Space of isospectral periodic tridiagonal matrices

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A periodic tridiagonal matrix is a tridiagonal matrix with an additional two entries at the corners. We study the space $X_{n,\lambda}$ of Hermitian periodic tridiagonal $n \times n$ matrices with a fixed simple spectrum λ. Using the discretized Schrödinger operator we describe all spectra λ for which $X_{n,\lambda}$ is a topological manifold. The space $X_{n,\lambda}$ carries a natural effective action of a compact $(n-1)$–torus. We describe the topology of its orbit space and, in particular, show that whenever the isospectral space is a manifold, its orbit space is homeomorphic to $S^4 \times T^{n-3}$. There is a classical dynamical system: the flow of the periodic Toda lattice, acting on $X_{n,\lambda}$. Except for the degenerate locus $X_{n,\lambda}^0$, the Toda lattice exhibits Liouville–Arnold behavior, so that the space $X_{n,\lambda} \setminus X_{n,\lambda}^0$ is fibered into tori. The degenerate locus of the Toda system is described in terms of combinatorial geometry: its structure is encoded in the special cell subdivision of a torus, which is obtained from the regular tiling of the euclidean space by permutohedra. We apply methods of commutative algebra and toric topology to describe the cohomology and equivariant cohomology modules of $X_{n,\lambda}$.

1 Introduction

Let $\Gamma = (V, E)$ be a simple graph on a set $V = [n] = \{1, \dots, n\}$. Let M_Γ be the vector space of Hermitian $n \times n$ matrices $A = (a_{ij})$ such that $a_{ij} = 0$ for $(i, j) \notin E$. We consider the space $M_{\Gamma,\lambda} \subset M_\Gamma$ of all such matrices, which have a given simple spectrum $\lambda = (\lambda_1 < \lambda_2 < \cdots < \lambda_n)$. Note that each space $M_{\Gamma,\lambda}$ carries the conjugation action of a compact torus T^n. The action is noneffective: scalar matrices commute with every matrix, hence the diagonal subgroup of T^n acts trivially.

Several examples are well studied. The complete graph $\Gamma = K_n$ corresponds to the space of all isospectral matrices, which is diffeomorphic to the variety Fl_n of complete flags in $\mathbb{C}^n$. The path graph $\Gamma = \mathbb{I}_n$ with $n + 1$ vertices produces the space $M_{\mathbb{I}_n,\lambda}$ of isospectral tridiagonal matrices, which is known to be a smooth $2n$–manifold; its

smooth type is independent of λ. The real version of $M_{\mathbb{I}_n,\lambda}$ is called the Tomei manifold; it was introduced and studied in [31]. The T^n–action on $M_{\mathbb{I}_n,\lambda}$ is locally standard and its orbit space is diffeomorphic to a simple polytope, the permutohedron; see Tomei [31] and Davis and Januszkiewicz [16]. Note that $M_{\mathbb{I}_n,\lambda}$ is not a toric variety, although it is closely related to the permutohedral variety; see Bloch, Flaschka and Ratiu [8].

More generally, the spaces $M_{\Gamma_h,\lambda}$ corresponding to indifferent graphs Γ_h are the spaces of staircase matrices. It is more convenient to encode this type of space by Hessenberg functions. The Hessenberg function is a function $h\colon[n]\to[n]$ such that $h(i)\geqslant i$ and $h(i+1)\geqslant h(i)$. The space M_{Γ_h} is the space of Hermitian matrices A such that $a_{ij}=0$ for $j>h(i)$. Every space $M_{\Gamma_h,\lambda}$ is a smooth manifold independent of a simple spectrum λ. Its odd-degree cohomology modules vanish; therefore, $M_{\Gamma_h,\lambda}$ is equivariantly formal in the sense of Goresky, Kottwitz and MacPherson (see Definition 9.5). The equivariant cohomology ring of $M_{\Gamma_h,\lambda}$ can be described using GKM theory [19]; see also Kuroki [21]. See Ayzenberg and Buchstaber [6] for details on the spaces $M_{\Gamma_h,\lambda}$ and their relation to regular semisimple Hessenberg varieties.

For the star graph $\Gamma=\mathrm{St}_n$ (see Figure 1), the space $M_{\mathrm{St}_n,\lambda}$ is also a smooth manifold, and its diffeomorphism type does not depend on λ. The effective action of $T=T^{n+1}/\Delta(T^1)$ on $M_{\mathrm{St}_n,\lambda}$ is locally standard, therefore the orbit space $Q_{\mathrm{St}_n,\lambda}=M_{\mathrm{St}_n,\lambda}/T$ is a manifold with corners. Unlike the case of tridiagonal matrices, the orbit space $Q_{\mathrm{St}_n,\lambda}$ for $n\geqslant 3$ is not a simple polytope. The topology of $Q_{\mathrm{St}_n,\lambda}$ itself is quite complicated, and it is difficult to state any general result about the manifold $M_{\mathrm{St}_n,\lambda}$ itself. However, the topology can be described in detail for $n=4$, which was done in Ayzenberg and Buchstaber [5].

In this paper we consider the case $\Gamma=\mathrm{Cy}_n$, the cyclic graph on n vertices. The Hermitian matrices corresponding to Cy_n have the form

$$(1\text{-}1)\qquad L=L(\underline{a},\underline{b})=\begin{pmatrix} a_1 & b_1 & 0 & \cdots & & \bar{b}_n \\ \bar{b}_1 & a_2 & b_2 & 0 & & \vdots \\ 0 & \bar{b}_2 & a_3 & \ddots & & 0 \\ \vdots & 0 & \ddots & \ddots & & b_{n-1} \\ b_n & \cdots & 0 & & \bar{b}_{n-1} & a_n \end{pmatrix},$$

where $a_i\in\mathbb{R}$ and $b_i\in\mathbb{C}$. Such matrices are called *periodic tridiagonal matrices* or *periodic Jacobi matrices*. We will simply call them *periodic*. It is assumed throughout the paper that $n\geqslant 3$.
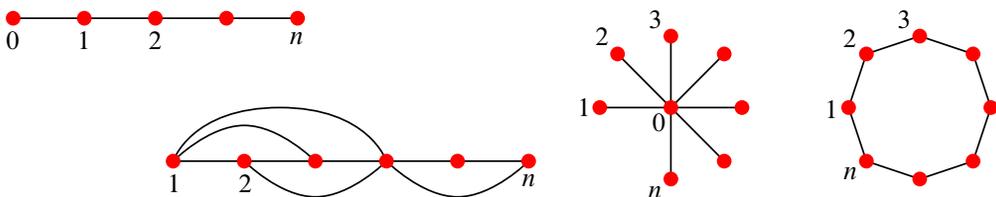
Figure 1: Particular graphs, encoding important isospectral matrix spaces: the path graph $\mathbb{I}_n$, indifferent graphs Γ_h, the star graph St_n, and the cycle graph Cy_n

The space $X_{n,\lambda} = M_{\mathrm{Cy}_n,\lambda}$ of all periodic matrices with a simple spectrum λ has dimension $2n$, and carries an effective action of $T = T^{n-1}$. Hence the torus action has complexity one. The difference between half the real dimension of a manifold and the dimension of a torus is called *the complexity of the action*; this terminology naturally comes from both algebraic geometry and symplectic geometry.

We prove that under certain conditions on a simple spectrum, the space $X_{n,\lambda}$ is not a smooth manifold, nor even a homology manifold; see Theorem 3.9. This gives a negative answer to our question, posed in [5]. This also settles a certain inaccuracy appearing in the work of van Moerbeke [23], who studied the real analogue of $X_{n,\lambda}$.

For any simple spectrum λ, we describe the topology of the orbit space $X_{n,\lambda}/T$; see Corollary 3.6. If $X_{n,\lambda}$ is a topological manifold, we prove that the orbit space $X_{n,\lambda}/T$ is homeomorphic to the product $S^4 \times T^{n-3}$. When $n = 3$, the space $X_{3,\lambda}$ is the space of all Hermitian matrices with the given spectrum λ. This space is diffeomorphic to the full complex flag variety Fl_3. Hence, for $n = 3$, we recover the result of Buchstaber and Terzić [11; 12; 13], which states that $\mathrm{Fl}_3/T^2 \cong S^4$. Note that the action is not free, however the orbit space is still a topological manifold. This fact is consistent with the general theory developed in Ayzenberg [4].

The main ingredient of our arguments is the product of off-diagonal elements

$$B = \prod_{i=1}^{n} b_i \in \mathbb{C}$$

of the periodic matrix $L(\underline{a}, \underline{b})$ and the corresponding map $p\colon X_{n,\lambda} \to \mathbb{C}$ given by $p(L(\underline{a}, \underline{b})) = B$. We show that with the matrix spectrum fixed, the number B takes values inside a compact convex subset $\mathbb{B} \subset \mathbb{C}$, lying between two confocal parabolas; see Theorem 3.4. This statement may be considered folklore: its real version was proved in van Moerbeke [23] and Krichever and Vaninsky [20], and the complex version

is not more complicated. In Section 4 we briefly review the facts about the discrete Schrödinger operator needed for this result.

The value B is preserved by the torus action, hence there is a map $\widetilde{p}\colon X_{n,\lambda}/T \to \mathbb{C}$ from the orbit space, evaluating the number B. The set $\widetilde{p}^{-1}(\mathbb{C} \setminus \{0\})$ consists of free orbits. However, the torus action has nontrivial T–equivariant skeleton, which is a proper subset of $\widetilde{p}^{-1}(0)$. To describe the structure of the equivariant skeleton, we use combinatorial geometry.

It is well known that euclidean space can be tiled by parallel copies of a regular permutohedron. Taking quotients by lattices in a euclidean space, we may produce many interesting permutohedral cell subdivisions of a torus. We show that a certain lattice produces a regular cell subdivision $\mathcal{PT}^{n-1}$ of an $(n-1)$–dimensional torus, which we called *the wonderful subdivision*. It has several interesting properties. First, it models the equivariant skeleton of the torus action on $X_{n,\lambda}$. Second, this wonderful subdivision minimizes the number of facets among all possible regular cell subdivisions of a torus. Such subdivisions and their dual simplicial cell subdivisions for general PL manifolds are known in combinatorial topology under the name of crystallizations; see Ferri, Gagliardi and Grasselli [18]. We briefly recall the required combinatorial geometry in Section 5.

Next we describe the topology of the whole space $X_{n,\lambda}$. Let $X_{n,\lambda}^0 = p^{-1}(0)$ denote the subset of matrices with $B = 0$. The space $X_{n,\lambda}$ is smooth in vicinity of $X_{n,\lambda}^0$: this actually follows from the properties of nonperiodic Toda lattice, see Proposition 3.1. Using the result of [4] concerning the topological classification of complexity one torus actions, we describe the topology of a small neighborhood $X_{n,\lambda}^{\leq \varepsilon}$ of $X_{n,\lambda}^0$. It happens that, up to homeomorphism, the T^{n-1}–action on $X_{n,\lambda}^{\leq \varepsilon}$ can be extended to a locally standard T^n–action on this space. The necessary notions related to complexity one torus actions are given in Section 6.

In a series of works — see our [2; 1; 3] and Ayzenberg, Masuda, Park and Zeng [7] — we developed a toolbox to compute cohomology and equivariant cohomology of manifolds with locally standard torus action whose orbit spaces have acyclic proper faces. This toolbox is applied to the subspace $X_{n,\lambda}^{\leq \varepsilon}$. The T^n–orbit space of $X_{n,\lambda}^{\leq \varepsilon}$ is a manifold with corners, whose face structure is the wonderful cell subdivision of a torus, hence all its proper faces are acyclic, so we are in position to apply the general technique. The algebrotopological invariants of $X_{n,\lambda}^{\leq \varepsilon}$ are computed in terms of combinatorial invariants of the wonderful cell subdivision $\mathcal{PT}^{n-1}$. We recall the theory of $h-$,

h'– and h''–numbers of simplicial posets and compute these invariants for the dual simplicial poset of the wonderful subdivision in Section 7.

In Section 8 we describe the additive structure of T^{n-1}–equivariant cohomology modules of the neighborhood $X_{n,\lambda}^{\leq \varepsilon}$. The ordinary Betti numbers of $X_{n,\lambda}$ are calculated in Section 9. There we also prove that $X_{n,\lambda}$ is not equivariantly formal for $n \geq 4$ by comparing the equivariant and ordinary Betti numbers of $X_{n,\lambda}$.

2 Torus action and Toda flow

The element $t = (t_1, \ldots, t_n) \in T^n$ acts on a cyclic matrix by the formula

$$(2\text{-}1) \qquad t L(\underline{a}; b_1, \ldots, b_{n-1}, b_n) = L(\underline{a}; t_1 t_2^{-1} \cdot b_1, \ldots, t_{n-1} t_n^{-1} \cdot b_{n-1}, t_n t_1^{-1} \cdot b_n).$$

It is easy to see that the torus action preserves the quantity $B = \prod_1^n b_i$. The action is noneffective: the scalar matrices act trivially. Hence we consider the effective action of the quotient torus $T^{n-1} = T^n / \Delta(S^1)$ on $X_{n,\lambda}$.

Apart from the torus action, there is a classical dynamical system acting on the space of periodic matrices: the *periodic Toda lattice*. We now briefly recall the definition and properties of this dynamical system.

Construction 2.1 For a matrix $L = L(\underline{a}, \underline{b})$, consider the skew-Hermitian matrix

$$P = P(L) = \begin{pmatrix} 0 & b_1 & & & -\bar{b}_n \\ -\bar{b}_1 & 0 & b_2 & & \\ & -\bar{b}_2 & 0 & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ b_n & & & -\bar{b}_{n-1} & 0 \end{pmatrix}.$$

The Toda flow (the flow of the periodic Toda lattice) is the flow

$$(2\text{-}2) \qquad \dot{L} = [L, P] = LP - PL.$$

The solution $L(t)$ to (2-2) remains similar to the initial matrix $L(0)$ at all times $t \in \mathbb{R}$, so the Toda flow preserves the spectrum. Therefore the flow acts on the isospectral space $X_{n,\lambda}$.

Remark 2.2 The Toda flow commutes with the torus action. Indeed, the action of T on L is given by DLD^{-1} for some diagonal Hermitian matrix D. We have $P(DLD^{-1}) = DP(L)D^{-1}$ and therefore $[DLD^{-1}, P(DLD^{-1})] = D[L, P(L)]D^{-1}$.

The periodic Toda system is well studied for real symmetric matrices. We need a more general Hermitian version of periodic Toda system in order to incorporate torus actions. However, the complex case is not more complicated than the real one. The equations of the flow have the coordinate form

$$
(2\text{-}3) \qquad \begin{cases} \dot{a}_i = 2(|b_{i-1}|^2 - 2|b_i|^2) & \text{for } i = 1, \ldots, n, \\ \dot{b}_i = b_i(a_i - a_{i+1}) & \text{for } i = 1, \ldots, n, \end{cases}
$$

where a_i and b_i are assumed to be cyclically ordered. Since $b_i \in \mathbb{C}$, each expression in the second line represents two real equations. We see that the arguments of $b_i \in \mathbb{C}$ remain constant along the flow. The equations on a_i and $|b_i|$ have the form

$$
(2\text{-}4) \qquad \begin{cases} \dot{a}_i = 2(|b_{i-1}|^2 - 2|b_i|^2) & \text{for } i = 1, \ldots, n, \\ \frac{d}{dt}|b_i| = |b_i|(a_i - a_{i+1}) & \text{for } i = 1, \ldots, n, \end{cases}
$$

which coincides with the real form of the periodic Toda flow.

Construction 2.3 It is a simple exercise that the quantity $B = \prod_1^n b_i$ is preserved along the flow. In what follows we consider the exceptional subspace

$$
X_{n,\lambda}^0 = \{L \in X_{n,\lambda} \mid B = 0\}.
$$

This subspace can be represented as the union $X_{n,\lambda}^0 = \bigcup_1^n Y_i$, where $Y_i \subset X_{n,\lambda}$ is the subset of matrices having $b_i = 0$ for a particular index $i \in [n]$. The set Y_n is just the set of isospectral tridiagonal Hermitian matrices, which is known to be a smooth manifold whose smooth type is independent of a simple spectrum λ [31]. Moreover, it is known that Y_n is a quasitoric manifold over a permutohedron [8; 16] (the reader is advised to consult [10] concerning the terminology of quasitoric manifolds). Each Y_i for $i \neq n$ is diffeomorphic to Y_n. This follows from the fact that the matrix with $b_i = 0$ can be transformed to a tridiagonal Hermitian matrix by a cyclic permutation of rows and columns.

Therefore $X_{n,\lambda}^0$ is the union of n submanifolds of dimension $2n - 2$; however, these submanifolds intersect nontrivially. In the intersection of Y_i and Y_j there lies the submanifold of matrices with $b_i = b_j = 0$, which is a torus-invariant codimension 2 submanifold of both Y_i and Y_j. The combinatorial structure of these intersections will be described in detail in Section 5.

The Toda flow degenerates to a Toda flow of nonperiodic Toda lattice on the exceptional set $X_{n,\lambda}^0$. Each submanifold Y_i is preserved by the flow. The Toda flow on Y_i is a

gradient flow (see eg [32] or [14]), which means that asymptotically each trajectory on Y_i tends to an equilibrium point. The equilibrium points are the diagonal matrices

$$L_\sigma = \mathrm{diag}(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}), \quad \sigma \in S_n.$$

A direct check shows that the subspace $X_{n,\lambda}$ is a smooth manifold in a neighborhood of each equilibrium point L_σ [31]. The asymptotical properties of the flow on the exceptional set imply that $X_{n,\lambda}$ is a smooth manifold in a neighborhood of $X_{n,\lambda}^0$. It will be shown in Section 3 that $X_{n,\lambda}$ is not always smooth in points with large values of B.

Remark 2.4 For a generic spectrum λ the whole space $X_{n,\lambda}$ is a smooth manifold. This easily follows from Sard's theorem applied to the map sending the periodic tridiagonal matrix L to the tuple $(\mathrm{tr}\, L, \mathrm{tr}\, L^2, \ldots, \mathrm{tr}\, L^n)$.

3 The orbit space of the torus action

The action of $T = T^{n-1}$ on $X_{n,\lambda}$ has $n!$ fixed points, L_σ for $\sigma \in S_n$, which coincide with the equilibria of the Toda flow.

Proposition 3.1 The orbit space $Q_{n,\lambda} = X_{n,\lambda}/T$ is a topological manifold in a neighborhood of $X_{n,\lambda}^0/T$. The space $Q_{n,\lambda}$ is a topological manifold for generic λ.

Proof Note that $\dim X_{n,\lambda} = 2n$ and $\dim T = n - 1$. Consider any fixed point L_σ. The tangent representation of the action at a point L_σ has the weight decomposition

$$T_{L_\sigma} X_{n,\lambda} = V(\alpha_{1,\sigma}) \oplus \cdots \oplus V(\alpha_{n,\sigma}), \quad \alpha_{i,\sigma} \in \mathrm{Hom}(T^n, S^1),$$

where $V(\alpha)$ is the 1–dimensional complex representation

$$tz = \alpha(t) \cdot z.$$

In terms of the noneffective action of the n–dimensional torus T^n, we have

$$\alpha_{i,\sigma} = \epsilon_i - \epsilon_{i+1} \quad \text{for any } \sigma \in S_n,$$

where $\{\epsilon_1 = \epsilon_{n+1}, \epsilon_2, \ldots, \epsilon_n\}$ is the standard basis of $\mathrm{Hom}(T^n, S^1) \cong \mathbb{Z}^n$, as follows from the explicit expression (2-1) for the action.

The following fact was proved in [4]. Suppose a torus T of dimension $n - 1$ acts effectively on a smooth manifold X of dimension $2n$, and assume that each connected

component of each equivariant skeleton X_j contains a fixed point. Assume, moreover, that the action has finitely many fixed points, and, at each fixed point, any $n-1$ of n weights $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}^{n-1}$ of the tangent representation are linearly independent. Then X/T is a closed topological $(n+1)$–manifold. Applying this result to $X_{n,\lambda}$ in a neighborhood of $X_{n,\lambda}^0$, we get the first part of the proposition.

The second part follows easily from Remark 2.4, since the action of T outside $X_{n,\lambda}^0$ is free. Therefore, whenever $X_{n,\lambda}$ is a smooth manifold, the orbit space $X_{n,\lambda}/T$ is smooth outside $X_{n,\lambda}^0/T$, thus it is a topological manifold. $\qquad\square$

To describe the topology of $Q_{n,\lambda}$ and $X_{n,\lambda}$, we formulate a result of independent interest. Let $p\colon X_{n,\lambda} \to \mathbb{C}$ be the map which associates the number $B = \prod_{i=1}^{n} b_i$ to a periodic tridiagonal matrix $L(\underline{a}, \underline{b})$. Since the T–action preserves B, there is an induced continuous map $\widetilde{p}\colon Q_{n,\lambda} \to \mathbb{C}$.

The aim of the following constructions is to describe the image of $\widetilde{p}$ and all its preimages. The description is given in Theorem 3.4 below.

Construction 3.2 Let a simple spectrum $(\lambda_1 < \ldots < \lambda_n)$ be given. Consider the characteristic polynomial $F(x) = \prod_{i=1}^{n}(x - \lambda_i)$. Since the polynomial has n real roots, we have the sequence of real numbers

$$x_1 < x_2 < \cdots < x_{n-2} < x_{n-1},$$

where $x_{n-1}, x_{n-3}, x_{n-5}, \ldots$ are the local minima and $x_{n-2}, x_{n-4}, \ldots$ are the local maxima of F. Let

(3-1) $$M = \min_{i \text{ is even}} F(x_{n-i}), \quad m = \min_{i \text{ is odd}} -F(x_{n-i}).$$

We obviously have $m, M > 0$.

Remark 3.3 The interval $[-m, M]$ represents the set of all $s \in \mathbb{R}$ such that all roots of the polynomial $F(x) - s$ are real; see Figure 2.

Let n_+ be the number of local maxima at which M is achieved and, similarly, n_- be the number of local minima at which $-m$ is achieved. For generic λ there holds $n_+ = n_- = 1$. Figure 2 shows the case $n_+ = 1$ and $n_- = 2$.

Theorem 3.4 *The image of $p\colon X_{n,\lambda} \to \mathbb{C}$ is the set*

(3-2) $$\mathbb{B} = \left\{ z \in \mathbb{C} \ \middle| \ |z| \leqslant \tfrac{1}{2} \min\left(\frac{m}{1 + \cos \operatorname{Arg} z}, \frac{M}{1 - \cos \operatorname{Arg} z} \right) \right\}.$$
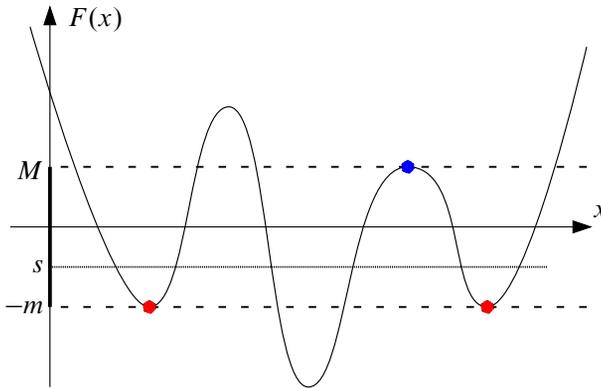
Figure 2: The values M and $-m$ on the plot of a characteristic polynomial.

The preimages of the map $\widetilde{p}\colon Q_{n,\lambda} \to \mathbb{B}$ are as follows. If $z \in \mathbb{B}^{\circ}$, then $\widetilde{p}^{-1}(z)$ is homeomorphic to a compact torus $\mathcal{T}^{n-1}$. If $z \in \partial\mathbb{B}$ and the minimum in (3-2) is achieved at $M/(1 - \cos \operatorname{Arg} z)$, then $\widetilde{p}^{-1}(z)$ is a torus of dimension $n - 1 - n_{+}$. If $z \in \partial\mathbb{B}$ and the minimum in (3-2) is achieved at $m/(1 + \cos \operatorname{Arg} z)$, then $\widetilde{p}^{-1}(z)$ is a torus of dimension $n - 1 - n_{-}$. If $z \in \partial\mathbb{B}$ and $m/(1 + \cos \operatorname{Arg} z) = M/(1 - \cos \operatorname{Arg} z)$, then $\widetilde{p}^{-1}(z)$ is a torus of dimension $n - 1 - n_{+} - n_{-}$.

The convex set $\mathbb{B}$ is shown in Figure 3; it is bounded by arcs of two confocal parabolas. The set $\mathbb{B}$ is a 2–dimensional manifold with corners; we denote by F_{+} and F_{-} its left and right sides, respectively, and $F_{+} \cap F_{-} = \{z_{\text{top}}, z_{\text{bot}}\}$. Note that the minimum is achieved at $M/(1 - \cos \operatorname{Arg} z)$ whenever z lies on the left side of the figure, which explains the notation.
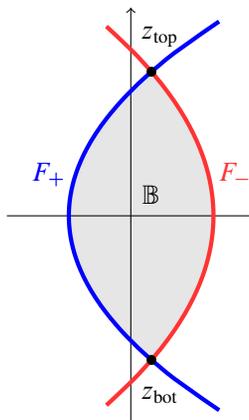

Figure 3: The set $\mathbb{B}$.

Remark 3.5 It will be convenient to distinguish between the torus, which acts on spaces, and the geometrical tori arising in Theorem 3.4. Hence, toric groups are denoted by the symbol T, and tori appearing in geometrical considerations are denoted by the symbol $\mathcal{T}$.

Corollary 3.6 With parameters n_+ and n_- as above, the orbit space $Q_{n,\lambda}$ is homeomorphic to $\Sigma(\mathcal{T}^{n_-} * \mathcal{T}^{n_+}) \times \mathcal{T}^{n-1-n_--n_+}$.

Proof The space $Q_{n,\lambda}$ is foliated over the contractible space $\mathbb{B}$ by tori. Hence,

$$Q_{n,\lambda} \cong \mathbb{B} \times \mathcal{T}^{n-1}/\sim,$$

where a certain n_+–dimensional subtorus $\mathcal{T}_+$ is collapsed over F_+ and another n_-–dimensional subtorus $\mathcal{T}_-$ is collapsed over F_- (the nature of these tori and their independence is clarified in Section 4). We have $\mathcal{T} = \mathcal{T}_+ \times \mathcal{T}_- \times \mathcal{T}^{n-1-n_--n_+}$. The subtorus $\mathcal{T}^{n-1-n_--n_+}$ separates as a direct factor of $Q_{n,\lambda}$. The remaining factor is the suspension space, with the suspension points being the preimages of the points z_{top} and z_{bot}. This suspension is taken over the space $\tilde{p}^{-1}(\mathbb{B} \cap \mathbb{R})$, which is homeomorphic to the join of $\mathcal{T}_+$ and $\mathcal{T}_-$. $\qquad \square$

Corollary 3.7 For a generic spectrum λ there is a homeomorphism $Q_{n,\lambda} \cong S^4 \times \mathcal{T}^{n-3}$.

Proof In the generic case we have $n_+ = n_- = 1$. Therefore, $\Sigma(\mathcal{T}^1 * \mathcal{T}^1) \cong \Sigma S^3 \cong S^4$. $\qquad \square$

Corollary 3.8 (theorem of Buchstaber and Terzić [11; 13]) Consider the effective action of $T = T^3/\Delta(T^1)$ on the manifold Fl_3 of complete complex flags in $\mathbb{C}^3$. The orbit space Fl_3/T is homeomorphic to S^4.

Proof Note that $X_{3,\lambda}$ is just the set of all Hermitian matrices with the given spectrum. This manifold is diffeomorphic to the flag manifold Fl_3. Notice that, if $n = 3$, the characteristic polynomial is cubic, so it has exactly one local maximum and one local minimum: $n_+ = n_- = 1$. Hence the statement is the particular case of Corollary 3.7 with $n = 3$. $\qquad \square$

Theorem 3.9 If λ is a simple spectrum such that either $n_+ > 1$ or $n_- > 1$, then $X_{n,\lambda}$ is not a homology manifold. In particular, this space is not a smooth manifold.

Proof The space $Q_{n,\lambda} \cong \Sigma(T^{n-} * T^{n+}) \times T^{n-1-n_--n_+}$ is not a homology manifold unless $n_+ = n_- = 1$. Indeed, assume $n_- > 1$ (the case $n_+ > 1$ is completely similar). Let $q \in Q_{n,\lambda}$ be a point such that $\tilde{p}(q)$ lies in the interior of F_+. There exists a neighborhood $U_q \subset Q_{n,\lambda}$ of q homeomorphic to Cone $T^{n-} \times \mathbb{R}^{n-n_-}$, according to the structure of the join. We have

$$H_i(U_q, U_q \setminus \{q\}) \cong \tilde{H}_{i-1}(T^{n-} * S^{n-n_--1}) \cong \tilde{H}_{i-1-n+n_-}(T^{n-}) \neq 0$$

for $n - n_- + 1 < i < n + 1 = \dim Q_{n,\lambda}$, that is, at least for one value of i. This argument shows that $Q_{n,\lambda}$ fails to be a homology manifold at q.

The torus action on $X_{n,\lambda}$ is free over $\partial\mathbb{B}$. Hence, for any point $x \in X_{n,\lambda}$ lying in the orbit q, its neighborhood $U_x \ni x$ is homeomorphic to $U_q \times \mathbb{R}^{n-1}$. Therefore,

$$H_i(U_x, U_x \setminus \{x\}) \cong H_{i-n+1}(U_q, U_q \setminus \{q\}) \neq 0$$

for some $i < 2n$, so that $X_{n,\lambda}$ fails to be a homology manifold at x. □

Remark 3.10 Van Moerbeke [23] proves the real analogue of Theorem 3.4. In the real case, there is a family of tori, parametrized by real numbers from the interval $\left[-\frac{1}{4}M, 0\right) \subset \mathbb{B} \cap \mathbb{R}$. The dimension of all tori is $n - 1$, except for the torus over the endpoint $-\frac{1}{4}M$: its dimension decreases by n_+. Van Moerbeke calls the union of such a family "an open n–dimensional torus". This naming seems misleading, since this union is not even a manifold for $n_+ > 1$, which is proved similarly to Theorem 3.9.

Remark 3.11 The most degenerate case appears in the situation when the values at all local minima of the characteristic polynomial $F(x) = \prod_{i=1}^{n}(x - \lambda_i)$ coincide, and the values at all local maxima coincide. For example, this holds for the Chebyshev polynomials $T_n(x)$ (which are defined on the interval $[-1, 1]$ by $T_n(x) = \cos(n \arccos x)$). It can be seen that a polynomial gives the maximal possible degeneration if and only if it coincides with Chebyshev polynomial up to affine transformation of the image and the domain:

$$F(x) = \gamma T_n(\alpha x + \beta) + \delta,$$

with the natural requirement that $F(x)$ is monic and has n distinct real roots.

4 Schrödinger equation and the spectral curve

In this section we prove the first part of Theorem 3.4. It will be assumed that $B = \prod_1^n b_i \neq 0$, ie $L \notin X_{n,\lambda}^0$. The action of T is free on such matrices. We may identify

$Q_{n,\lambda} = X_{n,\lambda}/T$ with the set of isospectral Hermitian matrices of the form

(4-1)
$$L(w) = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & w^{-1}b_n \\ b_1 & a_2 & b_2 & & \vdots \\ 0 & b_2 & a_3 & \ddots & \\ \vdots & & \ddots & \ddots & b_{n-1} \\ wb_n & \cdots & & b_{n-1} & a_n \end{pmatrix},$$

where $b_1, \ldots, b_n$ are positive real numbers and $w \in \mathbb{C}$ with $|w| = 1$. Indeed, the arguments of any $n - 1$ off-diagonal terms of a periodic tridiagonal matrix $L(\underline{a}, \underline{b})$ can be rotated to zero by the torus action (2-1). We continue denoting $\prod_{i=1}^{n} b_i$ by the letter B, although in the new notation B is a positive real number.

Proposition 4.1 *For a matrix $L(w)$ with a simple spectrum $\lambda_1 < \cdots < \lambda_n$ there holds*

$$B \leqslant \tfrac{1}{2} \min\left(\frac{M}{1 - \cos \operatorname{Arg} w}, \frac{m}{1 + \cos \operatorname{Arg} w} \right),$$

where M and m are defined by (3-1).

Proof Matrices of the form $L(w)$ can be studied using algebrogeometric methods in mathematical physics (we refer to [20] for a brief exposition of this subject in relation to periodic Toda flow). Let ℓ be the space of two-sided infinite sequences $\{\psi_k\}$,

$$\psi_k \in \mathbb{C}, \quad k \in \mathbb{Z}.$$

Consider the *periodic discrete Schrödinger* operator given by

$$H: \ell \to \ell, \quad H(\psi)_k = b_{k-1}\psi_{k-1} + a_k\psi_k + b_k\psi_{k+1},$$

where we assume $a_{k+n} = a_k$ and $b_{k+n} = b_k$. The eigenfunction $\psi \in \ell$ of the Schrödinger operator with eigenvalue x satisfies the equation

(4-2)
$$H(\psi) = x\psi.$$

Since $b_i \neq 0$, every eigenfunction is determined by its initial values $(\psi_0, \psi_1) \in \mathbb{C}^2$. We can define the *monodromy operator* along the period,

(4-3)
$$M(x): \mathbb{C}^2 \to \mathbb{C}^2, \quad M(x): (\psi_0, \psi_1) \mapsto (\psi_n, \psi_{n+1}).$$

Note that the matrix $L(w)$ has eigenvalue x if and only if there exists a solution ψ to (4-2) such that

$$\psi_{k+n} = w\psi_k.$$

Such functions are called *Bloch solutions*. We see that whenever there exists a nonzero Bloch solution with parameter w, the number w is the eigenvalue of the monodromy operator $M(x)$, so we get a relation

$$(4\text{-}4) \qquad \det(wI - M(x)) = 0.$$

This equation defines a so-called *spectral curve* of the periodic Schrödinger equation in the space of parameters $(w, x) \in \mathbb{C}^2$. We have $\det M(x) = 1$. Indeed, the operator $M_i \colon (\psi_{i-1}, \psi_i) \mapsto (\psi_i, \psi_{i+1})$ has determinant b_{i-1}/b_i, therefore

$$M(x) = M_n M_{n-1} \cdots M_1 = \frac{b_{n-1}}{b_n} \frac{b_{n-2}}{b_{n-1}} \cdots \frac{b_0}{b_1} = 1.$$

Hence, the equation (4-4) of the spectral curve can be rewritten in the form

$$(4\text{-}5) \qquad w^2 - \operatorname{tr} M(x) w + 1 = 0.$$

We have $\operatorname{tr} M(x) = \frac{1}{B} P(x)$, where $B = \prod_1^n b_i$ as before and $P(x)$ is a monic polynomial in x. Indeed, decomposing $M(x)$ as the product of operators M_i along the period for $i = 1, \ldots, n$, and counting the terms of highest degree of x, we see that the leading coefficient of $\operatorname{tr} M(x)$ is $1/B$. Dividing (4-5) by w and letting $t = \operatorname{Re} w = \frac{1}{2}(w + w^{-1})$, we get $2t = \frac{1}{B} P(x)$.

The polynomial $P(x) - 2Bt$ is monic and has the given sequence $\lambda_1, \ldots, \lambda_n$ as its roots, therefore

$$(4\text{-}6) \qquad P(x) - 2Bt = \prod_{i=1}^n (x - \lambda_i) = F(x), \qquad P(x) = F(x) + 2Bt.$$

Consider the set

$$\mathcal{A} = \{s \in \mathbb{R} \mid P(x) = 2Bs \text{ has } n \text{ real roots}\}.$$

Recalling the definition of m and M and Remark 3.3 as well as the relation (4-6), we see that $\mathcal{A}$ is the closed interval $\left[-\frac{m}{2B} + t, \frac{M}{2B} + t\right]$.

Note that the polynomial $P(x) - 2Bs$ is the characteristic polynomial of the matrix $L(w_s)$, where $|w_s| = 1$, $\operatorname{Re} w_s = s$. The matrix $L(w_s)$ is Hermitian, therefore all its eigenvalues are real. Hence, for any $s \in [-1, 1]$, all roots of the equation $P(x) = 2Bs$ are real. Therefore,

$$[-1; 1] \subseteq \left[-\frac{m}{2B} + t, \frac{M}{2B} + t\right],$$

from which we deduce $B \leqslant \frac{1}{2} \min\left(\frac{M}{1-t}, \frac{m}{1+t}\right)$. Remembering $t = \operatorname{Re} w = \cos \operatorname{Arg} w$, we get the required inequality. $\qquad \square$

Proposition 4.2 *For $z \in \mathbb{B}$, $z \neq 0$, the preimage $\widetilde{p}^{-1}(z)$ is homeomorphic to a torus. The dimension of a torus is $n - 1$ if z lies in the interior of $\mathbb{B}$, $n - 1 - n_+$ if z lies in the relative interior of F_+, $n - 1 - n_-$ if z lies in the relative interior of F_-, and $n - 1 - n_+ - n_-$ if z is either z_{top} or z_{bot}.*

Proof In short, this follows from the fact that the periodic Toda lattice is an integrable dynamical system and its energy levels are the compact submanifolds. The Liouville–Arnold theorem then implies that these preimages $\widetilde{p}^{-1}(z)$ are tori. To specify the dimensions we give more details on the theory, related to periodic tridiagonal matrices.

As before, consider $P(x) = B \operatorname{tr} M(x)$, the monic polynomial in x with coefficients depending on $a_i, b_i \in \mathbb{R}$. As follows from the considerations above, the eigenvalues of matrices $L(1)$ and $L(-1)$ are the roots of the polynomials $P(x) - 2B$ and $P(x) + 2B$, respectively. Let

$$x_1 < x_2 \leqslant x_3 < \cdots < x_{2n-2} \leqslant x_{2n-1} < x_{2n}$$

be the union of all these roots, so that $x_{2n}, x_{2n-3}, x_{2n-4}, x_{2n-7}, x_{2n-8}, \ldots$ are the roots of $P(x) - 2B$ and $x_{2n-1}, x_{2n-2}, x_{2n-5}, x_{2n-6}, \ldots$ are the roots of $P(x) + 2B$. The intervals

$$I_1 = [x_2, x_3], \quad I_2 = [x_4, x_5], \quad \ldots, \quad I_{n-1} = [x_{2n-2}, x_{2n-1}]$$

are called the *forbidden zones*. We will call $I_{n-1}, I_{n-3}, \ldots$ lower forbidden zones, and $I_{n-2}, I_{n-4}, \ldots$ upper forbidden zones, as motivated by Figure 4.

Consider the Riemannian surface Θ_g of the multivalued function

$$g(x) = \sqrt{\prod_{i=1}^{2n} (x - x_i)}.$$

Over each forbidden interval I_k for $k = 1, \ldots, n - 1$, there lies a circle S_k on Θ_g. If an interval I_k degenerates to a point (ie $x_{2k} = x_{2k+1}$), the circle S_k also collapses to a point. Van Moerbeke [23] proved:

Proposition 4.3 *Real periodic tridiagonal symmetric matrices $L(\underline{a}, \underline{b})$ with the given spectrum λ, given $B = \prod_1^n b_i$ and $b_i > 0$ are in one-to-one correspondence with $(n-1)$–tuples $(\mu_1, \ldots, \mu_{n-1})$, where $\mu_k \in S_k$.*

Therefore, for real z, the preimage $\widetilde{p}^{-1}(z)$ is diffeomorphic to a torus $\mathcal{T} = \prod_1^{n-1} S_i$. The dimension of this torus equals $n - 1$ in general; however, when some forbidden
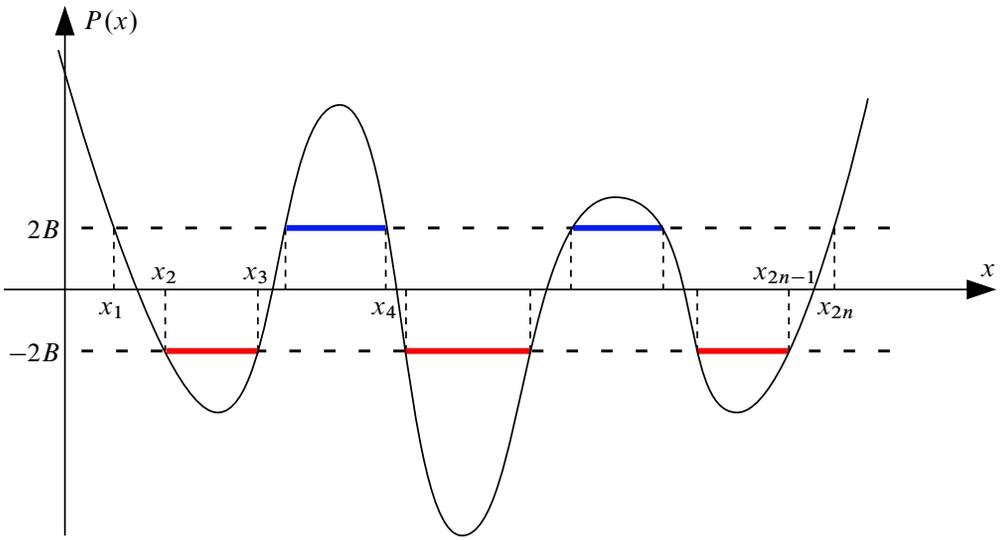
Figure 4: Upper and lower forbidden zones.

intervals are collapsed, the dimension reduces by the number of collapsed intervals. The upper forbidden intervals collapse if and only if the value $2B$ reaches M. The number of collapses among upper intervals equals n_+. Similarly, the lower intervals collapse if $2B$ reaches m, and the number of collapses among lower intervals is n_-.

Now let $L(w)$ be an arbitrary matrix with $w \in \mathbb{C}$, $|w| = 1$ and the given spectrum λ. Let $\widetilde{\lambda}$ be the set of roots of the polynomial $\prod(x - \lambda_i) + 2B \operatorname{Re} w$. It was mentioned in the proof of Proposition 4.1 that the matrix $L(w)$ has spectrum λ if and only if $L(1)$ has spectrum $\widetilde{\lambda}$. Thus Proposition 4.3 implies the required statement for all matrices. $\qquad\square$

5 Permutohedral tilings

In this section we study the degenerate locus of the periodic Toda lattice. Recall that $X_{n,\lambda}^0 = p^{-1}(0) \subset X_{n,\lambda}$ is the set of all isospectral matrices with $B = \prod_1^n b_i = 0$, and $Q_{n,\lambda}^0 = X_{n,\lambda}^0 / T = \widetilde{p}^{-1}(0)$.

We recall some standard facts from combinatorial geometry. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis of $\mathbb{Z}^n \cong \operatorname{Hom}(T^n, S^1)$. We assume that $\mathbb{Z}^n \subset \mathbb{R}^n$ and there is a fixed inner product on $\mathbb{R}^n$ such that $\epsilon_1, \ldots, \epsilon_n$ are orthonormal.

Consider the sublattice $N \subset \mathbb{Z}^n$ of rank $n-1$ given by

$$N = \left\{ \sum_{i=1}^{n} a_i \epsilon_i \ \middle|\ a_i \in \mathbb{Z}, \ \sum a_i = 0 \text{ and } a_i - a_j \equiv 0 \bmod n \right\},$$

and let $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ be its real span. Consider the vectors $\alpha_1, \ldots, \alpha_n$,

$$\alpha_i = (n-1)\epsilon_i - \sum_{j \neq i} \epsilon_j, \quad i = 1, \ldots, n.$$

We see that

(5-1)
$$\sum_{i=1}^{n} \alpha_i = 0,$$

and any $n-1$ of $\alpha_1, \ldots, \alpha_n$ generate the lattice N. One can think about the α_i as the inward normal vectors to the facets of a regular simplex in $\mathbb{R}^{n-1}$.

For any subset $S \subset [n] = \{1, \ldots, n\}$ such that $S \neq \varnothing, [n]$, consider the vector

(5-2)
$$\alpha_S = \sum_{i \in S} \alpha_i.$$

Let $\mathcal{P}_{n-1}$ be the Voronoi cell decomposition of $N_\mathbb{R} \cong \mathbb{R}^{n-1}$ generated by the lattice N. In other words, for any $\alpha \in N$ we consider the Voronoi cell

$$P_\alpha = \{x \in N_\mathbb{R} \mid \mathrm{dist}(x, \alpha) \leqslant \mathrm{dist}(x, \beta) \text{ for any } \beta \in N, \ \beta \neq \alpha\},$$

where dist is the distance determined by the inner product on $N_\mathbb{R} \subset \mathbb{R}^n$. Each P_α is a convex $(n-1)$–dimensional polytope and all these polytopes are the parallel copies of each other, $P_\alpha = P_0 + \alpha$.

Construction 5.1 It can be shown (see details in [15, Section 3.8, Theorem 7]) that P_0 is the $(n-1)$–dimensional permutohedron Pe^{n-1} determined by the inequalities

$$\mathrm{Pe}^{n-1} = P_0 = \left\{ x \in N_\mathbb{R} \mid \langle \alpha_S, x \rangle \leqslant \tfrac{1}{2}\langle \alpha_S, \alpha_S \rangle, \ S \in 2^{[n]}, \ S \neq \varnothing, [n] \right\},$$

where $\langle \cdot, \cdot \rangle$ is the inner product on N. We recall the basic facts about the combinatorics of a permutohedron. The polytope Pe^{n-1} is simple, which means that every codimension k face is contained in exactly k facets. For a proper subset $S \subset [n]$ let F_S denote the facet of Pe^{n-1} determined by the support hyperplane $\langle \alpha_S, x \rangle = \tfrac{1}{2}\langle \alpha_S, \alpha_S \rangle$. Note that Pe^{n-1} is centrally symmetric: the facets F_S and $F_{\bar{S}}$ are opposite to each other whenever $\bar{S} = [n] \setminus S$.

Facets $F_{S_1}, \ldots, F_{S_k}$ have nonempty intersection in Pe^{n-1} if and only if the subsets $\{S_1, \ldots, S_k\}$ form a chain in the Boolean lattice $2^{[n]}$. If $\sigma = (S_1 \subset \cdots \subset S_k)$ is such a chain, we denote by F_σ the face $F_{S_1} \cap \cdots \cap F_{S_k}$ of the permutohedron. Each face of Pe^{n-1} is known to be a product of permutohedra of smaller dimensions.

We denote by $F_S(P_\alpha)$ (resp. $F_\sigma(P_\alpha)$) the corresponding facets (resp. faces) of the Voronoi cell P_α to distinguish different copies of a permutohedron in the Voronoi diagram. It can be seen that

$$(5\text{-}3) \qquad\qquad F_S(P_\alpha) = F_{\overline{S}}(P_{\alpha + \alpha_S}).$$

A facet of each cell is adjacent to an opposite facet of a neighboring cell.

We formulate a general construction to precede a particular case needed in the proof of Theorem 3.4.

Construction 5.2 Let $\widehat{N} \subseteq N$ be a sublattice of finite index, ie $q = |N/\widehat{N}| < \infty$. Consider the quotient $N_\mathbb{R}/\widehat{N}$. Since $\widehat{N}$ is a cocompact lattice, this quotient is a torus $\mathcal{T}^{n-1}$. The action of $\widehat{N}$ by parallel shifts preserves the Voronoi diagram, therefore we have a cell subdivision of the torus $\mathcal{T}^{n-1} \cong N_\mathbb{R}/\widehat{N}$. There are q maximal cells in this subdivision; each is a parallel copy of a permutohedron.

Example 5.3 A natural example is $\widehat{N} = N$. In this case the torus is given by identifying the opposite facets of a single permutohedron. The cell structure on a torus given by this identification is known: the corresponding partially ordered set was introduced and studied by Panina [26] under the name of *cyclopermutohedron*. This poset has a natural combinatorial description.

For the considerations of this paper we need another sublattice.

Construction 5.4 Let $N' \subset N$ be the sublattice generated by the vectors

$$\beta_k = \alpha_k - \alpha_{k+1}, \quad k = 1, \ldots, n-1.$$

Note that $\beta_{n-1} = \alpha_{n-1} - \alpha_n = 2\alpha_{n-1} + \alpha_1 + \cdots + \alpha_{n-2}$ according to (5-1). It can be shown that N/N' is the cyclic group of order n. Indeed, in the quotient group N/N' we have the identities

$$(5\text{-}4) \qquad [\alpha_1] = \cdots = [\alpha_n], \quad n[\alpha_1] = [\alpha_1] + \cdots + [\alpha_{n-2}] + 2[\alpha_{n-1}] = 0.$$

Definition 5.5 Let $\mathcal{PT}^{n-1}$ be the cell decomposition of a torus $\mathcal{T}^{n-1}$ obtained as a quotient of the Voronoi diagram of the space $N_{\mathbb{R}}$ by the sublattice N'. We call $\mathcal{PT}^{n-1}$ the *wonderful cell decomposition* of a torus.

The wonderful decomposition $\mathcal{PT}^{n-1}$ has n maximal cells. The cells P_α and $P_{\alpha+\beta}$ are identified in $\mathcal{PT}^{n-1}$ whenever $\beta \in N'$. We denote the resulting cell of $\mathcal{PT}^{n-1}$ by $P_{[\alpha]}$. The relations (5-4) imply

$$[\alpha_S] = |S|[\alpha_1], \quad [n\alpha_1] = [0].$$

Lemma 5.6 *Let* $1 \leqslant k < m \leqslant n$. *In the cell complex* $\mathcal{PT}^{n-1}$ *we have*

$$F_S(P_{k[\alpha_1]}) = F_{\overline{S}}(P_{m[\alpha_1]}),$$

where S *is any subset of* $[n]$ *of cardinality* $m - k$.

Proof Choose any subset S' such that $|S'| = k$ and S' is disjoint from S. According to (5-3) we have

$$F_S(P_{k[\alpha_1]}) = F_S(P_{[\alpha_{S'}]}) = F_{\overline{S}}(P_{[\alpha_{S'}+\alpha_S]}) = F_{\overline{S}}(P_{[\alpha_{S' \sqcup S}]}) = F_{\overline{S}}(P_{m[\alpha_1]}),$$

which proves the statement. □

In the following, we denote the maximal cells $P_{k[\alpha_1]}$ by $\mathcal{PT}_k$. Now we return to the space of tridiagonal matrices. Recall that Y_k denotes the space of all isospectral matrices $L(\underline{a}, \underline{b})$ with $b_k = 0$ for $k = 1, \ldots, n$. Let Q_k denote the orbit space Y_k/T. We have $Q^0_{n,\lambda} = \bigcup_1^n Q_k$.

Theorem 5.7 *The space* $Q^0_{n,\lambda}$ *can be identified with* $\mathcal{PT}^{n-1}$ *so that the subspaces* Q_k *are identified with* $\mathcal{PT}_k$.

Proof The orbit space $Q_0 = Q_n$ is identified with the space of all tridiagonal symmetric real matrices

$$L = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & & 0 \\ b_1 & a_2 & b_2 & & & 0 \\ 0 & b_2 & a_3 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & b_{n-1} \\ 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix}$$

with $b_i \geqslant 0$ and the given simple spectrum λ. It is known (see [31]) that Q_0 is diffeomorphic to a permutohedron Pe^{n-1} as a manifold with corners. The facet

$F_S(\mathrm{Pe}^{n-1})$ corresponds to the subset of Q_0, which consists of matrices L such that $b_{|S|} = 0$ and the eigenvalues $\{\lambda_i \mid i \in S\}$ are distributed in the first $(|S| \times |S|)$ block.

Similar considerations are valid for other spaces Q_k: this can be shown by cyclic permutation of rows and columns of L. Indeed, the set Q_k can be identified with Pe^{n-1} in such a way that the facet $F_S(\mathrm{Pe}^{n-1})$ consists of all matrices with the property

$$b_k = 0, \quad b_{k+|S|} = 0,$$

and the block between the kth and $(k + |S|)^{\mathrm{th}}$ rows and columns has eigenvalues $\{\lambda_i \mid i \in S\}$.

It can be seen that the faces $F_S(Q_k)$ and $F_{\overline{S}}(Q_m)$ represent the same set of matrices for $1 \leqslant k < m \leqslant n$ and $|S| = m - k$. Therefore, $F_S(Q_k) = F_{\overline{S}}(Q_m)$ in $Q^0_{n,\lambda}$. These gluing rules for the cells in $Q^0_{n,\lambda}$ coincide with the gluing rules for $\mathcal{PT}_k$ in $\mathcal{PT}^{n-1}$ according to Lemma 5.6. □

For convenience introduce the cyclic notation $Q_k = Q_{k+n}$ for any $k \in \mathbb{Z}$.

Example 5.8 Figure 5, right, shows the space $Q^0_{3,\lambda} = \mathcal{PT}^2$. This example was described in details by van Moerbeke [23]. The 1–skeleton of $\mathcal{PT}^2$ is shown on the left. As an abstract graph, it is isomorphic to the complete bipartite graph $K_{3,3}$. This graph is a GKM-graph of the complete flag variety Fl_3; see details in [4].

Remark 5.9 Let us briefly sketch the phase portrait of the Toda flow on the degenerate set of orbits $Q^0_{n,\lambda}$. Let $v \in N^*_{\mathbb{R}}$ be a generic linear function on $N_{\mathbb{R}}$. Take any face of any permutohedron of the Voronoi diagram in $N_{\mathbb{R}}$. On each such polytope consider a flow, which moves all points in the interior of P to the vertex maximizing the linear function v. The flow looks the same on all Voronoi cells, thus we have an induced flow on the torus $\mathcal{PT}^{n-1} = N_{\mathbb{R}}/N'$.
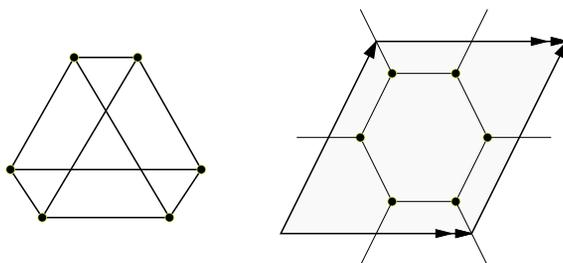

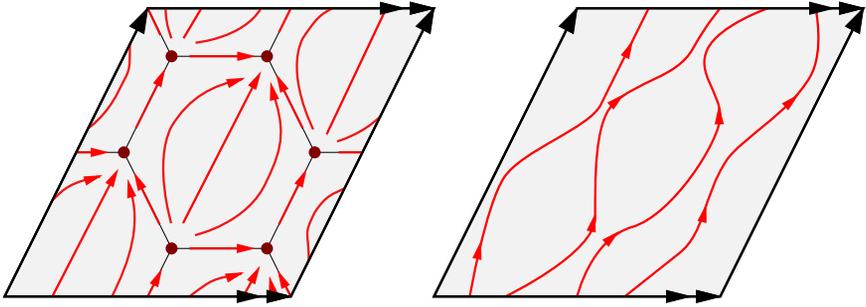Figure 5: The wonderful cell decomposition $Q^0_{3,\lambda} = \mathcal{PT}^2$.

Figure 6: The Toda flow on the level set $B = \text{const}$ of the orbit space $Q^0_{3,\lambda}$.
Left: the case $B = 0$. Right: the case $B = \text{const} \neq 0$.

This picture describes the Toda flow on $Q^0_{n,\lambda} \cong \mathcal{PT}^{n-1}$. Indeed, Toda flow degenerates to the flow of a nonperiodic Toda lattice on each permutohedron Q_i, and its Morse-like behavior is well known (see [17]). For any block tridiagonal matrix, the Toda flow "sorts" the diagonal elements within each block [31].

The phase portrait for $n = 3$ is shown in Figure 6, left. The oddity of the phase portrait near equilibrium points is explained by the fact that the orbit space $Q_{n,\lambda}$ is not smooth at these points.

Note that for $B \neq 0$, the Toda flow exhibits Liouville–Arnold behavior. The equilibrium points disappear, however the flow still follows some direction v on a torus; see Figure 6, right.

Remark 5.10 Each k–dimensional cell of the cell subdivision $\mathcal{PT}^{n-1}$ lies in exactly $n - k$ different maximal cells. This means there exists a dual simplicial cell subdivision $K^{n-1}_{\mathcal{PT}}$. In Section 7 we recall the definition of a simplicial poset, which is a useful combinatorial notion to study simplicial cell subdivisions.

Note that the simplicial poset $K^{n-1}_{\mathcal{PT}}$ minimizes the number of vertices among all simplicial cell subdivisions of the torus $\mathcal{T}^{n-1}$. Indeed, any $(n-1)$–dimensional simplex of such a subdivision has n distinct vertices, therefore a simplicial cell subdivision of $\mathcal{T}^{n-1}$ should have at least n vertices. This number is achieved at $K^{n-1}_{\mathcal{PT}}$.

Remark 5.11 Any closed connected $(n-1)$–manifold admits a simplicial cell subdivision with exactly n vertices. This result was proved in [18], where such subdivisions (or their equivalent combinatorial representations) were called *the crystallizations*. The previous remark shows that $K^{n-1}_{\mathcal{PT}}$ provides an explicit crystallization for the torus $\mathcal{T}^{n-1}$.

Propositions 4.1 and 4.2 and Theorem 5.7 conclude the proof of Theorem 3.4.

6 Topology near degeneration locus

In this section we study the topology of a small neighborhood of $X_{n,\lambda}^0$. The space $X_{n,\lambda}$ is a smooth manifold in the vicinity of $X_{n,\lambda}^0$; see Construction 2.3.

Remark 6.1 The T–action is free outside $X_{n,\lambda}^0$ and admits a section given by the formula (4-1). However, the free part of the action is larger than the outside of $X_{n,\lambda}^0$: the action is also free over the interiors of facets of $Q_{n,\lambda}^0 \cong \mathcal{PT}^{n-1}$. The whole free action $X_{n,\lambda}^{\text{free}} \to X_{n,\lambda}^{\text{free}}/T$ does not admit a section, as explained below.

Recall that $p \colon X_{n,\lambda} \to \mathbb{C}$ maps a matrix $L(\underline{a}, \underline{b})$ to the product $B = \prod_1^n b_i$. For a small ε consider the preimage of points close to zero,

$$X_{n,\lambda}^{\leqslant \varepsilon} = p^{-1}\big(\{z \in \mathbb{C} \mid |z| \leqslant \varepsilon\}\big).$$

According to Proposition 4.2, $X_{n,\lambda}^{\leqslant \varepsilon}$ is a manifold with boundary, the boundary $\partial X_{n,\lambda}^{\leqslant \varepsilon}$ being the subset

$$(6\text{-}1) \qquad X_{n,\lambda}^{=\varepsilon} = p^{-1}\big(\{|z| = \varepsilon\}\big) \cong S_\varepsilon^1 \times \mathcal{T}^{n-1} \times T^{n-1} \cong T^{2n-1},$$

where $S_\varepsilon^1 = \{z \mid |z| = \varepsilon\}$, $\mathcal{T}^{n-1}$ is the Liouville–Arnold torus and T^{n-1} is the acting torus.

It will be useful to incorporate the circle S_ε^1 into the action to obtain a T^n–action on $X_{n,\lambda}^{\leqslant \varepsilon}$.

Construction 6.2 Consider a topological manifold with boundary $W = \mathcal{T}^{n-1} \times [0, 1]$. Its boundary consists of two connected components

$$\partial W = \partial_0 W \sqcup \partial_1 W, \quad \partial_0 W = \mathcal{T}^{n-1} \times \{0\}, \quad \partial_1 W = \mathcal{T}^{n-1} \times \{1\}.$$

On the left component $\partial_0 W$, we introduce the wonderful cell structure $\mathcal{PT}^{n-1}$, constructed in Section 5. This procedure subdivides $\partial_0 W$ into n permutohedra $\mathcal{PT}_1, \ldots, \mathcal{PT}_n$ of dimension $n-1$ so that every cell of dimension k lies in $n-k$ top-dimensional cells. This makes W a manifold with corners (understood in a broad topological sense). We leave the right boundary component $\partial_1 W$ unchanged: no face structure is imposed on $\partial_1 W$.

Let $T^n = \{t = (t_1, \ldots, t_n) \mid |t_i| = 1\}$ be a compact n–torus and T_I with $I \subseteq [n]$ be its coordinate subtorus,

$$T_I = \{t \in T^n \mid t_j = 1, j \notin I\}.$$

Consider the space

$$Y = W \times T^n / \sim,$$

where (r, t) and (r', t') are identified whenever $r = r'$ lies in the intersection of facets $\{\mathcal{PT}_i \mid i \in I\}$ and $t^{-1}t' \in T_I$ for some subset $I \subseteq [n]$. This construction can be considered as a particular case of either the moment-angle manifold construction for simplicial posets (see [22; 10]) or the construction of locally standard actions (see [33]). The space Y is a particular case of the collar models introduced in [3].

The space Y is a manifold with boundary $\partial Y = \partial_1 W \times T^n \cong \mathcal{T}^{n-1} \times T^n$. It carries the action of T^n which is free on the boundary and its orbit space is W.

Consider the induced action of the subtorus

$$T^{n-1} = \{t_1 t_2 \cdots t_n = 1\} \subset T^n$$

on the space Y. It can be checked (details can be found in [4, Construction 5.9]) that the orbit space Y / T^{n-1} is homeomorphic to $\mathcal{T}^{n-1} \times D^2$.

Theorem 6.3 *The space $X_{n,\lambda}^{\leq \varepsilon}$ is T^{n-1}–equivariantly homeomorphic to the collar model Y.*

Proof In [4] we developed a topological theory of complexity one torus actions. The main concepts are recalled here. By definition, an effective action of $T \cong T^{n-1}$ on $X = X^{2n}$ is called a *strictly appropriate action in general position* if the following conditions hold:

(1) The action has finitely many fixed points.

(2) Stabilizers of all points are connected.

(3) Each connected component of each equivariant skeleton X_j contains a fixed point.

(4) For every fixed point x, the weights $\alpha_1, \ldots, \alpha_n \in \mathrm{Hom}(T^{n-1}, S^1) \cong \mathbb{Z}^{n-1}$ of the tangent representation are in general position, which means that every $n-1$ of them are linearly independent.

For such actions we proved that the orbit space $Q = X / T^{n-1}$ is a closed topological manifold of dimension $n + 1$. The orbits of dimensions less than $n - 1$ form a subset $Z \subset Q$ which is called a *sponge*. A sponge is an $(n-2)$–dimensional subset of Q locally modeled by an $(n-2)$–skeleton of $\mathbb{R}_{\geq 0}^n$. The free part of the action gives the

principal T^{n-1}–bundle

$$X^{\text{free}} \to Q \setminus Z.$$

This bundle is classified by the cohomology class $e \in H^2(Q \setminus Z; H_1(T^{n-1}))$, which is called the *Euler class* of the action. Proposition 3.7 of [4] asserts that equivariant topological type of X is uniquely determined by the triple (Q, Z, e) (which essentially means that the information on stabilizers of the action can be recovered from the class e).

Let $x \in Z$ be an orbit with nontrivial stabilizer and U_x be a small open disc neighborhood of x in Q. The inclusion $i_x \colon U_x \to Q$ induces a homomorphism

$$i_x^* \colon H^2(Q, Q \setminus Z; H_1(T^{n-1})) \to H^2(U_x, U_x \setminus Z; H_1(T^{n-1})).$$

The class $e_x = i_x^*(e) \in H^2(U_x, U_x \setminus Z; H_1(T^{n-1}))$ is called the local Euler class at x. It was noted in [4] that local Euler classes are always nonzero. In particular, the global Euler class is always nonzero for suitable actions of complexity one.

These constructions work similarly if X is a manifold with boundary, and the torus action is free on the boundary. In this case, $Q = X/T^{n-1}$ is a manifold with boundary $\partial X/T^{n-1}$. The sponge of the action lies in the interior of Q. Under certain conditions the local Euler classes at fixed points determine the space X uniquely.

Assume Q has the form $Q_M = M \times D^2$, where M is a closed $(n-1)$–manifold with a fixed simple cell decomposition. Assume that the sponge Z_M is the $(n-2)$–skeleton of this cell structure, and we have

$$Z_M = M^{(n-2)} = M^{(n-2)} \times \{0\} \subset M \times D^2 = Q_M.$$

Proposition 6.4 [4, Proposition 5.7] *Let X be a manifold with boundary, which carries a strictly appropriate torus action in general position such that the orbit space and the sponge of the action are given by (Q_M, Z_M). Assume that the free action of T on the boundary is a trivial principal bundle. Then the local Euler classes at fixed points uniquely determine the T^{n-1}–equivariant homeomorphism type of X.*

Apply this proposition to spaces $X_{n,\lambda}^{\leq \varepsilon}$ and Y. The orbit space is $\mathcal{T}^{n-1} \times D^2$ in both cases. The sponge of the action is the $(n-2)$–skeleton of the wonderful cell subdivision $\mathcal{PT}^{n-1}$, defined earlier. The free action on the boundary is a trivial principal bundle. This is true for $X_{n,\lambda}^{\leq \varepsilon}$ since there is a section of the action; see Remark 6.1. This is true for Y since $Y = W \times T^n/\sim$, and the T^n–action over $\partial_1 W$ is a trivial principal bundle.

Finally, consider any fixed point $x = L_\sigma$ of $X_{n,\lambda}^{\leq\varepsilon}$. The tangent representation at x is isomorphic to the standard action of

$$T^{n-1} = \{t_1 \cdots t_n = 1\} \subset T^n = \{(t_1, \ldots, t_n)\}$$

on $\mathbb{C}^n$ (the infinitesimal action just rotates off-diagonal entries, so that the angles of rotation sum to zero). This action coincides with the action of T^{n-1} in the neighborhood of the corresponding fixed point of Y, by the definition of Y. Therefore the local Euler classes of $X_{n,\lambda}^{\leq\varepsilon}$ and Y coincide at each fixed point.

Proposition 6.4 then implies the existence of T^{n-1}–homeomorphism $X_{n,\lambda}^{\leq\varepsilon} \cong Y$.   □

7 Enumerative combinatorics of the wonderful subdivision

In this section we study the enumerative invariants of the permutohedral cell complex $\mathcal{PT}^{n-1}$ or, equivalently, its dual simplicial poset $K_{\mathcal{PT}}^{n-1}$. These invariants will be used further to describe the homological structure of $X_{n,\lambda}$. At first, we recall several standard definitions from commutative algebra and combinatorics.

Definition 7.1 A finite partially ordered set S is called *simplicial* if it has the minimal element $\hat{0} \in S$ and, for any $I \in S$, the order interval $\{J \in S \mid J \leqslant I\}$ is isomorphic to the poset of faces of a k–dimensional simplex for some number $k \geqslant 0$.

The elements of S are called *simplices*. The number k from the definition is called the dimension of a simplex I. A simplex of dimension 0 is called a *vertex*. The geometrical realization of S is the simplicial cell complex obtained by gluing geometrical simplices according to the order relation in S; see [9] for details. In the following we only consider *pure* simplicial posets, which means that all maximal elements of S have the same dimension. A simplicial poset is called *a homology sphere* (resp. *a homology manifold*) if its geometrical realization is a homology sphere (resp. a homology manifold).

Construction 7.2 Let f_j denote the number of j–dimensional simplices of S for $j = -1, 0, \ldots, n-1$; in particular, $f_{-1} = 1$ (the empty simplex $\hat{0}$ has dimension -1). The h–numbers of S are defined from the relation

$$\text{(7-1)} \qquad\qquad \sum_{j=0}^{n} h_j t^{n-j} = \sum_{j=0}^{n} f_{j-1}(t-1)^{n-j},$$

where t is a formal variable. Let $\tilde{\beta}_j(S) = \dim \tilde{H}_j(S)$ be the reduced Betti number of the geometric realization of S. The $h'-$ and h''–numbers of S are defined as

$$(7\text{-}2) \qquad h_j' = h_j + \binom{n}{j}\left(\sum_{s=1}^{j-1}(-1)^{j-s-1}\tilde{\beta}_{s-1}(S)\right) \quad \text{for } 0 \leqslant j \leqslant n,$$

$$(7\text{-}3) \qquad h_j'' = h_j' - \binom{n}{j}\tilde{\beta}_{j-1}(S) = h_j + \binom{n}{j}\left(\sum_{s=1}^{j}(-1)^{j-s-1}\tilde{\beta}_{s-1}(S)\right)$$

for $0 \leqslant j \leqslant n-1$, and $h_n'' = h_n'$. Sums over empty sets are assumed to be zero.

Let $[m] = \{1, \dots, m\}$ be the vertex set of S, with $m = f_0$. Let R be a field or the ring $\mathbb{Z}$, and let $R[m] = R[v_1, \dots, v_m]$, with $\deg v_i = 2$, denote the graded polynomial algebra with m generators, corresponding to the vertices of S. Slightly abusing the terminology, we call the elements of degree 2 linear when working with such polynomial rings. For a graded R–module $V^* = \bigoplus_{j=0}^{\infty} V_j$ we denote by $\mathrm{Hilb}(V^*; t)$ its Hilbert–Poincaré function $\sum_{j=0}^{\infty} t^j \, \mathrm{rk}_R V_j \in \mathbb{Z}[[t]]$.

Definition 7.3 (see [30]) The face ring of a simplicial poset S is the commutative associative graded algebra $R[S]$ over a ring R generated by formal variables v_I, one for each simplex $I \in S$, with relations

$$v_{I_1} \cdot v_{I_2} = v_{I_1 \cap I_2} \cdot \sum_{J \in I_1 \vee I_2} v_J, \quad v_{\hat{0}} = 1.$$

Here $I_1 \vee I_2$ denotes the set of least upper bounds of $I_1, I_2 \in S$, and $I_1 \cap I_2 \in S$ is the intersection of simplices (it is well defined and unique when $I_1 \vee I_2 \neq \hat{0}$). We take the doubled grading on the ring, in which v_I has degree $2(\dim I + 1)$. The natural graded ring homomorphism $R[m] = R[v_1, \dots, v_m] \to R[S]$ defines the structure of the $R[m]$–module on $R[S]$.

If R is an infinite field and $\dim S = n-1$, then a generic set of linear elements $\theta_1, \dots, \theta_n \in R[S]_2$ is a linear system of parameters (we remark that linear systems of parameters can be constructed using characteristic functions on S; see eg [10, Lemma 3.5.8]). Let Θ denote the parametric ideal of $R[S]$ generated by $\theta_1, \dots, \theta_n$.

Proposition 7.4 (Reisner [27] and Stanley [29]) *For a pure simplicial poset S of dimension $n-1$, there holds*

$$\mathrm{Hilb}(R[S]; t) = \frac{h_0 + h_1 t^2 + \cdots + h_n t^n}{(1 - t^2)^n}.$$

For a homology sphere S, there holds $\mathrm{Hilb}(R[S]/\Theta; t) = \sum_i h_i t^{2i}$.

Proposition 7.5 (Schenzel [28] and Novik and Swartz [25; 24]) (1) *For a homology manifold S, there holds*

$$\mathrm{Hilb}(R[S]/\Theta; t) = \sum_i h'_i t^{2i}.$$

(2) *Let S be a connected R–orientable homology manifold of dimension $n-1$. The $2j^{\text{th}}$ graded component of the module $R[S]/\Theta$ contains a vector subspace $(I_{\mathrm{NS}})_{2j} \cong \binom{n}{j}\tilde{H}^{j-1}(S; R)$, which is a trivial $R[m]$–submodule (that is, $R[m]_+(I_{\mathrm{NS}})_{2j} = 0$). Let $I_{\mathrm{NS}} = \bigoplus_{j=0}^{n-1}(I_{\mathrm{NS}})_{2j}$ be the sum of all these submodules except the top-degree component. Then the quotient module $R[S]/\Theta/I_{\mathrm{NS}}$ is a Poincaré duality algebra, and there holds*

$$\mathrm{Hilb}(R[S]/\Theta/I_{\mathrm{NS}}; t) = \sum_i h''_i t^{2i}.$$

We now compute the combinatorial characteristics of the simplicial poset $K_{\mathcal{PT}}^{n-1}$ dual to $\mathcal{PT}^{n-1}$. Combinatorially, the simplicial cell complex $K_{\mathcal{PT}}^{n-1}$ can be defined as a poset, whose elements are the faces of the wonderful cell decomposition $\mathcal{PT}^{n-1}$ and the order is given by the reversed inclusion. It can be seen that $K_{\mathcal{PT}}^{n-1}$ is a simplicial poset. Recall that $\left\{{n \atop k}\right\}$ denotes the Stirling number of the second kind, that is, the number of unordered partitions of the set $[n]$ into k nonempty subsets.

Proposition 7.6 *For the simplicial poset $K_{\mathcal{PT}}^{n-1}$, there holds*

$$f_{k-1} = n(k-1)!\left\{{n \atop k}\right\} \quad \text{for } k = 1, 2, \ldots, n, \qquad f_{-1} = 1,$$

(7-4)　$\displaystyle h_l = (-1)^l\binom{n}{n-l} + \sum_{k=1}^{l}(-1)^{l-k}\binom{n-k}{n-l}n(k-1)!\left\{{n \atop k}\right\}$　for $l = 0, 1, \ldots, n$,

(7-5)　$\displaystyle h'_l = (-1)^l\binom{n}{n-l} + \sum_{k=1}^{l}(-1)^{l-k}\binom{n-k}{n-l}n(k-1)!\left\{{n \atop k}\right\}$

$$+ \binom{n}{l}\sum_{k=2}^{l-1}(-1)^{l-k-1}\binom{n-1}{k-1} \quad \text{for } l = 0, 1, \ldots, n,$$

(7-6)　$\displaystyle h''_l = (-1)^l\binom{n}{n-l} + \sum_{k=1}^{l}(-1)^{l-k}\binom{n-k}{n-l}n(k-1)!\left\{{n \atop k}\right\}$

$$+ \binom{n}{l}\sum_{k=2}^{l}(-1)^{l-k-1}\binom{n-1}{k-1} \quad \text{for } l = 0, 1, \ldots, n-1,$$

$$h''_n = 1.$$

Proof From the combinatorial description of a permutohedron it follows that the number $f_{n-k}(\mathrm{Pe}^{n-1})$ is equal to $k!\left\{{n \atop k}\right\}$. The wonderful subdivision $\mathcal{PT}^{n-1}$ consists of n permutohedra and each $(n-k)$–dimensional face of $\mathcal{PT}^{n-1}$ lies in exactly k permutohedral cells, since the subdivision is simple. Therefore,

$$f_{k-1}(K_{\mathcal{PT}}^{n-1}) = f_{n-k}(\mathcal{PT}^{n-1}) = \frac{n}{k} f_{n-k}(\mathrm{Pe}^{n-1}) = n(k-1)!\left\{{n \atop k}\right\}$$

for $k \geqslant 1$. The identity $f_{-1} = 1$ holds automatically.

Since $K_{\mathcal{PT}}^{n-1}$ is a simplicial cell subdivision of the torus $\mathcal{T}^{n-1}$, we have $\tilde{\beta}_j(K_{\mathcal{PT}}^{n-1}) = \binom{n-1}{j}$ for $j \geqslant 1$. Expressions (7-4), (7-5) and (7-6) follow from the general definitions of $h-$, $h'-$ and h''–numbers. $\square$

8 Equivariant cohomology

Let X be a $2n$–manifold with a locally standard action of T^n. The orbit space $P = X/T^n$ is a manifold with faces. It means that every codimension k face of P lies in exactly k different facets of P. Let S_P denote the simplicial poset dual to the poset of faces of P. In [7] we proved:

Proposition 8.1 *Assume that all proper faces of P are acyclic and the projection map $X \to P$ admits a section. Then $H_{T^n}^*(X;\mathbb{Z}) \cong \mathbb{Z}[S_P] \oplus H^*(P;\mathbb{Z})$ as rings, and as modules over $\mathbb{Z}[n] \cong H^*(BT^n)$. The components of degree 0 are identified in the direct sum $\mathbb{Z}[S_P] \oplus H^*(P;\mathbb{Z})$. The ring $H^*(P;\mathbb{Z})$ is considered a trivial $\mathbb{Z}[n]$–module.*

Now we apply this statement to the space $X_{n,\lambda}^{\leqslant\varepsilon}$ which is T^{n-1}–equivariantly homeomorphic to Y (see Construction 6.2).

Theorem 8.2 *The Hilbert–Poincaré series of the T^{n-1}–equivariant cohomology ring of $X_{n,\lambda}^{\leqslant\varepsilon}$ is given by*

$$\mathrm{Hilb}(H_{T^{n-1}}^*(X_{n,\lambda}^{\leqslant\varepsilon});t) = \frac{1}{(1-t^2)^{n-1}} \sum_{i=0}^{n} h_i t^{2i} + (1+t)^n - 1 - t.$$

Here the h_i, the h–numbers of the simplicial poset $K_{\mathcal{PT}}^{n-1}$, are given by (7-4).

Proof Recall that Y is the collar model, that is, the locally standard T^n–space over $\mathcal{T}^{n-1} \times [0,1]$. Proposition 8.1 implies the isomorphism, for the T^n–equivariant

cohomology,

$$H^*_{T^n}(Y) \cong \mathbb{Z}[K^{n-1}_{\mathcal{PT}}] \oplus H^*(\mathcal{T}^{n-1}).$$

There is an induced action of the $(n-1)$–dimensional subtorus

(8-1)
$$T^{n-1} = \{t_1 \cdots t_n = 1\}$$

on Y, and Theorem 6.3 states that Y and $X^{\leq \varepsilon}_{n,\lambda}$ are T^{n-1}–equivariantly homeomorphic. To compute the T^{n-1}–equivariant cohomology of Y, we first note that there is a Serre fibration

$$Y_{T^{n-1}} \xrightarrow{S^1} Y_{T^n}, \quad S^1 = T^n/T^{n-1},$$

where $Y_{T^{n-1}}$ and Y_{T^n} are the Borel constructions of T^{n-1}– and T^n–actions on Y, respectively. Consider the corresponding Serre spectral sequence

$$E^{p,q}_2 = H^p_{T^n}(Y) \otimes H^q(S^1) \Rightarrow H^{p+q}_{T^{n-1}}(Y).$$

The sequence has only two nonzero rows, hence it collapses at the E_3–term. Let ω denote a generator of $H^1(S^1)$. The second differential $d_2 \colon H^1(S^1) \to H^2_{T^n}(Y)$ of the spectral sequence coincides with the composition

$$H^1(T^n/T^{n-1}) \cong H^2(B(T^n/T^{n-1})) \to H^2(BT^n) \to H^2_{T^n}(Y),$$

where the middle map is induced by the projection $T^n \to T^n/T^{n-1}$ and the right map is the defining map for the $H^*(BT^n)$–module structure on $H^*_{T^n}(Y)$. It follows that

$$d_2(\omega) = \eta \in \mathbb{Z}[K^{n-1}_{\mathcal{PT}}]_2 \subset H^2_{T^n}(Y), \quad \text{where } \eta = \sum_{i=1}^{n} v_i,$$

according to the definition (8-1) of the subtorus T^{n-1}.

Lemma 8.3 η *is not a zero-divisor in the face ring* $\mathbb{Z}[K^{n-1}_{\mathcal{PT}}]$, *or, equivalently,* η *is a regular element.*

Proof We use the standard argument in the theory of face rings. For any nonempty simplex $I \cong \Delta^k$ in $K^{n-1}_{\mathcal{PT}}$, consider the epimorphism

$$\varphi_I \colon \mathbb{k}[K^{n-1}_{\mathcal{PT}}] \to \mathbb{k}[I]$$

defined by sending v_J to 0 for all $J \not\leqslant I$. Notice that $\mathbb{k}[I]$ is just the polynomial algebra in $\dim I + 1$ generators. The map φ_I is a homomorphism of $\mathbb{k}[n]$–algebras, with the $\mathbb{k}[n]$–structure on $\mathbb{k}[I]$ defined by an epimorphism ψ_I sending the excess variables to zeroes.

Assume that there exists $\beta \in \Bbbk[K_{\mathcal{PT}}^{n-1}]$ such that $\eta \cdot \beta = 0$. Then $\psi_I(\eta) \cdot \varphi_I(\beta) = 0$ in $\Bbbk[I]$. Since there are no zero-divisors in $\Bbbk[I]$ and $\psi_I(\eta) = \sum_{i \in I} v_i \neq 0$, we have $\varphi_I(\beta) = 0$ for any simplex I of $K_{\mathcal{PT}}^{n-1}$. The homomorphism

$$\bigoplus_I \varphi_I \colon \Bbbk[K_{\mathcal{PT}}^{n-1}] \to \bigoplus_I \Bbbk[I]$$

is known to be injective [10, Theorem 3.5.6]. Therefore $\beta = 0$. $\qquad\square$

According to the lemma, for any nonzero element $\beta \in \mathbb{Z}[K_{\mathcal{PT}}^{n-1}] \subset H_{T^n}^*(Y)$, there holds

$$d_2(\omega\beta) = (d_2\omega)\beta \pm \omega d_2(\beta) = \eta\beta \neq 0.$$

In other words, d_2 is injective on the submodule $\mathbb{Z}[K_{\mathcal{PT}}^{n-1}] \otimes H^1(S^1) \subset E_2^{*,1}$.

On the other hand, for any element $\alpha \in H^i(\mathcal{T}^{n-1}) \subset H_{T^n}^i(Y)$ with $i > 0$, we have

$$d_2(\omega\alpha) = (d_2\omega)\alpha \pm \omega d_2(\alpha) = \eta \cdot \alpha = 0,$$

since the products of elements from the components $H^+(\mathcal{T}^{n-1})$ and $\mathbb{Z}[K_{\mathcal{PT}}^{n-1}]_+$ of the ring $H_{T^n}^*(Y)$ vanish. Therefore the differential d_2 vanishes on $H^+(\mathcal{T}^{n-1}) \otimes H^1(S^1)$. Finally, we have

$$E_3^{p,q} \cong \begin{cases} (\mathbb{Z}[K_{\mathcal{PT}}^{n-1}]/(\eta)_p) \oplus H^p(\mathcal{T}^{n-1}) & \text{for } q = 0, \\ H^p(\mathcal{T}^{n-1}) & \text{for } q = 1, p > 0. \end{cases}$$

Two things should be noted here. First, the components of degree zero are identified in $(\mathbb{Z}[K_{\mathcal{PT}}^{n-1}]/(\eta)_p) \oplus H^p(\mathcal{T}^{n-1})$. Second, the term $E_3^{0,1}$ vanishes, since the generator ω of $E_2^{0,1}$ maps to $\eta \neq 0$ by d_2. The Hilbert–Poincaré series of $\mathbb{Z}[K_{\mathcal{PT}}^{n-1}]$ is given by $\left(\sum_{i=0}^n h_i t^{2i}\right)/(1-t^2)^n$ according to Proposition 7.4, so we have

$$\mathrm{Hilb}(\mathbb{Z}[K_{\mathcal{PT}}^{n-1}]/(\eta); t) = \frac{1}{(1-t^2)^{n-1}} \sum_{i=0}^n h_i t^{2i}$$

by Lemma 8.3. The statement of the theorem now follows from the degeneration of the spectral sequence at the E_3–term. $\qquad\square$

Corollary 8.4 *For the whole isospectral space $X_{n,\lambda}$, there holds*

$$\mathrm{Hilb}(H_{T^{n-1}}^*(X_{n,\lambda}); t) = \frac{1}{(1-t^2)^{n-1}} \sum_{i=0}^n h_i t^{2i} + R(t),$$

where $R(t)$ is a polynomial and the h_i are given by (7-4).

Proof The space $X_{n,\lambda}$ is patched from $X_{n,\lambda}^{\leq\varepsilon}$ and $X_{n,\lambda}^{\geq\varepsilon} = \{p^{-1}(|z| \geq \varepsilon)\}$. Both subsets are preserved by the torus action, hence the equivariant cohomology groups can be computed via the Mayer–Vietoris exact sequence. However, the torus action on $X_{n,\lambda}^{\geq\varepsilon}$ and $X_{n,\lambda}^{\geq\varepsilon} \cap X_{n,\lambda}^{\leq\varepsilon}$ is free, so the equivariant cohomology groups of these subsets coincide with the ordinary cohomology groups of their orbit spaces. Hence they are concentrated in a finite range of degrees. The statement now follows from Theorem 8.2. □

Example 8.5 We check the calculations for the case $n = 3$. The isospectral space $X_{3,\lambda}$ coincides with the manifold Fl_3 of complete complex flags. Its equivariant cohomology is known to satisfy

$$\mathrm{Hilb}(H_{T^2}^*(\mathrm{Fl}_3); t) = \frac{\mathrm{Hilb}(H^*(\mathrm{Fl}_3); t)}{(1-t^2)^2} = \frac{1 + 2t^2 + 2t^4 + t^6}{(1-t^2)^2}.$$

Using formulas (7-4), compute the h–numbers of $K_{\mathcal{PT}}^2$: $(h_0, h_1, h_2, h_3) = (1, 0, 6, -1)$. There holds

$$\frac{1 + 2t^2 + 2t^4 + t^6}{(1-t^2)^2} = \frac{1 + 6t^4 - t^6}{(1-t^2)^2} + 2t^2,$$

which confirms Corollary 8.4.

9   Betti numbers

In this section we describe the additive structure of the cohomology ring of $X_{n,\lambda}$. As a first step, we compute the homological structure of the subset $X_{n,\lambda}^{\leq\varepsilon}$, containing the essential information on the torus action. It is assumed in this section that all coefficients are taken in a fixed field. The next proposition follows from the general technique developed in [2].

Proposition 9.1 The homology modules of $Y \cong X_{n,\lambda}^{\leq\varepsilon}$ admit the double grading $H_j(Y) \cong \bigoplus_{p+q=j} H_{p,q}(Y)$. There holds:

(1) $H_{p,q}(Y) \cong H_p(\mathcal{T}^{n-1}) \otimes H_q(T^n)$ for $q < p < n$.

(2) $H_{p,q}(Y) = 0$ for $q > p$.

(3) The dimension of $H_{p,p}(Y)$ equals

$$h_p + \binom{n}{p} \sum_{k=2}^{p+1} (-1)^{p+k-1} \binom{n-1}{k-1}$$

for $p = 0, 1, \ldots, n-1$. In particular, for $p \geqslant 2$ there holds $\dim H_{p,p}(Y) = h''_p + \binom{n}{p}\binom{n-1}{p}$.

The inclusion map $i \colon \mathcal{T}^{n-1} \times T^n \cong \partial Y \to Y$ induces a homomorphism in homology, which respects the double grading,

$$i_* \colon H_p(\mathcal{T}^{n-1}) \otimes H_q(T^n) \to H_{p,q}(Y).$$

This homomorphism is an isomorphism for $q < p$, injective for $q = p$, and zero for $q > p$.

Note that the subspace $X_{n,\lambda}^{\leqslant \varepsilon}$ does not depend on the parameters n_+ and n_- discussed in the previous sections. Now we are in position to compute the Betti numbers of $X_{n,\lambda}$. Homology of $X_{n,\lambda}$ will certainly depend on the parameters n_+ and n_-, which encode "the degree of degeneration" of this space.

Theorem 9.2 *The Hilbert–Poincaré series for homology of $X_{n,\lambda}$ is given by the formula*

$$(9\text{-}1) \qquad \sum_{i=0}^{2n} \beta_i(X_{n,\lambda}) t^i = H^{\geqslant \varepsilon}(t) + H^{\leqslant \varepsilon}(t) - H^{=\varepsilon}(t) + (1+t) \cdot H^{\mathrm{Ker}}(t),$$

where

$(9\text{-}2) \quad H^{=\varepsilon}(t) = (1+t)^{2n-1},$

$(9\text{-}3) \quad H^{\geqslant \varepsilon}(t) = (1+t)^{2n-n_+-n_--2}(1-t+t(1+t)^{n_+}+t(1+t)^{n_-}),$

$(9\text{-}4) \quad H^{\leqslant \varepsilon}(t) = \displaystyle\sum_{p=0}^{n-1} \left(h_p + \binom{n}{p} \sum_{k=2}^{p+1} (-1)^{p+k-1} \binom{n-1}{k-1} \right) t^{2p}$

$$+ \sum_{q<p<n} \binom{n-1}{p}\binom{n}{q} t^{p+q},$$

$(9\text{-}5) \quad H^{\mathrm{Ker}}(t) = \displaystyle\sum_{(p,e,q,s,r)\in\Upsilon} \binom{n-1-n_+-n_-}{p}\binom{1}{e}\binom{n_+}{q}\binom{n_-}{s}\binom{n-1}{r} t^{p+e+q+s+r}$

and the indexing subset Υ in the last expression is defined by the conditions

$$0 \leqslant p \leqslant n-1-n_+-n_-, \quad 0 \leqslant e \leqslant 1,$$

$$0 \leqslant q \leqslant n_+, \quad 0 \leqslant s \leqslant n_-, \quad 0 \leqslant r \leqslant n-1,$$

$(9\text{-}6)$
$$r+e > p+q+s,$$

$$(e=0 \quad \text{and} \quad q+s>0) \qquad \text{or} \qquad (e=1 \quad \text{and} \quad q>0 \text{ and } s>0).$$

The h–numbers are given by (7-4).

Proof Although the result looks awkward, the idea behind this calculation is straightforward: we analyze the Mayer–Vietoris sequence for the union $X_{n,\lambda} = X_{n,\lambda}^{\leqslant \varepsilon} \cup X_{n,\lambda}^{\geqslant \varepsilon}$. Let $X_{n,\lambda}^{=\varepsilon}(t)$ denote the intersection $X_{n,\lambda}^{\leqslant \varepsilon} \cap X_{n,\lambda}^{\geqslant \varepsilon}$. Then there is a long exact sequence

$$(9\text{-}7) \quad \cdots \to H_i(X_{n,\lambda}^{=\varepsilon}) \xrightarrow{\iota_i} H_i(X_{n,\lambda}^{\leqslant \varepsilon}) \oplus H_i(X_{n,\lambda}^{\geqslant \varepsilon}) \to H_i(X_{n,\lambda}) \to H_{i-1}(X_{n,\lambda}^{=\varepsilon})$$
$$\xrightarrow{\iota_{i-1}} H_{i-1}(X_{n,\lambda}^{\leqslant \varepsilon}) \oplus H_{i-1}(X_{n,\lambda}^{\geqslant \varepsilon}) \to \cdots .$$

Note that given an exact sequence

$$A_i \xrightarrow{\alpha_i} B_i \to C_i \to A_{i-1} \xrightarrow{\alpha_{i-1}} B_{i-1},$$

the dimension of the vector space in the middle is given by

$$(9\text{-}8) \qquad \dim C_i = \dim B_i - \dim A_i + \dim \operatorname{Ker} \alpha_i + \dim \operatorname{Ker} \alpha_{i-1}.$$

After introducing the natural notation

$$H^{=\varepsilon}(t) = \sum_i \dim H_i(X_{n,\lambda}^{=\varepsilon}) t^i,$$

$$H^{\geqslant \varepsilon}(t) = \sum_i \dim H_i(X_{n,\lambda}^{\geqslant \varepsilon}) t^i,$$

$$H^{\leqslant \varepsilon}(t) = \sum_i \dim H_i(X_{n,\lambda}^{\leqslant \varepsilon}) t^i,$$

$$H^{\operatorname{Ker}}(t) = \sum_i \dim(\operatorname{Ker} \iota_i \colon H_i(X_{n,\lambda}^{=\varepsilon}) \to H_i(X_{n,\lambda}^{\leqslant \varepsilon}) \oplus H_i(X_{n,\lambda}^{\geqslant \varepsilon})) t^i,$$

formula (9-8) implies (9-1). We need to check formulas (9-2)–(9-5):

(1) According to (6-1), $X_{n,\lambda}^{=\varepsilon} \cong T^{2n-1}$, which implies (9-2).

(2) The space $X_{n,\lambda}^{\geqslant \varepsilon}$ supports a free action of T^{n-1}, which admits a section. Therefore,

$$(9\text{-}9) \qquad\qquad X_{n,\lambda}^{\geqslant \varepsilon} = Q_{n,\lambda}^{\geqslant \varepsilon} \times T^{n-1},$$

where $Q_{n,\lambda}^{\geqslant \varepsilon} = X_{n,\lambda}^{\geqslant \varepsilon}/T^{n-1}$. The structure of the orbit space was described in detail in Section 3: $Q_{n,\lambda}^{\geqslant \varepsilon} = X_{n,\lambda}^{\geqslant \varepsilon}/T^{n-1}$ is fibered by Liouville–Arnold tori over $\mathbb{B} \setminus \{|z| < \varepsilon\}$ — the biangle with a hole. The homotopy type of the latter space is

$$(9\text{-}10) \qquad\qquad Q_{n,\lambda}^{\geqslant \varepsilon} \simeq \mathcal{T}^{n-n_+ - n_- - 1} \times (S^1 \vee \Sigma \mathcal{T}^{n_+} \vee \Sigma \mathcal{T}^{n_-}).$$

Indeed, the space $Q_{n,\lambda}^{\geqslant \varepsilon}$ can be collapsed to the subspace $\tilde{p}^{-1}(\partial \mathbb{B})$. The latter space is foliated over the circle $\partial \mathbb{B} = F_+ \cup F_-$ by tori: the preimages of points in the interior of F_+ are $\mathcal{T}^{n-n_+ - n_- - 1} \times \mathcal{T}^{n_-}$, the preimages of points in the interior of F_- are

$\mathcal{T}^{n-n+-n--1} \times \mathcal{T}^{n+}$, and the preimages of points in $F_+ \cap F_-$ are the tori $\mathcal{T}^{n-n+-n--1}$. It can be seen that $\mathcal{T}^{n-n+-n--1}$ splits as a direct factor. The remaining space is the suspension over the disjoint union $\mathcal{T}^{n+} \sqcup \mathcal{T}^{n-}$. By a standard property of the suspension, we have $\Sigma(\mathcal{T}^{n+} \sqcup \mathcal{T}^{n-}) \simeq S^1 \vee \Sigma \mathcal{T}^{n+} \vee \Sigma \mathcal{T}^{n-}$, which proves (9-10).

Formula (9-3) follows from (9-9) and (9-10).

(3) Homology of $X_{n,\lambda}^{\leq \varepsilon}$ is described by Proposition 9.1. Formula (9-4) is its simple consequence.

(4) To derive (9-5), we need to count the dimension of the vector space of all homology cycles of $X_{n,\lambda}^{=\varepsilon}$, annihilated by both maps

$$\iota_*^{\leq} \colon H_*(X_{n,\lambda}^{=\varepsilon}) \to H_*(X_{n,\lambda}^{\leq \varepsilon}), \quad \iota_*^{\geq} \colon H_*(X_{n,\lambda}^{=\varepsilon}) \to H_*(X_{n,\lambda}^{\geq \varepsilon}).$$

The torus $X_{n,\lambda}^{=\varepsilon}$ decomposes into the product

$$X_{n,\lambda}^{=\varepsilon} \cong S_\varepsilon^1 \times \mathcal{T}^{n-1} \times T^{n-1} \cong \mathcal{T}^{n-1-n+-n-} \times S_\varepsilon^1 \times \mathcal{T}^{n+} \times \mathcal{T}^{n-} \times T^{n-1},$$

where

(a) the component T^{n-1} corresponds to the acting torus;

(b) $\mathcal{T}^{n+}$ and $\mathcal{T}^{n-}$ are the components of Liouville–Arnold tori, collapsing over the sides of the biangle $\mathbb{B}$;

(c) $\mathcal{T}^{n-1-n+-n-}$ is the surviving component of the Liouville–Arnold torus;

(d) S^1 is the circle $\{|z| = \varepsilon\}$, lifted to the total space.

These five components of the torus explain the appearance of the five-element indexing subset in the statement. Let $\theta = \omega_p \otimes \alpha_e \otimes \omega_q^+ \otimes \omega_s^- \otimes \nu_r$ be a homology cycle from

$$H_*(X_{n,\lambda}^{=\varepsilon}) \cong H_*(\mathcal{T}^{n-1-n+-n-}) \otimes H_*(S^1) \otimes H_*(\mathcal{T}^{n+}) \otimes H_*(\mathcal{T}^{n-}) \otimes H_*(T^{n-1}),$$

where the degrees of the factors are indicated in their subscripts.

Lines 1–2 of condition (9-6) simply indicate the range of the indices: from zero to the dimension of the corresponding torus. We have $\iota_*^{\leq}(\theta) = 0$ if and only if $r + e > p + q + s$ according to the last part of Proposition 9.1. This explains line 3 of condition (9-6). Finally, we investigate the map $\iota_*^{\geq} \colon H_*(X_{n,\lambda}^{=\varepsilon}) \to H_*(X_{n,\lambda}^{\geq \varepsilon})$ and describe its kernel. Note that the tori T^{n-1} and $\mathcal{T}^{n-1-n+-n-}$ split as direct factors in the map $\iota^{\geq} \colon X_{n,\lambda}^{=\varepsilon} \to X_{n,\lambda}^{\geq \varepsilon}$. Hence we have

$$\operatorname{Ker} \iota_*^{\geq} \cong H_*(T^{n-1} \times \mathcal{T}^{n-1-n+-n-}) \otimes \operatorname{Ker} \tilde{\iota}_*^{\geq},$$

$$S^1 \times \mathcal{T}^{n+} \times \mathcal{T}^{n-} \qquad \Sigma(\mathcal{T}^{n+} \times \mathcal{T}^{n-} \sqcup \mathcal{T}^{n+} \times \mathcal{T}^{n-}) \qquad S^1 \vee \Sigma\mathcal{T}^{n+} \vee \Sigma\mathcal{T}^{n-}$$
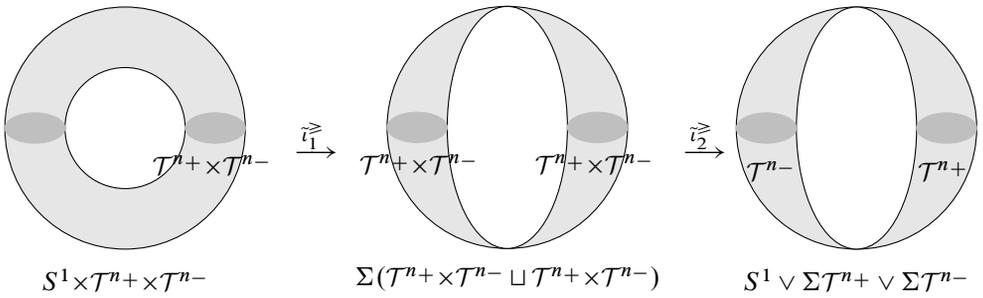
Figure 7: The map $\tilde{\iota}^{\geqslant}$ pinches the product $S^1 \times (\mathcal{T}^{n+} \times \mathcal{T}^{n-})$ at two points of S^1, then collapses $\mathcal{T}^{n+}$ on the left side and $\mathcal{T}^{n-}$ on the right side.

where $\tilde{\iota}^{\geqslant}$ is the inclusion map from $Q_{n,\lambda}^{=\varepsilon}/\mathcal{T}^{n-1-n+-n-} \cong S^1 \times \mathcal{T}^{n+} \times \mathcal{T}^{n-}$ to $Q_{n,\lambda}^{\geqslant\varepsilon}/\mathcal{T}^{n-1-n+-n-} \simeq \Sigma(\mathcal{T}^{n+} \sqcup \mathcal{T}^{n-}) \simeq S^1 \vee \Sigma\mathcal{T}^{n+} \vee \Sigma\mathcal{T}^{n-}$. We have

$$(9\text{-}11) \qquad \tilde{\iota}_*^{\geqslant}(\alpha_e \otimes \omega_q^+ \otimes \omega_s^-) = \begin{cases} 1 & \text{if } e = 0,\ q + s = 0, \\ 0 + 0 + 0 & \text{if } e = 0,\ q + s > 0, \\ \alpha_e + 0 + 0 & \text{if } e = 1,\ q + s = 0, \\ 0 + \Sigma\omega_q^+ + 0 & \text{if } e = 1,\ q > 0,\ s = 0, \\ 0 + 0 + \Sigma_s\omega^- & \text{if } e = 1,\ q = 0,\ s > 0, \\ 0 + 0 + 0 & \text{if } e = 1,\ q > 0,\ s > 0. \end{cases}$$

A convenient way to see these formulas is to replace $\tilde{\iota}^{\geqslant}$ by a homotopy equivalent map illustrated on Figure 7. For example, a homology element $\alpha_e \otimes \omega_q^+ \otimes \omega_s^-$ with $e = 1$, $q > 0$ and $s = 0$ is represented by a subtorus $S^1 \times \mathcal{T}^q \times \mathrm{pt}$. The map $\tilde{\iota}_1^{\geqslant}$ sends this cycle to $\Sigma\mathcal{T}^q$ in the right half of $\Sigma(\mathcal{T}^{n+} \times \mathcal{T}^{n-} \sqcup \mathcal{T}^{n+} \times \mathcal{T}^{n-})$. The left half vanishes since $\mathcal{T}^{n+}$ is collapsed, together with $\mathcal{T}^q$ over the left part. The same reasoning explains all other formulas (9-11).

This calculation shows that $\theta \in \operatorname{Ker} \iota_*^{\geqslant}$ if and only if the 4th line of condition (9-6) holds. Therefore, both maps $\iota_*^{\leqslant}$ and $\iota_*^{\geqslant}$ annihilate θ if and only if the 5–tuple (p, e, q, s, r) satisfies the conditions (9-6). This proves (9-5). $\square$

Example 9.3 The Betti numbers computed for small values of n are shown in Tables 1 and 2.

Corollary 9.4 *The space $X_{n,\lambda}$ has nonzero Betti numbers in odd degrees for $n \geqslant 4$.*

Proof Theorem 9.2 implies that $\beta_1(X_{n,\lambda}) = n - 1 - n_+ - n_-$. This number is nonzero unless n_+ and n_- are the maximal possible, representing the most degenerate case. For the most degenerate case, and $n \geqslant 4$, $\beta_{2n-1}(X_{n,\lambda})$ is nonzero. To prove this, it is

n	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
3	1	0	2	0	2	0	1						
4	1	1	6	2	16	2	6	1	1				
5	1	2	13	9	65	16	65	9	13	2	1		
6	1	3	23	25	203	67	456	67	203	25	23	3	1

Table 1: Manifold case, $n_+ = 1, n_- = 1$.

sufficient to estimate the coefficient at t^{2n-1} in (9-5) by 2 from below. Indeed, the only negative term in (9-1) comes from $H^{\mathrm{Ker}}(t)$, and its coefficient at t^{2n-1} is one. The estimation is straightforward. $\square$

Recall that with any action of a torus T on a space X one can associate a fibration $r\colon X \times_T ET \xrightarrow{X} BT$, where ET is a contractible space, carrying the free action of T and BT is the classifying space of a torus. It can be assumed that $ET = (S^\infty)^k$ and $BT = (\mathbb{C}P^\infty)^k$, where $k = \dim T$.

Definition 9.5 The space X is called *equivariantly formal* in the sense of Goresky, Kottwitz and Macpherson [19] if the Serre spectral sequence

(9-12) $\qquad E_2^{*,*} \cong H^*(BT) \otimes H^*(X) \Rightarrow H^*(X \times_T ET) = H_T^*(X),$

ie the spectral sequence of the fibration r degenerates at its second page.

Proposition 9.6 The space $X_{n,\lambda}$ is not equivariantly formal for $n \geqslant 4$.

Proof Assume that $X_{n,\lambda}$ is equivariantly formal. The degeneration of the Serre spectral sequence (9-12) at a second page then implies

$$\mathrm{Hilb}(H_T^*(X_{n,\lambda});t) = \mathrm{Hilb}(H^*(X_{n,\lambda});t) \cdot \mathrm{Hilb}(H^*(BT^{n-1});t)$$
$$= \mathrm{Hilb}(H^*(X_{n,\lambda});t) \cdot (1-t^2)^{-(n-1)}.$$

n	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}
4	1	0	3	1	16	3	9	2	1				
5	1	0	4	2	57	16	77	22	24	4	1		
6	1	0	5	4	167	55	471	115	276	61	39	5	1

Table 2: Most degenerate case: n_+ and n_- are the maximal possible for the given n; that is, $n_+ + n_- = n - 1$.

This identity and Corollary 9.4 imply that $H_T^*(X_{n,\lambda})$ has nontrivial components of arbitrarily high odd degree. This contradicts Corollary 8.4.   □

Remark 9.7 The fundamental group of $X_{n,\lambda}$ can be explicitly described as well. As in the computations above, we consider the decomposition $X_{n,\lambda} = X_{n,\lambda}^{\leq \varepsilon} \cup X_{n,\lambda}^{\geq \varepsilon}$ and apply the van Kampen theorem. For the intersection there holds

$$\pi_1(X_{n,\lambda}^{=\varepsilon}) = \pi_1(T^{n+} \times T^{n-} \times T^{n-1-n_+ - n_-} \times S_\varepsilon^1 \times T^{n-1})$$
$$= \mathbb{Z}^{n+} \oplus \mathbb{Z}^{n-} \oplus \mathbb{Z}^{n-1-n_+ - n_-} \oplus \mathbb{Z}^n.$$

The summands $\mathbb{Z}^{n+}$ and $\mathbb{Z}^{n-}$ vanish in $\pi_1(X_{n,\lambda}^{\geq \varepsilon})$ since the corresponding components of Liouville–Arnold tori are collapsed over the facets of biangle $\mathbb{B}$. The summand $\mathbb{Z}^n = \pi_1(S_\varepsilon^1 \times T^{n-1})$ vanishes in $\pi_1(X_{n,\lambda}^{\leq \varepsilon})$ according to [34]. The result of this paper asserts that for a locally standard action of T with $\dim T = n$ on M with $\dim M = 2n$, having a fixed point, there holds $\pi_1(M) \cong \pi_1(M/T)$. In other words, any loop on the acting torus can be contracted via a fixed point. This result is applied to $X_{n,\lambda}^{\leq \varepsilon} \cong Y$, carrying the extended action of T^n.

No other loops appear in $X_{n,\lambda}^{\geq \varepsilon}$ and $X_{n,\lambda}^{\leq \varepsilon}$, therefore

$$\pi_1(X_{n,\lambda}) \cong \mathbb{Z}^{n-1-n_+ - n_-}.$$

Hence $X_{n,\lambda}$ is simply connected if and only if the spectrum λ satisfies $n_- + n_+ = n - 1$. This corresponds to the most degenerate situation, considered in Remark 3.11.

Acknowledgements

References

[1] **A Ayzenberg**, *Homology cycles in manifolds with locally standard torus actions*, Homology Homotopy Appl. 18 (2016) 1–23 MR

[2] **A Ayzenberg**, *Locally standard torus actions and h′–numbers of simplicial posets*, J. Math. Soc. Japan 68 (2016) 1725–1745 MR

[3] **A Ayzenberg**, *Topological model for h″ vectors of simplicial manifolds*, Bol. Soc. Mat. Mex. 23 (2017) 413–421 MR

[4] **A A Ayzenberg**, *Torus actions of complexity 1 and their local properties*, Tr. Mat. Inst. Steklova 302 (2018) 23–40 MR In Russian; translated in Proc. Steklov Inst. Math. 302 (2018), 16–32

[5] **A Ayzenberg**, **V Buchstaber**, *Manifolds of isospectral arrow matrices*, preprint (2018) arXiv

[6] **A Ayzenberg**, **V Buchstaber**, *Manifolds of isospectral matrices and Hessenberg varieties*, Int. Math. Res. Not. (online publication April 2020)

[7] **A Ayzenberg**, **M Masuda**, **S Park**, **H Zeng**, *Cohomology of toric origami manifolds with acyclic proper faces*, J. Symplectic Geom. 15 (2017) 645–685 MR

[8] **A M Bloch**, **H Flaschka**, **T Ratiu**, *A convexity theorem for isospectral manifolds of Jacobi matrices in a compact Lie algebra*, Duke Math. J. 61 (1990) 41–65 MR

[9] **V M Buchstaber**, **T E Panov**, *Combinatorics of simplicial cell complexes, and torus actions*, Tr. Mat. Inst. Steklova 247 (2004) 41–58 MR In Russian; translated in Proc. Steklov Inst. Math. 247 (2004), 33–49

[10] **V M Buchstaber**, **T E Panov**, *Toric topology*, Math. Surv. Monogr. 204, Amer. Math. Soc., Providence, RI (2015) MR

[11] **V M Buchstaber**, **S Terzić**, $(2n, k)$–*manifolds and applications*, Oberwolfach Rep. 11 (2014) 1469–1472 Part of the conference report "Okounkov bodies and applications"

[12] **V M Buchstaber**, **S Terzić**, *Topology and geometry of the canonical action of T^4 on the complex Grassmannian $G_{4,2}$ and the complex projective space $\mathbb{C}P^5$*, Mosc. Math. J. 16 (2016) 237–273 MR

[13] **V M Buchstaber**, **S Terzić**, *Toric topology of the complex Grassmann manifolds*, Mosc. Math. J. 19 (2019) 397–463 MR

[14] **Y B Chernyakov**, **G I Sharygin**, **A S Sorin**, *Bruhat order in full symmetric Toda system*, Comm. Math. Phys. 330 (2014) 367–399 MR

[15] **J H Conway**, **N J A Sloane**, *Sphere packings, lattices and groups*, 3rd edition, Grundl. Math. Wissen. 290, Springer (1999) MR

[16] **M W Davis**, **T Januszkiewicz**, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991) 417–451 MR

[17] **P Deift**, **T Nanda**, **C Tomei**, *Ordinary differential equations and the symmetric eigenvalue problem*, SIAM J. Numer. Anal. 20 (1983) 1–22 MR

[18] **M Ferri**, **C Gagliardi**, **L Grasselli**, *A graph-theoretical representation of PL-manifolds: a survey on crystallizations*, Aequationes Math. 31 (1986) 121–141 MR

[19] **M Goresky**, **R Kottwitz**, **R MacPherson**, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131 (1998) 25–83 MR

[20] **I Krichever**, **K L Vaninsky**, *The periodic and open Toda lattice*, from "Mirror symmetry, IV" (E D'Hoker, D Phong, S-T Yau, editors), AMS/IP Stud. Adv. Math. 33, Amer. Math. Soc., Providence, RI (2002) 139–158 MR

[21] **S Kuroki**, *Introduction to GKM theory*, Trends Math. 11 (2009) 113–129

[22] **Z Lü**, **T Panov**, *Moment-angle complexes from simplicial posets*, Cent. Eur. J. Math. 9 (2011) 715–730 MR

[23] **P van Moerbeke**, *The spectrum of Jacobi matrices*, Invent. Math. 37 (1976) 45–81 MR

[24] **I Novik**, **E Swartz**, *Gorenstein rings through face rings of manifolds*, Compos. Math. 145 (2009) 993–1000 MR

[25] **I Novik**, **E Swartz**, *Socles of Buchsbaum modules, complexes and posets*, Adv. Math. 222 (2009) 2059–2084 MR

[26] **G Y Panina**, *Cyclopermutohedron*, Tr. Mat. Inst. Steklova 288 (2015) 149–162 MR In Russian; translated in Proc. Steklov Inst. Math. 288 (2015), 132–144

[27] **G A Reisner**, *Cohen–Macaulay quotients of polynomial rings*, Adv. Math. 21 (1976) 30–49 MR

[28] **P Schenzel**, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Z. 178 (1981) 125–142 MR

[29] **R P Stanley**, *Combinatorics and commutative algebra*, Progr. Math. 41, Birkhäuser, Boston (1983) MR

[30] **R P Stanley**, *f–vectors and h–vectors of simplicial posets*, J. Pure Appl. Algebra 71 (1991) 319–331 MR

[31] **C Tomei**, *The topology of isospectral manifolds of tridiagonal matrices*, Duke Math. J. 51 (1984) 981–996 MR

[32] **C Tomei**, *The Toda lattice, old and new*, J. Geom. Mech. 5 (2013) 511–530 MR

[33] **T Yoshida**, *Local torus actions modeled on the standard representation*, Adv. Math. 227 (2011) 1914–1955 MR

[34] **H Zeng**, *The fundamental group of locally standard T–manifolds*, Algebr. Geom. Topol. 18 (2018) 3031–3035 MR

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