V}_1 and $\mathcal{V}_2(r)$. The fact that \mathcal{V}_1 and $\mathcal{V}_2(r)$ are asymptotically isomorphic implies that $\pi_2(r) \rightarrow \pi_1$ as $r \rightarrow \infty$ in the operator norm.

In the families setting the various cohomology groups that we are considering define vector bundles over B . In particular, we think of $H_{L^2}^2(\widehat{M})$, $H_{L^2}^2(\widehat{N})$ and $H^2(\widehat{X}(r))$ as being vector bundles over B and $H_{L^2}^+(\widehat{M})$, $H_{L^2}^+(\widehat{N})$ and $H^+(\widehat{X}(r))$ as being subbundles. Furthermore the walls $\mathcal{W}_{\widehat{M}}, \mathcal{W}_{\widehat{N}}, \mathcal{W}_{\widehat{X}(r)}$ define sections of $H_{L^2}^+(\widehat{M})$, $H_{L^2}^+(\widehat{N})$ and $H^+(\widehat{X}(r))$, respectively. Similarly, the cohomology classes of the perturbations $\eta_{\widehat{M}}, \eta_{\widehat{N}}$ and $\eta_{\widehat{X}(r)}$ also define sections of these vector bundles.

Here we note that if at least one of given perturbations on \widehat{M} and \widehat{N} does not lie on the wall, the glued perturbation also does not lie on the wall after stretching the neck sufficiently:

Lemma 7.1 *Let $\eta_{\widehat{M}}$ and $\eta_{\widehat{N}}$ be perturbations on \widehat{M} and \widehat{N} supported away from the necks and set $\eta_{\widehat{X}(r)} = \eta_{\widehat{M}} \# \eta_{\widehat{N}}$. Suppose that $([\eta_{\widehat{M}}], [\eta_{\widehat{N}}]) \neq (\mathcal{W}_{\widehat{M}}, \mathcal{W}_{\widehat{N}})$, that is, suppose that at least one of $\eta_{\widehat{M}}$ and $\eta_{\widehat{N}}$ does not lie on the wall. Then there exists r_0 such that $\eta_{\widehat{X}(r)}$ does not lie on the wall $\mathcal{W}_{\widehat{X}(r)}$ for all $r \geq r_0$. The result also holds in the families setting over a compact base B .*

Proof Let $\tilde{\varphi}: H^2(\widehat{X}(r), \mathbb{R}) \rightarrow H_{L^2}^2(\widehat{M}) \oplus H_{L^2}^2(\widehat{N})$ be the isomorphism defined as the composition of the maps appearing in the definition of φ , namely

$$\tilde{\varphi}: H^2(\widehat{X}(r), \mathbb{R}) \cong H^2(M, \mathbb{R}) \oplus H^2(N, \mathbb{R}) \cong H_{L^2}^2(\widehat{M}) \oplus H_{L^2}^2(\widehat{N}).$$

Since $\widehat{X}(r)$ is independent of r as a topological space, the map $\tilde{\varphi}$ is also independent of r . At the level of cohomology classes supported away from the necks, we have that φ_r is just the Mayer–Vietoris isomorphism. In particular, we have

$$\varphi_r[\eta_{\widehat{X}(r)}] = ([\eta_{\widehat{M}}], [\eta_{\widehat{N}}]) \quad \text{and} \quad \tilde{\varphi}(c_1(\mathfrak{s}_X)) = (c_1(\mathfrak{s}_M), c_1(\mathfrak{s}_N)).$$

Applying $(2\pi) \cdot \pi_1$ to the second of these, we get

$$(\mathcal{W}_{\widehat{M}}, \mathcal{W}_{\widehat{N}}) = (2\pi) \cdot \pi_1(\tilde{\varphi}(c_1(\mathfrak{s}_X))) = \varphi(\mathcal{W}_{\widehat{X}(r)}) + u_r,$$

where

$$u_r = (2\pi) \cdot (\pi_1 - \pi_2(r))(\tilde{\varphi}(c_1(\mathfrak{s}_X))).$$

Now suppose that $([\eta_{\widehat{M}}], [\eta_{\widehat{N}}]) \neq (\mathcal{W}_{\widehat{M}}, \mathcal{W}_{\widehat{N}})$. Let δ be the distance between these two points. Or, in the families setting, let δ be the infimum of

$$\|([\eta_{\widehat{M}}], [\eta_{\widehat{N}}]) - (\mathcal{W}_{\widehat{M}}, \mathcal{W}_{\widehat{N}})\|$$

over $b \in B$. Since B is compact, we have $\delta > 0$. Then

$$\begin{aligned} \|\varphi_r([\eta_{\widehat{X}(r)}]) - \varphi_r(\mathcal{W}_{\widehat{X}(r)})\| &= \|([\eta_{\widehat{M}}], [\eta_{\widehat{N}}]) - (\mathcal{W}_{\widehat{M}}, \mathcal{W}_{\widehat{N}}) + u_r\| \\ &\geq \|([\eta_{\widehat{M}}], [\eta_{\widehat{N}}]) - (\mathcal{W}_{\widehat{M}}, \mathcal{W}_{\widehat{N}})\| - \|u_r\| \\ &\geq \delta - \|u_r\|. \end{aligned}$$

The result follows by observing that $u_r \rightarrow 0$ (uniformly with respect to B), since $\pi_2(r) \rightarrow \pi_1$ (uniformly with respect to B). So there exists an r_0 such that $\|u_r\| < \frac{1}{2}\delta$ for all $r \geq r_0$ and all $b \in B$. □

Now we are almost ready to consider families gluing. To make the analysis much simpler we will show that the metrics and perturbations can be chosen sufficiently nicely. The setting of functional spaces will be given just after the statement of [Proposition 7.2](#). Recall, by [Section 5](#), that it was sufficient to assume $\dim B = d = b^+(N)$.

Proposition 7.2 *Let $\delta > 0$ be given. It is possible to choose the families metrics $\widehat{g}_{\widehat{M}}$ and $\widehat{g}_{\widehat{N}}$, families perturbations $\eta_{\widehat{M}}$ and $\eta_{\widehat{N}}$ and positive number $R_{\text{vanish}} > 0$ satisfying the following conditions:*

- (A1) $[\eta_{\widehat{N}}]$ meets the wall $\mathcal{W}_{\widehat{N}}$ transversely in a finite set of points $R = \{b_1, \dots, b_l\} \subseteq B$.
- (A2) For each $b_i \in R$, we have that $\eta_{\widehat{M}}(b_i)$ does not lie on the wall $\mathcal{W}_{\widehat{M}}$ and is moreover a generic perturbation in the unparametrised sense (so that the fibre over b_i of the families moduli space for the family $E_{\widehat{M}}$ is smooth).

- (A3) There are disjoint open subsets U_1, \dots, U_l of B satisfying that $b_i \in U_i$, each U_i is diffeomorphic to the standard disk D^d , and the families $E_{\widehat{M}}$ and $E_{\widehat{N}}$ admit trivialisations over each U_i .
- (A4) With respect to the given trivialisations $E_{\widehat{M}}|_{U_i} \cong \widehat{M} \times U_i$ and $E_{\widehat{N}}|_{U_i} \cong \widehat{N} \times U_i$, we have that $g_{\widehat{M}}$, $g_{\widehat{N}}$ and $\eta_{\widehat{M}}$ are constant on each U_i .
- (A5) Let us set

$$\eta_i = \eta_{\widehat{N}}(b_i) \in L_{\mu}^{1,2}(\widehat{N}, \wedge_+^2 T^* \widehat{N})$$

and define $L_i(t) \in L_{\mu}^{1,2}(\widehat{N}, \wedge_+^2 T^* \widehat{N})$ for $t \in U_i$ by

$$\eta_{\widehat{N}}(t) = \eta_i + L_i(t).$$

Then $L_i(t)$ satisfies the following conditions with respect to the given trivialisation $E_{\widehat{N}}|_{U_i} \cong \widehat{N} \times U_i$ and an identification of U_i with an open neighbourhood of the origin in $T_{b_i} B$ (so we view U_i as a subset of $T_{b_i} B$) and a fixed metric on $T_{b_i} B$:

- (a) The linear map $T_{b_i} B \rightarrow H_{L^2}^+(\widehat{N})$ defined as the projection of the differential $L_i'(0)$ to $H_{L^2}^+(\widehat{N})$ is an isomorphism.
- (b) There exists some $\delta' \in (0, \delta)$ such that

$$\|(d^+)^*(\eta_{\widehat{N}}(t) - \eta_{\widehat{N}}(t'))\|_{L_{\mu}^2(\widehat{N})} \leq \delta' \|t - t'\|$$

holds for all $t, t' \in U_i$. (In particular, by setting $t' = b_i$, we have

$$(7-3) \quad \|(d^+)^* L_i(t)\|_{L_{\mu}^2(\widehat{N})} \leq \delta' \|t\|$$

for any $t \in U_i$.)

- (c) There exists some $\epsilon' \in (0, \epsilon)$ such that

$$\|\pi^{\perp} L_i(t)\|_{L_{\mu}^2(\widehat{N})} \leq \epsilon' \|\pi L_i(t)\|_{L_{\mu}^2(\widehat{N})}$$

holds for any $t \in U_i$. Here ϵ is a constant introduced in [Lemma 7.16](#), and

$\pi: L_{\mu}^2(\widehat{N}, \wedge_+^2 T^* \widehat{N}) \rightarrow H_{L^2}^+(\widehat{N})$, $\pi^{\perp}: L_{\mu}^2(\widehat{N}, \wedge_+^2 T^* \widehat{N}) \rightarrow H_{L^2}^+(\widehat{N})^{\perp}$ are the L_{μ}^2 -orthogonal projections, where $H_{L^2}^+(\widehat{N})^{\perp}$ is the L_{μ}^2 -orthogonal complement of $H_{L^2}^+(\widehat{N})$ in $L_{\mu}^2(\widehat{N}, \wedge_+^2 T^* \widehat{N})$.

- (d) There exists $\epsilon' \in (0, 1)$ such that

$$\|P_-^r \circ \Psi_r \circ \pi^{\perp} L_i(t)\|_{L^2(\widehat{X}(r))} \leq \epsilon' \|P_-^r \circ \Psi_r \circ \pi L_i(t)\|_{L^2(\widehat{X}(r))}$$

holds for any $t \in U_i$ and any $r > 1$. Here P_-^r is as defined in [Definition 6.11](#) and Ψ_r is the map defined in [\(7-10\)](#).

(e) We have

$$\|\eta_{\widehat{N}}(t) - \eta_{\widehat{N}}(t')\|_{L^2_{\mu}(\widehat{N})} \leq \delta' \|t - t'\|$$

for all $t, t' \in U_i$. In particular, setting $t' = b_i$ gives

$$\|L_i(t)\|_{L^2_{\mu}(\widehat{N})} \leq \delta' \|t\|$$

for any $t \in U_i$.

(f) There exists $\delta'' > \frac{1}{2}\delta$ such that

$$\|\pi \circ \eta_{\widehat{N}}(t) - \pi \circ \eta_{\widehat{N}}(t')\|_{L^2_{\mu}(\widehat{N})} \geq \delta'' \|t - t'\|$$

holds for all $t, t' \in U_i$. In particular, setting $t' = b_i$ gives

$$\|\pi \circ L_i(t)\|_{L^2_{\mu}(\widehat{N})} \geq \delta'' \|t\|$$

for any $t \in U_i$.

(A6) For each $b_i \in R$, let $(A, 0)$ be the unique (under the gauge fixing) reducible solution of the Seiberg–Witten equations perturbed by $\eta_{\widehat{N}}(b_i) = \eta_i$. Then $\text{Coker}(D_A) = 0$, where $D_A: L^2_{\mu^{2,2}}(\widehat{N}, S^+) \rightarrow L^2_{\mu^{1,2}}(\widehat{N}, S^-)$ is the associated Dirac operator.

(A7) The perturbation $\eta_{\widehat{N}}$ is generic in the families sense: for every irreducible solution of the perturbed Seiberg–Witten moduli space we have that its families deformation theory is unobstructed (ie H^2 of the families deformation complex vanishes).

(A8) For any $b \in B$, $\eta_{\widehat{N}}(b)$ is supported outside the subspace $[R_{\text{vanish}}, \infty) \times S^3$ of the neck of \widehat{N} .

The main part of the proof of Proposition 7.2 is to ensure that we can take $\eta_{\widehat{N}}$ satisfying the desired conditions. To do this, we need to show some lemmas. To state the lemmas, let us fix some notation and exhibit the functional space where our desired perturbation $\eta_{\widehat{N}}$ lives. The open sets U_1, \dots, U_l and families metrics $g_{\widehat{M}}$ and $g_{\widehat{N}}$ appearing in the statement of Proposition 7.2 are given here. Fix a generic section of $H^1_{L^2}(\widehat{N}) \rightarrow B$. Then the zero set of this section consists of finitely many points x_1, \dots, x_l for some $l \geq 0$. For each of these points x_i for $1 \leq i \leq l$, choose a small open neighbourhood U_i in B such that U_i is diffeomorphic to the standard disk D^d ; E_M and E_N admit local trivialisations over U_i ; and $U_i \cap U_j = \emptyset$ if $i \neq j$. Let $U \subset B$ be the union of these disk-like neighbourhoods U_i . We can assume the sections ι_M and ι_N are constant on U with respect to these trivialisations. Then we obtain also trivialisations of $E_{\widehat{M}}$ and $E_{\widehat{N}}$. We also assume the families of metrics g_M and g_N are chosen so

as to be constant over U . Then similarly we can carry out the construction of $E_{\widehat{M}}$ and $E_{\widehat{N}}$ in such a way that $g_{\widehat{M}}$ and $g_{\widehat{N}}$ are also constant on U . Let $\mathcal{E} \rightarrow B$ be the Hilbert bundle whose fibre on $b \in B$ is given by

$$L_{\mu}^{1,2}(\widehat{N}, i \wedge_{+g_{\widehat{N}}(b)}^2 T^* \widehat{N}).$$

Let $\mathcal{C}^1(B, \mathcal{E})$ denotes the space of \mathcal{C}^1 -class sections of \mathcal{E} on B , where our desired perturbation $\eta_{\widehat{N}}$ lives. Note that \mathcal{E} has a natural trivialisation on U because of the choice of $g_{\widehat{N}}$ above, and hence the restricted sections belonging to $\mathcal{C}^1(B, \mathcal{E})$ on each U_i modelled by the space of functions

$$\mathcal{C}^1(U_i, L_{\mu}^{1,2}(\widehat{N}, i \wedge_{+g_{\widehat{N}}(x_i)}^2 T^* \widehat{N})).$$

Lemma 7.3 *Let $\mathcal{O} \subset \mathcal{C}^1(B, \mathcal{E})$ be the set of $\eta_{\widehat{N}} \in \mathcal{C}^1(B, \mathcal{E})$ satisfying the following conditions (i)–(iv):*

- (i) $\eta_{\widehat{N}}$ meets the wall transversally.
- (ii) $\eta_{\widehat{N}}$ meets the wall once over U_i for each i and does not meet the wall over $B \setminus U$.
- (iii) For any point $b_i \in B$ where $\eta_{\widehat{N}}$ lies on the wall, condition (A5) holds.
- (iv) For any point $b_i \in B$ where $\eta_{\widehat{N}}$ lies on the wall, condition (A6) holds.

Then \mathcal{O} is an open set in $\mathcal{C}^1(B, \mathcal{E})$.

Proof We shall see that each of conditions (i)–(iv) on $\eta_{\widehat{N}} \in \mathcal{C}^1(B, \mathcal{E})$ is an open condition with respect to \mathcal{C}^1 -norm. Condition (i) is evidently an open condition. Since U is an open set in B , condition (ii) is also an open condition. The fact that condition (iv) is also an open condition follows from the fact that the map $\tau \mapsto \dim \text{Coker}(D_{\tau})$ is upper-semicontinuous for a continuous family of Fredholm operators $\{D_{\tau}\}_{\tau}$. So let us check the remaining part: (iii) is an open condition. First note that (A5)(a), (c), (d), (e) and (f) are obviously open conditions, so we have to care only about (A5)(b). Suppose that $\eta_{\widehat{N}}$ satisfies (A5)(b) and $\xi \in \mathcal{C}^1(B, \mathcal{E})$ satisfies

$$(7-4) \quad \|\xi\|_{\mathcal{C}^1} \leq \frac{\delta - \delta'}{2(1 + \|(d^+)^*\|_{\text{op}})},$$

where $\|\cdot\|_{\text{op}}$ denotes the operator norm. Then, for any $t, t' \in U_i$, we have

$$(7-5) \quad \|(d^+)^*((\eta_{\widehat{N}} + \xi)(t) - (\eta_{\widehat{N}} + \xi)(t'))\|_{L_{\mu}^2} \leq \delta' \|t - t'\| + \|(d^+)^*\|_{\text{op}} \|\xi(t) - \xi(t')\|_{L_{\mu}^{1,2}}.$$

Under the identification of U_i with D^d , we can find the line connecting t with t' , and therefore the mean value theorem implies that there exists $\bar{t} \in U_i$ satisfying that

$$(7-6) \quad \|\xi(t) - \xi(t')\|_{L_\mu^{1,2}} \leq \|\nabla \xi(\bar{t})\|_{L_\mu^{1,2}} \|t - t'\| \leq \|\xi\|_{C^1} \|t - t'\|.$$

The inequalities (7-4), (7-5) and (7-6) imply that

$$\|(d^+)^*((\eta_{\widehat{N}} + \xi)(t) - (\eta_{\widehat{N}} + \xi)(t'))\|_{L_\mu^2} \leq \frac{1}{2}(\delta + \delta') \|t - t'\|.$$

This proves that $\eta_{\widehat{N}} + \xi$ also meets (A5)(b), and hence (A5)(b) is an open condition. Therefore we can conclude that (A5) is also an open condition. \square

Lemma 7.4 *For each $i \in \{1, \dots, l\}$, there exists*

$$\eta_i \in L_\mu^{1,2}(\widehat{N}, i \wedge^2_{+g_{\widehat{N}}(x_i)} T^*\widehat{N})$$

such that the (unique) reducible solution $(A, 0)$ to the η_i -perturbed Seiberg–Witten equations satisfies that $\text{Coker}(D_A) = 0$.

Proof For a closed 4-manifold, the analogous statements can be found in, for example, Lemma 2.5 in [2]. For our case, namely, a cylindrical 4-manifold, this follows from Proposition 9.2 in Strle [23]. (Although Strle stated the proposition for a 4-manifold with $b^+ = 1$ and with some special cylinder, the proof of the proposition works without any change in more general setting, including our case. We also note that since we assumed that the index of the Dirac operator on N is zero, $\text{Coker}(D_A) = 0$ is equivalent to $\text{Ker}(D_A) = 0$.) \square

Lemma 7.5 *The data $\widehat{g}_{\widehat{N}}$ and U_1, \dots, U_l can be chosen in such a way that the set \mathcal{O} contains an element $\eta_{\widehat{N}}$ satisfying that there exists $R_{\text{vanish}} > 0$ such that $\eta_{\widehat{N}}(b)$ is supported outside the subspace $[R_{\text{vanish}}, \infty) \times S^3$ of the neck of \widehat{N} for any $b \in B$.*

Proof First pick some initial family of asymptotically cylindrical metrics $g' = \{g'_b\}$ on $E_{\widehat{N}}$, so that the bundle $H_{g', L^2}^+(\widehat{N})$ is defined. Choose some section η'_0 of $H_{g', L^2}^+(\widehat{N})$ which is transverse to the wall $w_{g'}$. Since $\dim B = b^+(N)$ and B is compact, it follows that $\eta'_0 = w_{g'}$ at finitely many points. Let $b_1, \dots, b_l \in B$ be the points where $\eta'_0 = w_{g'}$.

Choose disjoint open neighbourhoods U'_1, \dots, U'_l of b_1, \dots, b_l such that for each i , $E_{\widehat{N}}|_{U'_i}$ and $H_{g', L^2}^+(\widehat{N})|_{U'_i}$ are trivialisable, and fix trivialisations

$$E_{\widehat{N}}|_{U'_i} \cong \widehat{N} \times U'_i, \quad H_{g', L^2}^+(\widehat{N})|_{U'_i} \cong \mathbb{R}^{b^+(N)} \times U'_i.$$

We can also assume that for each i , there are local coordinates centred at i such that U'_i corresponds to the open disc of radius 2, $U'_i \cong \{t \in \mathbb{R}^{b^+(N)} \mid \|t\| < 2\}$. Let U_1, \dots, U_l be the open sets around b_1, \dots, b_l such that $U_i \subset U'_i$ corresponds to the open disc of radius 1, $U_i \cong \{t \in \mathbb{R}^{b^+(N)} \mid \|t\| < 1\}$. Now choose a new families metric $g_{\widehat{N}} = \{g_{\widehat{N},b}\}$ on $E_{\widehat{N}}$ such that

- for each i , the restriction $g_{\widehat{N}}|_{U_i}$ is constant (with respect to the trivialisation of $E_{\widehat{N}}|_{U_i}$),
- $g_{\widehat{N},b} = g'_b$ for any b outside of U'_1, \dots, U'_l .

Clearly such a family $\{g_{\widehat{N},b}\}$ can be constructed using cutoff functions. Let

$$\varphi: H_{g_{\widehat{N}},L^2}^+(\widehat{N}) \rightarrow H_{g',L^2}^+(\widehat{N})$$

be the isomorphism given by inclusion of $H_{g_{\widehat{N}},L^2}^+(\widehat{N})$ into $H^2(N, \mathbb{R})$ followed by L^2 -orthogonal projection to $H_{g',L^2}^+(\widehat{N})$ using the metric g' and let $\psi = \varphi^{-1}$. Let η_0 be the section of $H_{g_{\widehat{N}},L^2}^+(\widehat{N})$ defined by $\eta_0 = \psi(\eta'_0) + w_{g_{\widehat{N}}} - \psi(w_{g'})$. Then

- $\eta_0 = w_{g_{\widehat{N}}}$ at precisely the points b_1, \dots, b_l ,
- η_0 meets $w_{g_{\widehat{N}}}$ transversally.

To see this, note that $\eta_0 = w_{g_{\widehat{N}}}$ if and only if $\psi(\eta'_0) = \psi(w_{g'})$, hence if and only if $\eta'_0 = w_{g'}$. So $\eta_0 = w_{g_{\widehat{N}}}$ precisely at the points b_1, \dots, b_l . Moreover, η'_0 is transverse to $w_{g'}$, hence $\eta'_0 - w_{g'}$ is transverse to the zero section. Applying the linear isomorphism ψ , we see that $\psi(\eta'_0) - \psi(w_{g'})$ is transverse to the zero section. Hence $\eta_0 = \psi(\eta'_0) - \psi(w_{g'}) + w_{g_{\widehat{N}}}$ is transverse to $w_{g_{\widehat{N}}}$.

By restriction to the U_i , we obtain trivialisations

$$E_{\widehat{N}}|_{U_i} \cong \widehat{N} \times U_i, \quad H_{g_{\widehat{N}},L^2}^+(\widehat{N})|_{U_i} \cong \mathbb{R}^{b^+(N)} \times U_i.$$

Moreover, $g_{\widehat{N}}$ is constant on U_i with respect to these trivialisations. Next, choose a lift of η_0 to a section $\tilde{\eta}_0$ such that $\tilde{\eta}_0$ is valued in $L_{\mu}^{1,2}(\widehat{N}, \wedge_+^2 T^* \widehat{N})$ and is supported away from the neck. Let $\pi: L_{\mu}^2(\widehat{N}, \wedge_+^2 T^* \widehat{N}) \rightarrow H_{g_{\widehat{N}},L^2}^+(\widehat{N})$ be the L^2 -orthogonal projection. So $\eta_0 = \pi \tilde{\eta}_0$. With respect to the local trivialisation over U_i , we have that η_0 corresponds to a smooth map

$$\eta_{0,i}: D_1(\mathbb{R}^{b^+(N)}) \rightarrow \mathbb{R}^{b^+(N)},$$

where $D_1(\mathbb{R}^{b^+(N)}) = \{t \in \mathbb{R}^{b^+(N)} \mid \|t\| < 1\}$. In fact $\eta_{0,i}$ extends smoothly over the boundary of $D_1(\mathbb{R}^{b^+(N)})$ (because both η_0 and the local trivialisations

$H_{g_{\widehat{N}}, L^2}^+(\widehat{N})|_{U_i} \cong \mathbb{R}^{b^+(N)} \times U_i$ extend smoothly from U_i to U'_i). Moreover, $\eta_{0,i}$ is transverse to $w_{g_{\widehat{N}}}$ and equals $w_{g_{\widehat{N}}}$ only at $t = 0$. It follows that the restriction of $\eta_{0,i}$ to the unit sphere defines a map

$$\eta_{0,i}|_{S^{b^+(N)-1}}: S^{b^+(N)-1} \rightarrow \mathbb{R}^{b^+(N)} \setminus \{w_{g_{\widehat{N}}}\} \rightarrow S^{b^+(N)-1},$$

where the rightmost arrow is the deformation retraction of $\mathbb{R}^{b^+(N)} \setminus \{w_{g_{\widehat{N}}}\}$ to the unit sphere. This map has degree ± 1 , since $\eta_{0,i}$ meets w_g transversally at a single point. By changing the local trivialisation of $H_{g_{\widehat{N}}, L^2}^+(\widehat{N})|_{U'_i}$ if necessary, we can assume that the degree is $+1$. Now we will construct the desired perturbation $\eta_{\widehat{N}}$.

For each $i \in \{1, \dots, l\}$, identify U_i with a neighbourhood of the origin in $T_{x_i} B$. By Lemma 7.4, we can take

$$\eta_i \in L_{\mu}^{1,2}(\widehat{N}, i \wedge^2_{+g_{\widehat{N}}(x_i)} T^* \widehat{N})$$

such that the (unique) reducible solution $(A, 0)$ to the η_i -perturbed Seiberg–Witten equations satisfies $\text{Coker}(D_A) = 0$. Since the condition $\text{Coker}(D_A) = 0$ is an open condition with respect to η_i , by taking an approximating sequence consisting of smooth and compactly supported sections, we can assume that η_i is supported outside the subspace $[R_i, \infty) \times S^3$ of the neck for some $R_i > 0$. Set $R_{\text{vanish}} := \max_i R_i$. Let us take a linear isomorphism $L_i: T_{x_i} B \rightarrow H_{L^2}^+(\widehat{N})$ satisfying $\frac{2}{3}\delta\|t\| \leq \|L_i(t)\| \leq \frac{5}{6}\delta\|t\|$. Here δ is the positive number fixed in the statement of Proposition 7.2 and we think of $H_{L^2}^+(\widehat{N})$ as a subspace of $L_{\mu}^{1,2}(\widehat{N}, i \wedge^2_{+g_{\widehat{N}}(x_i)} T^* \widehat{N})$ via (7-1). Let us define

$$\eta_{\widehat{N},i} \in \mathcal{C}^1(U_i, L_{\mu}^{1,2}(\widehat{N}, i \wedge^2_{+g_{\widehat{N}}(x_i)} T^* \widehat{N}))$$

by $\eta_{\widehat{N},i}(t) := \eta_i + L_i(t)$ for $t \in U_i$. In addition, we assume L_i is chosen with $\det(L_i) > 0$. Hence the local degree of $\eta_{\widehat{N},i}: U_i \setminus \{b_i\} \rightarrow \mathbb{R}^{b^+(N)} \setminus \{w_g\}$ is $+1$. We now extend the definition of $\eta_{\widehat{N},i}$ from U_i to U'_i such that

- $\eta_{\widehat{N},i} = \eta_i + L_i(t)$ for $0 \leq \|t\| < \frac{4}{3}$,
- $\eta_{\widehat{N},i} = \tilde{\eta}_0$ for $\frac{5}{3} < \|t\| < 2$,
- $\eta_{\widehat{N},i}$ does not lie in the wall and vanishes outside $[R_{\text{vanish}}, \infty) \times S^3$ (after possibly increasing R_{vanish}).

To see that we can do this, note that both $\tilde{\eta}_0$ and $\eta_i + L_i(t)$ do not lie on the wall and vanish outside some $[R_{\text{vanish}}, \infty) \times S^3$. We can then find a smooth homotopy between them within the space of perturbations vanishing on $[R_{\text{vanish}}, \infty) \times S^3$ and not lying on

the wall because both sections (projected to $H_{g, L^2}^+(\widehat{N}) \setminus \{w_{g_{\widehat{N}}}\}$) have local degree $+1$. Use such a homotopy to extend over the annulus $1 \leq \|t\| \leq 2$.

Lastly, define $\eta_{\widehat{N}} = \widetilde{\eta}_0$ on the complement of U'_1, \dots, U'_l . So $\eta_{\widehat{N}}$ does not lie on the wall for any point outside of U_1, \dots, U_l . By construction we have that for any $b \in B$, $\eta_{\widehat{N}}(b)$ is supported outside $[R_{\text{vanish}}, \infty) \times S^3$. We shall check that $\eta_{\widehat{N}}$ belongs to \mathcal{O} . The reason why $\eta_{\widehat{N}}$ satisfies condition (i) is that L_i is an linear isomorphism. It is obvious that $\eta_{\widehat{N}}$ satisfies condition (ii). Since for $t, t' \in U_i$ we have

$$\|(d^+)^*(\eta_{\widehat{N}}(t) - \eta_{\widehat{N}}(t'))\|_{L^2_\mu} = \|(d^+)^*L(t) - (d^+)^*L(t')\|_{L^2_\mu} = 0,$$

we see that $\eta_{\widehat{N}}$ satisfies condition (A5)(b). Conditions (A5)(c)–(d) are obviously satisfied since $\pi^\perp L_i = 0$. Conditions (A5)(e)–(f) are satisfied by the assumption $\frac{2}{3}\delta\|t\| \leq \|L_i(t)\| \leq \frac{5}{6}\delta\|t\|$ and the fact that $\pi^\perp L_i = 0$. Because of the choice of η_i , it is obvious that $\eta_{\widehat{N}}$ satisfies condition (iv). □

Proof of Proposition 7.2 Let \mathcal{E}' be the Banach subbundle of $\mathcal{E} \rightarrow B$ whose fibre on $b \in B$ is given by

$$\{\eta \in L^{1,2}_\mu(\widehat{N}, i \wedge^2_{+g_{\widehat{N}}(b)} T^*\widehat{N}) \mid \eta|_{[R_{\text{vanish}}, \infty) \times S^3} = 0\}.$$

Set $\mathcal{O}' := \mathcal{O} \cap \mathcal{C}^1(B, \mathcal{E}')$. Because of Lemmas 7.3 and 7.5, \mathcal{O}' is a nonempty open subset of $\mathcal{C}^1(B, \mathcal{E}')$. Let $\mathcal{D} \subset \mathcal{C}^1(B, \mathcal{E})$ be the set of generic perturbations in the families sense, and define $\mathcal{D}' := \mathcal{D} \cap \mathcal{C}^1(B, \mathcal{E}')$. Note that \mathcal{D}' is a dense subset of $\mathcal{C}^1(B, \mathcal{E}')$. This follows from a similar argument which ensures that, in the unparametrised setting, one can take a generic perturbation supported on a given open set, found in, for example, Remark 8.18 in [22]. So the intersection $\mathcal{O}' \cap \mathcal{D}'$ is nonempty. Take $\eta_{\widehat{N}}$ from this intersection.

On each U_i choose $\eta_{\widehat{M}}$ to be a constant perturbation which is generic in the unparametrised setting and belongs outside the wall of \widehat{M} . Then choose some extension of $\eta_{\widehat{M}}$ to all of B (eg by using cutoff functions and extending by zero).

Then $\eta_{\widehat{N}}$ and $\eta_{\widehat{M}}$ satisfy the all conditions (A1)–(A8). □

Assume that the families metrics $g_{\widehat{M}}$ and $g_{\widehat{N}}$, families perturbations $\eta_{\widehat{M}}$ and $\eta_{\widehat{N}}$, points $b_1, \dots, b_l \in B$ and open sets $U_1, \dots, U_l \subset B$ have been chosen by Proposition 7.2.

Remark 7.6 By the implicit function theorem, if we replace U_i by possibly smaller open neighbourhoods, we can take coordinates of U_i so that the projection of $L_i(t)$ to

$H_{L^2}^+(\widehat{N})$ is linear in t with respect to the coordinate via the fixed identification between U_i and the origin of $T_{b_i}B$ given in (A5). We shall employ this coordinate system of U_i . Henceforth we fix all of these data $g_{\widehat{M}}, g_{\widehat{N}}, \eta_{\widehat{M}}, \eta_{\widehat{N}}, b_1, \dots, b_l$ and U_1, \dots, U_l , and shall consider only sufficiently large r in the sense that $r \geq R_{\text{vanish}} + 1$.

In the unparametrised setting, the results of Nicolaescu, Section 4.5.3 imply there exists an r_0 such that if $r > r_0$ then every genuine solution of the Seiberg–Witten equations is close to a glued solution. In particular, if the moduli space of Seiberg–Witten equations on \widehat{N} for a given $(g_{\widehat{N}}, \eta_{\widehat{N}})$ is empty, then the moduli space of Seiberg–Witten equations on $\widehat{X}(r)$ is empty for all $r \geq r_0$. In the families setting, the value of $r_0 = r_0(b)$ may depend on the point $b \in B$. However, we have:

Lemma 7.7 *Suppose that $C \subseteq B$ is a closed subset of B and suppose that for each $c \in C$, there are no solutions to the Seiberg–Witten equations on \widehat{N} for $(g_{\widehat{N}}(c), \eta_{\widehat{N}}(c))$. Then there exists an r_0 such that for all $r \geq r_0$, all solutions of the Seiberg–Witten equations for the family $E_{\widehat{X}(r)}$ lie over $B \setminus C$.*

Proof Since C is compact, one can find such r_0 by repeating Nicolaescu, Section 4.5.3 uniformly over C . □

Lemma 7.8 *Every element of $\mathcal{M}(E_{\widehat{N}}, g_{\widehat{N}}, \eta_{\widehat{N}}, \mathfrak{s}_N, \mu)$ is reducible.*

Proof Let $\widehat{C}_{\widehat{N}} \in \mathcal{M}(E_{\widehat{N}}, g_{\widehat{N}}, \eta_{\widehat{N}}, \mathfrak{s}_N, \mu)$. Let $\widehat{H}_{\widehat{C}_{\widehat{N}}}^i$ for $i = 0, 1, 2$ denote the cohomology groups of the families deformation complex at $\widehat{C}_{\widehat{N}}$. The virtual dimension of the families moduli space $\mathcal{M}(E_{\widehat{N}}, g_{\widehat{N}}, \eta_{\widehat{N}}, \mathfrak{s}_N, \mu)$ is

$$d(N, \mathfrak{s}_N) + d = -1 - d + d = -1,$$

and hence

$$-\dim \widehat{H}_{\widehat{C}_{\widehat{N}}}^0 + \dim \widehat{H}_{\widehat{C}_{\widehat{N}}}^1 - \dim \widehat{H}_{\widehat{C}_{\widehat{N}}}^2 = -1.$$

But the genericity assumption (A7) implies that $\widehat{H}_{\widehat{C}_{\widehat{N}}}^2 = 0$. Hence the only way the above equation can be satisfied is if $\dim \widehat{H}_{\widehat{C}_{\widehat{N}}}^0 \neq 0$, so $\widehat{C}_{\widehat{N}}$ is reducible. □

By Lemma 7.8 the moduli space $\mathcal{M}(E_{\widehat{N}}, g_{\widehat{N}}, \eta_{\widehat{N}}, \mathfrak{s}_N, \mu)$ is just a finite set of points, which is in bijection with the finite set $\{b_1, b_2, \dots, b_l\}$.

Corollary 7.9 *There exists an r_0 such that for all $r \geq r_0$ we have that every solution of the Seiberg–Witten equations for the family $E_{\widehat{X}(r)}$ lies over U .*

Proof Because of [Lemma 7.8](#), any solutions to the Seiberg–Witten equations on \widehat{N} appears on b_1, \dots, b_l . So this corollary immediately follows from [Lemma 7.7](#) by setting $C = B \setminus U$ in [Lemma 7.7](#). \square

Thus, to describe the families moduli space $\mathcal{M}(E_{\widehat{X}(r)}, g_{\widehat{X}(r)}, \eta_{\widehat{X}(r)}, \mathfrak{s}_X)$, we just need to study its restriction to each of the open sets U_1, \dots, U_l . We pick one of the open sets U_i and study the moduli space over U_i . The notation t indicates an element of U_i . Let \widehat{C}_N be the unique reducible solution of the Seiberg–Witten equations on \widehat{N} over the point b_i and let \widehat{C}_M be any solution of the Seiberg–Witten equations on \widehat{M} over b_i . We wish to study the solutions of families Seiberg–Witten equations on $E_{\widehat{X}(r)}|_{U_i}$ which are close to $\widehat{C}_r = \widehat{C}_M \#_r \widehat{C}_N$. By the assumptions given in [Proposition 7.2](#), we may assume some simplifications, summarised as follows:

- We have trivialisations $E_{\widehat{M}}|_{U_i} \cong \widehat{M} \times U_i$ and $E_{\widehat{N}}|_{U_i} \cong \widehat{N} \times U_i$. Note this also induces a trivialisation $E_{\widehat{X}(r)}|_{U_i} \cong \widehat{X}(r) \times U_i$.
- We may identify U_i with an open ball at the origin in $T_{b_i}B$ such that $b_i \in U_i$ corresponds to $0 \in T_{b_i}B$.
- To be specific, we can fix some Euclidean norm on $T_{b_i}B$ and take $U_i = \{t \in T_{b_i}B \mid \|t\| < \tau\}$ for some $\tau > 0$.
- The fibrewise metrics $g_{\widehat{M}}$ and $g_{\widehat{N}}$ are independent of $t \in U_i$.
- $\eta_{\widehat{M}}$ is a constant which does not lie on the wall $\mathcal{W}_{\widehat{M}}$. Moreover, $\eta_{\widehat{M}}$ is generic in the unparametrised sense. But this also implies that over each U_i , $\eta_{\widehat{M}}$ is generic in the parametrised sense as well. Thus we have an isomorphism

$$\mathcal{M}(E_{\widehat{M}}|_{U_i}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_M, \mu) \cong \mathcal{M}(\widehat{M}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_M) \times U_i,$$

where $\mathcal{M}(\widehat{M}, g_{\widehat{M}}, \eta_{\widehat{M}}, \mathfrak{s}_M)$ is the unparametrised moduli space on \widehat{M} , which, by our assumptions, is smooth and contains no reducibles.

- Over U_i , we have $\eta_{\widehat{N}}(t) = \eta_0 + L(t)$, where $L(t) = L_i(t)$ is as described in [Proposition 7.2](#) and $\eta_{\widehat{N}}(t)$ is generic in the families sense.

Now we proceed in much the same way as we did in the unparametrised case explained in [Section 6.2](#). For each $t \in U_i$, form the nonlinear map

$$\mathcal{N}(\cdot, t): L^{2,2}(\widehat{X}(r), S^+ \oplus iT^*\widehat{X}(r)) \rightarrow L^{1,2}(\widehat{X}(r), S^- \oplus i \wedge_+^2 T^*\widehat{X}(r) \oplus i\mathbb{R})$$

given by

$$\mathcal{N}(\widehat{C}, t) = \widehat{S\widehat{W}}_t(\widehat{C} + \widehat{C}_r) \oplus \mathcal{L}_{\widehat{C}_r}^*(\widehat{C}),$$

where \widehat{SW}_t denotes the Seiberg–Witten equations taken with respect to the perturbation $\eta_{\widehat{X}(r)}(t)$. Namely, \widehat{SW}_t is defined as

$$\widehat{SW}_t(\widehat{C} + \widehat{C}_r) = (F_{\widehat{A}_0 + \widehat{a} + \widehat{a}_r}^+ - q(\widehat{\psi} + \widehat{\psi}_r, \widehat{\psi} + \widehat{\psi}_r) - i\eta_{\widehat{X}(r)}(t), D_{\widehat{A}_0 + \widehat{a} + \widehat{a}_r}(\widehat{\psi} + \widehat{\psi}_r)),$$

where we write \widehat{C} and \widehat{C}_r as $\widehat{C} = (\widehat{a}, \widehat{\psi})$ and $\widehat{C}_r = (\widehat{a}_r, \widehat{\psi}_r)$. The term $\mathcal{L}_{\widehat{C}_r}^*(\widehat{C})$ is the gauge fixing condition exactly as in the unparametrised case considered in Section 6.2. Since this term does not involve the perturbation $\eta_{\widehat{X}(r)}(t)$, it is independent of t . As in the unparametrised case, we are interested in the zero set $\mathcal{N}^{-1}(0)$.

Write $\eta_{\widehat{X}(r)}(t) \in L^{1,2}(\widehat{X}(r), i \wedge_{+g_{\widehat{X}(r)}(b_i)}^2 T^* \widehat{X}(r))$ in the form

$$\eta_{\widehat{X}(r)}(t) = \eta_{\widehat{X}(r)}(0) + K_r(t).$$

Since $\eta_{\widehat{M}}$ is constant on U_i , we have $K_r(t) = 0 \#_r L(t)$.

We will usually omit the subscript and write this as $K(t)$. Here recall the estimate (7-3), which is a consequence of condition (A5): $L(t)$ is approximately harmonic in the sense that $\|(d^+)^* L(t)\| < \delta \|t\|$, where $\delta > 0$ is some specified real number. Then $K_r(t)$ will also be approximately harmonic. More precisely:

Lemma 7.10 *We have the estimate*

$$(7-7) \quad \|(d^+)^* K(t)\|_{L^2(\widehat{X}(r))} \leq C e^{-\mu r} + \delta \|t\|,$$

where C is a constant which is independent of r . (Note that C may depend on the choice of $\eta_{\widehat{N}}$.)

Proof Recall the gluing operation $\#_r$ given in Nicolaescu, page 305. This gives us the following description of $K(t) = 0 \#_r L(t)$ on the neck of $\widehat{X}(r)$. Let $\tau \in [0, r]$ denote the time-direction coordinate on the neck $[0, r] \times S^3$, regarded as a subset of $\widehat{X}(r)$. Then, on the neck, we have

$$(7-8) \quad K(t)(\tau, y) = \alpha_r(\tau) L(t)(\tau, y)$$

for $(\tau, y) \in [0, r] \times S^3$, where $\alpha_r: [0, \infty) \rightarrow [0, 1]$ is a cutoff function such that

$$\alpha_r(\tau) = 1 \quad \text{on } [0, r - 1] \quad \text{and} \quad \alpha_r(\tau) = 0 \quad \text{on } [r, \infty).$$

(Note that, although $L_{\mu}^{1,2}$ cannot be embedded in \mathcal{C}^0 , by taking an approximating sequence converging to $L(t)$ and consisting of smooth sections, we can define the pointwise multiplication of α_r with $L(t)$ as the limit of the sequence multiplied by α_r .)

Therefore we can define $\alpha(\tau)L(t)$ by (7-8) even if $L(t)$ is not embedded in \mathcal{C}^0 . Therefore, by the inequality (7-3), we have

$$\begin{aligned}
 (7-9) \quad \|(d^+)^*K(t)\|_{L^2(\widehat{X}(r))} &\leq \|\nabla\alpha_r \cdot L(t)\|_{L^2(\widehat{N}(r))} + \|(d^+)^*L(t)\|_{L^2(\widehat{N}(r))} \\
 &\leq C\|L(t)\|_{L^2([r-1,r]\times S^3)} + \|(d^+)^*L(t)\|_{L^2_\mu(\widehat{N}(r))} \\
 &< C\|L(t)\|_{L^2([r-1,r]\times S^3)} + \delta\|t\|.
 \end{aligned}$$

Recall that we take r to be $r \geq R_{\text{vanish}} + 1$. Therefore we have $L(t) = -\eta_i$. By setting $t = 0$ in (7-3), we have $(d^+)^*\eta_i = 0$. Since η_i is an L^2 -self-dual 2-form, this implies that η_i is harmonic. Therefore we get an exponential decay estimate,

$$\|\eta_i\|_{L^2([r-1,r]\times S^3)} \leq C'e^{-\mu r}\|\eta_i\|_{L^2(\widehat{N})} = Ce^{-\mu r}.$$

This inequality and (7-9) imply the desired inequality (7-7). □

Here we give a notation Ψ_r for the operation used to get $K(t)$ from $L(t)$, namely define

$$(7-10) \quad \Psi_r: H^+_{L^2}(\widehat{N}) \rightarrow L^2(\widehat{X}(r))$$

by

$$(\Psi_r(\eta))(\tau, y) := \alpha_r(\tau)\eta(\tau, y)$$

on the neck, and $\Psi_r(\eta)$ equals η outside the neck on the N side and is zero outside the neck on the M side.

Let \widehat{T}_r denote the differential of $\mathcal{N}(\cdot, 0)$ at the origin. Namely, \widehat{T}_r is the map given as $d\mathcal{N}(\cdot, 0)|_0: L^{2,2}(\widehat{X}(r), S^+ \oplus iT^*\widehat{X}(r)) \rightarrow L^{1,2}(\widehat{X}(r), S^- \oplus i \wedge^2_+ T^*\widehat{X}(r) \oplus i\mathbb{R})$.

We also define a map R whose domain and range are the same as \widehat{T}_r by

$$\mathcal{N}(\widehat{C}, 0) = \mathcal{N}(0, 0) + \widehat{T}_r(\widehat{C}) + R(\widehat{C}).$$

Now we define $D_r(\widehat{C}, t)$ by

$$\mathcal{N}(\widehat{C}, t) = \mathcal{N}(0, t) + D_r(\widehat{C}, t) + R(\widehat{C}).$$

Note that $D_r(\widehat{C}, t)$ is written as

$$D_r(\widehat{C}, t) = \widehat{T}_r(\widehat{C}) + iK_r(t).$$

Note that, compared with the unparametrised case, $D_r(\widehat{C}, t)$ is the only part of $\mathcal{N}(\widehat{C}, t)$ that has changed.

Let us define $X^k = X_+^k \oplus X_-^k$, $H_r = H_r^+ \oplus H_r^-$, $Y^k(r) = Y_+^k(r) \oplus Y_-^k(r)$ and the projections P_\pm^r and Q_\pm^r just as in Definitions 6.6, 6.7, 6.9 and 6.11. We again write P_\pm^r and Q_\pm^r simply as P_\pm and Q_\pm .

For each $\widehat{C} \in X_+^0$, we decompose it as $\widehat{C} = \widehat{C}_0 + \widehat{C}_1$, where

$$\widehat{C}_0 = P_+ \widehat{C}, \quad \widehat{C}_1 = Q_+ \widehat{C}.$$

Let $S: Y_-^k \rightarrow Y_+^{k+1}$ be defined exactly as in Section 6.2.

Lemma 7.11 *We have an estimate of the form*

$$(7-11) \quad \|\widehat{T}_r^* iK(t)\|_{L^2(\widehat{X}(r))} \leq C e^{-\mu r} + \delta \|t\|$$

and an estimate of the form

$$\|Q_- iK(t)\|_{L^2(\widehat{X}(r))} \leq C r^2 (e^{-\mu r} + \delta \|t\|).$$

Proof Write $\widehat{C}_r = (\widehat{A}_r, \widehat{\psi}_r)$. We have

$$\widehat{T}_r^*(iK(t)) = ((d^+)^* iK(t), \beta^*(iK(t))),$$

where $\beta: S^+ \rightarrow i \wedge_+^2 T^* \widehat{X}(r)$ is given by

$$(7-12) \quad \beta(\varphi) = -\frac{1}{2} q(\widehat{\psi}_r, \varphi).$$

Here q is the quadratic term of spinors in the Seiberg–Witten equations. Here note that $\widehat{\psi}_r$ vanishes on $N(r)$, regarded as a subspace of $\widehat{X}(r)$, since $\widehat{\psi}_r$ was made from the reducible on \widehat{N} just by using cutoff functions. On the other hand, since we took r to be $r \geq R_{\text{vanish}} + 1$, $K(t)$ vanishes outside $N(r)$. Therefore the expression (7-12) of β implies that $\beta^*(iK(t)) = 0$. The estimate (7-11) then follows from the inequality (7-7).

Now decompose $iK(t)$ as $iK(t) = P_- K(t) + Q_- iK(t)$. Applying \widehat{T}_r^* , we get $\widehat{T}_r^*(iK(t)) = \widehat{T}_r^* P_-(iK(t)) + \widehat{T}_r^* Q_-(iK(t))$. But the two terms on the right are orthogonal, so the estimate (7-11) yields

$$\|\widehat{T}_r^* Q_-(iK(t))\| \leq C e^{-\mu r} + \delta \|t\|.$$

By combining this estimate with Lemma 6.12, we can get

$$\|Q_-(iK(t))\| \leq C' r^2 \|\widehat{T}_r^* Q_-(iK(t))\| \leq C' r^2 (C e^{-\mu r} + \delta \|t\|). \quad \square$$

Now we repeat the story of Section 6.2 as follows. The equation $\mathcal{N}(\widehat{C}, t) = 0$ is equivalent to the pair of equations

$$P_- \mathcal{N}(\widehat{C}, t) = 0, \quad Q_- \mathcal{N}(\widehat{C}, t) = 0.$$

Expanding \mathcal{N} and \widehat{C} , these equations become

$$\begin{aligned} Q_- \mathcal{N}(0) + \widehat{T}_r \widehat{C}_1 + Q_- R(\widehat{C}_0 + \widehat{C}_1) + Q_- iK(t) &= 0, \\ P_- \mathcal{N}(0) + \widehat{T}_r \widehat{C}_0 + P_- R(\widehat{C}_0 + \widehat{C}_1) + P_- iK(t) &= 0. \end{aligned}$$

Let $S: Y_-^k \rightarrow Y_+^{k+1}$ be the inverse operator of the restricted operator $\widehat{T}_r: Y_+^{k+1} \rightarrow Y_-^k$. Define $U \in Y_+^0$ to be

$$U = -SQ_- \mathcal{N}(0).$$

Applying S to the first of the two equations above, we end up with

$$\widehat{C}_1 = U - SQ_- R(\widehat{C}_0 + \widehat{C}_1) - SQ_- iK(t).$$

For the moment, fix some $\widehat{C}_0 \in H_r^+$ and some $t \in U_i$. Then define $\widetilde{\mathcal{F}}: Y_+^2 \rightarrow Y_+^2$ as

$$\widetilde{\mathcal{F}}(\widehat{C}_1) = U - SQ_- R(\widehat{C}_0 + \widehat{C}_1) - SQ_- iK(t).$$

Note that $\widetilde{\mathcal{F}}$ is well-defined as a map from Y_+^2 to itself. This is easy to see since $\widetilde{\mathcal{F}}$ differs from the map \mathcal{F} defined in Section 6.2 only by the term $-SQ_- iK(t)$. As in Section 6.2, define

$$B_1(r^{-4}) = \{\widehat{C}_1 \in Y_+^2 \mid \|\widehat{C}_1\|_{2,2} \leq r^{-4}\} \subset Y_+^2(r)$$

and, similarly,

$$B_0(r^{-4}) = \{\widehat{C}_0 \in H_r^+ \mid \|\widehat{C}_0\|_{2,2} \leq r^{-4}\} \subset H_r^+.$$

Lemma 7.12 *The operator*

$$P'_- \circ \Psi_r \circ L: T_{b_i} B \rightarrow H_r^-$$

is an isomorphism.

Proof First, note that an estimate given in the last paragraph of page 170 of Nicolaescu [17] implies that the map

$$P'_- \circ \Psi_r: H_{L^2}^+(\widehat{N}) \rightarrow H_r^-$$

is isomorphic. Because of Remark 7.6 and condition (A5)(a), the map $t \mapsto \pi \circ L(t)$ is also isomorphic. Therefore we can conclude that

$$P'_- \circ \Psi_r \circ \pi \circ L: H_{L^2}^+(\widehat{N}) \rightarrow H_r^-$$

is also isomorphic.

On the other hand, because of condition (A5)(d), we have

$$\|P'_- \circ \Psi_r \circ \pi \circ L_i(t) - P'_- \circ \Psi_r \circ L_i(t)\|_{L^2(\widehat{X}(r))} < \|P'_- \circ \Psi_r \circ \pi \circ L_i(t)\|_{L^2(\widehat{X}(r))}$$

for any t , therefore we have

$$\|P'_- \circ \Psi_r \circ \pi \circ L_i - P'_- \circ \Psi_r \circ L_i\| < \|P'_- \circ \Psi_r \circ \pi \circ L_i\|;$$

here the norms of both sides are the operator norms. Since $P'_- \circ \Psi_r \circ \pi \circ L$ is isomorphic, this inequality implies that $P'_- \circ \Psi_r \circ L_i$ is also isomorphic. \square

In Lemmas 7.13 to 7.16, we shall give the argument to determine the constant ϵ in condition (A5)(c). For the purpose we shall note that the constants appearing in Lemmas 7.13 to 7.16 do not depend on choice of $\eta_{\widehat{N}}$.

Lemma 7.13 *There exists a sequence $C(r)$, which is independent of the choice of $\eta_{\widehat{N}}$, of positive numbers such that $C(r) \rightarrow 0$ as $r \rightarrow \infty$ and*

$$\|Q'_- \circ \Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} \leq C(r) \|\pi \circ L(t)\|_{L_{\mu}^2(\widehat{N})}$$

holds for any $t \in U_i$.

Proof First recall that the definition of the term “asymptotic map” in Nicolaescu’s book. Let H_0 and H_1 be Hilbert spaces. Let U_r and V_r be families parametrised by $r > 0$ of closed subspaces of H_0 and H_1 , respectively. Let $f_r: U_r \rightarrow H_1$ be densely defined linear operators with closed ranges. For closed subspaces U and V of H_1 , let us define

$$\widehat{\delta}(U, V) := \sup\{\text{dist}(u, V) \mid u \in U, \|u\| = 1\} = \sup\{\|P^\perp(u)\| \mid u \in U, \|u\| = 1\}.$$

Here $P^\perp: H_1 \rightarrow V^\perp$ is the orthogonal projection onto the orthogonal complement V^\perp of V . We call the family of triples $(U_r, V_r, f_r)_r$ an asymptotic map if $\widehat{\delta}(\text{Im}(f_r), V_r) \rightarrow 0$ as $r \rightarrow 0$. In such a case, we use the notation $f_r: U_r \rightarrow^a V_r$.

As noted in Nicolaescu’s book, page 307, the map $\Psi_r: H_{L^2}^+(\widehat{N}) \rightarrow L^2(\widehat{X}(r))$ defines an asymptotic map $\Psi_r: H_{L^2}^+(\widehat{N}) \rightarrow^a H_r^-$. Therefore there exists a sequence $C(r)$ of

positive numbers such that $C(r) \rightarrow 0$ as $r \rightarrow \infty$ and

$$\left\| Q_-^r \circ \Psi_r \left(\frac{\pi \circ L(t)}{\|\pi \circ L(t)\|} \right) \right\| \leq C(r)$$

holds for any $t \in U_i$. This proves the lemma. □

Lemma 7.14 *We have*

$$\|P_-^r \circ \Psi_r\|_{\text{op}} \leq 1,$$

where $\|\cdot\|_{\text{op}}$ denotes the operator norm for operators whose domain is $L^2_\mu(\widehat{N}, \wedge^+ T^* \widehat{N})$ and codomain is $L^2(\widehat{X}(r), \wedge^+ T^* \widehat{X}(r))$.

Proof Let us take $\eta \in L^2_\mu(\widehat{N}, \wedge^+ T^* \widehat{N})$. Then we have

$$\|P_-^r \circ \Psi_r(\eta)\|_{L^2(\widehat{X}(r), \wedge^+ T^* \widehat{X}(r))} \leq \|P_-^r\|_{\text{op}} \|\Psi_r(\eta)\|_{L^2(\widehat{X}(r), \wedge^+ T^* \widehat{X}(r))}$$

and $\|P_-^r\|_{\text{op}} \leq 1$, since P_-^r is the L^2 -orthogonal projection. In addition, we have

$$\|\Psi_r(\eta)\|_{L^2(\widehat{X}(r))} = \|\alpha_r \eta\|_{L^2(\widehat{X}(r))} \leq \|\alpha_r \eta\|_{L^2(\widehat{N})} \leq \|\eta\|_{L^2(\widehat{N})} \leq \|\eta\|_{L^2_\mu(\widehat{N})}.$$

This proves the lemma. □

Lemma 7.15 *There exists $C_0 > 0$, which depends only on the geometry of \widehat{N} , such that*

$$C_0 \|\pi \circ L(t)\|_{L^2_\mu(\widehat{N})} \leq \|\Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))}$$

holds for any $t \in U_i$ and all sufficiently large r .

Proof First, since $H^+_{L^2}(\widehat{N})$ is finite-dimensional, the L^2 -norm on $H^+_{L^2}(\widehat{N})$ and L^2_μ -norm on $H^+_{L^2}(\widehat{N})$ are equivalent. Therefore it suffices to show that there exists $C > 0$, which depends only on the geometry of \widehat{N} , such that

$$C \|\pi \circ L(t)\|_{L^2(\widehat{N})} \leq \|\Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))}$$

holds for all sufficiently large r . We have

$$\begin{aligned} (7-13) \quad & \|\pi \circ L(t)\|_{L^2(\widehat{N})} - \|\Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} \\ &= \|\pi \circ L(t)\|_{L^2(\widehat{N})} - \|\Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{N})} \\ &\leq \|\pi \circ L(t) - \Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{N})} \\ &= \|(1 - \alpha_r) \cdot \pi \circ L(t)\|_{L^2(\widehat{N})} \\ &= \|(1 - \alpha_r) \cdot \pi \circ L(t)\|_{L^2([r, r+1] \times S^3)} \\ &\leq \|\pi \circ L(t)\|_{L^2([r, r+1] \times S^3)}. \end{aligned}$$

Note that, because of the exponential decay estimate for L^2 -harmonic forms, there exists $r_0 > 0$ such that

$$\|\eta\|_{L^2([r,r+1]\times S^3)} \leq \frac{1}{2}\|\eta\|_{L^2(\widehat{N})}$$

holds for any $\eta \in H_{L^2}^+(\widehat{N})$ and any $r \geq r_0$. Thus we have $\|\pi \circ L(t)\|_{L^2([r,r+1]\times S^3)} \leq \frac{1}{2}\|\pi \circ L(t)\|_{L^2(\widehat{N})}$ for sufficiently large r . This inequality and the inequality (7-13) imply

$$\|\pi \circ L(t)\|_{L^2(\widehat{N})} - \|\Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} \leq \frac{1}{2}\|\pi \circ L(t)\|_{L^2(\widehat{N})},$$

and thus we have

$$\frac{1}{2}\|\pi \circ L(t)\|_{L^2(\widehat{N})} \leq \|\Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))}$$

for sufficiently large r . □

Let C_0 be the constant appearing in the statement of Lemma 7.15. We now assume the constant ϵ in (A5)(c) is chosen with $\epsilon < \frac{1}{64}C_0$.

Lemma 7.16 *Let C_0 be the constant appearing in the statement of Lemma 7.15. Then*

$$\|P_-^r \circ \Psi_r \circ L(t)\|_{L^2(\widehat{X}(r))} \geq \frac{1}{4}C_0\|\pi \circ L(t)\|_{L_\mu^2(\widehat{N})}$$

holds for any $t \in U_i$ and all sufficiently large r .

Proof From the decomposition $L(t) = \pi \circ L(t) + \pi^\perp \circ L(t)$, we get

$$\begin{aligned} & \|P_-^r \circ \Psi_r \circ L(t)\|_{L^2(\widehat{X}(r))} \\ & \geq \|P_-^r \circ \Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} - \|P_-^r \circ \Psi_r \circ \pi^\perp \circ L(t)\|_{L^2(\widehat{X}(r))} \\ & \geq \|P_-^r \circ \Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} - \|\pi^\perp \circ L(t)\|_{L_\mu^2(\widehat{N})} \\ & \geq \|P_-^r \circ \Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} - \epsilon\|\pi \circ L(t)\|_{L_\mu^2(\widehat{N})} \\ & \geq \|\Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} - \|Q_-^r \circ \Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} - \epsilon\|\pi \circ L(t)\|_{L_\mu^2(\widehat{N})} \\ & \geq \|\Psi_r \circ \pi \circ L(t)\|_{L^2(\widehat{X}(r))} - (C(r) + \epsilon)\|\pi \circ L(t)\|_{L_\mu^2(\widehat{N})} \\ & \geq (C_0 - C(r) - \epsilon)\|\pi \circ L(t)\|_{L_\mu^2(\widehat{N})}. \end{aligned}$$

Here, in the second, third, fifth and sixth inequalities, we used Lemma 7.14, (A5)(c), Lemma 7.13 and Lemma 7.15. Henceforth let us focus only on r satisfying $C(r) \leq \frac{1}{2}C_0$. Then $C_0 - C(r) - \epsilon \geq \frac{1}{2}C_0 - \epsilon > \frac{1}{4}C_0$, since we take $\epsilon < \frac{1}{64}C_0$. Then the above estimates prove the lemma. □

Lemma 7.17 *Let C_0 be the constant appearing in the statement of Lemma 7.15. Then, for all sufficiently large r , we have that if $\widehat{C}_0 \in B_0(r^{-4})$ and $\widehat{C}_1 \in B_1(r^{-4})$, then*

$$\|t\| \leq \frac{16}{\delta C_0} r^{-6}$$

whenever $\mathcal{N}(\widehat{C}_0, \widehat{C}_1, t) = 0$.

Proof Expanding the equation $P_- \mathcal{N}(\widehat{C}, t) = 0$ out, we get

$$(7-14) \quad P_- \mathcal{N}(0) + \widehat{T}_r(\widehat{C}_0) + P_- R(\widehat{C}_0 + \widehat{C}_1) + P_-(iK(t)) = 0.$$

Let

$$\rho_r: T_{b_i} B \rightarrow H_r^-$$

be the operator defined by $\rho_r(t) = P'_- \circ \Psi_r \circ L(t) = P'_- \circ K_r(t)$. Because of Lemma 7.12, ρ_r is invertible. Let J_r be the inverse of ρ_r . Lemma 7.16 implies that

$$\|\rho_r(t)\|_{L^2(\widehat{X}(r))} \geq \frac{1}{4} C_0 \|\pi \circ L(t)\|_{L^2_\mu(\widehat{N})}$$

for any $t \in T_{b_i} B$. Therefore, using (A5)(f), we have

$$\|\rho_r(t)\|_{L^2(\widehat{X}(r))} \geq C_1 \|t\|,$$

where $C_1 = \frac{1}{8} \delta C_0$. Thus we have $\|\rho_r\|_{\text{op}} \geq C_1$, and hence $\|J_r\|_{\text{op}} \leq 1/C_1$.

Applying J_r to (7-14), we get

$$t = -J_r P_- \mathcal{N}(0) - J_r \widehat{T}_r(\widehat{C}_0) - J_r P_- R(\widehat{C}_0 + \widehat{C}_1).$$

Then we estimate

$$\begin{aligned} \|t\| &\leq \|J_r\|_{\text{op}} (\|P_- \mathcal{N}(0)\| + \|\widehat{T}_r(\widehat{C}_0)\| + \|P_- R(\widehat{C}_0 + \widehat{C}_1)\|) \\ &\leq C_1^{-1} (C e^{-\mu r} + r^{-2} \|\widehat{C}_0\| + \|R(\widehat{C}_0 + \widehat{C}_1)\|) \\ &\leq C_1^{-1} (C e^{-\mu r} + r^{-6} + C r^{3/2} (r^{-4} + r^{-4})^2) \end{aligned}$$

for some constant $C > 0$. The dominant term for large r is $C_1^{-1} r^{-6}$. Hence, for all large enough r we get

$$\|t\| \leq 2C_1^{-1} r^{-6} = \frac{16}{\delta C_0} r^{-6}. \quad \square$$

By this lemma, when considering solutions of the Seiberg–Witten equations over U_i such that $\|\widehat{C}_0\|, \|\widehat{C}_1\| < r^{-4}$, we may as well assume $\|t\| \leq C_2 r^{-6}$, where $C_2 = 16/(\delta C_0)$.

Lemma 7.18 For all sufficiently large r , all sufficiently small δ , all $\widehat{C}_0 \in B_0(r^{-4})$ and all $t \in U_i$ satisfying $\|t\| \leq C_2 r^{-6}$, the map $\widetilde{\mathcal{F}}$ sends $B_1(r^{-4})$ to itself and is a contraction mapping.

Proof Since $\widetilde{\mathcal{F}}$ differs from \mathcal{F} only by a constant term, we immediately have that $\widetilde{\mathcal{F}}$ acts as a contraction on $B_1(r^{-4})$ for all sufficiently large r , provided that we can show that it sends $B_1(r^{-4})$ to itself. Recall that we had obtained an estimate of the form

$$\|\mathcal{F}(\widehat{C}_1)\| \leq C(r^2 e^{-\mu r} + r^{-13/2}).$$

Hence, we immediately have

$$\|\widetilde{\mathcal{F}}(\widehat{C}_1)\| \leq C(r^2 e^{-\mu r} + r^{-13/2}) + \|SQ_{-i}K(t)\|.$$

Let C' denote the constant in the $k = 1$ case of Lemma 6.12. Then

$$\|SQ_{-i}K(t)\| \leq C' r^2 \|Q_{-}K(t)\|.$$

Also, since $K(t) = \Psi_r \circ L(t)$, we get

$$\begin{aligned} \|Q_{-}K(t)\| &= \|Q_{-} \circ \Psi_r \circ L(t)\| \\ &\leq \|Q_{-} \circ \Psi_r \circ \pi \circ L(t)\| + \|Q_{-} \circ \Psi_r \circ \pi^\perp \circ L(t)\| \\ &\leq C(r) \|\pi \circ L(t)\| + \|\pi^\perp \circ L(t)\| \\ &\leq (C(r) + \epsilon) \|\pi \circ L(t)\|, \end{aligned}$$

where we used Lemma 7.13 and (A5)(c). Then, since $C(r) \rightarrow 0$ as $r \rightarrow \infty$, we have $C(r) \leq \epsilon$ for all large enough r . This gives

$$\|SQ_{-i}K(t)\| \leq 2C' r^2 \epsilon \|\pi \circ L(t)\| \leq 2C' r^2 \epsilon \delta \|t\| \leq \frac{32}{C_0} \epsilon r^{-4} < \frac{1}{2} r^{-4},$$

by Lemma 7.17 and (A5)(e), and since we have taken $\epsilon < \frac{1}{64} C_0$. Hence $\widetilde{\mathcal{F}}$ sends $B_1(r^{-4})$ to itself for all large enough r and small enough ϵ . \square

Thus, for each $\widehat{C}_0 \in B_0(r^{-4})$ and each $t \in U_i$ with $\|t\| \leq C r^{-6}$, there is a uniquely determined fixed point of $\widetilde{\mathcal{F}}$, which will be denoted as $\widehat{C}_1 = \Phi(\widehat{C}_0, t) \in B_1(r^{-4})$. It can be shown that $\Phi(\widehat{C}_0, t)$ depends differentiably on \widehat{C}_0 and t (by the implicit function theorem).

Now consider again the pair of equations

$$P_{-}\mathcal{N}(\widehat{C}_0 + \widehat{C}_1, t) = 0, \quad Q_{-}\mathcal{N}(\widehat{C}_0 + \widehat{C}_1, t) = 0.$$

The second of these equations is solved by the substitution $\widehat{C}_1 = \Phi(\widehat{C}_0, t)$ and we are left with just the single equation for \widehat{C}_0 and t ,

$$P_- \mathcal{N}(\widehat{C}_0 + \Phi(\widehat{C}_0, t), t) = 0.$$

Expanding this out, the equation becomes

$$(7-15) \quad P_- \mathcal{N}(0) + \widehat{T}_r(\widehat{C}_0) + P_- R(\widehat{C}_0 + \Phi(\widehat{C}_0, t)) + P_-(iK(t)) = 0.$$

We apply J_r to (7-15) to obtain

$$(7-16) \quad \begin{aligned} t &= -J_r P_- \mathcal{N}(0) - J_r \widehat{T}_r(\widehat{C}_0) - J_r P_- R(\widehat{C}_0 + \Phi(\widehat{C}_0, t)) \\ &= -J_r \mathcal{N}(C_0 + \Phi(\widehat{C}_0, t)). \end{aligned}$$

Fix $\widehat{C}_0 \in B_0(r^{-4})$. We try to solve this as an equation for t using the contraction mapping principle. Define $G: U_i \rightarrow T_{b_i} B$ by

$$G(t) = -J_r P_- \mathcal{N}(C_0 + \Phi(\widehat{C}_0, t)).$$

Then (7-16) reduces to the fixed-point equation $G(t) = t$.

Lemma 7.19 *There is a constant C independent of r such that for all sufficiently large r we have that G sends $V = \{t \mid \|t\| \leq Cr^{-6}\}$ to itself.*

Proof This is just a consequence of Lemma 7.17. □

Thus for all large enough r we have that G defines a map $G: V \rightarrow V$. Next we want to show this is a contraction.

Lemma 7.20 *Let \mathcal{F}_t be a family of contractions on $B_1(r^{-4})$ of the form*

$$\mathcal{F}_t(\widehat{C}_1) = \mathcal{F}(\widehat{C}_1) + \lambda(t),$$

where \mathcal{F} is a contraction. Let $0 < \kappa < 1$ be such that

$$\|\mathcal{F}(\widehat{C}_1) - \mathcal{F}(\widehat{C}_2)\| \leq \kappa \|\widehat{C}_1 - \widehat{C}_2\|.$$

Let $\Phi(t)$ be the unique fixed point of \mathcal{F}_t . Then we have

$$\|\Phi(t_1) - \Phi(t_2)\| \leq \frac{1}{1-\kappa} \|\lambda(t_1) - \lambda(t_2)\|.$$

Proof We start with the defining relations $\mathcal{F}_{t_i}(\Phi(t_i)) = \Phi(t_i)$ for $i = 1, 2$. Thus

$$\begin{aligned} \|\Phi(t_1) - \Phi(t_2)\| &\leq \|\mathcal{F}_{t_1}(\Phi(t_1)) - \mathcal{F}_{t_2}(\Phi(t_2))\| \\ &\leq \|\mathcal{F}(\Phi(t_1)) - \mathcal{F}(\Phi(t_2)) + \lambda(t_1) - \lambda(t_2)\| \\ &\leq \|\mathcal{F}(\Phi(t_1)) - \mathcal{F}(\Phi(t_2))\| + \|\lambda(t_1) - \lambda(t_2)\| \\ &\leq \kappa \|\Phi(t_1) - \Phi(t_2)\| + \|\lambda(t_1) - \lambda(t_2)\|. \end{aligned}$$

Hence $(1 - \kappa)\|\Phi(t_1) - \Phi(t_2)\| \leq \|\lambda(t_1) - \lambda(t_2)\|$ and the result follows. □

Corollary 7.21 For any $t_1, t_2 \in V$, we have an estimate of the form

$$\|\Phi(\widehat{C}_0, t_1) - \Phi(\widehat{C}_0, t_2)\| \leq Cr^2\|t_1 - t_2\|.$$

Proof We apply Lemma 7.20, where \mathcal{F} is as usual and $\lambda(t) = -SQ_-(iK(t))$. Note that, by the proof of Lemma 6.14, we can take $\kappa = \frac{1}{2}$. Therefore, we get an estimate

$$\|\Phi(\widehat{C}_0, t_1) - \Phi(\widehat{C}_0, t_2)\| \leq 2\|SQ_-(iK(t_1) - iK(t_2))\| \leq Cr^2\|t_1 - t_2\|. \quad \square$$

Lemma 7.22 For all sufficiently large r , G is a contraction on V .

Proof Let $t_1, t_2 \in V$. We estimate

$$\begin{aligned} \|G(t_1) - G(t_2)\| &\leq \|J_r\| \|P_{-\mathcal{N}}(C_0 + \Phi(\widehat{C}_0, t_1)) - P_{-\mathcal{N}}(C_0 + \Phi(\widehat{C}_0, t_2))\| \\ &\leq C \|P_{-R}(\widehat{C}_0 + \Phi(\widehat{C}_0, t_1)) - P_{-R}(\widehat{C}_0 + \Phi(\widehat{C}_0, t_2))\| \\ &\leq C \|R(\widehat{C}_0 + \Phi(\widehat{C}_0, t_1)) - R(\widehat{C}_0 + \Phi(\widehat{C}_0, t_2))\| \\ &\leq Cr^{3/2}r^{-4} \|\Phi(\widehat{C}_0, t_1) - \Phi(\widehat{C}_0, t_2)\| \\ &\leq Cr^{-1/2} \|t_1 - t_2\| \quad (\text{by Corollary 7.21}). \end{aligned}$$

Hence, for all sufficiently large r , this is a contraction. □

Thus, for all large enough r , we see that for each $\widehat{C}_0 \in B_0(r^{-4})$ there is a unique $t = t(\widehat{C}_0)$ such that $(\widehat{C} = \widehat{C}_0 + \Phi(\widehat{C}_0, t(\widehat{C}_0)), t(\widehat{C}_0))$ is a solution of the Seiberg–Witten equations for the family $E_{\widehat{X}(r)}$. In particular, setting $\widehat{C}_0 = 0$, we get a solution $(\widehat{C}, t) = (\Phi(0, t(0)), t(0))$, which we call $\Psi_r(\widehat{C}_r)$.

Proposition 7.23 For all large enough r , the mapping $\widehat{C}_r \mapsto \Psi_r(\widehat{C}_r)$ defines a diffeomorphism between the disjoint union

$$\bigsqcup_{i=1}^l \mathcal{M}(\widehat{M}, g_{\widehat{M}}(b_i), \eta_{\widehat{M}}(b_i), \mathfrak{s}_M, \mu)$$

and the families moduli space

$$\mathcal{M}(E_{\widehat{X}(r)}, g_{\widehat{X}(r)}, \eta_{\widehat{X}(r)}, \mathfrak{s}_X).$$

Proof We just have to adapt the arguments of Nicolaescu’s book, Sections 4.5.3 and 4.5.4. □

Remark 7.24 Recall also that each moduli space $\mathcal{M}(\widehat{M}, g_{\widehat{M}}(b_i), \eta_{\widehat{M}}(b_i), \mathfrak{s}_M, \mu)$ is diffeomorphic to a smooth Seiberg–Witten moduli space for the compact manifold M .

End of proof of Theorem 4.1(1) Recall that we assume $\dim B = d = b^+(N)$. Therefore $\theta \in H^0(B, \mathbb{Z}_2)$ and we may as well take $\theta = 1$. For simplicity, we assume ρ is trivial and consider the \mathbb{Z}_2 -valued invariant. The gluing formula for the ρ -twisted and \mathbb{Z} -valued invariants are a straightforward extensions of this case. So we have to show that

$$\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, 1) = \text{SW}(M, \mathfrak{s}_M)\langle w_{b^+}(H^+(N)), [B] \rangle \in \mathbb{Z}_2.$$

But $\langle w_{b^+}(H^+(N)), [B] \rangle$ is just the count (mod 2) of the number of zeros of a generic section of $H^+(N)$. Thus $\langle w_{b^+}(H^+(N)), [B] \rangle = l$ and we must show

$$\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, 1) = \text{SW}(M, \mathfrak{s}_M)l.$$

But from Proposition 7.23 we have diffeomorphisms

$$\begin{aligned} \mathcal{M}(E_{\widehat{X}(r)}, g_{\widehat{X}(r)}, \eta_{\widehat{X}(r)}, \mathfrak{s}_X) &\cong \bigsqcup_{i=1}^l \mathcal{M}(\widehat{M}, g_{\widehat{M}}(b_i), \eta_{\widehat{M}}(b_i), \mathfrak{s}_M, \mu) \\ &\cong \bigsqcup_{i=1}^l \mathcal{M}(M, g_M(b_i), \eta_M(b_i), \mathfrak{s}_M) \end{aligned}$$

(for suitably defined metrics $g_M(b_i)$ and perturbations $\eta_M(b_i)$ on M). As in Nicolaescu, Section 4.5.4 this diffeomorphism can be seen to respect the isomorphism class of the line bundles over the moduli spaces. It follows immediately that

$$\text{FSW}_m^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, 1) = \sum_{i=1}^l \text{SW}(M, \mathfrak{s}_M) = \text{SW}(M, \mathfrak{s}_M)l$$

and the proof is complete. □

8 Proof of Theorem 4.1(2)

Let $\theta \in H^{s+2k}(B, \mathbb{Z}_2)$, where $s = \dim B - b^+(N) \geq 0$ and $k > 0$. By the solution to the Steenrod problem, θ is Poincaré dual to a homology class of the form $f_*[S] \in H_{b^+(N)-2k}(B, \mathbb{Z}_2)$, where S is a compact smooth manifold of dimension $\dim S = b^+(N) - 2k < b^+(N)$ and $f: S \rightarrow B$ is a smooth map. Then

$$\begin{aligned} \text{FSW}_{m-k}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \theta) &= \langle c_1(\mathcal{L})^{m-k} \cup \pi^*\theta, [\mathcal{M}] \rangle \\ &= \langle \pi_*(c_1(\mathcal{L})^{m-k} \cup \pi^*(\theta)), [B] \rangle \quad (\text{by definition of the wrong-way map } \pi_*) \\ &= \langle \pi_*(c_1(\mathcal{L})^{m-k}) \cup \theta, [B] \rangle \\ &= \langle \pi_*(c_1(\mathcal{L})^{m-k}), [S] \rangle \quad (\theta \text{ is Poincaré dual to } [S]) \\ &= \langle c_1(\mathcal{L})^{m-k}, [\mathcal{M}|_S] \rangle \quad (\text{by definition of the wrong-way map } \pi_*) \\ &= \text{FSW}_{m-k}^{\mathbb{Z}_2}(E_X|_S, \mathfrak{s}_X, 1). \end{aligned}$$

This reduces the problem to computing the families Seiberg–Witten invariant for the restricted family $E_X|_S$. We can analyse the families moduli space associated to $E_X|_S$ using gluing theory. However, this time since $\dim S < b^+(N)$, we have that $\eta_{\widehat{N}}$ can be chosen so as to never meet the wall, so we can take $l = 0$ in Proposition 7.2. Then, by Corollary 7.9, we have that the families moduli space for $E_{\widehat{X}(r)}|_S$ is empty for all sufficiently large r . Hence $\text{FSW}^{\mathbb{Z}_2}(E_X|_S, \mathfrak{s}_X, 1) = 0$.

9 Applications of the gluing formula

9.1 General setup

In this section we briefly recall the setup for the gluing formula from Section 4. In the following sections we consider various applications of this formula.

Let M and N be compact, smooth, oriented 4–manifolds and set $X = M \# N$. Let B be a compact smooth manifold of dimension d and suppose we have smooth fibrewise oriented families $\pi_M: E_M \rightarrow B$ and $\pi_N: E_N \rightarrow B$ whose fibres are M and N , respectively. Let $T(E_M/B) = \text{Ker}(\pi_{M*})$ and $T(E_N/B) = \text{Ker}(\pi_{N*})$ be the vertical tangent bundles. To form the connected sum family, suppose we are given the following additional data:

- (1) Sections $\iota_M: B \rightarrow E_M$ and $\iota_N: B \rightarrow E_N$.

- (2) An orientation-reversing vector bundle isomorphism $\phi: \iota_M^*(T(E_M/B)) \rightarrow \iota_N^*(T(E_N/B))$.

Given this data, we can construct a fibrewise connected sum family $E_X \rightarrow B$ as described in more detail in [Section 4](#).

Let \mathfrak{s}_M and \mathfrak{s}_N be Spin^c -structures on M and N and let \mathfrak{s}_X be the corresponding Spin^c -structure on X . Assume that:

- (3) The Spin^c -structures $\mathfrak{s}_M, \mathfrak{s}_N$ are monodromy-invariant under the monodromy action of $\pi_1(B)$, up to a common sign factor given by a homomorphism $\rho: \pi_1(B) \rightarrow \mathbb{Z}_2$.

Then \mathfrak{s}_X is also invariant up to the same sign factor ρ .

Assume further that:

- (4) $d(M, \mathfrak{s}_M) = 0$.
- (5) $c_1(\mathfrak{s}_N)^2 = \sigma(N)$.
- (6) $b_1(N) = 0$.
- (7) $0 < b^+(N) \leq \dim B = d$.
- (8) $b^+(M) > (\dim B - b^+(N)) + 1$.

Remark 9.1 The assumption that $d(M, \mathfrak{s}_M) = 0$ is not strictly necessary, but is useful for the following two reasons. First, it ensures that we can use [Theorem 4.1\(1\)](#) without any requirement that \mathfrak{s}_X extend to a Spin^c -structure on $T(E_X/B)$. Second, at the time of writing, all known examples of pairs (M, \mathfrak{s}_M) with $b^+(M) > 1$ and $\text{SW}(M, \mathfrak{s}_M) \neq 0$ satisfy $d(M, \mathfrak{s}_M) = 0$. A compact, oriented smooth 4-manifold is said to be of simple type if its Seiberg–Witten invariants vanish whenever the expected dimension of the moduli space is nonzero. The so-called simple type conjecture (see for example [\[9, Problem 4.131\]](#)) states that all compact, simply connected smooth 4-manifolds with $b^+ > 1$ have simple type.

By these assumptions, it follows that

$$d(N, \mathfrak{s}_N) = -1 - b^+(N) < 0$$

and that

$$d(X, \mathfrak{s}_X) = -b^+(N).$$

Note that assumption (8) ensures that we have a well-defined Seiberg–Witten invariant for the family E_X which takes values in $H^{b^+(N)}(B, \mathbb{Z}_2)$.

Theorem 9.2 (gluing formula, \mathbb{Z}_2 -valued case) *Under assumptions (1)–(8) we have*

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X) = \text{SW}(M, \mathfrak{s}_M) \cdot w_{b^+(N)}(H^+(N)) \in H^{b^+(N)}(B, \mathbb{Z}_2)$$

in the untwisted case. More generally, in the ρ -twisted case we have

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho) = \text{SW}(M, \mathfrak{s}_M) \cdot w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_\rho) \in H^{b^+(N)}(B, \mathbb{Z}_2),$$

where $\mathbb{R}_\rho \rightarrow B$ is the real line bundle classified by $\rho \in H^1(B, \mathbb{Z}_2)$.

9.2 A vanishing theorem

The gluing formula immediately gives us:

Proposition 9.3 *Suppose assumptions (1)–(8) of Section 9.1 hold.*

- (i) *If $\text{SW}(M, \mathfrak{s}_M) = 0 \pmod{2}$ or if $w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_\rho) = 0$, then we have $\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho) = 0$.*
- (ii) *Suppose B , $\det^+(E_M)$ and $\det^+(E_N) \cong \det(H^+(N))$ are oriented and ρ is trivial. If $\text{SW}(M, \mathfrak{s}_M) = 0$ or if $e(H^+(N)) = 0$, then $\text{FSW}^{\mathbb{Z}}(E_X, \mathfrak{s}_X) = 0$.*

Remark 9.4 *Suppose that B is simply connected. Then ρ is the trivial homomorphism and we have that $w_{b^+(N)}(H^+(N))$ and $e(H^+(N))$ both vanish. To see this, note that $H^2(N, \mathbb{R})$ defines a flat vector bundle with structure group $O(b^+(N), b^-(N))$. If B is simply connected, then $H^2(N, \mathbb{R})$ is trivialisable as an $O(b^+(N), b^-(N))$ -bundle. Hence $H^+(N)$ defines a trivialisable vector bundle on B .*

The above remark shows that in order to use the gluing formula to get a nonzero families Seiberg–Witten invariant, we must take B to have a nontrivial fundamental group.

9.3 Smooth isotopy classes of diffeomorphisms

In this section, we consider the case that B is 1-dimensional. The families Seiberg–Witten invariants for 1-dimensional families was studied by Ruberman in [19; 21] and a special case of the 1-parameter gluing formula was proven in [21]. Even for 1-parameter families, our gluing formula is more general.

We assume that $B = S^1$ is the circle. Any family $E_X \rightarrow S^1$ is the mapping cylinder of a diffeomorphism $f: X \rightarrow X$, which we denote by $\text{Cyl}(f)$. Note that up to isomorphism the family $\text{Cyl}(f)$ depends only on the smooth isotopy class of f .

Let $f: X \rightarrow X$ be a diffeomorphism. Suppose that $b^+(X) > 2$ and suppose that f preserves a Spin^c -structure \mathfrak{s}_X up to sign:

$$f^*(\mathfrak{s}_X) = (-1)^\rho \mathfrak{s}_X, \quad \rho \in \mathbb{Z}_2.$$

Then we can consider the families Seiberg–Witten invariant attached to the family $\text{Cyl}(f)$ and Spin^c -structure \mathfrak{s}_X . Assume also that

$$d(X, \mathfrak{s}_X) = -1.$$

Then we obtain a \mathbb{Z}_2 -valued invariant by setting

$$\text{SW}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, f, \rho) = \text{FSW}^{\mathbb{Z}_2}(\text{Cyl}(f), \mathfrak{s}_X, \rho) \in H^1(S^1, \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

This is the 1-parameter families Seiberg–Witten invariant considered by Ruberman in [19]. Note that $\text{SW}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, f, \rho)$ depends only on the smooth isotopy class of f . Note also that the condition $d(X, \mathfrak{s}_X) = -1$ forces $b^+(X)$ to be even. We can also think of $\text{SW}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, f, \rho)$ as the equivariant Seiberg–Witten invariant associated to the $G = \mathbb{Z}$ -action determined by f .

Suppose now that $X = M \# N$ is a connected sum and suppose that we have families $E_M = \text{Cyl}(f_M)$ and $E_N = \text{Cyl}(f_N)$ which are mapping cylinders associated to diffeomorphisms f_M and f_N . Assume that f_M acts as the identity in the neighbourhood of a point $m \in M$ and similarly that f_N acts as the identity in the neighbourhood of a point $n \in N$. Then we construct X as the connected sum by removing small balls around m and n and identifying their boundaries. In this way, f_M and f_N can be composed to give a diffeomorphism of X , which we denote as $f_X = f_M \# f_N$. Suppose that f_M and f_N preserve Spin^c -structures \mathfrak{s}_M and \mathfrak{s}_N up to a common sign $\rho \in \mathbb{Z}_2$ and set $\mathfrak{s}_X = \mathfrak{s}_M \# \mathfrak{s}_N$.

Observe that f_N acts as an orientation-preserving isometry of $H^2(N, \mathbb{R})$. As the space of maximal positive-definite subspace of $H^2(N, \mathbb{R})$ is connected, it makes sense to say whether f_N preserves or reverses the orientation of maximal definite subspaces and we can associate to f_N a corresponding sign factor ± 1 . Abusing notation, we will denote this sign factor as $\det(f_N|_{H^+(N)})$, even though f_N may not preserve any particular choice of maximal positive-definite subspace $H^+(N)$. Specialising the gluing formula to this setting, we obtain:

Theorem 9.5 *Let $(M, N, f_M, f_N, \mathfrak{s}_M, \mathfrak{s}_N, \rho)$ be as above. Assume also that:*

- (i) $d(M, \mathfrak{s}_M) = 0$.

- (ii) $c_1(\mathfrak{s}_N)^2 = \sigma(N)$.
- (iii) $b_1(N) = 0$.
- (iv) $b^+(N) = 1$.
- (v) $b^+(M) > 1$.

Then

$$SW^{\mathbb{Z}_2}(X, \mathfrak{s}_X, f_X, \rho) = \begin{cases} \overline{SW}(M, \mathfrak{s}_M) \pmod{2} & \text{if } \det(f_N|_{H^+(N)}) = (-1)^{\rho+1}, \\ 0 & \text{if } \det(f_N|_{H^+(N)}) = (-1)^\rho. \end{cases}$$

We consider two special cases for (N, \mathfrak{s}_N, f_N) :

- (i) Let $N = S^2 \times S^2$, let \mathfrak{s}_N be the unique Spin^c -structure with $c_1(\mathfrak{s}_N) = 0$ and let f_N be given by $f_N(x, y) = (rx, ry)$, where $r: S^2 \rightarrow S^2$ is a reflection. Observe that $c_1(\mathfrak{s}_N)^2 = \sigma(N)$ and that f_N preserves \mathfrak{s}_N .
- (ii) Let $N = \mathbb{C}\mathbb{P}^2 \# \#^2 \overline{\mathbb{C}\mathbb{P}^2} = (S^2 \times S^2) \# \overline{\mathbb{C}\mathbb{P}^2}$. Then $H^3(N, \mathbb{Z}) \cong \mathbb{Z}^3$, where the first two factors of \mathbb{Z} correspond to $H^2(S^2 \times S^2, \mathbb{Z})$ and the last factor corresponds to $H^2(\overline{\mathbb{C}\mathbb{P}^2}, \mathbb{Z})$. Let \mathfrak{s}_N be the unique Spin^c -structure with $c_1(\mathfrak{s}_N) = (0, 0, 1)$. Then $c_1(\mathfrak{s}_N)^2 = \sigma(N)$. We define f_N as an equivariant connected sum of a diffeomorphism $f_1: S^2 \times S^2 \rightarrow S^2 \times S^2$ and $f_2: \overline{\mathbb{C}\mathbb{P}^2} \rightarrow \overline{\mathbb{C}\mathbb{P}^2}$ as follows. Let $f_1(x, y) = (y, x)$. Then f_1 is an involution with fixed-point set S^2 . Let $f_2[z_0, z_1, z_2] = [z_0, z_1, -z_2]$. Then f_2 is an involution with fixed-point set $\mathbb{C}\mathbb{P}^1 \cup \text{pt}$. Since f_1 and f_2 are involutions whose fixed-point sets contain 2-dimensional components, it is possible to form their equivariant connected sum $f_N = f_1 \# f_2$. Observe that f_N preserves \mathfrak{s}_N .

In both of these cases the diffeomorphism f_N does not act as the identity in a neighbourhood of any point. However, we can replace f_N with a diffeomorphism which is smoothly isotopic to f_N and which acts as the identity in a neighbourhood of some point. By abuse of notation we let this new diffeomorphism be denoted as f_N .

Corollary 9.6 *Let M be a compact oriented smooth 4-manifold with $b^+(M) > 1$. Let \mathfrak{s}_M be a Spin^c -structure with $d(M, \mathfrak{s}_M) = 0$. Let (N, \mathfrak{s}_N, f_N) be given by either (i) or (ii) above. Then*

$$SW^{\mathbb{Z}_2}(M \# N, \mathfrak{s}_M \# \mathfrak{s}_N, \text{id}_M \# f_N, 0) = \overline{SW}(M, \mathfrak{s}_M) \pmod{2}.$$

Theorem 9.7 *Let M and M' be compact simply connected smooth 4-manifolds with indefinite intersection forms. Suppose that M and M' are homeomorphic and*

fix a homeomorphism $\phi: M \rightarrow M'$. Suppose that $b^+(M) > 1$ and that \mathfrak{s}_M is a Spin^c -structure on M with $d(M, \mathfrak{s}_M) = 0$ and that $\text{SW}(M, \mathfrak{s}_M) \not\equiv \text{SW}(M', \phi(\mathfrak{s}_M)) \pmod{2}$. Lastly, suppose that $X = M \# (S^2 \times S^2)$ is diffeomorphic to $M' \# (S^2 \times S^2)$. Then there exists a diffeomorphism on X which is continuously isotopic to the identity but not smoothly isotopic.

Proof Let (N, \mathfrak{s}_N, f_N) be as in (i) above. The homeomorphism $\phi: M \rightarrow M'$ determines a homeomorphism $\phi': M \# (S^2 \times S^2) \rightarrow M' \# (S^2 \times S^2)$. Since M is indefinite, using a theorem of Wall [24, Theorem 2], we can choose the diffeomorphism $\psi: M \# (S^2 \times S^2) \rightarrow M' \# (S^2 \times S^2)$ so that ψ and ϕ' induce the same action on cohomology groups. In particular, we have $\psi(\mathfrak{s}_M \# \mathfrak{s}_N) = \phi(\mathfrak{s}_M) \# \mathfrak{s}_N$.

To prove the result, it suffices to show there are diffeomorphisms f_1 and f_2 of X which induce the same map on $H^2(X, \mathbb{Z}_2)$ and are therefore continuously isotopic by [18], but which have different families Seiberg–Witten invariants. Then $f_1 \circ f_2^{-1}$ will be continuously isotopic to the identity but not smoothly.

Let $f_1 = \text{id}_M \# f_N$ and let $f_2 = \psi^{-1} \circ (\text{id}_{M'} \# f_N) \circ \psi$. Then f_1 and f_2 act the same way on $H^2(X, \mathbb{Z})$ since $\psi^* = (\phi')^*$ commutes with $(f_N)^*$. It remains to check that f_1 and f_2 have different Seiberg–Witten invariants. We have

$$\text{SW}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, f_1, 0) = \text{SW}(M, \mathfrak{s}_M) \pmod{2}.$$

On the other hand, since ψ gives an isomorphism between the mapping cylinders of $\psi^{-1} \circ (\text{id}_{M'} \# f_N) \circ \psi$ and $\text{id}_{M'} \# f_N$, we find

$$\begin{aligned} \text{SW}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, f_2, 0) &= \text{SW}^{\mathbb{Z}_2}(M \# (S^2 \times S^2), \mathfrak{s}_M \# \mathfrak{s}_N, \psi^{-1} \circ (\text{id}_{M'} \# f_N) \circ \psi, 0) \\ &= \text{SW}^{\mathbb{Z}_2}(M' \# (S^2 \times S^2), \phi(\mathfrak{s}) \# \mathfrak{s}_N, \text{id}_{M'} \# f_N, 0) \\ &= \text{SW}(M', \phi(\mathfrak{s}_M)) \pmod{2}. \end{aligned}$$

This completes the proof, since $\text{SW}(M, \mathfrak{s}_M) \not\equiv \text{SW}(M', \phi(\mathfrak{s}_M)) \pmod{2}$. □

Corollary 9.8 *The following 4-manifolds admit diffeomorphisms which are continuously isotopic to the identity, but are not smoothly isotopic to the identity:*

- (i) $X = \#^n(S^2 \times S^2) \# \#^n(K3)$ for any $n \geq 2$.
- (ii) $X = \#^{2n} \mathbb{C}P^2 \# \#^m \overline{\mathbb{C}P^2}$ for any $n \geq 2$ and any $m \geq 10n + 1$.

Proof We apply the previous corollary, where in case (i) M is the elliptic surface $E(2n)$ and $M' = \#^{n-1}(S^2 \times S^2) \# \#^n(K3)$ and in case (ii) $M = E(n) \# \#^{m-10n} \overline{\mathbb{C}P^2}$

and $M' = \#^{2n-1} \mathbb{C}P^2 \# \#^{m-1} \overline{\mathbb{C}P}^2$. Then use the fact that if V is a simply connected elliptic surface, then $V \# (S^2 \times S^2)$ and $V \# \mathbb{C}P^2$ both dissolve (see Corollaries 8 and 9 of [8]). □

9.4 The mod 2 Seiberg–Witten invariant for spin structures

In this section we will use the gluing formula to give a simple new proof of a theorem of Morgan and Szabó:

Theorem 9.9 [14] *Let M be smooth compact spin 4–manifold with $b_1(M) = 0$, $b^+(M) = 4n - 1$ and $b^-(M) = 20n - 1$, where $n \geq 1$. Let \mathfrak{s}_M be a Spin^c –structure on M which comes from a spin structure. Then $\text{SW}(M, \mathfrak{s}_M)$ is odd if $n = 1$ and is even otherwise.*

Remark 9.10 For a compact spin 4–manifold with $b_1(M) = 0$, the conditions $b^+(M) = 4n - 1$ and $b^-(M) = 20n - 1$ for some $n \geq 1$ are equivalent to requiring that $d(M, \mathfrak{s}_M) = 0$.

Proof Let $N = \#^{n-1} (S^2 \times S^2) \# \#^n (K3)$ and note that $b_1(N) = 0$, $b^+(N) = 4n - 1$ and $b^-(N) = 20n - 1$. Let \mathfrak{s}_N be the unique Spin^c –structure on N which comes from a spin structure. Then $\text{SW}(N, \mathfrak{s}_N)$ is odd if $n = 1$ (since in this case $N = K3$) and is zero if $n > 1$ (by the vanishing of the Seiberg–Witten invariants on connected sums with each summand having $b^+ > 0$). Thus it suffices to prove that $\text{SW}(M, \mathfrak{s}_M) = \text{SW}(N, \mathfrak{s}_N) \pmod{2}$. Let \bar{N} denote N with the opposite orientation and $\mathfrak{s}_{\bar{N}}$ the unique Spin^c –structure on \bar{N} which comes from the spin structure. Let $X = M \# \bar{N} \# N$ and $\mathfrak{s}_X = \mathfrak{s}_M \# \mathfrak{s}_{\bar{N}} \# \mathfrak{s}_N$. Consider the trivial family $E_X = X \times \mathbb{R}P^{24n-2}$ over $\mathbb{R}P^{24n-2}$ equipped with the twisting class ρ given by the unique nontrivial element of $H^1(\mathbb{R}P^{24n-2}, \mathbb{Z}_2)$. Identify $H^{24n-2}(\mathbb{R}P^{24n-2}, \mathbb{Z}_2)$ with \mathbb{Z}_2 . We will evaluate $\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho)$ using the gluing formula in two different ways. First, writing X as

$$X = M \# (\bar{N} \# N)$$

and noting that $\sigma(\bar{N} \# N) = 0$, $b^+(\bar{N} \# N) = 24n - 2$, we see from the gluing formula that

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho) = \text{SW}(M, \mathfrak{s}_M) \pmod{2}.$$

On the other hand, writing X as

$$X = N \# (\bar{N} \# M)$$

we similarly find that

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho) = \text{SW}(N, \mathfrak{s}_N) \pmod{2},$$

which completes the proof. □

9.5 Relations between mod 2 Seiberg–Witten invariants

The argument used in the previous section to compute mod 2 Seiberg–Witten invariants for spin structures can be applied more generally. As a result we find a surprising relation between the mod 2 Seiberg–Witten invariants on different 4–manifolds. To state the result, let us first describe the setup.

Let M, M' and N be compact, smooth, oriented 4–manifolds. Let B be a compact smooth manifold of dimension d and suppose we have smooth fibrewise oriented families $\pi_M: E_M \rightarrow B, \pi_{M'}: E_{M'} \rightarrow B$ and $\pi_N: E_N \rightarrow B$. Suppose we are given:

- (1) Sections $\iota_M: B \rightarrow E_M$ and $\iota_{M'}: B \rightarrow E_{M'}$ and a pair of sections $\iota_N, \iota'_N: B \rightarrow E_N$ with disjoint images.
- (2) Orientation-reversing isomorphisms between the normal bundles of (ι_M, ι_N) and $(\iota_{M'}, \iota'_N)$.

Let $X = M \# N \# M'$. From the above data, we can form the families connected sum of $E_M, E_{M'}$ and E_N .

Let $\mathfrak{s}_M, \mathfrak{s}_{M'}$ and \mathfrak{s}_N be Spin^c –structures on M, M' and N and let \mathfrak{s}_X be the corresponding Spin^c –structure on X . Assume that:

- (3) The Spin^c –structures $\mathfrak{s}_M, \mathfrak{s}_{M'}$ and \mathfrak{s}_N are monodromy-invariant up to a common sign factor $\rho: \pi_1(B) \rightarrow \mathbb{Z}_2$.

Assume further that:

- (4) $b^+(M) = b^+(M')$.
- (5) $b_1(M) = b_1(M') = b_1(N) = 0$.
- (6) $c_1(\mathfrak{s}_M)^2 - \sigma(M) = c_1(\mathfrak{s}_{M'})^2 - \sigma(M') = 4(b^+(M) + 1) = -c_1(\mathfrak{s}_N)^2 + \sigma(N)$.
- (7) $0 < b^+(N) + b^+(M) \leq \dim B = d$.
- (8) $2b^+(M) > (\dim B - b^+(N)) + 1$.

Then we have a well-defined Seiberg–Witten invariant for the family E_X which takes values in $H^{b^+(M)+b^+(N)}(B, \mathbb{Z}_2)$.

Proposition 9.11 Under the above assumptions we have the following equality in $H^{b^+(M)+b^+(N)}(B, \mathbb{Z}_2)$:

$$(9-1) \quad \text{SW}(M, \mathfrak{s}_M)w_{b^+(M)}(H^+(M') \otimes \mathbb{R}_\rho)w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_\rho) \\ = \text{SW}(M', \mathfrak{s}_{M'})w_{b^+(M)}(H^+(M) \otimes \mathbb{R}_\rho)w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_\rho).$$

Remark 9.12 Let $\pi_M: E_M \rightarrow B$ satisfy the above assumptions. Since

$$c_1(\mathfrak{s}_M)^2 - \sigma(M) > 0,$$

the arguments used in [2] implies that $w_{b^+(M)}(H^+(M)) = 0$ in the case that ρ is trivial. In the case that ρ is nontrivial, the same argument used in [2] can be adapted, provided $c_1(\mathfrak{s}_M)^2 - \sigma(M) = 8 \pmod{16}$, to show that $w_{b^+(M)}(H^+(M) \otimes \mathbb{R}_\rho) = 0$. Thus the statement of Proposition 9.11 is not vacuous only when ρ is nontrivial and $c_1(\mathfrak{s}_M)^2 - \sigma(M) = 0 \pmod{16}$. Note that this also implies $b^+(M) = 3 \pmod{4}$.

Proof We write $X = M \# N \# M'$ in two different ways, as $X = M \# (N \# M')$ and as $X = M' \# (N \# M)$. Applying the gluing formula to each of these and equating gives (9-1). □

Corollary 9.13 Let M and M' be compact, smooth, oriented 4–manifolds. Let B be a compact smooth manifold and suppose we have smooth fibrewise oriented families $\pi_M: E_M \rightarrow B$ and $\pi_{M'}: E_{M'} \rightarrow B$. Suppose there exist sections $\iota_M: B \rightarrow E_M$ and $\iota_{M'}: B \rightarrow E_{M'}$ whose normal bundles are trivial. Let \mathfrak{s}_M and $\mathfrak{s}_{M'}$ be Spin^c –structures on M and M' . Assume that \mathfrak{s}_M and $\mathfrak{s}_{M'}$ are monodromy-invariant up to a common sign factor $\rho: \pi_1(B) \rightarrow \mathbb{Z}_2$. Assume further that:

- (4) $b^+(M) = b^+(M')$.
- (5) $b_1(M) = b_1(M') = 0$.
- (6) $c_1(\mathfrak{s}_M)^2 - \sigma(M) = c_1(\mathfrak{s}_{M'})^2 - \sigma(M') = 4(b^+(M) + 1)$.
- (7) $b^+(M) = 3 \pmod{4}$.
- (8) $1 + \frac{1}{2} \dim B \leq b^+(M) \leq \dim B$.

Then we have the following equality in $H^{b^+(M)}(B, \mathbb{Z}_2)$:

$$\text{SW}(M, \mathfrak{s}_M)w_{b^+(M)}(H^+(M') \otimes \mathbb{R}_\rho) = \text{SW}(M', \mathfrak{s}_{M'})w_{b^+(M)}(H^+(M) \otimes \mathbb{R}_\rho).$$

Proof Since $b^+(M) = 3 \pmod{4}$, we have $4(b^+(M) + 1) = 16k$ for some $k \geq 0$. Let $N = \#^k \overline{K3}$. Let E_N be the trivial family over B . Set $B' = B \times \mathbb{R}P^{19k}$. Set

$\rho' = (\rho, 1) \in H^1(B', \mathbb{Z}_2) = H^1(B, \mathbb{Z}_2) \oplus \mathbb{Z}_2$. Pull the families $E_M, E_{M'}$ and E_N back to B' and apply Proposition 9.11. The result now follows, noting that in this case the factor $w_{b+(N)}(H^+(N) \otimes \mathbb{R}_\rho)$ can be cancelled from both sides of (9-1). \square

As an illustration of this result, we find, remarkably, that the existence of a certain type of involution on a 4–manifold M implies M has a nonvanishing Seiberg–Witten invariant:

Corollary 9.14 *Let M be a compact smooth 4–manifold with $b_1(M) = 0$ and $b^+(M) = 3$. Suppose that $f: M \rightarrow M$ is an involutive diffeomorphism of M whose fixed-point set contains an isolated point. Suppose \mathfrak{s}_M is a Spin^c –structure such that $f(\mathfrak{s}_M) = -\mathfrak{s}_M$ and $c_1(\mathfrak{s}_M)^2 - \sigma(M) = 16$. Then $\text{SW}(M, \mathfrak{s}_M) = 1 \pmod{2}$ if f acts trivially on $H^+(M)$ and $\text{SW}(M, \mathfrak{s}_M) = 0 \pmod{2}$ otherwise.*

Proof Let $B = \mathbb{R}\mathbb{P}^3$. We consider the family $E_M = M \times_{\mathbb{Z}_2} S^3$, where \mathbb{Z}_2 acts on S^3 by the antipodal map. Let $m \in M$ be an isolated fixed point. Then m defines a section of E_M whose normal bundle is isomorphic to \mathbb{R}_- , where \mathbb{R}_- is the unique nontrivial line bundle on $\mathbb{R}\mathbb{P}^3$. This is easily seen to be a trivialisable bundle. Now repeat this construction for $M' = K3$ equipped with an involution f' with isolated fixed points and such that f' acts on $H^+(M')$ trivially (such an involution exists; see eg [4]). Take $\mathfrak{s}_{M'}$ to be the Spin^c –structure with $c_1(\mathfrak{s}_{M'}) = 0$. Take ρ to be the unique nontrivial element of $H^1(\mathbb{R}\mathbb{P}^3, \mathbb{Z}_2)$. Then, applying Corollary 9.13, we get $\text{SW}(M, \mathfrak{s}_M) = \text{SW}(K3, \mathfrak{s}_{M'}) = 1 \pmod{2}$ if f acts trivially on $H^+(M)$ and $\text{SW}(M, \mathfrak{s}_M) = 0 \pmod{2}$ otherwise. \square

Proposition 9.15 *Let M be a compact, smooth 4–manifold with $b_1(M) = 0$, $b^+(M) = 3 \pmod{4}$ and $b^+(M) \neq 3$. Suppose that $f: M \rightarrow M$ is an involutive diffeomorphism of M whose fixed-point set contains an isolated point. Suppose \mathfrak{s}_M is a Spin^c –structure such that $f(\mathfrak{s}_M) = -\mathfrak{s}_M$ and $c_1(\mathfrak{s}_M)^2 - \sigma(M) = 4(b^+(M) + 1)$. Then $\text{SW}(M, \mathfrak{s}_M) = 0 \pmod{2}$.*

Proof First we construct an involution on $K3$ as follows. Recall the Kummer construction of $K3$. Start with the 4–torus $T^4 = \mathbb{C}^2/(\mathbb{Z}^2 + i\mathbb{Z}^2)$ equipped with the involution $i: T^4 \rightarrow T^4$ defined by $i((z_1, z_2)) = -(z_1, z_2)$. There are 16 fixed points of i . Take the blowup $Y = T^4 \# \#^{16} \mathbb{C}\mathbb{P}^2$ of T^4 at the fixed points. Then i extends to an involution on Y with only nonisolated fixed points (16 spheres each with self-intersection -1). The quotient $K3 = Y/\langle i \rangle$ can be given the structure of a smooth

manifold and is a $K3$ surface. The -1 curves in Y become -2 curves in $K3$. Let $f: T^4 \rightarrow T^4$ be the involution $f(z_1, z_2) = (z_1 + \frac{1}{2}, z_2)$. Then f commutes with i and acts freely on the fixed-point set of i . It follows that f determines an involution on Y which permutes the sixteen -1 curves in Y in pairs. Since f commutes with i , we have that f descends to an involution $f: K3 \rightarrow K3$ which permutes the sixteen -2 curves in $K3$. Note that f has 8 isolated fixed points and by [4] acts as the identity on $H^+(K3)$. Let S_1 and S_2 be two of the -2 curves in $K3$, which may be chosen so that $S_1, S_2, f(S_1)$ and $f(S_2)$ are pairwise disjoint. Let $c = 2[S_1] + 2[S_2] - 2[f(S_1)] - 2[f(S_2)]$. Then $c = c_1(\mathfrak{s}_{K3})$ for a uniquely determined characteristic on $K3$. Moreover, $f(c) = -c$, which implies that $f(\mathfrak{s}_{K3}) = -\mathfrak{s}_{K3}$. Also we have $c^2 = -32$, so that $-c_1(\mathfrak{s}_{K3})^2 + \sigma(K3) = 16$.

Now define $k \geq 2$ by $16k = 4(b^+(M) + 1)$ and set $N = \#^k K3$. We equip N with the involution f_N which is the equivariant connected sum of k copies of $f: K3 \rightarrow K3$. Also we take $\mathfrak{s}_N = \#^k \mathfrak{s}_{K3}$. Then $f_N(\mathfrak{s}_N) = -\mathfrak{s}_N$ and $-c_1(\mathfrak{s}_N)^2 + \sigma(N) = 16k$.

Next, suppose that $(M', f_{M'}, \mathfrak{s}_{M'})$ also satisfies the hypotheses of the proposition. In fact we will take $(M', f_{M'})$ to be an equivariant connected sum of the form $M' = \#^{k-1}(S^2 \times S^2) \# \#^k(K3)$, where on each factor of $K3$ we use the involution f and on each factor of $S^2 \times S^2$ we use the involution $\phi \times \phi$, where $\phi: S^2 \rightarrow S^2$ is a rotation by π about some axis. Note that $\phi \times \phi$ has 4 isolated fixed points. We also take $\mathfrak{s}_{M'}$ to be the Spin^c -structure with characteristic zero. Now we apply Proposition 9.11, where $B = \mathbb{R}P^{7k-1}$ and where $E_M, E_{M'}$ and E_N are the families over B associated to the involutions on M, M' and N . Also ρ is taken to be the unique nontrivial element of $H^1(\mathbb{R}P^{7k-1}, \mathbb{Z}_2)$. We perform the families connected sum using the isolated fixed points. Proposition 9.11 now gives $\text{SW}(M, \mathfrak{s}_M) = \text{SW}(M', \mathfrak{s}_{M'}) = 0 \pmod{2}$ if f acts trivially on $H^+(M)$ and $\text{SW}(M, \mathfrak{s}_M) = 0 \pmod{2}$ otherwise. In either case, we get $\text{SW}(M, \mathfrak{s}_M) = 0 \pmod{2}$. □

Remark 9.16 It is worth comparing Proposition 9.15 with the following similar but much more elementary result: Let M be a compact smooth 4-manifold with $b_1(M) = 0, b^+(M) = 1 \pmod{4}$ and $b^+(M) > 1$. Suppose that $f: M \rightarrow M$ is a diffeomorphism of M such that $\det(f|_{H^+(M)}) = 1$ and suppose \mathfrak{s}_M is a Spin^c -structure such that $f(\mathfrak{s}_M) = -\mathfrak{s}_M$ and $c_1(\mathfrak{s}_M)^2 - \sigma(M) = 4(b^+(M) + 1)$. Then $\text{SW}(M, \mathfrak{s}_M) = 0$. The proof is simple: First, by charge conjugation symmetry we have

$$\text{SW}(M, \mathfrak{s}_M) = (-1)^{(b^+(M)+1)/2} \text{SW}(M, -\mathfrak{s}_M) = -\text{SW}(M, -\mathfrak{s}_M).$$

On the other hand, since $\det(f|_{H^+(M)}) = 1$, we have by diffeomorphism-invariance that

$$SW(M, \mathfrak{s}_M) = SW(M, f(\mathfrak{s}_M)) = SW(M, -\mathfrak{s}_M).$$

Hence $SW(M, \mathfrak{s}_M) = 0$.

Using the blowup formula for Seiberg–Witten invariants [7, Theorem 14.1.1], we can strengthen Proposition 9.15 as follows:

Corollary 9.17 *Let M be a compact smooth 4–manifold with*

$$b_1(M) = 0, \quad b^+(M) = 3 \pmod{4} \quad \text{and} \quad b^+(M) \not\equiv 3.$$

Suppose that $f: M \rightarrow M$ is an involutive diffeomorphism of M whose fixed-point set contains an isolated point. Suppose \mathfrak{s}_M is a Spin^c –structure such that $f(\mathfrak{s}_M) = -\mathfrak{s}_M$ and $c_1(\mathfrak{s}_M)^2 - \sigma(M) = 0 \pmod{16}$. Then $SW(M, \mathfrak{s}_M) = 0 \pmod{2}$.

Proof We can assume $c_1(\mathfrak{s}_M)^2 - \sigma(M) \geq 4(b^+(M) + 1)$, otherwise $d(M, \mathfrak{s}_M) < 0$. Let $c_1(\mathfrak{s}_M)^2 - \sigma(M) - 4(b^+(M) + 1) = 16k$, where $k \geq 1$. Taking the blowup of M at k pairs of nonfixed points $(m_1, f(m_1)), \dots, (m_k, f(m_k))$ we obtain an involution f' on $M' = M \# \#^{2k} \overline{\mathbb{C}\mathbb{P}^2}$. Let $\mathfrak{s}_{M'} = \mathfrak{s}_M \# \mathfrak{s}$, where $c_1(\mathfrak{s}) = (3, -3, 3, -3, \dots, 3, -3)$. Then $(M', f', \mathfrak{s}_{M'})$ satisfies the conditions of Proposition 9.15, so $SW(M', \mathfrak{s}_{M'}) = 0 \pmod{2}$. But $SW(M, \mathfrak{s}_M) = SW(M', \mathfrak{s}_{M'})$ by the blowup formula for Seiberg–Witten invariants [7, Theorem 14.1.1]. □

Using the same blowup argument as above we can similarly strengthen Corollary 9.14:

Corollary 9.18 *Let M be a compact smooth 4–manifold with $b_1(M) = 0$ and $b^+(M) = 3$. Suppose that $f: M \rightarrow M$ is an involutive diffeomorphism of M whose fixed-point set contains an isolated point. Suppose \mathfrak{s}_M is a Spin^c –structure such that $f(\mathfrak{s}_M) = -\mathfrak{s}_M$, $c_1(\mathfrak{s}_M)^2 - \sigma(M) = 0 \pmod{16}$ and $c_1(\mathfrak{s}_M)^2 - \sigma(M) \geq 16$. Then $SW(M, \mathfrak{s}_M) = 1 \pmod{2}$ if f acts trivially on $H^+(M)$ and $SW(M, \mathfrak{s}_M) = 0 \pmod{2}$ otherwise.*

Example 9.19 Suppose M is a compact smooth spin 4–manifold with $b_1(M) = 0$, $b^+(M) = 3$ and $\sigma(M) \leq -32$. Suppose that $f: M \rightarrow M$ is a smooth involution which has an isolated fixed point. Then f acts nontrivially on $H^+(M)$. To see this, suppose on the contrary that f acts trivially on $H^+(M)$. Then, by Corollary 9.18, we would find that $SW(M, \mathfrak{s}_M) = 1 \pmod{2}$, where \mathfrak{s}_M is a spin structure on M . But, since $d(M, \mathfrak{s}_M) > 0$ and $b^+(M) = 3$, we can deduce that $SW(M, \mathfrak{s}_M) = 0 \pmod{2}$ by [3, Corollary 3.6].

9.6 Seiberg–Witten invariants of group actions

Let X be a smooth compact oriented 4–manifold and let G be a group acting on X by orientation-preserving diffeomorphisms. Suppose that G preserves the isomorphism class of a Spin^c –structure \mathfrak{s}_X . Suppose that $d(X, \mathfrak{s}_X) \leq 0$ and that $b^+(X) > -d(X, \mathfrak{s}_X) + 1$. Recall from Section 3 that in this situation we may define the G –equivariant Seiberg–Witten invariant $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X) \in H^{-d(X, \mathfrak{s}_X)}(BG, \mathbb{Z}_2)$. More generally, such an invariant is defined if G only preserves \mathfrak{s}_X up to a sign factor $\rho: G \rightarrow \mathbb{Z}_2$ and we again obtain an invariant $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) \in H^{-d(X, \mathfrak{s}_X)}(BG, \mathbb{Z}_2)$.

Recall also from Section 3 that $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho)$ is related to the families Seiberg–Witten invariant as follows. Let B be a smooth compact manifold with $\dim B < b^+(X) - 1$ and let $E \rightarrow B$ be a principal G –bundle over B . Let $E_X = E \times_G X \rightarrow B$ be the associated family of 4–manifolds over B . Then \mathfrak{s}_X defines

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X) = f^*(\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X)),$$

where $f: B \rightarrow BG$ is the classifying map of $E \rightarrow B$.

Next, we will use the gluing formula to compute $\text{SW}_G^{\mathbb{Z}_2}$ under some conditions. Suppose first that G acts smoothly on a smooth compact 4–manifold N . Then G acts on $H^2(N, \mathbb{R})$ preserving the intersection form. We therefore have an associated vector bundle $\mathbb{H}^2(N, \mathbb{R}) = EG \times_G H^2(N, \mathbb{R})$ with structure group $O(b^+(N), b^-(N))$. We can reduce this to the maximal compact subgroup $O(b^+(N)) \times O(b^-(N))$, which amounts to decomposing $\mathbb{H}^2(N, \mathbb{R})$ into the direct sum $\mathbb{H}^+(N) \oplus \mathbb{H}^-(N)$ of a maximal positive-definite subbundle and its orthogonal complement. We will abuse notation and write $H^+(N)$ instead of $\mathbb{H}^+(N)$. In particular, we may consider the top Stiefel–Whitney class of $H^+(N)$,

$$w_{b^+(N)}(H^+(N)) \in H^{b^+(N)}(BG, \mathbb{Z}_2).$$

Theorem 9.20 (gluing formula for $\text{SW}_G^{\mathbb{Z}_2}$) *Suppose that $X = M \# N$ is a G –equivariant connected sum, where M and N are smooth compact 4–manifolds equipped with an action of G by diffeomorphisms. Suppose that G preserves a Spin^c –structure $\mathfrak{s}_X = \mathfrak{s}_M \# \mathfrak{s}_N$ up to a sign factor given by a homomorphism $\rho: G \rightarrow \mathbb{Z}_2$. Suppose in addition that the following assumptions hold:*

- (i) $d(M, \mathfrak{s}_M) = 0$.
- (ii) $c_1(\mathfrak{s}_N)^2 = \sigma(N)$.

- (iii) $b_1(N) = 0$.
- (iv) $b^+(M) > 1$.

Then $\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) \in H^{b^+(N)}(BG, \mathbb{Z}_2)$ is defined and given by

$$(9-2) \quad \text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) = \text{SW}(M, \mathfrak{s}_M) \cdot w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_\rho),$$

where \mathbb{R}_ρ is the real line bundle over BG determined by $\rho \in H^1(BG, \mathbb{Z}_2)$.

Proof To show (9-2) it suffices by the universal coefficient theorem to show that

$$\langle \text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho), \beta \rangle = \langle \text{SW}(M, \mathfrak{s}_M) \cdot w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_\rho), \beta \rangle$$

for every $\beta \in H_{-d(X, \mathfrak{s}_X)}(G, \mathbb{Z}_2)$. By the solution to the Steenrod problem with \mathbb{Z}_2 -coefficients, there exists a compact smooth manifold B of dimension $-d(X, \mathfrak{s}_X)$ and a continuous map $f: B \rightarrow BG$ such that $f_*[B] = \beta$. Let $E = f^*(EG)$ be the pullback of the universal bundle $EG \rightarrow BG$ and let $E_X = E \times_G X$ be the family associated to E . Then

$$\langle \text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho), \beta \rangle = \langle f^*(\text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X), \rho), [B] \rangle = \langle \text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho), [B] \rangle.$$

But, since $X = M \# N$ is an equivariant connected sum, it follows that E_X is the families connected sum of $E_M = E \times_G M$ and $E_N = E \times_G N$. Thus we are in the setting of the families gluing formula, which gives

$$\text{FSW}^{\mathbb{Z}_2}(E_X, \mathfrak{s}_X, \rho) = \text{SW}(M, \mathfrak{s}_M) \cdot w_{b^+(N)}(f^*(H^+(N) \otimes \mathbb{R}_\rho)).$$

Hence we have that

$$\begin{aligned} \langle \text{SW}_G^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho), \beta \rangle &= \text{SW}(M, \mathfrak{s}_M) \langle f^*(w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_\rho)), [B] \rangle \\ &= \text{SW}(M, \mathfrak{s}_M) \langle w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_\rho), \beta \rangle, \end{aligned}$$

as required. □

9.7 Seiberg–Witten invariants of involutions

In this section we take $G = \mathbb{Z}_2 = \langle f \rangle$. The cohomology ring of $B\mathbb{Z}_2$ with \mathbb{Z}_2 -coefficients is given by

$$H^*(B\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2[u],$$

where $u \in H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$ classifies the unique nontrivial real line bundle \mathbb{R}_- over $B\mathbb{Z}_2$.

Let $\rho: G \rightarrow \mathbb{Z}_2$ be a homomorphism. Thus either ρ is the trivial homomorphism, which we write as $\rho = 0$, or ρ is the identity homomorphism, which we write as $\rho = 1$. Note that $\mathbb{R}_\rho = \mathbb{R}$ for $\rho = 0$ and $\mathbb{R} = \mathbb{R}_-$ for $\rho = 1$.

Let N be a smooth compact 4–manifold. An action of G on N by diffeomorphisms is equivalent to giving an involutive diffeomorphism $f_N: N \rightarrow N$. Since G is finite, we can always choose a G –invariant metric on N . Then G acts on $H^+(N)$. We then find that $w_{b^+(N)}(H^+(N))$ is nonzero if and only if f_N acts as $-\text{id}$ on $H^+(N)$ and similarly $w_{b^+(N)}(H^+(N) \otimes \mathbb{R}_-)$ is nonzero if and only if f_N acts as $+\text{id}$ on $H^+(N)$.

Proposition 9.21 *Suppose $X = M \# N$ is a \mathbb{Z}_2 –equivariant connected sum, where M and N are smooth, compact 4–manifolds equipped with involutions $f_M: M \rightarrow M$ and $f_N: N \rightarrow N$. Suppose that there are Spin^c –structures \mathfrak{s}_M and \mathfrak{s}_N and a $\rho \in \mathbb{Z}_2$ such that*

$$f_M^*(\mathfrak{s}_M) = (-1)^\rho \mathfrak{s}_M, \quad f_N^*(\mathfrak{s}_N) = (-1)^\rho \mathfrak{s}_N.$$

Set $f = f_M \# f_N$ and $\mathfrak{s}_X = \mathfrak{s}_M \# \mathfrak{s}_N$. Suppose in addition that the following assumptions hold:

- (i) $d(M, \mathfrak{s}_M) = 0$.
- (ii) $c_1(\mathfrak{s}_N)^2 = \sigma(N)$.
- (iii) $b_1(N) = 0$.
- (iv) $b^+(M) > 1$.

Then $\text{SW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) \in H^{b^+(N)}(B\mathbb{Z}_2, \mathbb{Z}_2)$ is defined and given by

$$\text{SW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) = \begin{cases} \text{SW}(M, \mathfrak{s}_M)u^{b^+(N)} & \text{if } \rho = 0 \text{ and } f_N|_{H^+(N)} = -\text{id}, \\ \text{SW}(M, \mathfrak{s}_M)u^{b^+(N)} & \text{if } \rho = 1 \text{ and } f_N|_{H^+(N)} = +\text{id}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 9.22 Let $f_M: M \rightarrow M$ be an orientation-preserving smooth involution on M which is not the identity map. Let F denote the fixed-point set of f_M . If $m \in F$ is a fixed point, then f_M acts on $T_m M$ with eigenvalues $+1$ and -1 . Since the orientation is preserved, the number of -1 eigenvalues must be 2 or 4. Thus, in some basis, f_M acts either as $\text{diag}(1, 1, -1, -1)$ or $\text{diag}(-1, -1, -1, -1)$. In the first case m lies on a 2–dimensional component of F and we say m is a nonisolated fixed point. In the second case m is an isolated point of F and we say m is an isolated fixed point.

Now suppose that $f_N: N \rightarrow N$ is an orientation-preserving smooth involution of N . We can form the equivariant connected sum $X = M \# N$ provided f_M and f_N have fixed points of the same type (either both isolated or both nonisolated) and we write $f = f_M \# f_N$ for the resulting involution on X . It is important to note that the isomorphism class of the pair (X, f) will in general depend on the choice of fixed points of M and N used to perform the connected sum. However, in the examples where we compute $\text{SW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho)$, the result will not depend on the choice of fixed point, since the gluing formula is insensitive to this choice.

To make use of [Proposition 9.21](#), we look for some examples of pairs (N, f_N) for which $w_{b+(N)}(H^+(N) \otimes \mathbb{R}_\rho) \neq 0$. By the above remark it is also important to keep track of the dimensions of the connected components of the fixed-point set of f_N . Our examples are listed in [Table 1](#). In this table we use some notation, r, ϕ, ι and e , as we now explain. We let r denote a reflection $r: S^2 \rightarrow S^2$, let $\phi: S^2 \rightarrow S^2$ denote a rotation by π around some axis and let $\iota: \overline{\mathbb{C}\mathbb{P}^2} \rightarrow \overline{\mathbb{C}\mathbb{P}^2}$ be the map $\iota[z_0, z_1, z_2] = [z_0, z_1, -z_2]$. In the last three rows of the table we view $\mathbb{C}\mathbb{P}^2 \# \#^3 \overline{\mathbb{C}\mathbb{P}^2}$ as $(S^2 \times S^2) \# \#^2 \overline{\mathbb{C}\mathbb{P}^2}$ and construct the involution f_N as follows. In row 4, f_N is constructed as an equivariant connected sum. Note that the fixed-point set of ι has components of dimension 0 and 2. We form the connected sum using only nonisolated fixed points. In rows 5 and 6, we start with an involution i on $(S^2 \times S^2)$. Let p be a nonfixed point of i . Then we attach a copy of $\overline{\mathbb{C}\mathbb{P}^2}$ at x and $i(x)$ and denote the resulting involution on the connected sum as i' . The last column of the table gives the characteristic of a Spin^c -structure \mathfrak{s}_N satisfying $f_N(\mathfrak{s}_N) = (-1)^\rho \mathfrak{s}_N$ and $c_1(\mathfrak{s}_N)^2 = \sigma(N)$. In the last three rows we use the isomorphism

$$H^2(\mathbb{C}\mathbb{P}^2 \# \#^3 \overline{\mathbb{C}\mathbb{P}^2}, \mathbb{Z}) \cong H^2(S^2 \times S^2, \mathbb{Z}) \oplus H^2(\mathbb{C}\mathbb{P}^2, \mathbb{Z}) \oplus H^2(\overline{\mathbb{C}\mathbb{P}^2}, \mathbb{Z})$$

and write e for a generator of $H^2(\mathbb{C}\mathbb{P}^2, \mathbb{Z})$.

Proposition 9.23 *Let M be a smooth compact 4-manifold with $b^+(M) > 1$ and let $f_M: M \rightarrow M$ be a smooth involution. Let \mathfrak{s}_M be a Spin^c -structure with $d(M, \mathfrak{s}_M) = 0$. Suppose that one of the following holds:*

- (i) f_M has an isolated fixed point and $f_M(\mathfrak{s}_M) = \mathfrak{s}_M$.
- (ii) f_M has a nonisolated fixed point and $f_M(\mathfrak{s}_M) = \mathfrak{s}_M$.
- (iii) f_M has an isolated fixed point and $f_M(\mathfrak{s}_M) = -\mathfrak{s}_M$.
- (iv) f_M has a nonisolated fixed point and $f_M(\mathfrak{s}_M) = -\mathfrak{s}_M$.

	N	f_N	dimensions of components of F	ρ	$c_1(\mathfrak{s}_N)$
1	$S^2 \times S^2$	$r \times r$	2	0	0
2	$S^2 \times S^2$	$\phi \times \phi$	0	1	0
3	$S^2 \times S^2$	$\phi \times \text{id}$	2	1	0
4	$\mathbb{C}P^2 \# \#^3 \overline{\mathbb{C}P}^2$	$(r \times r) \# \iota \# \iota$	0, 2	0	$(0, e, e)$
5	$\mathbb{C}P^2 \# \#^3 \overline{\mathbb{C}P}^2$	$(\phi \times \phi)'$	0	1	$(0, e, -e)$
6	$\mathbb{C}P^2 \# \#^3 \overline{\mathbb{C}P}^2$	$(\phi \times \text{id})'$	2	1	$(0, e, -e)$

Table 1: Some pairs (N, f_N) with $w_{b+(N)}(H^+(N) \otimes \mathbb{R}_\rho) \neq 0$.

Then, for any integer $k \geq 1$:

- In cases (ii), (iii) and (iv) the manifolds $X = M \# \#^k(S^2 \times S^2)$ and $X = M \# \#^k \mathbb{C}P^2 \# \#^{k+2} \overline{\mathbb{C}P}^2$ each admit a smooth involution f_X and a Spin^c -structure \mathfrak{s}_X such that $\text{FSW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) \in H^k(B\mathbb{Z}_2, \mathbb{Z}_2)$ is defined, where $\rho = 0$ in case (ii), $\rho = 1$ in case (iii) or (iv) and

$$\text{FSW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) = \text{SW}(M, \mathfrak{s}_M)u^k.$$

- In case (i) the manifold $X = M \# \#^k(\mathbb{C}P^2 \# \#^3 \overline{\mathbb{C}P}^2)$ admits a smooth involution f_X and a Spin^c -structure \mathfrak{s}_X such that $\text{FSW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) \in H^k(B\mathbb{Z}_2, \mathbb{Z}_2)$ is defined, where $\rho = 0$ and

$$\text{FSW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho) = \text{SW}(M, \mathfrak{s}_M)u^k.$$

Proof We apply Proposition 9.21, where (N, f_N, \mathfrak{s}_N) is an equivariant connected sum of k summands, where each summand is one of the entries in Table 1.

- In case (i), we use k copies of (4) as our summands.
- In case (ii), we either use k copies of (1) or $k - 1$ copies of (1) and one copy of (4).
- In case (iii), we either use k copies of (2) or $k - 1$ copies of (2) and one copy of (5).
- In case (iv), we either use k copies of (3) or $k - 1$ copies of (3) and one copy of (6). □

Theorem 9.24 Let M be a smooth compact 4-manifold with $b^+(M) > 1$ and let $f_M: M \rightarrow M$ be a smooth involution. Let \mathfrak{s}_M be a Spin^c -structure with $d(M, \mathfrak{s}_M) = 0$. Suppose that one of the following holds:

- (i) f_M has an isolated fixed point and $f_M(\mathfrak{s}_M) = \mathfrak{s}_M$.
- (ii) f_M has a nonisolated fixed point and $f_M(\mathfrak{s}_M) = \mathfrak{s}_M$.
- (iii) f_M has an isolated fixed point and $f_M(\mathfrak{s}_M) = -\mathfrak{s}_M$.
- (iv) f_M has a nonisolated fixed point and $f_M(\mathfrak{s}_M) = -\mathfrak{s}_M$.

Suppose also that $\text{SW}(M, \mathfrak{s}_M)$ is odd. Then, for any integer $k \geq 1$:

- In cases (ii), (iii) and (iv) the manifolds $X = M \# \#^k(S^2 \times S^2)$ and $X = M \# \#^k \mathbb{C}P^2 \# \#^{k+2} \overline{\mathbb{C}P^2}$ each admit a smooth involution f_X and a Spin^c -structure \mathfrak{s}_X such that $\text{FSW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho)$ is nonzero. Hence X does not admit an f_X -invariant metric of positive scalar curvature.
- In cases (i), (ii), (iii) and (iv) the manifold $X = M \# \#^k(\mathbb{C}P^2 \# \#^3 \overline{\mathbb{C}P^2})$ admits a smooth involution f_X and a Spin^c -structure \mathfrak{s}_X such that $\text{FSW}_{\mathbb{Z}_2}^{\mathbb{Z}_2}(X, \mathfrak{s}_X, \rho)$ is nonzero. Hence X does not admit an f_X -invariant metric of positive scalar curvature.

Corollary 9.25 For every $n \geq 4$ and $m \geq n + 18$, the 4-manifold

$$X = \#^n \mathbb{C}P^2 \# \#^m \overline{\mathbb{C}P^2}$$

admits a metric of positive scalar curvature and a smooth involution $f: X \rightarrow X$ such that there does not exist an f -invariant metric of positive scalar curvature on X .

Remark 9.26 In the course of the proof we construct an explicit example of such an involution f . Many more examples could be produced by varying the choice of the pair (M, f_M) used below in the proof. A similar result to this was obtained by LeBrun in [11]. However, the proof in [11] uses a covering space argument which can only be used to construct examples where the involution f acts freely (in particular, the Euler characteristic must be even in such examples).

Proof That X has a metric of positive scalar curvature is clear. Let M_0 be a $K3$ surface and let $M = M_0 \# \#^{m-n-18} \overline{\mathbb{C}P^2}$. Then $M \# \#^{n-3} \mathbb{C}P^2 \# \#^{n-1} \overline{\mathbb{C}P^2}$ is diffeomorphic to X . We will find an involution f_M on M which antipreserves a Spin^c -structure \mathfrak{s}_M with $d(M, \mathfrak{s}_M) = 0$ and $\text{SW}(M, \mathfrak{s}_M) = 1 \pmod{2}$ and such that f_M has nonisolated fixed points.

Let us view M_0 as the quartic hypersurface in $\mathbb{C}P^3$ defined by $z_1^4 - z_2^4 - z_3^4 - z_4^4 = 0$. Clearly complex conjugation $[z_1, z_2, z_3, z_4] \mapsto [\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4]$ defines an orientation-preserving involution on M_0 with nonisolated fixed points. Since M_0 is spin we can

take \mathfrak{s}_{M_0} to be the unique spin structure, then \mathfrak{s}_{M_0} is antipreserved by f_{M_0} . Next, we blow M_0 up at $m - n - 18$ nonisolated fixed points. Then f_{M_0} extends to an involution f_M on the blowup $M = M_0 \# \#^{m-n-18} \overline{\mathbb{C}\mathbb{P}^2}$. The blowup can be viewed as an equivariant connected sum in the following way. Each copy of $\overline{\mathbb{C}\mathbb{P}^2}$ is equipped with the involution given by complex conjugation. This acts as -1 on $H^2(\overline{\mathbb{C}\mathbb{P}^2}, \mathbb{Z})$. Let $\mathfrak{s}_{\overline{\mathbb{C}\mathbb{P}^2}}$ be the Spin^c -structure whose characteristic is a generator of $H^2(\overline{\mathbb{C}\mathbb{P}^2}, \mathbb{Z})$. Then we take our Spin^c -structure \mathfrak{s}_M on M to be \mathfrak{s}_{M_0} summed with a copy of $\mathfrak{s}_{\overline{\mathbb{C}\mathbb{P}^2}}$ for each copy of $\overline{\mathbb{C}\mathbb{P}^2}$. Then \mathfrak{s}_M is clearly antipreserved by f_M . By the blowup formula for Seiberg–Witten invariants we have $\text{SW}(M, \mathfrak{s}_M) = \text{SW}(M_0, \mathfrak{s}_{M_0}) = 1 \pmod{2}$. To complete the proof, we apply [Theorem 9.24\(iv\)](#) with $k = n - 3$. \square

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