

Monopole contributions to refined Vafa–Witten invariants

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We study the monopole contribution to the refined Vafa–Witten invariant recently defined by Maulik and Thomas (work in progress). We apply the results of Gholampour and Thomas (to appear in *Compos. Math.*) to prove a universality result for the generating series of contributions of Higgs pairs with 1–dimensional weight spaces. For prime rank, these account for the entire monopole contribution by a theorem of Thomas. We use toric computations to determine part of the generating series and find agreement with the conjectures of Göttsche and Kool (*Pure Appl. Math. Q.* 14 (2018) 467–513) for ranks 2 and 3.

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1. Introduction	2782
2. The moduli space	2790
3. Stability	2794
4. Tautological integrals	2799
5. Removing trace	2801
6. The leading term	2803
7. Universality	2807
8. Proof of Theorem A	2810
9. Toric computations	2813
10. The universal series A	2816
11. Smooth components	2820
12. Comparison to the Göttsche–Kool conjectures	2823
Appendix A. Functions appearing in the Göttsche–Kool conjectures	2825
Appendix B. Rank 3 results	2826
References	2827

1 Introduction

1.1 Vafa–Witten invariants

In [20], Yuuji Tanaka and Richard Thomas proposed a definition of an $SU(r)$ Vafa–Witten invariant [22]. Let (S, H) be a polarized smooth complex surface with canonical bundle ω_S . A Higgs pair is a pair

$$(E, \phi) \quad \text{with } E \in \text{Coh}(S) \text{ and } \phi: E \rightarrow E \otimes \omega_S.$$

Choose a rank r , Chern classes c_1 and c_2 on S , and a line bundle M on S with $c_1(M) = c_1$. Assume that r , c_1 and c_2 are chosen in such a way that stability and semistability of Higgs pairs coincide (see Section 3). Let

$$\mathcal{N}_{r,M,c_2}^\perp = \{(E, \phi) \mid \text{tr } \phi = 0, \text{rk}(E) = r, \det E \cong M, c_2(E) = c_2\}$$

be the moduli space of Gieseker stable trace-free Higgs pairs with fixed determinant. In [20] a symmetric perfect obstruction theory on $\mathcal{N}_{r,M,c_2}^\perp$ is constructed. Its dual complex is given by the cone

$$(1.1) \quad R\mathcal{H}om_\pi(E, E)_0 \xrightarrow{[\cdot, \phi]} R\mathcal{H}om_\pi(E, E \otimes \omega_S)_0 \rightarrow T,$$

where (E, ϕ) is a universal Higgs pair on $\mathcal{N}_{r,M,c_2}^\perp \times S$ and

$$\pi: \mathcal{N}_{r,M,c_2}^\perp \times S \rightarrow \mathcal{N}_{r,M,c_2}^\perp$$

denotes the projection. The \mathbb{C}^* -action on $\mathcal{N}_{r,M,c_2}^\perp$, which is given by scaling the Higgs field, can be lifted to an equivariant structure on E . It gives rise to a localized virtual class, which is used to define the Vafa–Witten invariant by

$$(1.2) \quad \text{VW}_{r,c_1,c_2}(S) = \int_{[(\mathcal{N}_{r,M,c_2}^\perp)^{\mathbb{C}^*}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})},$$

where N^{vir} is the virtual normal bundle to $(\mathcal{N}_{r,M,c_2}^\perp)^{\mathbb{C}^*}$ in $\mathcal{N}_{r,M,c_2}^\perp$, and $e(N^{\text{vir}})$ denotes its equivariant Euler class.

A Higgs pair (E, ϕ) in the fixed locus $(\mathcal{N}_{r,M,c_2}^\perp)^{\mathbb{C}^*}$ can be equipped with a \mathbb{C}^* -action and hence decomposes into weight spaces. We may assume the 0 is the highest weight appearing in the decomposition. As explained in [20], the Higgs field acts with weight -1 . Hence we can write

$$E = \bigoplus_{i=0}^k E_i \otimes \mathfrak{t}^{-i} \quad \text{and} \quad \phi = (\phi_1, \dots, \phi_k): E \rightarrow E \otimes \omega_S \otimes \mathfrak{t},$$

where the E_i are torsion-free sheaves of rank r_i , and ϕ decomposes into maps

$$\phi_i: E_{i-1} \rightarrow E_i \otimes \omega_S \otimes \mathfrak{t} \quad \text{for } i = 1, \dots, k.$$

We will write

$$\mathcal{M}_{(r_0, \dots, r_k)} = \mathcal{M}_{(r_0, \dots, r_k), c_1, c_2} \subset (\mathcal{N}_{r, \mathcal{M}, c_2}^\perp)^{\mathbb{C}^*}$$

for the open and closed locus of Higgs pairs with weight spaces of dimensions r_0, \dots, r_k . The locus

$$\mathcal{M}_{(r)} = \{(E, \phi) \in (\mathcal{N}_{r, \mathcal{M}, c_2}^\perp)^{\mathbb{C}^*} \mid \phi = 0\}$$

is called the *instanton branch*; see Lothar Göttsche and Martijn Kool [10]. It is isomorphic to the moduli space of torsion-free rank r sheaves, and its contribution to the Vafa–Witten invariant is the (localized) virtual Euler characteristic (up to a sign). Its complement in the \mathbb{C}^* -fixed locus is called the *monopole branch*. In this paper, we will discuss the contribution of the locus $\mathcal{M}_{1r} = \mathcal{M}_{(1 \dots 1)}$ of Higgs pairs with 1-dimensional weight spaces to the monopole branch. As an application of work of Gholampour and Thomas [7], we will describe the structure of the generating series of the contributions of \mathcal{M}_{1r} to the Vafa–Witten invariant and compute them in some cases.

Maulik and Thomas (work in progress; see also [21]) define a refined version of the Vafa–Witten invariant. It is a rational function in \sqrt{y} , rather than a rational number. It specializes to the unrefined invariant at $y = 1$. The instanton contribution to the refined Vafa–Witten invariant is given, up to a sign and a power of y , by the χ_y -genus — see Fantechi and Göttsche [3] — of the component $\mathcal{M}_{(r)}$, which refines the virtual Euler characteristic; see Göttsche and Kool [11]. We will discuss the contribution of \mathcal{M}_{1r} to the refined invariant.

1.2 Nested Hilbert schemes

Fix a rank r . For an r -tuple of nonnegative integers $n = (n_0, \dots, n_{r-1})$ and an $(r-1)$ -tuple $\beta = (\beta_1, \dots, \beta_{r-1})$ of classes in $H^2(S, \mathbb{Z})$, let

$$S^{[n_i]} := \text{Hilb}^{n_i}(S)$$

denote the Hilbert schemes of n_i points on S , and let $\text{Hilb}_{\beta_i}(S)$ be the Hilbert schemes of curves on S with class β_i . We will also write

$$\text{Hilb}_\beta^n(S) := S^{[n_0]} \times \dots \times S^{[n_{r-1}]} \times \text{Hilb}_{\beta_1}(S) \times \dots \times \text{Hilb}_{\beta_s}(S).$$

The *nested Hilbert scheme*

$$(1.3) \quad i: S_\beta^{[n]} \hookrightarrow \text{Hilb}_\beta^n(S)$$

is defined as the incidence locus

$$\{I_0, \dots, I_{r-1}, C_1, \dots, C_{r-1} \mid I_{i-1}(-C_i) \subset I_i\}.$$

The nested Hilbert schemes are studied by Gholampour, Sheshmani and Yau in [5], in which a perfect obstruction theory is constructed. Write $\mathcal{I}^{[n_i]}$ for the universal ideal sheaf on $S^{[n_i]} \times S$, and let

$$\mathcal{D}_i \rightarrow \text{Hilb}_{\beta_i}(S) \times S$$

be the universal curve with class β_i . Finally, write

$$\pi: S_\beta^{[n]} \times S \rightarrow S_\beta^{[n]}$$

for the projection.

Theorem 1.4 [5] *The nested Hilbert scheme $S_\beta^{[n]}$ admits a perfect obstruction theory, the dual of which is given by a cone on*

$$\left(\bigoplus_{i=0}^{r-1} R\mathcal{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}) \right)_0 \rightarrow \bigoplus_{i=1}^{r-1} R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\mathcal{D}_i)),$$

in which the left-hand side is the cocone of the trace map

$$\bigoplus_{i=0}^{r-1} R\mathcal{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}) \rightarrow R\pi_* \mathcal{O}_S.$$

In [7], Gholampour and Thomas give another construction of the perfect obstruction theory, using virtual resolutions of degeneracy loci of complexes. Moreover, they give a formula for the induced virtual class in the ambient space (1.3). We will give the statement in the following restricted setting.

Let S be a surface satisfying

$$H^1(\mathcal{O}_S) = 0 \quad \text{and} \quad p_g(S) > 0.$$

For $i = 0, \dots, r - 1$, let $\mathcal{O}_S(\beta_i)$ be the line bundle with $c_1(\mathcal{O}_S(\beta_i)) = \beta_i$, so that

$$\text{Hilb}_{\beta_i}(S) = |\mathcal{O}_S(\beta_i)| := \mathbb{P}(H^0(\mathcal{O}_S(\beta_i))).$$

We will write

$$\mathcal{F}(\beta_i) = \mathcal{F} \otimes \mathcal{O}_S(\beta_i)$$

for any sheaf \mathcal{F} on S .

Theorem 1.5 [7, Theorem 5.6] *After pushforward by i , the virtual class of $S_\beta^{[n]}$ is given by*

$$i_*[S_\beta^{[n]}]^{\text{vir}} = \prod_{i=1}^{r-1} e(R\pi_*\mathcal{O}_S(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))) \cap [S^{[n_0]} \times \dots \times S^{[n_{r-1}]}] \times \text{SW}(\beta_1) \times \dots \times \text{SW}(\beta_{r-1}) \in A_{n_0+n_s}(\text{Hilb}_\beta^n(S)),$$

in which

$$\text{SW}(\beta_i) \in A_0(|\mathcal{O}_S(\beta_i)|) \cong \mathbb{Z}$$

is the Seiberg–Witten invariant of β_i , considered as a 0–cycle.

Remark 1.6 We write

$$e(R\pi_*\mathcal{O}_S(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)))$$

for the $(n_{i-1}+n_i)^{\text{th}}$ Chern class of the K –theory class in the brackets, which behaves in some sense like a rank $n_{n-1}+n_i$ vector bundle. For example, by the generalized Carlsson–Okounkov vanishing — see Gholampour and Thomas [6] — its Chern classes vanish beyond its rank.

Remark 1.7 For the definition and some basic properties of Seiberg–Witten classes of algebraic surfaces with $H^1(S) = 0$ and $p_g > 0$, we refer the reader to Mochizuki [18, Section 6.3.1] or Gholampour, Sheshmani and Yau [4, Section 4].

Remark 1.8 It is Theorem 1.5 that allows us to compute the \mathcal{M}_{1^r} contributions to the Vafa–Witten invariant. A large part of our paper should be seen as an application of the result by Gholampour and Thomas.

1.3 Results

The moduli space \mathcal{M}_{1^r} is a union of nested Hilbert schemes $S_\beta^{[n]}$ [4; 20]. Moreover, the \mathbb{C}^* –localized virtual class from [20] agrees with the virtual class from Theorem 1.4. It follows that the contribution of each component \mathcal{M}_{1^r} to the Vafa–Witten invariant is *topological* [7]. The observation that the generating series of these contributions is multiplicative — see Göttsche [9] — leads to the following result.

Notation 1.9 We will write

$$\text{VW}_{1^r, c_1, c_2}(S, y)$$

for the contribution of $\mathcal{M}_{1^r} = \mathcal{M}_{1^r, c_1, c_2}$ to the refined Vafa–Witten invariant of Maulik and Thomas (work in progress) and

$$\mathbb{Z}_{S, r, c_1}(q, y) = \frac{q^{(1-r)/(2r)c_1^2}}{\#H^2(S, \mathbb{Z})[r]} \sum_{c_2 \in \mathbb{Z}} \text{VW}_{1^r, c_1, c_2}(S, y) q^{c_2}$$

for the generating series of such contributions. Here

$$H^2(S, \mathbb{Z})[r] := \ker(H^2(S, \mathbb{Z}) \xrightarrow{-r} H^2(S, \mathbb{Z}))$$

denotes the r -torsion subgroup of $H^2(S, \mathbb{Z})$.

Theorem A Fix a rank $r \geq 1$. There are universal Laurent series

$$A, B, C_{ij} \in \mathbb{Q}(\sqrt{y})((q^{1/(2r)})) \quad \text{for } 1 \leq i \leq j \leq r - 1,$$

depending **only** on r , such that for any surface S with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$ and any class $c_1 \in H^2(S, \mathbb{Z})$ such that semistability of Higgs pairs implies stability for all c_2 , we have

$$(1.10) \quad \mathbb{Z}_{S, r, c_1}(q, y) = A^{\chi(\mathcal{O}_S)} B^{K_S^2} \sum_{\beta} \text{SW}(\beta^1) \cdots \text{SW}(\beta^{r-1}) \prod_{i \leq j} C_{ij}^{\beta^i \beta^j},$$

where the sum is taken over tuples $\beta = (\beta^1, \dots, \beta^{r-1}) \in (H^2(S, \mathbb{Z}))^{r-1}$ with

$$c_1 \equiv \sum_i i \beta^i \pmod{rH^2(S, \mathbb{Z})}.$$

Remark 1.11 The condition $H^1(\mathcal{O}_S) = 0$ in [Theorem A](#) is superfluous. However, for expository reasons, we will work with this condition throughout the paper. Moreover, (1.10) can be shown to hold for all c_1 when we extend the definition of the left-hand side to the semistable case. This will be subject of future work.

Remark 1.12 For odd rank r , the Laurent series have coefficients in $\mathbb{Q}(y)$, rather than in $\mathbb{Q}(\sqrt{y})$ (see [Proposition 8.4](#)).

The following corollary is implicitly in the statement of [Theorem A](#).

Corollary 1.13 Fix a rank r , and let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$. Let c_1 be a Chern class for which semistability implies stability for all c_2 . Then $\mathbb{Z}_{S, r, c_1}(q, y)$ is independent of the choice of a polarization of the surface S .

We will define the Laurent series in the theorem explicitly in terms of tautological integrals over products of Hilbert schemes of points on the surface S (see [Sections 5](#),

7 and 8). Although for surfaces with $\deg(K_S) < 0$, the locus \mathcal{M}_{1r} is empty by stability, the Hilbert schemes and the integrals are still defined. We will prove universality of these integrals for *all* surfaces (Proposition 7.6). As usual [9], the coefficients of the power series can be determined by evaluating these integrals on \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$, where we have access to toric methods, as we explain in Section 9.

1.4 Göttsche–Kool conjectures

In [10], Göttsche and Kool conjecture a formula for the generating series of the χ_y -genus of the instanton branch for ranks 2 and 3. Moreover they conjecture, motivated by S -duality [22], that the generating series of refined Vafa–Witten invariants has modular properties that relate the contributions of the instanton branch to those of the monopole branch. Using this, they give a conjectural formula for the contribution of the monopole branch to refined Vafa–Witten invariants of ranks 2 and 3. For rank 2, their conjectures refine the predictions in the physics literature [22].

The formulas of [10] that predict the monopole contributions to the Vafa–Witten invariants in ranks 2 and 3 have precisely the structure of the generating series (1.10) of the \mathcal{M}_{1r} contributions. This suggests that \mathcal{M}_{1r} accounts for the entire monopole contribution.

Conjecture 1.14 For S and c_1 as in Theorem A and r prime, we have

$$\text{VW}_{r,c_1,c_2}^{\text{monopole}}(S, y) = \text{VW}_{1r,c_1,c_2}(S, y)$$

for all $c_2 \in \mathbb{Z}$.

The conjecture has now been proved by Thomas in [21].

Theorem 1.15 (Thomas) Conjecture 1.14 holds.

It follows that Theorems A and 1.15 prove the *structure* of [10, Conjecture 1.5], generalized to arbitrary prime rank. The rank 2 and 3 conjectures of [10] give the universal series appearing in the formula explicitly in terms of functions

$$\phi_{-2,1}(x, y), \Delta(x), \Theta_{A_2,(1,0)}(x, y), \eta(x), \theta_2(x, y), \theta_3(x, y) \text{ and } W_{\pm}(x, y),$$

which we give in Appendix A. The following conjectures imply [10, Remark 1.7 and Conjecture 1.5].

Notation 1.16 To emphasize the dependency on r , we will write $A^{(r)}, B^{(r)}, \dots$ for the series appearing in Theorem A.

Conjecture 1.17 For rank 2, the universal series appearing in [Theorem A](#), and defined in [Section 8](#), are given by

$$A^{(2)}(y) = \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^2, y^2)^{\frac{1}{2}} \tilde{\Delta}(q^2)^{\frac{1}{2}}}, \quad B^{(2)}(y) = \frac{\tilde{\eta}(q)^2}{\theta_3(q, y)}, \quad C_{11}^{(2)}(y) = \frac{-\theta_3(q, y)}{\theta_2(q, y)}.$$

Conjecture 1.18 For rank 3, we have

$$A^{(3)}(y) = \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^3, y^3)^{\frac{1}{2}} \tilde{\Delta}(q^3)^{\frac{1}{2}}}, \quad B^{(3)}(y) = \frac{\tilde{\eta}(q)^3 W_{-}(q^{\frac{1}{2}}, y)}{\Theta_{A_2, (1,0)}(q^{\frac{1}{2}}, y)},$$

$$C_{12}^{(3)}(y) = W_{+}(q^{\frac{1}{2}}, y) W_{-}(q^{\frac{1}{2}}, y), \quad C_{11}^{(3)}(y) = C_{22}^{(3)}(y) = \frac{1}{W_{-}(q^{\frac{1}{2}}, y)}.$$

1.5 Toric computations

As remarked before, the universality result of [Proposition 7.6](#) allows us to determine the first few terms of the power series of [Theorem A](#) by toric computations. I implemented the Atiyah–Bott localization formula for the surfaces \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ in Sage [\[19\]](#) and found agreement with [Conjectures 1.17](#) and [1.18](#).

Define multiplicative subgroups

$$U_N^{(r)} := 1 + q^N \mathbb{Q}(y^{\frac{1}{2}})[[q]] \subset \mathbb{Q}(y^{\frac{1}{2}})((q^{1/(2r)}))^*$$

for all $r, N \geq 1$, and consider series

$$P = cq^{z/(2r)}(1 + p_1q + p_2q^2 + \dots) \quad \text{and} \quad P' = c'q^{z'/(2r)}(1 + p'_1q + p'_2q^2 + \dots)$$

with $c, c' \in \mathbb{Q}(\sqrt{y})^*$, $p_i, p'_i \in \mathbb{Q}(\sqrt{y})$ and $z, z' \in \mathbb{Z}$. The Laurent series appearing in [Theorem A](#) and [Conjectures 1.17](#) and [1.18](#) are all of this form. Then we have

$$P \equiv P' \pmod{U_{N+1}^{(r)}} \quad (\text{ie } P'P^{-1} \in U_{N+1}^{(r)})$$

if and only if

$$c = c', \quad z = z' \quad \text{and} \quad p_1 = p'_1, \dots, p_N = p'_N.$$

Theorem B Let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$. The rank 2 conjectures of [\[10\]](#) correctly predict the first 15 terms of the universal series of [Theorem A](#). The rank 3 conjectures correctly predict the first 11 terms. In other words, the equations of [Conjecture 1.17](#) hold modulo $U_{15}^{(2)}$, and the equations of [Conjecture 1.18](#) hold modulo $U_{11}^{(3)}$.

1.6 K3 surfaces

Let S be a K3 surface. Then $0 \in H^2(S, \mathbb{Z})$ is the only Seiberg–Witten basic class of S , and for $c_1 = 0$, Equation (1.10) becomes

$$“\mathbb{Z}_{S,r,0}(q, y)” = (A^{(r)})^2.$$

Note that in our setting, the left-hand side has not been defined for $r > 1$, due to the existence of strictly semistable sheaves. Hence we cannot apply Theorem A directly to determine the power series $A^{(r)}$. We can, however, evaluate on S the tautological integrals that are used to define the universal series $A^{(r)}$. This leads to the following result, which we prove in Section 10.

Theorem C We have

$$(1.19) \quad A^{(r)}(y) = \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^r, y^r)^{\frac{1}{2}} \tilde{\Delta}(q^r)^{\frac{1}{2}}}$$

for any $r \geq 1$.

Remark 1.20 Refined Vafa–Witten invariants of K3 surfaces have been computed in [21]. In [15], we will extend Theorem A to the semistable case, so Theorem C will follow also from the results of [21].

1.7 A special case

Let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$, and assume the Picard group

$$\text{Pic}(S) = \mathbb{Z} \cdot [C]$$

of S is generated by a smooth very ample canonical curve $C \in |K_S|$. Let $c_1 = K_S$. For rank 3, only $\beta = (K_S, 0)$ contributes to the right-hand side of (1.10) (see Lemma 11.1). In rank 2, and in a slightly more general setting [20], the only contribution is given by $\beta = (K_S)$. By Theorem A, we have

$$(1.21) \quad \mathbb{Z}_{S,r,K_S}(q, y) = (-A^{(r)}(y))^{\chi(\mathcal{O}_S)} (B^{(r)}(y) C_{11}^{(r)}(y))^{K_S^2}$$

for $r = 2, 3$. Here we have used the equation

$$(1.22) \quad \text{SW}(K_S) = (-1)^{\chi(\mathcal{O}_S)},$$

by eg Mochizuki [18, Proposition 6.3.4]. In this setting, our computations are slightly faster, and we find the following result.

Theorem B' Let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$, and assume that the Picard group of S is generated by a smooth very ample canonical curve. Then

$$\mathbb{Z}_{S,2,K_S}(q,y) \equiv \left(\frac{-(y^{\frac{1}{2}} - y^{-\frac{1}{2}})}{\phi_{-2,1}(q^2,y^2)^{\frac{1}{2}} \tilde{\Delta}(q^2)^{\frac{1}{2}}} \right)^{\chi(\mathcal{O}_S)} \left(\frac{-\tilde{\eta}(q)^2}{\theta_2(q,y)} \right)^{K_S^2} \pmod{U_{18}^{(2)}},$$

$$\mathbb{Z}_{S,3,K_S}(q,y) \equiv \left(\frac{-(y^{\frac{1}{2}} - y^{-\frac{1}{2}})}{\phi_{-2,1}(q^3,y^3)^{\frac{1}{2}} \tilde{\Delta}(q^3)^{\frac{1}{2}}} \right)^{\chi(\mathcal{O}_S)} \left(\frac{\tilde{\eta}(q)^3}{\Theta_{A_2,(1,0)}(q^{\frac{1}{2}},y)} \right)^{K_S^2} \pmod{U_{14}^{(3)}}.$$

For S a surface as in **Theorem B'** and rank $r = 2$, the moduli space $\mathcal{M}_{1^2,K_S,c_2}$ is smooth for $c_2 \leq 3$. In [20] and [21], this is used to compute the Vafa–Witten invariant by direct intersection-theoretic calculations. The rank 2 equation of **Theorem B'** is proved modulo $U_3^{(2)}$ in [21]. In [20], it is proved modulo $U_4^{(2)}$ in the unrefined case.

For rank 3, the moduli space $\mathcal{M}_{1^3,K_S,c_2}$ is smooth if and only if $c_2 \leq 2$ (see **Proposition 11.2**). This allows us to compute the Vafa–Witten invariants by the methods of [20] and [21]. As a result, we obtain an *alternative proof* by direct calculations, discussed in **Section 11**, for the rank 3 equation of **Theorem B'**, modulo $U_3^{(3)}$.

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2 The moduli space

Let S be a smooth projective surface with $H^1(\mathcal{O}_S) = 0$, and fix a rank r . As mentioned in the introduction, the locus \mathcal{M}_{1^r} of Higgs pairs with 1–dimensional weight spaces is a union of nested Hilbert schemes. In this section, we will introduce some notation and describe universal Higgs pairs over the components.

Write $s := r - 1$ and let $L = (L_0, \dots, L_s)$ be an r –tuple of line bundles on S .

Notation 2.1 Define classes

$$\beta_i = c_1(L_i \otimes L_{i-1}^* \otimes \omega_S) \in H^2(S, \mathbb{Z})$$

for $i = 1, \dots, s$, and write

$$\beta = \beta(L) = (\beta_1, \dots, \beta_s).$$

We will also write

$$\beta^i = K_S - \beta_i$$

for $i = 1, \dots, s$ and an s -tuple $\beta = (\beta_1, \dots, \beta_s) \in (H^2(S, \mathbb{Z}))^s$. In particular, when $\beta = \beta(L)$, we have

$$\beta^i = c_1(L_{i-1} \otimes L_i^*)$$

for $i = 1, \dots, s$.

Remark 2.2 We will use the convention

$$s := r - 1$$

throughout the paper. Furthermore, L will always denote an $(s+1)$ -tuple of line bundles on S , and β an s -tuple of classes in $H^2(S, \mathbb{Z})$.

Consider the product of complete linear systems

$$\text{Hilb}_\beta(S) = \text{Hilb}_{\beta_1}(S) \times \dots \times \text{Hilb}_{\beta_s}(S) = |\mathcal{O}_S(\beta_1)| \times \dots \times |\mathcal{O}_S(\beta_s)|,$$

and write

$$\begin{array}{ccc} |\mathcal{O}_S(\beta_i)| \times S & \xleftarrow{\text{pr}_i} & \text{Hilb}_\beta(S) \times S & \xrightarrow{q} & S \\ & & \downarrow \pi & & \\ & & \text{Hilb}_\beta(S) & & \end{array}$$

for the projections, where $i = 1, \dots, s$. We will write $\mathcal{O}_{\beta_i}(1)$ for the canonical line bundle on $|\mathcal{O}_S(\beta_i)|$.

Define the following line bundles on $\text{Hilb}_\beta(S) \times S$:

$$\begin{aligned} \mathcal{L}_0 &:= L_0, \\ \mathcal{L}_1 &:= L_1 \otimes \text{pr}_1^* \mathcal{O}_{\beta_1}(1), \\ &\vdots \\ \mathcal{L}_s &:= L_s \otimes \text{pr}_1^* \mathcal{O}_{\beta_1}(1) \otimes \dots \otimes \text{pr}_s^* \mathcal{O}_{\beta_s}(1). \end{aligned}$$

The tautological sections

$$\mathcal{O}_{|\mathcal{O}_S(\beta_i)| \times S} \rightarrow \mathcal{O}_{\beta_i}(1) \boxtimes \mathcal{O}_S(\beta_i)$$

induce maps

$$\phi_{\mathcal{L},i}: \mathcal{L}_{i-1} \rightarrow \mathcal{L}_i \otimes q^* \omega_S$$

for $i = 1, \dots, s$.

Let \mathfrak{t} be an equivariant parameter for the trivial \mathbb{C}^* -action on a point. Define the locally free sheaf

$$E_{\mathcal{L}} := (\mathcal{L}_0 \otimes \mathfrak{t}^0) \oplus \dots \oplus (\mathcal{L}_s \otimes \mathfrak{t}^{-s}).$$

The maps $\phi_{\mathcal{L},i}$ define a \mathbb{C}^* -equivariant Higgs field

$$\phi_{\mathcal{L}} = (\phi_{\mathcal{L},1}, \dots, \phi_{\mathcal{L},s}): E_{\mathcal{L}} \rightarrow E_{\mathcal{L}} \otimes q^* \omega_S \otimes \mathfrak{t}.$$

Now choose nonnegative integers $n = (n_0, \dots, n_s)$ and write

$$\text{Hilb}_{\beta}^n(S) = S^{[n_0]} \times \dots \times S^{[n_s]} \times \text{Hilb}_{\beta}(S),$$

as in the introduction. Let $\mathcal{I}^{[n_i]}$ denote the universal ideal sheaf on $S^{[n_i]} \times S$. Define the following sheaf on $\text{Hilb}_{\beta}^n(S) \times S$, suppressing obvious pullbacks:

$$E_{\mathcal{L}}^{[n]} := (\mathcal{L}_0 \otimes \mathcal{I}^{[n_0]} \otimes \mathfrak{t}^0) \oplus \dots \oplus (\mathcal{L}_s \otimes \mathcal{I}^{[n_s]} \otimes \mathfrak{t}^{-s}).$$

The nested Hilbert scheme is by definition the maximal subscheme

$$i: S_{\beta}^{[n]} \hookrightarrow \text{Hilb}_{\beta}^n(S)$$

over which $\phi_{\mathcal{L}}$ restricts to a Higgs field

$$\phi_{\mathcal{L}}^{[n]}: E_{\mathcal{L}}^{[n]} \rightarrow E_{\mathcal{L}}^{[n]} \otimes \omega_S \otimes \mathfrak{t};$$

ie there exist (necessarily unique) dotted arrows completing the diagram

$$\begin{array}{ccc} \mathcal{L}_{i-1} & \xrightarrow{\phi_{\mathcal{L},i}} & \mathcal{L}_i \otimes q^* \omega_S \\ \downarrow & & \downarrow \\ \mathcal{L}_{i-1} \otimes \mathcal{I}^{[n_{i-1}]} & \xrightarrow{\dots \phi_{\mathcal{L},i}^{[n]} \dots} & \mathcal{L}_i \otimes \mathcal{I}^{[n_i]} \otimes q^* \omega_S \end{array}$$

of sheaves on $S_{\beta}^{[n]} \times S$ (where we suppress obvious pullbacks), defining the Higgs field $\phi_{\mathcal{L}}^{[n]}$, and the subscheme $S_{\beta}^{[n]} \subseteq \text{Hilb}_{\beta}^n(S)$ is maximal with this property.

Remark 2.3 Throughout the paper, the letter n is reserved for $(s+1)$ -tuples of nonnegative integers, and $\text{Hilb}_{\beta}^n(S)$ will always denote a product of Hilbert schemes as above.

Proposition 2.4 [4; 20] *The scheme \mathcal{M}_{1r} is a disjoint union of components that are uniquely represented by a triple*

$$(S_\beta^{[n]}, E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]}),$$

as constructed above.

By Proposition 2.4, \mathcal{M}_{1r} consists of components that are naturally indexed by tuples $L = (L_0, \dots, L_s)$ and $n = (n_0, \dots, n_s)$.

Definition 2.5 We denote a component of \mathcal{M}_{1r} represented by a triple $(S_\beta^{[n]}, E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]})$ by $\mathcal{M}_L^{[n]}$.

Obviously, not every pair (L, n) corresponds to a component of \mathcal{M}_{1r} . The nested Hilbert scheme might be empty, or the Higgs pairs in the family $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]})$ might be unstable. The other restriction is the Chern data of the Higgs pairs. We will address stability in Section 3. We finish this section with a lemma regarding the second issue.

Lemma 2.6 *The total Chern class (in cohomology) of any fibre E of $E_{\mathcal{L}}^{[n]}$ over $S_\beta^{[n]}$ is given by*

$$\begin{aligned} c(E) &= 1 + c_1 + |n| \cdot \text{pt} + \sum_{0 \leq i < j \leq s} c_1(L_i)c_1(L_j) \\ &= 1 + (s + 1)c_1(L_s) + \sum_{i=1}^s i\beta^i + |n| \cdot \text{pt} + \frac{s}{2(s + 1)}c_1^2 \\ &\quad - \sum_{1 \leq i < j \leq s} \frac{i(s + 1 - j)}{s + 1}\beta^i\beta^j - \sum_{1 \leq i \leq s} \frac{i(s + 1 - i)}{2(s + 1)}(\beta^i)^2, \end{aligned}$$

where

$$c_1 = c_1(L_0) + \dots + c_1(L_s) \quad \text{and} \quad |n| = n_0 + \dots + n_s,$$

and pt denotes the Poincaré dual of the homology class of a point.

Proof This is a straightforward computation. For the second equation, note that

$$(s + 1)c_1(L_i) = \sum_{k=1}^i -k\beta^k + \sum_{k=i+1}^s (s + 1 - k)\beta^k + c_1$$

for $i = 0, \dots, s$. Substituting this into

$$\sum_{0 \leq i < j \leq s} c_1(L_i)c_1(L_j)$$

and interchanging sums gives the result. □

3 Stability

By Proposition 2.4, \mathcal{M}_{1r} consists of components that are isomorphic to nested Hilbert schemes $S_\beta^{[n]}$, with

$$\beta = (\beta_1, \dots, \beta_s) \quad \text{and} \quad n = (n_0, \dots, n_s)$$

tuples of divisor classes on S and integers, respectively. The Hilbert scheme is empty if and only if one of the β_i is not effective, or $\beta_i = 0$ and $n_{i-1} < n_i$ for some i . Obviously, the virtual class of the nested Hilbert scheme vanishes in this case.

We will give dual conditions on β and n which hold whenever the Higgs pairs parametrized by $S_\beta^{[n]}$ are Gieseker unstable and which in turn imply the vanishing of the virtual class. We recall the definition of stability of Higgs pairs.

Definition 3.1 Let H be a polarization of the surface S . A Higgs pair (E, ϕ) is called *slope stable* (resp. *slope semistable*) if

$$\frac{\deg(F)}{\text{rk}(F)} < \frac{\deg(E)}{\text{rk}(E)} \quad \left(\text{resp.} \quad \frac{\deg(F)}{\text{rk}(F)} \leq \frac{\deg(E)}{\text{rk}(E)} \right)$$

for every ϕ -invariant subsheaf $0 \neq F \subsetneq E$ with $\text{rk}(F) < \text{rk}(E)$. It is called *Gieseker stable* (resp. *Gieseker semistable*) if we have inequalities of polynomials in m

$$\frac{\chi(F(mH))}{\text{rk}(F)} < \frac{\chi(E(mH))}{\text{rk}(E)} \quad \left(\text{resp.} \quad \frac{\chi(F(mH))}{\text{rk}(F)} \leq \frac{\chi(E(mH))}{\text{rk}(E)} \right)$$

for every proper ϕ -invariant subsheaf $0 \neq F \subsetneq E$. By “(semi)stable”, we will always mean Gieseker (semi)stable.

Let $E = E_0 \oplus \dots \oplus E_s$ be a sum of torsion-free rank 1 sheaves, equipped with a Higgs field

$$\phi = (\phi_1, \dots, \phi_s): E \rightarrow E \otimes \omega_S$$

given by homomorphisms

$$\phi_i: E_{i-1} \rightarrow E_i \otimes \omega_S \quad \text{for } i = 1, \dots, s.$$

Note that all Higgs pairs in \mathcal{M}_{1r} are of this form.

Lemma 3.2 Assume that (E, ϕ) is indecomposable, ie $\phi_i \neq 0$ for $i = 1, \dots, s$, and assume that

$$\deg(E_{i-1}) \geq \deg(E_i) \quad \text{for } i = 1, \dots, s.$$

Then the pair (E, ϕ) is slope semistable. It is slope stable unless

$$\deg(E_0) = \dots = \deg(E_s).$$

Proof Let $F \subset E$ be a ϕ -invariant Higgs field. Let j be maximal, such that

$$(3.3) \quad F \subset E_j \oplus \dots \oplus E_s.$$

I claim that F has rank $s + 1 - j$. It follows that if F is a destabilizing subsheaf, so is $E_j \oplus \dots \oplus E_s$.

In order to prove the claim, consider the filtration

$$F = F^0 \supset \dots \supset F^{s-j} \supset F^{s+1-j} = 0$$

of F given by

$$F^i = K_S^{-i} \otimes \phi^{oi}(F).$$

Then

$$F^i \subset E_{j+i} \oplus \dots \oplus E_s$$

for $i = 0, \dots, s + 1 - j$. Note that for $i = 0, \dots, s - j$, by injectivity of $\phi_{j+i} \dots \phi_{j+1}$ and by the choice of j , the composition

$$F^i \subset E_{j+i} \oplus \dots \oplus E_s \rightarrow E_{j+i}$$

is nonzero, and hence its image has rank 1, since E_{j+i} is torsion-free. On the other hand, its kernel contains F^{i+1} . It follows that we have

$$\text{rk } F > \text{rk } F^1 > \dots > \text{rk } F^{s+1-j} = 0$$

and hence $\text{rk}(F) = s + 1 - j$ by (3.3), proving the claim.

It follows that (E, ϕ) is slope semistable if and only if

$$\frac{\sum_{i=j}^s \deg(E_i)}{s + 1 - j} \leq \frac{\sum_{i=0}^s \deg(E_i)}{s + 1} = \frac{\deg(E)}{\text{rk}(E)}$$

for $j = 0, \dots, s$. This clearly holds when $\deg(E_i) \leq \deg(E_{i-1})$ for all i . Finally note that (E, ϕ) is slope stable if one of the inequalities is strict. \square

The hypothesis of Lemma 3.2 certainly holds when $c_1(E_{i-1}) - c_1(E_i)$ is effective for each i . In this case, the condition

$$\deg(E_0) = \dots = \deg(E_s)$$

implies that

$$c_1(E_0) = \cdots = c_1(E_s).$$

Although such a Higgs pair is not slope stable, it might still be Gieseker (semi)stable.

Lemma 3.4 *Assume that (E, ϕ) is indecomposable, and assume that*

$$c_1(E_0) = \cdots = c_1(E_s) \quad \text{and} \quad c_2(E_0) \leq \cdots \leq c_2(E_s).$$

Then the pair (E, ϕ) is Gieseker semistable. It is Gieseker stable unless

$$c_2(E_0) = \cdots = c_2(E_s).$$

Proof The proof is similar to that of Lemma 3.2. Simply note that by Grothendieck–Riemann–Roch, the hypothesis implies

$$\chi(E_{i-1}(m)) \geq \chi(E_i(m))$$

for $i = 1, \dots, s$, with equality whenever $n_{i-1} = n_i$. □

Now let S be a surface with $p_g(S) > 0$ and $H^1(\mathcal{O}_S) = 0$. Let L_0, \dots, L_s be line bundles on S and let $n = (n_0, \dots, n_s)$ be nonnegative integers. Let $\beta = \beta(L)$ (and β_i and β^i for $i = 1, \dots, s$) be given as in Notation 2.1, and consider the flat family of Higgs pairs $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]})$ over the base $S_{\beta}^{[n]}$, as defined in Section 2.

In terms of β and n , Lemmas 3.2 and 3.4 tell us that whenever the family $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]})$ is not Gieseker semistable, there is an $i \in \{1, \dots, s\}$ such that the divisor class β^i is not effective or such that $\beta^i = 0$ and $n_{i-1} > n_i$ (compare to the introduction of this section!). As we will see in the following proposition, this suffices to show that we have $i_*[S_{\beta}^{[n]}]^{\text{vir}} = 0$ in this case (recall that we write $i: S_{\beta}^{[n]} \hookrightarrow \text{Hilb}^n(S)$).

Proposition 3.5 *Assume that*

$$i_*[S_{\beta}^{[n]}]^{\text{vir}} \neq 0.$$

Then the family $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}})$ of Higgs pairs is (Gieseker) semistable for any polarization of S . It is stable unless $L_0 = \cdots = L_s$ and $n_0 = \cdots = n_s$.

Proof By Theorem 1.5, (1.22) and the hypothesis, we have

$$\text{SW}(\beta^1) \cdots \text{SW}(\beta^s) = (-1)^{s \cdot \chi(\mathcal{O}_S)} \text{SW}(\beta_1) \cdots \text{SW}(\beta_s) \neq 0.$$

It follows that $\beta^i \geq 0$ for $i = 0, \dots, s$, by definition of the Seiberg–Witten class. By Lemma 3.2, the fibres of $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}})$ are slope stable, and hence Gieseker stable, unless $L_0 = \dots = L_s$. Assume the latter. By Lemma 3.4, we need to show that $n_{i-1} \leq n_i$ for all i . Assume that $n_{i-1} > n_i$ for some i . Then the nested Hilbert scheme

$$i: S^{[n_i, n_{i-1}]} \hookrightarrow S^{[n_i]} \times S^{[n_{i-1}]}$$

is empty, and we have by Serre duality and Theorem 1.5

$$\begin{aligned} (3.6) \quad e(R\pi_*\omega_S - R\mathcal{H}om_{\pi}(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]} \otimes \omega_S)) \\ = (-1)^{n_{i-1} + n_i} e(R\pi_*(\mathcal{O}_S) - R\mathcal{H}om_{\pi}(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_{i-1}]}) \\ = (-1)^{n_{i-1} + n_i} i_*[S^{[n_i, n_{i-1}]}]^{\text{vir}} \\ = 0. \end{aligned}$$

By the assumption $L_0 = \dots = L_s$, we have in particular $\beta_i = K_S$. It follows that the expression of Theorem 1.5 has a factor as in (3.6). We find $i_*[S_{\beta}^{[n]}]^{\text{vir}} = 0$, which contradicts the hypothesis. \square

Recall from Proposition 2.4 that $\mathcal{M}_{1r, c_1, c_2}$ is a union of Hilbert schemes $S_{\beta}^{[n]}$. We will also write i for the morphism

$$i: \mathcal{M}_{1r, c_1, c_2} \rightarrow \bigsqcup_{\beta, n} \text{Hilb}_{\beta}^n(S)$$

which is given on each component of $\mathcal{M}_{1r, c_1, c_2}$ by the inclusion

$$i: S_{\beta}^{[n]} \rightarrow \text{Hilb}_{\beta}^n(S).$$

By the vanishing of Proposition 3.5, we can sum in the following proposition over all pairs (L, n) or (β, n) , rather than the ones that correspond to components of stable Higgs pairs. In particular, the pushforward by i of the virtual class does not depend on the polarization of the surface S .

Proposition 3.7 *Let S be a surface with $p_g(S) > 0$ and $H^1(\mathcal{O}_S) = 0$. Fix r, c_1 and c_2 such that Gieseker semistability of Higgs pairs implies Gieseker stability. Then*

$$\begin{aligned} (3.8) \quad i_*[\mathcal{M}_{1r, c_1, c_2}]^{\text{vir}} &= \sum_{L, n} i_*[S_{\beta(L)}^{[n]}]^{\text{vir}} \\ &= \sum_{\beta, n} i_*[S_{\beta}^{[n]}]^{\text{vir}} \cdot \#\ker(H^2(S, \mathbb{Z}) \xrightarrow{\cdot(s+1)} H^2(S, \mathbb{Z})), \end{aligned}$$

where the sums are taken over pairs (L, n) with

$$c_1 = \sum_{i=1}^s c_1(L_i) \quad \text{and} \quad c_2 = |n| + \sum_{0 \leq i < j \leq s} c_1(L_i)c_1(L_j),$$

and, respectively, pairs (β, n) with

$$c_1 \equiv \sum_{i=1}^s i\beta^i \pmod{(s+1)H^2(S, \mathbb{Z})},$$

$$c_2 = |n| + \frac{s}{2(s+1)}c_1^2 - \sum_{1 \leq i < j \leq s} \frac{i(s+1-j)}{s+1}\beta^i\beta^j - \sum_{1 \leq i \leq s} \frac{i(s+1-i)}{2(s+1)}(\beta^i)^2.$$

Proof The c_2 -conditions on the pairs (L, n) and (β, n) appearing in the sums are given by Lemma 2.6. Moreover, it is easy to see that for an s -tuple of curve classes $\beta = (\beta_1, \dots, \beta_s) \in (H^2(S, \mathbb{Z}))^s$, there is a tuple of vector bundles $L = (L_0, \dots, L_s)$ with

$$c_1 = \sum_{i=1}^s c_1(L_i) \quad \text{and} \quad \beta = \beta(L)$$

if and only if

$$c_1 \equiv \sum_{i=1}^s i\beta^i \pmod{(s+1)H^2(S, \mathbb{Z})}.$$

Now assume that $S_\beta^{[n]} \cong \mathcal{M}_L^{[n]} \subset \mathcal{M}_{1^r, c_1, c_2}$ is a component (see Definition 2.5). For a curve class

$$\gamma \in \ker(H^2(S, \mathbb{Z}) \xrightarrow{\cdot(s+1)} H^2(S, \mathbb{Z})) =: K$$

there is a component

$$S_\beta^{[n]} \cong \mathcal{M}_{L(\gamma)}^{[n]} \subset \mathcal{M}_{1^r, c_1, c_2},$$

where

$$L(\gamma) = (L_0 \otimes \mathcal{O}_S(\gamma), \dots, L_s \otimes \mathcal{O}_S(\gamma)).$$

In fact, there is a K -torsor of components of $\mathcal{M}_{1^r, c_1, c_2}$ that are isomorphic to $S_\beta^{[n]}$. This explains the second equation of (3.8).

Finally, note that pairs (β, n) for which the scheme $S_\beta^{[n]}$ is empty obviously do not contribute to the right-hand side of (3.8). By Proposition 3.5, for pairs (β, n) for which $S_\beta^{[n]}$ parametrizes unstable Higgs pairs, the pushforward of the virtual class $i_*[S_\beta^{[n]}]$ vanishes. Hence these pairs also do not contribute to the right-hand side of (3.8). For this reason, the sum can be taken over all pairs (β, n) rather than only over the ones corresponding to components of $\mathcal{M}_{1^r, c_1, c_2}$. □

4 Tautological integrals

Choose line bundles $L = (L_0, \dots, L_s)$ on S and let $\beta = \beta(L) = (\beta_1, \dots, \beta_s)$ and $\mathcal{L} = (\mathcal{L}_0, \dots, \mathcal{L}_s)$ be defined as in Section 2. Let $n = (n_0, \dots, n_s)$ be nonnegative integers. Recall that we write

$$E_{\mathcal{L}}^{[n]} = \mathcal{L}_0 \otimes \mathcal{I}^{[n_0]} \otimes \mathfrak{t}^0 \oplus \dots \oplus \mathcal{L}_s \otimes \mathcal{I}^{[n_s]} \otimes \mathfrak{t}^{-s}$$

for the sheaf on

$$\text{Hilb}_{\beta}^n(S) \times S = S^{[n_0]} \times \dots \times S^{[n_s]} \times |\mathcal{O}_S(\beta_1)| \times \dots \times |\mathcal{O}_S(\beta_s)| \times S$$

and for its restriction to the nested Hilbert scheme

$$i: S_{\beta}^{[n]} \hookrightarrow \text{Hilb}_{\beta}^n(S),$$

over which we have a canonically defined Higgs field $\phi_{\mathcal{L}}: E_{\mathcal{L}}^{[n]} \rightarrow E_{\mathcal{L}}^{[n]} \otimes \omega_S \otimes \mathfrak{t}$.

Define a class

$$T_{\mathcal{L}}^{[n]} := R\mathcal{H}om_{\pi}(E_{\mathcal{L}}^{[n]}, E_{\mathcal{L}}^{[n]} \otimes \omega_S \otimes \mathfrak{t})_0 - R\mathcal{H}om_{\pi}(E_{\mathcal{L}}^{[n]}, E_{\mathcal{L}}^{[n]})_0 \in K_0^{\mathbb{C}^*}(\text{Hilb}_{\beta}^n(S)),$$

and denote its pullback to $S_{\beta}^{[n]}$ by the same symbol. Note that $T_{\mathcal{L}}^{[n]}$ depends only on β , rather than on L (or on \mathcal{L}). We will write

$$N_{\mathcal{L}}^{[n]} := T_{\mathcal{L}}^{[n]} - (T_{\mathcal{L}}^{[n]})^{\mathbb{C}^*}$$

for its moving part. Let e denote the \mathbb{C}^* -equivariant Euler class, and define the rational number

$$(4.1) \quad \text{VW}_{\beta}^{[n]} := \int_{[S_{\beta}^{[n]}]_{\text{vir}}} \frac{1}{e(N_{\mathcal{L}}^{[n]})}.$$

If $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}})$ represents a component

$$\mathcal{M}_L^{[n]} = (S_{\beta}^{[n]}, E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}}^{[n]}) \subset \mathcal{M}_{1^r, c_1, c_2},$$

then $T_{\mathcal{L}}^{[n]}$ is the class in K -theory of the cone (1.1) in the introduction and hence equals the \mathbb{C}^* -localized perfect obstruction theory of [20] on $\mathcal{M}_L^{[n]}$. Over $\mathcal{M}_L^{[n]}$, the class $N_{\mathcal{L}}^{[n]}$ is the virtual normal bundle to the \mathbb{C}^* -fixed locus $(\mathcal{N}_{r, M, c_2}^{\perp})^{\mathbb{C}^*}$ in $\mathcal{N}_{r, M, c_2}^{\perp}$. By definition of the Vafa–Witten invariant (1.2), the contribution from the component $\mathcal{M}_L^{[n]}$ is given by $\text{VW}_{\beta}^{[n]}$.

If the Higgs pair $(E_{\mathcal{L}}^{[n]}, \phi_{\mathcal{L}})$ contains fibres that are not Gieseker semistable, it does not represent a component of any $\mathcal{M}_{1r, c_1, c_2}$ and hence does not contribute to the Vafa–Witten invariant. On the other hand, by Proposition 3.5, we have $\text{VW}_{\beta}^{[n]} = 0$ in this case. It follows that, using the notation from Proposition 3.7, we have

$$\begin{aligned} \text{VW}_{1r, c_1, c_2} &= \sum_{L, n} \text{VW}_{\beta(L)}^{[n]} \\ &= \sum_{\beta, n} \text{VW}_{\beta}^{[n]} \cdot \#\ker(H^2(S, \mathbb{Z}) \xrightarrow{\cdot(s+1)} H^2(S, \mathbb{Z})). \end{aligned}$$

Now define a line bundle

$$K_{\mathcal{L}}^{[n]} := \det(T_{\mathcal{L}}^{[n]\vee})$$

on $\text{Hilb}_{\beta}^{[n]}(S)$. Note that $T_{\mathcal{L}}^{[n]}$ is defined as the difference between a complex and its dual, up to a factor \mathfrak{t} . Hence its determinant is by construction a square, up to a factor \mathfrak{t} . Hence, after choosing once and for all a square root of \mathfrak{t} , the line bundle $K_{\mathcal{L}}^{[n]}$ has a canonical square root, denoted by $(K_{\mathcal{L}}^{[n]})^{\frac{1}{2}}$. Over $S_{\beta}^{[n]}$, the bundle $K_{\mathcal{L}}^{[n]}$ restricts to the virtual canonical bundle [21], and its square root restricts to the canonical square root of [21, Proposition 2.6].

By [21], the contribution to the refined invariant can be computed by

$$(4.2) \quad \text{VW}_{\beta}^{[n]}(y) := \left[\int_{[S_{\beta}^{[n]}]_{\text{vir}}} \frac{\text{ch}((K_{\mathcal{L}}^{[n]})^{\frac{1}{2}})}{\text{ch}(\Lambda^{\bullet}(N_{\mathcal{L}}^{[n]\vee}))} \text{Td}((T_{\mathcal{L}}^{[n]})^{\mathbb{C}^*}) \right]_{\text{ch}(\mathfrak{t})=y},$$

where ch and Td denote the \mathbb{C}^* -equivariant Chern character and Todd class respectively. Again, in the language of Proposition 3.7, we have

$$(4.3) \quad \begin{aligned} \text{VW}_{1r, c_1, c_2}(y) &= \sum_{L, n} \text{VW}_{\beta(L)}^{[n]}(y) \\ &= \sum_{\beta, n} \text{VW}_{\beta}^{[n]}(y) \cdot \#\ker(H^2(S, \mathbb{Z}) \xrightarrow{\cdot(s+1)} H^2(S, \mathbb{Z})). \end{aligned}$$

By Theorem 1.5, we have

$$i_*[S_{\beta}^{[n]}]_{\text{vir}} = \prod_{i=1}^s e(R\pi_*(\mathcal{O}_S(\beta_i)) - R\mathcal{H}om_{\pi}(\mathcal{I}^{[n_i-1]}, \mathcal{I}^{[n_i]}(\beta_i))) \cap [S^{[n_0]} \times \dots \times S^{[n_s]}] \times \text{SW}(\beta_1) \times \dots \times \text{SW}(\beta_s).$$

The factor

$$\text{SW}(\beta) := \text{SW}(\beta_1) \times \dots \times \text{SW}(\beta_s) \in A_0(|\mathcal{O}_S(\beta_1)| \times \dots \times |\mathcal{O}_S(\beta_s)|)$$

annihilates all Chern classes in the integrands of (4.1) and (4.2) that are pulled back from

$$|\mathcal{O}_S(\beta_1)| \times \cdots \times |\mathcal{O}_S(\beta_s)|.$$

It follows that we can rewrite (4.1) and (4.2) as integrals over

$$\text{Hilb}^n(S) = S^{[n_0]} \times \cdots \times S^{[n_s]}.$$

Define the sheaf

$$E_L^{[n]} := L_0 \otimes \mathcal{I}^{[n_0]} \otimes \mathfrak{t}^0 \oplus \cdots \oplus L_s \otimes \mathcal{I}^{[n_s]} \otimes \mathfrak{t}^{-s} \quad \text{on } \text{Hilb}^n(S) \times S$$

and classes

$$\begin{aligned} T_L^{[n]} &:= R\mathcal{H}om_\pi(E_L^{[n]}, E_L^{[n]} \otimes \omega_S \otimes \mathfrak{t})_0 - R\mathcal{H}om_\pi(E_L^{[n]}, E_L^{[n]})_0, \\ N_L^{[n]} &:= T_L^{[n]} - (T_L^{[n]})^{\mathbb{C}^*}, \\ K_L^{[n]} &:= \det(T_L^{[n]\vee}) \end{aligned}$$

in $K_0(\text{Hilb}^n(S))$. Again, note that since $H^1(\mathcal{O}_S) = 0$, the classes $T_L^{[n]}$, $N_L^{[n]}$ and $K_L^{[n]}$ depend on $\beta = \beta(L)$, rather than on L . We have, now considering $\text{SW}(\beta)$ as an integer,

$$\begin{aligned} (4.4) \quad \text{VW}_\beta^{[n]} &= \text{SW}(\beta) \int_{[\text{Hilb}^n(S)]} \frac{1}{e(N_L^{[n]})} \\ &\quad \times \prod_{i=1}^s e(R\pi_*\mathcal{O}(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))) \end{aligned}$$

and

$$\begin{aligned} (4.5) \quad \text{VW}_\beta^{[n]}(y) &= \text{SW}(\beta) \left[\int_{[\text{Hilb}^n(S)]} \frac{\text{ch}((K_L^{[n]})^{\frac{1}{2}})}{\text{ch}(\Lambda^\bullet(N_L^{[n]\vee}))} \text{Td}((T_L^{[n]})^{\mathbb{C}^*}) \right. \\ &\quad \left. \times \prod_{i=1}^s e(R\pi_*\mathcal{O}(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i))) \right]_{\text{ch}(\mathfrak{t})=y}. \end{aligned}$$

5 Removing trace

The generating series

$$\sum_n \text{VW}_\beta^{[n]} q^n$$

has leading term

$$\text{VW}_\beta^{[0]} = \text{SW}(\beta) \frac{1}{e(N_L^{[0]})}.$$

We can renormalize the series by dividing through the factor $1/e(N_L^{[0]})$ (which is $F(S, \beta)$ in the notation below). In terms of the integrals of (4.4), this comes down to considering “traceless” integrands. By this we mean the following. Note that the integrand of (4.4) can be written as product of (equivariant) Euler classes of terms of the form

$$R\mathcal{H}om_\pi(\mathcal{E}, \mathcal{F})$$

with \mathcal{E} and \mathcal{F} torsion-free rank 1 sheaves. Then a traceless version of the integral of (4.4) (denoted by $Q_n(S, \beta)$ below) is given by replacing each such term by

$$R\mathcal{H}om_\pi(\mathcal{E}, \mathcal{F})_0 = R\mathcal{H}om_\pi(\mathcal{E}, \mathcal{F}) - R\mathcal{H}om_\pi(\det \mathcal{E}, \det \mathcal{F}).$$

It is easy to see that the resulting expression computes the renormalized generating series (see (5.1) below). In Section 6, we will deal with the normalizing factor separately.

We keep the notation from the previous section. Moreover, we will write

$$E_L := E_L^{[0]} = L_0 \otimes \mathfrak{t}^0 \oplus \cdots \oplus L_s \otimes \mathfrak{t}^{-s}$$

for the vector bundle on S , and furthermore

$$T_L := T_L^{[0]} = R\text{Hom}(E_L, E_L \otimes \omega_S \otimes \mathfrak{t})_0 - R\text{Hom}(E_L, E_L)_0,$$

$$N_L := N_L^{[0]} = T_L - (T_L)^{\mathbb{C}^*},$$

$$K_L := K_L^{[0]} = \det T_L^\vee$$

for the classes in the equivariant K -group of a point. Finally, we will also use the notation

$$T_{L,0}^{[n]} = T_L^{[n]} - T_L, \quad N_{L,0}^{[n]} = N_L^{[n]} - N_L \quad \text{and} \quad K_{L,0}^{[n]} = K_L^{[n]} \otimes K_L^*$$

for the classes in $K_0(\text{Hilb}^n(S))$, where we suppress pullbacks from the point. Define

$$F(S, \beta) := \frac{1}{e(N_L)},$$

$$Q_n(S, \beta) := \int_{[\text{Hilb}^n(S)]} \frac{1}{e(N_{L,0}^{[n]})} \prod_{i=1}^s e(R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_i-1]}, \mathcal{I}^{[n_i]}(\beta_i))),$$

so that

$$(5.1) \quad \text{VW}_\beta^{[n]} = \text{SW}(\beta) F(S, \beta) Q_n(S, \beta).$$

In the refined case, define

$$\begin{aligned}
 F(S, \beta, y) &:= \left[\frac{\text{ch}(K_L^{\frac{1}{2}})}{\text{ch}(\Lambda^\bullet(N_L^\vee))} \text{Td}(T_L^{\mathbb{C}^*}) \right]_{\text{ch}(t)=y}, \\
 Q_n(S, \beta, y) &:= \left[\int_{[\text{Hilb}^n(S)]} \frac{\text{ch}((K_{L,0}^{[n]})^{\frac{1}{2}})}{\text{ch}(\Lambda^\bullet((N_{L,0}^{[n]})^\vee))} \text{Td}((T_{L,0}^{[n]})^{\mathbb{C}^*}) \right. \\
 &\quad \left. \times \prod_{i=1}^s e(R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_i-1]}, \mathcal{I}^{[n_i]}(\beta_i))) \right]_{\text{ch}(t)=y},
 \end{aligned}$$

so that

$$(5.2) \quad \text{VW}_\beta^{[n]}(y) = \text{SW}(\beta) F(S, \beta, y) Q_n(S, \beta, y).$$

Remark 5.3 A priori, $Q_n(S, \beta, y)$ is a rational function in \sqrt{y} , due to the fractional exponent of the virtual canonical bundle. However, an easy computation shows that the equivariant parameter t appears in $K_{\beta,0}^{[n]}$, with *even* exponent, and hence, $Q_n(S, \beta, y)$ is in fact a rational function in y .

In Section 6 we will compute $F(S, \beta)$ under the assumption $\text{SW}(\beta) \neq 0$. In Section 7 we will show that the numbers $Q_n(S, \beta)$ are given by universal polynomials $P_n(S, \beta)$ in the Chern numbers of S and $\beta = (\beta_1, \dots, \beta_s)$. We will deal with the refined version at the same time.

6 The leading term

In this section we compute the factor $F(S, \beta, y)$. Let $L = (L_0, \dots, L_s)$ be an $(s+1)$ -tuple of line bundles on S , and let $\beta = \beta(L)$ be as in Notation 2.1. Also recall

$$E_L = L_0 \otimes t^0 \oplus \dots \oplus L_s \otimes t^{-s}.$$

Assume that

$$\text{SW}(\beta) = \text{SW}(\beta_1) \cdots \text{SW}(\beta_s) \neq 0.$$

Then, by [18, Proposition 6.29], we have $(\beta^i)^2 = (\beta^i K_S)$ or, equivalently,

$$\chi(\beta^i) = \chi(\mathcal{O}_S),$$

for $i = 1, \dots, s$. Using Serre duality, we can express T_L as follows:

$$\begin{aligned}
 T_L &= R\text{Hom}(E_L, E_L \otimes \omega_s \otimes \mathfrak{t})_0 - R\text{Hom}(E_L, E_L)_0 \\
 &= \sum_{i=0}^s (\mathfrak{t}^{i+1} - \mathfrak{t}^{-i}) \left(\sum_{j=0}^{s-i} \chi(L_{i+j}^* \otimes L_j \otimes \omega_S) - \sum_{j=1}^{s-i} \chi(L_{i+j}^* \otimes L_{j-1}) \right) - (\mathfrak{t}-1) \cdot \chi(\mathcal{O}_S) \\
 &= \sum_{i=1}^s (\mathfrak{t}^{i+1} - \mathfrak{t}^{-i}) \left(\sum_{j=0}^{s-i} \chi(\beta^{1+j} + \dots + \beta^{i+j} + K_S) - \sum_{j=1}^{s-i} \chi(\beta^j + \dots + \beta^{i+j}) \right),
 \end{aligned}$$

where the second sum starts with $i = 1$, since the coefficient of $(\mathfrak{t}-1)$ equals

$$\left(\sum_{j=0}^s \chi(K_S) - \sum_{j=1}^s \chi(\beta^j) \right) - \chi(\mathcal{O}_S) = 0$$

by the assumption $\text{SW}(\beta) \neq 0$. Note that, in particular, we have

$$(6.1) \quad T_L = N_L.$$

Moreover, note that

$$\begin{aligned}
 \chi(\beta^{1+j} + \dots + \beta^{i+j} + K_S) &= \frac{1}{2}((\beta^{1+j} + \dots + \beta^{i+j} + K_S) \cdot (\beta^{1+j} + \dots + \beta^{i+j})) + \chi(\mathcal{O}_S) \\
 &= \sum_{j < k \leq l \leq i+j} \beta^k \beta^l + \chi(\mathcal{O}_S),
 \end{aligned}$$

and, similarly,

$$\begin{aligned}
 \chi(\beta^j + \dots + \beta^{i+j}) &= \frac{1}{2}((\beta^j + \dots + \beta^{i+j}) \cdot (\beta^j + \dots + \beta^{i+j} - K_S)) + \chi(\mathcal{O}_S) \\
 &= \sum_{j \leq k < l \leq i+j} \beta^k \beta^l + \chi(\mathcal{O}_S).
 \end{aligned}$$

It follows that for $k \leq l$, the multiplicity with which the term

$$(\mathfrak{t}^{i+1} - \mathfrak{t}^{-i}) \cdot \beta^k \beta^l$$

appears in T_L is given by

$$\begin{aligned}
 \mu(i, k, l) &:= \#\{j \mid 0 \leq j < k \leq l \leq i+j \leq s\} - \#\{j \mid 0 < j \leq k < l \leq i+j \leq s\} \\
 &= \#\{j \mid 0, l-i \leq j \leq k-1, s-i\} - \#\{j \mid 1, l-i \leq j \leq k, s-i \text{ and } k < l\} \\
 &= \begin{cases} \left\{ \begin{array}{l} -1 \text{ if } l-k \leq i < \min(l, s-k+1), \\ 1 \text{ if } \max(l, s-k+1) \leq i \leq s, \\ 0 \text{ otherwise,} \end{array} \right\} & \text{if } k < l, \\ \min\{i, s-k+1, k, s-i+1\} & \text{if } k = l, \end{cases}
 \end{aligned}$$

so we have

$$(6.2) \quad N_L = T_L = \sum_{i=1}^s \left(\chi(\mathcal{O}_S) + \sum_{k \leq l} \mu(i, k, l) \cdot \beta^k \beta^l \right) \cdot (t^{i+1} - t^{-i}).$$

We define the following rational numbers:

$$\begin{aligned} F_0^{(s+1)} &:= \frac{(-1)^s}{s+1}, \\ F_{kk}^{(s+1)} &:= \frac{(-1)^{sk}}{\binom{s+1}{k}} \quad \text{for } 1 \leq k \leq s, \\ F_{kl}^{(s+1)} &:= \frac{l(s+1-k)}{(l-k)(s+1)} \quad \text{for } 1 \leq k < l \leq s. \end{aligned}$$

Proposition 6.3 Recall that we assume $SW(\beta) \neq 0$. We have

$$F(S, \beta) = (F_0^{(s+1)})^{\chi(\mathcal{O}_S)} \prod_{k \leq l} (F_{kl}^{(s+1)})^{\beta^k \beta^l}.$$

Proof Applying $1/e(\cdot)$ to (6.2), we obtain

$$\begin{aligned} F(S, \beta) &= \frac{1}{e(N_L)} \\ &= e \left(\sum_{i=1}^s t^{i+1} - t^{-i} \right)^{-\chi(\mathcal{O}_S)} \prod_{k \leq l} e \left(\sum_{i=1}^s \mu(i, k, l) (t^{i+1} - t^{-i}) \right)^{-\beta^k \beta^l}. \end{aligned}$$

Note that we have

$$\frac{1}{e(t^{i+1} - t^{-i})} = \frac{-i}{i+1},$$

and hence

$$e \left(\sum_{i=1}^s (t^{i+1} - t^{-i}) \right)^{-1} = \frac{-1 \cdots -i}{2 \cdots (i+1)} = \frac{(-1)^s}{s+1}.$$

For $k < l$ we find

$$\begin{aligned} e \left(\sum_{i=1}^s \mu(i, k, l) \cdot (t^{i+1} - t^{-i}) \right)^{-1} &= \prod_{i=l-k}^{\min(l-1, s-k)} \left(\frac{-i}{i+1} \right)^{-1} \cdot \prod_{i=\max(l, s-k+1)}^s \frac{-i}{i+1} \\ &= \frac{l(s+1-k)}{(l-k)(s+1)}. \end{aligned}$$

Finally, write

$$a := \min(k, s+1-k) \quad \text{and} \quad b := \max(k, s+1-k),$$

so that

$$\begin{aligned}
 e\left(\sum_{i=1}^s \mu(i, k, k) \cdot (t^{i+1} - t^{-i})\right)^{-1} &= \prod_{i=1}^a \left(\frac{-i}{i+1}\right)^i \prod_{i=a+1}^{b-1} \left(\frac{-i}{i+1}\right)^a \prod_{i=b}^s \left(\frac{-i}{i+1}\right)^{s+1-i} \\
 &= (-1)^{sk} \frac{1 \cdots a}{(b+1) \cdots (s+1)} = \frac{(-1)^{sk}}{\binom{s+1}{k}}. \quad \square
 \end{aligned}$$

Notation 6.4 We will use *quantum integers*, which are given by

$$[i]_y := \frac{y^{\frac{1}{2}i} - y^{-\frac{1}{2}i}}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}.$$

We will also use the notation

$$\binom{i}{j}_y := \frac{[i]_y \cdots [i-j+1]_y}{[1]_y \cdots [j]_y} \quad \text{for nonnegative integers } i \leq j.$$

Define the following rational functions in $y^{\frac{1}{2}}$:

$$\begin{aligned}
 F_0^{(s+1)}(y) &:= \frac{(-1)^s}{[s+1]_y}, \\
 F_{kk}^{(s+1)}(y) &:= \frac{(-1)^{sk}}{\binom{s+1}{k}_y} \quad \text{for } 1 \leq k \leq s, \\
 F_{kl}^{(s+1)}(y) &:= \frac{[l]_y [s+1-k]_y}{[l-k]_y [s+1]_y} \quad \text{for } 1 \leq k < l \leq s.
 \end{aligned}$$

Proposition 6.5 Assume that $\text{SW}(\beta) \neq 0$. Then we have

$$F(S, \beta, y) = (F_0^{(s+1)}(y))^{\chi(\mathcal{O}_S)} \prod_{k \leq l} (F_{kl}^{(s+1)}(y))^{\beta^k \beta^l}.$$

Proof Recall (6.1) that T_L has no fixed part, so we have

$$F(S, \beta, y) = \left[\frac{\text{ch}(K_L^{\frac{1}{2}})}{\text{ch}(\Lambda^\bullet(N_L^\vee))} \text{Td}((T_L)^{\mathbb{C}^*}) \right]_{\text{ch}(t)=y} = \left[\frac{\text{ch}(\det(N_L^\vee))^{\frac{1}{2}}}{\text{ch}(\Lambda^\bullet(N_L^\vee))} \right]_{\text{ch}(t)=y}.$$

Note that we have

$$\left[\frac{\text{ch}(\det((t^{i+1} - t^{-i})^\vee)^{\frac{1}{2}})}{\text{ch}(\Lambda^\bullet((t^{i+1} - t^{-i})^\vee))} \right]_{\text{ch}(t)=y} = -y^{\frac{1}{2}} \frac{y^i - 1}{y^{i+1} - 1} = -\frac{[i]_y}{[i+1]_y}.$$

Now follow the proof of [Proposition 6.3](#). □

Remark 6.6 If $r = s + 1$ is odd, $F(S, \beta, y)$ is a function in y , rather than in \sqrt{y} , for any $\beta = (\beta_0, \dots, \beta_s)$ with $\text{SW}(\beta) \neq 0$.

Example 6.7 For rank 2 we have

$$F(S, \beta, y) = \left(\frac{-y^{\frac{1}{2}}}{1+y} \right)^{\chi(\mathcal{O}_S) + \beta^1 \beta^1},$$

and for rank 3

$$F(S, \beta, y) = \left(\frac{y}{1+y+y^2} \right)^{\chi(\mathcal{O}_S) + \beta^1 \beta^1 + \beta^2 \beta^2} \left(\frac{(y+1)^2}{1+y+y^2} \right)^{\beta^1 \beta^2}.$$

7 Universality

Let S be a smooth projective surface, not necessarily with $H^1(\mathcal{O}_S) = 0$ or $p_g > 0$. For nonnegative integers $n = (n_0, \dots, n_s)$ and classes $\beta = (\beta_1, \dots, \beta_s)$, consider the rational number $Q_n(S, \beta)$ defined in Section 5 as an integral over

$$\text{Hilb}^n(S) = S^{[n_0]} \times \dots \times S^{[n_s]}.$$

Using the notation

$$q^n := q_0^{n_0} \dots q_s^{n_s}$$

we form the generating series

$$\sum_n Q_n(S, \beta) q^n.$$

The following universality result, Proposition 7.3, or rather its refinement, Proposition 7.6, is the main ingredient for the proof of Theorem A.

Remark 7.1 In Section 5, the integrals $Q_n(S, \beta)$ were defined in terms of a lift of β to a vector of line bundles L , such that $\beta = \beta(L)$ (see Notation 2.1). Since we do not assume $H^1(\mathcal{O}_S) = 0$ in this section, this lift involves a lift of the β_i to divisor classes of S . We assume we have made such choice, and we consider β as a vector of classes in $A^1(S)$. By Proposition 7.3, $Q_n(S, \beta)$ does not depend on this choice.

Notation 7.2 In the following we will use the formal symbols

$$\underline{\chi(\mathcal{O}_S)}, \underline{K_S^2}, \underline{K_S \beta^i}, \underline{\beta^i \beta^j} \quad \text{for } 1 \leq i \leq j \leq s.$$

Formally, they form a dual basis of the vector space of cobordism classes (with \mathbb{Q} -coefficients) of surfaces together with an s -tuple of divisors. In particular they can be evaluated on (classes of) such pairs. For example, we have

$$\underline{K_S} \beta^2(\mathbb{P}^2, (H, H, 0)) = K_{\mathbb{P}^2} H = -3,$$

where H is the class of a hyperplane. In general we write, a bit informally,

$$\underline{\mathfrak{N}}(S, \beta) = \mathfrak{N}.$$

Proposition 7.3 *For each symbol*

$$\underline{\mathfrak{N}} \in \{\underline{\chi}(\mathcal{O}_S), \underline{K_S}^2, \underline{K_S} \beta^i, \beta^i \beta^j\}_{1 \leq i \leq j \leq s},$$

there is a power series $A_{\underline{\mathfrak{N}}}^{(s+1)} \in \mathbb{Q}[[q_0, \dots, q_s]]$, starting with 1 and depending only on s such that

$$\sum_n Q_n(S, \beta) q^n = \prod_{\underline{\mathfrak{N}}} (A_{\underline{\mathfrak{N}}}^{(s+1)})^{\mathfrak{N}}$$

for **any** smooth projective surface S and classes $\beta_1, \dots, \beta_s \in A^1(S)$.

Proof By the techniques of [2] (see also [12]), the integral $Q_n(S, \beta)$ can be universally expressed as a polynomial $P_n(S, \beta)$ in the Chern numbers of S and the classes β^1, \dots, β^s . Following [9, Proposition 2.3], it suffices to show that the generating series is multiplicative, ie

$$\sum Q_n(S \sqcup S', \beta + \beta') q^n = \sum Q_n(S, \beta) q^n \cdot \sum Q_n(S', \beta') q^n$$

for surfaces S and S' and s -tuples β and β' of classes in $A^1(S)$ and $A^1(S')$ respectively.

Note that

$$\begin{aligned} \text{Hilb}^n(S \sqcup S') &= (S \sqcup S')^{[n_0]} \times \dots \times (S \sqcup S')^{[n_s]} \\ &= \bigsqcup_{i_0 + j_0 = n_0} (S^{[i_0]} \times S'^{[j_0]}) \times \dots \times \bigsqcup_{i_s + j_s = n_s} (S^{[i_s]} \times S'^{[j_s]}) \\ &= \bigsqcup_{\substack{i_0 + j_0 = n_0, \\ \vdots \\ i_s + j_s = n_s}} S^{[i_0]} \times S'^{[j_0]} \times \dots \times S^{[i_s]} \times S'^{[j_s]} \\ (7.4) \quad &= \bigsqcup_{i+j=n} \text{Hilb}^i(S) \times \text{Hilb}^j(S'), \end{aligned}$$

in which the last sum is taken over $(s+1)$ -tuples $i = (i_0, \dots, i_s)$ and $j = (j_0, \dots, j_s)$ of nonnegative integers with $n = i + j$. Consider the universal ideal sheaves

$$\mathcal{I}_{S \sqcup S'}^{[n_k]} \quad \text{for } k = 0, \dots, s$$

on

$$\text{Hilb}^n(S \sqcup S') \times (S \sqcup S').$$

For fixed i and j with $i + j = n$ and for $k = 0, \dots, s$, we will write

$$p_k: \text{Hilb}^i(S) \times \text{Hilb}^j(S') \times S \rightarrow S^{[i_k]} \times S,$$

$$q_k: \text{Hilb}^i(S) \times \text{Hilb}^j(S') \times S' \rightarrow S'^{[j_k]} \times S'$$

for the projections. Over the components in the decomposition (7.4), the universal sheaves are given by

$$\mathcal{I}_{S \sqcup S'}^{[n_k]}|_{\text{Hilb}^i(S) \times \text{Hilb}^j(S') \times (S \sqcup S')} = p_k^* \mathcal{I}_S^{[i_k]} \oplus q_k^* \mathcal{I}_{S'}^{[j_k]}.$$

Write

$$\pi: S \rightarrow *, \quad \pi': S' \rightarrow * \quad \text{and} \quad \pi \sqcup \pi': S \sqcup S' \rightarrow *$$

for the projections. Let M and M' be line bundles on S and S' respectively. It follows that for $0 \leq k, l \leq s$ we have

$$\begin{aligned} (7.5) \quad R\mathcal{H}om_{\pi \sqcup \pi'}(\mathcal{I}_{S \sqcup S'}^{[n_k]}, \mathcal{I}_{S \sqcup S'}^{[n_l]} \otimes (M \oplus M')) \\ = \sum_{i+j=n} R\mathcal{H}om_{\pi}(p_k^* \mathcal{I}_S^{[i_k]}, p_l^* \mathcal{I}_S^{[i_l]} \otimes M) \oplus R\mathcal{H}om_{\pi'}(q_k^* \mathcal{I}_{S'}^{[j_k]}, q_l^* \mathcal{I}_{S'}^{[j_l]} \otimes M') \end{aligned}$$

in the ring

$$K_0(\text{Hilb}^n(S \sqcup S')) = \bigoplus_{i+j=n} K_0(\text{Hilb}^i(S) \times \text{Hilb}^j(S')).$$

For any pair i and j of $(s+1)$ -tuples of nonnegative integers, write

$$p: \text{Hilb}^i(S) \times \text{Hilb}^j(S') \rightarrow \text{Hilb}^i(S) \quad \text{and} \quad q: \text{Hilb}^i(S) \times \text{Hilb}^j(S') \rightarrow \text{Hilb}^j(S').$$

Let L and L' be $(s+1)$ -tuples of line bundles on S and S' , respectively, such that $\beta = \beta(L)$ and $\beta' = \beta(L')$ (see Notation 2.1). Consider K -theory classes

$$N_{L \oplus L', 0}^{[n]}, \quad N_{L, 0}^{[i]} \quad \text{and} \quad N_{L', 0}^{[j]}$$

as defined in Section 5. (Note that these classes may depend on the choice of L and L' , but will only depend on β and β' after passing to numerical K -theory. See also

Remark 7.1.) By definition, $N_{L+L',0}^{[n]}$ is a linear combination of classes of the form (7.5), and we find

$$N_{L+L',0}^{[n]} = \sum_{i+j=n} p^* N_{L,0}^{[i]} + q^* N_{L',0}^{[j]}.$$

It follows that

$$\frac{1}{e(N_{L+L',0}^{[n]})} = \sum_{i+j=n} \frac{1}{e(N_{L,0}^{[i]})} \cdot \frac{1}{e(N_{L',0}^{[j]})}.$$

Finally, the corresponding multiplicative property of the factor

$$\prod_{i=k}^s e(R\pi_* \mathcal{O}(\beta_k) - R\mathcal{H}om_{\pi}(\mathcal{I}^{[n_{k-1}]}, \mathcal{I}^{[n_k]}(\beta_k)))$$

in the integrand of $Q_n(S, \beta)$ follows from the generalized Carlsson–Okounkov vanishing of [6] (see Remark 1.6). Integrating gives the result. □

The proof of Proposition 7.3 also gives the following refined result.

Proposition 7.6 *For each symbol*

$$\mathfrak{N} \in \{\chi(\mathcal{O}_S), \underline{K}_S^2, \underline{K}_S \beta^i, \beta^i \beta^j\}_{1 \leq i \leq j \leq s}$$

there is a power series $A_{\mathfrak{N}}^{(s+1)}(y) \in \mathbb{Q}(y)[[q_0, \dots, q_s]]$, starting with 1, such that

$$\sum_n Q_n(S, \beta, y) q^n = \prod_{\mathfrak{N}} (A_{\mathfrak{N}}^{(s+1)}(y))^{\mathfrak{N}}$$

for any smooth projective surface S and classes $\beta_1, \dots, \beta_s \in A^1(S)$.

Proof A similar proof holds, using the multiplicative properties of ch , Λ^\bullet , \det and Td . Note that, by Remark 5.3, the universal series take coefficients in $\mathbb{Q}(y)$, rather than in $\mathbb{Q}(\sqrt{y})$. □

8 Proof of Theorem A

As usual, we fix a rank $r = s + 1$. In this section, we will make the identifications

$$q := q_0 = \dots = q_s,$$

so the equation in Proposition 7.6 becomes

$$(8.1) \quad \sum_n \mathcal{Q}_n(S, \beta, y) q^{|n|} = \prod_{\mathfrak{N}} (A_{\mathfrak{N}}^{(r)}(y))^{\mathfrak{N}}$$

in the ring $\mathbb{Q}(y)[[q]]$, where we use the notation $|n| = n_0 + \dots + n_s$.

Let $L = (L_0, \dots, L_s)$ be line bundles on a surface S , and let

$$\beta = \beta(L) = (\beta_1, \dots, \beta_s) = (K_S - \beta^1, \dots, K_S - \beta^s)$$

be given as in Notation 2.1. For nonnegative integers $n = (n_0, \dots, n_s)$ and ideal sheaves $I_i \in S^{[n_i]}$, consider the sheaf

$$E = L_0 \otimes I_0 \oplus \dots \oplus L_s \otimes I_s.$$

In the notation of Section 2, E is a fibre of the family $E_{\mathcal{L}}^{[n]}$ of sheaves on S over $\text{Hilb}_{\beta}^n(S)$. By Lemma 2.6 we have

$$c_2(E) = |n| + \frac{r-1}{2r} c_1(E)^2 - \sum_{i < j} \frac{i(r-j)}{r} \beta^i \beta^j - \sum_i \frac{i(r-i)}{2r} (\beta^i)^2.$$

We will write

$$d(\beta) := - \sum_{i < j} \frac{i(r-j)}{r} \beta^i \beta^j - \sum_i \frac{i(r-i)}{2r} (\beta^i)^2,$$

so that we have

$$(8.2) \quad q^{(1-r)/(2r)c_1(E)^2} q^{c_2(E)} = q^{|n|+d(\beta)}.$$

Finally, recall that for any surface S with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$ and any β with

$$\text{SW}(\beta) = \text{SW}(\beta_1) \cdots \text{SW}(\beta_s) \neq 0,$$

we have, by Proposition 6.5,

$$(8.3) \quad F(S, \beta, y) = (F_0^{(r)}(y))^{\chi(\mathcal{O}_S)} \prod_{k \leq l} (F_{kl}^{(r)}(y))^{\beta^k \beta^l},$$

where $F(S, \beta, y)$ is defined as in Section 5.

Define the following Laurent series in $q^{1/(2r)}$ with coefficients in $\mathbb{Q}(\sqrt{y})$:

$$\begin{aligned}
 A &:= F_0^{(r)}(y) A_{\underline{\chi(\mathcal{O}_S)}}^{(r)}(y) (-1)^{(r-1)}, \\
 B &:= A_{\underline{K_S^2}}^{(r)}(y), \\
 C_{ij} &:= q^{i(j-r)/r} F_{ij}^{(r)}(y) A_{\underline{\beta^i \beta^j}}^{(r)}(y) \quad \text{for } 1 \leq i < j \leq r-1, \\
 C_{ii} &:= q^{i(i-r)/(2r)} F_{ii}^{(r)}(y) A_{\underline{\beta^i \beta^i}}^{(r)}(y) A_{\underline{\beta^i K_S}}^{(r)}(y) \quad \text{for } 1 \leq j \leq r-1.
 \end{aligned}$$

Proof of Theorem A First note that, by definition, the Laurent series are universal in the sense that they *only* depend on r . Now let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$. We have

$$\begin{aligned}
 \mathbb{Z}_{S,r,c_1}(q, y) &= \frac{q^{(1-r)/(2r)c_1^2}}{\#H^2(S, \mathbb{Z})[r]} \sum_{c_2 \in \mathbb{Z}} \text{VW}_{1r, c_1, c_2}(S, y) q^{c_2} \\
 &= q^{(1-r)/(2r)c_1^2} \sum_{c_2 \in \mathbb{Z}} \sum_{\beta, n} \widehat{\text{VW}}_{\beta}^{[n]}(y) q^{c_2} && \text{by (4.3)} \\
 &= \sum_{\beta} \sum_{n \in (\mathbb{Z}_{\geq 0})^r} \text{VW}_{\beta}^{[n]}(y) q^{|n|+d(\beta)} && \text{by (8.2)} \\
 &= \sum_{\beta} \text{SW}(\beta) F(S, \beta, y) \sum_{n \in (\mathbb{Z}_{\geq 0})^r} Q_n(S, \beta, y) q^{|n|+d(\beta)} && \text{by (5.2)} \\
 &= \sum_{\beta} \text{SW}(\beta^\vee) A^{\chi(\mathcal{O}_S)} B_{K_S^2} \prod_{i \leq j} C_{ij}^{\beta^i \beta^j} && \text{by (8.1) and (8.3)}.
 \end{aligned}$$

Here the symbol \sum_{β} denotes a sum over $(r-1)$ -tuples $\beta = (\beta_1, \dots, \beta_{r-1})$ satisfying

$$c_1 \equiv \sum_{i=1}^{r-1} i \beta^i \pmod{rH^2(S, \mathbb{Z})}.$$

The sum $\widehat{\sum}_{\beta, n}$ is taken over β as above and $n \in (\mathbb{Z}_{\geq 0})^r$ with

$$c_2 = |n| + \frac{r-1}{2r} c_1^2 + d(\beta);$$

see Proposition 3.7. Finally, we have used the notation $\beta^\vee = (\beta^1, \dots, \beta^{r-1})$ and the equation

$$\text{SW}(\beta^\vee) = \text{SW}(\beta^1) \cdots \text{SW}(\beta^{r-1}) = (-1)^{(r-1)\chi(\mathcal{O}_S)} \text{SW}(\beta),$$

which follows from [18, Proposition 6.3.4]. □

Proposition 8.4 *Let S be surface with $H^1(\mathcal{O}_S) = 0$ and $p_g(S) > 0$, and let r, c_1 and c_2 be Chern classes such that semistability implies stability. If r is odd, we have*

$$\text{VW}_{1^r, c_1, c_2}(S, y) \in \mathbb{Q}(y) \subset \mathbb{Q}(\sqrt{y}).$$

Proof By Proposition 7.6 and Remark 6.6, the Laurent series A, B and C_{ij} have coefficients in $\mathbb{Q}(y)$. □

9 Toric computations

We will see how to compute the coefficients of the series

$$A_{\underline{\mathfrak{N}}}^{(s+1)} \quad \text{for } \underline{\mathfrak{N}} \in \mathcal{N} := \{\chi(\mathcal{O}_S), \underline{K}_S^2, \underline{K}_S \beta^i, \beta^i \beta^j\}_{1 \leq i \leq j \leq s},$$

up to some degree N . In fact (see [9] and as we will explain in this section), it suffices to evaluate the integrals

$$(9.1) \quad Q_n(S, \beta) = \int_{[\text{Hilb}^n(S)]} \frac{\prod_{i=1}^s e(R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)))}{e(N_{L,0}^{[n]})}$$

for $|n| \leq N$ on \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ and sufficiently many different β (see Section 5 for notation and definitions).

Let $\omega_{2,1^s}$ denote the \mathbb{Q} -vector space of cobordism classes of surfaces with s -tuples of line bundles, as defined in [16], and let B be a basis. For rank $s + 1 = 2$, we could take

$$B = ([\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}], [\mathbb{P}^2, \mathcal{O}], [\mathbb{P}^2, \omega_S^{\pm 1}]),$$

and for rank $s + 1 = 3$,

$$B = ([\mathbb{P}^1 \times \mathbb{P}^1, (\mathcal{O}, \mathcal{O})], [\mathbb{P}^2, (\mathcal{O}, \mathcal{O})], [\mathbb{P}^2, (\omega_S^{\pm 1}, \mathcal{O})], [\mathbb{P}^2, (\mathcal{O}, \omega_S^{\pm 1})], [\mathbb{P}^2, (\omega_S, \omega_S)]).$$

We can view the symbols $\underline{\mathfrak{N}}$ as coordinate functions on $\omega_{2,1^s}$. For a surface S , with an s -tuple $\beta \in (H^2(S, \mathbb{Z}))^s$, we write

$$\underline{\mathfrak{N}}(S, \beta) = \mathfrak{N},$$

so we have

$$\underline{\chi(\mathcal{O}_S)}(S, \beta) = \chi(\mathcal{O}_S), \quad \underline{K_S^2}(S, \beta) = K_S^2, \quad \underline{K_S \beta^i}(S, \beta) = K_S \beta^i, \quad \dots$$

Then the fact that B is a basis can be expressed by the fact that the matrix

$$M^{(s+1)} := [\underline{\mathfrak{N}}(S, \beta)]_{[S, \beta] \in B, \underline{\mathfrak{N}} \in \mathcal{N}}$$

is invertible. For the bases for ranks 2 and 3 given above, we have, respectively,

$$M^{(2)} = \begin{pmatrix} 1 & 8 & 0 & 0 \\ 1 & 9 & 0 & 0 \\ 1 & 9 & 9 & 9 \\ 1 & 9 & -9 & 9 \end{pmatrix} \quad \text{and} \quad M^{(3)} = \begin{pmatrix} 1 & 8 & 0 & 0 & 0 & 0 & 0 \\ 1 & 9 & 0 & 0 & 0 & 0 & 0 \\ 1 & 9 & 9 & 9 & 0 & 0 & 0 \\ 1 & 9 & -9 & 9 & 0 & 0 & 0 \\ 1 & 9 & 0 & 0 & 9 & 9 & 0 \\ 1 & 9 & 0 & 0 & -9 & 9 & 0 \\ 1 & 9 & 9 & 9 & 9 & 9 & 9 \end{pmatrix}.$$

Recall that, by Proposition 7.3, we have

$$\sum_n Q_n(S, \beta) q^n = \prod_{\mathfrak{N}} (A_{\mathfrak{N}}^{(s+1)})^{\mathfrak{N}}$$

for any surface S and curve classes $\beta = (\beta_1, \dots, \beta_s)$. Taking the natural logarithm, we obtain

$$\log \sum_n Q_n(S, \beta) q^n = \sum_{\mathfrak{N}} \mathfrak{N} \log A_{\mathfrak{N}}^{(s+1)}.$$

By definition of M , we have

$$\left[\log \sum_n Q_n(S, \beta) q^n \right]_{[S, \beta] \in B} = M \cdot [\log A_{\mathfrak{N}}^{(s+1)}]_{\mathfrak{N} \in \mathcal{N}}.$$

Now assume we want to compute the power series $A_{\mathfrak{N}}^{(s+1)}$ up to order N . Since M is invertible, it suffices evaluate the integrals $Q_n(S, \beta)$ for all $n \in (\mathbb{Z}_{\geq 0})^{s+1}$ with $|n| \leq N$. Note that, by Proposition 7.6, the discussion above also applies to the refined case.

Let S be any toric surface with a torus T , and assume that we have equipped all line bundles appearing in the integral (9.1) with an equivariant structure. Then, by applying the Atiyah–Bott localization formula, we obtain

$$\begin{aligned} Q_n(S, \beta) &= \sum_{F \in (\text{Hilb}^n(S))^T} \int \frac{\prod_{i=1}^s e((R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_{\pi}(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)))|_F)}{e(N_{L,0}^{[n]}|_F) e(T_{\text{Hilb}^n(S), F})} \\ (9.2) \quad &= \sum_{F \in (\text{Hilb}^n(S))^T} \int e(-T_{L,0}^{[n]}|_F), \end{aligned}$$

in which $e(\cdot)$ denotes the equivariant Euler class for the torus $T \times \mathbb{C}^*$.

Remark 9.3 In the factor

$$e((R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_{\pi}(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)))|_F)$$

in the formula above, the Euler class $e(\cdot)$ should a priori be the T –equivariant Chern class $c_{n_{i-1}+n_i}^T(\cdot)$, but by [1, Lemma 6] and Lemma 9.5 below, the K –theory class

$$(R\pi_*\mathcal{O}(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)))|_F \in K_0^T(F)$$

can be represented by an honest $(n_{i-1} + n_i)$ –dimensional representation of the torus T . It follows that the T –equivariant top Chern class agrees with T –equivariant Euler class.

Remark 9.4 The compact form of the expression (9.2) is due to the fact that it is obtained by applying the Atiyah–Bott localization formula twice. The virtual version of the formula, due to Graber and Pandharipande [14], expresses by definition the contributions (4.1) of nested Hilbert schemes $S_\beta^{[n]}$ to the monopole branch of the Vafa–Witten invariant for a surface S with $p_g(S) > 0$. The second time, however, we applied the formula to Hilbert schemes of points on a toric surface.

Similarly, we have

$$\begin{aligned} Q_n(S, \beta, y) &= \sum_{F \in (\text{Hilb}^n(S))^T} \int \frac{\text{ch}((K_{L,0}^{[n]})^{\frac{1}{2}}|_F)}{\text{ch}(\Lambda^\bullet(N_{L,0}^{[n]\vee}|_F))} \text{Td}((T_{L,0}^{[n]})^{\mathbb{C}^*}|_F) \\ &\quad \times \frac{\prod_{i=1}^s e((R\pi_*\mathcal{O}(\beta_i) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)))|_F)}{e(T_{\text{Hilb}^n(S),F})} \\ &= \sum_{F \in (\text{Hilb}^n(S))^T} \int \frac{\text{ch}((K_{L,0}^{[n]})^{\frac{1}{2}}|_F)}{\text{ch}(\Lambda^\bullet(T_{L,0}^{[n]\vee}|_F))}, \end{aligned}$$

where ch and Td denote the $(T \times \mathbb{C}^*)$ –equivariant Chern character and Todd class respectively. By convention, we denote the Chern character of \mathfrak{t} by $y = \text{ch}(\mathfrak{t})$. Finally, note that the second equation follows from the identity

$$\text{ch}(\Lambda^\bullet(L^*)) = 1 - \exp(-\alpha) = \frac{e(L)}{\text{Td}(L)}$$

for any 1–dimensional T –representation L with $c_1(L) = \alpha$.

Let $F \in \text{Hilb}^n(S)$ be a T –fixed point. Let $0 \leq i, j \leq s$, and write

$$I = \mathcal{I}_F^{[n_i]} \quad \text{and} \quad J = \mathcal{I}_F^{[n_j]}.$$

The class $T_{L,0}^{[n]}|_F$ is a linear combination of classes of the form

$$(R\pi_*M - R\mathcal{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_j]} \otimes M))|_F = \chi(M) - R\text{Hom}_S(I, J \otimes M),$$

where M is a $(T \times \mathbb{C}^*)$ –equivariant line bundle on S .

Lemma 9.5 *Let $\{U_\sigma\}_{\sigma=1,\dots,e(S)}$ be the maximal open cover of S by affine T -fixed subsets; see [11, Section 4]. Then we have*

$$(9.6) \quad \chi(M) - R\mathrm{Hom}_S(I, J \otimes M) = \sum_{\sigma=1}^{e(S)} \Gamma(U_\sigma, M) - R\mathrm{Hom}_{U_\sigma}(I|_{U_\sigma}, J|_{U_\sigma} \otimes M|_{U_\sigma}).$$

Proof Write $U_{\sigma\tau} = U_\sigma \cap U_\tau$ for $\sigma < \tau$. Since I and J are ideal sheaves of \mathbb{C}^* -fixed 0-dimensional subschemes of S , and $U_{\sigma\tau}$ does not contain any fixed points, we have

$$\begin{aligned} \Gamma(U_{\sigma\tau}, \mathcal{E}xt^i(I, J \otimes M)) &= \Gamma(U_\sigma \cap U_\tau, \mathcal{E}xt^i(I|_{U_{\sigma\tau}}, (J \otimes M)|_{U_{\sigma\tau}})) \\ &= \Gamma(U_{\sigma\tau}, \mathcal{E}xt^i(\mathcal{O}, M)) \end{aligned}$$

for any i and a similar identity for intersections $U_\sigma \cap U_\tau \cap U_\nu$. Now use the local-to-global spectral sequence and the Čech complex for the covering $\{U_\alpha\}$ (see [17, Section 4.6]), to compare the classes $\chi(M)$ and $R\mathrm{Hom}_S(I, J \otimes M)$. □

Now [1, Lemma 6] gives — as does the proof of [11, Proposition 4.1] — an explicit expression for the right-hand side of (9.6). This allows us to compute the integrals $Q_n(S, \beta)$ and $Q_n(S, \beta, y)$. We have implemented the computation in Sage [19] for $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $S = \mathbb{P}^2$ and for any β , n and rank. Part of the results for rank 3 are listed in Appendix B.

10 The universal series A

Fix a K3 surface S and nonnegative integers $n = (n_0, \dots, n_s)$. Consider the inclusion

$$i: S^{[n]} \hookrightarrow \mathrm{Hilb}^n(S) = S^{[n_0]} \times \dots \times S^{[n_s]}$$

of the nested Hilbert scheme. In the case that $n = (k, \dots, k)$ for some nonnegative integer k , we write

$$\Delta_{S^{[k]} \times \dots \times S^{[k]}} \cong S^{[k]} \subset S^{[k]} \times \dots \times S^{[k]}$$

for the diagonal.

Lemma 10.1 *For $n = (k, \dots, k)$,*

$$i_*[S^{[n]}]^{\mathrm{vir}} = [\Delta_{S^{[k]} \times \dots \times S^{[k]}}].$$

Otherwise,

$$i_*[S^{[n]}]^{\mathrm{vir}} = 0.$$

Proof For the case $n = (k, \dots, k)$, see [5, Theorem 2] (note that in this case we have

$$S^{[n]} = S^{[k, \dots, k]} \cong \Delta_{S^{[k]} \times \dots \times S^{[k]}} \cong S^{[k]},$$

and the perfect obstruction theory is just the cotangent bundle). Otherwise, note that, by Theorem 1.5 and Serre duality,

$$\begin{aligned} i_*[S^{[n_0, \dots, n_s]}]^{vir} &= \prod_i e(\chi(\mathcal{O}_S) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]})) \cap [\text{Hilb}^n(S)] \\ &= (-1)^{n_0 + n_s} \prod_i e(\chi(\mathcal{O}_S) - R\mathcal{H}om_\pi(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_{i-1}]}) \cap [\text{Hilb}^n(S)] \\ &= (-1)^{n_0 + n_s} j_*[S^{[n_s, \dots, n_0]}]^{vir}, \end{aligned}$$

where j is the inclusion

$$j: S^{[n_s, \dots, n_0]} \hookrightarrow \text{Hilb}^n(S).$$

Now note that $S^{[n_0, \dots, n_s]}$ or $S^{[n_s, \dots, n_0]}$ is empty unless $n_0 = \dots = n_s$. □

Set $\beta = (0, \dots, 0)$, so that $\beta^i = K_S - \beta_i = 0$ for $i = 1, \dots, s$. By Proposition 7.3,

$$\sum_n Q_n(S, \beta) q^n = (A_{\chi(\mathcal{O}_S)}^{(s+1)})^2.$$

Recall that $Q_n(S, \beta)$ is by definition integral over the virtual class of $S_\beta^{[n]} = S^{[n]}$. Hence, by Lemma 10.1, we have $Q_n(S, \beta) = 0$ or $n_0 = \dots = n_s$. Assume that we have $n = (k, \dots, k)$ for a nonnegative integer k . We will compute $Q_n(S, \beta)$.

We let $L = (\mathcal{O}_S, \dots, \mathcal{O}_S)$ be the $(s+1)$ -tuple of copies of the trivial line on S , so that $\beta = \beta(L)$. We write

$$E^{[n]} := E_L^{[n]} = \text{pr}_0^* \mathcal{I}^{[k]} \otimes \mathfrak{t}^0 \oplus \dots \oplus \text{pr}_s^* \mathcal{I}^{[k]} \otimes \mathfrak{t}^{-s}$$

for the sheaf on $\text{Hilb}^n(S) \times S$, where pr_i denotes the i^{th} projection

$$\text{pr}_i: \text{Hilb}^n(S) = S^{[k]} \times \dots \times S^{[k]} \rightarrow S^{[k]}$$

and its base change to S . We will write $\Delta: S^{[k]} \rightarrow S^{[k]} \times \dots \times S^{[k]}$ for the diagonal embedding (which of course can be identified with the embedding i) and denote its base change to S by the same symbol. We have an isomorphism

$$\Delta^* E^{[n]} \cong \mathcal{I}^{[k]} \otimes \mathfrak{t}^0 \oplus \dots \oplus \mathcal{I}^{[k]} \otimes \mathfrak{t}^{-s}$$

of sheaves on $S^{[k]} \times S$. Write

$$E = (\mathfrak{t}^0 \oplus \dots \oplus \mathfrak{t}^{-s}) \otimes \mathcal{O}_S$$

for the vector bundle on S . Using the notation of Section 5, we have the following equality in the \mathbb{C}^* -equivariant K-group of $S^{[k]}$:

$$\begin{aligned} \Delta^* T_{L,0}^{[n]} &= R\mathcal{H}om_{\pi}(\Delta^* E^{[n]}, \Delta^* E^{[n]} \otimes \mathfrak{t})_0 - R\mathcal{H}om_{\pi}(\Delta^* E^{[n]}, \Delta^* E^{[n]})_0 \\ &\quad - (R\mathrm{Hom}(E, E \otimes \mathfrak{t})_0 - R\mathrm{Hom}(E, E)_0) \\ &= \sum_{i=1}^{s+1} R\mathcal{H}om_{\pi}(\mathcal{I}^{[k]}, \mathcal{I}^{[k]})_0 \otimes \mathfrak{t}^i + \sum_{i=0}^s (-R\mathcal{H}om_{\pi}(\mathcal{I}^{[k]}, \mathcal{I}^{[k]})_0 \otimes \mathfrak{t}^{-i}) \\ (10.2) \quad &= T_{S^{[k]}} + \sum_{i=1}^s (T_{S^{[k]}} \otimes \mathfrak{t}^{-i} - T_{S^{[k]}} \otimes \mathfrak{t}^i) - T_{S^{[k]}} \otimes \mathfrak{t}^{s+1}. \end{aligned}$$

Note that $T_{S^{[k]}}$ has even rank, so we have

$$e(T_{S^{[k]}} \otimes \mathfrak{t}^{-i}) = e(T_{S^{[k]}} \otimes \mathfrak{t}^i),$$

and hence

$$e\left(\sum_{i=1}^s (T_{S^{[k]}} \otimes \mathfrak{t}^{-i} - T_{S^{[k]}} \otimes \mathfrak{t}^i)\right) = \prod_{i=1}^s \frac{e(T_{S^{[k]}} \otimes \mathfrak{t}^{-i})}{e(T_{S^{[k]}} \otimes \mathfrak{t}^i)} = 1.$$

It follows that we have

$$\begin{aligned} Q_n(S, \beta) &= \int_{[S_{\beta}^{[n]}]_{\mathrm{vir}}} \frac{1}{e(N_{L,0}^{[n]})} \\ &= \int_{S^{[k]}} \frac{1}{e(\Delta^*(T_{L,0}^{[n]})^{\mathrm{mov}})} \\ &= \int_{S^{[k]}} e(T_{S^{[k]}} \otimes \mathfrak{t}^{s+1}) \\ &= \int_{S^{[k]}} e(T_{S^{[k]}}) \\ &= e(S^{[k]}). \end{aligned}$$

By Göttsche’s formula [8], we have

$$\begin{aligned} (A_{\chi(\mathcal{O}_S)}^{(s+1)})^2 &= \sum_{n \in (\mathbb{Z}_{\geq 0})^{s+1}} Q_n(S, \beta) q^{|n|} \\ &= \sum_{k \in \mathbb{Z}_{\geq 0}} e(S^{[k]}) q^{k(s+1)} \\ &= \left(\prod_{k \geq 1} \frac{1}{1 - q^{k(s+1)}}\right)^{24}. \end{aligned}$$

It follows that

$$F_0^{(s+1)} A_{\chi(\mathcal{O}_S)}^{(s+1)} (-1)^{s+1} = \frac{1}{s+1} \left(\prod_{k \geq 1} \frac{1}{1 - q^{k(s+1)}} \right)^{12},$$

which is the specialization of the right-hand side of (1.19) at $y = 1$, as expected.

Now consider the integral

$$Q_n(S, \beta, y) = \left[\int_{[S_\beta^{[n]}]_{\text{vir}}} \frac{\text{ch}((K_{L,0}^{[n]})^{\frac{1}{2}})}{\text{ch}(\Lambda^\bullet((N_{L,0}^{[n]})^\vee))} \text{Td}((T_{L,0}^{[n]})^{\mathbb{C}^*}) \right]_{\text{ch}(t)=y}.$$

Since $T_{S^{[k]}}$ is self-dual and has even rank, we have by [21, Equation 2.28] the following equation in the \mathbb{C}^* -equivariant K -group on $S^{[k]}$:

$$\frac{(\det(T_{S^{[k]}} \otimes t^{-i} - T_{S^{[k]}} \otimes t^i)^\vee)^{\frac{1}{2}}}{\Lambda^\bullet((T_{S^{[k]}} \otimes t^{-i} - T_{S^{[k]}} \otimes t^i)^\vee)} = \frac{\det(T_{S^{[k]}}^* \otimes t^i) \otimes \Lambda^\bullet(T_{S^{[k]}}^* \otimes t^{-i})}{\Lambda^\bullet(T_{S^{[k]}}^* \otimes t^i)} = 1.$$

It follows that, as above, the middle terms of (10.2) do not contribute to $Q_n(S, \beta, y)$.

Remark 10.3 Taking the square root involves a choice. First of all, our choice here is consistent with the one we made before. More importantly, after choosing a root \sqrt{t} , the choice is unique up to 2-torsion in $\text{Pic}(S_\beta^{[n]})$, which is killed by ch .

Since $T_{S^{[k]}}$ is self-dual, so is $K_{S^{[k]}}$, and hence $\text{ch}(K_{S^{[k]}}) = 1$. Writing $r = s + 1$, we find

$$\begin{aligned} Q_n(S, \beta, y) &= \left[\int_{S^{[k]}} \frac{\text{ch}((\det(T_{S^{[k]}} - T_{S^{[k]}} \otimes t^r)^\vee)^{\frac{1}{2}})}{\text{ch}(\Lambda^\bullet(-T_{S^{[k]}} \otimes t^r)^\vee)} \text{Td}(T_{S^{[k]}}) \right]_{\text{ch}(t)=y} \\ &= \left[\int_{S^{[k]}} \text{ch}(K_{S^{[k]}} \otimes t^{rk}) \text{ch}(\Lambda^\bullet(T_{S^{[k]}} \otimes t^{-r})) \text{Td}(T_{S^{[k]}}) \right]_{\text{ch}(t)=y} \\ &= y^{rk} \sum_{i=0}^{2k} (-1)^i y^{-ri} \int_{S^{[k]}} \text{ch}(\Lambda^i T_{S^{[k]}}) \text{Td}(T_{S^{[k]}}) \\ &= y^{rk} \sum_{i=0}^{2k} (-1)^i y^{-ri} \chi(\Lambda^i T_{S^{[k]}}) \\ &=: \chi_{-y^r}(S^{[k]}). \end{aligned}$$

The generating series of χ_y -genera of the Hilbert schemes $S^{[k]}$ has been computed in [13]. It follows that we have

$$\begin{aligned} (A_{\underline{\chi}(\mathcal{O}_S)}^{(r)}(y))^2 &= \sum_{n \in (\mathbb{Z}_{\geq 0})^r} Q_n(S, \beta, y) q^{|n|} = \sum_{k \in \mathbb{Z}_{\geq 0}} \chi_{-y^r}(S^{[k]}) q^{rk} \\ &= \prod_{k \geq 1} \frac{1}{(1 - q^{rk})^{20} (1 - y^{-r} q^{rk})^2 (1 - y^r q^{rk})^2}. \end{aligned}$$

We conclude

$$\begin{aligned} A^{(r)}(y) &= F_0^{(r)}(y) A_{\underline{\chi}(\mathcal{O}_S)}^{(r)}(y) (-1)^r = \frac{1}{[r]_y} \prod_{k \geq 1} \frac{1}{(1 - q^{rk})^{10} (1 - y^{-r} q^{rk}) (1 - y^r q^{rk})} \\ &= \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^r, y^r)^{\frac{1}{2}} \tilde{\Delta}(q^r)^{\frac{1}{2}}}, \end{aligned}$$

proving Theorem C.

11 Smooth components

In the case that the monopole branch of the moduli space of \mathbb{C}^* -fixed Higgs pairs is smooth, there is a direct method to compute the Vafa–Witten invariants. Let S be a surface with $H^1(\mathcal{O}_S) = 0$ and $p_g > 0$, and assume that $\text{Pic}(S)$ is generated by a smooth very ample canonical curve C . In this case, the only Seiberg–Witten basic classes of S are 0 and K_S . For rank 2, the monopole branch

$$\mathcal{M}_{1^2} = \mathcal{M}_{1^2, K_S, c_2} \subset (\mathcal{N}_{2, \omega_S, c_2}^\perp)^{\mathbb{C}^*}$$

is smooth precisely when $c_2 = 0, 1, 2, 3$; see [20]. In particular, the virtual class is given by the Euler class of the obstruction bundle and the Vafa–Witten invariants can be computed using the intersection theory of (smooth) nested Hilbert schemes of points on the surface and the smooth canonical curve. This method, which is carried out in [20] (unrefined) and [21] (refined), can be generalized to rank 3, but only for $c_2 = 0, 1, 2$. We have done the computations in this setting and have found that they confirm our results (see the discussion after Theorem B’).

Let $(E, \phi) \in \mathcal{M}_{1^3, K_S, c_2}$ be a Higgs pair. Then E can be written as

$$E = I_0 \otimes \omega_S^a \oplus I_1 \otimes \omega_S^b \oplus I_2 \otimes \omega_S^c,$$

where $I_i \in S^{[n_i]}$ for $i = 0, 1, 2$ and $a, b, c \in \mathbb{Z}$ such that $a + b + c = 1$. Moreover, we have $\phi = (\phi_1, \phi_2)$ for nonzero homomorphisms

$$\phi_1: I_0 \otimes \omega_S^a \rightarrow I_1 \otimes \omega_S^{b+1} \quad \text{and} \quad \phi_2: I_1 \otimes \omega_S^b \rightarrow I_2 \otimes \omega_S^{c+1}.$$

Lemma 11.1 $(a, b, c) = (1, 0, 0).$

Proof Slope semistability of E implies that

$$c \leq \frac{1}{3} \quad \text{and} \quad \frac{1}{2}(b + c) \leq \frac{1}{3}.$$

On the other hand, by the existence of the maps ϕ_1 and ϕ_2 , we have

$$a \leq b + 1 \quad \text{and} \quad b \leq c + 1.$$

It is easy to see that the only integral solution to these inequalities together with $a + b + c = 1$ is $(a, b, c) = (1, 0, 0)$. □

Proposition 11.2 *Let S be given as above. Then $\mathcal{M}_{1^3, K_S, c_2}$ is smooth if and only if $c_2 \leq 2$. In particular, we have*

$$\begin{aligned} \mathcal{M}_{1^3, K_S, 1} &\cong (S^{[1]} \times |K_S|) \sqcup \mathcal{C}, \\ \mathcal{M}_{1^3, K_S, 2} &\cong (S^{[2]} \times |K_S|) \sqcup (S^{[1]} \times |K_S|) \sqcup (S^{[1]} \times \mathcal{C}) \sqcup \mathbb{C}_{|K_S|}^{[2]}, \end{aligned}$$

in which

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & S \\ \downarrow & & \\ |K_S| & & \end{array}$$

is the universal canonical curve and $\mathbb{C}_{|K_S|}^{[2]} \rightarrow |K_S|$ the relative Hilbert scheme of pairs of points.

Proof Note that for $I_i \in S^{[n_i]}$ for $i = 0, 1, 2$, we have

$$c_2(I_0 \otimes \omega_S \oplus I_1 \oplus I_2) = n_0 + n_1 + n_2.$$

By Lemma 11.1, we find

$$\mathcal{M}_{1^3, K_S, c_2} \cong \bigsqcup_{|n|=c_2, n_0 \geq n_1} S_{(0, K_S)}^{[n_0, n_1, n_2]},$$

where we have used that $S_{(0, K_S)}^{[n_0, n_1, n_2]}$ is empty whenever $n_0 < n_1$. In particular, we have

$$\mathcal{M}_{1^3, K_S, 1} = S_{(0, K_S)}^{[1, 0, 0]} \sqcup S_{(0, K_S)}^{[0, 0, 1]} \cong (S^{[1]} \times |K_S|) \sqcup \mathcal{C},$$

$$\begin{aligned} \mathcal{M}_{1^3, K_S, 2} &= S_{(0, K_S)}^{[2, 0, 0]} \sqcup S_{(0, K_S)}^{[1, 1, 0]} \sqcup S_{(0, K_S)}^{[1, 0, 1]} \sqcup S_{(0, K_S)}^{[0, 0, 2]} \\ &\cong (S^{[2]} \times |K_S|) \sqcup (S^{[1]} \times |K_S|) \sqcup (S^{[1]} \times \mathcal{C}) \sqcup \mathbb{C}_{|K_S|}^{[2]}. \end{aligned}$$

The total spaces of the universal canonical curve \mathcal{C} and the relative Hilbert scheme of points $\mathbb{C}_{|K_S|}^{[2]}$ are smooth by the assumption that K_S is very ample.

The component

$$S_{(0, K_S)}^{[1, 1, 1]} \cong (S \times \mathcal{C}) \cup (\Delta_S \times |K_S|) \xrightarrow{i} S \times S \times |K_S|$$

of $\mathcal{M}_{1^3, K_S, 3}$ has two irreducible components with nonempty intersection. More generally, let $c_2 \geq 3$. For an ideal sheaf I on S , let Z_I denote the corresponding subscheme. Then the component

$$S_{0, K_S}^{[1, 1, c_2 - 2]} \cong S_{K_S}^{[1, c_2 - 2]} = \{p \in S, I \in S^{[c_2 - 2]}, C \in |K_S| : Z_I \subset C \cup p\}$$

has two components given by the conditions $p \in Z_I$ and $Z_I \subset C$, respectively. Hence, it is singular at points in the intersection given by $p \in Z_I \subset C$. It follows that $\mathcal{M}_{1^r, K_S, c_2}$ is singular. □

The connected components of $\mathcal{M}_{1^3, K_S, 1}$, together with the restrictions of the universal sheaf on $\mathcal{M}_{1^3, K_S, 1} \times S$, are given as follows:

connected component	restriction
$S^{[1]} \times K_S $	$\mathcal{I}^{[1]} \otimes \omega_S \oplus \mathfrak{t}^{-1} \oplus \mathcal{O}_{ K_S }(1) \otimes \mathfrak{t}^{-2}$
\mathcal{C}	$\omega_S \oplus \mathfrak{t}^{-1} \oplus j^*(\mathcal{I}^{[1]} \otimes \mathcal{O}_{ K_S }(1)) \otimes \mathfrak{t}^{-2}$

We write

$$j: \mathcal{C} \times S \hookrightarrow S^{[1]} \times |K_S| \times S$$

for the inclusion. We have suppressed pullbacks along the several projections.

For $c_2 = 2$, we have:

connected component	restriction
$S^{[2]} \times K_S $	$(\mathcal{I}^{[2]} \otimes \omega_S) \oplus \mathfrak{t}^{-1} \oplus (\mathcal{O}_{ K_S }(1) \otimes \mathfrak{t}^{-2})$
$S^{[1]} \times K_S $	$(\mathcal{I}^{[1]} \otimes \omega_S) \oplus (\mathcal{I}^{[1]} \otimes \mathfrak{t}^{-1}) \oplus (\mathcal{O}_{ K_S }(1) \otimes \mathfrak{t}^{-2})$
$S^{[1]} \times \mathcal{C}$	$(\mathcal{I}^{[1]} \otimes \omega_S) \oplus \mathfrak{t}^{-1} \oplus (j^*(\mathcal{I}^{[1]} \otimes \mathcal{O}_{ K_S }(1)) \otimes \mathfrak{t}^{-2})$
$\mathbb{C}_{ K_S }^{[2]}$	$\omega_S \oplus \mathfrak{t}^{-1} \oplus (j_2^*(\mathcal{I}^{[2]} \otimes \mathcal{O}_{ K_S }(1)) \otimes \mathfrak{t}^{-2})$

We have written

$$j_2: \mathbb{C}_{|K_S|}^{[2]} \times S \hookrightarrow S^{[2]} \times |K_S| \times S$$

for the inclusion. Again, we have suppressed pullbacks along projections. Now define Higgs fields $\phi = (\phi_1, \phi_2)$ by the several natural inclusions of ideal sheaves.

As the moduli spaces are smooth, we can compute the virtual class of each component by taking the Euler class of the obstruction bundle. Write $H := c_1(\mathcal{O}_{|K_S|}(1))$. Using [Theorem 1.4](#), we find, for $c_2 = 1$,

$$\begin{aligned} [S^{[1]} \times |K_S|]^{\text{vir}} &= e(K_S^* + \Omega_{|K_S|}(1)) = (-1)^{p_g} \cdot [C], \\ [C]^{\text{vir}} &= e(\Omega_{|K_S|}(1)) = (-1)^{p_g-1} \cdot [C], \end{aligned}$$

and, for $c_2 = 2$,

$$\begin{aligned} [S^{[2]} \times |K_S|]^{\text{vir}} &= e(\omega_S^{[2]*} + \Omega_{|K_S|}(1)) \\ &= [\mathbb{C}_{|K_S|}^{[2]}] \cap (-H)^{p_g-1} = (-1)^{p_g-1} \cdot [C^{[2]}], \\ [S^{[1]} \times |K_S|]^{\text{vir}} &= e(H^0(\omega_S)) = 0, \\ [S^{[1]} \times C]^{\text{vir}} &= e(\omega_S^* + \Omega_{|K_S|}(1)) = (-1)^{p_g} \cdot [C \times C], \\ [\mathbb{C}_{|K_S|}^{[2]}]^{\text{vir}} &= e(\Omega_{|K_S|}(1)) = (-1)^{p_g-1} \cdot [C^{[2]}]. \end{aligned}$$

It follows that the computation of the contribution of the Vafa–Witten invariant reduces to computations in the intersection rings of C and $C^{[2]}$. Using Grothendieck–Riemann–Roch to compute the Chern classes of the relative Hom complexes, this is a straightforward computation. The details are similar to the computations in [\[20\]](#) and [\[21\]](#).

12 Comparison to the Göttsche–Kool conjectures

For rank 2, the Laurent series that appear in [Theorem A](#), and are defined in [Section 8](#), are given by

$$\begin{aligned} A^{(2)} &= \frac{1}{y^{-\frac{1}{2}} + y^{\frac{1}{2}}} A_{\underline{\chi}(\mathcal{O}_S)}^{(2)}(y), \quad B^{(2)} = A_{\underline{K}_S^2}^{(2)}(y), \\ C_{11}^{(2)} &= -q^{-\frac{1}{4}} \frac{1}{y^{-\frac{1}{2}} + y^{\frac{1}{2}}} A_{\underline{\beta}^1 \beta^1}^{(2)}(y) A_{\underline{\beta}^1 \underline{K}_S}^{(2)}(y), \end{aligned}$$

and for rank 3 by

$$A^{(3)} = \frac{1}{y^{-1} + 1 + y} A_{\underline{\chi}(\mathcal{O}_S)}^{(3)}(y), \quad B^{(3)} = A_{\underline{K}_S^2}^{(3)}(y),$$

$$C_{11}^{(3)} = q^{-\frac{1}{3}} \frac{1}{y^{-1} + 1 + y} A_{\beta^1 \beta^1}^{(3)}(y) A_{\beta^1 K_S}^{(3)}(y),$$

$$C_{22}^{(3)} = q^{-\frac{1}{3}} \frac{1}{y^{-1} + 1 + y} A_{\beta^2 \beta^2}^{(3)}(y) A_{\beta^2 K_S}^{(3)}(y), \quad C_{12}^{(3)} = q^{-\frac{1}{3}} \frac{(1 + y)^2}{1 + y + y^2} A_{\beta^1 \beta^2}^{(3)}(y).$$

In Section 9, we have discussed a method for computing the terms of the generating series $A_{\mathfrak{M}}^{(r)}$ appearing above. In Appendix B we have listed the first few terms of the rank 3 power series. The computations allow us to check the equations of Conjectures 1.17 and 1.18 term by term, leading to Theorem B. As an example, let us just verify one term of $C_{12}^{(3)}$. We have

$$\begin{aligned} C_{12}^{(3)} &= q^{-\frac{1}{3}} \frac{(1 + y)^2}{1 + y + y^2} A_{\beta^1 \beta^2}^{(3)}(y) \\ &= q^{-\frac{1}{3}} \frac{(1 + y)^2}{1 + y + y^2} \left(1 + \frac{y^4 + 6y^3 + 6y^2 + 6y + 1}{(y + 1)^2 y} q + \dots \right). \end{aligned}$$

On the other hand we have

$$\begin{aligned} W(q^{\frac{1}{2}}, y) &= \frac{\Theta_{A_2, (0,0)}(q^{\frac{1}{2}}, y)}{\Theta_{A_2, (1,0)}(q^{\frac{1}{2}}, y)} \\ &= \frac{1 + (y^2 + 2y + 2y^{-1} + y^{-2})q + \dots}{(y + 1 + y^{-1})q^{\frac{1}{3}} + (y^2 + 1 + y^{-2})q^{\frac{4}{3}} + \dots} \\ &= q^{-\frac{1}{3}} \frac{1}{y + 1 + y^{-1}} \frac{1 + (y^2 + 2y + 2y^{-1} + y^{-2})q + \dots}{1 + (y - 1 + y^{-1})q + \dots} \\ &= q^{-\frac{1}{3}} \left(\frac{1}{y + 1 + y^{-1}} + \frac{y^2 + y + 1 + y^{-1} + y^{-2}}{y + 1 + y^{-1}} q + \dots \right), \end{aligned}$$

and hence

$$W(q^{\frac{1}{2}}, 1) = q^{-\frac{1}{3}} \frac{1}{3} (1 + 5q + \dots).$$

It follows that

$$\begin{aligned} W_+(q^{\frac{1}{2}}, y) W_-(q^{\frac{1}{2}}, y) &= W(q^{\frac{1}{2}}, y) + 3W(q^{\frac{1}{2}}, 1) \\ &= q^{-\frac{1}{3}} \left(1 + \frac{1}{y + 1 + y^{-1}} + \left(\frac{y^2 + y + 1 + y^{-1} + y^{-2}}{y + 1 + y^{-1}} + 5 \right) q + \dots \right) \\ &= q^{-\frac{1}{3}} \left(\frac{y + 2 + y^{-1}}{y + 1 + y^{-1}} + \left(\frac{y^2 + y + 1 + y^{-1} + y^{-2} + 5(y + 1 + y^{-1})}{y + 1 + y^{-1}} \right) q + \dots \right) \end{aligned}$$

$$\begin{aligned}
 &= q^{-\frac{1}{3}} \frac{(1+y)^2}{1+y+y^2} \left(1 + \frac{y^4 + 6y^3 + 6y^2 + 6y + 1}{(y+1)^2 y} q + \dots \right) \\
 &\equiv C_{12}^{(3)}(y) \pmod{U_2^{(3)}},
 \end{aligned}$$

where we have used the notation

$$U_2^{(3)} = 1 + q^2 \mathbb{Q}(y^{\frac{1}{2}})[[q]] \subset \mathbb{Q}(y^{\frac{1}{2}})((q^{\frac{1}{6}}))^*$$

from the introduction.

Appendix A Functions appearing in the Göttsche–Kool conjectures

These functions appear in the rank 2 and 3 conjectures of [10]:

$$\phi_{-2,1}(x, y) := (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2 \prod_{n=1}^{\infty} \frac{(1-x^n y)^2 (1-x^n y^{-1})^2}{(1-x^n)^4},$$

$$\tilde{\eta}(x) := \prod_{n \in \mathbb{Z}_{>0}} (1-x^n),$$

$$\tilde{\Delta}(x) := \prod_{n \in \mathbb{Z}_{>0}} (1-x^n)^{24},$$

$$\theta_2(x, y) := \sum_{n \in \mathbb{Z} + \frac{1}{2}} x^{n^2} y^n,$$

$$\theta_3(x, y) := \sum_{n \in \mathbb{Z}} x^{n^2} y^n,$$

$$\Theta_{A_2, (0,0)}(x, y) := \sum_{(m,n) \in \mathbb{Z}^2} x^{2(m^2 - mn + n^2)} y^{m+n},$$

$$\Theta_{A_2, (1,0)}(x, y) := \sum_{(m,n) \in \mathbb{Z}^2} x^{2(m^2 - mn + n^2 + m - n + \frac{1}{3})} y^{m+n},$$

$$W(x, y) := \frac{\Theta_{A_2, (0,0)}(x, y)}{\Theta_{A_2, (1,0)}(x, y)}.$$

The functions $W_{\pm}(x, y)$ are defined by the polynomial equation in ω

$$\omega^2 - (W(x, y)^2 + 3W(x, y)W(x, 1))\omega + W(x, y) + 3W(x, 1) = 0.$$

We will use the convention that $W_-(x, y)$ is the one with leading term

$$x^{\frac{2}{3}}(y^{-1} + 1 + y),$$

so we have

$$W_-(q^{\frac{1}{2}}, y) = \frac{y^2 + y + 1}{y} q^{\frac{1}{3}} \left(1 + \frac{2y^2 + 3y + 2}{(y + 1)^2} q + \dots \right),$$

$$W_+(q^{\frac{1}{2}}, y) = \frac{(y + 1)^2 y}{y^2 + y + 1} q^{-\frac{2}{3}} \left(1 + \frac{y^4 + 4y^3 + 3y^2 + 4y + 1}{(y + 1)^2} q + \dots \right).$$

Appendix B Rank 3 results

We set $q := q_0 = q_1 = q_2$, and for

$$\underline{\mathfrak{N}} \in \{ \underline{\chi}(\mathcal{O}_S), \underline{K}_S^2, \beta^1 K_S, \beta^2 K_S, \beta^1 \beta^1, \beta^2 \beta^2, \beta^1 \beta^2 \},$$

we print the first few terms of $A_{\underline{\mathfrak{N}}}^{(3)}(y)$, each modulo q^4 :

$$A_{\underline{\chi}(\mathcal{O}_S)}^{(3)}(y) \equiv 1 + \frac{y^6 + 10y^3 + 1}{y^3} q^3,$$

$$A_{\underline{K}_S^2}^{(3)}(y) \equiv 1 - \frac{(y^2 + y + 1)^2}{(y + 1)^2 y} q - \frac{(2y^4 + 7y^3 + 12y^2 + 7y + 2)(y^2 + y + 1)^2}{y^2(y + 1)^4} q^2$$

$$+ \frac{1}{y^3(y + 1)^6} (5y^{12} + 39y^{11} + 150y^{10} + 382y^9 + 705y^8 + 1002y^7$$

$$+ 1121y^6 + 1002y^5 + 705y^4 + 382y^3 + 150y^2 + 39y + 5) q^3,$$

$$A_{\beta^1 K_S}^{(3)}(y) \equiv A_{\beta^2 K_S}^{(3)}(y)$$

$$\equiv 1 + \frac{1}{2} \frac{(y^2 + y + 1)(y - 1)^2}{(y + 1)^2 y} q$$

$$+ \frac{1}{8} \frac{(23y^4 + 68y^3 + 142y^2 + 68y + 23)(y^2 + y + 1)^2}{(y + 1)^4 y^2} q^2$$

$$- \frac{y^2 + y + 1}{16y^3(y + 1)^6} (15y^{10} + 244y^9 + 1006y^8 + 2790y^7 + 4719y^6$$

$$+ 5780y^5 + 4719y^4 + 2790y^3 + 1006y^2 + 244y + 15) q^3,$$

$$A_{\beta^1 \beta^1}^{(3)}(y) \equiv A_{\beta^2 \beta^2}^{(3)}(y)$$

$$\begin{aligned}
 &\equiv 1 - \frac{1}{2} \frac{y^4 + 3y^3 + 6y^2 + 3y + 1}{(y+1)^2 y} q \\
 &\quad - \frac{1}{8} \frac{1}{(y+1)^4 y^2} (5y^8 + 30y^7 + 109y^6 + 218y^5 + 280y^4 + 218y^3 \\
 &\quad\quad\quad + 109y^2 + 30y + 5) q^2 \\
 &\quad + \frac{1}{16y^3 (y+1)^6} (11y^{12} + 115y^{11} + 571y^{10} + 1868y^9 + 4205y^8 \\
 &\quad\quad\quad + 6845y^7 + 8026y^6 + 6845y^5 + 4205y^4 \\
 &\quad\quad\quad + 1868y^3 + 571y^2 + 115y + 11) q^3, \\
 A_{\underline{\beta^1 \beta^2}}^{(3)}(y) &\equiv 1 + \frac{y^4 + 6y^3 + 6y^2 + 6y + 1}{(y+1)^2 y} q - \frac{y^6 + y^5 + 8y^4 + 8y^3 + 8y^2 + y + 1}{(y+1)^2 y^2} q^2 \\
 &\quad + \frac{y^6 + 3y^4 + 4y^3 + 3y^2 + 1}{y^2 (y+1)^2} q^3.
 \end{aligned}$$

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