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The Segal conjecture for infinite discrete groups

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We formulate and prove a version of the Segal conjecture for infinite groups. For finite groups it reduces to the original version. The condition that G is finite is replaced in our setting by the assumption that there exists a finite model for the classifying space $\underline{E}G$ for proper actions. This assumption is satisfied for instance for word hyperbolic groups or cocompact discrete subgroups of Lie groups with finitely many path components. As a consequence we get for such groups G that the zeroth stable cohomotopy of the classifying space BG is isomorphic to the I–adic completion of the ring given by the zeroth equivariant stable cohomotopy of $\underline{E}G$ for I the augmentation ideal.

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0 Introduction

We first recall the Segal conjecture for a finite group G. The equivariant stable cohomotopy groups $\pi_G^n(X)$ of a G–CW–complex are modules over the ring $\pi_G^0 = \pi_G^0(*)$, which can be identified with the Burnside ring $A(G)$. Here and elsewhere $*$ denotes the one-point space. The augmentation homomorphism $A(G) \to \mathbb{Z}$ is the ring homomorphism sending the class of a finite set to its cardinality. The augmentation ideal $\mathbb{I}_G \subseteq A(G)$ is its kernel. Let $\pi_G^m(X)_{\widehat{\mathbb{I}_G}}$ be the $\mathbb{I}_G$–adic completion $\operatorname{invlim}_{n\to\infty} \pi_G^m(X)/\mathbb{I}_G^n \cdot \pi_G^m(X)$ of $\pi_G^m(X)$.

The following result was formulated as a conjecture by Segal and proved by Carlsson [6].

Theorem 0.1 (Segal conjecture for finite groups) *For every finite group G and finite G–CW–complex X there is an isomorphism*

$$\pi_G^m(X)_{\widehat{\mathbb{I}_G}} \xrightarrow{\cong} \pi_s^m(EG \times_G X).$$

In particular we get for $X = *$ and $m = 0$ an isomorphism

$$A(G)_{\widehat{\mathbb{I}_G}} \xrightarrow{\cong} \pi_s^0(BG).$$

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The purpose of this paper is to formulate and prove a version of it for infinite (discrete) groups, ie we will show:

Theorem 0.2 (Segal conjecture for infinite groups) *Let G be a (discrete) group. Let X be a finite proper G–CW–complex. Let L be a proper finite-dimensional G–CW–complex with the property that there is an upper bound on the order of its isotropy groups. Let $f\colon X \to L$ be a G–map.*

Then the map of pro-$\mathbb{Z}$–modules

$$\lambda_G^m(X)\colon \{\pi_G^m(X)/\mathbb{I}_G(L)^n \cdot \pi_G^m(X)\}_{n\geq 1} \xrightarrow{\cong} \{\pi_s^m((EG \times_G X)_{(n-1)})\}_{n\geq 1}$$

defined in (3.3) *is an isomorphism of pro-$\mathbb{Z}$–modules.*

In particular we obtain an isomorphism

$$\pi_G^m(X)_{\widehat{\mathbb{I}_G(L)}} \cong \pi_s^m(EG \times_G X).$$

If there is a finite G–CW–model for $\underline{E}G$, we obtain an isomorphism

$$\pi_G^m(\underline{E}G)_{\widehat{\mathbb{I}_G(\underline{E}G)}} \cong \pi_s^m(BG).$$

Here $\underline{E}G$ is the classifying space for proper G–actions and $\pi_G^*(X)$ is equivariant stable cohomotopy as defined in Lück [11, Section 6]. The ideal $\mathbb{I}_G(L)$ is the augmentation ideal in the ring $\pi_G^0(L)$; see Definition 3.1. We view $\pi_G^*(X)$ as a $\pi_G^0(L)$–module by the multiplicative structure on equivariant stable cohomotopy and the map f. We denote by $\pi_G^m(X)_{\widehat{\mathbb{I}_G(L)}}$ its $\mathbb{I}_G(L)$–completion. More explanations will follow in the main body of the text.

In [11] various mutually distinct notions of a Burnside ring of a group G are introduced, which all agree with the standard notion for finite G. If there is a finite G–CW–model for $\underline{E}G$, then the homotopy theoretic definition is $A(G) := \pi_G^0(\underline{E}G)$; we define the ideal $\boldsymbol{I}_G \subseteq A(G)$ to be $\boldsymbol{I}_G(\underline{E}G)$, and we get in this notation from Theorem 0.2 an isomorphism

$$A(G)_{\widehat{\boldsymbol{I}_G}} \cong \pi_s^0(BG).$$

We will actually formulate for every equivariant cohomology theory $\mathcal{H}_?^*$ with multiplicative structure a "completion theorem"; see Problem 3.4. It is not expected to be true in all cases. We give a strategy for its proof in Theorem 4.1. We show that this applies to equivariant stable cohomotopy, thus proving Theorem 0.2. It also applies to equivariant topological K–theory, where the completion theorem for infinite groups has already been proved in Lück and Oliver [16].

If G is finite, we can take $L = \underline{E}G = *$, then Theorem 0.2 reduces to Theorem 0.1. We will not give a new proof of Theorem 0.1, but use it as input in the proof of Theorem 0.2.

This paper is part of a general program to systematically study equivariant homotopy theory, which is well-established for finite groups and compact Lie groups, for infinite groups and noncompact Lie groups. The motivation comes among other things from the Baum–Connes conjecture and the Farrell–Jones conjecture.

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1 Equivariant cohomology with multiplicative structure

We briefly recall the axioms of a (proper) equivariant cohomology $\mathcal{H}^*_?$ theory with values in R–modules and with multiplicative structure. More details can be found in [12].

Let G be a (discrete) group. Let R be a commutative ring with unit. A (*proper*) *G–cohomology theory $\mathcal{H}^*_G$ with values in R–modules* assigns to any pair (X, A) of (proper) G–CW–complexes a $\mathbb{Z}$–graded R–module $\{\mathcal{H}^n_G(X, A) \mid n \in \mathbb{Z}\}$ such that G–homotopy invariance holds and there exist long exact sequences of pairs and long exact Mayer–Vietoris sequences. Often one also requires the disjoint union axiom, which we will need not here since all our disjoint unions will be over finite-index sets.

A *multiplicative structure* is given by a collection of R–bilinear pairings

$$\cup\colon \mathcal{H}^m_G(X, A) \otimes_R \mathcal{H}^n_G(X, B) \to \mathcal{H}^{m+n}_G(X, A \cup B).$$

This product is required to be graded commutative, to be associative, to have a unit $1 \in \mathcal{H}^0_G(X)$ for every (proper) G–CW–complex X, to be compatible with boundary homomorphisms and to be natural with respect to G–maps.

Let $\alpha\colon H \to G$ be a group homomorphism. Given an H–space X, define the *induction of X with α* to be the G–space $\operatorname{ind}_\alpha X$ which is the quotient of $G \times X$ by the right H–action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$. If $\alpha\colon H \to G$ is an inclusion, we also write ind_H^G instead of $\operatorname{ind}_\alpha$.

A *(proper) equivariant cohomology theory* $\mathcal{H}^*_?$ *with values in* R*–modules* consists of a collection of (proper) G–cohomology theories $\mathcal{H}^*_G$ with values in R–modules for each group G, together with the following so-called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a (proper) H–CW–pair (X, A) there are for each $n \in \mathbb{Z}$ natural homomorphisms

$$\text{(1.1)} \qquad \operatorname{ind}_\alpha\colon \mathcal{H}^n_G(\operatorname{ind}_\alpha(X, A)) \to \mathcal{H}^n_H(X, A).$$

If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha\colon \mathcal{H}^n_G(\operatorname{ind}_\alpha(X, A)) \to \mathcal{H}^n_H(X, A)$ is bijective for all $n \in \mathbb{Z}$. The induction structure is required to be compatible with the boundary homomorphisms, to be functorial in α and to be compatible with inner automorphisms.

Sometimes we will need the following lemma, whose elementary proof is analogous to the one in [10, Lemma 1.2].

Lemma 1.2 *Consider finite subgroups* $H, K \subseteq G$ *and an element* $g \in G$ *with* $gHg^{-1} \subseteq K$. *Let* $R_{g^{-1}}\colon G/H \to G/K$ *be the* G*–map sending* $g'H$ *to* $g'g^{-1}K$ *and let* $c(g)\colon H \to K$ *be the group homomorphism sending* h *to* ghg^{-1}. *Denote by* $\operatorname{pr}\colon (\operatorname{ind}_{c(g)\colon H\to K} *) \to *$ *the projection to the one-point space* $*$.

Then the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{H}^n_G(G/K) & \xrightarrow{\mathcal{H}^n_G(R_{g^{-1}})} & \mathcal{H}^n_G(G/H) \\
\Big\downarrow{\scriptstyle \operatorname{ind}^G_K} & & \Big\downarrow{\scriptstyle \operatorname{ind}^G_H} \\
\mathcal{H}^n_K(*) & \xrightarrow[\operatorname{ind}_{c(g)} \circ \mathcal{H}^n_K(\operatorname{pr})]{} & \mathcal{H}^n_H(*)
\end{array}$$

Let $\mathcal{H}^*_?$ be a (proper) equivariant cohomology theory. A *multiplicative structure* on it assigns a multiplicative structure to the associated (proper) G–cohomology theory $\mathcal{H}^*_G$ for every group G such that for each group homomorphism $\alpha\colon H \to G$, the maps given by the induction structure of (1.1) are compatible with the multiplicative structures on $\mathcal{H}^*_G$ and $\mathcal{H}^*_H$.

Example 1.3 (equivariant cohomology theories coming from nonequivariant ones) Let $\mathcal{K}^*$ be a (nonequivariant) cohomology theory with multiplicative structure, for instance singular cohomology or topological K–theory. We can assign to it an equivariant cohomology theory with multiplicative structure $\mathcal{H}^*_?$ in two ways. Namely, for a group G and a pair of G–CW–complexes (X, A), we define $\mathcal{H}^n_G(X, A)$ by $\mathcal{K}^n(G\backslash(X, A))$ or by $\mathcal{K}^n(EG \times_G (X, A))$.

Example 1.4 (equivariant topological K–theory) In [16], equivariant topological K–theory is defined for finite proper equivariant CW–complexes in terms of equivariant vector bundles. It reduces to the classical notion, which appears for instance in [2]. Its relation to equivariant KK–theory is explained in [17]. This definition is extended to (not necessarily finite) proper equivariant CW–complexes in [16] in terms of equivariant spectra using Γ–spaces and yields a proper equivariant cohomology theory $K_?^*$ with multiplicative structure, as explained in [12, Example 1.7]. It has the property that for any finite subgroup H of a group G we have

$$K_G^n(G/H) = K_H^n(*) = \begin{cases} R_{\mathbb{C}}(H) & \text{for } n \text{ even,} \\ \{0\} & \text{for } n \text{ odd,} \end{cases}$$

where $R_{\mathbb{C}}(H)$ denotes the complex representation ring of H.

Example 1.5 (equivariant stable cohomotopy) In [11, Section 6], equivariant stable cohomotopy $\pi_?^*$ is defined for finite proper equivariant CW–complexes in terms of maps of sphere bundles associated to equivariant vector bundles. For finite G it reduces to the classical notion. This definition is extended to arbitrary proper G–CW–complexes by Degrijse–Hausmann–Lück–Patchkoria–Schwede [7], where a systematic study of equivariant homotopy theory for (not necessarily compact) Lie groups and proper G–CW–complexes is developed.

Let $H \subseteq G$ be a finite subgroup. Recall that by the induction structure we have $\pi_G^n(G/H) = \pi_H^n(*)$. The equivariant stable homotopy groups π_H^n are computed in terms of the splitting due to Segal [18, Proposition 2] and tom Dieck [8, Chapter II, Theorem 7.7 on page 154] by

$$\pi_n^H = \pi_{-n}^H = \bigoplus_{(K)} \pi_{-n}^s(BW_H K),$$

where π_{-n}^s denotes (nonequivariant) stable homotopy and (K) runs through the conjugacy classes of subgroups of H. In particular, we get

$$|\pi_G^n(G/H)| < \infty \quad \text{for } n \le -1, \quad \pi_G^0(G/H) = A(H), \quad \pi_G^n(G/H) = \{0\} \quad \text{for } n \ge 1,$$

where $A(H)$ is the Burnside ring.

2 Some preliminaries about pro-modules

It will be crucial to handle pro-systems and pro-isomorphisms and not to pass directly to inverse limits. In this section we fix our notation for handling pro-R–modules for a commutative ring with unit R. For the definitions in full generality see for instance [1, Appendix] or [4, Section 2].

For simplicity, all pro-R–modules dealt with here will be indexed by the positive integers. We write $\{M_n,\alpha_n\}$, or briefly $\{M_n\}$, for the inverse system

$$M_0 \xleftarrow{\alpha_1} M_1 \xleftarrow{\alpha_2} M_2 \xleftarrow{\alpha_3} M_3 \xleftarrow{\alpha_4} \cdots ,$$

and also write $\alpha_n^m := \alpha_{m+1}\circ\cdots\circ\alpha_n\colon M_n \to M_m$ for $n > m$, and put $\alpha_n^n = \mathrm{id}_{M_n}$. For the purposes here, it will suffice (and greatly simplify notation) to work with "strict" pro-homomorphisms $\{f_n\}\colon \{M_n,\alpha_n\} \to \{N_n,\beta_n\}$, ie a collection of homomorphisms $f_n\colon M_n \to N_n$ for $n \ge 1$ such that $\beta_n \circ f_n = f_{n-1}\circ\alpha_n$ holds for each $n \ge 2$. Kernels and cokernels of strict homomorphisms are defined in the obvious way, namely levelwise.

A pro-R–module $\{M_n,\alpha_n\}$ will be called *pro-trivial* if, for each $m \ge 1$, there is some $n \ge m$ such that $\alpha_n^m = 0$. A strict homomorphism $f\colon \{M_n,\alpha_n\} \to \{N_n,\beta_n\}$ is a *pro-isomorphism* if and only if $\ker(f)$ and $\operatorname{coker}(f)$ are both pro-trivial, or, equivalently, for each $m \ge 1$ there is some $n \ge m$ such that $\operatorname{im}(\beta_n^m) \subseteq \operatorname{im}(f_m)$ and $\ker(f_n) \subseteq \ker(\alpha_n^m)$. A sequence of strict homomorphisms

$$\{M_n,\alpha_n\} \xrightarrow{\{f_n\}} \{M_n',\alpha_n'\} \xrightarrow{g_n} \{M_n'',\alpha_n''\}$$

will be called *exact* if the sequences of R–modules $M_n \xrightarrow{f_n} N_n \xrightarrow{g_n} M_n''$ is exact for each $n \ge 1$, and it is called *pro-exact* if $g_n \circ f_n = 0$ holds for $n \ge 1$ and the pro-R–module $\{\ker(g_n)/\operatorname{im}(f_n)\}$ is pro-trivial.

The elementary proofs of the following two lemmas can be found for instance in [14, Section 2].

Lemma 2.1 *Let* $0 \to \{M_n',\alpha_n'\} \xrightarrow{\{f_n\}} \{M_n,\alpha_n\} \xrightarrow{\{g_n\}} \{M_n'',\alpha_n''\} \to 0$ *be a pro-exact sequence of pro-*R*–modules. Then there is a natural exact sequence*

$$0 \to \operatorname*{invlim}_{n\ge1} M_n' \xrightarrow{\operatorname{invlim}_{n\ge1} f_n} \operatorname*{invlim}_{n\ge1} M_n \xrightarrow{\operatorname{invlim}_{n\ge1} g_n} \operatorname*{invlim}_{n\ge1} M_n''$$
$$\xrightarrow{\delta} \operatorname*{invlim}_{n\ge1}{}^1 M_n' \xrightarrow{\operatorname{invlim}^1{}_{n\ge1} 1 f_n} \operatorname*{invlim}_{n\ge1}{}^1 M_n \xrightarrow{\operatorname{invlim}^1{}_{n\ge1} 1 g_n} \operatorname*{invlim}_{n\ge1}{}^1 M_n'' \to 0.$$

In particular, a pro-isomorphism $\{f_n\}\colon \{M_n,\alpha_n\} \to \{N_n,\beta_n\}$ *induces isomorphisms*

$$\operatorname*{invlim}_{n\ge1} f_n\colon \operatorname*{invlim}_{n\ge1} M_n \xrightarrow{\cong} \operatorname*{invlim}_{n\ge1} N_n, \quad \operatorname*{invlim}_{n\ge1}{}^1 f_n\colon \operatorname*{invlim}_{n\ge1}{}^1 1 M_n \xrightarrow{\cong} \operatorname*{invlim}_{n\ge1}{}^1 N_n.$$

Lemma 2.2 *Fix any commutative Noetherian ring R, and any ideal $I \subseteq R$. Then for any exact sequence $M' \to M \to M''$ of finitely generated R–modules, the sequence*

$$\{M'/I^n M'\} \to \{M/I^n M\} \to \{M''/I^n M''\}$$

of pro-R–modules is pro-exact.

3 The formulation of a completion theorem

Consider a proper equivariant G–cohomology theory $\mathcal{H}^*_?$ with multiplicative structure. In the sequel, $\mathcal{H}^*$ is the nonequivariant cohomology theory with multiplicative structure given by $\mathcal{H}^*_G$ for $G = \{1\}$. Notice that $\mathcal{H}^0(*)$ is a commutative ring with unit and $\mathcal{H}^n_G(X)$ is a $\mathcal{H}^0(*)$–module. In some future applications $\mathcal{H}^0(*)$ will be just $\mathbb{Z}$. In the sequel, $[Y, X]^G$ denotes the set of G–homotopy classes of G–maps $Y \to X$. Notice that evaluation at the unit element of G induces a bijection $[G, X]^G \xrightarrow{\cong} \pi_0(X)$. It is compatible with the left G–actions, which are induced on the source by precomposing with right multiplication $r_g\colon G \to G$, $g' \mapsto g'g$, and on the target by the given left G–action on X.

So we can represent elements in $G\backslash\pi_0(X)$ by classes $\bar{x}$ of G–maps $x\colon G \to X$, where $x\colon G \to X$ and $y\colon G \to X$ are equivalent if for some $g \in G$, the composite $y \circ r_g$ is G–homotopic to x.

Definition 3.1 (augmentation ideal) For any proper G–CW–complex X, define the *augmentation module* $\mathbb{I}^n_G(X) \subseteq \mathcal{H}^n_G(X)$ to be the kernel of the map

$$\mathcal{H}^n_G(X) \xrightarrow{\prod_{\bar{x} \in G\backslash\pi_0(X)} \operatorname{ind}_{\{1\}\to G} \circ \mathcal{H}^n_G(x)} \prod_{\bar{x} \in G\backslash\pi_0(X)} \mathcal{H}^n(*).$$

(The composite above is independent of the choice of $x \in \bar{x}$ by G–homotopy invariance and Lemma 1.2.) If $n = 0$, the map above is a ring homomorphism and $\mathbb{I}_G(X) := \mathbb{I}^0_G(X)$ is an ideal called *the augmentation ideal.*

Given a G–map $f\colon X \to Y$, the induced map $\mathcal{H}^n_G(f)\colon \mathcal{H}^n_G(Y) \to \mathcal{H}^n_G(X)$ restricts to a map $\mathbb{I}^n_G(Y) \to \mathbb{I}^n_G(X)$.

We will need the following elementary lemma:

Lemma 3.2 *Let X be a CW–complex of dimension $n-1$. Then any n–fold product of elements in $\mathbb{I}^*_G(X)$ is zero.*

Proof Write $X = Y \cup A$, where Y and A are closed subsets, Y contains $X^{(n-2)}$ as a homotopy deformation retract, and A is a disjoint union of $(n-1)$–disks. Fix elements $v_1, v_2, \dots, v_n \in \mathbb{I}_G^*(X)$. We can assume by induction that $v_1 \cdots v_{n-1}$ vanishes after restricting to Y, and hence that it is the image of an element $u \in \mathcal{H}_G^*(X, Y)$. Also, v_n clearly vanishes after restricting to A, and hence is the image of an element $v \in \mathcal{H}_G^*(X, A)$. The product of $v_1 \cdots v_{n-1}$ and v_n is the image in $\mathcal{H}_G^*(X)$ of the element $uv \in \mathcal{H}_G^*(X, Y \cup A) = \mathcal{H}_G^*(X, X) = 0$, and so $v_1 \cdots v_n = 0$. □

Now fix a map $f\colon X \to L$ between G–CW–complexes. Consider $\mathcal{H}_G^*(X)$ as a module over the ring $\mathcal{H}_G^0(L)$. Consider the composition

$$\mathbb{I}_G(L)^n \cdot \mathcal{H}_G^m(X) \xrightarrow{i} \mathcal{H}_G^m(X) \xrightarrow{\mathcal{H}_G^m(\mathrm{pr})} \mathcal{H}_G^m(EG \times X) \xrightarrow{(\mathrm{ind}_{G\to\{1\}})^{-1}} \mathcal{H}^m(EG \times_G X) \xrightarrow{\mathcal{H}^m(j)} \mathcal{H}^m((EG \times_G X)_{(n-1)}),$$

where i and j denote the inclusions, pr the projection and $(EG \times_G X)_{(n-1)}$ is the $(n-1)$–skeleton of $EG \times_G X$. This composite is zero because of Lemma 3.2 since its image is contained in $\mathbb{I}^n((EG \times_G X)_{(n-1)})$. Thus we obtain a homomorphism of pro-$\mathcal{H}^0(*)$–modules,

$$\textbf{(3.3)} \quad \lambda_G^m(f\colon X \to L)\colon \{\mathcal{H}_G^m(X)/\mathbb{I}_G(L)^n \cdot \mathcal{H}_G^m(X)\}_{n\ge 1} \to \{\mathcal{H}_G^m((EG \times_G X)_{(n-1)})\}_{n\ge 1}.$$

We will sometimes write λ_G^m or $\lambda_G^m(X)$ instead of $\lambda_G^m(f\colon X \to L)$, if the map $f\colon X \to L$ is clear from the context. Notice that the target of $\lambda_G^m(f\colon X \to L)$ depends only on X and not on the map $f\colon X \to L$, whereas the source does depend on f.

Problem 3.4 (completion problem) *Under which conditions on $\mathcal{H}_?^*$ and L is the map of pro-$\mathcal{H}^0(*)$–modules $\lambda_G^m(f\colon X \to L)$, defined in* (3.3), *an isomorphism of pro-$\mathcal{H}^0(*)$–modules?*

Remark 3.5 (consequences of the completion theorem) Suppose that the map of pro-$\mathcal{H}^0(*)$–modules $\lambda_G^m(X)$ defined in (3.3) is an isomorphism of pro-$\mathcal{H}^0(*)$–modules. Obviously the pro-module $\{\mathcal{H}_G^m(X)/\mathbb{I}_G(L)^n \cdot \mathcal{H}_G^m(X)\}_{n\ge 1}$ satisfies the Mittag-Leffler condition since all structure maps are surjective. This implies that its $\lim^1$–term vanishes. We conclude from Lemma 2.1 that

$$\operatorname*{invlim}_{n\to\infty}{}^1\, \mathcal{H}^m((EG \times_G X)_{(n-1)}) = 0,$$

$$\operatorname*{invlim}_{n\to\infty} \mathcal{H}^m((EG \times_G X)_{(n-1)}) \cong \operatorname*{invlim}_{n\to\infty} \mathcal{H}_G^m(X)/\mathbb{I}_G(L)^n \cdot \mathcal{H}_G^m(X).$$

Milnor's exact sequence

$$0 \to \operatorname*{invlim}_{n\to\infty}{}^{1} \mathcal{H}^{m-1}((EG \times_G X)_{(n-1)}) \to \mathcal{H}^m(EG \times_G X) \to \operatorname*{invlim}_{n\to\infty} \mathcal{H}^m((EG \times_G X)_{(n-1)}) \to 0$$

implies that we obtain an isomorphism

$$\mathcal{H}^m(EG \times_G X) \cong \operatorname*{invlim}_{n\to\infty} \mathcal{H}^m_G(X)/\mathbb{I}_G(L)^n \cdot \mathcal{H}^m_G(X).$$

Remark 3.6 (taking $L = \underline{E}G$) The *classifying space* $\underline{E}G$ *for proper* G*–actions* is a proper G–CW–complex such that the H–fixed point set is contractible for every finite subgroup $H \subseteq G$. It has the universal property that for every proper G–CW–complex X there is, up to G–homotopy, precisely one G–map $f\colon X \to \underline{E}G$. Recall that a G–CW–complex is proper if and only if all its isotropy groups are finite, and is finite if and only if it is cocompact. There is a cocompact G–CW–model for the classifying space $\underline{E}G$ for proper G–actions if, for instance, G is word-hyperbolic in the sense of Gromov, or if G is a cocompact subgroup of a Lie group with finitely many path components, or if G is a finitely generated one-relator group, or if G is an arithmetic group, a mapping class group of a compact surface or the group of outer automorphisms of a finitely generated free group. For more information about $\underline{E}G$ we refer for instance to [5] and [13].

Suppose that there is a finite model for the classifying space of proper G–actions $\underline{E}G$. Then we can apply this to id: $\underline{E}G \to \underline{E}G$ and obtain an isomorphism

$$\mathcal{H}^m(BG) \cong \operatorname*{invlim}_{n\to\infty} \mathcal{H}^m_G(\underline{E}G)/\mathbb{I}_G(\underline{E}G)^n \cdot \mathcal{H}^m_G(\underline{E}G).$$

Remark 3.7 (the free case) The statement of the completion theorem as stated in Problem 3.4 is always true for trivial reasons if X is a free finite G–CW–complex. Then induction induces an isomorphism

$$\operatorname{ind}_{G\to\{1\}}\colon \mathcal{H}^m(G\backslash X) \xrightarrow{\cong} \mathcal{H}^m_G(X).$$

Since $\mathbb{I}(G\backslash X)^n = 0$ for large enough n by Lemma 3.2, the canonical map

$$\{\mathcal{H}^m(G\backslash X)\}_{n\geq 1} \xrightarrow{\cong} \{\mathcal{H}^m(G\backslash X)/\mathbb{I}_G(L)^n \cdot \mathcal{H}^m(G\backslash X)\}_{n\geq 1}$$

with the constant pro-$\mathcal{H}^0(*)$–module as source is an isomorphism. Hence the source of $\lambda^m_G(f\colon G \to X)$ can be identified with constant pro-$\mathcal{H}^0(*)$–module $\{\mathcal{H}^m(G\backslash X)\}_{n\geq 1}$.

The projection $EG \times_G X \to G\backslash X$ is a homotopy equivalence and induces an isomorphism of pro-$\mathbb{Z}$–modules

$$\{\mathcal{H}^m((G\backslash X)_{(n-1)})\}_{n\geq 1} \xrightarrow{\cong} \{\mathcal{H}^m((EG \times_G X)_{(n-1)})\}_{n\geq 1}.$$

Since $G\backslash X$ is finite-dimensional, the canonical map

$$\{\mathcal{H}^m(G\backslash X)\}_{n\geq 1} \xrightarrow{\cong} \{\mathcal{H}^m((G\backslash X)_{(n-1)})\}_{n\geq 1}$$

is an isomorphism of pro-$\mathbb{Z}$–modules. Hence also the target of $\lambda_G^m(f\colon G \to X)$ can be identified with the constant pro-$\mathcal{H}^0(*)$–module $\{\mathcal{H}^m(G\backslash X)\}_{n\geq 1}$. One easily checks that under these identifications, $\lambda_G^m(f\colon G \to X)$ is the identity.

Hence the completion theorem is only interesting in the case where G contains torsion.

4 A strategy for a proof of a completion theorem

Theorem 4.1 (strategy for the proof of Theorem 0.2) *Let $\mathcal{H}_*^?$ be an equivariant cohomology theory with values in R–modules and with a multiplicative structure. Let L be a proper G–CW–complex. Let $\mathcal{F}(L)$ be the family of subgroups of G given by $\{H \subseteq G \mid L^H \neq \varnothing\}$, and suppose that the following conditions are satisfied:*

(1) *The ring $\mathcal{H}^0(*)$ is Noetherian.*

(2) *For any $H \in \mathcal{F}(L)$ and $m \in \mathbb{Z}$, the $\mathcal{H}^0(*)$–module $\mathcal{H}_H^m(*)$ is finitely generated.*

(3) *Let $H \in \mathcal{F}(L)$, let $\mathcal{P} \subseteq \mathcal{H}_H^0(*)$ be a prime ideal, and let $f\colon G/H \to L$ be any G–map. Then the augmentation ideal*

$$\mathbb{I}(H) = \ker\bigl(\mathcal{H}_H^0(*) \xrightarrow{\mathcal{H}_H^0(\mathrm{pr})} \mathcal{H}_H^0(H) \xrightarrow{\mathrm{ind}_{\{1\}\to H}} \mathcal{H}^0(*)\bigr)$$

is contained in $\mathcal{P}$ if $\mathcal{H}_G^0(L) \xrightarrow{\mathcal{H}_G^0(f)} \mathcal{H}_G^0(G/H) \xrightarrow{\mathrm{ind}_{G\to\{1\}}} \mathcal{H}_H^0(*)$ maps $\mathbb{I}_G(L)$ into $\mathcal{P}$.

(4) *The completion theorem is true for every finite group H with $H \in \mathcal{F}(L)$ in the case where $X = L = *$ and $f = \mathrm{id}\colon * \to *$. In other words, for every finite group H with $L^H \neq \varnothing$, the map of pro-$\mathcal{H}^0(*)$–modules*

$$\lambda_H^m(*)\colon \{\mathcal{H}_H^m(*)/\mathbb{I}(H)^n\}_{n\geq 1} \to \{\mathcal{H}^m((BH)_{(n-1)})\}_{n\geq 1}$$

defined in (3.3) is an isomorphism of pro-$\mathcal{H}^0(*)$–modules.

Then, under the conditions above, the completion theorem is true for $\mathcal{H}^?_*$ *and every* G*–map* $f\colon X \to L$ *from a finite proper* G*–CW–complex* X *to* L*; ie the map of pro-*$\mathcal{H}^0(*)$*–modules*

$$\lambda^m_G(X)\colon \{\mathcal{H}^m_G(X)/\mathbb{I}_G(L)^n \cdot \mathcal{H}^m_G(X)\}_{n\ge 1} \to \{\mathcal{H}^m_G((EG\times_G X)_{(n-1)})\}_{n\ge 1}$$

defined in (3.3) *is an isomorphism of pro-*$\mathcal{H}^0(*)$*–modules.*

Proof We first prove the completion theorem for $X = G/H$, ie for any a G–map $f\colon G/H \to L$. Obviously H belongs to $\mathcal{F}(L)$. The following diagram of pro-modules commutes:

$$\begin{array}{ccc}
\{\mathcal{H}^m_G(G/H)/\mathbb{I}_G(L)^n\cdot\mathcal{H}^m_G(G/H)\}_{n\ge 1} & \xrightarrow{\lambda^m_G(f\colon G/H\to L)} & \{\mathcal{H}^m((EG\times_G G/H)_{(n-1)})\}_{n\ge 1} \\
\downarrow{\scriptstyle \{\mathrm{ind}_{H\to G}\}_{n\ge 1}} & & \\
\{\mathcal{H}^m_H(*)/\mathbb{I}_G(L)^n\cdot\mathcal{H}^m_H(*)\}_{n\ge 1} & & \uparrow{\scriptstyle \{\mathcal{H}^m_G(\mathrm{pr})\}_{n\ge 1}} \\
\downarrow{\scriptstyle \mathrm{pr}} & & \\
\{\mathcal{H}^m_H(*)/\mathbb{I}(H)^n\cdot\mathcal{H}^m_H(*)\}_{n\ge 1} & \xrightarrow[\lambda^m_H(\mathrm{id}\colon *\to *)]{} & \{\mathcal{H}^m((BH)_{(n-1)})\}_{n\ge 1}
\end{array}$$

where pr denotes the obvious projection. The lower horizontal arrow is an isomorphism of pro-modules by condition (4). The right vertical arrow and the upper-left vertical arrow are obviously isomorphisms of pro-modules. Hence the upper horizontal arrow is an isomorphism of pro-modules if we can show that the lower-left vertical arrow is an isomorphism of pro-modules.

Let I_f be the image of $\mathbb{I}_G(L)$ under the composite of ring homomorphisms,

$$\mathcal{H}^0_G(L) \xrightarrow{\mathcal{H}^0_G(f)} \mathcal{H}^0_G(G/H) \xrightarrow{\mathrm{ind}_{H\to G}} \mathcal{H}^0_H(*).$$

Let J_f be the ideal in $\mathcal{H}^0_H(*)$ generated by I_f. Obviously $I_f \subseteq J_f \subseteq \mathbb{I}(H)$. Then the lower-left vertical arrow is the composite

$$\mathcal{H}^m_H(*)/\mathbb{I}_G(L)^n\cdot\mathcal{H}^m_H(*) \to \mathcal{H}^m_H(*)/(J_f)^n\cdot\mathcal{H}^m_H(*) \to \mathcal{H}^m_H(*)/\mathbb{I}(H)^n\cdot\mathcal{H}^m_H(*),$$

where the first map is already levelwise an isomorphism, and in particular an isomorphism of pro-modules. In order to show that the second map is an isomorphism of pro-modules, it remains to show that $\mathbb{I}(H)^k \subseteq J_f$ for an appropriate integer $k \ge 1$. Equivalently, we want to show that the ideal $\mathbb{I}(H)/J_f$ of the quotient ring $\mathcal{H}^0_H(*)/J_f$

is nilpotent. Since $\mathcal{H}^0_H(*)$ is Noetherian by conditions (1) and (2), the ideal $\mathbb{I}(H)/J_f$ is finitely generated. Hence it suffices to show that $\mathbb{I}(H)/J_f$ is contained in the nilradical, ie the ideal consisting of all nilpotent elements, of $\mathcal{H}^0_H(*)/J_f$. The nilradical agrees with the intersection of all the prime ideals of $\mathcal{H}^0_H(*)/J_f$ by [3, Proposition 1.8]. The preimage of a prime ideal in $\mathcal{H}^0_H(*)/J_f$ under the projection $\mathcal{H}^0_H(*) \to \mathcal{H}^0_H(*)/J_f$ is again a prime ideal. Hence it remains to show that any prime ideal of $\mathcal{H}^0_H(*)$ which contains I_f also contains $\mathbb{I}(H)$. But this is guaranteed by condition (3). This finishes the proof in the case $X = G/H$.

The general case of a G–map $f\colon X \to L$ from a finite G–CW–complex X to a G–CW–complex L is done by induction over the dimension r of X and subinduction over the number of top-dimensional equivariant cells. For the induction step we write X as a G–pushout

$$\begin{array}{ccc} G/H \times S^{r-1} & \xrightarrow{q} & Y \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\ G/H \times D^r & \xrightarrow{Q} & X \end{array}$$

In the sequel we equip $G/H \times S^{r-1}$, Y and $G/H \times D^r$ with the maps to L given by the composite of $f\colon X \to L$ with $k \circ q$, k and Q. The long exact Mayer–Vietoris sequence of the G–pushout above is a long exact sequence of $\mathcal{H}^0_G(L)$–modules and looks like

$$\cdots \to \mathcal{H}^{m-1}(G/H \times D^r) \oplus \mathcal{H}^{m+1}_G(Y) \to \mathcal{H}^{m-1}_G(G/H \times S^{r-1}) \to \mathcal{H}^m_G(X)$$
$$\to \mathcal{H}^m_G(G/H \times D^r) \oplus \mathcal{H}^m_G(Y) \to \mathcal{H}^m_G(G/H \times S^{r-1}) \to \cdots .$$

Condition (2) implies that $\mathcal{H}^m_G(G/H)$ and $\mathcal{H}^m_G(G/H \times D^r)$ are finitely generated as $\mathcal{H}^0(*)$–modules. Since $\mathcal{H}^0(*)$ is Noetherian by condition (1), the $\mathcal{H}^0(*)$–module $\mathcal{H}^m_G(X)$ is finitely generated provided that the $\mathcal{H}^0(*)$–module $\mathcal{H}^m_G(Y)$ is finitely generated. Thus we can show inductively that the $\mathcal{H}^0(*)$–module $\mathcal{H}^m_G(X)$ is finitely generated for every $m \in \mathbb{Z}$. In particular, the ring $\mathcal{H}^0_G(X)$ is Noetherian. Let $J \subseteq \mathcal{H}^0_G(X)$ be the ideal generated by the image of $\mathbb{I}_G(L)$ under the ring homomorphism $\mathcal{H}^0_G(L) \to \mathcal{H}^0_G(X)$. Then for every $\mathcal{H}^0_G(X)$–module, the obvious map

$$\{M/\mathbb{I}_G(L)^n \cdot M\}_{n\geq 1} \to \{M/J^n \cdot M\}_{n \geq 1}$$

is levelwise an isomorphism, and in particular an isomorphism of $\mathcal{H}^0_G(X)$–modules. We conclude from Lemma 2.2 that the following sequence of pro-$\mathcal{H}^0(*)$–modules is exact, where M/I stands for $M/I \cdot M$:

$$\begin{aligned}\textbf{(4.2)}\quad \cdots \to \{\mathcal{H}^{m-1}(G/H \times D^r)/\mathbb{I}_G(L)\}_{n\ge 1} \oplus \{\mathcal{H}^{m-1}_G(Y)/\mathbb{I}_G(L)\}_{n\ge 1} \\ \to \{\mathcal{H}^{m-1}_G(G/H \times S^{r-1})/\mathbb{I}_G(L)\}_{n\ge 1} \to \{\mathcal{H}^m_G(X)/\mathbb{I}_G(L)\}_{n\ge 1} \\ \to \{\mathcal{H}^m_G(G/H \times D^r)/\mathbb{I}_G(L)\}_{n\ge 1} \oplus \{\mathcal{H}^m_G(Y)/\mathbb{I}_G(L)\}_{n\ge 1} \\ \to \{\mathcal{H}^m_G(G/H \times S^{r-1})/\mathbb{I}_G(L)\}_{n\ge 1} \to \cdots .\end{aligned}$$

Applying $EG_{(n-1)} \times_G -$ to the G–pushout above yields a pushout and thus a long exact Mayer–Vietoris sequence

$$\begin{aligned}\cdots \to \mathcal{H}^{m-1}\big(EG_{(n-1)} \times_G (G/H \times D^r)\big) \oplus \mathcal{H}^{m-1}_G(EG_{(n-1)} \times_G Y) \\ \to \mathcal{H}^{m-1}_G\big(EG_{(n-1)} \times_G (G/H \times S^{r-1})\big) \to \mathcal{H}^m_G(EG_{(n-1)} \times_G X) \\ \to \mathcal{H}^m_G\big(EG_{(n-1)} \times_G (G/H \times D^r)\big) \oplus \mathcal{H}^m_G(EG_{(n-1)} \times_G Y) \\ \to \mathcal{H}^m_G\big(EG_{(n-1)} \times_G (G/H \times S^{r-1})\big) \to \cdots .\end{aligned}$$

For any finite-dimensional G–CW–complex Z, the obvious map

$$\{\mathcal{H}^m_G(EG_{(n-1)} \times_G Z)\}_{n\ge 1} \xrightarrow{\cong} \{\mathcal{H}^m_G((EG \times_G Z)_{(n-1)})\}_{n\ge 1}$$

is an isomorphism of pro-$\mathcal{H}^0(*)$–modules. Hence we obtain a long exact sequence of pro-$\mathcal{H}^0(*)$–modules

$$\begin{aligned}\textbf{(4.3)}\quad \cdots\to \big\{\mathcal{H}^{m-1}_G\big(\big(EG\times_G(G/H\times D^r)\big)_{(n-1)}\big)\big\}_{n\ge 1} \oplus \big\{\mathcal{H}^m_G\big((EG\times_G Y)_{(n-1)}\big)\big\}_{n\ge 1} \\ \to \big\{\mathcal{H}^{m-1}_G\big(\big(EG\times_G(G/H\times S^{r-1})\big)_{(n-1)}\big)\big\}_{n\ge 1} \to \big\{\mathcal{H}^m_G\big((EG\times_G X)_{(n-1)}\big)\big\}_{n\ge 1} \\ \to \big\{\mathcal{H}^m_G\big(\big(EG\times_G(G/H\times D^r)\big)_{(n-1)}\big)\big\}_{n\ge 1} \oplus \big\{\mathcal{H}^m_G\big((EG\times_G Y)_{(n-1)}\big)\big\}_{n\ge 1} \\ \to \big\{\mathcal{H}^m_G\big(\big(EG\times_G(G/H\times S^{r-1})\big)_{(n-1)}\big)\big\}_{n\ge 1} \to \cdots .\end{aligned}$$

Now the various maps λ^m_G induce a map from the long exact sequence of pro-$\mathcal{H}^0(*)$–modules (4.2) to the long exact sequence of pro-$\mathcal{H}^0(*)$–modules (4.3). The maps for $G/H\times S^{r-1}$, $G/H\times D^r$ and Y are isomorphisms of pro-$\mathcal{H}^0(*)$–modules by induction hypothesis and by G–homotopy invariance applied to the G–homotopy equivalence $G/H \times D^r \to G/H$. By the five lemma for maps of pro-modules, the map

$$\lambda^m_G(X)\colon \{\mathcal{H}^m_G(X)/\mathbb{I}_G(L)^n \cdot \mathcal{H}^m_G(X)\}_{n\ge 1} \to \big\{\mathcal{H}^m_G\big((EG \times_G X)_{(n-1)}\big)\big\}_{n\ge 1}$$

is an isomorphism of pro-$\mathcal{H}^0(*)$–modules. This finishes the proof of Theorem 4.1. □

The next lemma will be needed to check condition (3) appearing in Theorem 4.1.

Given a G–cohomology theory $\mathcal{H}_G^*$, there is an equivariant version of the Atiyah–Hirzebruch spectral sequence of $\mathcal{H}^0(*)$–modules which converges to $\mathcal{H}^{p+q}(L)$ in the usual sense provided that L is finite-dimensional, and whose E_2–term is

$$E_2^{p,q} := H_G^p(L; \mathcal{H}_G^q(G/?)),$$

where $H_G^p(X; \mathcal{H}_G^q(G/?))$ is the *Bredon cohomology* of L with coefficients in the $\mathbb{Z}\mathrm{Or}(G)$–module sending G/H to $\mathcal{H}_G^q(G/H)$. If $\mathcal{H}_G^*$ comes with a multiplicative structure, then this spectral sequence comes with a multiplicative structure.

Lemma 4.4 *Suppose that L is an l–dimensional proper G–CW–complex for some positive integer l. Suppose that for $r = 2, 3, \dots, l$, the differential appearing in the Atiyah–Hirzebruch spectral sequence for L and $\mathcal{H}_G^*$,*

$$d_r^{0,0}\colon E_r^{0,0} \to E_r^{r,1-r},$$

vanishes rationally.

(1) *Then we can find for a given $x \in H_G^0(L; \mathcal{H}_G^0(G/?))$ a positive integer k such that x^k is contained in the image of the edge homomorphism*

$$\mathrm{edge}^{0,0}\colon \mathcal{H}_G^0(L) \to H_G^0(L; \mathcal{H}_G^0(G/?)).$$

(2) *Let $H \in \mathcal{F}(L)$, let $\mathcal{P} \subseteq \mathcal{H}_H^0(*)$ be a prime ideal and let $f\colon G/H \to L$ be any G–map. Suppose that the augmentation ideal*

$$\mathbb{I}(H) = \ker\big(\mathcal{H}_H^0(*) \xrightarrow{\mathcal{H}_H^0(\mathrm{pr})} \mathcal{H}_H^0(H) \xrightarrow{\mathrm{ind}_{\{1\}\to H}} \mathcal{H}^0(*)\big)$$

is contained in $\mathcal{P}$ if $\mathcal{P}$ contains the image of the inverse limit over the orbit category $\mathrm{Or}(G; \mathcal{F}(L))$ associated to the family $\mathcal{F}(L)$, under the structure map for H,

$$\phi_H\colon \operatorname*{invlim}_{G/K \in \mathrm{Or}(G;\mathcal{F}(L))} \mathbb{I}(K) \to \mathbb{I}(H).$$

Then condition (3) *appearing in Theorem 4.1 is satisfied for H, $\mathcal{P}$ and f.*

Proof (1) Consider $x \in H_G^0(L; \mathcal{H}_G^0(G/?))$. We inductively construct positive integers $k_1, k_2, \dots, k_l$ such that

$$x^{\prod_{i=1}^r k_i} \in E_{r+1}^{0,0} \quad \text{for } r = 1, 2, \dots, l.$$

Put $k_1 = 1$. We have $H_G^0(L; \mathcal{H}_G^0(G/?)) = E_2^{0,0}$ and hence $x = x^1 = x^{\prod_{i=1}^1 k_i} \in E_2^{0,0}$. This finishes the induction base step at $r = 1$.

In the induction step from $r-1$ to $r \geq 2$, we can assume that we have already constructed $k_1, \dots, k_{r-1}$ and shown that $x^{\prod_{i=1}^{r-1} k_i}$ belongs to $E_r^{0,0}$. Now choose k_r with $k_r \cdot d_r^{0,0}\bigl(x^{\prod_{i=1}^{r-1} k_i}\bigr) = 0$. This is possible since by assumption $d_r^{0,0} \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}} = 0$. For any element $y \in E_r^{0,0}$, one checks inductively for $j = 1, 2, \dots$ that

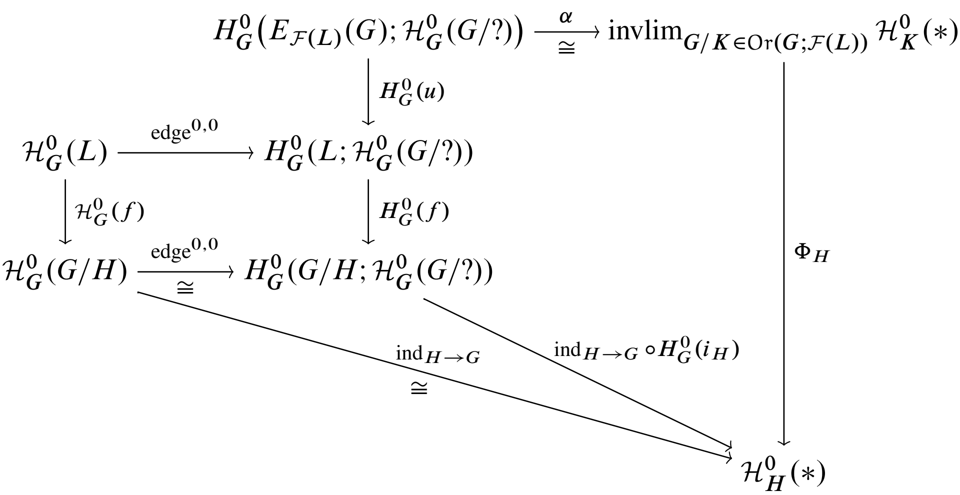
$$d_r^{0,0}(y^j) = j \cdot d_r^{0,0}(y) \cdot y^{j-1}.$$

This implies

$$d_r^{0,0}\bigl(x^{\prod_{i=1}^{r} k_i}\bigr) = d_r^{0,0}\bigl(\bigl(x^{\prod_{i=1}^{r-1} k_i}\bigr)^{k_r}\bigr) = k_r \cdot d_r^{0,0}\bigl(x^{\prod_{i=1}^{r-1} k_i}\bigr) \cdot \bigl(x^{\prod_{i=1}^{r-1}}\bigr)^{k_r - 1} = 0.$$

Since $E_{r+1}^{0,0}$ is the kernel of $d_r^{0,0}\colon E_r^{0,0} \to E_{r+1}^{0,0}$, we conclude $x^{\prod_{i=1}^{r} k_i} \in E_{r+1}^{0,0}$. Since L is l–dimensional, we get for $k = \prod_{i=1}^{l} k_i$ that $x^k \in E_\infty^{0,0}$. Since $E_\infty^{0,0}$ is the image of the edge homomorphism $\mathrm{edge}^{0,0}$, assertion (1) follows.

(2) Consider the commutative diagram



Here α is the isomorphism which sends $v \in H_G^0\bigl(E_{\mathcal{F}(L)}(G); \mathcal{H}_G^0(G/?)\bigr)$ to the system of elements that is, for $G/K \in \mathrm{Or}(G; \mathcal{F}(L))$, the image of v under the homomorphism

$$H_G^0\bigl(E_{\mathcal{F}(L)}(G); \mathcal{H}_G^0(G/?)\bigr) \xrightarrow{H_G^0(i_K)} H_G^0(G/K; \mathcal{H}_G^0(G/?)) \xrightarrow{(\mathrm{edge}^{0,0})^{-1}} \mathcal{H}_G^0(G/K) \xrightarrow{\mathrm{ind}_{\{1\} \to K}} \mathcal{H}_K^0(*)$$

for the unique (up to G–homotopy) G–map $i_K\colon G/K \to E_{\mathcal{F}(L)}(G)$. The G–map $u\colon L \to E_{\mathcal{F}(L)}(G)$ is the unique (up to G–homotopy) G–map from L to the classifying space of the family $\mathcal{F}(L)$, and Φ_H is the structure map of the inverse limit for H. We have to prove that $\mathbb{I}(H)$ is contained in the prime ideal $\mathcal{P}$ provided that $\mathcal{P}$ contains the image of $\mathbb{I}_G(L)$ under the composite

$$\mathcal{H}^0_G(L) \xrightarrow{\mathcal{H}^0_G(f)} \mathcal{H}^0_G(G/H) \xrightarrow{\mathrm{ind}_{H\to G}} \mathcal{H}^0_H(*).$$

Consider $a \in \operatorname{invlim}_{G/K \in \mathrm{Or}(G;\mathcal{F}(L))} \mathbb{I}(K)$. Let $x \in H^0_G(L;\mathcal{H}^0_G(G/?))$ be the image of a under the composite

$$\operatorname*{invlim}_{G/K \in \mathrm{Or}(G;\mathcal{F}(L))} \mathbb{I}(K) \longrightarrow \operatorname*{invlim}_{G/K \in \mathrm{Or}(G;\mathcal{F}(L))} \mathcal{H}^0_K(*) \xrightarrow{\alpha^{-1}} H^0_G(E_{\mathcal{F}(L)}(G);\mathcal{H}^0_G(G/?)) \xrightarrow{H^0_G(u;\mathcal{H}^0_G(G/?))} H^0_G(L;\mathcal{H}^0_G(G/?)).$$

We conclude from assertion (1) that for some positive number k, there is an element $y \in \mathcal{H}^0_G(L)$ with $\mathrm{edge}^{0,0}(y) = x^k$. One easily checks that y belongs to $\mathbb{I}_G(L)$, just inspecting the diagram above for $H = \{1\}$. Hence the composite

$$\mathcal{H}^0_G(L) \xrightarrow{\mathcal{H}^0_G(f)} \mathcal{H}^0_G(G/H) \xrightarrow{\mathrm{ind}_{H\to G}} \mathcal{H}^0_H(*)$$

maps y to $\mathcal{P}$ by assumption. An easy diagram chase shows that

$$\phi_H \colon \operatorname*{invlim}_{G/K \in \mathrm{Or}(G;\mathcal{F}(L))} \mathbb{I}(K) \to \mathbb{I}(H)$$

maps a^k to $\mathcal{P}$. Since $\mathcal{P}$ is a prime ideal and ϕ_H is multiplicative, ϕ_H sends a to $\mathcal{P}$. Hence the image of $\phi_H \colon \operatorname{invlim}_{G/K \in \mathrm{Or}(G;\mathcal{F}(L))} \mathbb{I}(K) \to \mathbb{I}(H)$ lies $\mathcal{P}$. Hence we get by assumption $\mathbb{I}(H) \subseteq \mathcal{P}$. This finishes the proof of Lemma 4.4. □

5 The Segal conjecture for infinite groups

In this section we prove the Segal conjecture for infinite groups, Theorem 0.2. It is just the completion theorem formulated in Problem 3.4 for equivariant stable cohomotopy $\mathcal{H}^*_? = \pi^*_?$ under the conditions that there is an upper bound on the orders of finite subgroups on L, and L has finite dimension.

Proof of Theorem 0.2 We want to apply Theorem 4.1 and therefore have to prove conditions (1), (2), (3) and (4) appearing there.

Condition (1) is satisfied because of $\pi^0_s(*) = \mathbb{Z}$.

Condition (2) is satisfied because of Example 1.5.

Next we prove condition (3). Recall the assumption that there is an upper bound on the orders of finite subgroups of L and that L is finite-dimensional. Recall that $\mathcal{F}(L)$ denotes the family of finite subgroups $H \subseteq G$ with $L^H \neq \varnothing$. By Example 1.5 we can find, for every $q \in \mathbb{Z}$ with $q \neq 0$, a positive integer $C(q)$ such that the order of $\pi_H^q(*)$ divides $C(q)$ for every $H \in \mathcal{F}(L)$. Furthermore recall that L is finite-dimensional. Consider the equivariant cohomological Atiyah–Hirzebruch spectral sequence converging to $\pi_G^{p+q}(L)$. Its E_2–term is given by

$$E_2^{p,q} = H_G^p(L;\pi_G^q(*)).$$

Therefore $E_r^{r,1-r}$ is annihilated by multiplication with $C(1-r)$ and hence rationally trivial for $r \geq 2$. Hence for $r \geq 2$ the differential

$$d_r^{0,0}\colon E_r^{0,0} \to E_r^{r,1-r}$$

vanishes rationally. We have shown that the conditions appearing in Lemma 4.4 are satisfied. Hence in order to verify condition (3), it suffices to prove, for any family $\mathcal{F}$ of subgroups of G with the property that there exists an upper bound on the orders of subgroups appearing $\mathcal{F}$, for any $H \in \mathcal{F}$, and for any prime ideal $\mathcal{P}$ of the Burnside ring $A(H)$, that $\mathcal{P}$ contains the augmentation ideal $\mathbb{I}_H$ provided $\mathcal{P}$ contains the image of the structure map for H of the inverse limit

$$\phi_H\colon \operatorname*{invlim}_{G/K \in \mathrm{Or}(G;\mathcal{F})} \mathbb{I}_K \to \mathbb{I}_H.$$

Fix a finite group H. We begin by recalling some basics about the prime ideals in the Burnside ring $A(H)$, taken from [9]. In the sequel, p is a prime number or $p = 0$. For a subgroup $K \subseteq H$, let $\mathcal{P}(K,p)$ be the preimage of $p\cdot\mathbb{Z}$ under the character map for K,

$$\mathrm{char}_K^H\colon A(H) \to \mathbb{Z}, \quad [S] \mapsto |S^K|.$$

This is a prime ideal, and each prime ideal of $A(H)$ is of the form $\mathcal{P}(K,p)$. If $\mathcal{P}(K,p) = \mathcal{P}(L,q)$, then $p = q$. If p is a prime, then $\mathcal{P}(K,p) = \mathcal{P}(L,p)$ if and only if $(K[p]) = (L[p])$, where $K[p]$ is the minimal normal subgroup of K with a p–group as quotient. Notice for the sequel that $K[p] = \{1\}$ if and only if K is a p–group. If $p = 0$, then $\mathcal{P}(K,p) = \mathcal{P}(L,p)$ if and only if $(K) = (L)$.

Fix a prime ideal $\mathcal{P} = \mathcal{P}(K,p)$. Choose a positive integer m such that $|H|$ divides m for all $H \in \mathcal{F}$. Fix $H \in \mathcal{F}$. Choose a free H–set S together with a bijection $u\colon S \xrightarrow{\cong} [m]$, where $[m] = \{1,2,\dots,m\}$. Such an S exists since $|H|$ divides m

and we can take for S the disjoint union of $m/|H|$ copies of H. Thus we obtain an injective group homomorphism

$$\rho_u\colon H \to S_m, \quad h \mapsto u \circ l_h \circ u^{-1},$$

where $l_h\colon S \to S$ is given by left multiplication with h and $S_m = \operatorname{aut}([m])$ is the group of permutations of $[m]$. Let $S_m[\rho_u]$ denote the H–set obtained from S_m by the H–action $h \cdot \sigma := \rho_u(h) \circ \sigma$. Let $\operatorname{Syl}_p(S_m)$ be the p–Sylow subgroup of S_m. Let $S_m/\operatorname{Syl}_p(S_m)[\rho_u]$ denote the H–set obtained from the homogeneous space $S_m/\operatorname{Syl}_p(S_m)$ by the H–action given by $h \cdot \overline{\sigma} = \overline{\rho_u(h) \circ \sigma}$. The H–action on $S_m[\rho_u]$ is free. If for $K \subseteq H$ we have $(S_m/\operatorname{Syl}_p(S_m)[\rho_u])^K \neq \varnothing$, then for some $\sigma \in S_m$ we get $\rho_u(K) \subseteq \sigma \cdot \operatorname{Syl}_p(S_m) \cdot \sigma^{-1}$, and hence K must be a p–group.

Suppose that T is another free H–set together with a bijection $v\colon T \xrightarrow{\cong} [m]$. Then we can choose an H–isomorphism $w\colon S \to T$. Let $\tau \in S_m$ be given by the composition $v \circ w \circ u^{-1}$. Then $c(\tau) \circ \rho_u = \rho_v$ holds, where $c(\tau)\colon S_m \to S_m$ sends σ to $\tau \circ \sigma \circ \tau^{-1}$. Moreover, left multiplication with τ induces isomorphisms of H–sets

$$S_m[\rho_u] \cong_H S_m[\rho_v] \quad \text{and} \quad S_m/\operatorname{Syl}_p(S_m)[\rho_u] \cong_H S_m/\operatorname{Syl}_p(S_m)[\rho_v].$$

Hence we obtain elements in $A(H)$,

$$[S_m] := [S_m[\rho_u]] \quad \text{and} \quad [S_m/\operatorname{Syl}_p(S_m)] := [S_m/\operatorname{Syl}_p(S_m)[\rho_u]],$$

which are independent of the choice of S and $u\colon S \xrightarrow{\cong} [m]$. If $i\colon H_0 \to H_1$ is an injective group homomorphism between elements in $\mathcal{F}$, then one easily checks that the restriction homomorphism $A(i)\colon A(H_1) \to A(H_0)$ sends $[S_m]$ to $[S_m]$ and $[S_m/\operatorname{Syl}_p(S_m)]$ to $[S_m/\operatorname{Syl}_p(S_m)]$. Thus we obtain elements

$$[[S_m]], [[S_m/\operatorname{Syl}_p(S_m)]] \in \operatorname*{invlim}_{G/K \in \operatorname{Or}(G;\mathcal{F})} A(K).$$

Define elements

$$|S_m| \cdot 1, |S_m/\operatorname{Syl}_p(S_m)| \cdot 1 \in \operatorname*{invlim}_{G/K \in \operatorname{Or}(G;\mathcal{F})} A(K)$$

by the collection of elements $|S_m| \cdot [K/K]$ and $|S_m/\operatorname{Syl}_p(S_m)| \cdot [K/K]$ in $A(K)$ for $K \in \mathcal{F}$. Thus we get elements

$$[[S_m]] - |S_m| \cdot 1, [[S_m/\operatorname{Syl}_p(S_m)]] - |S_m/\operatorname{Syl}_p(S_m)| \cdot 1 \in \operatorname*{invlim}_{G/K \in \operatorname{Or}(G;\mathcal{F})} \mathbb{I}_K.$$

The image of $[[S_m]] - |S_m| \cdot 1$ (resp. $[[S_m/\operatorname{Syl}_p(S_m)]] - |S_m/\operatorname{Syl}_p(S_m)| \cdot 1$) under the structure map of the inverse limit $\operatorname{invlim}_{G/K \in \operatorname{Or}(G;\mathcal{F})} \mathbb{I}_K$ for the object $G/H \in$

$\mathrm{Or}(G;\mathcal{F})$ is $[S_m] - |S_m| \cdot [H/H]$ (resp. $[S_m/\mathrm{Syl}_p(S_m)] - |S_m/\mathrm{Syl}_p(S_m)| \cdot [H/H]$). Hence, by assumption,

$$[S_m] - |S_m| \cdot [H/H] \in \mathcal{P}(K, p),$$
$$[S_m/\mathrm{Syl}_p(S_m)] - |S_m/\mathrm{Syl}_p(S_m)| \cdot [H/H] \in \mathcal{P}(K, p).$$

Therefore $\mathrm{char}_K^H\colon A(H) \to \mathbb{Z}$ sends both $[S_m] - |S_m| \cdot [H/H]$ and $[S_m/\mathrm{Syl}_p(S_m)] - |S_m/\mathrm{Syl}_p(S_m)| \cdot [H/H]$ to elements in $p\mathbb{Z}$. Since $\mathrm{char}_K^H([S_m] - |S_m| \cdot [H/H]) = 0 - |S_m|$ for $K \neq \{1\}$, we conclude that $K = \{1\}$ or that $p \neq 0$. If $K = \{1\}$, then $\mathbb{I}(H) = \mathcal{P}(\{1\}, 0)$ is contained in $\mathcal{P}(K, p)$. Suppose that $K \neq \{1\}$. Then p is a prime. We have

$$\mathrm{char}_K^H\big([S_m/\mathrm{Syl}_p(S_m)] - |S_m/\mathrm{Syl}_p(S_m)| \cdot [H/H]\big) = |(S_m/\mathrm{Syl}_p(S_m))^K| - |S_m/\mathrm{Syl}_p(S_m)|.$$

Since this integer must belong to $p\mathbb{Z}$ and $|S_m/\mathrm{Syl}_p(S_m)|$ is relatively prime to p, we get $(S_m/\mathrm{Syl}_p(S_m))^K \neq \varnothing$. Hence K must be a p–group. This implies $\mathcal{P}(K, p) = \mathcal{P}(\{1\}, p)$ and therefore $\mathbb{I}(H) = \mathcal{P}(\{1\}, 0) \subseteq \mathcal{P}(K, p)$. This finishes the proof of condition (3).

Condition (4) follows from the proof of the Segal conjecture for a finite group H due to Carlsson [6]. This finishes the proof of Theorem 0.2. □

6 An improved strategy for a proof of a completion theorem

The next result follows from Theorem 4.1, Lemma 4.4 and a construction of a modified Chern character analogous to the one in [12, Theorem 4.6 and Lemma 6.2], which will ensure that the condition about the differentials in the equivariant Atiyah–Hirzebruch spectral sequence appearing in Lemma 4.4 is satisfied. We do not give more details here, since the interesting cases of the Segal conjecture and of the Atiyah–Segal completion theorem are already covered by Theorem 0.2 and [16].

Let G be a (discrete) group. Let $\mathcal{F}$ be a family of subgroups of G such that there is an upper bound on the orders of the subgroups appearing $\mathcal{F}$. Let $\mathcal{H}_*^?$ be an equivariant cohomology theory with values in R–modules which satisfies the disjoint-union axiom. Define a contravariant functor

(6.1) $$\mathcal{H}_?^q(*)\colon \mathsf{FGINJ} \to R\text{–}\mathsf{MODULES},$$

with the category FGINJ of finite groups with injective group homomorphism as source, by sending an injective homomorphism $\alpha\colon H \to K$ to the composite

$$\mathcal{H}^q_K(*) \xrightarrow{\mathcal{H}^q(\mathrm{pr})} \mathcal{H}^q_K(K/H) \xrightarrow{\mathrm{ind}_\alpha} \mathcal{H}^q_H(*),$$

where pr: $H/K = \mathrm{ind}_\alpha(*) \to *$ is the projection and ind_α comes from the induction structure of $\mathcal{H}^*_?$. Assume that $\mathcal{H}^?_*$ comes with a multiplicative structure.

Theorem 6.2 (improved strategy for the proof of Theorem 0.2) *Suppose that the following conditions are satisfied.*

(1) *The ring* $\mathcal{H}^0(*)$ *is Noetherian.*

(2) *Let* $H \subseteq G$ *be any finite subgroup and* $m \in \mathbb{Z}$ *be any integer. Then the* $\mathcal{H}^0(*)$*–module* $\mathcal{H}^m_H(*)$ *is finitely generated, there exists an integer* $C(H,m)$ *such that multiplication with* $C(H,m)$ *annihilates the torsion submodule* $\mathrm{tors}_{\mathbb{Z}}(\mathcal{H}^m_H(*))$ *of the abelian group* $\mathcal{H}^m_H(*)$*, and the* R*–module* $\mathcal{H}^m_H(*)/\mathrm{tors}_{\mathbb{Z}}(\mathcal{H}^m_H(*))$ *is projective.*

(3) *Let* H *be any element of* $\mathcal{F}$*. Let* $\mathcal{P} \subseteq \mathcal{H}^0_H(*)$ *be any prime ideal. Then the augmentation ideal*

$$\mathbb{I}(H) = \ker\big(\mathcal{H}^0_H(*) \to \mathcal{H}^0_H(H) \xrightarrow{\cong} \mathcal{H}^0(*)\big)$$

is contained in $\mathcal{P}$ *if* $\mathcal{P}$ *contains the image of the inverse limit under the structure map for* H*,*

$$\phi_H\colon \operatorname*{invlim}_{G/K \in \mathrm{Or}(G;\mathcal{F})} \mathbb{I}(K) \to \mathbb{I}(H).$$

(4) *The completion theorem is true for every finite group* H *in the case* $X = L = *$ *and* $f = \mathrm{id}\colon * \to *$*, ie for every finite group* H*, the map of pro-*$\mathcal{H}^0(*)$*–modules*

$$\lambda^m_H(*)\colon \{\mathcal{H}^m_H(*)/\mathbb{I}(H)^n\}_{n\geq 1} \to \{\mathcal{H}^m((BH)_{(n-1)})\}_{n\geq 1}$$

defined in (3.3) *is an isomorphism of pro-*$\mathcal{H}^0(*)$*–modules.*

(5) *The covariant functor* (6.1) *extends to a Mackey functor.*

Then the completion theorem is true for $\mathcal{H}^?_*$ *and every* G*–map* $f\colon X \to L$ *from a finite proper* G*–CW–complex* X *to a proper finite-dimensional* G*–CW–complex* L *with the property that there is an upper bound on the order of its isotropy groups; ie the map of pro-*$\mathcal{H}^0(*)$*–modules*

$$\lambda^m_G(X)\colon \{\mathcal{H}^m_G(X)/\mathbb{I}_G(L)^n \cdot \mathcal{H}^m_G(X)\}_{n\geq 1} \to \big\{\mathcal{H}^m_G\big((EG \times_G X)_{(n-1)}\big)\big\}_{n\geq 1}$$

defined in (3.3) *is an isomorphism of pro-*$\mathcal{H}^0(*)$*–modules.*

Remark 6.3 The advantage of Theorem 6.2 in comparison with Theorem 4.1 is that the conditions do not involve L and $f\colon X \to L$ anymore, and only depend on the functor $\mathcal{H}^q_?(*)\colon \mathsf{FGINJ} \to \mathbb{Z}\text{-}\mathsf{MODULES}$. If one considers the case $R = \mathbb{Z}$ and assumes $\mathcal{H}^0(*) = \mathbb{Z}$, then condition (1) is obviously satisfied and condition (2) reduces to the condition that for any finite subgroup $H \subseteq G$ and any integer $m \in \mathbb{Z}$ the abelian group $\mathcal{H}^m_H(*)$ is finitely generated.

Remark 6.4 (family version) We mention without proof that there is a also a family version of Theorem 0.2. Its formulation is analogous to the one of the family version of the Atiyah–Segal completion theorem for infinite groups; see [15, Section 6].

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