The prism manifold realization problem

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The spherical manifold realization problem asks which spherical three-manifolds arise from surgeries on knots in S^3. In recent years, the realization problem for C–, T–, O– and I–type spherical manifolds has been solved, leaving the D–type manifolds (also known as the prism manifolds) as the only remaining case. Every prism manifold can be parametrized as $P(p, q)$ for a pair of relatively prime integers $p > 1$ and q. We determine a list of prism manifolds $P(p, q)$ that can possibly be realized by positive integral surgeries on knots in S^3 when $q < 0$. Based on the forthcoming work of Berge and Kang, we are confident that this list is complete. The methodology undertaken to obtain the classification is similar to that of Greene for lens spaces.

1 Introduction

There are many notions of simplicity for closed three-manifolds. Perhaps the simplest is that of a manifold with a finite fundamental group. One of the most prominent problems in three-manifold topology is to indicate the list of the simplest closed three-manifolds that can be realized by the simplest three-dimensional topological operations. Given that every closed three-manifold can be obtained by performing surgery on a link in S^3 (see Lickorish [14] and Wallace [23]), the aforementioned realization problem may be stated as follows:

Question 1.1 *Which closed* 3*–manifolds with finite fundamental groups can be realized by surgeries on nontrivial knots in* S^3 *?*

By the work of Thurston [22], any knot in S^3 is precisely one of a torus knot, a satellite knot or a hyperbolic knot. Moser [15] classified all finite surgeries on torus knots. Later, Boyer and Zhang [4, Corollary 1.4] showed that if surgery on a satellite knot $K \subset S^3$ results in a manifold with a finite fundamental group, then K must be a cable of a torus knot. Such surgeries are classified by Bleiler and Hodgson in [3, Theorem 7]. In regard to the surgery coefficient, Culler–Gordon–Luecke–Shalen [5] proved that any cyclic surgery must be integral. As further proved by Boyer and Zhang [4, Theorem 1.1], the coefficient of any finite surgery is either p or $p/2$ for some integer p. Li and Ni [13] showed that if half-integral surgery on a hyperbolic knot results in a manifold Y with a finite fundamental group, then Y is homeomorphic to $p/2$ surgery on either a torus knot or a cable of a torus knot. As a result, we henceforth restrict attention to integral surgeries.

Using Perelman's geometrization theorem, closed three-manifolds with finite fundamental groups can be characterized as those three-manifolds that admit *spherical geometry*. A *spherical 3–manifold* (also known as an *elliptic 3–manifold*) has the form

$$Y = S^3/G,$$

where G is a finite subgroup of SO(4) that acts freely on S^3 by rotations. The center $Z = Z(G)$ of $G \cong \pi_1(Y)$ is necessarily a cyclic group. According to the structure of G/Z, spherical manifolds (besides S^3) are divided into five types: C or cyclic, D or dihedral, T or tetrahedral, O or octahedral, I or icosahedral. In particular, if G/Z is the dihedral group

$$D_{2p} = \langle x, y \mid x^2 = y^2 = (xy)^p = 1 \rangle$$

for some integer $p > 1$, we get the D–type manifolds. These manifolds are also known as the *prism manifolds*.

Greene [10] solved the integer surgery realization problem (that is, Question 1.1 when the surgery coefficient is integral) for lens spaces, namely, the C–type manifolds. Later, Gu [12] provided the solution for T–, O– and I–type manifolds. This leaves the D–type manifolds as the only remaining case, and that is the theme of the present work.

There are many more prism manifolds than any other type of spherical manifolds. It is straightforward to check that for each integer $m > 0$, there are only finitely many spherical manifolds Y of other types with $|H_1(Y)| = m$. However, for each m divisible by 4, there are infinitely many prism manifolds with the order of the first singular homology equal to m. To justify, let $P(p, q)$ be the oriented prism manifold with

Seifert invariants

(1) $(-1; (2, 1), (2, 1), (p, q))$,

where $p > 1$ and q are a pair of relatively prime integers. These manifolds satisfy

(2) $|H_1(P(p, q))| = 4|q|$.

Therefore, any integer $p > 1$ relatively prime to q will give a prism manifold $P(p, q)$ with the desired order of the first singular homology. In regard to the realization problem, however, we still have a finiteness result. It was first proved by Doig in [7] that, for a fixed $|q|$, there are only finitely many p for which $P(p, q)$ may be realized by surgery on a knot $K \subset S^3$. Later, Ni and Zhang [16] proved an explicit bound for p in terms of q:

$$p < 4|q|.$$

We are now in a position to state the main result of the paper:

Theorem 1.2 *Given a pair of relatively prime integers $p > 1$ and $q < 0$, if the prism manifold $P(p, q)$ can be obtained by $4|q|$–Dehn surgery on a knot K in S^3, then $P(p, q)$ belongs to one of the six families in* Table 1. *Moreover, the knot Floer homology group of K is determined by $P(p, q)$ and the family containing it.*

Remark 1.3 The six families in Table 1 are divided so that each changemaker vector (see Definition 1.5) corresponds to a unique family. However, a prism manifold $P(p, q)$ could belong to different families, and not just a unique one. In particular, given $P(p, q)$, there may be several knots with surgery yielding $P(p, q)$, and such that these knots have different knot Floer homology groups. We will address the overlaps between these families in Section 13.1. See Table 3.

All the known examples of integral cyclic surgeries (lens space surgeries) come from Berge's primitive/primitive (or simply P/P) construction [1]. There is a generalization of this construction to Seifert-fibered surgeries due to Dean [6], called the primitive/Seifert-fibered (or P/SF) construction. See Definition 13.3. The *surface slope* Dehn surgery on a hyperbolic P/SF knot results in a Seifert-fibered space. Berge and Kang [2] classified all P/SF knots in S^3. Further, they specified the indices of the singular fibers of the Seifert-fibered manifolds resultant from the surface slope surgeries on such knots. Since prism manifolds are Seifert-fibered spaces over S^2 with three singular fibers of indices $(2, 2, p)$, following from the work of Berge and Kang, we obtain a list of prism

manifolds that are realizable by knot surgeries. It turns out that this list coincides with Table 1 when $q < 0$. However, since Berge and Kang's work is not publicly available, we will not claim that we have resolved the realization problem for prism manifolds when $q < 0$. Instead, we will state this part of the realization problem as a conjecture.

Conjecture 1.4 *For any* $P(p, q)$ *in Tables 1 and 2, there exists a knot* K *such that the* $4|q|$*–Dehn surgery on* K *results in* $P(p, q)$.

Our study (not included in this paper) of the Berge–Kang classification indicates that the knot K in Conjecture 1.4 can be taken to be a Berge–Kang knot.

The methodology undertaken to prove Theorem 1.2 is inspired from that of Greene [10]. A prism manifold $P(p, q)$ with $q < 0$ naturally bounds a negative definite four-manifold $X(p, q)$. See Section 2. Suppose that $P(p, q)$ is realized by $4|q|$–surgery on a knot $K \subset S^3$. In particular, $P(p, q)$ bounds the two-handle cobordism $W_{4|q|} = W_{4|q|}(K)$, obtained by attaching a two-handle to D^4 along $K \subset S^3$ with framing $4|q|$. Note that the surgery coefficient is dictated by homology considerations: see equation (2). The four-manifold $Z := X(p, q) \cup -W_{4|q|}$ is a smooth, closed, negative definite 4–manifold with $b_2(Z) = n + 4$, where $n + 3 = b_2(X(p, q))$ for some $n \geq 1$. Donaldson's Theorem A [8] implies that the intersection pairing on $H_2(Z)$ is isomorphic to $-\mathbb{Z}^{n+4}$, the negative of the standard $(n+4)$–dimensional Euclidean integer lattice. Consequently, the negative of the intersection pairing on $X(p, q)$, denoted $\Delta(p, q)$, embeds as a codimension-one sublattice of $\mathbb{Z}^{n+4}$. For the prism manifold $P(p, q)$ to arise from a knot surgery, this already gives a restriction on the pair (p, q). We then appeal to the innovative work of Greene that provides even more constraints on $\Delta(p, q)$. To state this essential step, we first need to make a combinatorial definition.

Definition 1.5 A vector $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{n+3}) \in \mathbb{Z}^{n+4}$ that satisfies $0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_{n+3}$ is a *changemaker vector* if for every k with $0 \leq k \leq \sigma_0 + \sigma_1 + \cdots + \sigma_{n+3}$, there exists a subset $S \subset \{0, 1, \ldots, n + 3\}$ such that $k = \sum_{i \in S} \sigma_i$.

The lattice embedding theorem of Greene [11, Theorem 3.3] now reads as follows: if $P(p, q)$ with $q < 0$ is realized by $4|q|$–surgery on $K \subset S^3$, then $\Delta(p, q)$ embeds into $\mathbb{Z}^{n+4}$ as the orthogonal complement of a changemaker vector $\sigma \in \mathbb{Z}^{n+4}$. By determining the pairs (p, q) which pass this refined embedding restriction, we get the list of all prism manifolds that could possibly be realized by positive integral surgeries on knots.

Theorem 1.6 *Given a pair of relatively prime integers $p > 1$ and $q < 0$, we have that $\Delta(p, q) \cong (\sigma)^{\perp}$ for a changemaker vector $\sigma \in \mathbb{Z}^{n+4}$ if and only if $P(p, q)$ belongs to one of the six families in* Theorem 1.2. *If we further assume that* Conjecture 1.4 *holds true in this case, then there exists a knot $K \subset S^3$ and an isomorphism of lattices*

$$\phi \colon (\mathbb{Z}^{n+4}, I) \to (H_2(Z), -Q_Z)$$

satisfying the property that $\phi(\sigma)$ is a generator of $H_2(-W_{4|q|})$. Here I is the standard inner product on $\mathbb{Z}^{n+4}$ and Q_Z is the intersection form of $Z = X(p, q) \cup -W_{4|q|}$.

1.1 Prism manifolds $P(p, q)$ with $q > 0$

As discussed, digging up the list of P/SF knots in S^3 given in [2], we obtain a family of knots with prism manifold surgeries. We can get a list $\mathcal{P}^+$ of prism manifolds with $q > 0$ arising from surgeries on knots in S^3. See Table 2. In light of Theorem 1.2, we make the following conjecture.

Conjecture 1.7 *Given a pair of relatively prime integers $p > 1$ and $q > 0$, if $P(p, q)$ can be obtained by $4q$–Dehn surgery on a knot K in S^3, then $P(p, q) \in \mathcal{P}^+$.*

Theorem 1.2 leaves open the integer surgery realization problem for manifolds $P(p, q)$ with $q > 0$, and Conjecture 1.7 proposes the solution. A natural direction to pursue is to indicate the list of all knots in S^3 that admit surgeries to spherical manifolds. In [1], Berge proposed a complete list of knots in S^3 with lens space surgeries. Indeed, Berge's conjecture states that the P/P knots form a complete list of knots in S^3 that admit lens space surgeries. All the known examples of spherical manifolds arising from knot surgeries will provide supporting evidence to the following conjecture:

Conjecture 1.8 *Let K be a knot in S^3 that admits a surgery to a spherical manifold. Then K is either a P/SF or a P/P knot.*

We point out that Conjecture 1.8 implies Conjecture 1.7.

When $q > p$, $P(p, q)$ is the double branched cover of S^3 with branching locus being an alternating Montesinos link, thus it is the boundary of a sharp 4–manifold; see Ozsváth and Szabó [20]. Greene's strategy can still be used to study the realization problem in this case, but the lattices will not be of D–type. We plan to address this case in a future paper.

1.2 Organization

In Section 2, we give the basic topological properties of prism manifolds, and explain how to reduce the realization problem to a problem about lattices. In Section 3, we study the D–type lattices which are central in our paper. There is a natural *vertex basis* for a D–type lattice. Every vector in the vertex basis is *irreducible*. A classification of irreducible vectors is given in Proposition 3.9. In Section 4, we endow a *changemaker lattice* $(\sigma)^{\perp}$ with a *standard basis* S, and study the question when such a lattice is isomorphic to a D–type lattice. From the standard basis elements of a changemaker lattice we can form an intersection graph; see Definition 4.16. We collect many structural results about this graph.

Section 5 addresses some technical lemmas regarding the nonexistence of certain edges in the intersection graph associated to a changemaker lattice. The elements of a standard basis S, viewed as an ordered set, are of three types: *tight*, *just right*, and *gappy* (Definition 4.1). As it turns out, the classification of changemaker lattices that are isomorphic to D–type lattices relies highly on the properties of one specific element v_{f-1} of the standard basis: whether it is tight, just right or gappy, together with its placement in S. Accordingly, we will do a case-by-case analysis to enumerate the possible standard bases for such a lattice. This occupies Sections 6–11. Section 12 is devoted to converting these standard bases into vertex bases to get a list of pairs (p, q) corresponding to the D–type lattices found in Sections 6–11.

In Section 13 we discuss how one may tabulate all the Berge–Kang P/SF knots that admit prism manifold surgeries. Thus we finish the proofs of Theorems 1.2 and 1.6.

2 Background

In this section, we start with recalling some basic facts about prism manifolds, then provide a concise strategy to translate the prism manifold realization problem into a
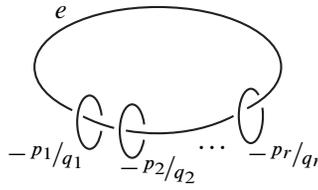
Figure 1: The surgery diagram of a Seifert-fibered space.

lattice theory question. Meanwhile, the necessary background from Heegaard Floer homology will be cited.

2.1 Prism manifolds

It is well known that every spherical manifold is a Seifert-fibered space, that is, a three-manifold with a surgery diagram as depicted in Figure 1.[1] The data

$$(3) \qquad\qquad (e; (p_1, q_1), (p_2, q_2), \dots, (p_r, q_r))$$

are called the *Seifert invariants*, where e is an integer, and $(p_1, q_1), \dots, (p_r, q_r)$ are pairs of relatively prime integers such that $p_i > 1$. The rational number

$$e_{\mathrm{orb}} := e + \sum_{i=1}^{r} \frac{q_i}{p_i}$$

is called the *orbifold Euler number*. The oriented homeomorphism type of a Seifert-fibered space is determined by the multiset

$$\left\{ \frac{q_i}{p_i} - \left\lfloor \frac{q_i}{p_i} \right\rfloor \;\middle|\; i = 1, \dots, r \right\}$$

together with e_{orb}. It is elementary to verify that if a Seifert-fibered space is a rational homology sphere, it must be the case that $e_{\mathrm{orb}} \neq 0$, and

$$(4) \qquad\qquad |H_1(Y)| = p_1 p_2 \cdots p_r |e_{\mathrm{orb}}|.$$

For a pair of relatively prime integers $p > 1$ and q, a prism manifold $P(p, q)$ is a Seifert-fibered space over S^2 with three singular fibers of indices $(2, 2, p)$, having the Seifert invariants

$$(-1; (2, 1), (2, 1), (p, q)).$$

It is well known that $P(p, q)$ has exactly two Seifert fibrations, and the above one is the only fibration over an orientable base orbifold [21]. As a result, the orbifold

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[1] We only consider Seifert-fibered spaces whose base orbifold has genus zero.

Euler number of the above Seifert fibration, which is q/p, is a topological invariant for $P(p,q)$. Hence $P(p_1,q_1) \cong P(p_2,q_2)$ if and only if $(p_1,q_1) = (p_2,q_2)$. Here "$\cong$" denotes orientation-preserving homeomorphism.

Following from their Seifert-fibered presentations, prism manifolds enjoy the symmetry

$$P(p,-q) \cong -P(p,q),$$

where $-P(p,q)$ is the manifold $P(p,q)$ with opposite orientation. The fundamental group of $P(p,q)$ has presentation

$$(5) \qquad \pi_1(P(p,q)) = \langle x, y \mid xyx^{-1} = y^{-1}, x^{2|q|} = y^p \rangle.$$

The center of this group is a cyclic group of order $2|q|$ generated by x^2. It follows from (5) that $H_1(P(p,q))$ is cyclic if and only if p is odd. Hence if $P(p,q)$ is obtained by surgery on a knot in S^3, p must be odd.

Lemma 2.1 *Suppose that $P(p,q)$ is obtained by $4|q|$–surgery on a knot $K \subset S^3$. If K is a torus knot, then $(p,q) = (2k+1,k)$ or $(2k+1,-k-1)$ for some $k > 0$, and K is $T(2k+1,2)$. If K is a satellite knot, then either $(p,q) = (2k+1,9k+4)$ for some $k > 0$, and K is the $(12k+5,3)$–cable of $T(2k+1,2)$, or $(p,q) = (2k+1,-9k-5)$ for some $k > 0$, and K is the $(12k+7,3)$–cable of $T(2k+1,2)$.*

Proof First suppose that $P(p,q)$ is obtained by $4|q|$–surgery on a torus knot $T(r,s)$. On one hand, it follows from [15, Proposition 3.1] that if $4|q|$ Dehn surgery along $T(r,s)$ results in a Seifert-fibered manifold with three singular fibers, then the fibers have indices r, s and $|rs - 4|q||$. On the other hand, the prism manifold $P(p,q)$ has fibers of indices 2, 2, p. By comparing the indices we find that the torus knot is $T(2k+1,2)$. This implies that $p = 2k+1$, and we can deduce that $|q| = k$ or $k+1$. Note that the slope of the Seifert fiber of the complement of K is $4k+2$. If $|q| = k$, since $0 < 4k < 4k+2$, the orbifold Euler number of the resulting manifold is positive; that is, $q = k$. If $|q| = k+1$, since $0 < 4k+2 < 4k+4$, the orbifold Euler number of the resulting manifold is negative; that is, $q = -k-1$.

When K is a satellite knot, we deduce from [4, Corollary 1.4] that K must be a cable of a torus knot. Therefore, we can use the classification of finite surgeries on such knots in [3, Theorem 7]: the classification is given in [3, Table 1]. It follows that K is the $(12k+6 \mp 1, 3)$–cable of $T(2k+1,2)$, and $|q| = 9k+4$ or $9k+5$. When $|q| = 9k+4$, the $(36k+16)$–surgery on the $(12k+5,3)$–cable of $T(2k+1,2)$ is the
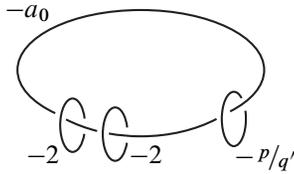
Figure 2: A surgery diagram of $P(p,q)$.

same as the $\frac{1}{9}(36k+16)$–surgery on $T(2k+1,2)$. Since the resulting manifold has a singular fiber of index $2k+1$, $p=2k+1$. Similarly, we can deal with the case $|q|=9k+5$. □

Remark 2.2 In [16], it is proved that if $P(2k+1,k)$ or $P(2k+1,-k-1)$ can be obtained by positive surgery on a knot $K \subset S^3$, then K must be $T(2k+1,2)$.

One key step in the proof of Theorem 1.2 is that every prism manifold $P(p,q)$ with $q<0$ bounds a negative definite four-manifold. Writing

$$k = \left\lfloor \frac{q}{p} \right\rfloor < 0, \quad q' = q - kp > 0,$$

then $P(p,q)$ has an equivalent Seifert fibration with Seifert invariants

$$(-1+k; (2,1),(2,1),(p,q')).$$

Correspondingly, we get a surgery diagram for $P(p,q)$ as in Figure 2, where $a_0 = 1-k \geq 2$. Having $p/q' > 1$, expand p/q' in a continued fraction

$$(6) \qquad \frac{p}{q'} = [a_1,\ldots,a_n]^- := a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}},$$

where the a_i are integers satisfying $a_i \geq 2$. Equivalently, we can write

$$(7) \qquad \frac{-q}{p} = [a_0 - 1, a_1, \ldots, a_n]^-.$$

Let $X(p,q)$ be the four-manifold that $P(p,q)$ bounds, obtained by attaching two-handles to D^4 instructed by the framed link in Figure 3. More precisely, each unknot component in Figure 3 denotes a disk bundle over S^2 with Euler number specified by its coefficient. The manifold $X(p,q)$ is obtained from plumbing these disk bundles together: two disk bundles are plumbed if the corresponding unknot components are linked.
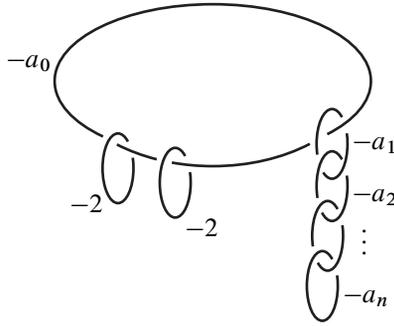
Figure 3: An integral surgery diagram of $P(p,q)$.

Let

$$Q_X \colon H_2(X) \times H_2(X) \to \mathbb{Z}$$

denote the intersection pairing on $X = X(p,q)$. The second homology of X has rank $n+3$, generated by elements $x_*, x_{**}, x_0, \ldots, x_n$. Note that x_* and x_{**} correspond to the vertices with weights -2 in Figure 3.

Lemma 2.3 $X(p,q)$ *is a negative definite four-manifold.*

Proof We will show that $-Q_X$ is positive definite. Given a vector $v \in H_2(X)$, for each $i = 1, \ldots, n$, it is easy to check that $-Q_X(v, v)$ is an increasing function of the a_i. In particular it suffices to prove the claim when each a_i satisfies $a_i = 2$. Proceeding by induction on $b_2(X) = n + 3$ with $n \geq 1$, we get that

$$\det(-Q_X) = 4.$$

Since all principal minors are positive by induction, $-Q_X$ is positive definite. □

2.2 The realization problem: from correction terms to lattice theory

In what follows, we will present the methodology we apply to prove Theorem 1.2. One main ingredient is the *correction terms* in Heegaard Floer homology.

In [17], Ozsváth and Szabó defined the correction term $d(Y, \mathfrak{t})$ that associates a rational number to an oriented rational homology sphere Y equipped with a Spin^c structure $\mathfrak{t}$. They showed that this invariant obeys the relation

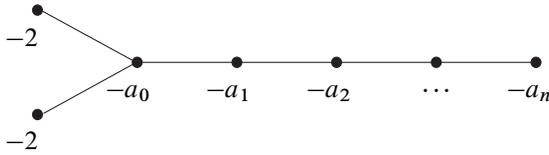$$d(-Y, \mathfrak{t}) = -d(Y, \mathfrak{t}),$$

Figure 4: A negative definite plumbing diagram of $P(p,q)$.

where $-Y$ is the manifold Y with the reversed orientation. If Y is boundary of a negative definite four-manifold X, then

$$(8) \qquad\qquad c_1(\mathfrak{s})^2 + b_2(X) \le 4d(Y, \mathfrak{t})$$

for any $\mathfrak{s} \in \mathrm{Spin}^c(X)$ that extends $\mathfrak{t} \in \mathrm{Spin}^c(Y)$.

Definition 2.4 A smooth, compact, negative definite 4–manifold X is *sharp* if for every $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, there exists some $\mathfrak{s} \in \mathrm{Spin}^c(X)$ extending $\mathfrak{t}$ such that the equality is realized in equation (8).

The manifold $X = X(p,q)$ is an example of a sharp four–manifold. In order to prove this, it will be profitable to view the plumbing diagram of X, depicted in Figure 4, as a *weighted graph*, that is a graph equipped with an integer-valued function m on its vertices. Given a weighted graph G, let V be the abelian group freely generated by the vertices of G. Define a quadratic form

$$Q_G \colon V \otimes V \to \mathbb{Z}$$

as follows. For each vertex v, $Q_G(v \otimes v) = m(v)$; for each pair of distinct vertices v and w, $Q_G(v \otimes w)$ is 1 if v and w are connected by an edge, and 0 otherwise.

Definition 2.5 A weighted graph G is said to be a *negative definite graph* if

- G is a disjoint union of trees, and
- the quadratic form associated to G is negative definite.

The *degree* of a vertex v, denoted $d(v)$, is the number of edges incident to v. A vertex v is said to be a *bad vertex* of the weighted graph if

$$m(v) > -d(v).$$

Given a weighted graph G, we can get a four-manifold X_G obtained from the plumbing construction instructed by G. In [18], Ozsváth and Szabó showed that if G is a negative definite weighted graph with at most two bad vertices, then X_G is sharp. In summary, using Lemma 2.3:

Lemma 2.6 $X(p,q)$ *is a sharp four-manifold.*

We end this subsection by presenting how the integer surgery realization problem for prism manifolds translates to a lattice theory question.

Definition 2.7 A *lattice* is a finitely generated free abelian group L together with a positive definite quadratic form

$$\langle\cdot,\cdot\rangle\colon L\times L\to\mathbb{R}.$$

The lattice is *integral* if the value of the quadratic form is in $\mathbb{Z}$.

Throughout this paper, we will only consider integral lattices.

Definition 2.8 Suppose $p>1$ and $q<0$ are a pair of relatively prime integers. The *D–type lattice* $\Delta(p,q)$ is the lattice freely generated by elements

(9) $x_*,x_{**},x_0,x_1,\ldots,x_n$

with inner product given by

(10) $\langle x_i,x_j\rangle=\begin{cases}-1 & \text{if }\{i,j\}\text{ is either }\{*,0\}\text{ or }\{**,0\},\\ -1 & \text{if }|i-j|=1\text{ with }0\leq i,j\leq n,\\ 2 & \text{if }i=j\in\{*,**\},\\ a_i & \text{if }0\leq i=j\leq n,\\ 0 & \text{in other cases,}\end{cases}$

where the coefficients $a_0,\ldots,a_n$ satisfy $a_j\geq 2$ and also (7). We then call (9) a *vertex basis* of $\Delta(p,q)$.

The inner product space $(H_2(X),-Q_X)$ equals $\Delta(p,q)$, where $X=X(p,q)$ is the four-manifold with Kirby diagram as in Figure 3. See also Figure 4. Now suppose that $4|q|$–surgery along a knot $K\subset S^3$ produces $P(p,q)$ with $q<0$. Let $W_{4|q|}$ denote the associated two-handle cobordism from S^3 to $P(p,q)$, capped off with D^4. Form the closed, oriented, smooth four-manifold $Z=X(p,q)\cup-W_{4|q|}$. Since $b_1(S^3_{4|q|}(K))=0$, it follows that

$$b_2(Z)=b_2(X)+b_2(-W_{4|q|})=n+4.$$

The 4–manifold $-W_{4|q|}$ is negative definite. Using this together with Lemma 2.3 and the fact that $H_2(X)\oplus H_2(-W_{4|q|})\hookrightarrow H_2(Z)$, it follows at once that Z is negative definite. Using Lemma 2.6, and with the notation of this section in place, the following is immediate from [11, Theorem 3.3].

Theorem 2.9 *Suppose $P(p,q)$ with $q < 0$ arises from positive integer surgery on a knot in S^3. The D–type lattice $\Delta(p,q)$ is isomorphic to the orthogonal complement $(\sigma)^\perp$ of some changemaker vector $\sigma \in \mathbb{Z}^{n+4}$, and the generator of $H_2(W_{4|q|})$ maps to σ.*

Using techniques that will be developed in the next sections in tandem with Theorem 2.9, we will find a classification of all D–type lattices $\Delta(p,q)$ isomorphic to $(\sigma)^\perp$ for some changemaker vector σ in $\mathbb{Z}^{n+4}$. If the corresponding prism manifold $P(p,q)$ does indeed arise from surgery on a knot $K \subset S^3$, we are able to compute the Alexander polynomial of K from the values of the components of σ. Giving an algorithmic method to compute the coefficients of the polynomial occupies the rest of this section.

Let Σ be the closed surface obtained from capping off a Seifert surface for K in $W_{4|q|}$. It is straightforward to check that the class $[\Sigma]$ generates $H_2(W_{4|q|})$. It follows from Theorem 2.9 that, under the embedding $H_2(X) \oplus H_2(-W_{4|q|}) \hookrightarrow H_2(Z)$, $[\Sigma]$ gets mapped to a changemaker vector σ. Let $\{e_0, e_1, \ldots, e_{n+3}\}$ be the standard orthonormal basis for $\mathbb{Z}^{n+4}$, and write

$$\sigma = \sum_{i=0}^{n+3} \sigma_i e_i.$$

Also, define the *characteristic covectors* of $\mathbb{Z}^{n+4}$ to be

$$\mathrm{Char}(\mathbb{Z}^{n+4}) = \left\{ \sum_{i=0}^{n+3} \mathfrak{c}_i e_i \,\middle|\, \mathfrak{c}_i \text{ odd for all } i \right\}.$$

We remind the reader that, writing the Alexander polynomial of K as

$$(11) \qquad\qquad \Delta_K(T) = b_0 + \sum_{i>0} b_i(T^i + T^{-i}),$$

the kth torsion coefficient of K is

$$t_k(K) = \sum_{j \geq 1} j b_{k+j},$$

where $k \geq 0$. With the preceding notation in place, the following lemma is immediate from [11, Lemma 2.5].

Lemma 2.10 *The torsion coefficients satisfy*

$$t_i(K) = \begin{cases} \min_{\mathfrak{c}} \frac{1}{8}(\mathfrak{c}^2 - n - 4) & \text{for each } i \in \{0, 1, \ldots, 2|q|\}, \\ 0 & \text{for } i > 2|q|, \end{cases}$$

where $\mathfrak{c}$ is subject to

$$\mathfrak{c} \in \mathrm{Char}(\mathbb{Z}^{n+4}), \quad \langle \mathfrak{c}, \sigma \rangle + 4|q| \equiv 2i \pmod{8|q|}.$$

For $i > 0$,

$$b_i = t_{i-1} - 2t_i + t_{i+1} \quad \text{and} \quad b_0 = 1 - 2\sum_{i>0} b_i,$$

where the b_i are as in (11).

3   D–type lattices

Let L be a lattice. Given $v \in L$, let $|v| = \langle v, v \rangle$ be the *norm* of v. Following Greene [10], say that an element $\ell \in L$ is *reducible* if $\ell = x + y$ for some nonzero $x, y \in L$ with $\langle x, y \rangle \geq 0$, and *irreducible* otherwise. We say ℓ is *breakable* if $\ell = x + y$ with $|x|, |y| \geq 3$ and $\langle x, y \rangle = -1$, and *unbreakable* otherwise.

The main goal of this section will be to characterize the irreducible and unbreakable elements of a D–type lattice $\Delta(p, q)$. Since any isomorphism of lattices must send irreducible elements to irreducible elements, and similarly unbreakables to unbreakables, this will let us constrain the form of an isomorphism between $\Delta(p, q)$ and a changemaker lattice.

The pairing graph, also introduced in [10], will be one of the main tools we use to study lattices:

Definition 3.1   Given a lattice L and a subset $V \subset L$, the *pairing graph* is the graph $\widehat{G}(V) = (V, E)$, where $e = (v_i, v_j) \in E$ if $\langle v_i, v_j \rangle \neq 0$.

Proposition 3.2   *For each* $i \in \{*, **, 0, \ldots, n\}$, x_i *is an irreducible element of* $\Delta(p, q)$.

Proof   Take some $w \in \Delta(p, q)$ with $w \neq 0, x_i$. Write

$$w = w_* x_* + w_{**} x_{**} + \sum_{j=0}^{n} w_j x_j$$

and consider $\langle w, x_i - w \rangle$. This is

$$a_i w_i (1 - w_i) - \sum_{j \neq i} a_j w_j^2 + (\text{terms not involving the } a_j).$$

Here the a_j are as defined in (7) when $j \geq 0$, and $\langle x_j, x_j \rangle = a_j$. We also let $a_* = a_{**} = 2$. The above expression is nonincreasing with respect to each a_j. Therefore,

to show that $\langle w, x_i - w \rangle$ is negative, it suffices to show this when all of the a_j are 2. In this case, $\Delta(p,q)$ is isomorphic to the standard D_{n+3} lattice, the lattice of elements of $\mathbb{Z}^{n+3} = \langle e_{-2}, e_{-1}, e_0, \ldots, e_n \rangle$ the sum of whose coefficients is even. This isomorphism sends

$$(12) \qquad x_* \mapsto e_{-2} + e_{-1}, \quad x_{**} \mapsto -e_{-2} + e_{-1}, \quad x_j \mapsto -e_{j-1} + e_j \quad (j \geq 0).$$

If $\pm e_i \pm e_j \in \mathbb{Z}^{n+3}$ is written as a sum of two other elements of $\mathbb{Z}^{n+3}$ with nonnegative pairing, these must be $\pm e_i$ and $\pm e_j$. However, these are not in D_{n+3}. Therefore, the image of x_i is an irreducible element of D_{n+3}, so x_i is an irreducible element of $\Delta(p,q)$. $\qquad\square$

Corollary 3.3 *The lattice $\Delta(p,q)$ is indecomposable, namely, $\Delta(p,q)$ is not the direct sum of two nontrivial lattices.*

Proof Suppose that $\Delta(p,q) \cong L_1 \oplus L_2$. Then each x_i, being irreducible, must be in either L_1 or L_2. However, any element of L_1 has zero pairing with any element of L_2. Since $x_i \sim x_{i+1}$, $\widehat{G}(\{*, **, 0, \ldots, n\})$ is connected. This means that all of the x_i are in the same part of the decomposition, and the other is trivial. $\qquad\square$

This gives us some information about irreducibility in the D–type lattices, but we want something more complete. Another important class of irreducible elements is that of the intervals:

Definition 3.4 For $A \subset \{*, **, 0, \ldots, n\}$, let $[A] := \sum_{i \in A} x_i$. We say $x \in \Delta(p,q)$ is an *interval* if $x = [A]$ for some A where $\widehat{G}(A)$ is connected and A does not contain both $*$ and $**$.

Proposition 3.5 *Intervals are irreducible.*

Proof As in the proof of Proposition 3.2 we may assume that all of the $a_i = 2$, and consider the image of an interval x in $\mathbb{Z}^{n+3}$ under the embedding (12). (Note that the reduction to the case $a_i = 2$ only works because all the coefficients c_i of $x = \sum c_i x_i$ are 0 or ± 1.) Since x is an interval, the image of x is again of the form $\pm e_i \pm e_j$, which is irreducible as in Proposition 3.2. $\qquad\square$

For linear lattices, this is essentially the end of the story — Greene proves that every irreducible is either an interval or the negation of an interval. The situation here is somewhat richer, but is similar in spirit: every irreducible is either an interval, or can

be obtained from one by applying some involution of the lattice. To prove this, we first
need to know a few involutions of $\Delta(p, q)$.

Definition 3.6 Let m be the smallest index for which $a_m \geq 3$. For any $0 \leq j \leq m$,
let $y_j = x_* + x_{**} + 2\sum_{0 \leq i < j} x_i$, and let $\tau_j \colon \Delta(p, q) \to \Delta(p, q)$ be the reflection in
the subspace spanned by $y_0, \ldots, y_j$. More explicitly, τ_j acts as

$$\tau_j(x_i) = \begin{cases} x_{**} & \text{if } i = *, \\ x_* & \text{if } i = **, \\ x_i & \text{if } 0 \leq i < j, \\ -x_* - x_{**} - 2x_0 - \cdots - 2x_{j-1} - x_j & \text{if } i = j, \\ -x_i & \text{if } i > j. \end{cases}$$

Lemma 3.7 *Each τ_j satisfies $\langle \tau_j(x), \tau_j(y) \rangle = \langle x, y \rangle$ and $\tau_j^2(x) = x$. Furthermore,
for any $v, w \in \Delta(p, q)$, $\langle v, w \rangle \equiv \langle \tau_j(v), w \rangle \pmod 2$.*

Proof Each τ_j is a reflection, which gives the first two properties. For the last property,
it suffices to check that $\langle \tau_j(v_i), v_k \rangle \equiv \langle v_i, v_k \rangle \pmod 2$ for all i, j, k, which is entirely
straightforward.   □

The following special case of Lemma 3.7 will be used very heavily in what follows:

Lemma 3.8 *If $x \in \Delta(p, q)$, then $\langle x, x_* \rangle \equiv \langle x, x_{**} \rangle \pmod 2$.*

Proof This follows from Lemma 3.7 since $x_{**} = \tau_0(x_*)$.   □

The τ_j preserve the pairings on $\Delta(p, q)$, so $\tau_j(x)$ is irreducible whenever x is. This
gives some noninterval examples of irreducible elements of $\Delta(p, q)$. For example,

$$x_* + x_{**} + x_0 + \cdots + x_i = -\tau_0(x_0 + \cdots + x_i)$$

and (as long as $m \geq 2$)

$$x_* + x_{**} + 2x_0 + 2x_1 + x_2 + \cdots + x_i = -\tau_2(x_2 + \cdots + x_i)$$

are both irreducible. However, these are essentially the only examples:

Proposition 3.9 *If $x \in \Delta(p, q)$ is any irreducible element, either x or $-x$ is an
interval, or there is some j such that either $\tau_j(x)$ or $-\tau_j(x)$ is an interval.*

Proof Suppose $x = \sum c_i x_i$ is irreducible. Replacing x by $-x$ if necessary, we can assume that $c_0 \geq 0$ and there is at least one $c_i > 0$. Let $A \subset \{*, **, 0, \ldots, n\}$ be the set of indices i with $c_i < 0$, and let $y = \sum_{i \in A} c_i x_i$. Clearly, $\langle y, x - y \rangle$ is a sum of terms of the form $-c_i c_{i'}$ with $c_i < 0$ and $c_{i'} \geq 0$, so is nonnegative. Since x is irreducible, then $y = 0$, so actually all of the c_i are nonnegative.

Now, let $B \subset \{*, **, 0, \ldots, n\}$ be the set of indices i with $c_i \neq 0$. The pairing graph $\widehat{G}(B)$ must be connected, since otherwise x could be written as a sum of two nonzero elements that pair to zero.

If $c_i \in \{0, 1\}$ for all i, then $x = [B]$. Since $\widehat{G}(B)$ is connected, either x is already an interval, or $B = \{*, **, 0, \ldots, j\}$ for some j, but then $-\tau_0(x) = x_0 + \cdots + x_j$ is an interval.

If $c_i > 1$ for some i, let $z = [B]$, and consider the pairing $\langle z, x - z \rangle$. If $0 \notin B$, then let $j_0 = \min B$ and $j_1 = \max B$, so

$$\langle z, x - z \rangle = \left(\sum_{j_0 \leq i \leq j_1} (a_i - 2)(c_i - 1) \right) + c_{j_0} + c_{j_1} - 2.$$

Since $c_i \geq 1$ for each $i \in B$, and $a_i \geq 2$ for each i, this is nonnegative, contradicting the irreducibility of x. Therefore, $0 \in B$, so $c_0 \geq 1$. Since $\widehat{G}(B)$ is connected, then, $c_i \geq 1$ for all $0 \leq i \leq \max B$.

Now let j be the largest index with $c_j > 1$. Suppose there were some largest k, $0 \leq k < j$, with $c_k < c_j$, and let $w = x_{k+1} + \cdots + x_j$. Then

$$\langle w, x - w \rangle = \left(\sum_{k+1 \leq i \leq j} (a_i - 2)(c_i - 1) \right) + (c_{k+1} - c_k) + (c_j - c_{j+1}) - 2.$$

Since all of the c_i are at least 1, each of these terms is nonnegative, so this cannot happen. Therefore, $c_i \geq c_j > 1$ for any $0 \leq i < j$.

Finally, let $w_0 = x_0 + \cdots + x_j$, and let $w_1 = x_* + x_{**} + w_0$. Consider

$$\langle w_0, x - w_0 \rangle = \left(\sum_{0 \leq i \leq j} (a_i - 2)(c_i - 1) \right) - (c_* + c_{**} - c_0) + (c_j - 2) - c_{j+1},$$

$$\langle w_1, x - w_1 \rangle = \left(\sum_{0 \leq i \leq j} (a_i - 2)(c_i - 1) \right) + (c_* + c_{**} - c_0) + (c_j - 2) - c_{j+1}.$$

The first term of each of these sums is nonnegative and, for at least one of these sums, the second term is nonnegative. Moreover, $c_j \geq 2$, so the third term is also nonnegative.

The last term c_{j+1} is 0 or 1. Therefore, for both of these to be negative, the first three terms must be zero, and the last one must be negative. Since $c_i > 1$ for $i \leq j$, the first term can only be zero when $a_i = 2$ for $0 \leq i \leq j$, so $j < m$, where m is as in Definition 3.6. For the remaining terms to be zero or negative, $c_j = 2$, $c_0 = c_* + c_{**}$, and $c_{j+1} = 1$.

Since $j < m$, we can consider $-\tau_{j+1}(x) = \sum c_i' x_i$. For $i > j$, $c_i' = c_i \leq 1$ and $c_j' = 2c_{j+1} - c_j = 0$. Note that $c_{j+1}' = c_{j+1} = 1 \neq 0$. So c_i' vanishes for $i < j$ as well, since $-\tau_{j+1}(x)$ is still irreducible. Therefore, $-\tau_{j+1}(x) = \sum_{i>j} c_i x_i$, which is an interval. $\qquad\square$

We will also want to know which irreducible elements of $\Delta(p, q)$ are breakable. Since negation and the τ_j preserve breakability, it will suffice to know this for intervals:

Lemma 3.10 *An interval $x = [A]$ is breakable if there are at least two indices $i, k \in A$ with $a_i, a_k \geq 3$.*

Proof Suppose that $i, k \in A$ with $a_i, a_k \geq 3$. Fix some index J with $i \leq J < k$, and let $T_0 = \{j \in A \mid j \leq J\}$ and $T_1 = \{j \in A \mid J < j\}$. Since $i \in T_0$ and $k \in T_1$, $|[T_0]| \geq 3$ and $|[T_1]| \geq 3$. Also, since x is an interval, $j \in A$ for all $i \leq j \leq k$, so $J \in T_0$ and $J + 1 \in T_1$. Therefore, $\langle [T_0], [T_1] \rangle = -1$, and x is breakable. $\qquad\square$

Definition 3.11 When $[A_j]$ is unbreakable and has norm at least 3, let $z_j \in A_j$ be the unique element with $|z_j| \geq 3$.

Finally, let us determine when two D–type lattices are isomorphic.

Proposition 3.12 *If $\Delta(p, q) \cong \Delta(p', q')$, then $p = p'$ and $q = q'$.*

Proof Suppose L is a lattice isomorphic to $\Delta(p, q)$ for some p and q. To recover p and q from L, it suffices to recover the ordered sequence of norms $(|x_0|, |x_1|, \ldots, |x_n|)$ of the vertex basis, or equivalently to recover the weighted tree $\mathcal{T}$ corresponding to $\Delta(p, q)$. As in the proof of [10, Proposition 3.6], we will first recover the vertices with weight at least 3, and then fill in the vertices of weight 2. Let $R \subset L$ be the sublattice generated by the vectors of norm 2 in L. Every such element is irreducible as in Proposition 3.2. It then follows from Proposition 3.9 that R is generated by vertices with weight 2, since the involution τ_j does not change the norm, and that an interval of norm 2 merely has vertices of norm 2.

Let V_0 be the set of irreducible, unbreakable elements of L of norm at least 3, and let V be the quotient of V_0 by the equivalence $v \sim u$ whenever either $v - u \in R$ or $v + u \in R$. By Proposition 3.9 and Lemma 3.10, each irreducible, unbreakable element of L corresponds (up to sign and applications of the τ_j) to an interval containing a unique vertex of weight at least 3. We claim that the set V is in bijection with the vertices in $\mathcal{T}$ of weight at least 3. In fact, if $v - u$ or $v + u$ can be written as a sum of vectors of norm 2, the corresponding high-weight vertices must be the same: if we write $v = \sum v_i x_i$ and $\pm \tau_j(v) = \sum v_i' x_i$ (where $\pm \tau_j(v)$ is an interval), then the coefficient v_i' is nonzero for exactly one i with $|x_i| \geq 3$. Since applying any τ_j does not change the coefficient v_i when $|x_i| \geq 3$ except by a sign, the original coefficient v_i is also nonzero (in fact, ± 1) for exactly one i with $|x_i| \geq 3$. Similarly, there is exactly one j with $|x_j| \geq 3$ with u_j nonzero and equal to ± 1, where $u = \sum u_j x_j$. Therefore, one of $v - u$ or $v + u$ can be written as a sum of vertices of weight 2 if and only if the high-weight vertices are the same. This finishes the proof of the claim. Also, if $v \in L$ is a representative of some class in V, then $|v|$ is the weight of the corresponding vertex.

Let W be the set of indecomposable sublattices of R. Each element of W corresponds to a connected subgraph of $\mathcal{T}$, all of whose vertices have weight 2. As a lattice, each element $w \in W$ is isomorphic to either the root lattice A_{n_w} for some n_w, or to D_{n_w} if the corresponding subgraph contains all of x_*, x_{**}, x_0 and x_1. This last case happens for at most one $w \in W$. Now, form a bipartite graph $\mathcal{B}$ as follows: the vertex set is $V \cup W$, and there is an edge between $v \in V$ and $w \in W$ if, for some representative $\tilde{v} \in L$ of v and some element $\tilde{w} \in w$, $\langle \tilde{v}, \tilde{w} \rangle \neq 0$. This happens if and only if there is an edge in $\mathcal{T}$ between the vertex corresponding to v and some vertex of the subgraph corresponding to w. In $\mathcal{B}$, each $w \in W$ neighbors at most two vertices $v \in V$, or at most one if the corresponding sublattice is isomorphic to D_{n_w}.

We say an edge connecting $v \in V$ and $w \in W$ is *special* if w is isomorphic to A_3, and there exists a representative $\tilde{v}$ of v such that there exist exactly 4 vectors $\tilde{w} \in w$ satisfying $|\tilde{w}| = 2$ and $\langle \tilde{v}, \tilde{w} \rangle = -1$. It is easy to check that an edge is special if and only if $a_0 = 2$, $a_1 > 2$, w is generated by x_*, x_{**} and x_0, and v corresponds to x_1.

The graph $\mathcal{B}$ contains all of the information about how the blocks of vertices of weight 2 fit together with the other vertices in $\mathcal{T}$. Since the part of $\mathcal{T}$ corresponding to one of those blocks must be the Dynkin diagram of the corresponding sublattice, reconstructing $\mathcal{T}$ is straightforward. The vertex set of $\mathcal{T}$ is a copy of V together with a copy of V_w for each $w \in W$, where V_w is a set containing $\mathrm{rank}(w)$ vertices. The edges

in $\mathcal{T}$ are constructed as follows. If the sublattice w is isomorphic to A_{n_w}, connect the vertices in V_w together in a path P_w. For each nonspecial edge in $\mathcal{B}$ connecting v and w, connect v to one of the ends of P_w. For each w, at most two vertices in V need to be connected in this way, so they can always be connected so that no end of P_w has valency 3 in $\mathcal{T}$. If there is a special edge between v and w, connect the middle vertex of P_w to v. If the sublattice w is isomorphic to D_{n_w}, connect $n_w - 2$ of the vertices in V_w into a path, connect the remaining two to one end of the path, and if there is a $v \in V$ neighboring w in $\mathcal{B}$, connect it to the other end. Finally, for $v, v' \in V$, put an edge between them if there are representatives $\tilde{v}, \tilde{v}'$ with $\langle \tilde{v}, \tilde{v}' \rangle \neq 0$ and v, v' do not have a common neighbor in $\mathcal{B}$. For the weights, put a 2 on any vertex in V_w, and $|\tilde{v}|$ on a vertex $v \in V$, where $\tilde{v}$ is a representative of the class v. □

4 Changemaker lattices

According to Theorem 2.9, whenever $P(p, q)$ comes from positive integer surgery on a knot, $\Delta(p, q)$ is isomorphic to the orthogonal complement $(\sigma)^\perp$ for some changemaker vector $\sigma \in \mathbb{Z}^{n+4}$. A lattice is called a *changemaker lattice* if it is isomorphic to the orthogonal complement of a changemaker vector. In this section, we will prove some basic structural results about D–type lattices which are also changemaker lattices.

Write $(e_0, e_1, \ldots, e_{n+3})$ for the orthonormal basis of $\mathbb{Z}^{n+4}$, and write $\sigma = \sum_i \sigma_i e_i$. Since $\Delta(p, q)$ is indecomposable (Corollary 3.3), $\sigma_0 \neq 0$, otherwise $(\sigma)^\perp$ would have a direct summand $\mathbb{Z}$. So $\sigma_0 = 1$.

We will need several results from [10, Section 3] about changemaker lattices:

Definition 4.1 [10, Definition 3.11] The *standard basis* of $(\sigma)^\perp$ is the collection $S = \{v_1, \ldots, v_{n+3}\}$, where

$$v_j = \left(2e_0 + \sum_{i=1}^{j-1} e_i \right) - e_j$$

whenever $\sigma_j = 1 + \sigma_0 + \cdots + \sigma_{j-1}$, and

$$v_j = \left(\sum_{i \in A} e_i \right) - e_j$$

whenever $\sigma_j = \sum_{i \in A} \sigma_i$, with $A \subset \{0, \ldots, j-1\}$ chosen to maximize the quantity $\sum_{i \in A} 2^i$. A vector $v_j \in S$ is called *tight* in the first case, *just right* in the second case

as long as $i + 1 \in A$ whenever $i < j - 1$ and $i \in A$, and *gappy* if there is some index i with $i \in A$, $i < j - 1$, and $i + 1 \notin A$. In this situation, call i a *gappy index* for v_j.

Definition 4.2 For $v \in \mathbb{Z}^{n+4}$, $\mathrm{supp}\, v = \{i \mid \langle e_i, v \rangle \neq 0\}$ and $\mathrm{supp}^+ v = \{i \mid \langle e_i, v \rangle > 0\}$.

Lemma 4.3 [10, Lemma 3.12(3)] *If $|v_{k+1}| = 2$, then k is not a gappy index for any $v_j \in S$.*

Proof This follows from the maximality of the set A: if this did not hold, we could remove k from A and add $k + 1$, increasing the sum $\sum_{i \in A} 2^i$ but leaving the sum $\sum_{i \in A} \sigma_i$ unchanged. $\square$

Lemma 4.4 [10, Lemma 3.13] *Each $v_j \in S$ is irreducible.*

Lemma 4.5 [10, Lemma 3.15] *If $v_j \in S$ is breakable, it is tight.*

Lemma 4.6 [10, Lemma 3.14(2)] *If $v_t \in S$ is tight, $j > t$ and $v_j = e_t + e_{j-1} - e_j$, then $v_t + v_j$ is irreducible.*

Lemma 4.7 *If $v_t \in S$ is tight, $j > t$ and $v_j = e_0 + \cdots + e_{t-1} + \left(\sum_{i \in A} e_i\right) - e_j$ for some $A \subseteq \{t + 1, \ldots, j - 1\}$, then $v_j - v_t$ is irreducible.*

Proof For contradiction, suppose there exist x and y with

$$v_j - v_t = -e_0 + e_t + \left(\sum_{i \in A} e_i\right) - e_j = x + y$$

such that $\langle x, y \rangle \geq 0$. Write $x = \sum x_i e_i$ and $y = \sum y_i e_i$. Since $|x_i + y_i| \leq 1$ for all i, $x_i y_i \leq 0$ for all i. Since $\langle x, y \rangle = \sum x_i y_i \geq 0$, we must have $x_i y_i = 0$ for each i. Observe only e_0 and e_j have negative coefficients in $v_j - v_t$. If $x_0 = x_j = 0$ (resp. $y_0 = y_j = 0$), then since x (resp. y) is in $\langle \sigma \rangle^\perp$ and has nonnegative coefficients, $x = 0$ (resp. $y = 0$). If $x_0 = 0$ and $x_j = -1$, since $\sum \sigma_i y_i = 0$, $y = e_s - e_0$ for some $0 < s < t$. Since $y = v_j - v_t - x$, $x_s = -1$ and $x_s y_s \neq 0$, a contradiction. If $x_0 = -1$ and $x_j = 0$, we get a contradiction just as in the previous case. $\square$

If we have a lattice L isomorphic to both a D–type lattice $\Delta(p, q)$ and a changemaker lattice $(\sigma)^\perp$, it has both the vertex basis $x_*, x_{**}, x_0, \ldots, x_n$ and the standard basis $v_1, \ldots, v_{n+3}$. Since each of the v_i is irreducible, we can use Proposition 3.9 to constrain its expression in terms of the x_i. In some cases, we can say more:

Lemma 4.8 If $\Delta(p,q)$ is isomorphic to $(\sigma)^\perp$ for some changemaker $\sigma \in \mathbb{Z}^{n+4}$, then there is an isomorphism $\phi\colon \Delta(p,q) \xrightarrow{\sim} (\sigma)^\perp$ that sends x_* to v_1 and x_{**} to v_f for some index f. Furthermore, if $a_0 = 2$, then $f = 3$ and we can also choose ϕ to send x_0 to v_2.

Proof Let $\phi_0\colon \Delta(p,q) \to (\sigma)^\perp$ be an isomorphism. Both x_* and x_{**} have norm 2, and $\langle x_*, x_{**} \rangle = 0$. The only elements of $(\sigma)^\perp$ of norm 2 have the form $e_i - e_j$ for indices i and j with $\sigma_i = \sigma_j$. Since the entries of σ are nondecreasing, actually $\sigma_i = \sigma_k = \sigma_j$ for any $\min(i,j) \le k \le \max(i,j)$, so in particular $\sigma_i = \sigma_j = \sigma_m = \sigma_{m+1}$ for $m = \min(i,j)$. We define an automorphism ψ of $\mathbb{Z}^{n+4}$ as follows. If $|j-i| = 1$, let

$$\psi(e_i) = e_m, \quad \psi(e_j) = e_{m+1}, \quad \psi(e_l) = e_l \quad \text{when } l \notin \{i,j\}.$$

If $|j-i| > 1$, then $\psi(e_i)$ and $\psi(e_j)$ are as above, and

$$\psi(e_{m+1}) = e_{\max(i,j)}, \quad \psi(e_l) = e_l \quad \text{when } l \notin \{i,j,m+1\}.$$

Since ψ fixes σ, it restricts to an automorphism $\bar\psi$ of $(\sigma)^\perp$. Consider $\bar\psi \circ \phi_0$, which sends $\phi_0^{-1}(e_i - e_j)$ to the standard basis vector v_{m+1}. Since $\langle x_*, x_{**} \rangle = 0$, this process can be done separately for x_* and x_{**}, making them both sent to standard basis vectors v_e, v_f. By precomposing with the automorphism of $\Delta(p,q)$ that switches x_* and x_{**}, we can assume $e < f$, and then Lemma 3.8 ensures that $e = 1$, because otherwise $\langle v_{e-1}, x_* \rangle = -1$ while $\langle v_{e-1}, x_{**} \rangle = 0$.

When $a_0 = 2$, x_0 has norm 2 and pairing -1 with both x_* and x_{**}, so $x_0 = e_1 - e_{f-1}$ or $-e_0 + e_f$. It follows that $\sigma_0 = \sigma_1 = \cdots = \sigma_{f-1} = \sigma_f$. Therefore, $v_2 = e_1 - e_2$, so $\langle v_2, v_1 \rangle = -1$. By Lemma 3.8, $\langle v_2, v_f \rangle$ is odd. This can happen only if $f = 3$, so $v_f = e_3 - e_2$, and $x_0 = e_1 - e_2$ or $-e_0 + e_3$. Since $\sigma_0 = \sigma_1 = \sigma_2 = \sigma_3$, any permutation of $\{0,1,2,3\}$ induces a permutation of $\{e_0, e_1, e_2, e_3\}$, hence gives an automorphism of $(\sigma)^\perp$. If $x_0 = e_1 - e_2$, we are done. If $x_0 = -e_0 + e_3$, we can precompose ϕ_0 with the automorphism exchanging x_* and x_{**}, and postcompose ϕ_0 with the automorphism given by the permutation $(02)(13)$. $\qquad\square$

From now on, let $L = (\sigma)^\perp$ be a changemaker lattice isomorphic to some D–type lattice, and identify $x_*, x_{**}, x_0, \ldots, x_n$ with elements of L according to an isomorphism chosen as in Lemma 4.8. Since $\sigma_0 = 1$, by Lemma 4.8, we see that

(13) $\sigma_1 = \sigma_0 = 1.$

Lemma 4.9 *For any $v_j \in S$, we have $j - 1 \in \operatorname{supp} v_j$.*

Proof This is just the second part of [10, Lemma 3.12(1)], plus the fact that $v_1 = e_0 - e_1$. □

Lemma 4.10 *If $a_0 = 2$, then $a_1 \geq 3$.*

Proof Suppose $a_0 = 2$. Note that x_1 satisfies $\langle x_1, x_0 \rangle = -1$ and $\langle x_1, x_* \rangle = \langle x_1, x_{**} \rangle = 0$. Therefore, if we write $x_1 = \sum_{i=0}^{n+3} c_i e_i$, then $c_0 = c_1 = c_2 - 1 = c_3 - 1$. Also, x_1 is not in the span of x_0, x_* and x_{**}, so there is some $j > 3$ with $c_j \neq 0$. Therefore, at least three of the c_i are nonzero, so $a_1 = |x_1| \geq 3$. □

In particular, this means that $m \leq 1$ in the notation of Definition 3.6. So we only ever need to use the involution τ_j for $j \in \{0, 1\}$ to make irreducible elements into intervals.

Lemma 4.11 *If v_j is a standard basis vector for $j \neq 1, f$ and v_j is not tight, then $\langle v_j, v_1 \rangle = \langle v_j, v_f \rangle \in \{-1, 0\}$. If v_j is tight, $j = f - 1$ and $\langle v_j, v_1 \rangle = -\langle v_j, v_f \rangle = 1$.*

Proof Using Lemma 4.3, $\operatorname{supp} v_j \cap \{0, 1\} \neq \{0\}$, so $\langle v_j, v_1 \rangle \in \{0, -1\}$ unless v_j is tight. Similarly, either $j \neq f - 1$ and $\operatorname{supp} v_j \cap \{f - 1, f\} \neq \{f - 1\}$, or $j = f - 1$ and $\operatorname{supp} v_j \cap \{f - 1, f\} = \{f - 1\}$. In either case, $\langle v_j, v_f \rangle \in \{0, -1\}$. By Lemma 3.8, these two values have the same parity, so they must be equal if v_j is not tight. If v_j is tight, $\langle v_j, v_1 \rangle$ is 1, so $\langle v_j, v_f \rangle$ is -1. Therefore, $j = f - 1$. □

Corollary 4.12 *There is at most one tight vector in S. If $a_0 = 2$, there is no tight vector.*

Proposition 4.13 *If $a_0 \geq 3$, then for any standard basis vector v_j, either v_j is an interval or $\tau_0(v_j)$ is. If $a_0 = 2$, the same conclusion holds with τ_1 instead of τ_0. Furthermore, this interval contains $*$ or $**$ if and only if j is 1 or f or v_j is tight. If $a_0 = 2$, then this interval contains 0 if and only if $j = 2$.*

Proof First, suppose that $a_0 \geq 3$, so $m = 0$ and Proposition 3.9 says that one of v_j, $-v_j$, $\tau_0(v_j)$ or $-\tau_0(v_j)$ is an interval. It remains only to show that we do not ever need to use a negation. If $j \in \{1, f\}$, $v_j \in \{x_*, x_{**}\}$ is an interval. Therefore, suppose $j \neq 1, f$. If v_j is tight, by Lemma 4.11 $\langle v_j, y_0 \rangle = 0$, where $y_0 = x_* + x_{**}$ is defined in Definition 3.6. Since τ_0 is the reflection in the line spanned by y_0, we get that

$\tau_0(v_j) = -v_j$ and $-\tau_0(v_j) = v_j$, and we are done because we do not need to use the negation or $-\tau_0$. If v_j is not tight, Lemma 4.11 says that $\langle v_j, x_* \rangle = \langle v_j, x_{**} \rangle \in \{0, -1\}$. If $\langle v_j, x_* \rangle = 0$, we get $\langle v_j, y_0 \rangle = 0$. We are done by the same argument as before. If $\langle v_j, x_* \rangle = \langle v_j, x_{**} \rangle = -1$, we have

$$\langle \tau_0(v_j), x_* \rangle = \langle \tau_0(v_j), \tau_0(x_{**}) \rangle = \langle v_j, x_{**} \rangle = -1$$

and similarly $\langle \tau_0(v_j), x_{**} \rangle = -1$. Therefore, if $-v_j$ or $-\tau_0(v_j)$ were an interval $[A]$, it would satisfy

$$\langle [A], x_* \rangle = \langle [A], x_{**} \rangle = 1.$$

This can only happen if $*$, $**$ and 0 are all in A, so $[A]$ is not an interval.

If $a_0 = 2$, then $(v_1, v_2, v_3) = (x_*, x_0, x_{**})$, all of which are intervals, so we can assume $j > 3$. By Corollary 4.12, v_j is not tight, so Lemma 4.11 implies that $\langle v_j, v_1 \rangle = \langle v_j, v_3 \rangle$. Since $|v_1| = |v_2| = |v_3| = 2$, Lemma 4.3 implies that $\operatorname{supp} v_j \cap \{0, 1, 2, 3\}$ is either $\{0, 1, 2, 3\}$, $\{2, 3\}$ or $\varnothing$. In any case, $\langle v_j, v_1 \rangle = \langle v_j, v_3 \rangle = 0$. Therefore, $\tau_0(v_j) = -v_j$. This means that, for $j > 3$, one of $v_j, -v_j, \tau_1(v_j)$ or $-\tau_1(v_j)$ must be an interval. We can assert that 0 is never in this interval, since v_j has zero pairing with $v_1 = x_*$ and $v_3 = x_{**}$. If 1 is not in this interval, then also $\langle v_j, x_0 \rangle = 0$, so $\langle v_j, y_1 \rangle = 0$. Since τ_1 is the reflection in the plane spanned by y_0, y_1, $\tau_1(v_j) = -v_j$, and we are done. It remains only to show that if $[A]$ is an interval containing 1 but not 0, then neither $-[A]$ nor $-\tau_1([A])$ can be a standard basis vector, which holds because in this case

$$\langle -[A], x_0 \rangle = \langle -\tau_1([A]), x_0 \rangle = 1,$$

and by Lemma 4.11 for any nontight standard basis vector this pairing is either 0 or -1. □

From now on, let τ denote τ_0 if $a_0 \geq 3$, or τ_1 if $a_0 = 2$. For each standard basis vector v_j, let $[A_j]$ be an interval for which either $v_j = [A_j]$ or $v_j = \tau([A_j])$, and let $\epsilon_j = 1$ if $v_j = [A_j]$, and -1 otherwise. When $a_0 \geq 3$, for all j except 1 and f, ϵ_j is uniquely determined; when $a_0 = 2$, for $j \neq 1, 2, 3$, ϵ_j is uniquely determined. However, ϵ_1 and ϵ_f can be freely chosen since τ exchanges v_1 and v_f. To resolve this ambiguity, we will choose $[A_1]$ and $[A_f]$ to make $\epsilon_1 = -\epsilon_f = \epsilon_{f-1}$, and, when $a_0 = 2$, choose $\epsilon_2 = 1$. This makes $A_1 = A_f$, and, when v_{f-1} is tight, $A_1 = A_f = \{*\}$ whenever $* \in A_{f-1}$, and $\{**\}$ otherwise. By Proposition 4.13, this means that either $*$ or $**$ is in none of the A_j, which gives the following fact:

Lemma 4.14 *If A_j and A_k are disjoint, $\langle[A_j],[A_k]\rangle$ is either 0 or -1. Otherwise, $\langle[A_j],[A_k]\rangle$ is either $\|[A_j \cap A_k]\| - 1$ or $\|[A_j \cap A_k]\| - 2$, where $\|[A_j \cap A_k]\|$ is either $\|[A_j]\|$ or 2 if $[A_j]$ is unbreakable.*

Proof For simplicity in this proof, call whichever of $*$ or $**$ occurs in some of the A_j -1. If A_j and A_k are disjoint, either there is no index i with $x_i \in A_j$ and $x_{i\pm1} \in A_k$, in which case they pair to 0, or there is exactly one, in which case they pair to -1. If A_j and A_k do have nonempty intersection, either $\min A_j = \min A_k$ or $\max A_j = \max A_k$, in which case $\langle[A_j],[A_k]\rangle$ is $\|[A_j \cap A_k]\| - 1$, or this does not hold, in which case $\langle[A_j],[A_k]\rangle = \|[A_j \cap A_k]\| - 2$. To compute $\|[A_j \cap A_k]\|$, note that for any $A \subset \{-1, 0, \ldots, n\}$ with $\hat{G}(A)$ connected,

$$(14) \qquad\qquad \|[A]\| = 2 + \sum_{i \in A}(a_i - 2),$$

so removing indices with $a_i = 2$ from A does not change $\|[A]\|$. If $[A_j]$ is unbreakable, by Lemma 3.10 A_j has at most one element i with $a_i \geq 3$, so $\|[A_j \cap A_k]\|$ is either $\|[A_j]\|$ if A_k contains the index for which $a_i \geq 3$, or 2 if it does not. $\qquad\square$

Definition 4.15 Following Greene [10], say that two intervals $[A_j]$ and $[A_k]$ are *distant* if they do not intersect and $\langle[A_j],[A_k]\rangle = 0$, that they are *consecutive* and write $A_j \dagger A_k$ if they do not intersect and $\langle[A_j],[A_k]\rangle = -1$, and that they share a *common endpoint* if they intersect and $\langle[A_j],[A_k]\rangle = \|[A_j \cap A_k]\| - 1$. If A_j and A_k share a common endpoint and $A_j \subset A_k$, write $A_j \prec A_k$. Say that two intervals *abut* if they are either consecutive or share a common endpoint. Write $A_j \pitchfork A_k$ if $A_j \cap A_k \neq \varnothing$ and they do not share a common end point.

Definition 4.16 Let the *intersection graph* $G(T)$, where $T \subset S$ is a subset of the standard basis, be the graph with vertex set T, and with an edge between v_i and v_j (write $v_i \sim v_j$) whenever $[A_i]$ and $[A_j]$ abut. If $v_i \sim v_j$ and $i < j$, we say v_i is a *smaller neighbor* of v_j.

For the intersection graph to be a useful concept, we need to somehow relate abutment of intervals to pairings in the lattice. First, we need to know how the pairings between intervals relate to the pairings between standard basis vectors:

Lemma 4.17 *For any two standard basis vectors v_i, v_j with $\{i, j\} \neq \{1, f\}$, we have $\langle v_i, v_j\rangle = \pm\langle[A_i],[A_j]\rangle$. Furthermore, if both v_i and v_j have norm at least 3, then $\langle v_i, v_j\rangle = \epsilon_i\epsilon_j\langle[A_i],[A_j]\rangle$.*

Proof If $\epsilon_i = \epsilon_j$, $\langle v_i, v_j \rangle = \langle [A_i], [A_j] \rangle$ since τ is a reflection. Otherwise,

$$\langle [A_i], [A_j] \rangle = \langle \tau(v_i), v_j \rangle = \langle v_i, \tau(v_j) \rangle,$$

so the desired results will hold as long as either $\tau(v_i) = -v_i$ or $\tau(v_j) = -v_j$. It follows from the proof of Proposition 4.13 that the only intervals (corresponding to standard basis vectors) that are not simply negated by τ are x_*, x_{**}, intervals with left endpoint 0 and, if $a_0 = 2$, intervals with left endpoint 1. Note that any interval with left endpoint 0 is not tight since the tight vector pairs differently with x_* and x_{**}.

First, consider the case $a_0 \geq 3$. If one of $[A_i]$ or $[A_j]$ is either x_* or x_{**} and the other is an interval with left end 0 (which in particular is neither tight nor x_* or x_{**}), then the result follows from the facts that $\tau(x_*) = x_{**}$ and Lemma 4.11. If both $[A_i]$ and $[A_j]$ are intervals starting at 0, then neither one is tight, by Lemma 4.5 both are unbreakable. Therefore, since $a_0 \geq 3$ and 0 is in both A_i and A_j, every other $k \in A_i \cup A_j$ has $a_k = 2$; see Lemma 3.10. Assume $\langle v_i, v_j \rangle \neq \pm \langle [A_i], [A_j] \rangle$, then it follows from (14) that $|v_i| = |v_j| = a_0$. Note that

$$\langle v_i, v_j \rangle = \langle \tau([A_i]), [A_j] \rangle = \langle [A_i], \tau([A_j]) \rangle$$

and $\tau([A_i]) = -[A_i] - x_* - x_{**}$, and we can compute that $\langle v_i, v_j \rangle$ is either $3 - a_0$ if $A_i \neq A_j$, or $2 - a_0$ if $A_i = A_j$. We can check that neither of these can occur: for any two standard basis vectors, $\langle v_i, v_j \rangle \geq -1$, so a_0 is either 3 or 4, and $\langle v_i, v_j \rangle$ is either 0 or -1. By Lemma 4.11, $\langle v_i, v_1 \rangle = \langle v_j, v_1 \rangle = \langle v_i, v_f \rangle = \langle v_j, v_f \rangle = -1$, so $1 \in \text{supp}^+ v_i \cap \text{supp}^+ v_j$. Therefore, $\langle v_i, v_j \rangle = 0$, and $a_0 = 3$. However, using Lemma 4.9, the only possible standard basis vectors of norm 3 that have pairing -1 with both v_1 and v_f are $e_1 + e_{f-2} - e_{f-1}$ and $e_1 + e_f - e_{f+1}$, but these have pairing 1 with each other. This contradiction shows that there cannot be two standard basis vectors corresponding to intervals with left endpoint 0. Therefore, given any two standard basis vectors of norm at least 3 (so neither one is x_* or x_{**}), one of them must be negated by τ, which proves the last statement in this case.

When $a_0 = 2$, the situation is similar. Using Proposition 4.13, the only standard basis intervals that are not negated by τ are x_*, x_{**}, x_0 and intervals that start at 1. The vector x_0 is fixed by τ. For any interval $[A]$ with left end 1, $\tau([A]) = -[A] - x_* - x_{**} - 2x_0$, so $\langle x_*, [A] \rangle = \langle x_*, \tau([A]) \rangle = 0$, and this also holds for x_{**}. It only remains to rule out the case in which $[A_i]$ and $[A_j]$ are both intervals with left end 1. In this case, again v_i and v_j are unbreakable, so 1 is the only index $k \in A_i \cup A_j$ with $a_k \geq 3$, $|v_i| = |v_j| = a_1$, and $\langle v_i, v_j \rangle$ is either $3 - a_1$ or $2 - a_1$, always ≤ 0.

However,

$$\langle v_i, x_* \rangle = \langle v_i, x_{**} \rangle = \langle v_j, x_* \rangle = \langle v_j, x_{**} \rangle = 0$$

and

$$\langle v_i, x_0 \rangle = \langle v_j, x_0 \rangle = -1,$$

so $\mathrm{supp}^+(v_i) \cap \{0, 1, 2, 3\} = \mathrm{supp}^+(v_j) \cap \{0, 1, 2, 3\} = \{2, 3\}$. This forces $\langle v_i, v_j \rangle \geq 1$, a contradiction. Again, since x_*, x_{**}, and x_0 all have norm 2, at least one of any pair of standard basis vectors of norm at least 3 is negated by τ, and the conclusion follows. □

Because all of the A_j are subintervals of one interval and the pairings between the $[A_j]$ are, up to sign, the same as the pairings between standard basis vectors, many of the results from [10] about the intersection graph carry over unchanged to this situation. Most importantly, [10, Lemma 4.4] holds unchanged:

Lemma 4.18 [10, Lemma 4.4] *If v_i and v_j are unbreakable standard basis vectors of norm at least 3, then $|\langle v_i, v_j \rangle| \leq 1$, with equality if and only if $[A_i]$ and $[A_j]$ are consecutive and $\langle v_i, v_j \rangle = -\epsilon_i \epsilon_j$.*

The proof of Lemma 4.18, which will not be repeated here, is identical to the one Greene gives. The overall strategy is similar to the one used in Lemma 4.17: showing that, for unbreakable standard basis vectors v_i, v_j of norm at least 3, if $A_i \cap A_j$ contains an index k with $a_k \geq 3$ then the pairing $\langle v_i, v_j \rangle$ will be too large given the form of the standard basis vectors. The proof actually shows:

Corollary 4.19 *If v_i and v_j are unbreakable standard basis vectors of norm at least 3, then $z_i \neq z_j$, where z_i is as defined in Definition 3.11.*

Corollary 4.20 *If $\{i, j\} \neq \{1, f\}$ and $v_i \sim v_j$, then $\langle v_i, v_j \rangle \neq 0$. If $v_i \nsim v_j$ and v_i and v_j are both unbreakable, then $\langle v_i, v_j \rangle = 0$.*

Proof If $v_i \sim v_j$, then $\langle [A_i], [A_j] \rangle$ is equal to either -1 or $|[A_i \cap A_j]| - 1 \neq 0$. It follows from Lemma 4.17 that $\langle v_i, v_j \rangle \neq 0$.

If $v_i \nsim v_j$, by Lemma 4.17 we only need to show $\langle [A_i], [A_j] \rangle = 0$. If A_i and A_j are distant, we are done. If $A_i \pitchfork A_j$, by Lemma 4.14 and its proof

$$\langle [A_i], [A_j] \rangle = |[A_i \cap A_j]| - 2.$$

By Corollary 4.19, $A_i \cap A_j$ does not contain any vertex with norm ≥ 3, so $|[A_i \cap A_j]| = 2$ and our conclusion holds. □

In particular, when S contains no breakable vectors, $G(S)$ is almost the same as $\widehat{G}(S)$, with the only differences being that $G(S)$ has an edge between v_1 and v_f while $\widehat{G}(S)$ does not.

Corollary 4.21 *At most one unbreakable vector other than v_1 or v_f neighbors v_1 in $G(S)$. The same holds for v_f and, when $a_0 = 2$, for v_2.*

Proof If $v_j \sim v_1$ or $v_j \sim v_f$, then using Proposition 4.13 and Corollary 4.20 we get either $j \in \{1, f\}$ or $0 \in A_j$. If $a_0 \geq 3$ and v_j is unbreakable, then $z_j = 0$, so no other unbreakable standard basis vector corresponds to an interval containing 0. When $a_0 = 2$, by Proposition 4.13 and Corollary 4.20 v_2 is the only neighbor of v_1 or v_3, and when $v_j \sim v_2$ for some $j \notin \{1, f\}$ and v_j is unbreakable, $z_j = 1$.   □

Once we have Lemma 4.18, almost all of the remaining results of [10, Section 4] carry over. In particular, we have versions of [10, Lemmas 4.8 and 4.10], with identical proofs.

Definition 4.22 A *claw* in a graph G is a set of four vertices $(v; x, y, z)$ with v adjacent to x, y and z, and no two of x, y and z adjacent to each other.

Lemma 4.23 [10, Lemma 4.8] *$G(S)$ has no claws.*

Let $\overline{S}$ be the set of unbreakable elements of S, and let $S_g = \{v_1, v_2, \ldots, v_g\}$.

Definition 4.24 A triple $(v_i, v_j, v_k) \in (\overline{S})^3$ is a *heavy triple* if each of v_i, v_j and v_k have norm at least 3, and any two of them are connected by a path in $G(\overline{S})$ that does not pass through the third. If the heavy triple (v_i, v_j, v_k) spans a triangle, we say it is a *heavy triangle*.

Lemma 4.25 [10, Lemma 4.10] *No triple is heavy.*

Proof Suppose that (v_i, v_j, v_k) is heavy, with $z_i < z_j < z_k$. Then any path from v_i to v_k in $G(\overline{S})$ would contain an unbreakable interval containing z_j, contradicting Corollary 4.19.   □

The last of Greene's results that we will need to characterize the D–type lattices that are isomorphic to changemaker lattices is a description of the cycles that can occur in $G(S)$. However, this will require more modification since Greene's proof relies on the intervals $[A_i]$ being linearly independent, which fails in this situation since $[A_1] = [A_f]$. Luckily, this is the only problem:

Lemma 4.26 *$[A_2], [A_3], \ldots, [A_{n+2}]$ and $[A_{n+3}]$ are linearly independent.*

Proof We remind the reader we are using the notation that $L = (\sigma)^{\perp}$ is a changemaker lattice isomorphic to some D–type lattice; also that we identify $x_*, x_{**}, x_0, \ldots, x_n$ with elements of L according to an isomorphism chosen as in Lemma 4.8. Consider the map $\pi \colon L \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$ given by $\pi(x) = x - \tau(x)$. For any $x \in L \otimes \mathbb{Q}$, $\pi(\tau(x)) = -\pi(x)$, so in particular $\pi([A_i]) = \pm\pi(v_i)$ for each standard basis vector v_i.

When $a_0 \geq 3$, τ is the reflection in a line, so $\operatorname{im} \pi$ has dimension $n + 2$. Since $(v_1, v_2, \ldots, v_{n+3})$ spans $L \otimes \mathbb{Q}$, we get that $(\pi(v_1), \pi(v_2), \ldots, \pi(v_{n+3}))$ spans $\operatorname{im} \pi$. However, $\pi(v_1) = -\pi(v_f)$, so actually $\pi(v_2), \ldots, \pi(v_{n+3})$ suffice. Since $\operatorname{im} \pi$ has dimension $n + 2$, this means that $\pi(v_2), \ldots, \pi(v_{n+3})$ are linearly independent. Since $\pi([A_i]) = \pm\pi(v_i)$ for each i, a linear dependence among $[A_2], \ldots, [A_{n+3}]$ would induce one among $\pi(v_2), \ldots, \pi(v_{n+3})$, and the conclusion follows.

When $a_0 = 2$, τ is the reflection in a plane, so $\operatorname{im}(\pi)$ has dimension $n + 1$, and a basis is $(\pi(v_3), \pi(v_4), \ldots, \pi(v_{n+3}))$. The same argument as before gives that $[A_3], [A_4], \ldots, [A_{n+3}]$ are linearly independent. It remains only to be seen that $[A_2] = v_2$ is not in the span of $([A_3], \ldots, [A_{n+3}])$. Note that $[A_3] = v_1$ and, for $j > 3$, $\tau(v_j) = -v_j$ or $-v_1 - 2v_2 - v_3 - v_j$, so

$$\operatorname{span}([A_3], \ldots, [A_{n+3}]) \subset \operatorname{span}(v_1, v_1 + 2v_2 + v_3, v_4, \ldots, v_{n+3}),$$

and the right side does not contain v_2. □

Once we have this, a proof identical to that of Greene's Lemma 3.8 [10] gives:

Lemma 4.27 *Any simple cycle in $G(\{v_2, v_3, \ldots, v_{n+3}\})$ induces a complete subgraph.*

Actually, since $[A_1] = [A_f]$, for any $j \notin \{1, f\}$, $v_j \sim v_1$ if and only if $v_j \sim v_f$, so the same statement holds for $G(\{v_1, v_2, \ldots, v_{f-1}, v_{f+1}, \ldots, v_{n+3}\})$.

Lemma 4.28 *For any simple cycle $C \subset G(S)$, one of the following holds:*

- *The vertex set $V(C)$ induces a complete subgraph of $G(S)$.*
- *C has exactly four vertices, of which three are v_1, v_f and v_{f-1}, and v_{f-1} is breakable.*

Proof If C contains exactly 3 vertices, C is a complete subgraph. So we assume C contains at least 4 vertices. If either v_1 or v_f is not in C, the previous lemma says that

C induces a complete subgraph. Otherwise, $v_1, v_f \in V(C)$. Since any neighbor (other than v_1) of v_f also neighbors v_1, v_1 must have two neighbors v_j and v_k also in C, which will also neighbor v_f. By Corollary 4.21, one of these, say v_k, is breakable. By Lemmas 4.5 and 4.11, $k = f - 1$, and neither v_1 nor v_f has any other neighbors. This means that the vertices of C other than v_1, v_f, v_j and v_{f-1} all lie on a path γ for which one end neighbors v_j and the other neighbors v_{f-1}. It remains to see only that γ is empty. Since v_f neighbors both v_j and v_{f-1}, if γ were nonempty we could form a new simple cycle going from v_j to v_{f-1} along γ, then to v_f and back to v_j. This does not contain v_1, so induces a complete subgraph. This means that any vertex of γ neighbors v_f, so γ is empty.   □

Lemma 4.29 *Any cycle in $G(S)$ has three vertices, unless it contains a breakable vector, in which case it can contain* 4.

Proof By Corollary 4.21 and Lemma 4.28, we can assume that $V(C)$ does not contain both v_1 and v_f and that $V(C)$ induces a complete subgraph. If v_{f-1} is breakable and in $V(C)$, then $V(C) \setminus \{v_{f-1}\}$ will still induce a complete subgraph, so it suffices to assume that $V(C)$ contains no breakable vector. If $V(C)$ had more than two vectors of norm at least 3, any three of them would form a heavy triple. Since $V(C)$ induces a complete subgraph, any two vectors of norm 2 in $V(C)$ are of the form $v_i = e_{i-1} - e_i$ and $v_{i+1} = e_i - e_{i+1}$ for some index i. However, any other standard basis vector v_j that neighbored both v_i and v_{i+1} would have to have a gappy index at either $i-1$ or i, which cannot happen by Lemma 4.3 because v_i and v_{i+1} both have norm 2. Therefore, $V(C)$ has at most one vector of norm 2, and at most two vectors of norm at least 3, so has at most 3 vectors overall. Since C is a cycle, it must have exactly 3. □

The main use we will have for this characterization of cycles is the following lemma that restricts the possible forms of gappy vectors, which is [10, Lemma 7.1]. As usual, the proof will not be repeated.

Lemma 4.30 [10, Lemma 7.1] *Suppose that $v_g \in S$ is gappy, and that S contains no breakable vector. Then v_g is the unique gappy vector, $v_g = -e_g + e_{g-1} + \cdots + e_j + e_k$ for some $k + 1 < j < g$, and v_k and v_{k+1} belong to distinct components of $G(S_{g-1})$.*

5 Blocked vectors

In this section, we define blocked vectors and prove some lemmas about blocked vectors. These lemmas will be useful in the later sections.

Definition 5.1 Let $m \in \{1, 2, \ldots, n+2\}$. A vector $v_i \in S_m$ is (m, N)–*blocked* if $v_i \nsim v_j$ for any $m < j \leq N$. Otherwise, v_i is (m, N)–*open*. When $N = n + 3$ and $S_N = S$, we simply say v_i is m–*blocked* or m–*open*.

Lemma 5.2 When v_{f-1} is unbreakable, v_1 and v_f are f–blocked.

Proof If $a_0 \geq 3$, by Corollary 4.19 A_{f-1} is the unique unbreakable interval containing x_0, so v_1, v_f are f–blocked. If $a_0 = 2$, then $f = 3$ and $|v_1| = |v_2| = |v_3| = 2$ by Lemma 4.8. Since $0, 1, 2$ cannot be gappy indices, no vector other than v_2 can neighbor v_1 and v_3 at the same time, so v_1 and v_3 are 3–blocked by Lemma 4.11. □

Below, we introduce the notion of an (m, N)–blocking neighbor. We will see in Lemma 5.5 that a standard basis vector is (m, N)–blocked if it has two (m, N)–blocking neighbors that do not abut each other.

Definition 5.3 Let $v_i, v_j \in S_m$ be such that $v_j \sim v_i$. We say v_j is an (m, N)–*blocking neighbor of* v_i if one of the following holds:

(1) v_j is (m, N)–blocked.

(2) $|v_i| \geq 3$, $|v_j| \geq 3$, both unbreakable; v_ℓ is unbreakable for $m < \ell \leq N$.

(3) $j = i \pm 1$ and $|v_i| = |v_j| = 2$.

(4) $j = i \pm 1$, $|v_{\max(i,j)}| = 2$ and $v_{\min(i,j)}$ is unbreakable; v_ℓ is (m, N)–blocked for $\ell < \min(i, j)$ and v_ℓ is unbreakable for $m < \ell \leq N$.

When $N = n + 3$ and $S_N = S$, we simply say v_j is an m–*blocking neighbor of* v_i.

Lemma 5.4 Suppose v_j is an (m, N)–blocking neighbor of v_i. If $v_i \sim v_k$ for some $m < k \leq N$, then $v_j \nsim v_k$.

Proof We prove this case by case.

(1) Since v_j is (m, N)–blocked, by definition $v_k \nsim v_j$.

(2) For contradiction, suppose $v_k \sim v_j$. Since $v_k \sim v_i$ and $v_k \sim v_j$, $|v_k| \geq 3$. Otherwise, if $|v_k| = 2$, v_k has only one smaller neighbor. Since we assume v_i, v_j and v_k are unbreakable, (v_i, v_j, v_k) is a heavy triangle; see Lemma 4.25.

(3) Suppose $v_k \sim v_j$ for contradiction. Assume $j = i + 1$. This assumption is made without loss of generality because the situation $v_k \sim v_i, v_j$ and $v_i \sim v_j$ is symmetric

in i and j. Since $v_k \sim v_j$ and $v_j = e_i - e_{i+1}$ ($i > 0$), exactly one of i and $i+1$ is in supp v_k. Since $|v_j| = 2$, $j - 1 = i$ cannot be a gappy index by Lemma 4.3. Therefore, $i \notin$ supp v_k and $i + 1 \in$ supp v_k. Since $v_k \sim v_i$ and $v_i = e_{i-1} - e_i$, we get $i - 1 \in$ supp v_k. This contradicts Lemma 4.3 since $|v_i| = 2$.

(4) Again, suppose $v_k \sim v_j$ for contradiction. Assume $j = i + 1$ without loss of generality, since the situation is symmetric in i and j. From item (3), we may assume $|v_i| \geq 3$ and $|v_j| = 2$. As in item (3), $i \notin$ supp v_k and $i + 1 \in$ supp v_k. Let $\ell = \min$ supp v_k. Since $v_k \sim v_i$ and $i \notin$ supp v_k, we get $\ell < i$.

If $\ell = 0$, let s be the first gappy index of v_k; then $s < i$, and we claim that $v_k \sim v_{s+1}$. Since $\langle v_k, v_{s+1} \rangle > 0$, if $v_k \nsim v_{s+1}$, it follows from Lemmas 4.14 and 4.17 and Corollary 4.20 that v_{s+1} is breakable, $A_k \pitchfork A_{s+1}$ and $\epsilon_k = \epsilon_{s+1}$. Thus, up to applying τ, $v_k - v_{s+1}$ becomes $[A_k] - [A_{s+1}]$, which is reducible since it is the signed sum of two distant intervals, a contradiction to Lemma 4.7. Since $v_k \sim v_{s+1}$ and $s + 1 \leq i$, we have $s = i - 1$ by our assumption. Since $|v_i| \geq 3$, we must have $\langle v_i, v_k \rangle \geq 2$. However, since v_i and v_k are both unbreakable, this contradicts Lemma 4.18.

If $\ell > 0$, then $\langle v_\ell, v_k \rangle = -1$. Since $v_k \nsim v_\ell$, it follows from Corollary 4.20 that v_ℓ is breakable, so $\ell = f - 1$. By Lemmas 4.17 and 4.14, $|v_k| = 3$. Using Lemma 4.9, we get $v_k = -e_k + e_{k-1} + e_\ell$. Since $\ell < i < m < k$, $k > \ell + 2$ so $k - 1 > \ell + 1$. Since $v_{\ell+1} = v_f = -e_{\ell+1} + e_\ell$, we have $\langle v_k, v_{\ell+1} \rangle = 1$, so $v_k \sim v_{\ell+1}$. If $\ell + 1 < i$, then $v_{\ell+1}$ is m–blocked by assumption, a contradiction. If $\ell + 1 = i$, $|v_i| = 2$, a contradiction to our assumption that $|v_i| \geq 3$. $\qquad\square$

Lemma 5.5 *If $v_i \in S_m$ has two (m, N)–blocking neighbors $v_{j_1}, v_{j_2} \in S_m$ such that $v_{j_1} \nsim v_{j_2}$, then v_i is (m, N)–blocked.*

Proof Suppose $v_k \sim v_i$ for some $m < k \leq N$. By Lemma 5.4, $v_{j_1} \nsim v_k$ and $v_{j_2} \nsim v_k$. However, this would imply $\{v_i; v_{j_1}, v_{j_2}, v_k\}$ is a claw. $\qquad\square$

Lemma 5.6 *Suppose that v_k is just right for all $k > m$, and that $v_j \in S_m$ is unbreakable. Then $v_i \in S_m$ is m–blocked if either of the following holds:*

(1) supp v_j *has at least 4 elements* $\geq i$.

(2) supp v_j *has at least 3 elements* $\geq i$, *and v_j is m–blocked.*

Proof In the first case, any v_k neighboring v_i with $k > m$ would have min supp $v_k \leq i$, hence $|$supp $v_k \cap$ supp $v_j| \geq 4$ and $\langle v_k, v_j \rangle \geq 2$, contradicting Lemma 4.18. In the second case if $v_i \sim v_k$ for some $k > m$, $\langle v_k, v_j \rangle \geq 1$, so $v_k \sim v_j$, a contradiction. $\qquad\square$

Lemma 5.7 *Suppose v_i is (m, N)–blocked for all $i < m$. If S_{m-1} contains at least one unbreakable vector of norm ≥ 3, then $|v_j| = 2$ for all $m < j \leq N$.*

Proof We induct on $N - m$. The base case $N - m = 0$ is trivial. Inductively, suppose the lemma holds whenever $N - m \leq k$ for any nonnegative integer k. The following shows that the lemma also holds when $N - m = k + 1$.

Since v_i is (m, N)–blocked for all $i < m$, either $\min \operatorname{supp} v_{m+1} = 0$ or $|v_{m+1}| = 2$. Since S_{m-1} contains some unbreakable vector v_i of norm ≥ 3, $\min \operatorname{supp} v_{m+1} = 0$ would imply $v_{m+1} \sim v_i$, but v_i is m–blocked. Therefore, $|v_{m+1}| = 2$.

Since S_{m-1} contains a high norm vector, v_m has some smaller neighbor v_s, which is (m, N)–blocked by assumption. By Definition 5.3(1) and (4), v_s and v_{m+1} are $(m+1, N)$–blocking neighbors of v_m. Since $|v_{m+1}| = 2$ and $s < m$, $v_s \nsim v_{m+1}$. Hence, by Lemma 5.5, v_m is $(m+1)$–blocked.

Since $N - (m + 1) \leq k$ and v_i is $(m+1, N)$–blocked for all $i < m + 1$, the inductive assumption implies $|v_j| = 2$ for all $m + 1 < j \leq N$. Since we have also shown $|v_{m+1}| = 2$, this finishes the induction proof. $\square$

Lemma 5.8 *If $G(S_m)$ is disconnected, every connected component has at least one m–open vector.*

Proof Otherwise, a component of $G(S_m)$ would still be a component in $G(S_{n+3})$, disconnected from the rest of $G(S_{n+3})$. However, this contradicts Corollary 3.3, which says the lattice is indecomposable and $G(S_{n+3})$ is connected. $\square$

Lemma 5.9 *Suppose S contains no breakable vector. If S_m contains at least one vector of norm ≥ 3, and contains exactly one m–open vector v_j with $j < m$, then v_{m+1} neighbors v_j and has no other smaller neighbors, and $|v_k| = 2$ for all $k > m+1$.*

Proof Since S_m has a high norm vector, v_{m+1} has some smaller neighbor, which must be v_j. By Lemma 5.8, $G(S_m)$ is connected, so v_j neighbors some $v_i \in S_m$. Since v_j is the only m–open vector in S_m, v_i is m–blocked, and v_i is an m–blocking neighbor of v_j by Definition 5.3(1). Since $v_{m+1} \sim v_j$ and $j < m$, $|v_{m+1}| \geq 3$. By Definition 5.3(2), v_{m+1} is an $(m+1)$–blocking neighbor of v_j. By Lemma 5.5, v_j is $(m+1)$–blocked. The rest follows by Lemma 5.7. $\square$

6 $a_0 = 2$

In this section, we assume $a_0 = 2$. By Lemma 4.8, $f = 3$ and $v_2 = e_1 - e_2$.

Lemma 6.1 *If $v_s \sim v_2$ for $s > 3$, then v_2 is s–blocked.*

Proof By Lemmas 4.8 and 4.10, $v_2 = x_0$ and $|x_1| = a_1 \geq 3$. If $v_s \sim v_2$, then, by Corollary 4.19, A_s is the unique interval containing x_1. Therefore, v_1, v_3 and v_s are the only neighbors of v_2. □

Lemma 6.2 *The vector v_4 is just right and $|v_4| \in \{3, 5\}$.*

Proof By Lemma 5.2, v_1 and v_3 are 3–blocked. We can then use Lemmas 4.3 and 4.9 to get our conclusion. □

Lemma 6.3 *Suppose $|v_4| = 3$. Unless $4 = n + 3$, we have $|v_5| \in \{2, 6\}$ and $|v_j| = 2$ for $j > 5$.*

Proof By Lemma 5.2, v_1 and v_3 are 3–blocked. Lemma 6.1 implies that v_2 is 4–blocked since $v_4 \sim v_2$. If v_5 exists, then to avoid pairing with v_1, v_2 and v_3, either $|v_5| = 2$ or $|v_5| = 6$. In either case, v_4 is 5–blocked by Lemma 5.5 and Definition 5.3(1) and (4) or (1)–(2). By Lemma 5.7, $|v_j| = 2$ for all $j > 5$. □

Lemma 6.4 *Suppose $|v_4| = 5$. There is an index s for which v_s is just right, $|v_s| = s - 1$, and $|v_j| = 2$ for $4 < j < s$. Either $s = n + 3$, or $|v_{s+1}| = 3$ and $|v_j| = 2$ for $j > s + 1$.*

Proof Note that $G(S)$ is connected while $G(S_4)$ has two components with vertex sets $\{v_1, v_2, v_3\}$ and $\{v_4\}$. Since v_1 and v_3 are 4–blocked, v_2 must be 4–open. Let $s > 4$ be the (unique) index with $v_s \sim v_2$. Since $|v_2| = 2$, 1 cannot be a gappy index for v_s. Hence, $v_s \sim v_2$ implies $\{1, 2\} \cap \text{supp } v_s = \{2\}$, which then implies $3 \in \text{supp } v_s$. To avoid having pairing 2 with v_4 (which would contradict Lemma 4.18), $4 \in \text{supp } v_s$ and $v_s \sim v_4$.

For contradiction, suppose there exists $|v_j| \geq 3$ with $4 < j < s$ chosen minimally. Since $|v_5| = \cdots = |v_{j-1}| = 2$, we have $v_4 \sim v_5 \sim \cdots \sim v_{j-1}$. Since v_j does not neighbor v_1, v_2, v_3, and $|\langle v_4, v_j \rangle| \leq 1$ by Lemma 4.18, we must have $\min \text{supp } v_j \geq 4$. Thus, v_j pairs nontrivially with one of $v_4, v_5, \ldots, v_{j-1}$. In other words, in $G(S_{s-1})$,

v_j is in the same connected component as v_4. Since $|v_i| = 2$ for $4 < i < j$, no i can be a gappy index for $4 \leq i < j - 1$. Since $2, 3, 4 \in \operatorname{supp} v_s$, $\{2, 3, \ldots, j - 1\} \subset \operatorname{supp} v_s$. Hence, $\langle v_s, v_j \rangle \geq |v_j| - 2$, which in turn implies $v_s \sim v_j$, creating a heavy triple (v_4, v_j, v_s) and resulting in a contradiction.

Therefore, $|v_j| = 2$ for $4 < j < s$. It follows from Lemma 4.3 that v_s is just right, and $|v_s| = s - 1$. So all the adjacency relations in $G(S_s)$ are

$$v_1 \sim v_2, \quad v_1 \sim v_3, \quad v_2 \sim v_3, \quad v_2 \sim v_s \sim v_4 \sim v_5 \sim \cdots \sim v_{s-1}.$$

By Lemma 6.1, since $v_2 \sim v_s$, v_2 is s–blocked. By Lemma 5.5 and Definition 5.3(3) and (4), v_j is s–blocked when $4 < j < s - 1$. By Lemma 5.5 and Definition 5.3(1) and (2), v_4 is s–blocked. By Lemma 5.5 and Definition 5.3(1), v_s is s–blocked. So v_{s-1} is the only possible s–open vector. If $s < n + 3$, the connectivity of $G(S)$ implies that v_{s-1} is s–open, so we can use Lemma 5.9 to conclude that v_{s-1} is the only smaller neighbor of v_{s+1}, which implies that $v_{s+1} = e_{s-1} + e_s - e_{s+1}$, and $|v_j| = 2$ for $j > s + 1$. □

To summarize, we have the following proposition:

Proposition 6.5 *When $a_0 = 2$, one of the following holds:*

(1) $|v_4| = 3$, $|v_i| = 2$ for $i \geq 5$.

(2) $|v_4| = 3$, $|v_5| = 6$, and $|v_i| = 2$ for $i > 5$.

(3) $|v_4| = 5$, $|v_i| = 2$ for $5 \leq i \leq s - 1$, $|v_s| = s - 1$, and $s = n + 3$.

(4) $|v_4| = 5$, $|v_i| = 2$ for $5 \leq i \leq s - 1$, $|v_s| = s - 1$, $|v_{s+1}| = 3$, and $|v_j| = 2$ for $j \geq s + 2$.

The corresponding changemakers are

- $(1, 1, 1, 1, 2^{[s]})$ with $s > 0$,
- $(1, 1, 1, 1, 4^{[s]}, 4s + 2, (4s + 6)^{[t]})$,

where $a^{[s]}$ means s copies of a.

Remark 6.6 We obtain the corresponding changemaker vectors by using Definition 4.1. Here, we do one example in detail for clarification: Proposition 6.5(1). We remind the reader that $\sigma_0 = \sigma_1 = 1$; see equation (13). In the beginning of this section we obtained that $v_2 = e_1 - e_2$, that is, $\sigma_2 = \sigma_1 = 1$; also that $f = 3$, which in turn implies

that $|v_3| = 2$ and therefore $v_3 = e_2 - e_3$. That is, $\sigma_3 = \sigma_2 = 1$. In item (1), $|v_4| = 3$, which implies that $v_4 = e_2 + e_3 - e_4$. This gives us that $\sigma_4 = \sigma_3 + \sigma_2 = 2$. We also have $|v_i| = 2$ for $i \geq 5$. That is, $v_i = e_{i-1} - e_i$ and so $\sigma_i = 2$ for $i \geq 5$. Therefore, $\sigma = (1, 1, 1, 1, 2^{[s]})$, and that $s > 0$ since the index 4 exists.

Remark 6.7 We combine degenerate cases when listing changemakers at the end of each section, and in our parametrization $s, t \geq 0$ unless otherwise specified. The parameter s is not the previous s in this section. The same convention is also used in later sections.

7   Classification of S_f

From now on, we assume $a_0 \geq 3$. In this section, we classify the possible forms of $v_1, \ldots, v_f$. Always, $v_1 = e_0 - e_1$, and $v_f = e_{f-1} - e_f$.

Lemma 7.1 For $1 < i < f - 1$, we have $v_i \sim v_1$. Moreover, $v_{f-1} \sim v_1$.

Proof By Lemma 4.11, $\langle v_i, v_1 \rangle = \langle v_i, v_f \rangle = 0$ for $1 < i < f - 1$. By Lemma 3.8, $\langle v_{f-1}, v_1 \rangle \equiv \langle v_{f-1}, v_f \rangle = -1 \pmod 2$, so $v_{f-1} \sim v_1$. $\square$

Proposition 7.2 When $f = 3$, $v_2 = 2e_0 + e_1 - e_2$.

Proof It follows from Lemma 7.1 that $v_2 \sim v_1$, so $v_2 = e_1 - e_2$ or $v_2 = 2e_0 + e_1 - e_2$. Since $\langle v_2, v_1 \rangle \neq 0$, we have $x_0 \in A_2$, so $|v_2| \geq |x_0| = a_0 \geq 3$. $\square$

Proposition 7.3 When $f > 3$, we have $v_2 = e_0 + e_1 - e_2$, and v_j is just right for all $j < f - 1$.

(1)   If v_{f-1} is tight, then $|v_j| = 2$ for $2 < j < f - 1$.

(2)   If v_{f-1} is just right, then $v_{f-1} = e_1 + \cdots + e_{f-2} - e_{f-1}$, and one of the following holds:

     (a) $|v_j| = 2$ for $2 < j < f - 1$.

     (b) $f = 5$ and $|v_3| = 4$.

     (c) $f = 6$, $|v_3| = 2$ and $|v_4| = 3$.

(3)   If v_{f-1} is gappy, then $f \geq 5$ and $v_{f-1} = e_1 + e_3 + \cdots + e_{f-2} - e_{f-1}$, and one of the following holds:

(a) $|v_j| = 2$ for $2 < j < f - 1$.

(b) $|v_3| = 4$, and $|v_j| = 2$ for $3 < j < f - 1$.

(c) $|v_3| = 2$, $|v_4| = 3$ and $|v_j| = 2$ for $4 < j < f - 1$.

Proof By Lemma 7.1, $v_2 \sim v_1$, so $v_2 = e_0 + e_1 - e_2$. Again using Lemma 7.1, to avoid a claw, any two smaller neighbors of v_{f-1} other than v_1 neighbor each other. Therefore, to avoid a cycle of length 4, which would violate Lemma 4.29, v_{f-1} has at most two smaller neighbors other than v_1. Furthermore, if v_{f-1} is not tight and has two smaller neighbors, then one of them must have norm 2, otherwise these three vectors would form a heavy triple. (Recall that v_{f-1} is the only possible tight or breakable vector by Lemma 4.5 and Lemma 4.11.)

When there exists $|v_j| > 2$ for some $2 < j < f - 1$, define

$$s = \min\{2 < j < f - 1 \mid |v_j| \geq 3\}.$$

The only possible gappy index for v_s is 1, which cannot occur since then either $\langle v_s, v_1 \rangle = -1$ or $\langle v_s, v_2 \rangle = 2$, contradicting Lemma 7.1 or Lemma 4.18. Thus, v_s is just right. Let

$$k = \min \operatorname{supp} v_s \leq s - 2.$$

To avoid a claw $(v_k; v_{k-1}, v_{k+1}, v_s)$, we must have $k \leq 2$. Since $\langle v_s, v_1 \rangle = 0$,

$$k \in \{0, 2\}.$$

So

(15) $$\langle v_s, v_2 \rangle = \pm 1 \neq 0.$$

Suppose v_{f-1} is tight; then $\langle v_{f-1}, v_2 \rangle = 2 = |v_2| - 1$. It follows from Lemmas 4.14 and 4.17 that $\epsilon_2 = \epsilon_{f-1}$, and A_2 shares an end with A_{f-1}. Since the left endpoint of A_{f-1} is v_* (see Proposition 4.13 and the paragraph following it), A_2 and A_{f-1} share right endpoints. By Lemma 4.18 and (15), $A_s \dagger A_2$. Therefore, either $A_s \dagger A_{f-1}$ and $\langle v_s, v_{f-1} \rangle = \pm 1$, or $A_s \pitchfork A_{f-1}$ and $\langle v_s, v_{f-1} \rangle = |v_s| - 2$. If $k = 0$, then $\langle v_s, v_{f-1} \rangle = |v_s| - 1$, which does not fall into either case. Hence, $k = 2$ and $\langle v_s, v_{f-1} \rangle = |v_s| - 2$. We claim that A_s and A_{f-1} are not consecutive. Otherwise, $\langle v_s, v_{f-1} \rangle = 1$ and $\epsilon_s = -\epsilon_{f-1} = -\epsilon_2$. Hence $\langle v_s, v_2 \rangle = -\langle [A_s], [A_2] \rangle = 1$, contradicting the fact that $k = 2$. So $A_s \pitchfork A_{f-1}$. However, since $v_s \sim v_{f-1}$, $(v_2; v_s, v_{f-1}, v_3)$ is a claw. Thus, v_s cannot exist when v_{f-1} is tight.

Suppose v_{f-1} is just right. From the first paragraph we know that v_{f-1} has at most one smaller neighbor with norm ≥ 3. Since $v_{f-1} \sim v_1$, $\min \operatorname{supp} v_{f-1} = 1$. Therefore,

$v_{f-1} \sim v_j$ for $2 < j < f - 1$ if and only if $|v_j| \geq 3$. This implies v_1 and v_s are the only smaller neighbors of v_{f-1}, and $|v_i| = 2$ for $s < i < f - 1$. Since $\langle v_{f-1}, v_s \rangle \leq 1$, either $s = 3$ and $v_s = e_0 + e_1 + e_2 - e_3$, or $s = 4$ and $v_s = e_2 + e_3 - e_4$. In either case, unless $s + 1 = f - 1$, $(v_s; v_2, v_{s+1}, v_{f-1})$ forms a claw. Thus, $f = s + 2$.

Suppose v_{f-1} is gappy. By Lemmas 4.5 and 4.11, S contains no breakable vector. Since $v_{f-1} \sim v_1$, $1 = \min \operatorname{supp} v_{f-1}$. By Lemma 4.30,

$$v_{f-1} = e_1 + e_j + \cdots + e_{f-2} - e_{f-1}$$

for some $j \geq 3$, and all other standard basis vectors are just right. By Lemma 4.9, $f - 2 \in \operatorname{supp} v_{f-1}$, so $f - 2 \geq j \geq 3$. Since $v_{f-1} \sim v_2$, then, by the discussion in the first paragraph of this proof,

$$(16) \qquad\qquad v_i \sim v_{f-1} \quad \text{and} \quad 2 < i < f - 1 \quad \Longrightarrow \quad |v_i| = 2.$$

In particular, $v_s \nsim v_{f-1}$. If $|v_i| = 2$ for any $2 < i < f - 1$, to avoid a claw $(v_j; v_{j-1}, v_{j+1}, v_{f-1})$ we must have either $j = 3$ or $j = f - 2$. If $j = f - 2 > 3$, we would have a cycle

$$v_2 \sim v_3 \sim \cdots \sim v_{f-1} \sim v_2$$

of length $f - 2 > 3$. So $j = 3$ if v_s does not exist. Now assume v_s exists. Note $v_s \sim v_2$ by (15), $v_{f-1} \sim v_2$, and $v_3 \sim v_2$ if $3 < s$. To avoid a claw $(v_2; v_3, v_s, v_{f-1})$, either $s = 3$ or $v_{f-1} \sim v_3$. In the first case, $v_3 = e_0 + e_1 + e_2 - e_3$. Since $|v_3| > 2$, $v_{f-1} \nsim v_3$, forcing $j = 3$. If $|v_i| \geq 3$ for some i with $3 < i < f - 1$, by (16) we have $v_i \nsim v_{f-1}$. Hence $i = 4$ and $v_4 = e_2 + e_3 - e_4$, which would create a claw $(v_2; v_3, v_4, v_{f-1})$. Therefore, $|v_i| = 2$ for all $3 < i < f - 1$. In the second case, by (16) we have $|v_3| = 2$. Since $v_{f-1} \sim v_3$, $j = 3$. Since $v_s \nsim v_{f-1}$, we have $s = 4$ and $v_s = e_2 + e_3 - e_4$. If $|v_l| > 2$ for some l with $4 < l < f - 1$, we would have $\langle v_l, v_{f-1} \rangle > 0$, contradicting (16). $\qquad\square$

8 $f > 3$, v_{f-1} gappy

Recall from Proposition 7.3 that when $f > 3$ and v_{f-1} is gappy, we have $f \geq 5$ and

$$v_2 = e_0 + e_1 - e_2, \quad v_{f-1} = e_1 + e_3 + \cdots + e_{f-2} - e_{f-1}.$$

Proposition 8.1   *When v_{f-1} is gappy, one of the following holds:*

  (1) $f = n + 3$.

(2) $f = 5$, $|v_3| = 2$, $|v_6| = 5$, *and* $|v_j| = 2$ *for* $j > f + 1$.

(3) $|v_{f+1}| = 4$, *and* $|v_j| = 2$ *for* $j > f + 1$. *Either* $|v_4| = 3$, *or* $|v_3| = 4$.

Proof When $f = n + 3$, we get item (1). If $f < n + 3$, however, as before we will find it convenient to study the f–blocked vectors. First note that, by Lemmas 4.5 and 4.11, S contains no breakable vector. Since v_{f-1} is unbreakable, v_1 and v_f are f–blocked by Lemma 5.2. Since $v_{f-1} \sim v_1$ and v_2, v_{f-1} is also f–blocked by Lemma 5.5 and Definition 5.3(1)–(2). Lemma 4.30 then implies that v_j is just right for $j > f$.

We argue that when either $f = 5$ and $|v_{f-2}| = 4$ or when $f > 5$, the only f–open vector is v_{f-2}. When $f > 5$, v_i is f–blocked for $i < f - 2$ by Lemma 5.6, (the j in the statement of Lemma 5.6 is $f - 1$) and the only possible f–open vector is v_{f-2}. When $f = 5$ and $|v_3| = 4$,

$$v_3 = e_0 + e_1 + e_2 - e_3, \quad v_4 = e_1 + e_3 - e_4.$$

Since $v_3 \sim v_2 \sim v_4$ and $v_3 \nsim v_4$, we have that v_2 is f–blocked by Lemma 5.5 and Definition 5.3(1)–(2), and again the only possible f–open vector is v_{f-2}. This justifies the claim.

In the two cases (when $f = 5$ and $|v_{f-2}| = 4$ or when $f > 5$), by Lemma 5.9 $v_{f+1} \sim v_{f-2}$, and $|v_j| = 2$ for $j > f + 1$. Since v_{f-2} is the only smaller neighbor of v_{f+1} and v_{f+1} is just right, $|v_{f+1}| = 4$. This will give us a subset of the cases in the first statement in item (3). (The item also includes the case $f = 5$ and $|v_{f-2}| = 2$, which will be obtained in the last paragraph; see below.) To see the second statement in item (3), note that when $|v_3| = \cdots = |v_{f-2}| = 2$ and $|v_{f+1}| = 4$, (v_{f-1}, v_2, v_{f+1}) forms a heavy triple, so either case (3b) or case (3c) in Proposition 7.3 happens.

When $f = 5$ and $|v_3| = 2$, both v_2 and v_3 are possibly f–open. Since $v_6 \nsim v_1, v_4$ or v_5, we have $|v_6| \in \{4, 5\}$. In either case, v_2 and v_3 are 6–blocked by Lemma 5.6, and by Lemma 5.7 $|v_j| = 2$ for $j > f + 1$. In the case $|v_6| = 5$, this gives us item (2). In the case $|v_6| = 4$, we would have a heavy triple (v_2, v_4, v_6), contradicting Lemma 4.25. $\square$

The corresponding changemakers are

- $(1, 1, 2, 2, 2^{[s]}, 2s + 3, 2s + 3)$,
- $(1, 1, 2, 4, 4^{[s]}, 4s + 5, 4s + 5, (8s + 14)^{[t]})$,
- $(1, 1, 2, 2, 4^{[s]}, 4s + 3, 4s + 3, (8s + 10)^{[t]})$.

9 $f > 3$, v_{f-1} just right

Assume $f > 3$ and v_{f-1} is just right. Recall from Proposition 7.3 that

$$v_2 = e_0 + e_1 - e_2, \quad v_{f-1} = e_1 + \cdots + e_{f-2} - e_{f-1}.$$

First, we consider cases (2b) and (2c) in Proposition 7.3.

Lemma 9.1 *If there exists $|v_j| \geq 3$ for some $2 < j < f - 1$, then $f = n + 3$.*

Recall that in this case $f = j + 2 = 5$ or 6 by Proposition 7.3(2b)–(2c).

Proof of Lemma 9.1 By Lemma 5.2, v_1 and v_f are f–blocked, since v_{f-1} is unbreakable. Therefore, v_{f-1} is f–blocked by Lemma 5.5, since it neighbors v_1 and v_j. Because $G(S_f)$ is connected, any $G(S_g)$ is also connected for $g > f$. So there are no gappy vectors by Lemma 4.30. By Lemma 5.6, v_i is f–blocked for $i < f - 2$. It remains only to be shown that $v_{f-2} = v_j$ is f–blocked, but this follows from Lemma 5.5 and the fact that it neighbors v_2 and v_{f-1}, both of which are f–blocked but do not neighbor each other. $\qquad\square$

From now on in this section, we assume $|v_i| = 2$ for $2 < i < f - 1$, ie case (2a) in Proposition 7.3.

Lemma 9.2 *The vector v_{f+1} is just right. Furthermore:*

  (1) *If $f = 4$, then $|v_5| \in \{3, 4, 6\}$.*
  (2) *If $f = 5$, then $|v_6| \in \{3, 4, 5\}$.*
  (3) *If $f > 5$, then $|v_{f+1}| \in \{3, 4\}$.*

Proof The only vectors of norm at least 3 in S_f are v_2 and v_{f-1}, so the only possible gappy indices for v_{f+1} are 1 and $f - 2$. However, 1 cannot be a gappy index because then either $\langle v_{f+1}, v_1 \rangle = -1$ or $\langle v_{f+1}, v_2 \rangle = 2$, contradicting Lemma 5.2 or Lemma 4.18, and $f - 2$ cannot be a gappy index because then $\langle v_{f+1}, v_f \rangle = -1$, contradicting Lemma 5.2. Therefore, v_{f+1} is just right.

Let $j = \min \operatorname{supp} v_{f+1}$. To avoid pairing with v_1 and v_f, either $j = 0$ or $1 < j < f$. When $f = 4$, we get $j \in \{0, 2, 3\}$ and $|v_5| \in \{3, 4, 6\}$. When $f > 4$, $j \neq 0$ because otherwise $\langle v_{f-1}, v_{f+1} \rangle = f - 3 > 1$, contradicting Lemma 4.18. So $1 < j < f$. Since $\langle v_{f-1}, v_{f+1} \rangle = f - 2 - j \leq 1$, we have $j \geq f - 3$. If $j = f - 3$, then $(v_{f-3}; v_{f-4}, v_{f-2}, v_{f+1})$ is a claw unless $f = 5$. Therefore, j is either $f - 1$ or $f - 2$, unless $f = 5$ in which case j can be $f - 3$. $\qquad\square$

Lemma 9.3 *If* $f = 4$ *and* $|v_5| = 6$, *or* $f = 5$ *and* $|v_6| = 5$, *then* $f + 1 = n + 3$.

Proof In either case, $G(S_{f+1})$ is connected, so v_{f+2} is just right by Lemma 4.30. Since $\langle v_{f+2}, v_{f+1} \rangle \leq 1$ and $v_{f+2} \nsim v_f$, we get $\min \operatorname{supp} v_{f+2} \in \{f - 1, f + 1\}$. If $\min \operatorname{supp} v_{f+2} = f - 1$, then we have a heavy triple $(v_{f-1}, v_{f+1}, v_{f+2})$. If $\min \operatorname{supp} v_{f+2} = f + 1$, then one obtains the claw $(v_{f+1}; v_2, v_{f-1}, v_{f+2})$. $\square$

Now, assume we are not in the above two cases, ie

$$\tag{17} |v_{f+1}| \in \{3, 4\}.$$

Then $G(S_{f+1})$ is disconnected. Let $s > f + 1$ be the smallest index with $|v_s| \geq 3$. Note that such an index must exist, since otherwise $\hat{G}(S)$ would be disconnected, contradicting Corollary 3.3.

Lemma 9.4 *The only possible* $(f+1)$*–open vectors are* v_{f-2}, v_{f-1} *and* v_{f+1}.

Proof Since v_{f-1} is unbreakable, by Lemma 5.2 v_1 and v_f are f–blocked. As the statement is obvious for $f = 4$, we may assume $f > 4$ without loss of generality. Since $|v_j| = 2$ for $3 \leq j \leq f - 2$, v_j is f–blocked for $3 \leq j < f - 2$ by Definition 5.3(3)–(4) and Lemma 5.5. The following shows v_2 is $(f+1)$–blocked. Suppose $v_j \sim v_2$ for some $j > f + 1$. If $2 \notin \operatorname{supp} v_j$, then $v_j \nsim v_1$ implies $0, 1 \in \operatorname{supp} v_j$, which contradicts $|\langle v_j, v_2 \rangle| \leq 1$. If $2 \in \operatorname{supp} v_j$, then $\{2, 3, \ldots, f - 2\} \subset \operatorname{supp} v_j$ because none of $2, \ldots, f - 3$ can be a gappy index. We also have $f - 1 \in \operatorname{supp} v_j$, because otherwise $\langle v_j, v_{f-1} \rangle \geq f - 3 > 1$. Since $|v_f| = 2$, $f - 1$ is not a gappy index, so $f \in \operatorname{supp} v_j$. To satisfy (17) and $|\langle v_{f+1}, v_j \rangle| \leq 1$, it must be the case that $|v_{f+1}| = 3$, but this results in a heavy triangle (v_{f-1}, v_{f+1}, v_j). $\square$

Lemma 9.5 *If* v_s *is just right, then* $s = f + 2$. *Furthermore, either* $|v_{f+1}| = 4$ *and* $|v_{f+2}| = 4$, *or* $|v_{f+1}| = 3$ *and* $|v_{f+2}| = 5$. *In either case,* $f + 2 = n + 3$.

Proof Since $|v_s| \geq 3$, we have $j := \min \operatorname{supp} v_s < s - 1$. By assumption, $|v_{f+2}| = \cdots = |v_{s-1}| = 2$. If $j \geq f + 2$, we would have a claw $(v_j; v_{j-1}, v_{j+1}, v_s)$. If $j = f + 1$, we would have a claw $(v_{f+1}; v_{f+2}, v_s, v_{f-1} \text{ or } v_{f-2})$. Since $\langle v_s, v_f \rangle = 0$, in fact $j < f$. In particular, $v_s \sim v_{f+1}$. By (17), v_{f+1} has exactly one smaller neighbor, which is either v_{f-1} or v_{f-2}. In either case, this is not a neighbor of v_s — in the first case, this would form a heavy triple, and in the second, it would make $\langle v_s, v_{f+1} \rangle \geq 2$. To avoid a claw $(v_{f+1}; v_s, v_{f+2}, v_{f-1} \text{ or } v_{f-2})$, we must have $s = f + 2$. By Lemma 9.4,

min supp $v_{f+2} \in \{f-1, f-2\}$. By (17), min supp $v_{f+1} \in \{f-1, f-2\}$. As we showed earlier in this paragraph, the smaller neighbor of v_{f+1} is not adjacent to v_{f+2}, so min supp $v_{f+2} \neq$ min supp v_{f+1}. The conclusion about $|v_{f+2}|$ follows.

To see v_{f+2} is the last standard basis vector, note that $G(S_{f+2})$ is connected. By Lemma 4.30, v_{f+3} is just right. Since $|v_{f+2}| \geq 4$ and $\langle v_{f+2}, v_{f+3}\rangle \leq 1$, we see $|v_{f+3}| \in \{2, 3, 4\}$. Since $v_{f+3} \sim v_f$, we have $v_{f+3} \neq 4$. In $G(S_{f+2})$, there is a path containing 4 vertices $v_{f-2}, v_{f+1}, v_{f+2}, v_{f-1}$. Here v_{f-2}, v_{f-1} are the two ends, and v_{f+1}, v_{f+2} are in the interior. Since $|v_{f+3}| \in \{2, 3\}$, v_{f+3} is adjacent to exactly one of v_{f+1} and v_{f+2}, and it does not neighbor v_{f-2} or v_{f-1}. So we get a claw centered at the neighbor of v_{f+3}. □

Lemma 9.6 *If v_s is gappy, then $f = 4$, $|v_5| = 4$, and*

$$v_s = e_2 + e_5 + \cdots + e_{s-1} - e_s.$$

Furthermore, if $n + 3 > s$, then $|v_{s+1}| = 3$, and $|v_j| = 2$ for all $j > s + 1$.

Proof The possible gappy indices for v_s are 1, f and $f - 2$. However, 1 (resp. f) cannot be a gappy index, because otherwise $\langle v_s, v_1\rangle = -1$ (resp. $\langle v_s, v_f\rangle = -1$) or $\langle v_s, v_2\rangle = 2$ (resp. $\langle v_s, v_{f+1}\rangle \geq 2$). Therefore,

$$v_s = e_{f-2} + e_j + \cdots + e_{s-1} - e_s$$

for $f < j < s$. We have $v_s \sim v_{f-2}$ and $v_s \sim v_{f-1}$. If $|v_{f+1}| = 3$, then $v_{f+1} \sim v_{f-1}$, so either $(v_{f-1}; v_1, v_{f+1}, v_s)$ is a claw, or (v_{f-1}, v_{f+1}, v_s) is a heavy triple. Therefore, $|v_{f+1}| = 4$, and $v_{f+1} \sim v_{f-2}$. If $j > f + 1$, then $(v_s, v_{f-2}, v_{f+1}, \ldots, v_j, v_s)$ is a cycle of length ≥ 4. Therefore, $j = f + 1$, and $v_s \sim v_{f+1}$. To avoid a claw $(v_{f-2}; v_{f-3}, v_{f+1}, v_s)$, we must have $f = 4$ and $j = 5$.

Since $v_{f-1} \sim v_1, v_s$ and $v_2 \sim v_{f+1}, v_s$, we get that v_{f-1} and v_2 are both s–blocked by Lemma 5.5 and Definition 5.3(1)–(2), hence v_s is s–blocked by Lemma 5.5 and Definition 5.3(1). Lemma 5.5 and Definition 5.3(1) and (3) can further imply that v_j is s–blocked for $f + 1 \leq j < s - 1$. Thus, v_{s-1} is the only s–open vector, and the result follows from Lemma 5.9. □

Summarizing, we have:

Proposition 9.7 *Suppose that $f > 3$ and v_{f-1} is just right. Then $|v_{f-1}| = f - 1$, and one of the following holds (other than v_s in case (1e), all vectors are just right):*

(1) $|v_j| = 2$ for $2 < j < f - 1$.

    (a) $f = 4$, $n = 2$, $|v_5| = 6$.

    (b) $f = 5$, $n = 3$, $|v_6| = 5$.

    (c) $n = f - 1$, $|v_{f+1}| = 3$, $|v_{f+2}| = 5$.

    (d) $n = f - 1$, $|v_{f+1}| = 4$, $|v_{f+2}| = 4$.

    (e) $f = 4$, $|v_5| = 4$, $|v_j| = 2$ for $5 < j < s$ or $j > s + 1$, $|v_{s+1}| = 3$, *and*

$$v_s = e_2 + e_5 + \cdots + e_{s-1} - e_s.$$

(2) $f = 5$, $n = 2$, $|v_3| = 4$.

(3) $f = 6$, $n = 3$, $|v_3| = 2$, $|v_4| = 3$.

The corresponding changemakers are

- $(1, 1, 2, 3, 3, 10)$,

- $(1, 1, 2, 2, 5, 5, 14)$,

- $(1, 1, 2^{[s+1]}, 2s + 3, 2s + 3, 4s + 6, 8s + 14)$,

- $(1, 1, 2^{[s+1]}, 2s + 3, 2s + 3, 4s + 8, 8s + 14)$,

- $(1, 1, 2, 3, 3, 8, 8^{[s]}, 8s + 10, (8s + 18)^{[t]})$,

- $(1, 1, 2, 4, 7, 7)$,

- $(1, 1, 2, 2, 4, 9, 9)$.

10 $f > 3$, v_{f-1} tight

In this section, we assume $f > 3$ and v_{f-1} is tight. By Proposition 7.3, $v_2 = e_0 + e_1 - e_2$ and $|v_j| = 2$ for $2 < j < f - 1$. Since $\langle v_{f-1}, v_2 \rangle = 2 = |v_2| - 1$,

(18) $$\epsilon_2 = \epsilon_{f-1} \quad \text{and} \quad A_2 \prec A_{f-1}$$

by Lemmas 4.14 and 4.17. Since the left endpoint of A_{f-1} is x_* (which is only contained in A_1, A_{f-1} and A_f), A_2 and A_{f-1} must share the right endpoint.

Lemma 10.1 *The vector v_{f-1} is f–blocked. For $j > f$, we have $|\langle v_j, v_{f-1} \rangle| \in \{0, |v_j| - 2\}$. The only possible f–open vectors are v_1, v_{f-2} and v_f.*

Proof Suppose for contradiction that $v_j \sim v_{f-1}$ for some $j > f$. Since the left endpoint of A_{f-1} is x_*, A_j cannot share its left endpoint with A_{f-1}. Since $j > f$ and $\langle v_j, v_{f-1} \rangle \neq 0$, we get $|v_j| \geq 3$. Since $z_j \neq z_2$ (Corollary 4.19) and $A_2 \prec A_{f-1}$, A_j does not share its right endpoint with A_{f-1}. Thus, $A_j \dagger A_{f-1}$. Since the left endpoint of A_{f-1} is $*$, A_j is distant from A_1, so $\langle v_j, v_1 \rangle = 0$. Since $\epsilon_2 = \epsilon_{f-1}$, we have $\langle v_j, v_2 \rangle = \langle v_j, v_{f-1} \rangle = \pm 1$. This implies supp $v_j \cap \{0, 1, 2\} \neq \varnothing$, so $\langle v_j, v_{f-1} \rangle \geq 0$. Hence, $\langle v_j, v_2 \rangle = \langle v_j, v_{f-1} \rangle = 1$. However, if $\langle v_j, v_2 \rangle = 1$, using $\langle v_j, v_1 \rangle = 0$ we get $0, 1, 2 \in \text{supp } v_j$ and $\langle v_j, v_{f-1} \rangle \geq 3$, a contradiction. Thus, for all $j > f$, we have $v_j \nsim v_{f-1}$, and immediately $|\langle v_j, v_{f-1} \rangle| \in \{0, |v_j| - 2\}$ by Lemma 4.14.

Since $|v_i| = 2$ for $2 < i < f - 1$, v_i is f–blocked for $3 < i < f - 2$ by Lemma 5.5. When $3 < f - 1$, by Definition 5.3(1) and (4) v_{f-1} and v_3 are f–blocking neighbors of v_2. By Lemma 5.5, v_2 is f–blocked. When $3 < f - 2$, v_3 is also f–blocked by Lemma 5.5, since v_2 and v_4 are f–blocking neighbors.   □

Definition 10.2 $V_j := \text{supp } v_j \cap \{0, 1, \ldots, f\}$.

Lemma 10.3 $V_{f+1} = \{1, 2, \ldots, f - 2, f\}$, $V_{f+2} = \{f - 2\}$ or $\varnothing$, and $V_j = \varnothing$ for $j > f + 2$.

Proof Suppose $j > f$ and $V_j \neq \varnothing$. Then $\min \text{supp } v_j \in \{0, 1, f - 2, f - 1\}$ by Lemma 10.1 and the parity condition (Lemma 3.8). We will discuss each of the four possibilities of $\min \text{supp } v_j$. Since $|v_1| = |v_f| = 2$, either 0 or $f - 1$ is not a gappy index. Since $|v_i| = 2$ for $2 < i < f - 1$, no i is a gappy index for $2 \leq i < f - 2$. The following shows 1 is not a gappy index of v_j. For contradiction, suppose $1 \in \text{supp } v_j$ and $2 \notin \text{supp } v_j$. Since $|\langle v_j, v_2 \rangle| \leq 1$, $0 \notin \text{supp } v_j$. Since $\langle v_j, v_2 \rangle = 1$, by Lemma 4.18 and (18), $\epsilon_j = -\epsilon_2 = -\epsilon_{f-1}$. Since $\langle v_j, v_1 \rangle = -1$, A_j has x_0 as its left endpoint, so $z_j \in A_{f-1}$. However, since $\langle v_j, v_f \rangle \equiv \langle v_j, v_1 \rangle \equiv -1 \pmod 2$, we see supp $v_j \cap \{f - 1, f\} = \{f\}$, so $\langle v_j, v_{f-1} \rangle \geq 0$, contradicting $z_j \in A_{f-1}$ and $\epsilon_j = -\epsilon_{f-1}$. Thus, the only possible gappy index in $\{0, 1, \ldots, f - 1\}$ is $f - 2$.

If $\min \text{supp } v_j = 0$, then $0, 1, 2, \ldots, f - 2 \in \text{supp } v_j$, and $\langle v_j, v_2 \rangle = 1$, so $\epsilon_j = -\epsilon_2$ and $A_j \dagger A_2$ by Lemma 4.18. However, since $\langle v_j, v_{f-1} \rangle \geq f - 1 \geq 3$, we have that $\epsilon_j = \epsilon_{f-1}$, contradicting (18).

If $\min \text{supp } v_j = f - 1$, then $\langle v_j, v_{f-1} \rangle = -1$. By Lemma 10.1, $|\langle v_j, v_{f-1} \rangle| = |v_j| - 2$, so $|v_j| = 3$, $A_j \pitchfork A_{f-1}$ and $\epsilon_j = -\epsilon_{f-1}$. However, $V_j = \{f - 1, f\}$ by the parity condition, so by Lemma 4.9, $j = f + 1$. Now $v_{f+1} = -e_{f+1} + e_f + e_{f-1}$, so

$v_{f+1} + v_{f-1}$ is irreducible by Lemma 4.6. Since $\langle v_{f+1}, v_1 \rangle = 0$, A_{f+1} does not contain x_0, so $\tau([A_{f+1}]) = -[A_{f+1}]$. Since $A_{f+1} \pitchfork A_{f-1}$ and $\epsilon_{f+1} = -\epsilon_{f-1}$, up to applying τ, $v_{f-1} + v_{f+1}$ becomes $[A_{f-1}] - [A_{f+1}]$, which is a signed sum of two distant intervals, contradicting the irreducibility of $v_{f-1} + v_{f+1}$.

If $\min \operatorname{supp} v_j = 1$, then $1, 2, \ldots, f-2 \in \operatorname{supp} v_j$. Since $\langle v_j, v_f \rangle \equiv \langle v_j, v_1 \rangle \equiv -1 \pmod 2$ and $f-1$ is not a gappy index, $f-1 \notin \operatorname{supp} v_j$ and $f \in \operatorname{supp} v_j$. Hence, $\langle v_j, v_{f-1} \rangle = f-2$. By Lemma 10.1, $\langle v_j, v_{f-1} \rangle = |v_j| - 2$, so $|v_j| = f$. By Lemma 4.9, $j-1, j \in \operatorname{supp} v_j$. So $|v_j| = f$ implies that $j = f+1$ and $V_{f+1} = \{1, 2, \ldots, f-2, f\}$.

If $\min \operatorname{supp} v_j = f - 2$, then $\langle v_j, v_f \rangle = \langle v_j, v_1 \rangle = 0$. Suppose $\langle v_j, v_{f-1} \rangle = 0$. Then $z_j \notin A_{f-1}$, so z_j is to the right of z_2 by (18). However, since $v_2 \sim v_3$ and $v_3 \nsim v_{f-1}$ when $f > 4$, A_3 is consecutive to A_2 on the left, so $A_3, \ldots, A_{f-2}$ (all of norm 2) lie inside A_{f-1} to the left of z_2. This contradicts $v_{f-2} \sim v_j$ when $f > 4$. When $f = 4$, the contradiction instead comes via a similar argument from the fact that $v_j \sim v_2$ but neither $z_j \in A_{f-1}$ nor $v_j \sim v_{f-1}$. The above shows $\langle v_j, v_{f-1} \rangle \neq 0$, hence $f - 1 \notin \operatorname{supp} v_j$. Since $\langle v_j, v_f \rangle = 0$, we have $V_j = \{f-2\}$.

Since $f \in \operatorname{supp} v_{f+1}$, we may conclude that $V_{f+1} = \{1, 2, \ldots, f-2, f\}$.

For $j > f+1$, the above shows either $V_j = \varnothing$ or $V_j = \{f-2\}$. If $V_j = \{f-2\}$, then since $|v_j| - 2 = \langle v_j, v_{f-1} \rangle = 1$, we have that $|v_j| = 3$ and $v_j = -e_j + e_{j-1} + e_{f-2}$. Since $\langle v_j, v_{f-1} \rangle > 0$ and $\langle v_{f+1}, v_{f-1} \rangle > 0$, using Lemma 10.1 we have $\epsilon_j = \epsilon_{f-1}$ and $\epsilon_{f+1} = \epsilon_{f-1}$. Unless $j = f+2$ we have $\langle v_j, v_{f+1} \rangle = 1$, thus $A_j \dagger A_{f+1}$ and $\epsilon_j = -\epsilon_{f+1}$, a contradiction. $\qquad \square$

It follows from Lemma 10.3 that

(19)
$$v_{f+1} = e_1 + \cdots + e_{f-2} + e_f - e_{f+1}$$

and that

$$v_{f+2} = e_{f+1} - e_{f+2} \quad \text{or} \quad e_{f-2} + e_{f+1} - e_{f+2}.$$

Proposition 10.4 *We have* $|v_j| = 2$ *for* $j > f + 2$.

Proof By (19), $v_{f+1} \sim v_1, v_f$ and so z_{f+1} is the leftmost vertex with norm ≥ 3. In particular, $z_{f+1} \in A_{f-1}$. We also know that z_2 is the rightmost vertex in A_{f-1} with norm ≥ 3 from the beginning of this section. Thus, $A_{f+1} \subset A_{f-1}$.

By Lemma 10.3, $V_{f+2} = \{f-2\}$ or $\varnothing$. If $V_{f+2} = \varnothing$, then $v_{f+2} \sim v_{f+1}$. Since $A_{f+1} \subset A_{f-1}$ and $z_2 \in A_{f-1}$ is to the right of A_{f+1}, $A_{f+2} \subset A_{f-1}$. If $V_{f+2} = \{f-2\}$, we claim that $z_{f+2} \in A_{f-1}$. When $f > 4$, since $v_3 \nsim v_{f-1}$, $v_3 \sim v_2$ and A_2

shares the right endpoint with A_{f-1}, we see that A_3 must abut A_2 on its left. Then since $v_{f+2} \sim v_{f-2} \sim \cdots \sim v_3$ and z_{f+2} is the unique element with norm ≥ 3 in the intervals $A_{f+2}, A_{f-2}, \ldots, A_3$, it follows that z_{f+2} must be to the left of z_2. When $f = 4$, this still holds because $v_{f+2} \nsim v_{f-1}$ and $v_{f+2} \sim v_2$. Hence $z_{f+2} \in A_{f-1}$. In either case, since $z_2 \in A_{f-1}$ is to the right of z_{f+2}, we get $A_{f+2} \subset A_{f-1}$.

For any $j > f + 2$, $V_j = \varnothing$ by Lemma 10.3. Since $\langle v_j, v_{f-1} \rangle = 0$, Lemma 4.14 implies that either A_j is distant from A_{f-1} or $|v_j| = 2$. If A_j and A_{f-1} are distant, by the result in the first paragraph in this proof $A_{f+1} \subset A_{f-1}$ and $v_j \nsim v_{f+1}$. When $f > 4$, the argument in the second paragraph shows that $A_{f-2} \subset A_{f-1}$, and when $f = 4$, we get $A_{f-2} = A_2 \prec A_{f-1}$ by (18). So we also have $v_j \nsim v_{f-2}$. If $|v_j| = 2$, then $\langle v_j, v_{f+1} \rangle = \langle v_j, v_{f-2} \rangle = 0$, so $v_j \nsim v_{f+1}, v_{f-2}$. Thus, v_{f+1} and v_{f-2} are $(f+2)$–blocked. Since $\langle v_1, v_i \rangle = 0$ for all $i > f + 2$, we get that v_1 and v_f are $(f+2)$–blocked. The result follows from Lemmas 10.1 and 5.7. $\square$

The corresponding changemakers are

- $(1, 1, 2^{[s+1]}, 2s + 5, 2s + 5, (4s + 8)^{[t]})$,
- $(1, 1, 2^{[s+1]}, 2s + 5, 2s + 5, 4s + 8, (4s + 10)^{[t]})$ for $t > 0$.

11 $f = 3$

In this case, $a_0 \geq 3$, $v_2 = -e_2 + e_1 + 2e_0$ is tight by Proposition 7.2, and the left endpoint of A_2 is x_*.

Lemma 11.1 If A_2 contains m distinct vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ of norm ≥ 3, then S contains at least $m - 1$ vectors v_j such that $j > 3$, $z_j \in A_2$ and $V_j \neq \varnothing$, where V_j is as defined in Definition 10.2.

Proof Let $\Delta = \Delta(p, q)$, $\bar{\Delta}$ be the quotient of Δ by the sublattice spanned by vectors of norm 2, and let $\pi \colon \Delta \to \bar{\Delta}$ be the quotient map. By the definition of $\tau = \tau_0$ in Definition 3.6, we have

$$(20) \qquad\qquad\qquad \pi \circ \tau = -\pi.$$

Since S spans Δ, it follows from Proposition 4.13 and (20) that $\{\pi([A_i])\}_{i=1}^{n+3}$ spans $\bar{\Delta}$. In particular, $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ are all linear combinations of the $\pi([A_i])$. Hence there are at least $m - 1$ intervals other than A_2 containing one x_{i_t}. Since A_2 is the only

possibly breakable interval, Corollary 4.19 implies that at least $m - 1$ vertices among $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ are each contained in an interval A_j for some $j > 3$. For each $z_j \in A_2$, we get $\langle v_j, v_2 \rangle \neq 0$ and $V_j \neq \varnothing$. □

Lemma 11.2 *We have $v_4 = -e_4 + e_3 + e_2$ or $v_4 = -e_4 + e_3 + e_1$. For all $j \geq 5$, $V_j = \varnothing$ or $\{0, 1\}$ or $\{2, 3\}$.*

Proof Take any $j \geq 4$. Suppose $0, 1 \notin V_j$. Then Lemma 4.11 implies that $\langle v_j, v_3 \rangle = 0$, so $V_j = \varnothing$ or $V_j = \{2, 3\}$.

Suppose $0 \in V_j$. Then $1 \in V_j$ because $|v_1| = 2$ implies 0 cannot be a gappy index, so $V_j = \{0, 1, 2, 3\}$ or $\{0, 1\}$ by Lemma 4.11. However, if $V_j = \{0, 1, 2, 3\}$, then, using Lemma 4.14, $\langle v_j, v_2 \rangle = 2$ implies $z_j \in A_2$ and $|v_j| = 3$ or 4, which contradicts $V_j = \{0, 1, 2, 3\}$. Hence, $V_j = \{0, 1\}$.

Suppose $0 \notin V_j$ and $1 \in V_j$. Then Lemma 4.11 implies that $V_j = \{1, 3\}$. Since $\langle v_j, v_1 \rangle = -1$, x_0 is the left endpoint of A_j, and $z_j = x_0 \in A_2$. Hence, $|\langle v_j, v_2 \rangle| = |v_j| - 2$, and $|v_j| = 3$, which only happens when $j = 4$ by Lemma 4.9.

Since $3 \in \operatorname{supp} v_4$ by Lemma 4.9, $v_4 = -e_4 + e_3 + e_2$ or $v_4 = -e_4 + e_3 + e_1$. □

Lemma 11.3 *There is at most one $j \geq 4$ such that $V_j = \{2, 3\}$. If $V_j = \{2, 3\}$, then $A_j \dagger A_2$ and $\epsilon_j = \epsilon_2$. So $v_j \sim v_2$ and $z_j \notin A_2$.*

Proof Suppose $V_j = \{2, 3\}$. Since $\langle v_j, v_2 \rangle = -1$, either $A_j \dagger A_2$ and $\epsilon_j = \epsilon_2$, or $A_2 \pitchfork A_j$, $|v_j| = 3$ and $\epsilon_j = -\epsilon_2$. If $j > 4$, the latter is impossible because $|v_j| \geq 4$ by Lemma 4.9. If $j = 4$ and $A_2 \pitchfork A_4$, notice that A_4 does not contain x_0 since $\langle v_4, v_1 \rangle = 0$, so $\tau([A_4]) = -[A_4]$. Up to applying τ, $v_4 + v_2$ becomes $([A_2] - [A_4])$, which is reducible, contradicting Lemma 4.6.

Since $j = \min\{i \mid |z_i| \geq 3 \text{ and } z_i \notin A_2\}$, there is at most one such j. □

Lemma 11.4 *If $V_j = \varnothing$ for $4 < j \leq m$, then $|v_j| = 2$ for $4 < j \leq m$.*

Proof By assumption, v_1, v_2, v_3 are $(4, m)$–blocked. Unless $m = 4$ (the lemma is vacuously true when $m = 4$), $|v_5| = 2$ and $v_5 \sim v_4$ by Lemma 4.9. If $v_4 = -e_4 + e_3 + e_2$, then v_2 and v_5 are $(5, m)$–blocking neighbors of v_4 by Definition 5.3(1) and (4), so v_4 is $(5, m)$–blocked by Lemma 5.5. If $v_4 = -e_4 + e_3 + e_1$, then $v_4 \sim v_1$, hence $z_4 = x_0$, and $v_i \nsim v_4$ unless $|v_i| = 2$ or $z_i \in A_2$. If $|v_i| = 2$, then $v_i \nsim v_4$ for $i > 5$. For $5 < i \leq m$, since $\langle v_i, v_2 \rangle = 0$ we get $z_i \notin A_2$. Hence, v_4 is also $(5, m)$–blocked in the case $v_4 = -e_4 + e_3 + e_1$. The result follows from Lemma 5.7. □

Lemma 11.5 *If $v_4 = -e_4 + e_3 + e_1$, then $z_4 = x_0$, and A_2 contains at least two vertices with norm ≥ 3. Moreover, $\epsilon_2 = \epsilon_4$ and $v_2 \nsim v_4$.*

Proof We have $\langle v_4, v_1 \rangle \neq 0$, so A_4 is an interval containing $x_0 \in A_2$, $z_4 = x_0$, and $|v_4| = a_0 = 3$. In particular, A_2 and A_4 are not consecutive. Since $|v_2| = 6 > a_0$, A_2 contains another vertex with norm ≥ 3. Since $\langle v_4, v_2 \rangle = 1 = |v_4| - 2$, we get $\epsilon_2 = \epsilon_4$, $A_2 \pitchfork A_4$ and $v_2 \nsim v_4$. $\square$

Lemma 11.6 *If $V_j = \{0, 1\}$ for some $j > 4$, then we must have $j = 5$ and $v_5 = -e_5 + e_4 + e_1 + e_0$.*

Proof Suppose $V_j = \{0, 1\}$ for some $j > 4$. Then $\langle v_j, v_2 \rangle = 3$ and $\epsilon_j = \epsilon_2$. By Lemma 4.7, $v_j - v_2$ is irreducible, so $A_j \prec A_2$. Hence $|\langle v_j, v_2 \rangle| = |v_j| - 1$, and $|v_j| = 4$. Depending on whether $v_4 = -e_4 + e_3 + e_1$ or $v_4 = -e_4 + e_3 + e_2$, we can use Lemma 11.5 or Lemma 11.3 to conclude that $\epsilon_j = \epsilon_2 = \epsilon_4$. So $\langle v_j, v_4 \rangle \leq 0$ by Lemma 4.18, and hence $4 \in \operatorname{supp} v_j$. Since $|v_j| = 4$, Lemma 4.9 implies that v_j must be v_5. $\square$

First, we consider the case where $v_4 = -e_4 + e_3 + e_2$. Lemma 11.3 implies that $A_2 \dagger A_4$. We split the case $v_4 = -e_4 + e_3 + e_2$ into two subcases according to whether A_2 contains multiple high norm vertices.

Lemma 11.7 *Suppose $v_4 = -e_4 + e_3 + e_2$ and $a_0 = 6$. Then $|v_i| = 2$ for all $i > 4$.*

Proof Since $a_0 = |x_0| = |v_2|$, x_0 is the only vertex of norm ≥ 3 in A_2. By Lemmas 11.2 and 11.3, $v_5 = -e_5 + e_4$ or $-e_5 + e_4 + e_1 + e_0$. If $v_5 = -e_5 + e_4 + e_1 + e_0$, then $\langle v_5, v_2 \rangle = 3$, contradicting $z_5 \notin A_2$. Therefore, $v_5 = -e_5 + e_4$. By Lemmas 11.2, 11.3 and 11.6, $V_j = \varnothing$ for all $j > 4$. The result follows from Lemma 11.4. $\square$

Lemma 11.8 *Suppose $v_4 = -e_4 + e_3 + e_2$ and $a_0 < 6$. Then, $v_5 = -e_5 + e_4 + e_1 + e_0$, and $|v_i| = 2$ for all $i > 5$, except possibly one $v_m = -e_m + e_{m-1} + \cdots + e_4$.*

Proof Since $|x_0| < |v_2|$, there are at least 2 vertices of norm ≥ 3 in A_2. By Lemma 11.3, $z_4 \notin A_2$. Therefore, by Lemma 11.1, there exists $V_i \neq \varnothing$ for some $i > 4$. By Lemmas 11.2, 11.3 and 11.6, $v_5 = -e_5 + e_4 + e_1 + e_0$, and $V_i = \varnothing$ for all $i > 5$. This implies v_2 is 5–blocked by Corollary 4.20. Observe that $v_4 \sim v_5$. Since $\langle v_5, v_2 \rangle = 3$, we have $z_5 \in A_2$. By Lemma 11.3, $A_2 \dagger A_4$, so $z_4 \notin A_2$. If $|v_i| = 2$ for

some $i > 6$, then $\langle v_i, v_5 \rangle = 0$, so $v_5 \nsim v_i$. If $|v_i| \geq 3$, since $z_5 \in A_2$ and $A_2 \dagger A_4$, $v_5 \sim v_i$ would imply that $z_i \in A_2$. However, for $i > 6$, $V_i = \varnothing$ and $\langle v_i, v_2 \rangle = 0$, so $z_i \notin A_2$. Thus, v_5 is 6–blocked.

Let $m > 5$ be the minimal index such that $|v_m| \geq 3$, if such m exists. If $m = 6$, $V_m = \varnothing$ implies $v_6 = -e_6 + e_5 + e_4$. Assume $m > 6$; then v_5 is $(m-1)$–blocked. By Lemma 5.5, v_i is $(m-1)$–blocked for any $5 < i < m-1$. Thus, $\min \operatorname{supp} v_m = 4$. Since $v_m \nsim v_5$, 4 is not a gappy index of v_m. Since $|v_i| = 2$ for $5 < i < m$, i is not a gappy index for $5 \leq i < m-1$. Hence, v_m is just right.

Since v_2 and v_m are m–blocking neighbors of v_4 by Definition 5.3(1)–(2), v_4 is m–blocked by Lemma 5.5. We claim v_{m-1} is also m–blocked. Suppose $v_i \sim v_{m-1}$ for some $i > m$, then $|v_i| \geq 3$. Observe that v_i is connected to v_5 through a path of norm-2 vectors $(v_i \sim v_{m-1} \sim v_{m-2} \sim \cdots \sim v_5)$. Since $z_5 \in A_2$ and $A_2 \dagger A_4$, we must have $z_i \in A_2$, which contradicts $V_i = \varnothing$ and $\langle v_i, v_2 \rangle = 0$. Thus, v_i is m–blocked for all $i < m$. By Lemma 5.7, $|v_i| = 2$ for all $i > m$. □

Next, we consider the case where $v_4 = -e_4 + e_3 + e_1$.

Lemma 11.9 *Suppose $v_4 = -e_4 + e_3 + e_1$. If $v_5 = -e_5 + e_4 + e_1 + e_0$, then $|v_i| = 2$ for all $i > 5$.*

Proof Assume $v_5 = -e_5 + e_4 + e_1 + e_0$. By Lemmas 11.2 and 11.6, $V_i = \varnothing$ or $\{2, 3\}$ for $i > 5$. If $V_i = \{2, 3\}$, then by Lemma 11.3, $v_i \sim v_2$, $\epsilon_i = \epsilon_2$ and $z_i \notin A_2$. Since $\langle v_5, v_2 \rangle = 3 = |v_5| - 1$, A_2 and A_5 share the right endpoint and $\epsilon_5 = \epsilon_2$, so $v_i \sim v_5$. Since $z_i \notin A_2$ and $z_4 = x_0$ (which follows from Lemma 11.5), $v_i \nsim v_4$. Since $\langle v_i, v_4 \rangle = 0$, we have $4 \in \operatorname{supp} v_i$. Since $v_i \sim v_5$, we have $5 \notin \operatorname{supp} v_i$ and $\langle v_i, v_5 \rangle = 1$, so $\epsilon_5 = -\epsilon_i$. However, this contradicts $\epsilon_5 = \epsilon_2$ and $\epsilon_i = \epsilon_2$. Thus, $V_i = \varnothing$ for $i > 5$. Therefore, v_1, v_2 and v_3 are 5–blocked by Corollary 4.20. For $i > 5$, since $\langle v_i, v_2 \rangle = 0$, $z_i \notin A_2$. Since $z_4 = x_0$, we get $v_4 \nsim v_i$ for $i > 5$. Since v_1, v_2, v_3 and v_4 are 5–blocked, $|v_i| = 2$ for all $i > 5$ by Lemma 5.7. □

Lemma 11.10 *Suppose $v_4 = -e_4 + e_3 + e_1$. If there exists $V_m = \{2, 3\}$, then $v_m = -e_m + e_{m-1} + \cdots + e_2$ and $|v_i| = 2$ for all $i > 4$, $i \neq m$.*

Proof By Lemmas 11.2, 11.3, 11.6 and 11.9, $V_i = \varnothing$ for all $i > 4$, $i \neq m$. By Lemma 11.4, $|v_i| = 2$ for $4 < i < m$. Hence, i is not a gappy index for $4 \leq i < m-1$. Recall that $z_4 = x_0$ from Lemma 11.5. By Lemma 11.3, $z_m \notin A_2$, so $v_m \nsim v_4$, and $4 \in \operatorname{supp} v_m$. Therefore, $v_m = -e_m + e_{m-1} + \cdots + e_2$.

Since $V_i = \varnothing$ for $i > m$, we get that v_1, v_2, v_3 are m–blocked. Take any $4 \leq j < m$. Observe that v_j is connected to v_4 through a path of norm-2 vectors, and recall that $z_4 = x_0$, so each A_j is contained in the interior of A_2. For any $i > m$, $v_i \sim v_j$ unless $|v_i| = 2$ or $z_i \in A_2$. If $|v_i| = 2$, we have $\langle v_i, v_j \rangle = 0$, so $v_i \sim v_j$. If $z_i \in A_2$, then $\langle v_i, v_2 \rangle \neq 0$, so $V_i \neq \varnothing$, a contradiction. So v_j is m–blocked for $4 \leq j < m$. By Lemma 5.7, $|v_i| = 2$ for all $i > m$.   □

Proposition 11.11 *When $f = 3$ and v_2 is tight, one of the following holds:*

(1) $v_4 = -e_4 + e_3 + e_2$ *and* $|v_i| = 2$ *for all* $i > 4$.

(2) $v_4 = -e_4 + e_3 + e_2$, $v_5 = -e_5 + e_4 + e_1 + e_0$ *and* $|v_i| = 2$ *for all* $i > 5$ *except possibly one* $v_m = -e_m + e_{m-1} + \cdots + e_4$.

(3) $v_4 = -e_4 + e_3 + e_1$, $v_5 = -e_5 + e_4 + e_1 + e_0$ *and* $|v_i| = 2$ *for all* $i > 5$.

(4) $v_4 = -e_4 + e_3 + e_1$ *and* $|v_i| = 2$ *for all* $i > 4$ *except possibly one* $v_m = -e_m + e_{m-1} + \cdots + e_2$.

Proof When $v_4 = -e_4 + e_3 + e_2$ and $a_0 = 6$, item (1) holds by Lemma 11.7. When $v_4 = -e_4 + e_3 + e_2$ and $a_0 < 6$, item (2) holds by Lemma 11.8. When $v_4 = -e_4 + e_3 + e_1$ and $V_5 = \{0, 1\}$, item (3) holds by Lemma 11.9. When $v_4 = -e_4 + e_3 + e_1$ and $V_5 = \varnothing$ or $\{2, 3\}$, item (4) holds by Lemma 11.10 if there exists $V_m = \{2, 3\}$, or by Lemma 11.4 if there is no $V_m = \{2, 3\}$.   □

The corresponding changemakers are

- $(1, 1, 3, 3, 4^{[s]}, (4s + 6)^{[t]})$ for $s + t > 0$,

- $(1, 1, 3, 3, 6, 8^{[s]}, (8s + 6)^{[t]})$,

- $(1, 1, 3, 3, 4, 6^{[s]})$ for $s > 0$.

12   Determining p and q

Having classified all the $(n+3)$–dimensional D–type lattices which are changemaker lattices, we now aim to concretely compute the pairs (p, q) for these lattices. More precisely, we will give a list $\mathcal{P}^-$ of prism manifolds with the property that if positive surgery on some knot in S^3 results in a prism manifold $P(p, q)$ with $q < 0$, then $P(p, q) \in \mathcal{P}^-$.

Recall from Section 2 that a prism manifold $P(p, q)$ is the boundary of a sharp 4–manifold $X(p, q)$ when $q < 0$. The homology group $H_2(X(p, q))$ equipped with the

inner product $-Q_{X(p,q)}$ becomes a lattice isomorphic to $\Delta(p,q)$. From the integers $a_0, a_1, \ldots, a_n$ in (10), we can recover p and q using (7).

Our strategy to determine p and q is as follows. Let $S = \{v_1, \ldots, v_{n+3}\}$ be the standard basis for a changemaker lattice that is isomorphic to a D–type lattice. Convert S into a vertex basis, denoted $S^* = \{v_1^*, \ldots, v_{n+3}^*\}$; that is, we need to do a change of basis to convert S to a vertex basis; see the example below for instance. (Also, all the linear transformations needed to change the standard bases corresponding to the changemaker vectors to vertex bases are presented in Table 4, part I.) Using the results of Section 4, the a_i will be recovered. We can then get the pair (p,q) using (7). The following lemma helps to simplify some involved computations of continued fractions that we will face. These properties are the first two items of [10, Lemma 9.5].

Lemma 12.1 *For integers* $r, s, t \geq 2$,

(1) $[\ldots, r, 2^{[s]}, t, \ldots]^- = [\ldots, r - 1, -(s + 1), t - 1, \ldots]^-$, *and*

(2) $[\ldots, s, 2^{[t]}]^- = [\ldots, s - 1, -(t + 1)]^-$.

Example 12.2 We illustrate how to obtain the pair (p,q) in the case when $f = 3$, v_2 is tight, and $|v_4| = 3$ (cf Proposition 11.11(2)). We point out that the chosen changemaker corresponds to the parameters $t = 0$ and $s \geq 1$. Let $S = \{v_1, \ldots, v_{n+3}\}$ denote the standard basis for the changemaker lattice $L = (\sigma)^\perp$. One easily computes

$$\sigma = (1, 1, 3, 3, 6, 8^{[n-1]}), \quad n \geq 2.$$

From the pairing graph it follows that S is not a vertex basis. Taking

$$v_2^* = v_2 - v_1 - v_5 - \cdots - v_{n+3},$$

we get that

$$S^* = \{v_1, v_3\} \cup \{v_2^*, v_{n+3}, \ldots, v_5, v_4\}$$

is indeed a vertex basis with norms given by the tuple

$$V^* = (2, 2, 4, 2^{[n-2]}, 4, 3), \quad n \geq 2.$$

Using Lemma 12.1 together with (7), we get

$$(21) \qquad \frac{-q}{p} = [4 - 1, 2^{[n-2]}, 4, 3]^- = \frac{16(n-1) + 14}{8(n-1) + 3}.$$

Observe that $-q = 2(p + 4)$. Moreover, looking at the denominator of the right-hand side of (21), we get that $p \equiv 3 \pmod 8$. Since $n \geq 2$, we have $p \geq 11$. The case in the example is in Table 1, type 4 with $r = 2(n-1) + 1$, $p = 4r - 1$ and $-q = 8r + 6$.

type	$P(p,q)$	range of parameters (p and r are always odd, $p > 1$)
1A	$P\left(p, -\frac{1}{2}(p^2 - 3p + 4)\right)$	$p \geq 7$
1B	$P\left(p, -\frac{1}{22}(p^2 - 3p + 4)\right)$	$p \equiv 17$ or $19 \pmod{22}$, $\quad p > 22$
2	$P\left(p, -\frac{1}{\lvert 4r+2 \rvert}(r^2 p + 1)\right)$	$r \equiv -1 \pmod 4$, $\quad r \neq -1, 3$, $p \equiv 2r - 3 \pmod{4r + 2}$
3A	$P\left(p, -\frac{1}{2r}(p+1)(p+4)\right)$	$r \geq 1$, $p \equiv -1 \pmod{2r}$, $\quad p \geq 4r - 1$
3B	$P\left(p, -\frac{1}{2r}(p+1)(p+4)\right)$	$r \geq 5$, $p \equiv r - 4 \pmod{2r}$, $\quad p \geq 3r - 4$
4	$P\left(p, -\frac{1}{2r^2}((2r+1)^2 p + 1)\right)$	$r \neq -3, -1, 1$, $p \equiv 4r - 1 \pmod{2r^2}$, $\quad p \geq 4r - 1$
5	$P\left(p, -\frac{1}{r^2 - 2r - 1}(r^2 p + 1)\right)$	$r \neq 1$, $p \equiv 2r - 5 \pmod{r^2 - 2r - 1}$, $\quad p \geq 2r - 5$
sporadic	$P(11, -30)$, $P(17, -31)$, $P(13, -47)$, $P(23, -64)$	

Table 1: $\mathcal{P}^-$, table of $P(p,q)$ that are realizable, $q < 0$.

Similar computations for the D–type changemaker lattices give prism manifolds $P(p,q)$, so that up to reparametrization, each falls into one of the families in Table 1. We shall denote the collection of these families by $\mathcal{P}^-$. Here we divide the families so that each changemaker vector corresponds to a unique family. The detailed correspondence between the changemaker vectors and $P(p,q)$ can be found in Table 4. It should be noted that, as discussed in Section 2, the positive integer p is always odd.

13 Primitive/Seifert-fibered knots admitting prism manifold surgeries

In the previous sections, we provided a list of changemaker vectors in $\mathbb{Z}^{n+4}$ whose orthogonal complements are isomorphic to D–type lattices. This list gave rise to a collection $\mathcal{P}^-$ of prism manifolds. Let σ be a changemaker vector that corresponds to $P(p,q) \in \mathcal{P}^-$. We want find a knot $K_\sigma \subset S^3$ on which some surgery yields $P(p,q)$. All known such knots are P/SF knots, which are classified in the forthcoming work of Berge and Kang [2]; we outline their work here based on the preprint Berge shared with us. To start, we need to recall some definitions.

Definition 13.1 A *Seifert-fibered* 3–*manifold* is a 3–manifold together with a decomposition into disjoint simple closed curves, called fibers, such that each fiber has a tubular neighborhood that forms a standard fibered torus. A *standard fibered torus* corresponding to a pair of coprime integers (m, n), with m positive, is the surface bundle of the diffeomorphism of a disk given by rotation by an angle of $2\pi n/m$, with the natural fibering by circles. If $m > 1$, then the middle fiber is called *singular*.

Definition 13.2 Let H be a genus-two handlebody. A simple closed curve $c \subset \partial H$ is *primitive* in H if $H[c]$, the manifold obtained by adding a two-handle to H along c, is a solid torus. A simple closed curve $c \subset \partial H$ is *Seifert-fibered* in H if $H[c]$ is a Seifert-fibered space.

Definition 13.3 Let Σ denote the genus-two Heegaard surface of the standard genus-two Heegaard splitting of $S^3 = H \cup_\Sigma H'$. A knot $K \subset S^3$ is called *primitive/Seifert-fibered*, denoted P/SF, if it has a presentation as a simple closed curve on Σ such that K is primitive in H' and Seifert-fibered in H. The isotopy class in $\partial \nu(K)$ of the curves in $\partial \nu(K) \cap \Sigma$ is called the *surface slope of K with respect to Σ*. Below, we will use $\gamma \in \mathbb{Z}$ to denote the surface slope.

P/SF knots are generalizations of doubly primitive knots (ie knots that can be isotoped to lie on Σ and are primitive in both handlebodies H and H') studied by Berge [1]. In [6], Dean studied the Dehn surgery on P/SF knots along their surface slopes. He proved that the surface slope surgery on a P/SF knot results in either a Seifert-fibered manifold or a connected sum of lens spaces [6, Proposition 2.3]. Also, Eudave Muñoz proved that the latter case does not happen for hyperbolic P/SF knots [9, Theorem 4].

Berge and Kang, in [2], classified all P/SF knots. Furthermore, given a P/SF knot K, they specified the indices of the singular fibers of the Seifert-fibered manifold obtained from the surface slope surgery on K.

Recall that for a pair of relatively prime integers $p > 1$ and q, a prism manifold $P(p, q)$ is a Seifert-fibered manifold with base orbifold S^2 and three singular fibers of index

$$(22) \qquad\qquad\qquad\qquad 2, 2, p.$$

Using (2), if $P(p, q)$ arises from the surface slope surgery on a P/SF knot, then

$$(23) \qquad\qquad\qquad\qquad \gamma = \text{surface slope} = \pm 4q.$$

type	$P(p,q)$	range of parameters (p and r are always odd, $p > 1$)
1A	$P\left(p, \frac{1}{2}(p^2 + 3p + 4)\right)$	
1B	$P\left(p, \frac{1}{22}(p^2 + 3p + 4)\right)$	$p \equiv 5$ or $3 \pmod{22}$
2	$P\left(p, \frac{1}{\|4r+2\|}(r^2 p - 1)\right)$	$r \equiv -1 \pmod 4$, $p \equiv -2r + 3 \pmod{4r+2}$
3A	$P\left(p, \frac{1}{2r}(p-1)(p-4)\right)$	$p \equiv 1 \pmod{2r}$
3B	$P\left(p, \frac{1}{2r}(p-1)(p-4)\right)$	$p \equiv r + 4 \pmod{2r}$
4	$P\left(p, \frac{1}{2r^2}\left((2r+1)^2 p - 1\right)\right)$	$p \equiv -4r + 1 \pmod{2r^2}$
5	$P\left(p, \frac{1}{r^2 - 2r - 1}(r^2 p - 1)\right)$	$r \neq 1$, $p \equiv -2r + 5 \pmod{r^2 - 2r - 1}$
sporadic	$P(11, 19)$, $P(13, 34)$	

Table 2: $\mathcal{P}^+$, table of $P(p,q)$ that are realizable, $q > 0$.

The surface slope given in [2] is not necessarily positive, while we only consider positive surgery. To get a knot with positive prism manifold surgery, we need to take the mirror image of the Berge–Kang knot if the surface slope is negative. Suppose the surface slope surgery on K results in $P(p,q)$. Then p is the index of the singular fiber with odd index, and q satisfies (23). With a bit more work, we can determine the sign of q. Since Berge and Kang's paper [2] is not publicly available, we will omit the

prism manifold	type	changemaker	braid word
$P(3, -14)$	3A	$(1, 1, 2, 5, 5)$	$(\sigma_1 \cdots \sigma_{11})^5 (\sigma_1)^{-2}$
	5	$(1, 1, 3, 3, 6)$	$(\sigma_3\sigma_4\sigma_5\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3)^3 (\sigma_1\sigma_2)^{10}$ ((19, 3)–cable of $T(3,2)$)
$P(11, -18)$	2	$(1, 1, 2, 4, 5, 5)$	$(\sigma_1 \cdots \sigma_{13})^5 \sigma_1 \sigma_2$
	3B	$(1, 1, 3, 3, 4, 6)$	$(\sigma_1 \cdots \sigma_5)^{10} (\sigma_1\sigma_2\sigma_3)^3$
$P(8s + 3, -(16s + 14))$ $s \geq 1$	3A	$(1, 1, 2, 5, 5, 8^{[s]})$	$(\sigma_1 \cdots \sigma_7)^{5+8s}(\sigma_7 \cdots \sigma_1)^2$
	4	$(1, 1, 3, 3, 6, 8^{[s]})$	$(\sigma_1 \cdots \sigma_{13})^8 (\sigma_1 \cdots \sigma_7)^{8s-7}$
	spor $s = 1$	$(1, 1, 2, 4, 7, 7)$	$(\sigma_1 \cdots \sigma_{17})^7 (\sigma_2\sigma_1)^{-2}$

Table 3: Prism manifolds arising from multiple changemaker vectors.

Prop^n	changemaker vector	vertex basis (with x_*, x_{**} omitted) $\{x_0, \ldots, x_n\}$
9.7	$(1, 1, 2, 4, 7, 7)$	$\{v_4, -v_3, v_2\}$
	$(1, 1, 2, 3, 3, 10)$	$\{v_3, -v_5, v_2\}$
	$(1, 1, 2, 2, 4, 9, 9)$	$\{v_5, -v_4, -v_2, -v_3\}$
	$(1, 1, 2, 2, 5, 5, 14)$	$\{v_4, -v_6, -v_2, -v_3\}$
	$(1, 1, 2^{[s+1]}, (2s+3)^{[2]}, 4s+6, 8s+14)$	$\{v_{s+3}, v_{s+5}, -v_{s+6}, -v_{s+2}, \ldots, -v_2\}$
	$(1, 1, 2^{[s+1]}, (2s+3)^{[2]}, 4s+8, 8s+14)$	$\{v_{s+3}, v_{s+6}, -v_{s+5}, -v_{s+2}, \ldots, -v_2\}$
	$(1, 1, 2, 3, 3, 8, 8^{[s]}, 8s+10, (8s+18)^{[t]})$	$\{v_3, -v_{s+6}, -v_2, -v_5, \ldots, -v_{s+5}, -v_{s+7}, \ldots, -v_{s+t+6}\}$
6.5	$(1, 1, 1, 1, 2^{[s]}), \quad s > 0$	$\{v_2, v_4, v_5, \ldots, v_{s+3}\}$
	$(1, 1, 1, 1, 4^{[s]}, 4s+2, (4s+6)^{[t]})$	$\{v_2, v_{s+4}, -v_4, \ldots, -v_{s+3}, -v_{s+5}, \ldots, -v_{t+s+4}\}$
10.4	$(1, 1, 2^{[s+1]}, (2s+5)^{[2]}, (4s+8)^{[t]}),$ $t > 0$	$\{v_{s+5}, \ldots, v_{t+s+4}, v_{s+3} - v_{[1,s+2]} - v_{[s+5, t+s+4]}, v_{s+2}, \ldots, v_2\}$
	$(1, 1, 2^{[s+1]}, (2s+5)^{[2]})$	$\{v_{s+3} - v_{[1,s+2]}, v_{s+2}, \ldots, v_2\}$
	$(1, 1, 2^{[s+1]}, (2s+5)^{[2]}, 4s+8, (4s+10)^{[t]}),$ $t > 0$	$\{v_{s+5}, v_{s+3} - v_{[1,s+2]} - v_{[s+5, s+t+5]}, v_{s+t+5}, \ldots, v_{s+6}, v_{s+2}, \ldots, v_2\}$
11.11	$(1, 1, 3, 3, 4^{[s]}, (4s+6)^{[t]}), \quad s > 0$	$\{v_4, \ldots, v_{s+3}, v_2 - v_1 - v_{[4, s+3]}, v_{s+4}, \ldots, v_{t+s+3}\}$
	$(1, 1, 3, 3, 6^{[t]}), \quad t > 0$	$\{v_2 - v_1, v_4, \ldots, v_{t+3}\}$
	$(1, 1, 3, 3, 4, 6^{[s]}), \quad s > 0$	$\{v_4, v_2 - v_1 - v_{[4, s+4]}, v_{s+4}, \ldots, v_5\}$
	$(1, 1, 3, 3, 6, 8^{[s]}, (8s+6)^{[t]}), \quad s > 0$	$\{v_2 - v_1 - v_{[5, s+4]}, v_{s+4}, \ldots, v_4, v_{s+5}, \ldots, v_{t+s+4}\}$
8.1	$(1, 1, 2, 2, 2^{[s]}, (2s+3)^{[2]}), \quad s > 0$	$\{v_{s+4}, v_3, v_4, \ldots, v_{s+3}, -v_{[2, s+3]}\}$
	$(1, 1, 2, 4, 4^{[s]}, (4s+5)^{[2]}, (8s+14)^{[t]})$	$\{v_{s+4}, -v_2, v_3, \ldots, v_{s+3}, v_{s+6}, \ldots, v_{t+s+5}\}$
	$(1, 1, 2, 2, 4^{[s]}, (4s+3)^{[2]}, (8s+10)^{[t]}),$ $s > 0$	$\{v_{s+4} + v_3, -v_3, -v_2, -v_4, \ldots, -v_{s+3}, -v_{s+6}, \ldots, -v_{t+s+5}\}$
	$(1, 1, 2, 2, 3^{[2]}, 10^{[t]})$	$\{v_4 + v_3, -v_3, -v_2, -v_6, \ldots, -v_{t+5}\}$

Table 4: D–type changemakers vs prism manifolds, I.

process of determining the sign here. Instead, we directly give the list $\mathcal{P}^+$ of realizable $P(p, q)$ with $q > 0$ in Table 2 without proof, and will justify this list in future work. Unlike Table 1, we do not have the notion of changemaker vector here. We divide these

Propⁿ	vertex norms $\{a_0, \ldots, a_n\}$	prism manifold parameters	$\mathcal{P}^-$ type
9.7	$\{4, 4, 3\}$	$p = 11, \quad -q = 30$	spor
	$\{3, 6, 3\}$	$p = 17, \quad -q = 31$	spor
	$\{5, 3, 3, 2\}$	$p = 13, \quad -q = 47$	spor
	$\{4, 5, 3, 2\}$	$p = 23, \quad -q = 64$	spor
	$\{s + 3, 3, 5, 2^{[s]}, 3\}$	$p = 22s + 39, \quad -q = 22s^2 + 75s + 64$	1B
	$\{s + 3, 4, 4, 2^{[s]}, 3\}$	$p = 22s + 41, \quad -q = 22s^2 + 79s + 71$	1B
	$\{3, s + 3, 3, 4, 2^{[s]}, 3, 2^{[t-1]}\}$	$p = 2r^2 t + 2r^2 + 4r - 1,$ $-q = (2r + 1)^2 (t + 1) + 8r + 6,$ $r = -(2s + 5)$	4
6.5	$\{2, 3, 2^{[s-1]}\}$	$p = 2s + 1, \quad -q = s + 1, \quad r = -1$	5
	$\{2, s + 3, 5, 2^{[s-1]}, 3, 2^{[t-1]}\}$	$p = r^2 - 6 + (r^2 - 2r - 1)t,$ $-q = r^2(t + 1) + 2r - 1,$ $r = -(2s + 3)$	5
10.4	$\{s + 4, 2^{[t-1]}, 4, 2^{[s]}, 3\}$	$p = (2t + 1)(2s + 4) - 1,$ $-q = (s + 2)((2t + 1)(2s + 4) + 3),$ $r = 2t + 1$	3A
	$\{s + 6, 2^{[s]}, 3\}$	$p = 2s + 3, -q = 2s^2 + 11s + 14,$ $r = 1$	3A
	$\{s + 4, 3, 2^{[t-1]}, 3, 2^{[s]}, 3\}$	$p = (2t + 3)(2s + 5) - 4,$ $-q = (2s + 5)((2t + 3)s + 5t + 6),$ $r = 2t + 3$	3B

Table 4: D–type changemakers vs prism manifolds, II.

prism manifolds into families in such a way that it reflects the "symmetry" between the two tables, noting that a knot may correspond to more than one family in Table 2. We take $\mathcal{P} = \mathcal{P}^- \cup \mathcal{P}^+$.

13.1 Manifolds corresponding to distinct changemaker vectors

As we mentioned before, the families in Table 1 are divided so that every changemaker vector corresponds to a unique family. However, a lattice $\Delta(p, q)$ may be isomorphic to the orthogonal complements of different changemaker vectors. Thus the corresponding prism manifold $P(p, q)$ lies in different families, and not just one. In Table 3, we list all such $P(p, q)$. Each of these prism manifolds is contained in two families, except that $P(11, -30)$ is contained in three families. If a P/SF knot admits a surgery to such

Prop^n	vertex norms $\{a_0,\ldots,a_n\}$	prism manifold parameters	$\mathcal{P}^-$ type
11.11	$\{3, 2^{[s-1]}, 5, s+3, 2^{[t-1]}\}$	$p = (r^2-2r-1)t+2r-5,$ $-q = r^2t+2r-1, \quad r = 2s+3$	5
	$\{6, 3, 2^{[t-1]}\}$	$p = 2t+1, \quad -q = 9t+5, \quad r = 3$	5
	$\{3, 3, 2^{[s-1]}, 4\}$	$p = 6s+5, \quad -q = 9s+9, \quad r = 2s+3$	3B
	$\{4, 2^{[s-1]}, 4, 3, s+2, 2^{[t-1]}\}$	$p = 2r^2t+4r-1$ $-q = (2r+1)^2t+8r+6, \quad r = 2s+1$	4
8.1	$\{s+3, 2^{[s+1]}, 3\}$	$p = 2s+5, \quad -q = 2s^2+7s+7$	1A
	$\{s+3, 3, 4, 2^{[s]}, 4, 2^{[t-1]}\}$	$p = 16ts+30t+8s+11,$ $-q = 16ts^2+56ts+8s^2+24s+49t+18,$ $r = 4s+7$	2
	$\{s+3, 2, 3, 3, 2^{[s-1]}, 4, 2^{[t-1]}\}$	$p = 16ts+18t+8s+5,$ $-q = 16ts^2+40ts+8s^2+25t+16s+7,$ $r = -4s-5$	2
	$\{3, 2, 3, 5, 2^{[t-1]}\}$	$p = 18t+5, \quad -q = 25t+7, \quad r = -5$	2

Table 4: D–type changemakers vs prism manifolds, III.

a $P(p,q)$, we require extra information in order to detect the changemaker vector that corresponds to this knot. The information we will collect is the Alexander polynomial.

Let σ be a changemaker vector such that $\Delta(p,q) \cong (\sigma)^{\perp}$. Assume that σ corresponds to a knot K admitting a surgery to $P(p,q)$. Using Lemma 2.10, we can compute the Alexander polynomial $\Delta_K(T)$. We will explicitly exhibit a P/SF knot K_0 admitting a surgery to $P(p,q)$, and directly compute $\Delta_{K_0}(T)$ to check that it is equal to the predicted Alexander polynomial $\Delta_K(T)$. So K_0 matches with σ.

13.2 Proofs of the main results

This subsection is devoted to the proof of Theorems 1.2 and 1.6.

Proof of Theorem 1.6 If $\Delta(p,q)$ is isomorphic to a changemaker lattice L then it belongs to the families classified in Sections 6–11. As in Section 12, we can find a pair (p',q') such that L is isomorphic to $\Delta(p',q')$, and $P(p',q')$ is in $\mathcal{P}^-$. Using Proposition 3.12, the result follows. □

Table 4 parts I, II and III collect all results, matching each of the changemakers deduced from Propositions 6.5, 8.1, 9.7, 10.4 and 11.11 to one pair $(p,q) \in \mathcal{P}^-$. In all families of changemaker vectors, the parameters s and t are assumed to be nonnegative, unless

otherwise indicated. In Table 4, $v_{[a,b]}$ means $v_a + v_{a+1} + \cdots + v_b$ for $a < b$. All vertex bases are presented in the form $\{x_0, x_1, \ldots, x_n\}$. A superscript $[-1]$ at an element in the sequence of vertex norms means that the sequence is truncated just before the element preceding it. For example, the sequence $\{3, s+3, 3, 4, 2^{[s]}, 3, 2^{[t-1]}\}$ becomes $\{3, s+3, 3, 4, 2^{[s]}\}$ when $t = 0$.

Proof of Theorem 1.2 It follows from the discussion in this and the last subsections together with Theorem 1.6 that if $P(p,q) \cong S^3_{4|q|}(K)$, then $P(p,q)$ is in $\mathcal{P}^-$, and $\Delta_K(t)$ is determined by $P(p,q)$ and the family containing it. By [19], the knot Floer homology group of K is also determined. $\square$

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