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Simplifying Weinstein Morse functions

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We prove that the minimum number of critical points of a Weinstein Morse function on a Weinstein domain of dimension at least six is at most two more than the minimum number of critical points of a smooth Morse function on that domain; if the domain has nonzero middle-dimensional homology, these two numbers agree. There is also an upper bound on the number of gradient trajectories between critical points in smoothly trivial Weinstein cobordisms. As an application, we show that the number of generators for the Grothendieck group of the wrapped Fukaya category is at most the number of generators for singular cohomology and hence vanishes for any Weinstein ball. We also give a topological obstruction to the existence of finite-dimensional representations of the Chekanov–Eliashberg DGA for Legendrians.

57R17; 53D37, 53D40, 57R80

1 Introduction and main results

Weinstein domains are exact symplectic manifolds equipped with Morse functions compatible with their symplectic structures. These domains encompass a large class of symplectic manifolds, eg cotangent bundles, and are closely related to Stein manifolds in complex geometry; see Cieliebak and Eliashberg [7]. The Weinstein Morse function gives a symplectic handlebody presentation of the domain and allows one to study its symplectic geometry via high-dimensional Legendrian knot theory. This handlebody presentation is not unique and, like a smooth handlebody presentation, a Weinstein handlebody presentation can be modified by a series of moves, or *Weinstein homotopies*, that preserve the symplectic structure of the ambient domain; see Section 2. In this paper, we study how these moves can be used to simplify an arbitrary Weinstein presentation.

Abouzaid and Seidel [1] introduced the *complexity* $\mathrm{WCrit}(W)$ of a Weinstein structure W as the minimal number of critical points of a Weinstein Morse function on W, up to Weinstein homotopy. The corresponding notion for Stein domains was introduced by Eliashberg [17]. Complexity is tautologically a Weinstein homotopy invariant. The analog of WCrit in the smooth setting is $\mathrm{Crit}(M)$, the minimal number of critical points of *any* Morse function on a smooth manifold M. This is a classical invariant of

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smooth manifolds and we will study the relationship between $\mathrm{WCrit}(W)$ and $\mathrm{Crit}(W)$ as a way of investigating the difference between symplectic and smooth topology and the corresponding handlebody moves.

We first recall some results about $\mathrm{Crit}(M)$. A priori $\mathrm{Crit}(M)$ is just a smooth invariant of M. Morse proved that there is a lower bound for $\mathrm{Crit}(M)$ in terms of the integral homology $H_*(M;\mathbb{Z})$. Smale [39] showed in the proof of the h–cobordism theorem that if M^n is simply connected and $n \geq 6$, then this lower bound is in fact sharp. More precisely, it is possible to simplify an arbitrary Morse function on M^n to another Morse function whose number of critical points agrees with the homological lower bound. So in this case, $\mathrm{Crit}(M)$ is actually a homotopy invariant of M^n. To simplify an arbitrary Morse function, Smale uses certain moves called handle-slides and the Whitney trick, which requires M^n to be simply connected and $n \geq 6$. The h–cobordism theorem generally fails without these assumptions.

In this paper, we will study how much of Smale's h–cobordism theorem holds in the symplectic setting. Since any Weinstein Morse function is a smooth Morse function, we have the inequality $\mathrm{WCrit}(W) \geq \mathrm{Crit}(W)$ and Eliashberg [17] asked whether there are examples where $\mathrm{WCrit}(W)$ and $\mathrm{Crit}(W)$ differ. As first shown by Seidel and Smith [37], such examples do exist. For example, $\mathrm{Crit}(B^{2n}) = 1$ but any Weinstein structure Σ^{2n} on B^{2n} that is not symplectomorphic to (the completion of) B^{2n}_{std} must have $\mathrm{WCrit}(\Sigma^{2n}) \geq 2$; see [7, Corollary 11.27]. In fact, $\mathrm{WCrit}(\Sigma^{2n}) \geq 3$, since the Euler characteristic of B^{2n} is 1. Seidel and Smith constructed such an exotic Σ^{2n} and distinguished it from B^{2n}_{std} by the presence of a Floer-theoretically essential Lagrangian torus. Hence the proof of the inequality $\mathrm{WCrit}(\Sigma) \geq \mathrm{Crit}(\Sigma) + 2$ depends crucially on J–holomorphic curve type invariants. From a Weinstein homotopy point of view, WCrit and Crit differ because the Whitney trick, the key part of Smale's proof of the h–cobordism theorem, does not generally work in the symplectic setting; more precisely, smoothly isotopic Legendrian submanifolds are not necessarily Legendrian isotopic.

Given that Crit and WCrit can indeed be different, it is natural to ask how big this difference can be. We first note that for domains of dimension at least six, there are infinitely many different Weinstein structures in the same *almost* Weinstein class [1; 7; 32]. So, in principle, $\mathrm{WCrit}(W)$ can be arbitrarily larger than $\mathrm{Crit}(W)$. The first construction of infinitely many exotic Weinstein structures is due to McLean [32]. He constructed a single exotic ball Σ^{2n}_1 and then showed that $\Sigma^{2n}_k := \natural_{i=1}^{k} \Sigma^{2n}_1$, the boundary connected sum of k copies of Σ^{2n}_1, are pairwise nonsymplectomorphic, distinguished by a J–holomorphic curve invariant called symplectic homology. In particular, Σ_k^{2n} has a natural Weinstein presentation with at least $4k-1$ handles ($3k$ handles for $\coprod_{i=1}^k \Sigma_1^{2n}$ and $k-1$ index 1 handles) making it seem that these structures have unbounded complexity. Later, Abouzaid and Seidel [1] constructed infinitely many exotic Weinstein structures that do have bounded complexity.

On the other hand, recent work has shown that certain Weinstein structures have minimal complexity, ie $\mathrm{WCrit}(W) = \mathrm{Crit}(W)$. Cieliebak and Eliashberg [7] proved that *flexible* Weinstein structures, which satisfy an h–principle that reduces their symplectic topology to the underlying algebraic topology, have minimal complexity. Later Eliashberg, Ganatra, and the author [18] constructed infinitely many examples of exotic (nonflexible) Weinstein structures on T^*S^n and showed that they also have minimal complexity. We will show that minimal complexity holds quite generally.

1.1 Almost minimal Weinstein presentations

The above examples due to Seidel and Smith and to McLean show that there exist W for which $\mathrm{WCrit}(W) \geq \mathrm{Crit}(W) + 2$. This lower bound comes from J–holomorphic curve invariants (and some mild use of h–principles). Our main result shows that this is the only constraint on WCrit. In the following, we say a smooth domain W^{2n} (with the homotopy type of an n–dimensional CW complex) is *smoothly critical* if every smooth proper Morse function has a critical point of index n; for example, if $H^n(W^{2n}; \mathbb{Z})$ is nonzero. A smooth domain W^{2n} is *smoothly subcritical* if W^{2n} admits a smooth Morse function all of whose critical points have index strictly less than n. A (smoothly subcritical) Weinstein domain is *Weinstein subcritical* if it admits a Weinstein Morse function all of whose critical points have index strictly less than n. Subcritical Weinstein domains are flexible and hence have minimal complexity as mentioned above [7]; see Section 2.2 for details.

Theorem 1.1 *If W^{2n}, where $n \geq 3$, is a Weinstein domain, then* $\mathrm{WCrit}(W) \leq \mathrm{Crit}(W) + 2$. *Furthermore, if W is smoothly critical, then* $\mathrm{WCrit}(W) = \mathrm{Crit}(W)$. *If W is smoothly subcritical and $\pi_1(W) = 0$, then* $\mathrm{WCrit}(W) = \mathrm{Crit}(W)$ *if and only if W is a subcritical Weinstein domain; otherwise,* $\mathrm{WCrit}(W) = \mathrm{Crit}(W) + 2$.

More precisely, let $\mathrm{WCrit}_k(W^{2n})$ denote the minimum number of index k critical points of a Weinstein Morse function on W^{2n}; let $\mathrm{Crit}_k(W)$ denote the same for a smooth Morse function. Then the proof of Theorem 1.1 actually shows that

$$\mathrm{WCrit}_k(W^{2n}) = \mathrm{Crit}_k(W^{2n}) \quad \text{for } k \leq n-2$$

and either

$$\mathrm{WCrit}_{n-1}(W^{2n}) = \mathrm{Crit}_{n-1}(W^{2n}) \quad \text{and} \quad \mathrm{WCrit}_n(W^{2n}) = \mathrm{Crit}_n(W^{2n})$$

or

$$\mathrm{WCrit}_{n-1}(W^{2n}) = \mathrm{Crit}_{n-1}(W^{2n}) + 1 \quad \text{and} \quad \mathrm{WCrit}_n(W^{2n}) = 1.$$

The second case can only happen when $\mathrm{Crit}_n(W^{2n}) = 0$, ie W is smoothly subcritical. So we always have $\mathrm{WCrit}_n(W^{2n}) \le \max\{1, \mathrm{Crit}_n(W^{2n})\}$.

Now we give some examples illustrating Theorem 1.1.

Example 1.2 If M^n, where $n \ge 3$, is a closed smooth manifold, then

$$\mathrm{WCrit}(T^*M) = \mathrm{Crit}(T^*M) \le \mathrm{Crit}(M)$$

for *any* Weinstein structure on T^*M because it is smoothly critical; if $n \ge 6$ and $\pi_1(M) = 0$, then the second inequality is also an equality. In particular, all Weinstein structures on T^*S^n have $\mathrm{WCrit}(T^*S^n) = 2$; this generalizes the result in [18], where it was proven that this holds for a particular infinite collection of exotic structures on T^*S^n.

Example 1.3 Any Weinstein ball Σ^{2n} that is smoothly subcritical with $\mathrm{Crit}(\Sigma^{2n}) = 1$ has either $\mathrm{WCrit}(\Sigma^{2n}) = 1$ or 3. Since $\pi_1(\Sigma^{2n}) = 0$, the structure is Weinstein homotopic to the standard structure B^{2n}_{std} if and only if $\mathrm{WCrit}(\Sigma^{2n}) = 1$. In particular, McLean's exotic structures Σ^{2n}_k, which have natural presentations with at least $4k-1$ critical points, can be Weinstein homotoped to presentations with just 3 critical points, corresponding to handles of index 0, $n-1$, and n. They are all nonstandard structures and so $\mathrm{WCrit}(\Sigma^{2n}_k) = 3$.

Our proof of Theorem 1.1 relies on Murphy's h–principle for loose Legendrians [33] (and its consequences for flexible domains) as well as the smooth Whitney trick. Both of these results hold only for $n \ge 3$, hence our restriction on dimension.

Question 1.4 Is $\mathrm{WCrit}(W^4) \le \mathrm{Crit}(W^4) + 2$ for any Weinstein domain W^4?

1.2 Flexible subdomains

Our main result Theorem 1.1 essentially follows from the following theorem. For a Weinstein domain W^{2n} where $n \ge 3$, let W^{2n}_{flex} be the unique flexible Weinstein structure almost symplectomorphic to W^{2n}; see Section 2.2.

Theorem 1.5 *Any Weinstein domain W^{2n} where $n \geq 3$ can be Weinstein homotoped to $W^{2n}_{\text{flex}} \cup C^{2n}$, where C^{2n} is a smoothly trivial Weinstein cobordism with two critical points of respective indices $n-1$ and n.*

This result implies that the smooth topology and the symplectic topology can be separated in the sense that all the smooth topology can be put into a symplectically trivial (flexible) domain W^{2n}_{flex} while all the symplectic topology can be put into a smoothly trivial cobordism C^{2n}, which is a smooth collar of the boundary of W^{2n}. In particular, Theorem 1.5 shows that W_{flex} is a Weinstein subdomain of W. This extends previous work of Eliashberg and Murphy [19], who proved that W_{flex} is a *Liouville* subdomain of W, ie $W \backslash W_{\text{flex}}$ is an exact symplectic cobordism, perhaps without a compatible Weinstein Morse function. The decomposition in Theorem 1.5 has several applications, explored in later work; for example, it is used to prove an existence h–principle for regular Lagrangians with boundary in arbitrary Weinstein domains as well as regular Lagrangian caps [29] and construct "maximal" Weinstein domains that contain a complicated set of Lagrangians [30]. Theorem 1.5 implies most of Theorem 1.1. The presentation in Theorem 1.5 shows that $\text{WCrit}(W) \leq \text{WCrit}(W_{\text{flex}}) + 2$. Since flexible structures have minimal complexity [7], $\text{WCrit}(W_{\text{flex}}) = \text{Crit}(W)$. Combining these results, we get $\text{WCrit}(W) \leq \text{Crit}(W) + 2$, the first claim in Theorem 1.1. The proof of the smoothly critical case of Theorem 1.1 is similar.

Flexible Weinstein domains are defined only for $n \geq 3$. The analog of these domains for $n = 2$ are Weinstein domains whose index 2 handles are attached along stabilized Legendrians; we will call these *stabilized domains*. However, neither stabilized Legendrians nor stabilized domains satisfy an h–principle and so we do not know whether Theorem 1.1 holds for $n = 2$. However an analog of Theorem 1.5 holds for $n = 2$ if we replace flexible domains and loose Legendrians with these analogous domains and Legendrians respectively.

Theorem 1.6 *Any Weinstein domain W^4 can be Weinstein homotoped to $V^4 \cup H^2$, where V^4 is a stabilized domain that is simply homotopy equivalent to $W^4 \cup H^1$.*

The notation H^n_Λ denotes a Weinstein handle attached along an isotropic attaching sphere Λ, and we write H^n if we do not specify the attaching sphere; see Section 2. Theorem 1.6 cannot be improved so that V^4 is *diffeomorphic* to $W^4 \cup H^1$. For example, there is a unique Weinstein structure on T^*T^2 and it has nonvanishing symplectic homology — see Eliashberg [16] and Wendl [40]; the same holds for $T^*T^2 \cup H^1$ [7].

On the other hand, stabilized domains have vanishing symplectic homology and so $T^*T^2 \cup H^1$ does not admit a stabilized Weinstein structure. The reason for this is that stabilizing a 1–dimensional Legendrian knot changes its Thurston–Bennequin invariant, which affects the framing used to attach the Weinstein handle and hence the intersection form of the resulting Weinstein domain.

Theorem 1.5 shows that any Weinstein domain W^{2n} where $n \geq 3$ can be presented as a flexible domain $W^{2n}_{\text{flex}} \cup H^{n-1}$ plus a single critical handle. In fact, the proof of Theorem 3.1 is a bit more explicit about the single extra handle.

Corollary 1.7 *Every Weinstein domain W^{2n} where $n \geq 3$ can be Weinstein homotoped to a subcritical domain V_{sub} with handles attached to the Legendrian link $\Lambda_1 \amalg \cdots \amalg \Lambda_{k-1} \amalg \Lambda_k \subset \partial V_{\text{sub}}$ such that $\Lambda_1 \amalg \cdots \amalg \Lambda_{k-1}$ is a loose link and Λ_k is a loose Legendrian.*

Even though all of the Legendrians in Corollary 1.7 are individually loose, the entire link $\Lambda_1 \amalg \cdots \amalg \Lambda_{k-1} \amalg \Lambda_k$ may not be loose, ie the loose charts of Λ_i intersect Λ_k and the loose chart of Λ_k intersects Λ_i. Otherwise all Weinstein domains would be flexible. So the attaching Legendrians are themselves symplectically trivial but their linking is symplectically nontrivial, ie the symplectic topology of the domain is captured in this linking. Of course, Λ_k becomes nonloose once we attach handles to $\Lambda_1, \ldots, \Lambda_{k-1}$ (and vice versa).

Now we present an example demonstrating Theorem 1.5.

Example 1.8 Any Weinstein structure on T^*S^n where $n \geq 3$ can be Weinstein homotoped to $T^*S^n_{\text{flex}} \cup H^{n-1} \cup H^n_\Lambda$ for some Legendrian Λ in the contact manifold $\partial(B^{2n}_{\text{std}} \cup H^{n-1})$. A slightly modified version of Theorem 1.5 shows that T^*S^n can also be homotoped to $B^{2n}_{\text{std}} \cup H^n_\Lambda$; this is why we always have $\operatorname{WCrit}(T^*S^n) = 2$ in Example 1.2. We can reformulate this as follows. Let $\mathfrak{Legendrian}((Y,\xi);\Lambda_0)$ denote parametrized Legendrians in the contact manifold (Y,ξ), up to Legendrian isotopy, that are in some fixed Legendrian formal isotopy class Λ_0. Let X^{2n} be an almost Weinstein domain, ie an almost complex domain with the homotopy type of an n–dimensional CW complex; see Section 2. Then let $\mathfrak{Weinstein}(X^{2n})$ denote Weinstein structures on X^{2n} up to Weinstein homotopy. There is a natural map

$$\mathcal{H}_{\text{crit}}\colon \mathfrak{Legendrian}((S^{2n-1},\xi_{\text{std}});\Lambda_{\text{unknot}}) \to \mathfrak{Weinstein}(T^*S^n) \tag{1-1}$$

taking a Legendrian $\Lambda \subset (S^{2n-1},\xi_{\text{std}}) = \partial B^{2n}_{\text{std}}$ which is formally isotopic to Λ_{unknot}

to the Weinstein structure $B^{2n}_{\mathrm{std}} \cup H^n_\Lambda$ on T^*S^n. The statement that $\mathrm{WCrit} = 2$ for any Weinstein structure on T^*S^n implies that this map is surjective, ie the class of connected Legendrians is as complicated as the class of Weinstein structures.

Although our main result shows that Weinstein homotopy moves are more flexible than they might seem, there are limits to this flexibility. For example, Theorem 1.5 shows that any Weinstein domain can be presented as a flexible domain plus a single extra handle, which is possibly nonflexible. As we now explain, it is crucial that the nonflexible critical handle is attached last, and in general, it is impossible to first attach nonflexible handles and then attach flexible handles. So *order* of flexibility/nonflexibility matters, which is a sign of rigidity. As expected, this rigidity ultimately comes from J–holomorphic curves.

Example 1.9 By Theorem 1.5, $T^*S^n_{\mathrm{std}}$ is Weinstein homotopic to

$$T^*S^n_{\mathrm{flex}} \cup H^{n-1} \cup H^n_\Lambda = (B^{2n}_{\mathrm{std}} \cup H^n_{\mathrm{flex}}) \cup H^{n-1} \cup H^n_\Lambda$$

for some Legendrian Λ. In this case, we attach flexible handles first and then nonflexible handles. However, $T^*S^n_{\mathrm{std}}$ cannot be presented as $(B^{2n}_{\mathrm{std}} \cup H^{n-1} \cup H^n_\Lambda) \cup H^n_{\mathrm{flex}}$, where we first attach nonflexible handles and then flexible handles. This presentation is equivalent to a Weinstein structure of the form $\Sigma^{2n} \cup H^n_{\mathrm{flex}}$, for some exotic ball Σ^{2n}. We claim that $T^*S^n_{\mathrm{std}}$ is not symplectomorphic to $\Sigma^{2n} \cup H^n_{\mathrm{flex}}$ for any Σ^{2n}. To see this, let $C \subset \Sigma^{2n} \cup H^n_{\mathrm{flex}}$ be the Lagrangian cocore of H^n_{flex}. Since H^n_{flex} is attached along a loose Legendrian in $\partial\Sigma^{2n}$, the wrapped Floer homology $\mathrm{WH}(C, C; T^*S^n_{\mathrm{std}})$ vanishes. But C generates $H_n(T^*S^n, \partial T^*S^n) \cong \mathbb{Z}$ and so $C \cdot S^n = 1$, where $S^n \subset T^*S^n_{\mathrm{std}}$ is the zero section, a closed exact Lagrangian. But $\mathrm{WH}(C, C; T^*S^n_{\mathrm{std}}) = 0$ implies that $\mathrm{WH}(C, S^n; T^*S^n_{\mathrm{std}}) = 0$ and so $C \cdot S^n = \chi(\mathrm{WH}(C, S^n; T^*S^n_{\mathrm{std}})) = 0$, a contradiction.

Since $T^*S^n_{\mathrm{std}}$ is not of the form $\Sigma^{2n} \cup H^n_{\mathrm{flex}}$, the map

$$\mathcal{H}_{\mathrm{loose}} \colon \mathfrak{Weinstein}(B^{2n}) \to \mathfrak{Weinstein}(T^*S^n) \tag{1-2}$$

obtained by attaching a critical handle along a *loose* Legendrian unknot to an exotic Weinstein ball is not surjective. This map is well defined since any contact structure $\partial\Sigma^{2n}$ in the almost contact structure $(S^{2n-1}, J_{\mathrm{std}})$ has a unique loose Legendrian in the standard formal class. Furthermore, it has infinite image; for example, $\mathcal{H}_{\mathrm{loose}}$ is injective on the exotic structures Σ^{2n}_k constructed by McLean [32]. We contrast the nonsurjectivity of $\mathcal{H}_{\mathrm{loose}}$, a rigidity result, to the surjectivity of the map $\mathcal{H}_{\mathrm{crit}}$ in (1-1), a flexibility result.

Now we sketch the proof of Theorem 1.5, which implies the main result Theorem 1.1. The key idea is that certain Weinstein homotopy moves called handle-slides can be used to make a Legendrian loose; see Section 2. More precisely, given two Legendrians and a local chart intersecting them, the handle-slide produces another Legendrian, which was described by Casals and Murphy [4]. We will show that there is a special choice of local chart such that the handle-slid Legendrian is loose (not all choices of charts result in loose Legendrians). For an arbitrary Weinstein domain, we fix one Legendrian and handle-slide the rest of the Legendrians over that fixed Legendrian. For appropriate choices of local charts, the resulting Legendrians form a loose link except for the fixed Legendrian, which will in general intersect the loose charts of the other Legendrians; this is the content of Theorem 1.5.

1.3 Weinstein presentations with few gradient trajectories

As mentioned before, our goal is to study to what extent Smale's h–cobordism theorem holds in the symplectic setting. This theorem has two main steps. The first step is to apply handle-slides to make handles with consecutive indices cancel algebraically, ie for the belt sphere of a k–handle and the attaching sphere of a $(k+1)$–handle to have algebraic intersection number one. The second step is to use the Whitney trick to reduce the number of intersection points between algebraically canceling handles to make them geometrically canceling, ie have *geometric* intersection number one. Since Weinstein handles can be handle-slid in the same way as smooth handles, the first step can be done in the Weinstein setting. However the second step necessarily fails since $\mathrm{WCrit}(W) \neq \mathrm{Crit}(W)$ in general. By Theorem 1.5, any smoothly trivial Weinstein cobordism W can be Weinstein homotoped to have two Weinstein handles of respective indices $n-1$ and n that cancel algebraically, ie $W = H^{n-1} \cup H^n_\Lambda$. The Whitney trick shows that in this case, it is possible to *smoothly* isotope the attaching sphere Λ so it intersects the belt sphere of H^{n-1} in exactly one point. However, if Λ intersects the belt sphere of H^{n-1} in a single point, it is loose [7] and the Weinstein cobordism is flexible. Hence, in general it is impossible to realize this smooth isotopy by a Legendrian isotopy and to reduce the geometric intersection number to one. The minimal possible number is therefore three; it is greater than one and must be odd for homological reasons. Although we do not know whether the geometric intersection number can always be reduced to three, in the following result we reduce this number to some universal constant independent of the Weinstein structure. So we can get uniformly close to realizing the second step of Smale's h–cobordism proof.

Theorem 1.10 *There exists a constant $C_n \geq 3$ depending only on n such that any smoothly trivial Weinstein cobordism W^{2n} where $n \geq 3$ can be Weinstein homotoped to a presentation with two handles of respective indices $n-1$ and n such that the belt sphere of the $(n-1)$–handle and the attaching sphere of the n–handle intersect C_n times.*

This is equivalent to having a Weinstein Morse function with two critical points of respective indices $n-1$ and n such that there are C_n gradient trajectories from the index n to the index $n-1$ critical point. The proof of Theorem 1.10 actually shows that it is possible in principle to compute C_n but this depends on a good understanding of a certain (local) Legendrian isotopy which comes from an h–principle and is therefore not very explicit. As we explain in the following example, the situation is more complicated when the Weinstein cobordism is not smoothly trivial. Namely, in the presence of multiple $(n-1)$–handles, the attaching Legendrian for the n–handle might have to pass through *all* $(n-1)$–handles, even when this is topologically unnecessary. Again this rigidity comes from J–holomorphic curves.

Example 1.11 Consider a subflexible Weinstein structure W^{2n} on $B^{2n} \cup H^{n-1}$ that is not flexible. Such an example was constructed by Murphy and Siegel [34] and has zero symplectic homology $\mathrm{SH}(W^{2n})$ but nonzero *deformed* symplectic homology $\mathrm{SH}^{\alpha}(W^{2n})$; here α is the generator of $H^{n-1}(B^{2n} \cup H^{n-1}) \cong \mathbb{Z}$. So this domain is smoothly subcritical but is not symplectically subcritical and hence by Theorem 1.1 admits a Weinstein presentation of the form $B^{2n}_{\mathrm{std}} \cup H^{n-1}_1 \cup H^{n-1}_2 \cup H^n_\Lambda$. Here Λ has algebraic intersection number 1 with H^{n-1}_1 and 0 with H^{n-1}_2. However, Λ has geometric intersection number at least 3 with H^{n-1}_1 since otherwise Λ would be loose. Furthermore, Λ must have geometric intersection number at least 2 with H^{n-1}_2; therefore, Λ must interact with *both* H^{n-1}_1 and H^{n-1}_2. Otherwise, the domain would be of the form $(B^{2n}_{\mathrm{std}} \cup H^{n-1}_1 \cup H^n_\Lambda) \cup H^{n-1}_2 = \Sigma^{2n} \cup H^{n-1}$, for some exotic structure Σ^{2n} on B^{2n}. However $\Sigma^{2n} \cup H^{n-1}$ has zero deformed symplectic homology as we now show. Since H^{n-1} is a subcritical handle, the Viterbo transfer map $\mathrm{SH}^{\alpha}(\Sigma^{2n} \cup H^{n-1}) \to \mathrm{SH}^{i^*\alpha}(\Sigma^{2n})$ is an isomorphism, where $i^*\colon H^{n-1}(\Sigma^{2n} \cup H^{n-1}) \to H^{n-1}(\Sigma^{2n})$ is the induced map on cohomology. Since $i^*\alpha \in H^{n-1}(\Sigma^{2n}) = 0$, $\mathrm{SH}^{i^*\alpha}(\Sigma^{2n})$ agrees with the undeformed symplectic homology $\mathrm{SH}(\Sigma^{2n})$. Since Σ^{2n} is a subdomain of W^{2n}, which has vanishing SH, and the Viterbo map is unital, $\mathrm{SH}(\Sigma)$ also vanishes. Therefore $\mathrm{SH}^{\alpha}(\Sigma^{2n} \cup H^{n-1})$ is also zero and so $\Sigma^{2n} \cup H^{n-1}$ cannot be Weinstein homotopic to W^{2n}.

Since W is not of the form $\Sigma^{2n} \cup H^{n-1}$ for any exotic Weinstein ball Σ^{2n}, the map

$$\mathcal{H}_{\text{sub}}\colon \mathfrak{Weinstein}(B^{2n}) \to \mathfrak{Weinstein}(B^{2n} \cup H^{n-1}) \tag{1-3}$$

obtained by attaching a *subcritical* handle to an exotic Weinstein ball is not surjective; see Ghiggini, Niederkrüger, and Wendl [25] for an analog in the contact case. This rigidity result is similar to the nonsurjectivity of the map $\mathcal{H}_{\text{loose}}$ in (1-2) for flexible handle attachment and in contrast to the surjectivity of $\mathcal{H}_{\text{crit}}$ in (1-1) for critical handle attachment to the standard ball.

1.4 Results for the wrapped Fukaya category and the Chekanov–Eliashberg DGA

We now give some applications of the flexibility results in Sections 1.1 and 1.2 to certain J–holomorphic curve invariants. To a Weinstein (or Liouville) domain X^{2n} (with a choice of grading data), one can associate the wrapped Fukaya category $\mathcal{W}(X)$ of X, a certain A_∞–category. The objects of $\mathcal{W}(X)$ are (graded) exact Lagrangians in X^{2n} that are closed or have Legendrian boundary in ∂X^{2n}; the morphisms are wrapped Floer cochains. In homological mirror symmetry, one considers the derived Fukaya category $D^b\mathcal{W}(X) := H^0(\text{Tw}(\mathcal{W}(X))$, the cohomology category of twisted complexes over $\mathcal{W}(X)$. To obtain a more explicit description of the wrapped Fukaya category, it is useful to find a set of *generators*. The derived Fukaya category $D^b\mathcal{W}(X)$ is triangulated so mapping cones exist. A set of objects G_i are generators of $D^b\mathcal{W}(X)$ if every object of the category is isomorphic to an iterated mapping cone on them; equivalently, $D^b\mathcal{W}(X) \cong H^0(\text{Tw}(\mathcal{G}))$, where $\mathcal{G}$ is the A_∞–subcategory with objects G_i. Let $g(\mathcal{W}(X))$ denote the minimum number of generators for $D^b\mathcal{W}(X)$. Many proofs of homological mirror symmetry involve finding some collection of generators for $D^b\mathcal{W}(X)$ and then showing that the endomorphism algebra of these generators is quasi-isomorphic to the endomorphism algebra of some generating coherent sheaves on the mirror.

Theorem 1.1 can be used to bound the number of generators $g(\mathcal{W}(X))$ for $D^b\mathcal{W}(X)$. The unstable manifold of an index n critical point of a Weinstein Morse function, or *cocore*, is a Lagrangian disk with Legendrian boundary and hence defines an object in $D^b\mathcal{W}(X)$. As proven by Chantraine, Dimitroglou Rizell, Ghiggini, and Golovko [6] and Ganatra, Pardon, and Shende [23], the cocores of the index n critical points of any Weinstein Morse function on X generate $D^b\mathcal{W}(X)$, ie $g(\mathcal{W}(X^{2n})) \leq \text{WCrit}_n(X^{2n})$. Theorem 1.1 shows that there is a topological bound on $\text{WCrit}_n(X^{2n})$ and hence on the number of generators needed. In the following, let $g(A)$ denote the minimum number of generators of an abelian group A.

Corollary 1.12 *If X^{2n}, where $n \geq 3$, is a Weinstein domain, then*

$$g(\mathcal{W}(X)) \leq \max\{1, g(H^n(X;\mathbb{Z}))\}.$$

A related notion is that of *split-generation*: a set of objects are split-generators if every object is a *summand* of a twisted complex on these objects. This is a useful notion since there are closed symplectic manifolds whose Fukaya categories have finitely many split-generators but no finite collection of generators, eg the 2–torus. We emphasize that Corollary 1.12 concerns generation, not split-generation. Whenever there is a finite collection of generators (or split-generators), there is a single split-generator, namely the sum of all these objects. So the number of split-generators is not an interesting invariant.

The number of generators, on the other hand, is a meaningful invariant, and in certain cases, the inequality in Corollary 1.12 is sharp. For example, if X^{2n} is a Weinstein ball, then Corollary 1.12 shows that at most one generator is needed and if the Fukaya category of this ball is nontrivial (as is the case for the exotic structures constructed by McLean [32]), then at least one generator is needed. In certain cases, the number of generators needed for $\mathcal{W}(X)$ is greater than one. Since $D^b\mathcal{W}(X)$ is a triangulated category, we can consider its Grothendieck group $K_0(\mathcal{W}(X)) := K_0(D^b\mathcal{W}(X))$. For any triangulated category, the minimum number of generators for the Grothendieck group gives a lower bound on the number of generators of the category. In particular, Corollary 1.12 implies that for any Weinstein domain X^{2n} where $n \geq 3$ we have

$$g\big(K_0(\mathcal{W}(X))\big) \leq g(\mathcal{W}(X)) \leq \max\big\{1, g(H^n(X^{2n};\mathbb{Z}))\big\}. \tag{1-4}$$

There are Weinstein domains for which $g\big(K_0(\mathcal{W}(X))\big)$ is bigger than one. For example, consider the boundary connected sum $\natural^k T^*S^n$ of k copies of $T^*S^n_{\mathrm{std}}$. As explained to the author by Abouzaid, $K_0\big(\mathcal{W}(\natural^k T^*S^n)\big)$ has rank at least k. Namely, let $\varphi_i\colon K_0\big(\mathcal{W}(\natural^k T^*S^n)\big) \to \mathbb{Z}$ be $\chi(\mathrm{HW}(-, S^n_i))$, the Euler characteristic of morphisms from the i^{th} zero section S^n_i. Then $(\varphi_1, \dots, \varphi_k)\colon K_0\big(\mathcal{W}(\natural^k T^*S^n)\big) \to \mathbb{Z}^k$ is surjective, so $g\big(K_0\big(\mathcal{W}(\natural^k T^*S^n)\big)\big) \geq k$. On the other hand, $g\big(H^n\big(\natural^k T^*S^n;\mathbb{Z}\big)\big) = k$ and so all the inequalities in (1-4) are all actually equalities. The following result shows that (1-4) can actually be improved.

Corollary 1.13 *If X^{2n}, where $n \geq 3$, is a Weinstein domain, then*

$$g\big(K_0(\mathcal{W}(X))\big) \leq g(H^n(X;\mathbb{Z})).$$

In particular, if $H^n(X;\mathbb{Z}) = 0$, then $K_0(\mathcal{W}(X)) = 0$.

If $H^n(X;\mathbb{Z}) \neq 0$, then the result follows from (1-4). If $H^n(X;\mathbb{Z}) = 0$, we use an additional boundary connected sum argument, which was explained to the author by Ivan Smith in the case when X^{2n} is a ball. In particular, any Weinstein ball Σ^{2n} must have $K_0(\mathcal{W}(\Sigma)) = 0$. There are many exotic Weinstein balls Σ^{2n} with nonzero symplectic homology [32]. So their wrapped Fukaya categories are examples of triangulated categories with nonzero Hochschild cohomology but zero Grothendieck group; such *phantom* categories have been studied in algebraic geometry — see Galkin, Katzarkov, Mellit, and Shinder [21] and Gorchinskiy and Orlov [26] — and are possibly related to our wrapped Fukaya categories via mirror symmetry. The vanishing of $K_0(\mathcal{W}(\Sigma))$ implies that any object Q that has finite-dimensional morphism spaces with all other objects K has $\chi(\mathrm{HW}(Q, K)) = 0$, generalizing the geometric result that any closed exact Lagrangian $L \subset \Sigma^{2n}$ has $L \cdot K = 0$ for any other Lagrangian K; however the object Q need not be a twisted complex of closed exact Lagrangians. We also note that the inequality in Corollary 1.13 is sharp, eg consider $\natural^k T^*S^n_{\mathrm{std}}$. Conversely, for any integer $j \leq k = g\big(H^n(\natural^k T^*S^n;\mathbb{Z})\big)$, there is a Weinstein structure X_j^{2n} on $\natural^k T^*S^n$ so that $g\big(K_0(\mathcal{W}(X_j))\big) = j$, eg $X_j^{2n} = \natural^j T^*S^n_{\mathrm{std}} \natural \natural^{k-j} T^*S^n_{\mathrm{flex}}$.

One natural question is which triangulated categories can arise as the wrapped Fukaya category of Weinstein domains. For example, the wrapped Fukaya category of a Weinstein domain is a smooth category with a noncompact Calabi–Yau structure; see Chantraine, Dimitroglou Rizell, Ghiggini, and Golovko [6] and Ganatra [22]. Corollary 1.13 further restricts which categories can arise as the Fukaya categories of Weinstein domains and shows that in general the answer depends on the smooth topology of the domain.

Corollary 1.14 *There is no Weinstein ball Σ^{2n} such that $D^b(\mathcal{W}(\Sigma^{2n}))$ is exact equivalent $D^b(\mathcal{W}(T^*S^n_{\mathrm{std}}))$. There is no Weinstein structure X^{2n} on T^*S^n such that $D^b\mathcal{W}(X^{2n})$ is exact equivalent to $D^b\mathcal{W}(T^*S^n_{\mathrm{std}} \natural T^*S^n_{\mathrm{std}})$.*

Proof As noted above,

$$g\big(K_0(\mathcal{W}(T^*S^n_{\mathrm{std}}))\big) = 1 \quad \text{and} \quad g\big(K_0(\mathcal{W}(T^*S^n_{\mathrm{std}} \natural T^*S^n_{\mathrm{std}}))\big) = 2.$$

However, if Σ^{2n} is a ball, then $g\big(K_0(\mathcal{W}(\Sigma^{2n}))\big) = 0$; if $H^n(X;\mathbb{Z}) \cong \mathbb{Z}$, then $g\big(K_0(\mathcal{W}(X))\big) \leq 1$. □

On the other hand, for any Weinstein ball Σ^{2n}, the Weinstein structure $T^*S^n_{\mathrm{flex}} \natural \Sigma^{2n}$ on T^*S^n has the same Fukaya category as Σ^{2n}. So the class of categories arising as Fukaya categories of Weinstein structures on T^*S^n is genuinely larger than that for a ball B^{2n}.

Since Weinstein domains are constructed by attaching handles along Legendrians, Corollary 1.13 has implications for J–holomorphic curve invariants of Legendrians. Given a Legendrian sphere Λ^{n-1} in a contact manifold (Y^{2n-1}, ξ) with a Weinstein filling W^{2n}, there are (at least) two associated Legendrian isotopy invariants: the Chekanov–Eliashberg algebra $\mathrm{CE}(\Lambda)$ of Λ (augmented by the filling W^{2n}) and the wrapped Floer cochains $\mathrm{CW}(C, C)$ of the cocore C^n of the Weinstein n–handle H^n_Λ in the Weinstein domain $W^{2n} \cup H^n_\Lambda$. For both invariants, we work over a common ground field $\mathbb{K}$. The former invariant is only rigorously defined when (Y^{2n-1}, ξ) is $P^{2n-2} \times \mathbb{R}$ for some exact symplectic manifold P — see Ekholm, Etnyre, and Sullivan [12]; the latter is always defined. A proof was sketched bu Bourgeois, Ekholm, and Eliashberg [3] that these two invariants are quasi-isomorphic and for the results in the rest of this section, we will assume this.

Remark 1.15 Alternatively, let $\mathrm{CF}(D^n, D^n; (W, \Lambda))$ denote the Floer cochains of the linking disk D^n of Λ in the partially wrapped Fukaya category of W^{2n} stopped at Λ; a proof was sketched by Ekholm and Lekili [14] that this is quasi-isomorphic to the version of $\mathrm{CE}(\Lambda)$ with coefficients in $C(\Omega S^{n-1})$, chains on the based loop space of S^{n-1}. Without any reference to $\mathrm{CE}(\Lambda)$, it was proven by Ganatra, Pardon, and Shende [23] that $\mathrm{CF}(D^n, D^n; (W, \Lambda)) \otimes_{C_*(\Omega S^{n-1})} C_*(\Omega D^n) = \mathrm{CF}(D^n, D^n; (W, \Lambda)) \otimes_{C_*(\Omega S^{n-1})} \mathbb{K}$ is quasi-isomorphic to $\mathrm{CW}(C, C)$ and so this invariant can be considered as a rigorous replacement for $\mathrm{CE}(\Lambda)$; using this alternative invariant, all our results have complete proofs.

Certain geometric properties of a Legendrian have algebraic consequences for its Chekanov–Eliashberg DGA. For example, an exact Lagrangian filling of Λ induces an *augmentation* of $\mathrm{CE}(\Lambda)$, ie a differential graded algebra (DGA) map $\mathrm{CE}(\Lambda) \to \mathbb{K}$, where the latter has the zero differential and is concentrated in degree zero; see Ekholm, Honda, and Kálmán [13]. However, not all augmentations come from exact Lagrangian fillings — see Etnyre and Ng [20] — and furthermore, there are examples of Legendrians such that $\mathrm{CE}(\Lambda)$ is not acyclic but admits no augmentations. More generally, we can consider n–dimensional *representations* of $\mathrm{CE}(\Lambda)$, ie DGA maps $\mathrm{CE}(\Lambda) \to \mathrm{Mat}(n, \mathbb{K})$. There are examples [9; 38] of Legendrians for which $\mathrm{CE}(\Lambda)$ has a 2–dimensional representation but no augmentations. This is a useful notion since Dimitroglou Rizell and Golovko [9] showed that Legendrians with finite-dimensional representations have an Arnold-type lower bound on the number of Reeb chords. On the other hand, they showed that for each $n \geq 1$, there is a Legendrian $\Lambda \subset (\mathbb{R}^{2n-1}, \xi_{\text{std}})$ such that $\text{CE}(\Lambda)$ is not acyclic but has no finite-dimensional representations (although any nonacyclic DGA has an infinite-dimensional "representation" to its characteristic algebra; see Ng [35]). These examples are obtained by spinning a particular 1–dimensional Legendrian studied by Sivek [38], who proved that it has no finite-dimensional representations by explicit calculation. We now show that such Legendrians occur generally.

Consider a Legendrian Λ in $(S^{n-1} \times S^n, \xi_{\text{std}}) = \partial(B^{2n}_{\text{std}} \cup H^{n-1})$, where $n \geq 3$, that has algebraic intersection number one with $\{p\} \times S^n$ for some $p \in S^{n-1}$, ie $[\Lambda] = \pm 1 \in H_{n-1}(S^{n-1} \times S^n; \mathbb{Z}) \cong \mathbb{Z}$ is primitive in homology. This implies that $[\Lambda] = 1 \in H_{n-1}(B^{2n}_{\text{std}} \cup H^{n-1}; \mathbb{Z}) \cong \mathbb{Z}$ and hence Λ has no exact Lagrangian fillings in $B^{2n}_{\text{std}} \cup H^{n-1}$ for purely topological reasons. So there are no augmentations of $\text{CE}(\Lambda)$ that come from fillings. Using Corollary 1.13, we show that $\text{CE}(\Lambda)$ has no augmentations at all and, in fact, no finite-dimensional representations.

Corollary 1.16 *If a Legendrian* $\Lambda^{n-1} \subset (S^{n-1} \times S^n, \xi_{\text{std}})$, *where* $n \geq 3$, *is primitive in homology*, $\text{CE}(\Lambda)$ *has no finite-dimensional representations and no DGA maps to a commutative ring.*

If Λ intersects $\{p\} \times S^n$ *geometrically* once, then Λ is a loose Legendrian; see Casals and Murphy [4] and Section 1.3. In this case, $\text{CE}(\Lambda)$ is acyclic and hence has no finite representations for trivial reasons. Corollary 1.16 generalizes this to the case of *algebraic* intersection one, a topological condition. Although our proof of Corollary 1.16 holds only for $n \geq 3$, the $n = 2$ case for augmentations was proven by Leverson [31] using a different approach. We also note that a homological condition is necessary since the Chekanov–Eliashberg DGA of Legendrians in $(S^{n-1} \times S^n, \xi_{\text{std}})$ that have Lagrangian fillings in $B^{2n}_{\text{std}} \cup H^{n-1}$ have augmentations.

Corollary 1.16 has applications to the C^0–topology of the space of Legendrians. Murphy [33] proved that any Legendrian can be C^0–approximated by a loose Legendrian. On the other hand, Dimitroglou Rizell and Sullivan [10] recently used persistent homology to show that loose Legendrians cannot be C^0–approximated by certain nonloose Legendrians: if $\Lambda \subset (\mathbb{R}^{2n-1}, \xi_{\text{std}})$ is in a contact neighborhood $N(\Lambda_{\text{loose}})$ of a loose Legendrian Λ_{loose} and the map $i_*\colon H_{n-1}(\Lambda; \mathbb{Z}/2) \to H_{n-1}(N(\Lambda_{\text{loose}}); \mathbb{Z}/2) \cong \mathbb{Z}/2$ is nonzero, then $\text{CE}(\Lambda)$ has no augmentations. Using Corollary 1.16, we give a different proof of a slightly different result.

Corollary 1.17 *If $\Lambda \subset (S^{2n-1}, \xi_{\mathrm{std}})$, where $n \geq 3$, is in a contact neighborhood of a loose Legendrian Λ_{loose} and is primitive in $H_{n-1}(\Lambda_{\mathrm{loose}}; \mathbb{Z})$, then $\mathrm{CE}(\Lambda)$ has no finite-dimensional representations or DGA maps to a commutative ring.*

So the size of contact neighborhoods depends on the Legendrian isotopy class. In the proof of Corollary 1.17, the condition that Λ is in $N(\Lambda_{\mathrm{loose}})$ is used to show that a related Legendrian is disjoint from the loose chart of another loose Legendrian; the homological condition is needed to apply Corollary 1.16. Some homology condition is necessary since otherwise any Legendrian in $(S^{2n-1}, \xi_{\mathrm{std}})$ could be isotoped into a neighborhood of any other Legendrian.

Corollaries 1.16 and 1.17 place strong restrictions on the Chekanov–Eliashberg DGAs of certain Legendrians. Furthermore, if these Legendrians satisfy stronger conditions, eg have *geometric* intersection one with $\{p\} \times S^n$ instead of *algebraic* intersection one, then they are loose, showing there is not much room for interesting Legendrians. Nonetheless, we show there are many examples of such Legendrians with nontrivial DGAs, essentially one for each exotic Weinstein ball; this shows that Corollaries 1.16 and 1.17 are sharp.

Corollary 1.18 *For $n \geq 4$, there exist infinitely many different Legendrian spheres $\Lambda_k \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$ for which $\mathrm{CE}(\Lambda_k)$ is not acyclic but has no finite-dimensional representations. The same holds for $(S^{2n-1}, \xi_{\mathrm{std}})$ for $n \geq 4$. Furthermore, these Legendrians are C^0–close to loose Legendrians Λ_{loose} and are primitive in $H_{n-1}(\Lambda_{\mathrm{loose}}; \mathbb{Z})$.*

The restriction $n \geq 4$ is because we currently have examples of exotic Weinstein balls only in such dimensions [32]. The Legendrians Λ_k are distinguished by the Hochschild homology of $\mathrm{CE}(\Lambda_k)$, which is isomorphic to the symplectic cohomology of these Weinstein balls.

In Section 2, we provide some background material on Weinstein domains, loose Legendrians, and handle-slides. In Section 3, we give proofs of the results stated in the introduction.

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2 Background

In this section, we present some background material, including necessary definitions and theorems that were assumed in the introduction.

2.1 Liouville and Weinstein domains

2.1.1 Definitions A *Liouville domain* is a pair (W^{2n}, λ) such that

- W^{2n} is a compact manifold with boundary,
- $d\lambda$ is a symplectic form on W, and
- the Liouville field X_λ, defined by $i_X\, d\lambda = \lambda$, is outward transverse along ∂W.

A *Weinstein domain* is a triple $(W^{2n}, \lambda, \varphi)$ such that

- (W, λ) is a Liouville domain,
- $\varphi\colon W \to \mathbb{R}$ is a Morse function with maximal level set ∂W, and
- X_λ is a gradient-like vector field for φ.

Liouville and Weinstein cobordisms are defined similarly.

Because W is compact and φ is a Morse function with maximal level set ∂W, φ has finitely many critical points. We will call φ a Weinstein Morse function. Note that if c is any regular value, $W^c = \{\varphi \le c\}$ is also a Weinstein domain and is called a Weinstein *subdomain*.

If $\Sigma^{2n-1} \subset (W^{2n}, \lambda)$ is a hypersurface such that X_λ is transverse to Σ, then $\ker(\lambda|_\Sigma)$ is a contact structure on Σ. In the Weinstein case, a regular level set $Y^c = \varphi^{-1}(c)$ of φ is such a hypersurface and so $(Y^c, \lambda|_{Y^c})$ is a contact manifold. In particular, the boundary ∂W of a Liouville or Weinstein domain W has a natural contact structure given by $\xi = \ker(\lambda|_{\partial W})$. The *completion* $\widehat{W}$ of W is the noncompact, exact symplectic manifold obtained by attaching the symplectization $\big(\partial W \times [0,\infty), d(e^t \lambda|_{\partial W})\big)$ of $(\partial W, \xi)$ to W. Whenever we speak of the symplectomorphism type of a Weinstein domain, we will mean the symplectomorphism type of its completion.

2.1.2 Weinstein handle attachment A Weinstein structure yields a special handlebody decomposition for W. First, recall that λ vanishes on the X_λ–stable disc D_p of a critical point p; see [7]. In particular, D_p is isotropic with respect to $d\lambda$ and so all critical points of φ have index less than or equal to n. If all critical points of φ have index *strictly less* than n, then the Weinstein domain is *subcritical*.

Since λ vanishes on D_p, then $\Lambda_p := D_p \cap Y^c \subset (Y^c, \lambda|_{Y^c})$ is an isotropic sphere, where $c = \varphi(p) - \varepsilon$ for sufficiently small ε. Furthermore, Λ_p comes with a parametrization and framing, ie a trivialization of its normal bundle. Note that a framing of Λ_p is equivalent to the framing of the conformal symplectic normal bundle of Λ_p; see [24]. Hence parametrized Legendrians come with a canonical framing.

Suppose that $c_1 < c_2$ are regular values of φ and $W^{c_2} \setminus W^{c_1}$ contains a unique critical point p of φ. Then $W^{c_2} \setminus W^{c_1}$ is an elementary Weinstein cobordism between Y^{c_1} and Y^{c_2} and the symplectomorphism type of W^{c_2} is determined by the symplectomorphism type of W^{c_1} along with the framed isotopy class of the isotropic sphere $\Lambda_p \subset Y^{c_1}$. If φ is an arbitrary Weinstein Morse function on W with distinct critical values, then W can be viewed as the concatenation of such elementary Weinstein cobordisms.

On the other hand, one can explicitly construct such elementary cobordisms and use them to modify Liouville domains. Given a Liouville domain X and a framed isotropic sphere Λ in its contact boundary $Y = \partial X$, we can attach an elementary Weinstein cobordism with critical point p and $\Lambda_p = \Lambda$ to X to obtain a new Liouville domain that we denote by X_Λ or $X \cup H^k_\Lambda$, where $k = \operatorname{ind} p = \dim \Lambda + 1$. This operation is called *Weinstein handle attachment* and Λ is called the *attaching sphere* of the Weinstein handle. If X is Weinstein, then so is X_Λ. If the dimension of $\Lambda \subset Y^{2n-1}$ is less than $n-1$, the handle attachment operation and Λ itself are all called *subcritical*. So any (subcritical) Weinstein domain can be obtained by attaching (subcritical) Weinstein handles to the standard Weinstein structure on B^{2n}.

The corresponding modification of contact manifolds by Weinstein handle attachment is called *contact surgery*. If $\Lambda \subset (Y, \xi)$ is a framed isotropic sphere, then there exists an elementary Weinstein cobordism W with $\partial_- W = (Y, \xi)$ and attaching sphere Λ. Then we say $\partial_+ W$ is the result of contact surgery on Λ and denote this by Y_Λ or $Y \cup H^k_\Lambda$. In particular, the contact boundary of any (subcritical) Weinstein domain can be obtained by doing (subcritical) contact surgery to $(S^{2n-1}, \xi_{\mathrm{std}}) = \partial B^{2n}$.

2.1.3 Weinstein homotopies The natural notion of equivalence between Weinstein structures $(W, \lambda_0, \varphi_0)$ and $(W, \lambda_1, \varphi_1)$ on a fixed manifold W is a *Weinstein homotopy*, ie a 1–parameter family of Weinstein structures $(W, \lambda_t, \varphi_t)$ for $t \in [0, 1]$ connecting them, where φ_t is allowed to have birth-death critical points. Weinstein homotopic domains have exact symplectomorphic completions [7].

We will prove our main result Theorem 3.1 by starting with an arbitrary Weinstein domain and then applying a special Weinstein homotopy. As in the smooth setting, Weinstein homotopies consist of three elementary moves: doing an isotopy of the

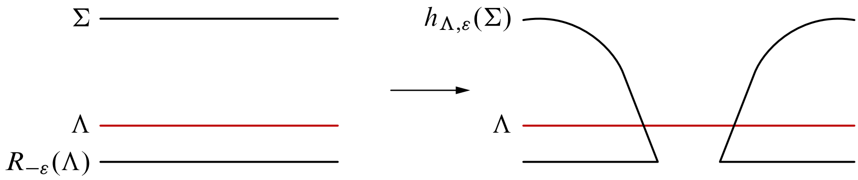

Figure 1: Front projection of handle-slide $h_\Lambda(\Sigma)$ of Σ over Λ.

attaching spheres through isotropic submanifolds, moving critical points that are not connected by gradient trajectories past each other, and sliding handles of the same index over each other. The only difference between the Weinstein and smooth setting is the first move: in the Weinstein case, the isotopies of attaching spheres must be through isotropics instead of arbitrary embedded spheres. Since subcritical handles satisfy an h–principle [7], Weinstein domains are essentially characterized by their index n handles, in particular the Legendrian attaching spheres of these critical handles. Therefore, it suffices to see how these moves affect Legendrians.

The first move implies that if Λ_1 and Λ_2 are isotopic Legendrians in ∂W, then $W \cup H^n_{\Lambda_1}$ and $W \cup H^n_{\Lambda_2}$ are Weinstein homotopic. The second move implies that if Λ_1 and Λ_2 are disjoint Legendrians in ∂W (which is true by dimension reasons if they are in general position), then $(W \cup H^n_{\Lambda_1}) \cup H^n_{\Lambda_2}$ and $(W \cup H^n_{\Lambda_2}) \cup H^n_{\Lambda_1}$ are Weinstein homotopic. In particular, we can write the resulting Weinstein domain as $W \cup H^n_{\Lambda_1} \cup H^n_{\Lambda_2}$ without any parentheses and it will be well defined up to Weinstein-homotopy.

We now discuss the last move, the handle-slide, which will be the most important for us. We will study Legendrians via their front projection. If $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{\mathrm{std}}) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^1$, then the front projection of Λ is the image of Λ in $\mathbb{R}^{n+1}$ under the projection to the first $\mathbb{R}^n$ and $\mathbb{R}^1$ components. Handles-slides were described in terms of front projections by Casals and Murphy [4].

Proposition 2.1 [4, Proposition 2.4] *Let (Y, ξ) be a contact manifold, and let $\Lambda, \Sigma \subset (Y, \xi)$ be two disjoint Legendrian submanifolds such that Λ is a sphere. Suppose there exists a Darboux chart U where the front projections of Σ and Λ look as in the left-hand side of Figure 1. Then, for sufficiently small $\varepsilon > 0$, the Legendrians Σ and $h_{\Lambda,\varepsilon}(\Sigma)$ presented in Figure 1 are Legendrian isotopic in the surgered contact manifold Y_Λ.*

Here $R_{-\varepsilon}(\Lambda)$ is the image of Λ under the negative time ε Reeb flow. We also note that the Legendrians in Figure 1 are extended by spherical symmetry out of the page. Furthermore, we note that the Darboux chart must have sufficient size so that front projections depicted in Figure 1 make sense; in particular, the size of the chart in the y_i direction must be at least as big as the slope of the front projection of $h_{\Lambda,\varepsilon}(\Lambda)$. For us, the key implication of Proposition 2.1 is that $W \cup H^n_\Lambda \cup H^n_\Sigma$ is Weinstein homotopic to $W \cup H^n_\Lambda \cup H^n_{h_\Sigma(\Lambda)}$ (and also to $W \cup H^n_{h_\Sigma(\Lambda)} \cup H^n_\Lambda$ by the above discussion).

Remark 2.2 Proposition 2.1 also holds if $\Sigma = \Sigma_1 \amalg \cdots \amalg \Sigma_k$ is a Legendrian link with several components. We inductively construct the new handle-slid link and show that it is isotopic to Σ in Y_Λ. We first take $\varepsilon_1 > 0$ sufficiently small that Σ is disjoint from an ε_1–neighborhood of Λ in $J^1(\Lambda) \subset Y$. We also take U_1 so that $\Sigma_1 \cap U_1$ and $\Lambda \cap U_1$ look as in the left-hand side of Figure 1 and $\Sigma_i \cap U_1 = \varnothing$ for $i \geq 2$. Then we can handle-slide Σ_1 over Λ via U_1 and the resulting Legendrian $h_{\Lambda,\varepsilon_1}(\Sigma_1)$ is isotopic to Σ_1 in Y_Λ by Proposition 2.1. In fact, something stronger holds. The isotopy in Proposition 2.1 is local since it is obtained by pushing a small disk of Σ_1 (starting from the chart U_1) past the belt sphere of Λ in Y_Λ. Therefore, since $\Sigma_2, \dots, \Sigma_k$ are disjoint from an ε_1–neighborhood of Λ in Y and the chart U_1, the handle-slid Legendrian $h_{\Lambda,\varepsilon_1}(\Sigma_1)$ is isotopic to Σ_1 in $Y_\Lambda \backslash (\Sigma_2 \amalg \cdots \amalg \Sigma_k)$, where we view $\Sigma_2, \dots, \Sigma_k$ as Legendrians of Y_Λ. Hence the link $h_{\Lambda,\varepsilon_1}(\Sigma_1) \amalg \Sigma_2 \amalg \cdots \amalg \Sigma_k$ is isotopic to $\Sigma_1 \amalg \Sigma_2 \amalg \cdots \amalg \Sigma_k$ in Y_Λ. Now we build the rest of the handle-slid link by induction and show that it is isotopic to the original link Σ at each stage. Namely, suppose we have constructed the i^{th} link $h_i(\Sigma) := h_{\Lambda,\varepsilon_1}(\Sigma_1) \amalg \cdots \amalg h_{\Lambda,\varepsilon_i}(\Sigma_i) \amalg \Sigma_{i+1} \amalg \cdots \amalg \Sigma_k$ and proved that it is isotopic to $h_{i-1}(\Sigma)$ in Y_Λ. Next we construct $h_{i+1}(\Sigma) := h_{\Lambda,\varepsilon_1}(\Sigma_1) \amalg \cdots \amalg h_{\Lambda,\varepsilon_i}(\Sigma_i) \amalg h_{\Lambda,\varepsilon_{i+1}}(\Sigma_{i+1}) \amalg \Sigma_{i+2} \amalg \cdots \amalg \Sigma_k$ by taking sufficiently small $\varepsilon_{i+1} < \varepsilon_j$ for all $j \leq i$ and a chart U_{i+1} disjoint from $h_i(\Sigma) \backslash \Sigma_{i+1}$ such that Σ_{i+1} and Λ appear in U_{i+1} as in Figure 1. As explained above, the new link $h_{i+1}(\Sigma)$ is Legendrian isotopic to the previous link $h_i(\Sigma)$ in Y_Λ since $h_i(\Sigma) \backslash \Sigma_{i+1}$ is disjoint from U_{i+1} and $h_i(\Sigma)$ is disjoint from an ε_{i+1}–neighborhood of Λ (since the Legendrians in $h_i(\Sigma)$ are at most ε_i–close to Λ), which proves the inductive $i+1$ case. For $i = k$, we get the desired Legendrian $h_k(\Sigma)$ which is isotopic to Σ in Y_Λ by induction. This implies that $W \cup H^n_\Lambda \cup H^n_{\Sigma_1} \cup \cdots \cup H^n_{\Sigma_k}$ is Weinstein homotopic to $W \cup H^n_\Lambda \cup H^n_{h_\Lambda(\Sigma_1)} \cup \cdots \cup H^n_{h_\Lambda(\Sigma_k)}$, a fact that we will use repeatedly later.

We also note that the handle-slide depends on more than just the data of Σ and Λ. The resulting Legendrian depends crucially on the choice of chart U, where Λ and Σ

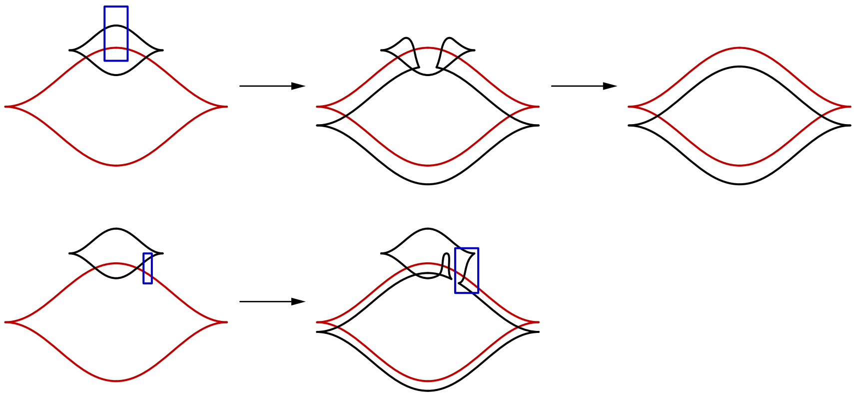

Figure 2: Handle-slides using different charts result in nonisotopic Legendrians.

appear as in the left-hand side of Figure 1. We will use the notation $h_{\Lambda,\varepsilon,U}(\Sigma)$ when we emphasize the dependence on U. In particular, different chart choices U_1 and U_2 can result in Legendrians $h_{\Lambda,\varepsilon,U_1}(\Sigma)$ and $h_{\Lambda,\varepsilon,U_2}(\Sigma)$ that are not Legendrian isotopic in Y (but are still smoothly isotopic in Y); however, $h_{\Lambda,\varepsilon,U_1}(\Sigma)$ and $h_{\Lambda,\varepsilon,U_2}(\Sigma)$ are Legendrian isotopic in Y_Λ. We also note that Σ and $h_{\Lambda,\varepsilon}(\Sigma)$ will generally not be *smoothly* isotopic in Y, while they are Legendrian isotopic in Y_Λ.

Example 2.3 We start with a Legendrian link consisting of two linked unknots in $(\mathbb{R}^{2n-1}, \xi_{\mathrm{std}})$, with one Legendrian the Reeb push-off of the other Legendrian; see Figure 2. The blue box in the left of each row is the Darboux chart used in the handle-slide. In the top row, the handle-slide produces a linked pair of Legendrian unknots (which can be seen by doing a Legendrian Reidemeister move), ie $h^{\mathrm{top}}_{\Lambda_{\mathrm{unknot}}}(\Lambda_{\mathrm{unknot}}) = \Lambda_{\mathrm{unknot}}$. In the bottom row, the handle-slide results in a link where one of the Legendrians is loose, ie $h^{\mathrm{bottom}}_{\Lambda_{\mathrm{unknot}}}(\Lambda_{\mathrm{unknot}}) = \Lambda_{\mathrm{loose}}$. The blue box on the right is the loose chart of this Legendrian; see Section 2.2 for definition. Since the Legendrian unknot is not loose, the handle-slid Legendrians $h^{\mathrm{top}}_{\Lambda_{\mathrm{unknot}}}(\Lambda_{\mathrm{unknot}})$ and $h^{\mathrm{bottom}}_{\Lambda_{\mathrm{unknot}}}(\Lambda_{\mathrm{unknot}})$ are not isotopic in the original contact manifold $(\mathbb{R}^{2n-1}, \xi_{\mathrm{std}})$. Of course, these Legendrians are both isotopic in the surgered manifold $Y_{\Lambda_{\mathrm{unknot}}}$ since they are both isotopic to the push-off of the attaching sphere there, ie the image of $\Lambda_{\mathrm{unknot}}$ in $Y_{\Lambda_{\mathrm{unknot}}}$.

2.2 Loose Legendrians and flexible Weinstein domains

There exist many Legendrians with rich symplectic topology invisible from the point of view of algebraic topology. On the other hand, Murphy [33] showed that exists a

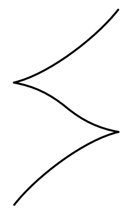

Figure 3: Front projection of Λ_0.

certain class of *loose* Legendrians which satisfy a h–principle and whose symplectic topology is governed by their underlying algebraic topology. These loose Legendrians are defined using a local model. We will use the following local model from Section 2.1 of [5]. Let $B^3 \subset (\mathbb{R}^3, \xi_{\mathrm{std}} = \ker \alpha_{\mathrm{std}})$ be a unit ball, and let Λ_0 be the 1–dimensional Legendrian whose front projection is shown in Figure 3. Let Q^{n-2}, where $n \geq 3$, be a closed manifold and U a neighborhood of the zero section $Q \subset T^*Q$. Then $\Lambda_0 \times Q \subset (B^3 \times U, \ker(\alpha_{\mathrm{std}} + \lambda_{\mathrm{std}}))$ is a Legendrian submanifold. This Legendrian is the *stabilization* over Q of the Legendrian $\{y = z = 0\} \times Q \subset (B^3 \times U, \ker(\alpha_{\mathrm{std}} + \lambda_{\mathrm{std}}))$.

Definition 2.4 A Legendrian $\Lambda^{n-1} \subset (Y^{2n-1}, \xi)$, where $n \geq 3$, is *loose* if there is a neighborhood $V \subset (Y, \xi)$ of Λ such that $(V, V \cap \Lambda)$ is contactomorphic to $(B^3 \times U, \Lambda_0 \times Q)$.

Remark 2.5 If $f\colon (U^{2n-1}, \xi_1) \to (V^{2n-1}, \xi_2)$ is an equidimensional contact embedding and $\Lambda \subset (U, \xi_1)$ is loose, then $f(\Lambda) \subset (V, \xi_2)$ is also loose.

A *formal Legendrian embedding* is an embedding $f\colon \Lambda \to (Y, \xi)$ together with a homotopy of bundle monomorphisms $F_s\colon T\Lambda \to TY$ covering f for all s such that $F_0 = df$ and $F_1(T\Lambda)$ is a Lagrangian subspace of ξ with its conformal symplectic structure. A *formal Legendrian isotopy* is an isotopy through formal Legendrian embeddings. Using these notions, we can state the h–principle for Legendrian embeddings, which has an existence and a uniqueness part:

- Any formal Legendrian of dimension at least two is formally Legendrian isotopic to a loose Legendrian [11; 15].
- Any two loose Legendrians that are formally Legendrian isotopic are genuinely Legendrian isotopic [33].

We now define a class of Weinstein domains introduced in [7] that are constructed by iteratively attaching Weinstein handles along loose Legendrians.

Definition 2.6 A Weinstein domain $(W^{2n}, \lambda, \varphi)$, where $n \geq 3$, is *flexible* if there exist regular values $c_1, \dots, c_k$ of φ such that $c_1 < \min \varphi < c_2 < \cdots < c_{k-1} < \max \varphi < c_k$ and, for all $i = 1, \dots, k-1$, $\{c_i \leq \varphi \leq c_{i+1}\}$ is a Weinstein cobordism with a single critical point p whose the attaching sphere Λ_p is either subcritical or a loose Legendrian in $(Y^{c_i}, \lambda|_{Y^{c_i}})$.

Flexible Weinstein *cobordisms* are defined similarly. Also, Weinstein handle attachment or contact surgery is called flexible if the attaching Legendrian is loose. So any flexible Weinstein domain can be constructed by iteratively attaching subcritical or flexible handles to $(B^{2n}, \omega_{\mathrm{std}})$. A Weinstein domain that is Weinstein homotopic to a Weinstein domain satisfying Definition 2.6 will also be called flexible. Finally, we note that subcritical domains are automatically flexible.

Our definition of flexible Weinstein domains is a bit different from the original definition in [7], where several critical points are allowed in $\{c_i \leq \varphi \leq c_{i+1}\}$. There are no gradient trajectories between these critical points and their attaching spheres form a loose *link* in $(Y^{c_i}, \lambda|_{Y^{c_i}})$, ie each Legendrian is loose in the complement of the others. These two definitions are the same up to Weinstein homotopy. Indeed if we have an ordered collection of Legendrians such that each one is loose in the complement of the previous ones, then we can use the loose Legendrian h–principle to move each Legendrian away from the loose charts of the previous ones so that all Legendrians are loose in the complement of each other.

Since they are built using loose Legendrians, which satisfy an h–principle, flexible Weinstein domains also satisfy an h–principle as proven by Cieliebak and Eliashberg [7]. Again, the h–principle has an existence and a uniqueness part:

- Any almost Weinstein domain of dimension at least six admits a flexible Weinstein structure in the same almost symplectic class.
- Any two flexible Weinstein domains that are almost symplectomorphic are Weinstein homotopic (and hence have exact symplectomorphic completions and contactomorphic boundaries).

3 Proofs of main results

In this section, we prove the results described in the introduction. We first prove a simpler version of Theorem 1.5 without as much control on the topology of the flexible subdomain.

Theorem 3.1 *Any Weinstein domain W^{2n} where $n \geq 3$ can be Weinstein homotoped to a Weinstein domain $V^{2n}_{\text{flex}} \cup H^n$ obtained by attaching a single n–handle to a flexible Weinstein domain V^{2n}_{flex}.*

Remark 3.2 Theorem 3.1 also holds for Weinstein cobordisms.

Proof of Theorem 3.1 Let $W^{2n} = (W^{2n}, \lambda, \varphi)$, where $n \geq 3$, be a Weinstein domain. By Lemma 12.20 of [7], we can Weinstein homotope W so that φ is self-indexing, ie if p is a critical point of index k, then $\varphi(p) = k$. In particular, we can assume that W is the result of attaching k index n handles to a subcritical Weinstein domain W_{sub} along disjoint Legendrians $\Lambda_1, \dots, \Lambda_k$.

If $k = 0$, then $W = W_{\text{sub}} = W_{\text{sub}} \cup H^{n-1} \cup H^n$, where H^{n-1} and H^n are two canceling handles of respective indices $n-1$ and n; the domain $W_{\text{sub}} \cup H^{n-1}$ is subcritical and hence flexible. If $k = 1$, then $W = W_{\text{sub}} \cup H^n_{\Lambda_1}$; again W_{sub} is subcritical and hence flexible. Therefore we can assume $W = W_{\text{sub}} \cup H^n_{\Lambda_1} \cup \dots \cup H^n_{\Lambda_k}$ for some $k \geq 2$.

The key step is to handle-slide $H_{\Lambda_2}, \dots, H_{\Lambda_k}$ over H_{Λ_1}. We will do this by induction. More precisely, we will prove that for every j with $2 \leq j \leq k$, W is Weinstein homotopic to $W_{\text{sub}} \cup H^n_{\Lambda'_1} \cup \dots \cup H^n_{\Lambda'_k}$ for some Legendrian link $\coprod_{i=1}^k \Lambda'_i$ such that $\coprod_{i=2}^j \Lambda'_i$ is a loose link in ∂W_{sub}. Then the case $j = k$ completes the proof since then W is Weinstein homotopic to the flexible domain $W_{\text{sub}} \cup H^n_{\Lambda'_2} \cup \dots \cup H^n_{\Lambda'_k}$ with the single handle $H^n_{\Lambda'_1}$ attached. The proof shows that we can assume that Λ_1 actually stays fixed throughout.

We first prove the base case $j = 2$. We begin by modifying Λ_1 and Λ_2 by Legendrian isotopies that move only a small neighborhood of a single point, ie the resulting Legendrians are the Legendrian connected sum of Λ_1 and Λ_2 with certain Legendrian unknots. More precisely, let U_2 be a Darboux ball in the contact manifold ∂W_{sub} that is disjoint from $\Lambda_1 \cup \dots \cup \Lambda_k$. Let S_2 be a Legendrian unknot in U_2 and let T_2 be a negative Reeb push-off of S_2 also contained in U_2 so that S_2 and T_2 are symplectically unlinked. We apply a Legendrian "Reidemeister move" to S_2 so that it appears as in Figure 4; this move is a Legendrian isotopy which is contained in U_2 and the resulting Legendrian, which we also call S_2, is still symplectically unlinked with T_2. For 1–dimensional Legendrians, this isotopy is the first Reidemeister move and in higher dimensions (as in our situation) it results in a spherically rotated version of this

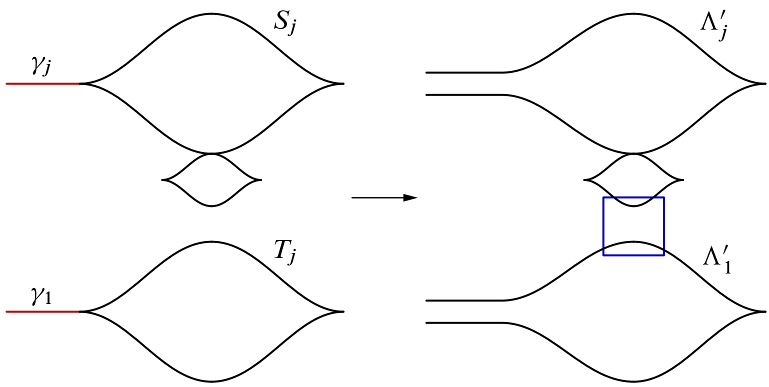

Figure 4: Left: front projections of S_j and T_j and isotropic arcs γ_1 and γ_j (in red) in U_j; right: front projections of the connected sums $\Lambda'_j := \Lambda_j \sharp S_j$ and $\Lambda'_1 := \Lambda_1 \sharp T_j$ formed along γ_j and γ_1, respectively; the blue box is the chart we will use to handle-slide Λ'_j over Λ'_1.

Reidemeister move. Note that the isotopy is not obtained by spherically rotating the 1–dimensional isotopy; see [4] for details on this isotopy.

Now we choose isotropic arcs γ_1 and γ_2 connecting Λ_1 to T_2 and Λ_2 to S_2 respectively. Since these arcs are subcritical, we can assume that they are disjoint; furthermore, we can assume that γ_1 is disjoint from Λ_i for $i \neq 1$ and γ_2 is disjoint from Λ_i for $i \neq 2$. We can also ensure that they intersect U_2 as depicted in the left-hand side of Figure 4. Let $\Lambda'_1 := \Lambda_1 \sharp T_2$ be the Legendrian connected sum of Λ_1 and T_2 along γ_1; see [8] for details about the connected sum operation. Similarly, let $\Lambda'_2 := \Lambda_2 \sharp S_2$ be the Legendrian connected sum of Λ_2 and S_2 along γ_2. By choice of γ_1 and γ_2, the Legendrians $\Lambda'_1 \cap U_2$ and $\Lambda'_2 \cap U_2$ look as in right-hand side of Figure 4. Since U_2 is disjoint from Λ_1 and T_2 is a Legendrian unknot in U_2, Λ'_1 is isotopic to Λ_1; we pull the unknot T_2 to Λ_1 using the isotropic arc γ_1. Similarly, Λ'_2 is Legendrian isotopic to Λ_2. In fact, the entire Legendrian link $\Lambda'_1 \amalg \Lambda'_2 \amalg \Lambda_3 \amalg \cdots \amalg \Lambda_k$ is Legendrian isotopic to the link $\Lambda_1 \amalg \Lambda_2 \amalg \Lambda_3 \amalg \cdots \amalg \Lambda_k$ because γ_1 and γ_2 are disjoint from $\Lambda_3, \dots, \Lambda_k$ and S_2 and T_2 are symplectically unlinked in U_2.

Now we handle-slide Λ'_2 over Λ'_1. We first take sufficiently small $\varepsilon_2 > 0$ so that an ε_2–neighborhood of Λ'_1 is disjoint from all other Legendrians. The ball U_2 contains a smaller chart V_2 where Λ'_1 and Λ'_2 look as in Figure 1; see the blue box in the right-hand side of Figure 4. So we can use this chart to handle-slide Λ'_2 over Λ'_1 and produce $h_{\Lambda'_1,\varepsilon_2}(\Lambda'_2)$; see the Legendrian in black in the right-hand side of Figure 5. Then $h_{\Lambda'_1,\varepsilon_2}(\Lambda'_2)$ is isotopic to the Legendrian Λ'_2 in $\partial(W_{\text{sub}} \cup H^n_{\Lambda'_1})$; in fact, the entire link $h_{\Lambda_1',\varepsilon_2}(\Lambda_2') \amalg \Lambda_3 \amalg \cdots \amalg \Lambda_k$ is Legendrian isotopic to $\Lambda_2' \amalg \Lambda_3 \amalg \cdots \amalg \Lambda_k$ in $\partial(W_{\mathrm{sub}} \cup H^n_{\Lambda_1'})$, as explained in Remark 2.2. In particular,

$$W_{\mathrm{sub}} \cup H^n_{\Lambda_1'} \cup H^n_{h_{\Lambda_1',\varepsilon_2}(\Lambda_2')} \cup H^n_{\Lambda_3} \cup \cdots \cup H^n_{\Lambda_k}$$

is Weinstein homotopic to $W_{\mathrm{sub}} \cup H^n_{\Lambda_1'} \cup H^n_{\Lambda_2'} \cup H^n_{\Lambda_3} \cup \cdots \cup H^n_{\Lambda_k}$ and hence to W. Finally, we note that the size requirement of the Darboux chart for the handle-slide is satisfied in our situation. We can take the bottom branch of S_2 and the top branch of T_2 to be close enough that the slope of the front projection of the handle-slid Legendrian is arbitrarily small; hence the y_i coordinate of the chart can be arbitrarily small for our handle-slide.

We observe that $h_{\Lambda_1'}(\Lambda_2')$ is loose in $\partial W_{\mathrm{sub}}$. The blue box in Figure 5 is the loose chart of $h_{\Lambda_1',\varepsilon_2}(\Lambda_2')$ in U_2. Recall that we have spherical symmetry in the handle-slide region so it is loose with $Q^{n-2} = S^{n-2}$; see Definition 2.4. However, $h_{\Lambda_1',\varepsilon_2}(\Lambda_2')$ is not loose *in the complement of* Λ_1' since Λ_1' intersects the loose chart of $h_{\Lambda_1',\varepsilon_2}(\Lambda_2')$. This completes the case $j = 2$. Note that we can extend the Legendrian isotopy of Λ_1' back to Λ_1 to an ambient contact isotopy and hence assume that $\Lambda_1' = \Lambda_1$.

Now suppose that the $j-1$ case holds for some $j \geq 3$. So we have Weinstein homotoped W to $W_{\mathrm{sub}} \cup H^n_{\Lambda_1} \cup \cdots \cup H^n_{\Lambda_k}$ (relabeling the Legendrians) so that $\coprod_{i=2}^{j-1} \Lambda_i$ is a loose link (but not loose in the complement of Λ_1). Again we take a Darboux ball U_j that is disjoint from all the Legendrians and unlinked Legendrian unknots $S_j, T_j \subset U_j$. Then we form $\Lambda_1' := \Lambda_1 \natural S_j$ and $\Lambda_j' := \Lambda_j \natural T_j$ using arcs γ_1 and γ_j that are disjoint from the other Legendrians. Then we take sufficiently small ε_j (smaller than the previous ε_{j-1}) and use the chart in U_j to handle-slide Λ_j' over Λ_1' and get a new Legendrian $h_{\Lambda_1'}(\Lambda_j')$. Then by Proposition 2.1 (and Remark 2.2),

$$W_{\mathrm{sub}} \cup H^n_{\Lambda_1'} \cup H^n_{\Lambda_2} \cup \cdots \cup H^n_{\Lambda_{j-1}} \cup H^n_{h_{\Lambda_1'}(\Lambda_j')} \cup H^n_{\Lambda_{j+1}} \cup \cdots \cup H^n_{\Lambda_k}$$

is Weinstein homotopic to $W_{\mathrm{sub}} \cup H^n_{\Lambda_1'} \cup H^n_{\Lambda_2} \cup \cdots \cup H^n_{\Lambda_{j-1}} \cup H^n_{\Lambda_j'} \cup H^n_{\Lambda_{j+1}} \cup \cdots \cup H^n_{\Lambda_k}$ and hence to W. As before, we can see explicitly that $h_{\Lambda_1'}(\Lambda_j')$ is loose in $\partial W_{\mathrm{sub}}$ (but not in the complement of Λ_1', which intersects its loose chart). Most importantly the loose chart of $h_{\Lambda_1'}(\Lambda_j')$ is contained in U_j, which is disjoint from $\Lambda_2, \ldots, \Lambda_{j-1}$. Therefore $h_{\Lambda_1'}(\Lambda_j')$ is loose in the complement of these Legendrians, which form a loose link by the induction hypothesis. So $\Lambda_2 \amalg \cdots \amalg \Lambda_{j-1} \amalg h_{\Lambda_1'}(\Lambda_j')$ is also a loose link, which proves the j^{th} inductive case. Again by applying an ambient contact isotopy to all the Legendrians, we can assume that $\Lambda_1' = \Lambda_1$. □

Now we give an example illustrating the entire procedure in Theorem 3.1.

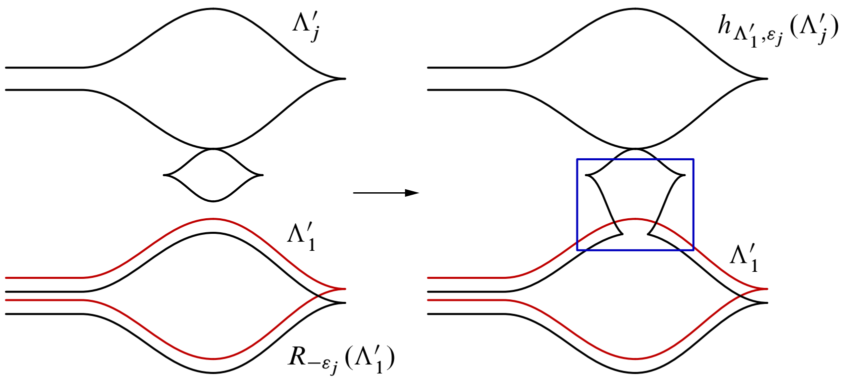

Figure 5: Left: front projection of Λ'_j, Λ'_1, and $R_{-\varepsilon_j}(\Lambda'_1)$ in U_j; right: front projection of $h_{\Lambda'_1,\varepsilon_j}(\Lambda'_j)$ in U_j; the blue box is a loose chart of $h_{\Lambda'_1,\varepsilon_j}(\Lambda'_j)$ in U_j.

Example 3.3 This example shows that $T^*S^n \natural T^*S^n \natural T^*S^n$, the boundary connected sum of three copies of T^*S^n, can be Weinstein homotoped to $W_{\text{flex}} \cup H^n$ for some flexible domain W_{flex}. We begin with the "natural" presentation of $T^*S^n \natural T^*S^n \natural T^*S^n$ of the form $B^{2n} \cup H^n_{\Lambda_1} \cup H^n_{\Lambda_2} \cup H^n_{\Lambda_3}$, where Λ_1, Λ_2 and Λ_3 are three unlinked Legendrian unknots in $(S^{2n-1}, \xi_{\text{std}})$. In Figure 6, Λ_1 is in red, Λ_2 (and its image after handle-slides) is in black, and Λ_3 (and its image after handle-slides) is in blue. The top diagram in Figure 6 denotes the setup after one iteration of the construction; the Legendrians are now Λ_1, $h_{\Lambda_1}(\Lambda_2)$, and Λ_3. The middle diagram in Figure 6 is the first part of the second iteration when we change Λ_1 to Λ'_1 and it bring it closer to Λ_3. The bottom diagram in Figure 6 shows the three Legendrians Λ_1, $h_{\Lambda_1}(\Lambda_2)$, and $h_{\Lambda'_1}(\Lambda_3)$ after the second iteration of the construction, ie handle-sliding Λ_3 over Λ'_1. Then $h_{\Lambda_1}(\Lambda_2)$ and $h_{\Lambda'_1}(\Lambda_3)$ form a loose link since $h_{\Lambda_1}(\Lambda_2)$ is a loose Legendrian and $h_{\Lambda'_1}(\Lambda_3)$ is loose in the complement of $h_{\Lambda_1}(\Lambda_2)$. We take W_{flex} to be $B^{2n} \cup H^n_{h_{\Lambda_1}(\Lambda_2)} \cup H^n_{h_{\Lambda'_1}(\Lambda_3)}$. Thus the original domain $T^*S^n \natural T^*S^n \natural T^*S^n$ is homotopic to $W_{\text{flex}} \cup H^n_{\Lambda'_1}$. Note that $h_{\Lambda_1}(\Lambda_2)$ and $h_{\Lambda'_1}(\Lambda_3)$ are not loose in the complement of Λ'_1, which intersects their loose charts. For simplicity's sake, W_{flex} in this example is not actually $(T^*S^n \natural T^*S^n)_{\text{flex}}$; it will have the wrong intersection form (in some dimensions n) and so will not even be diffeomorphic to $T^*S^n \natural T^*S^n$. However it is possible to do the construction so that W_{flex} is $(T^*S^n \natural T^*S^n)_{\text{flex}} \cup H^n$.

Although the order in which handles are attached does not affect the ambient domain (up to homotopy), it does affect which Weinstein subdomains are produced by a particular Weinstein presentation. To emphasize this, in Figure 7 we have depicted the Cerf

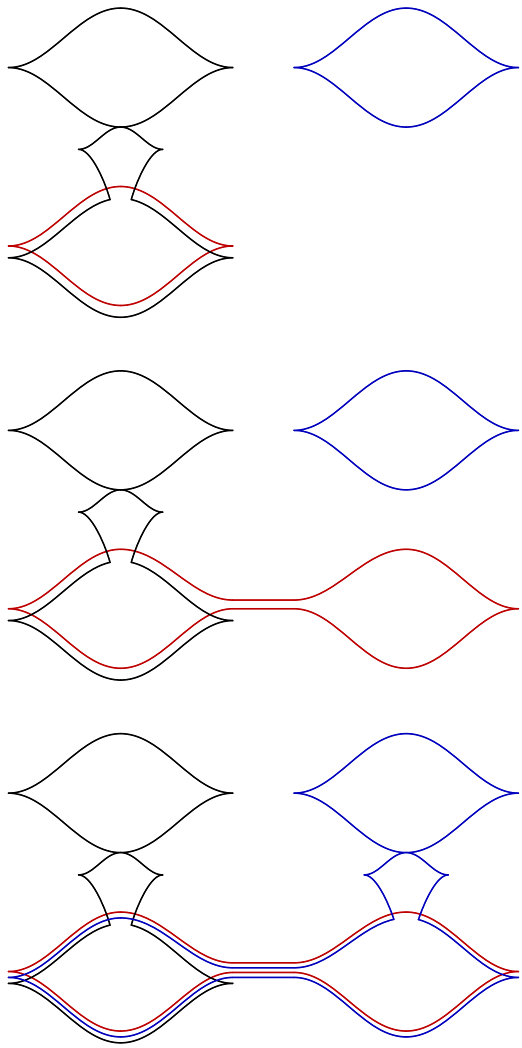

Figure 6: Theorem 3.1 applied to $T^*S^n_{\mathrm{std}} \natural T^*S^n_{\mathrm{std}} \natural T^*S^n_{\mathrm{std}}$.

diagram of the Weinstein homotopy for $T^*S^n \natural T^*S^n \natural T^*S^n$ discussed above, ie the graph of critical values of the index n critical points of the Weinstein Morse functions φ_t over the parameter space $t \in [0,1]$. That is, if p_i for $i = 1, 2, 3$ are the critical points with respective attaching spheres Λ_i in the regular level set $(S^{2n-1}, \xi_{\mathrm{std}})$, then the three line graphs depict $\varphi_t(p_i)$ for $t \in [0,1]$. In Figure 7, we have labeled the graph of $\varphi_t(p_i)$ by its attaching sphere. Handles are attached in order of the critical values of the corresponding critical points, from lowest to highest. At the beginning of the

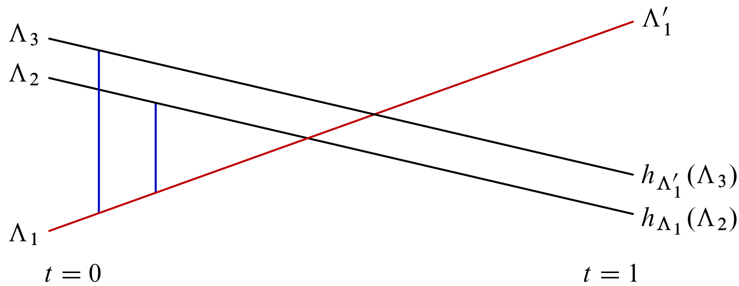

Figure 7: Cerf diagram of the Weinstein homotopy for $T^*S^n_{\mathrm{std}} \natural T^*S^n_{\mathrm{std}} \natural T^*S^n_{\mathrm{std}}$.

homotopy, $\varphi_0(p_2)$ and $\varphi_0(p_3)$ are greater than $\varphi_0(p_1)$ since we need to handle-slide the Λ_2 and Λ_3 handles over Λ_1. These handle-slide moments are depicted by the two vertical blue lines in Figure 7. After the two handle-slides are performed, the attaching spheres of p_2 and p_3 become $h_{\Lambda_1}(\Lambda_2)$ and $h_{\Lambda_1'}(\Lambda_3)$ respectively, as shown on the right-hand side of Figure 7. Away from the handle-slide moments, the homotopy changes the Legendrian attaching spheres just by Legendrian isotopy. Finally, the homotopy makes the critical value of p_1 greater than the critical values of p_2 and p_3, which is possible by the second Weinstein homotopy move (see Section 2.1.3). As a result, the Weinstein domain W_{flex} with attaching spheres $h_{\Lambda_1}(\Lambda_2)$ and $h_{\Lambda_1'}(\Lambda_3)$ is a sublevel set of φ_1 and hence a Weinstein subdomain of $T^*S^n \natural T^*S^n \natural T^*S^n$.

Note that the Weinstein homotopy in Theorem 3.1 involved just handle-slides. If we first create a pair of symplectically canceling handles and then handle-slide, we can achieve better control over the topology of the flexible subdomain. This is the approach we will take in the following proof of Theorem 1.5, which shows that W can be homotoped to $W_{\mathrm{flex}} \cup C^{2n}$ for some smoothly trivial Weinstein cobordism C^{2n} with two Weinstein handles. For example, this result shows that $T^*S^n \natural T^*S^n \natural T^*S^n$ can be Weinstein homotoped to $(T^*S^n \natural T^*S^n \natural T^*S^n)_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda}$, where the last two handles are smoothly canceling.

Proof of Theorem 1.5 We will assume $W = W_{\mathrm{sub}} \cup H^n_{\Lambda_1} \cup \cdots \cup H^n_{\Lambda_k}$ for $k \geq 1$. First, we attach a symplectically canceling pair of index $n-1$ and n handles H^{n-1} and $H^n_{\Lambda_0}$ to W in a small Darboux chart B^{2n} so that $W = W \natural (B^{2n} \cup H^{n-1} \cup H^n_{\Lambda_0}) = W_{\mathrm{sub}} \cup (H^{n-1} \cup H^n_{\Lambda_0}) \cup H^n_{\Lambda_1} \cup \cdots \cup H^n_{\Lambda_k}$. Now we proceed as in the proof of Theorem 3.1, with slight modifications. We first bring all the Λ_i for $i \geq 1$ close to Λ_0 by taking U_i in the proof of Theorem 3.1 to be contained in ∂B^{2n}. The main difference from before is that now we do two handle-slides of Λ_i, for each $i \geq 1$, over Λ_0,

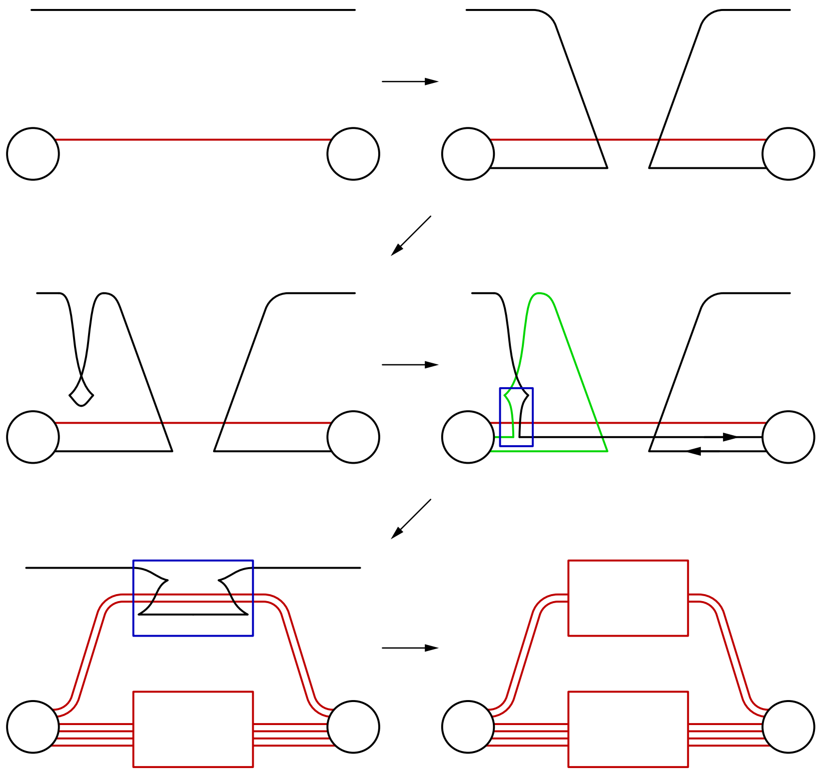

Figure 8: Front projections of Λ_0^{n-1} (in red) and Λ_i^{n-1} (in black) and their subsequent images under the moves in Theorems 1.5 and 1.10, for n even; the blue box appearing in the fourth and fifth diagrams is the loose chart of $h^2_{\Lambda_0}(\Lambda_i)$ and $\varphi(h^2_{\Lambda_0}(\Lambda_i))$ respectively; the green portion of the Legendrian in the fourth diagram is the boundary of the Whitney disk between $h^2_{\Lambda_0}(\Lambda_i)$ and the belt sphere of H^{n-1}.

which produces the Legendrian $h^2_{\Lambda_0}(\Lambda_i)$. Before doing the second handle-slide, we perform a Reidemeister move. This move depends slightly on the parity of n. For n even, we do the usual Reidemeister move which modifies the Legendrian in just a point; see Figure 4. As a result, $h^2_{\Lambda_0}(\Lambda_i)$ is loose. Note that $h^2_{\Lambda_0}(\Lambda_i)$ intersects the belt sphere of H^{n-1} two times. We will now show that $h^2_{\Lambda_0}(\Lambda_i)$ has algebraic intersection number zero with this belt sphere. Indeed, consider the orientation of the two branches of $h^2_{\Lambda_0}(\Lambda_i)$ as they approach the belt sphere. The tangent space of these branches can be decomposed into a 1–dimensional part in the "page" (as depicted in Figure 8) and an $(n-2)$–dimensional part transverse to the page. The tangent spaces parallel to the page have opposite orientations for the two branches; see the arrows in Figure 8. The tangent spaces transverse to the page differ by the antipodal map on D^{n-1} due to passage through the crossing point of the Reidemeister move. Hence if n is even, the two branches of $h^2_{\Lambda_0}(\Lambda_i)$ have opposite orientations and so $h^2_{\Lambda_0}(\Lambda_i)$ has algebraic intersection zero with the belt sphere of H^{n-1} as desired. If n is odd, the Legendrian $h^2_{\Lambda_0}(\Lambda_i)$ as described above has algebraic intersection two with the belt sphere. So instead of doing the Reidemeister move as in the even case, we perform the 1–dimensional Reidemeister move spun by $S^{n-2} \subset \Lambda^{n-1}$; so this move modifies Λ^{n-1} in a neighborhood of S^{n-2}. Then we form $h^2_{\Lambda_0}(\Lambda_i)$ by handle-sliding using a chart that intersects the bottom branch of this Legendrian. Now there is no crossing point and so $h^2_{\Lambda_0}(\Lambda_i)$ has algebraic intersection zero with the belt sphere of H^{n-1}; this modified procedure works for the n even case as well but it is more complicated to depict, which is why we have explained the n even case separately. Finally, we note that $h^2_{\Lambda_0}(\Lambda_i)$ is loose, even though we have used a different Reidemeister move and so a loose chart as defined in Definition 2.4 does not obviously appear. Namely, $h^2_{\Lambda_0}(\Lambda_i)$ has a 1–dimensional zigzag arc and since this arc is in a Darboux ball, it has arbitrary thickness and so defines a loose chart; see [4]. In conclusion, $h^2_{\Lambda_0}(\Lambda_i)$ is loose for all $n \geq 3$ and has algebraic intersection number zero with the belt sphere of H^{n-1}. We do this procedure for all the Legendrian Λ_i and so, as in Theorem 3.1, $h^2_{\Lambda_0}(\Lambda_1) \amalg \cdots \amalg h^2_{\Lambda_0}(\Lambda_k)$ forms a loose link; more precisely, the i^{th} Legendrian is loose in the complement of the previous $(i-1)$ Legendrians, which implies that the link is loose. Hence $W' := W_{\text{sub}} \cup H^{n-1} \cup H^n_{h^2_{\Lambda_0}(\Lambda_1)} \cup \cdots \cup H^n_{h^2_{\Lambda_0}(\Lambda_k)}$ is flexible and $W = W' \cup H^n_{\Lambda_0}$.

Since the algebraic intersection number of $h^2_{\Lambda_0}(\Lambda_i)$ with the belt sphere of H^{n-1} is zero, $n \geq 3$, and $\pi_1(\partial(B^{2n} \cup H^{n-1})) = 0$, we can use the Whitney trick to smoothly isotope $h^2_{\Lambda_0}(\Lambda_i)$ away from this belt sphere. In fact, we can assume that this smooth isotopy is supported in $\partial(B^{2n} \cup H^{n-1})$. To see this, note that we can take the boundary of the Whitney disk to lie in this region; see the green portion of Legendrian in the fourth diagram of Figure 8. This region is simply connected and hence the Whitney disk also lies in this region; so the isotopy is also supported in this region. Since $n \geq 3$, the Whitney disks will be generically disjoint for different i and so we can smoothly isotope the entire link $h^2_{\Lambda_0}(\Lambda_1) \amalg \cdots \amalg h^2_{\Lambda_0}(\Lambda_k)$ off the belt sphere of H^{n-1} (again via an isotopy supported in $\partial(B^{2n} \cup H^{n-1})$).

The Legendrian link $h^2_{\Lambda_0}(\Lambda_1) \amalg \cdots \amalg h^2_{\Lambda_0}(\Lambda_k)$ is loose and so the smooth isotopy can be approximated by a Legendrian isotopy. Since the smooth isotopy is supported in $\partial(B^{2n} \cup H^{n-1})$ and the Legendrians are loose in this region, the Legendrian isotopy is also supported in this region. Let φ_t be the ambient contact isotopy inducing this Legendrian isotopy and supported in a small neighborhood of the Legendrian isotopy; in particular φ_t is also supported in $\partial(B^{2n} \cup H^{n-1})$. Since $h^2_{\Lambda_0}(\Lambda_1) \amalg \cdots \amalg h^2_{\Lambda_0}(\Lambda_k)$ is a loose link, so is $\varphi(h^2_{\Lambda_0}(\Lambda_1)) \amalg \cdots \amalg \varphi(h^2_{\Lambda_0}(\Lambda_k))$, where $\varphi := \varphi_1$. Furthermore, we can assume that this link is loose in the complement of H^{n-1} and Λ_0 but not in the complement of $\varphi(\Lambda_0)$. See the fifth diagram in Figure 8. The upper Legendrian in black is $\varphi(h^2_{\Lambda_0}(\Lambda_i))$ and the blue box is its loose chart. The red Legendrian is $\varphi(\Lambda_0)$. This fifth diagram is purely schematic and is meant to demonstrate that $\varphi(\Lambda_0)$ intersects the belt sphere of H^{n-1} some number of times and is linked with $\varphi(h^2_{\Lambda_0}(\Lambda_1))$ in some way such that $\varphi(\Lambda_0)$ intersects the loose chart of $\varphi(h^2_{\Lambda_0}(\Lambda_i))$ (since Λ_0 intersected the loose chart of $h^2_{\Lambda_0}(\Lambda_i)$).

Now we apply the contact isotopy φ to all attaching Legendrians; see the transition from the fourth to the fifth diagram in Figure 8. As a result, we get that $W = W_{\mathrm{sub}} \cup H^{n-1} \cup H^n_{h^2_{\Lambda_0}(\Lambda_1)} \cup \cdots \cup H^n_{h^2_{\Lambda_0}(\Lambda_k)} \cup H^n_{\Lambda_0}$ is Weinstein homotopic to

$$W_{\mathrm{sub}} \cup H^{n-1} \cup H^n_{\varphi(h^2_{\Lambda_0}(\Lambda_1))} \cup \cdots \cup H^n_{\varphi(h^2_{\Lambda_0}(\Lambda_k))} \cup H^n_{\varphi(\Lambda_0)}.$$

The key point is that the latter presentation is Weinstein homotopic to

$$W_{\mathrm{sub}} \cup H^n_{\varphi(h^2_{\Lambda_0}(\Lambda_1))} \cup \cdots \cup H^n_{\varphi(h^2_{\Lambda_0}(\Lambda_k))} \cup H^{n-1} \cup H^n_{\varphi(\Lambda_0)}$$

because we can attach the handles $H^n_{\varphi(h^2_{\Lambda_0}(\Lambda_1))} \cup \cdots \cup H^n_{\varphi(h^2_{\Lambda_0}(\Lambda_k))}$ before H^{n-1} since $\varphi(h^2_{\Lambda_0}(\Lambda_1)) \amalg \cdots \amalg \varphi(h^2_{\Lambda_0}(\Lambda_k))$ is disjoint from the belt sphere of H^{n-1}. Let W'' be the domain

$$W_{\mathrm{sub}} \cup H^n_{\varphi(h^2_{\Lambda_0}(\Lambda_1))} \cup \cdots \cup H^n_{\varphi(h^2_{\Lambda_0}(\Lambda_k))}$$

obtained by viewing $\varphi(h^2_{\Lambda_0}(\Lambda_1)) \amalg \cdots \amalg \varphi(h^2_{\Lambda_0}(\Lambda_k))$ as a Legendrian link in $\partial W_{\mathrm{sub}}$. Then W is Weinstein homotopic to $W'' \cup H^{n-1} \cup H^n_{\varphi(\Lambda_0)}$. We note that W'' is flexible since $\varphi(h^2_{\Lambda_0}(\Lambda_1)) \amalg \cdots \amalg \varphi(h^2_{\Lambda_0}(\Lambda_k))$ is loose in the complement of H^{n-1}.

Finally, we show that the Weinstein cobordism $W \backslash W'' = H^{n-1} \cup H^n_{\varphi(\Lambda_0)}$ is smoothly trivial. Since φ is smoothly isotopic to the identity, $\varphi(\Lambda_0)$ is smoothly isotopic to Λ_0 in $\partial(W_{\mathrm{sub}} \cup H^{n-1})$. Since Λ_0 intersects the belt sphere of H^{n-1} exactly once, this isotopy gives Whitney disks that cancel out all intersection points between $\varphi(\Lambda_0)$ and the belt sphere of H^{n-1} (except for one). Since $n \geq 3$, the Whitney disks will be generically disjoint from the link $\varphi(h^2_{\Lambda_0}(\Lambda_1)) \amalg \cdots \amalg \varphi(h^2_{\Lambda_0}(\Lambda_k))$. So $\varphi(\Lambda_0)$ can be smoothly isotoped in the complement of this link to a sphere that intersects the belt sphere of H^{n-1} exactly once. This means that $\varphi(\Lambda_0)$ can be smoothly isotoped in $\partial(W'' \cup H^{n-1})$ to intersect this belt sphere exactly once, which proves that $W \backslash W'' = H^{n-1} \cup H^n_{\varphi(\Lambda_0)}$ is smoothly trivial.

Any almost symplectic structure on a smoothly trivial cobordism can be deformed relative to the negative end to the product almost symplectic structure. In particular, W and W'' are almost symplectomorphic. Since W'' is flexible, by the uniqueness h–principle [7] it is the flexibilization W_{flex} of W. □

Now we prove the 4–dimensional analog of Theorem 1.5.

Proof of Theorem 1.6 We take V^4 to be W' from the proof of Theorem 1.5, so that $W = V \cup H^2_{\Lambda_0}$. Note that V^4 is obtained by attaching a 1–handle and some 2–handles along $h^2_{\Lambda_0}(\Lambda_k)$ to W^4_{sub}. Each attaching knot for the 2–handles is stabilized in the complement of the previous ones; hence V^4 is a stabilized domain. Finally, we note that V^4 is simply homotopy equivalent to $W^4 \cup H^1$. To see this, we consider the 6–dimensional domain $V^4 \times B^2$; as can be seen explicitly, the attaching knots $h^2_{\Lambda_0}(\Lambda_k)$ are unknotted in the $B^6 \cup H^1$ region and hence can be smoothly isotoped to Λ_k. As a result, this domain is diffeomorphic to $(W^4 \cup H^1) \times B^2$. Here we do not use the Whitney trick directly since the region $B^6 \cup H^1$ is not simply connected. □

Using Theorem 1.5, we can prove Theorem 1.1, our result relating WCrit and Crit.

Proof of Theorem 1.1 By Theorem 1.5, we can Weinstein homotope any Weinstein domain W^{2n} where $n \geq 3$ to its flexibilization plus two smoothly canceling handles of respective indices $n-1$ and n, ie to $W_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_1}$, where Λ_1 can be smoothly isotoped to intersect the belt sphere of H^{n-1} exactly once. For any smooth Morse function f with critical points of index at most n on W, there is a Weinstein homotopy of W_{flex} to a Weinstein presentation with Weinstein Morse function f; see Theorem 14.1 of [7]. Furthermore, if f has $\partial W_{\mathrm{flex}}$ as a regular level set, then this Weinstein homotopy is fixed on $\partial W_{\mathrm{flex}}$ up to scaling. By Smale's handle-trading trick, there exists such a smooth function on W that minimizes the number of critical points, ie with $\mathrm{Crit}(W)$ critical points, and so we can Weinstein homotope W_{flex} to a Weinstein presentation with $\mathrm{Crit}(W)$ critical points. Since this homotopy is fixed up to scaling on $\partial W_{\mathrm{flex}}$, it extends to a Weinstein homotopy of $W_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_1}$, which is fixed up to scaling in $W \backslash W_{\mathrm{flex}}$. In particular, this homotopy on $W_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_1}$ does not alter the number of critical points in $W \backslash W_{\mathrm{flex}}$. Combining the homotopy of W to $W_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_1}$ and this second homotopy of $W_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_1}$ to a presentation with few critical points, we get a Weinstein homotopy of W to a Weinstein presentation with $\mathrm{Crit}(W)+2$ critical points: $\mathrm{Crit}(W)$ critical points in W_{flex} and 2 critical points in $W \backslash W_{\mathrm{flex}}$ due to the handles H^{n-1} and $H^n_{\Lambda_1}$. This proves the first claim in Theorem 1.1.

Now we prove the third claim in Theorem 1.1 about smoothly subcritical domains W^{2n}. If W^{2n} is Weinstein subcritical, then W^{2n} is flexible and so by the above discussion can be homotoped to a Weinstein presentation with $\mathrm{Crit}(W)$ critical points, ie $\mathrm{WCrit}(W)=\mathrm{Crit}(W)$. Conversely, suppose $\mathrm{WCrit}(W)=\mathrm{Crit}(W)$ and $\pi_1(W)=0$. If $\pi_1(W)=0$, the proof of Smale's h–cobordism theorem shows that $\mathrm{Crit}(W)$ equals the number of generators and relations for integral homology; see Theorem 6.1 of [39]. Then any minimizing smooth Morse function on W cannot have any critical points of index greater than $n-1$ since these critical points are algebraically unnecessarily; we can remove them and still have generators for integral homology since $H_n(W;\mathbb{Z})=0$ and $H_{n-1}(W;\mathbb{Z})$ is torsion-free for smoothly subcritical W. Hence if $\pi_1(W)=0$ and $\mathrm{WCrit}(W)=\mathrm{Crit}(W)$, then the minimal Weinstein presentation gives a minimal smooth presentation and so cannot have any critical points of index greater than $n-1$. Thus W is Weinstein subcritical. Finally, we note that if $\mathrm{WCrit}(W^{2n})\neq \mathrm{Crit}(W^{2n})$, then $\mathrm{WCrit}(W^{2n})=\mathrm{Crit}(W^{2n})+2$ since $\mathrm{WCrit}(W^{2n})\leq \mathrm{Crit}(W^{2n})+2$ by the first claim and $\mathrm{WCrit}(W^{2n})\equiv(\mathrm{Crit}(W^{2n})+2) \bmod 2$ by the Euler characteristic.

Now we prove the smoothly critical case. Suppose that ψ is a minimal smooth Morse function on W with $k=\mathrm{Crit}(W)$ critical points. By assumption, one of these critical points has index n (and the rest of the critical points have index at most n). By the previous discussion, we can assume that ψ is a Weinstein Morse function on W_{flex} and two other smoothly canceling handles H^{n-1} and $H^n_{\Lambda_1}$ are attached to W_{flex} to form W. The smooth isotopy from Λ_1 to canceling position gives some number of Whitney disks in $\partial(W_{\mathrm{flex}}\cup H^{n-1})$ pairing off all intersection points of Λ_1 and the belt sphere of H^{n-1} (except for one intersection point).

We can suppose that the index n critical point of ψ on W_{flex} is attached along a loose Legendrian Λ_0; so $W_{\mathrm{flex}}=W'_{\mathrm{flex}}\cup H^n_{\Lambda_0}$ and $W=W'_{\mathrm{flex}}\cup H^{n-1}\cup H^n_{\Lambda_0}\cup H^n_{\Lambda_1}$. Note that Λ_0 is disjoint from the belt sphere of H^{n-1} (since H^{n-1} is attached after $H^n_{\Lambda_0}$). We view $\Lambda_1\subset\partial(W'_{\mathrm{flex}}\cup H^{n-1})$ by taking any Legendrian in $\partial(W'_{\mathrm{flex}}\cup H^{n-1})$ that is isotopic to Λ_1 in $\partial(W'_{\mathrm{flex}}\cup H^{n-1}\cup H^n_{\Lambda_0})$; in general, there will be many such Legendrians, which are nonisotopic in $\partial(W'_{\mathrm{flex}}\cup H^{n-1})$. Since $n\geq 3$, we can assume that the Whitney disks of Λ_1 in $\partial(W_{\mathrm{flex}}\cup H^{n-1})$ are disjoint from the belt sphere of $H^n_{\Lambda_0}$ and hence lie in $\partial(W'_{\mathrm{flex}}\cup H^{n-1})$. In particular, Λ_1 can be smoothly isotoped in $\partial(W'_{\mathrm{flex}}\cup H^{n-1})$ to intersect the belt sphere of H^{n-1} in a single point. Furthermore, since the Whitney disks are disjoint from Λ_0 (since they are disjoint from its belt sphere), we can assume that this isotopy is supported away from Λ_0. We can also assume that this smooth isotopy of Λ_1 is the identity in a neighborhood of some point x in Λ_1. We take an isotropic path γ from x to Λ_0 and also assume that the isotopy is the identity in a neighborhood of this path.

Now we handle-slide Λ_1 over Λ_0 using the path γ. More precisely, we take the Legendrian connected sum of Λ_1 with a Legendrian unknot near Λ_0 via the isotropic arc γ and then handle-slide using a chart near this Legendrian unknot as in Theorem 3.1. We also do the handle-slide so that the resulting Legendrian $h_{\Lambda_0}(\Lambda_1)$ is loose in $\partial(W'_{\mathrm{flex}} \cup H^{n-1})$ (but not in the complement of Λ_0). Now we note that $h_{\Lambda_0}(\Lambda_1)$ can also be smoothly isotoped in $\partial(W'_{\mathrm{flex}} \cup H^{n-1})$ to a canceling sphere that intersects the belt sphere of H^{n-1} once. Namely, we can use exactly the same smooth isotopy that takes Λ_1 to a canceling sphere. This is because $h_{\Lambda_0}(\Lambda_1)$ is topologically the connected sum of Λ_0 and Λ_1. Since the previous isotopy is supported away from Λ_0 and the path γ used for the connected sum, we can extend it to the connected sum. Furthermore, Λ_0 is disjoint from the belt sphere of H^{n-1} and so after the smooth isotopy, $h_{\Lambda_0}(\Lambda_1)$ intersects this belt sphere once.

Since $h_{\Lambda_0}(\Lambda_1)$ is loose in $\partial(W'_{\mathrm{flex}} \cup H^{n-1})$ and smoothly cancels H^{n-1}, we can symplectically cancel H^{n-1} and $H^n_{h_{\Lambda_0}(\Lambda_1)}$. Thus $W'_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_0} \cup H^n_{h_{\Lambda_0}(\Lambda_1)}$ is Weinstein homotopic to $W'_{\mathrm{flex}} \cup H^n_{\Lambda'_0}$. Here Λ'_0 is the Legendrian obtained by handle-sliding Λ_0 off the canceling pair $H^{n-1} \cup H^n_{h_{\Lambda_0}(\Lambda_1)}$, ie Λ'_0 is the image of Λ_0 in $W'_{\mathrm{flex}} = W'_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{h_{\Lambda_0}(\Lambda_1)}$. Since W'_{flex} has a Weinstein presentation with $k-1$ critical points, $W'_{\mathrm{flex}} \cup H^n_{\Lambda'_0}$ has a presentation with $k = \mathrm{Crit}(W)$ critical points. This completes the proof since $W = W'_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_0} \cup H^n_{\Lambda_1}$ is Weinstein homotopic to $W'_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_0} \cup H^n_{h_{\Lambda_0}(\Lambda_1)}$, which is homotopic to $W'_{\mathrm{flex}} \cup H^n_{\Lambda'_0}$. □

The proof of Theorem 1.1 can be used to prove Corollary 1.7: all Legendrians in our Legendrian link can be made individually loose.

Proof of Corollary 1.7 The proof of Theorem 1.1 in the smoothly critical case shows that $W = W'_{\mathrm{flex}} \cup H^{n-1} \cup H^n_{\Lambda_0} \cup H^n_{h_{\Lambda_0}(\Lambda_1)}$ where Λ_0 and $h_{\Lambda_0}(\Lambda_1)$ are both loose; Λ_0 is loose by assumption and $h_{\Lambda_0}(\Lambda_1)$ is loose because of the handle-slide. Combining Λ_0 with the attaching spheres of the n–handles of $W'_{\mathrm{flex}} \cup H^{n-1}$ (which form a loose link for some presentation), we get the desired result. For general W, we first add a pair of symplectically canceling handles to W_{flex} and then proceed as in the smoothly critical case. □

Next we prove Theorem 1.10 about the number of intersection points between the belt and attaching spheres of smoothly canceling handles.

Proof of Theorem 1.10 By Theorem 3.1, we can assume that the smoothly trivial Weinstein cobordism W consists of two smoothly canceling handles H_1^{n-1} and $H_{\Lambda_1}^n$, ie Λ_1 is smoothly isotopic to a Legendrian that intersects the belt sphere of H_1^{n-1} in a single point. Now we follow the proof of Theorem 1.5. We first attach two canceling handles H_0^{n-1} and $H_{\Lambda_0}^n$ in a small Darboux ball and do two handle-slides (of opposite orientations) of Λ_1 over Λ_0 so that the resulting Legendrian $h_{\Lambda_0}^2(\Lambda_1)$ is loose. Then we use the contact isotopy φ to isotope $h_{\Lambda_0}^2(\Lambda_1)$ away from the belt sphere of H_0^{n-1}. The result is $W = H_0^{n-1} \cup H_1^{n-1} \cup H_{\varphi(h_{\Lambda_0}^2(\Lambda_1))}^n \cup H_{\varphi(\Lambda_0)}^n$; see the fifth diagram in Figure 8. The key observation is that this local diagram is independent of Λ_1 since all isotopies were done near $H_0^{n-1} \cup H_{\Lambda_0}^n$. In particular, let C_n be the number of times that $\varphi(\Lambda_0)$ intersects the belt sphere of H_0^{n-1}; in Figure 8, this number is 5 but since we do not compute this isotopy φ explicitly we do not know the exact number.

Next we note that the Legendrian $\varphi(h_{\Lambda_0}^2(\Lambda_1))$ is still smoothly isotopic to a Legendrian that intersects the belt sphere of H_1^{n-1} in a single point. This is because $\varphi(h_{\Lambda_0}^2(\Lambda_1))$ is exactly the same as Λ_1 except for a loose chart; see the blue box in the fifth diagram of Figure 8. Furthermore, we can assume that this smooth isotopy is supported away from $H^{n-1} \cup H_{\Lambda_0}^n$. Since $\varphi(h_{\Lambda_0}^2(\Lambda_1))$ is loose, there is a contact isotopy ψ taking it to a Legendrian that intersects the belt sphere of H_1^n in one point; since $\varphi(h_{\Lambda_0}^2(\Lambda_1))$ is loose away from $H^{n-1} \cup H_{\Lambda_0}^n$ and the smooth isotopy is supported away from this region, we can assume that this contact isotopy is also supported away from $H_0^{n-1} \cup H_{\Lambda_0}^n$. In particular, $\psi(\varphi(\Lambda_0))$ still intersects the belt sphere of H_0^{n-1} in C_n points. Finally, we handle-slide $\psi(\varphi(\Lambda_0))$ over $\psi(\varphi(h_{\Lambda_0}^2(\Lambda_1))$ and off H_1^{n-1}. This also does not change its geometric intersection number with the belt sphere of H_0^{n-1} since $\psi(\varphi(h_{\Lambda_0}^2(\Lambda_1))$ is disjoint from this belt sphere. We call the resulting Legendrian Λ_0'. Then $W = H_0^{n-1} \cup H_{\Lambda_0'}^n$ and Λ_0' intersects the belt sphere of H_0^{n-1} exactly C_n times as desired. The Legendrian Λ_0' is depicted in the sixth diagram of Figure 8. This diagram is also schematic and is meant to signify that Λ_0' has an upper and a lower part; the lower part of Λ_0' is close to H_0^{n-1} and is independent of Λ_1 while the upper part of Λ_0' depends on Λ_1 (and hence on W). □

Now we give proofs of the results in Section 1.4. We first prove Corollary 1.12 concerning the number of generators $g(\mathcal{W}(X))$ of the wrapped Fukaya category $\mathcal{W}(X)$.

Proof of Corollary 1.12 The proof of Theorem 1.1 shows that

$$\mathrm{WCrit}_n(X) \le \max\{1, \mathrm{Crit}_n(X)\}$$

for all X^{2n}. Combining this with the result from [6; 23], we get the inequality $g(\mathcal{W}(X)) \le \max\{1, \mathrm{Crit}_n(X)\}$. If X^{2n} is simply connected, then Smale's h–cobordism theorem (which holds since $n \ge 3$) implies that $\mathrm{Crit}_n(X) = g(H^n(X;\mathbb{Z}))$, which proves the result in that case. If X^{2n} is not simply connected, we attach some 2–handles to X^{2n} to get a simply connected Weinstein domain Y^{2n}. Since $n \ge 3$, we have $H^n(Y^{2n};\mathbb{Z}) \cong H^n(X^{2n};\mathbb{Z})$ and so $g(H^n(Y^{2n};\mathbb{Z})) = g(H^n(X^{2n};\mathbb{Z}))$. Furthermore, since $n \ge 3$, the 2–handles are subcritical and hence $D^b\mathcal{W}(Y)$ is exact equivalent to $D^b\mathcal{W}(X)$ by [23] and so $g(\mathcal{W}(X)) = g(\mathcal{W}(Y))$. Then the result for Y^{2n}, which is simply connected, implies the result for X^{2n}. □

Next we prove Corollary 1.13 that $g\big(K_0(\mathcal{W}(X))\big) \le g(H^n(X;\mathbb{Z}))$.

Proof of Corollary 1.13 The case $g(H^n(X;\mathbb{Z})) \ge 1$ is proven by (1-4) so it suffices to handle the case when $g(H^n(X;\mathbb{Z})) = 0$. Then $g\big(K_0(\mathcal{W}(X))\big) \le 1$ by (1-4) and if $g\big(K_0(\mathcal{W}(X))\big) = 0$, we are done. Otherwise, $g\big(K_0(\mathcal{W}(X))\big) = 1$ and therefore $K_0(\mathcal{W}(x)) \cong \mathbb{Z}/k\mathbb{Z}$ for some integer $k \ge 0$. Now we take the boundary connected sum and form the new Weinstein domain $X \natural X$. Since 1–handles are subcritical, $D^b\mathcal{W}(X \natural X) \cong D^b\mathcal{W}(X \amalg X)$ by [23] and $D^b\mathcal{W}(X \amalg X) \cong D^b\mathcal{W}(X) \oplus D^b\mathcal{W}(X)$. As a result, $K_0(\mathcal{W}(X \natural X)) \cong K_0(\mathcal{W}(X)) \oplus K_0(\mathcal{W}(X)) \cong \mathbb{Z}/k\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$. This implies that $g\big(K_0(\mathcal{W}(X \natural X))\big) = 2$ since $\mathbb{Z}/k\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$ is not a cyclic group. On the other hand, we also have $H^n(X \natural X;\mathbb{Z}) \cong H^n(X;\mathbb{Z}) \oplus H^n(X;\mathbb{Z}) \cong 0$ and therefore $g(H^n(X \natural X;\mathbb{Z})) = 0$. Again using the previous inequality, we get that $g\big(K_0(\mathcal{W}(X \natural X))\big) \le 1$, which contradicts $g\big(K_0(\mathcal{W}(X \natural X))\big) = 2$. Therefore, we must have that $g\big(K_0(\mathcal{W}(X))\big) = 0$ and so $K_0(\mathcal{W}(X)) = 0$ as desired. □

Remark 3.4 A similar boundary connected sum trick was used by Smith [36] to show that all exact symplectic fillings of $(S^{2n-1}, \xi_{\mathrm{std}})$ have vanishing symplectic cohomology; also see [41].

Next we prove our results about the Chekanov–Eliashberg algebra $\mathrm{CE}(\Lambda)$ of Legendrians. These results depend on the surgery formula [3]; alternatively, we can use the partially wrapped invariant $\mathrm{CF}(D, D; (W, \Lambda))$ and the rigorous proof of the surgery formula given in [23]. We first prove Corollary 1.16: the Chekanov–Eliashberg algebra of a Legendrian $\Lambda^{n-1} \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$ that is primitive in homology has no finite-dimensional representations.

Proof of Corollary 1.16 We first assume that Λ is a sphere and prove the general case later. Let $X^{2n} := B^{2n}_{\mathrm{std}} \cup H^{n-1} \cup H^n_\Lambda$. Since $[\Lambda] = 1 \in H_{n-1}(S^{n-1} \times S^n; \mathbb{Z}) \cong \mathbb{Z}$, $H^n(X^{2n}; \mathbb{Z}) = 0$ and so $K_0(\mathcal{W}(X)) = 0$ by Corollary 1.13. Let $C^n \subset X^{2n}$ be the cocore of H^n_Λ. Since C^n is the only index n cocore for X^{2n}, it generates $\mathcal{W}(X)$ and so $D^b\mathcal{W}(X) := H^0\big(\mathrm{Tw}(\mathrm{Fuk}(X))\big)$ is equivalent to $H^0\big(\mathrm{Tw}(\mathrm{CW}(C, C))\big)$, where we treat $\mathrm{CW}(C, C)$ is an A_∞–category with one object. By [3], $\mathrm{CW}(C, C)$ is quasi-isomorphic to $\mathrm{CE}(\Lambda)$ and hence $D^b\mathcal{W}(X)$ is exact equivalent to $H^0\big(\mathrm{Tw}(\mathrm{CE}(\Lambda))\big)$.

Suppose that $\mathrm{CE}(\Lambda)$ has a DGA map to $\mathrm{Mat}(n, \mathbb{K})$. Then there is an A_∞–functor

$$\mathrm{Tw}(\mathrm{CE}(\Lambda)) \to \mathrm{Tw}(\mathrm{Mat}(n, \mathbb{K}))$$

and an exact functor

$$H^0(\mathrm{Tw}(\mathrm{CE}(\Lambda)) \to H^0\big(\mathrm{Tw}(\mathrm{Mat}(n, \mathbb{K}))\big)$$

taking $\mathrm{CE}(\Lambda)$ to $\mathrm{Mat}(n, \mathbb{K})$ (considered as twisted complexes). Let $D(\mathrm{Mat}(n, \mathbb{K}))$ denote the classical derived category of $\mathrm{Mat}(n, \mathbb{K})$–modules and $D_\infty(\mathrm{Mat}(n, \mathbb{K}))$ its A_∞ analog, ie the homotopy category of A_∞–modules over $\mathrm{Mat}(n, \mathbb{K})$. There is an embedding $D(\mathrm{Mat}(n, \mathbb{K})) \to D_\infty(\mathrm{Mat}(n, \mathbb{K}))$; see [27]. Since $H^0\big(\mathrm{Tw}(\mathrm{Mat}(n, \mathbb{K}))\big)$ is equivalent to the subcategory of $D_\infty(\mathrm{Mat}(n, \mathbb{K}))$ generated by the free module $\mathrm{Mat}(n, \mathbb{K})$ and since the exact subcategory $D\mathrm{Mat}(n, \mathbb{K})$ contains this free module, $H^0\big(\mathrm{Tw}(\mathrm{Mat}(n, \mathbb{K}))\big)$ is also equivalent to the subcategory of $D\mathrm{Mat}(n, \mathbb{K})$ generated by the free module $\mathrm{Mat}(n, \mathbb{K})$. This subcategory is an exact subcategory of $D^b\mathrm{Proj}(\mathrm{Mat}(n, \mathbb{K}))$, the bounded derived category of projective $\mathrm{Mat}(n, \mathbb{K})$–modules. In summary, there is an exact functor $D^b\mathcal{W}(X) \to D^b\mathrm{Proj}(\mathrm{Mat}(n, \mathbb{K}))$ taking the cocore C^n to the free module $\mathrm{Mat}(n, \mathbb{K})$. This functor induces a map of Grothendieck groups $K_0(\mathcal{W}(X)) \to K_0(D^b\mathrm{Proj}(\mathrm{Mat}(n, \mathbb{K}))$, and the latter is just the usual Grothendieck group $K_0(\mathrm{Mat}(n, \mathbb{K}))$ of projective $\mathrm{Mat}(n, \mathbb{K})$–modules. It is well known that $[\mathrm{Mat}(n, \mathbb{K})] \in K_0(\mathrm{Mat}(n, \mathbb{K})) \cong \mathbb{Z}$ is nonzero. Therefore $K_0(\mathcal{W}(X))$ is also nonzero, which contradicts Corollary 1.13. Similarly, there are no DGA maps from $\mathrm{CE}(\Lambda)$ to a commutative ring R since $[R] \in K_0(R)$ is nonzero for commutative rings.

Now we prove the case when Λ^{n-1} is not a sphere. In this case, we cannot attach a standard n–handle along Λ but we can attach a generalized handle. Namely, let M^n be a smooth manifold with boundary Λ^{n-1}. Then we can construct the Weinstein domain $X^{2n} := B^{2n}_{\mathrm{std}} \cup H^{n-1} \cup_\Lambda T^*M$, where we glue T^*M to $B^{2n}_{\mathrm{std}} \cup H^{n-1}$ by identifying the Legendrian $\partial M \subset \partial T^*M$ with $\Lambda \subset \partial(B^{2n}_{\mathrm{std}} \cup H^{n-1})$; more precisely, we fix parametrized Legendrian embeddings

$$i\colon \Lambda \hookrightarrow \partial(B^{2n}_{\mathrm{std}} \cup H^{n-1}) \quad \text{and} \quad j\colon \Lambda \hookrightarrow \partial T^*M$$

which give us identifications of their neighborhoods with $J^1(\Lambda)$ that we use to glue $B^{2n}_{\mathrm{std}} \cup H^{n-1}$ to T^*M. Then $\mathrm{CE}(\Lambda; C_*(\Omega M^n))$, the Chekanov–Eliashberg algebra with coefficients in chains on the loop space of M^n, is quasi-isomorphic to $\mathrm{CW}(T^*_x M, T^*_x M)$, wrapped Floer cochains of the cotangent fiber $T^*_x M \subset T^*M \subset X^{2n}$; the partially wrapped analog of this result is proven in [23]. The cotangent fiber $T^*_x M$ is the cocore of the only index n handle of X^{2n} and hence generates $D^b\mathcal{W}(X)$. The condition that Λ is primitive in $H_{n-1}(S^{n-1} \times S^n; \mathbb{Z})$ again implies $H^n(X; \mathbb{Z}) = 0$. Therefore, $K_0(\mathcal{W}(X)) = 0$, and so by the same argument as when Λ is a sphere, $\mathrm{CE}(\Lambda; C_*(\Omega M^n))$ has no finite-dimensional representations or DGA maps to a commutative ring. On the other hand, there is a DGA map $\mathrm{CE}(\Lambda; C_*(\Omega M^n)) \to \mathrm{CE}(\Lambda)$ induced by the DGA map $C_*(\Omega M) \to C_*(\Omega D^n) = \mathbb{K}$. Any finite-dimensional representation or map to a commutative ring from $\mathrm{CE}(\Lambda)$ pulls back to such a map from $\mathrm{CE}(\Lambda; C_*(\Omega M^n))$, which we have proved cannot happen. So $\mathrm{CE}(\Lambda)$ also cannot have any finite-dimensional representations or DGA maps to commutative rings. □

Now we prove Corollary 1.17 concerning Legendrians that can be isotoped into a neighborhood of a loose Legendrian $\Lambda_{\mathrm{loose}} \subset (S^{2n-1}, \xi_{\mathrm{std}})$.

Proof of Corollary 1.17 We first prove the case when Λ_{loose} is the loose Legendrian unknot $\Lambda_{\mathrm{unknot,loose}}$ and then prove the general case. Consider a loose Legendrian sphere $A \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$ that is primitive in $H_n(S^{n-1} \times S^n; \mathbb{Z})$. Let $B \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$ be the stabilization of A, followed by a small Reeb push-off so that A and B are disjoint and form a loose link. The stabilization is done so that A and B are formally isotopic (and hence Legendrian isotopic). We can also assume that there exist disjoint contact neighborhoods U and V of A and B respectively so that A and B are loose in the complements of V and U respectively.

Since A is loose, $B^{2n}_{\mathrm{std}} \cup H^{n-1} \cup H^n_A$ is Weinstein homotopic to B^{2n}_{std}. By attaching the handle H^n_A using a neighborhood of A contained in U, we can assume B and its neighborhood V are disjoint from the attaching neighborhood and hence extend to a Legendrian $B' \subset (S^{2n-1}, \xi_{\mathrm{std}}) = \partial B^{2n}_{\mathrm{std}}$ and a contact neighborhood V' of B'. Since B is loose in the complement of U, its loose chart extends to $(S^{2n-1}, \xi_{\mathrm{std}})$ and so B' is loose. The belt sphere of H^n_A is the standard Legendrian unknot and so B' is formally isotopic to the Legendrian unknot. Since B' is loose, it is the loose Legendrian unknot $\Lambda_{\mathrm{unknot,loose}}$.

Let $\Lambda \subset (S^{2n-1}, \xi_{\mathrm{std}})$ be a Legendrian that can be isotoped into a neighborhood of $\Lambda_{\mathrm{unknot,loose}} = B'$ and is primitive in $H_{n-1}(\Lambda_{\mathrm{unknot,loose}}; \mathbb{Z})$, as in the statement of the corollary; we can assume that this neighborhood is V'. Using the identification between $V' \subset (S^{2n-1}, \xi_{\mathrm{std}})$ and $V \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$, $\Lambda \subset V'$ defines a Legendrian $\Lambda_0 \subset V \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$. In particular, $\Lambda \subset (S^{2n-1}, \xi_{\mathrm{std}})$ is obtained by trivially extending $\Lambda_0 \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$ through the Weinstein cobordism from $B^{2n}_{\mathrm{std}} \cup H^{n-1}$ to $B^{2n}_{\mathrm{std}} = B^{2n}_{\mathrm{std}} \cup H^{n-1} \cup H^n_A$ given by handle attachment along $A \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$. Since $\Lambda_0 \subset V$, $A \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$ is loose in the complement of Λ_0. Handle attachment along the loose Legendrian A does not change the Chekanov–Eliashberg algebras of Legendrians, like Λ_0, that are disjoint from the loose chart of A; see [3; 28]. Hence $\mathrm{CE}(\Lambda_0)$ and $\mathrm{CE}(\Lambda)$ are quasi-isomorphic; this is the key point where we use the fact that Λ is in a neighborhood of $\Lambda_{\mathrm{unknot,loose}} = B'$, which implies that Λ_0 is disjoint from the loose chart of A. Without this condition, $\mathrm{CE}(\Lambda_0)$ and $\mathrm{CE}(\Lambda)$ could be completely different and, in fact, $\mathrm{CE}(\Lambda_0)$ could be zero with $\mathrm{CE}(\Lambda)$ arbitrary.

That Λ is primitive in $H_{n-1}(\Lambda_{\mathrm{unknot,loose}}; \mathbb{Z})$ implies that $\Lambda_0 \subset (S^{n-1} \times S^n, \xi_{\mathrm{std}})$ is primitive in $H_{n-1}(B; \mathbb{Z})$ and therefore primitive in $H_{n-1}(S^{n-1} \times S^n; \mathbb{Z})$. Therefore $H^0(\mathrm{Tw}(\mathrm{CE}(\Lambda_0))$ is equivalent to $D^b\mathcal{W}(X)$, where X^{2n} is the Weinstein ball $B^{2n}_{\mathrm{std}} \cup H^{n-1} \cup H^n_{\Lambda_0}$. Then, as in Corollary 1.16, $\mathrm{CE}(\Lambda_0)$ has no finite-dimensional representations or DGA maps to commutative rings. Since $\mathrm{CE}(\Lambda)$ is quasi-isomorphic to $\mathrm{CE}(\Lambda_0)$ by the previous paragraph, $\mathrm{CE}(\Lambda)$ also has no finite-dimensional representations or DGA maps to a commutative ring. More precisely, this quasi-isomorphism implies that $H^0\big(\mathrm{Tw}(\mathrm{CE}(\Lambda))\big)$ and $H^0(\mathrm{Tw}(\mathrm{CE}(\Lambda_0))$ are equivalent and the rest of the proof is as in Corollary 1.16.

Next we prove the result when Λ is a neighborhood of an arbitrary loose Legendrian $\Lambda_{\mathrm{loose}} \subset (S^{2n-1}, \xi_{\mathrm{std}})$. Note that any Legendrian $\Lambda \subset (S^{2n-1}, \xi_{\mathrm{std}})$ can be Legendrian isotoped to a neighborhood of the Legendrian unknot $\Lambda_{\mathrm{unknot}}$ so that Λ and $\Lambda_{\mathrm{unknot}}$ agree on a small disk D^{n-1} (and hence Λ is primitive in $H_{n-1}(\Lambda_{\mathrm{unknot}}; \mathbb{Z})$). To see this, view Λ via its front projection in $\mathbb{R}^n$ and add a Reidemeister twist move to the topmost point of Λ, ie the one with the largest z–coordinate. Note that the smoothed-out twist is the front projection of $\Lambda_{\mathrm{unknot}}$. Taking this to be our copy of $\Lambda_{\mathrm{unknot}}$, we see that Λ and $\Lambda_{\mathrm{unknot}}$ agree on a disk and Λ is contained in a neighborhood of $\Lambda_{\mathrm{unknot}}$; most of Λ is contained in a neighborhood of the bottommost point of $\Lambda_{\mathrm{unknot}}$. See Figure 9. In particular, this construction holds for Λ_{loose}. We simultaneously stabilize $\Lambda_{\mathrm{unknot}}$ and Λ_{loose} using the disk D^{n-1} and get $\Lambda_{\mathrm{unknot,loose}}$ and $\Lambda'_{\mathrm{loose}}$. By construction, $\Lambda'_{\mathrm{loose}}$ is in a neighborhood of $\Lambda_{\mathrm{unknot,loose}}$ and is again primitive in

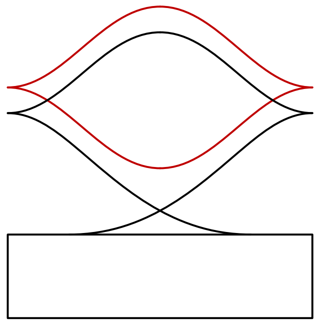

Figure 9: The black Legendrian is Λ with a Reidemeister twist added. The red Legendrian is a Legendrian unknot $\Lambda_{\mathrm{unknot}}$, which contains the black Legendrian in its neighborhood. They can be made to agree at the upper points of $\Lambda_{\mathrm{unknot}}$.

homology. Furthermore, $\Lambda'_{\mathrm{loose}}$ is formally Legendrian isotopic to Λ_{loose} and hence Legendrian isotopic to it. Combining these results, we can assume that Λ_{loose} is a neighborhood of $\Lambda_{\mathrm{unknot,loose}}$ and is primitive in $H_{n-1}(\Lambda_{\mathrm{unknot,loose}};\mathbb{Z})$. Therefore, since $\Lambda\subset(S^{2n-1},\xi_{\mathrm{std}})$ can be isotoped into a neighborhood of Λ_{loose} and is primitive in $H_{n-1}(\Lambda_{\mathrm{loose}};\mathbb{Z})$, it can also be isotoped into a neighborhood of $\Lambda_{\mathrm{unknot,loose}}$ and is primitive in $H_{n-1}(\Lambda_{\mathrm{unknot,loose}};\mathbb{Z})$ reducing this case to the previous case when Λ_{loose} is $\Lambda_{\mathrm{unknot,loose}}$. □

Combining Corollary 1.16 with the existence of infinitely many exotic Weinstein balls, we conclude that there are infinitely many Legendrian spheres in $(S^{n-1}\times S^n,\xi_{\mathrm{std}})$ or $(S^{2n-1},\xi_{\mathrm{std}})$ with no finite-dimensional representations; these Legendrians are also in a contact neighborhood of loose Legendrians and are primitive in their homology.

Proof of Corollary 1.18 McLean [32] showed that there are infinitely many exotic Weinstein balls Σ^{2n}_k for each $n\geq 4$, distinguished by symplectic cohomology. As explained in Example 1.3, $\mathrm{WCrit}(\Sigma^{2n}_k)=3$ and so Σ^{2n}_k can be presented as $B^{2n}_{\mathrm{std}}\cup H^{n-1}\cup H^n_{\Lambda_k}$ for some Legendrian $\Lambda_k\subset(S^{n-1}\times S^n,\xi_{\mathrm{std}})$. Since Σ^{2n}_k is a ball, Λ_k is primitive in homology, and so by Corollary 1.16, $\mathrm{CE}(\Lambda_k)$ has no finite-dimensional representations. By [3], the symplectic cohomology of Σ^{2n}_k is isomorphic to the Hochschild homology of $\mathrm{CE}(\Lambda_k)$ and hence the $\mathrm{CE}(\Lambda_k)$ are not acyclic and are different for different k, as desired.

Next we show that the Legendrian Λ_k can be isotoped into a contact neighborhood of a loose Legendrian and is primitive in its homology class. Note that $B^{2n}_{\mathrm{std}}\cup H^{n-1}$ is a subcritical Weinstein domain and hence Weinstein homotopic to $D^*S^{n-1}\times D^2$, where D^*S^{n-1} is the unit disk cotangent bundle. So $(S^{n-1}\times S^n,\xi_{\mathrm{std}})=\partial(B^{2n}_{\mathrm{std}}\cup H^{n-1})$ can be viewed as the boundary of the Lefschetz fibration $D^*S^{n-1}\times D^2$. By smoothing the corners of this Lefschetz fibration, $(S^{n-1}\times S^n,\xi_{\mathrm{std}})$ has an open book decomposition obtained by gluing $(T^*S^{n-1}\times S^1,\lambda+dz)$ to $(ST^*S^{n-1}\times D^2,\lambda+x\,dy-y\,dx)$ by identifying

$$ST^*S^{n-1}\times[1,\infty)\times S^1\subset T^*S^{n-1}\times S^1$$

with

$$ST^*S^{n-1}\times(D^2\backslash 0)\subset ST^*S^{n-1}\times D^2$$

via the contactomorphism $(x,r,\theta)\to(x,1/r^2,\theta)$. The pages of the open book decomposition are $T^*S^{n-1}\times\theta$, where $\theta\in S^1$. Akbulut and Arikan [2] showed that there is a Legendrian isotopy of Λ^{n-1} so that it becomes disjoint from the closure $T^*S^{n-1}\times\theta\amalg ST^*S^{n-1}\times(0,0)$ of the page $T^*S^{n-1}\times\theta$. The complement of the closure of this page is $T^*S^{n-1}\times(S^1\backslash\theta)$, which is a standard contact neighborhood of the Legendrian $S^{n-1}\times-\theta$. In particular, Λ_k can be isotoped into a neighborhood of $S^{n-1}\times-\theta$. Since $S^{n-1}\times-\theta$ and Λ_k are both primitive in $H_{n-1}(S^{n-1}\times S^n;\mathbb{Z})\cong\mathbb{Z}$, Λ_k is primitive in $H_{n-1}(S^{n-1}\times-\theta;\mathbb{Z})$. Finally, we note that $S^{n-1}\times-\theta$ is a loose Legendrian since it passes through the belt sphere of H^{n-1} exactly once.

For the second part of this corollary about Legendrians in $(S^{2n-1},\xi_{\mathrm{std}})$, we essentially reverse the procedure in the proof of Corollary 1.17. Take a loose Legendrian $A\subset(S^{n-1}\times S^n,\xi_{\mathrm{std}})$ disjoint from Λ_k and loose in the complement of Λ_k. Then $B^{2n}_{\mathrm{std}}\cup H^{n-1}\cup H^n_A$ is flexible and hence Weinstein homotopic to B^{2n}_{std}. Since Λ_k is disjoint from A, Λ_k defines a Legendrian sphere Λ'_k in $(S^{2n-1},\xi_{\mathrm{std}})=\partial B^{2n}_{\mathrm{std}}$. Since A is loose in the complement of Λ_k, $\mathrm{CE}(\Lambda'_k)$ is quasi-isomorphic to $\mathrm{CE}(\Lambda_k)$ by [3; 28], as discussed in the proof of Corollary 1.17. Therefore, $H^0\big(\mathrm{Tw}(\mathrm{CE}(\Lambda_k))\big)$ is equivalent to $H^0\big(\mathrm{Tw}(\mathrm{CE}(\Lambda'_k))\big)$ and so $\Lambda'_k\subset(S^{2n-1},\xi_{\mathrm{std}})$ has the same properties as $\Lambda_k\subset(S^{n-1}\times S^n,\xi_{\mathrm{std}})$, ie the $\mathrm{CE}(\Lambda'_k)$ have no finite-dimensional representations or DGA maps to a commutative ring and their Hochschild homology is different for different k. Finally, we observe that Λ'_k is in a contact neighborhood of a loose Legendrian in $(S^{2n-1},\xi_{\mathrm{std}})$ and is primitive in its homology. By the previous paragraph, $\Lambda_k\subset(S^{n-1}\times S^n,\xi_{\mathrm{std}})$ is in a contact neighborhood of the loose Legendrian $S^{n-1}\times-\theta$ and is primitive in its homology. The Legendrian $S^{n-1}\times-\theta$ is isotopic to the Legendrian B obtained by stabilizing A and taking a small Reeb push-off; so we assume from the start that Λ_k is in a neighborhood of B, is primitive in $H_{n-1}(B;\mathbb{Z})$, and is disjoint from A. So the extension Λ'_k of Λ_k is in a neighborhood of the extension B' of B to $(S^{2n-1}, \xi_{\mathrm{std}})$ and is primitive in $H_{n-1}(B';\mathbb{Z})$. Since B is loose in the complement of A, its extension $B' \subset (S^{2n-1}, \xi_{\mathrm{std}})$ is a loose Legendrian, in fact the loose Legendrian unknot, which proves the claim. □

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