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A relative 2–nerve

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We introduce a 2–categorical variant of Lurie's relative nerve functor. We prove that it defines a right Quillen equivalence which, upon passage to ∞–categorical localizations, corresponds to Lurie's scaled unstraightening equivalence. In this ∞–bicategorical context, the relative 2–nerve provides a computationally tractable model for the Grothendieck construction which becomes equivalent, via an explicit comparison map, to Lurie's relative nerve when restricted to 1–categories.

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Introduction

Let $F\colon C^{\mathrm{op}} \to \mathrm{Cat}$ be a contravariant functor from a category C into the category Cat of small categories. The *classical Grothendieck construction* of F is the category $\chi(F)$ with objects given by pairs (c, x), where $c \in C$ and $x \in F(c)$, and a morphism between objects (c, x) and (c', x') consisting of a pair (f, η), where $f\colon c \to c'$ is a morphism in C and $\eta\colon x \to F(f)(x')$ is a morphism in the category $F(c)$. This data admits an apparent composition law and the resulting category comes equipped with a forgetful functor $\chi(F) \to C$ which is a Cartesian fibration. It is a classical result that the Grothendieck construction χ establishes an equivalence between appropriately

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defined categories of

(I) pseudofunctors $C^{\mathrm{op}} \to \mathrm{Cat}$,

(II) Cartesian fibrations over C.

The construction originally appeared in the foundations of Grothendieck's theory of descent [3, Exposé VI], where descent data is described in terms of sections of Cartesian fibrations.

The relative nerve Let Set_Δ denote the category of simplicial sets and let $\mathrm{Cat}_\infty \subset \mathrm{Set}_\Delta$ denote the full subcategory spanned by the ∞–categories. Generalizing the above, we may define the Grothendieck construction of a contravariant functor

$$F\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

from C into the category Cat_∞ to be the simplicial set $\chi(F)$ in which an n–simplex is given by

(1) an n–simplex $\sigma\colon [n] \to C$ of the nerve of C,

(2) for every nonempty subset $I \subset [n]$, a map of simplicial sets $\Delta^I \to F(\sigma(\min(I)))$

such that, for every $I \subset I' \subset [n]$, the diagram

$$\begin{array}{ccc} \Delta^I & \longrightarrow & F(\sigma(\min(I))) \\ \downarrow & & \downarrow \\ \Delta^{I'} & \longrightarrow & F(\sigma(\min(I'))) \end{array}$$

commutes. In the case when F takes values in $\mathrm{Cat} \subset \mathrm{Cat}_\infty$, fully faithfully embedded via the nerve, then $\chi(F)$ will be the nerve of the classical Grothendieck construction. It is proven in Lurie [5, Section 3.2], where $\chi(F)$ is called the *relative nerve of* F, that this construction extends to a Quillen equivalence

$$\phi\colon (\mathrm{Set}_\Delta^+)_{/\,\mathrm{N}(C)} \rightleftarrows \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}_\Delta^+) :\chi,$$

which, upon passage to ∞–categorical localizations, identifies the ∞–categories of

(I) functors between the ∞–category $\mathrm{N}(C^{\mathrm{op}})$ and the ∞–category $\mathcal{C}\mathrm{at}_\infty$ of small ∞–categories,

(II) Cartesian fibrations over $\mathrm{N}(C)$.

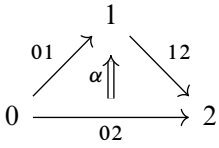
In fact, Lurie [5] proves a much more sophisticated variant of this equivalence, governed by certain constructions called straightening and unstraightening, where $\mathrm{N}(C)$ can be replaced by an arbitrary ∞–category $\mathcal{C}$.

The relative 2–nerve In the present work, we provide a generalization of the relative nerve to accommodate functors

$$F\colon \mathbb{C}^{(\mathrm{op},\mathrm{op})} \to \mathrm{Cat}_\infty,$$

where $\mathbb{C}$ is a Cat–enriched category and the functor F is Cat_∞–enriched (leaving the inclusion $\mathrm{Cat} \subset \mathrm{Cat}_\infty$ via the nerve implicit). To such a functor F, we associate a simplicial set $\between(F)$, equipped with a forgetful map $\pi\colon \between(F) \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ to the scaled nerve of $\mathbb{C}$, the latter being a 2–categorical version of the Street nerve as introduced in [7]. This *relative 2–nerve* construction $\between$ is an adaptation of χ to the given 2–categorical context; for example, a 2–simplex in $\between(F)$ consists of

(1) a 2–simplex $\sigma\colon \Delta^2 \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ of the scaled nerve of $\mathbb{C}$ whose data we label by

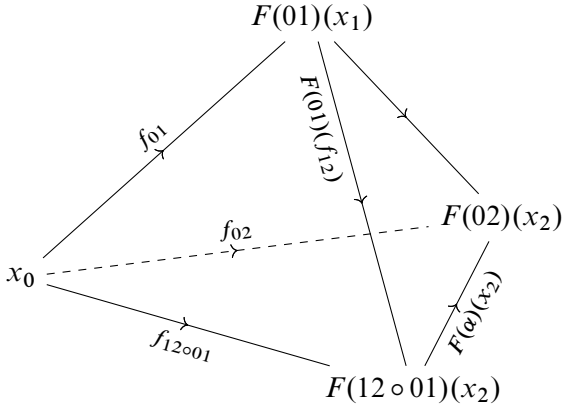


(2) vertices x_0, x_1 and x_2 in the ∞–categories $F(0)$, $F(1)$ and $F(2)$, respectively,

(3) edges

- $f_{01}\colon x_0 \to F(01)(x_1)$ in $F(0)$,
- $f_{02}\colon x_0 \to F(02)(x_2)$ in $F(0)$,
- $f_{12}\colon x_1 \to F(12)(x_2)$ in $F(1)$,
- $f_{12\circ 01}\colon x_0 \to F(12\circ 01)(x_2)$ in $F(0)$,

(4) a 3–simplex



in the ∞–category $F(0)$.

The description of the higher-dimensional n–simplices of $\mathbb{X}(F)$ requires some notational preparation and will be given in Section 2. Pictorially, just as in the case $n = 2$, they correspond to diagrams parametrized by blown-up versions of the standard n–simplices (see Figure 1).

Our main result concerning the relative 2–nerve $\mathbb{X}$ is the following:

Theorem 1 *The functor $\mathbb{X}$ extends to a Quillen equivalence*

$$\Phi\colon (\mathrm{Set}_\Delta^+)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \rightleftarrows \mathrm{Fun}_{\mathrm{Set}_\Delta^+}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}_\Delta^+)\ :\mathbb{X}$$

modelling an equivalence between ∞–categories of

(I) *functors between the ∞–bicategory $\mathrm{N}^{\mathrm{sc}}(\mathbb{C}^{(\mathrm{op},\mathrm{op})})$ and the ∞–bicategory $\mathfrak{C}\mathrm{at}_\infty$ of small ∞–categories,*

(II) *locally Cartesian fibrations over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ that are Cartesian over every scaled triangle.*

We emphasize that a much more general version of the equivalence between (I) and (II) in Theorem 1, where $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ is allowed to be an arbitrary ∞–bicategory, has been proven by Lurie in [6] using scaled variants of the straightening and unstraightening constructions from [5]. In fact, our proof of Theorem 1 reduces the statement to Lurie's equivalence by means of the following comparison result:

Theorem 2 *The functor Φ is weakly equivalent to a contravariant version of Lurie's scaled straightening functor.*

The benefit of the functor $\mathbb{X}$ over the rather unwieldy scaled unstraightening functor is its comparatively simple and intuitively clear form, which makes it more tractable in explicit computations. For example, it has already been utilized for precisely this reason in Dyckerhoff [2], and we expect it to become a useful tool in controlling ∞–bicategorical limits and colimits.

We conclude the paper with a comparison of the relative 2–nerve and the ordinary relative nerve.

Theorem 3 *Let C be an ordinary category. Then, for every functor $F\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$, there is an explicitly defined equivalence*

$$\chi(F) \xrightarrow{\simeq} \mathbb{X}(F)$$

of Cartesian fibrations over $\mathrm{N}(C)$, natural in F.

We note that a variant of the Grothendieck construction for Cat–enriched functors $\mathbb{C} \to \mathrm{Cat}$ has been introduced and related to the scaled unstraightening functor in Harpaz, Nuiten and Prasma [4]. Passing through appropriate op's, our posets D^I correspond to the categories $|\mathbf{\Delta}^n_{/i}|_1$ appearing in [4], so that our functor $\chi_{\mathbb{C}}$ can be regarded as a direct generalization.

Structure of the paper

In Section 1 we provide a brief review of the necessary background on marked simplicial sets and scaled simplicial sets, the model structures which play a role in the paper, as well as the scaled (un)straightening constructions. We then define the relative 2–nerve and attendant combinatorial constructions in Section 2. The proof of Theorem 1 occupies most of the rest of the paper and it is achieved by means of a comparison map with Lurie's straightening functor. The technical heart of the paper consists of Section 3, where we address the key ingredients necessary for the proof of our main theorem. We show that the left adjoint to the relative 2–nerve maps inner horn inclusions to projective trivial cofibrations, establishing a weak equivalence between said adjoint and Lurie's scaled straightening functor.

The subsequent arguments in Section 4 are then comparatively routine: We show Theorem 2, obtaining as a corollary the full statement of Theorem 1. We conclude in Section 4.4 with the comparison of the 1–categorical and 2–categorical relative nerves as stated in Theorem 3.

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1 Preliminaries

1.1 2–Categories

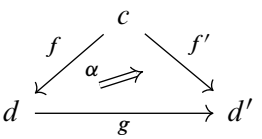
We denote by Cat the 1–category of small categories. We treat Cat as a symmetric monoidal category by virtue of the Cartesian product and define a 2–category to be a

Cat–enriched category. Further, a 2–functor between 2–categories is defined to be a Cat–enriched functor.

Since a 2–category has both 1– and 2–morphisms, there are two different ways of taking opposite categories. For a 2–category $\mathbb{C}$, we denote by $\mathbb{C}^{(\mathrm{op},-)}$ the 2–category in which the directions of the 1–morphisms have been reversed, and by $\mathbb{C}^{(-,\mathrm{op})}$ the 2–category in which the directions of the 2–morphisms have been reversed. Clearly, $(\mathbb{C}^{(\mathrm{op},-)})^{(-,\mathrm{op})} \cong (\mathbb{C}^{(-,\mathrm{op})})^{(\mathrm{op},-)}$, and we will denote this 2–category by $\mathbb{C}^{(\mathrm{op},\mathrm{op})}$.

Given a 2–category $\mathbb{C}$ and an object $c \in \mathbb{C}$, we define its *over 2–category* $\mathbb{C}_{c/}$ to have

- objects given by morphisms $f\colon c \to d$ in $\mathbb{C}$,
- 1–morphisms from $f\colon c \to d$ to $f'\colon c \to d'$ given by commutative diagrams



- 2–morphisms from (g, α) to (h, β) given by a 2–morphism $\mu\colon g \Rightarrow h$ in $\mathbb{C}$ satisfying $\alpha = \beta \circ (\mu * \mathrm{id}_f)$.

We moreover define the *homotopy category* $|\mathbb{C}|_1$ of a 2–category $\mathbb{C}$ to be the ordinary 1–category with

- the same objects as $\mathbb{C}$,
- for objects $c, c' \in \mathbb{C}$, the set of morphisms

$$|\mathbb{C}|_1(c, c') = \pi_0|\mathbb{C}(c, c')|,$$

where $|-|$ denotes the geometric realization of the nerve.

1.2 Marked simplicial sets

Following [5], we define a *marked simplicial set* to be a simplicial set X together with a chosen subset $E \subset X_1$ of *marked edges* containing all degenerate edges. We will sometimes use the notation $\overline{X} = (X, E)$ for a marked simplicial set. We denote the category of marked simplicial sets by Set_Δ^+. For a simplicial set K, we denote by $K^\flat$ (resp. $K^\sharp$) the marked simplicial set with marked edges given by the degenerate edges (resp. all edges). The category Set_Δ^+ is Cartesian closed, so that, for every pair of objects $X, Y \in \mathrm{Set}_\Delta^+$, there exists an internal mapping object, which we denote by $\mathrm{Map}(X, Y)$. There are two related ways to provide Set_Δ–enrichments of Set_Δ^+:

(1) The underlying simplicial set $\mathrm{Map}^{\flat}(X,Y)$ of $\mathrm{Map}(X,Y)$ provides a simplicial enrichment adjoint to the tensor structure

$$(K,\overline{X}) \mapsto K^{\flat} \times \overline{X}.$$

of $\mathrm{Set}^{+}_{\Delta}$ over Set_{Δ}.

(2) The simplicial subset $\mathrm{Map}^{\sharp}(X,Y) \subset \mathrm{Map}^{\flat}(X,Y)$ consisting of those simplices whose edges are marked in $\mathrm{Map}(X,Y)$ provides a simplicial enrichment adjoint to the tensor structure

$$(K,\overline{X}) \mapsto K^{\sharp} \times \overline{X}$$

of $\mathrm{Set}^{+}_{\Delta}$ over Set_{Δ}.

Given a 2–category $\mathbb{C}$, we may consider $\mathbb{C}$ as a $\mathrm{Set}^{+}_{\Delta}$–enriched category by virtue of taking nerves of the mapping 1–categories of $\mathbb{C}$ and marking those edges corresponding to 2–isomorphisms. In particular, we will speak of $\mathrm{Set}^{+}_{\Delta}$–enriched functors between a 2–category and a $\mathrm{Set}^{+}_{\Delta}$–enriched category.

1.3 Scaled simplicial sets

Scaled simplicial sets provide a model for a homotopy coherent theory of $(\infty,2)$–categories, called the theory of ∞–bicategories, parallelling the way ordinary simplicial sets model a homotopy coherent theory of $(\infty,1)$–categories. The idea goes back to work of Street [7], where the notion of a nerve of a strict n–category is defined as a simplicial set equipped with additional data that keeps track of the invertibility of higher morphisms. Verity's *complicial sets* [9; 8] are then obtained by weakening the horn filling properties of the Street nerve, leading to a candidate model for (∞,n)–categories. Along the same lines, Lurie [6] introduces scaled simplicial sets and proves that they indeed provide a model for $(\infty,2)$–categories equivalent to the other known models.

A *scaled simplicial set* consists of a simplicial set K together with a chosen subset $T \subset K_2$ of 2–simplices, containing all degenerate 2–simplices. The elements of T are called *thin*. Given a scaled simplicial set K, we denote by $K^{\mathrm{t}} \subset K$ the simplicial subset of K consisting of those simplices all of whose 2–simplices are thin. For a simplicial set K, we denote by $K_{\flat}$ the scaled simplicial set whose only thin simplices are the degenerate 2–simplices and by $K_{\sharp}$ the scaled simplicial set with all 2–simplices thin.

Given a scaled simplicial set (K,T), a marked simplicial set (X,M) and a map $p\colon X \to K$, we say that p is a *scaled Cartesian fibration* if

(1) p is a locally Cartesian fibration in the sense of [5, 2.4.2.6],

(2) M is the set of locally p–Cartesian edges of X, and

(3) the restriction of p to every thin 2–simplex $\Delta^2 \to K^{\mathrm{t}}$ is a Cartesian fibration.

1.4 Model structures

We recall certain model structures introduced in [5; 6] which will be relevant for us, as well as some facts about widely known model structures which will be of use in our coming arguments. Of particular importance are the model structures on the categories Set_Δ^+ and $(\mathrm{Set}_\Delta^+)_{/K}$, which are integral to the Quillen adjunction at the heart of this paper.

The basic building block for our construction will be the model structure on $(\mathrm{Set}_\Delta^+)_{/K}$ which models Cartesian fibrations; see [5, Proposition 3.1.3.7]:

Theorem 1.4.1 *Let K be a simplicial set. There exists a left-proper combinatorial simplicial model structure on $(\mathrm{Set}_\Delta^+)_{/K}$ with*

(1) *cofibrations those morphisms whose underlying maps of simplicial sets are monomorphisms,*

(2) *equivalences the Cartesian equivalences of* [5, Proposition 3.1.3.3],

(3) *fibrant objects $p\colon \overline{X} \to K$, where p is a Cartesian fibration with p–Cartesian edges marked,*

(4) *for objects X and Y, the simplicial enrichment is given by the simplicial set $\mathrm{Map}^\sharp(X, Y)$ from Section 1.2.*

We will refer to this model structure as the *Cartesian model structure*

Remark 1.4.2 A few facts about the Cartesian model structure will be of use in the sequel:

(1) There is a dual model structure on $(\mathrm{Set}_\Delta^+)_{/K^{\mathrm{op}}}$, the *co-Cartesian model structure*.

(2) As a special case of the Cartesian model structure, we obtain a model structure on $\mathrm{Set}_\Delta^+ \cong (\mathrm{Set}_\Delta^+)_{/*}$, whose fibrant objects are naturally marked ∞–categories. We will also refer to this model structure as the *marked model structure*.

For the right-hand side of the Grothendieck construction, we will need a model structure which models $(\infty, 2)$–functors into the $(\infty, 2)$–category $\mathfrak{Cat}_\infty$. This is provided by the *projective model structure* on the category $\mathrm{Fun}_{\mathrm{Set}_\Delta^+}(\mathbb{C}, \mathrm{Set}_\Delta^+)$ of Set_Δ^+–enriched functors with Set_Δ^+–enriched natural transformations:

Theorem 1.4.3 *For every 2–category $\mathbb{C}$, the category $\mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}, \mathrm{Set}^+_\Delta)$ carries a combinatorial left-proper model structure in which a natural transformation $\alpha\colon F \to G$ is*

(1) *a fibration if the induced map $\alpha_C\colon F(C) \to G(C)$ is a fibration in the marked model structure,*

(2) *a weak equivalence if the induced map $\alpha_C\colon F(C) \to G(C)$ is a weak equivalence in the marked model structure.*

Proof See eg [5, Proposition A.3.3.2]. □

Remark 1.4.4 The projective model structure on $\mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}, \mathrm{Set}^+_\Delta)$, equipped with mapping spaces $\mathrm{Map}^\sharp(-,-)$, is simplicial [6, Remark 3.8.2].

Finally, we need a model structure for the left-hand side of the Grothendieck construction, modelling scaled Cartesian fibrations over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$.

Theorem 1.4.5 *Let (K, T) be a scaled simplicial set. There exists a left-proper, combinatorial, simplicial model structure on $(\mathrm{Set}^+_\Delta)_{/K}$ such that*

(1) *a morphism is a cofibration if and only if it induces a monomorphism between underlying simplicial sets,*

(2) *an object $p\colon \overline{X} \to K$ is fibrant if and only if p is a scaled Cartesian fibration, and the marked edges are precisely the locally p–Cartesian edges.*

Proof This is (the dual of) a special case of [6, Theorem 3.2.6]. □

1.5 Scaled straightening and unstraightening

Given a scaled simplicial set $\overline{K} = (K, T)$ and a weak equivalence $\phi\colon \mathfrak{C}^{\mathrm{sc}}[(K, T)] \to \mathcal{C}$ of marked simplicially enriched categories, the $(\infty, 2)$–Grothendieck construction is presented by a Quillen equivalence

$$\mathrm{CSt}^{\mathrm{sc}}_\phi\colon (\mathrm{Set}^+_\Delta)_{/K} \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathrm{Set}^+_\Delta) :\mathrm{CUn}^{\mathrm{sc}}_\phi$$

constructed in [6, Theorem 3.8.1]. The left and right adjoints are called, respectively, scaled straightening and scaled unstraightening. As proved in [6], this Quillen equivalence relates the scaled *co*-Cartesian model structure to the projective model

structure.[1] In this paper, we consider the special case where $\mathbb{C}$ is a 2–category and $\phi\colon \mathfrak{C}^{\mathrm{sc}}[\mathrm{N}^{\mathrm{sc}}(\mathbb{C})] \to \mathbb{C}$ is the counit of the adjunction $\mathfrak{C}^{\mathrm{sc}} \vdash \mathrm{N}^{\mathrm{sc}}$ (see [6, Theorem 4.2.7]). In this context, we will denote the straightening and unstraightening by $\mathrm{CSt}^{\mathrm{sc}}_{\mathbb{C}}$ and $\mathrm{CUn}^{\mathrm{sc}}_{\mathbb{C}}$, respectively.

Since our relative nerve construction produces a scaled *Cartesian* fibration, it will relate to the duals of Lurie's scaled straightening and unstraightening functors. More precisely, it will related to the Quillen adjunction given by the *contravariant scaled straightening* $\mathrm{St}_{\mathbb{C}}$ and the *contravariant scaled unstraightening* $\mathrm{Un}_{\mathbb{C}}$, given respectively by the composites

$$(\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \xrightarrow{\mathrm{op}} (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})^{\mathrm{op}}} \xrightarrow{\mathrm{CSt}^{\mathrm{sc}}} \mathsf{Fun}(\mathbb{C}^{(\mathrm{op},-)}, \mathrm{Set}^+_\Delta) \xrightarrow{\mathrm{op}\circ -} \mathsf{Fun}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) \tag{1-1}$$

and

$$\mathsf{Fun}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) \xrightarrow{\mathrm{op}\circ -} \mathsf{Fun}(\mathbb{C}^{(\mathrm{op},-)}, \mathrm{Set}^+_\Delta) \xrightarrow{\mathrm{CUn}^{\mathrm{sc}}} (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})^{\mathrm{op}}} \xrightarrow{\mathrm{op}} (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}. \tag{1-2}$$

The aim of this section is to provide an explicit description of the former functor.

Before we give this description, however, it is worth pausing to comment on the appearance of op's in the above composites. Given marked simplicial sets $\overline{X}$ and $\overline{Y}$, there is a bijection

$$\mathrm{Hom}_{\mathrm{Set}^+_\Delta}(\overline{X}, \overline{Y}) \cong \mathrm{Hom}_{\mathrm{Set}^+_\Delta}(\overline{X}^{\mathrm{op}}, \overline{Y}^{\mathrm{op}}).$$

However, this does *not* extend to a simplicially enriched functor

$$\mathrm{op}\colon \mathrm{Set}^+_\Delta \to \mathrm{Set}^+_\Delta, \quad X \mapsto X^{\mathrm{op}}.$$

This is because $\mathrm{Map}^\flat(X, Y) \ncong \mathrm{Map}^\flat(X^{\mathrm{op}}, Y^{\mathrm{op}})$, but instead

$$\mathrm{Map}^\flat(X, Y) \cong \mathrm{Map}^\flat(X^{\mathrm{op}}, Y^{\mathrm{op}})^{\mathrm{op}}.$$

Consequently, we need to apply op to all of the Hom–spaces of Set^+_Δ to get an enriched functor

$$\mathrm{op}\colon \mathrm{Set}^+_\Delta \to (\mathrm{Set}^+_\Delta)^{(-,\mathrm{op})}, \quad X \mapsto X^{\mathrm{op}}.$$

This accounts for the removal of the second op from $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ in the composite above.

[1] The functors we denote by $\mathrm{CSt}^{\mathrm{sc}}$ and $\mathrm{CUn}^{\mathrm{sc}}$ appear in [6] as $\mathrm{St}^{\mathrm{sc}}$ and $\mathrm{Un}^{\mathrm{sc}}$. We will, however, use the latter symbols for the corresponding functors in the *contravariant* Grothendieck construction.

We now proceed with the definition of $\mathrm{St}_{\mathbb{C}}$ obtained by the following transcription of the constructions in [6, Section 3.5]:

Construction 1.5.1 For X a simplicial set, we define the *right cone over* X to be

$$C^{R}(X) := X \times \Delta^1 \amalg_{X\times\{1\}} \Delta^0.$$

Similarly, we set

$$\mathfrak{C}^{R}(X) := \mathfrak{C}[C^{R}(X)].$$

For $X \to K$ a morphism of simplicial sets, we define relative variants of both of these cones via eg

$$\mathfrak{C}^{R}_{K}(X) := \mathfrak{C}^{R}(X) \amalg_{\mathfrak{C}[X\times\{0\}]} \mathfrak{C}[K].$$

Now suppose that $\overline{X} = (X, M)$ is a marked simplicial set and $\overline{K} = (K, T)$ is a scaled simplicial set, with $X \to K$ a morphism. We can then define a scaling T_{R} on $X \times \Delta^1$ by requiring that:

- All degenerate 2–simplices are in T_{R}.
- For $\sigma\colon \Delta^2 \to X \times \Delta^1$, if the preimage of 1 under the composite

$$\Delta^2 \xrightarrow{\sigma} X \times \Delta^1 \to \Delta^1$$

  is $\{1, 2\}$ and $\sigma|_{\Delta^{\{1,2\}}} \in M$, then $\sigma \in T_{R}$.

We then define a scaled variant of the cone,

$$\overline{C}^{R}(\overline{X}) := (X \times \Delta^1, T_{R}) \amalg_{(X\times\{1\})_\flat} (\Delta^0)_\flat.$$

This, together with the scaling on K, yields a scaled relative cone $\overline{C}^{R}_{\overline{K}}(\overline{X})$. We then define marked simplicial categories $\overline{\mathfrak{C}}^{R}(\overline{X})$ and $\overline{\mathfrak{C}}^{R}_{\overline{K}}(\overline{X})$ by applying the scaled rigidification $\mathfrak{C}^{\mathrm{sc}}$ to the respective scaled cones.

Definition 1.5.2 Let $\mathbb{C}$ be a 2–category and $\overline{X} \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ a marked simplicial set over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$. We define

$$\mathbb{C}(X) := \mathbb{C} \amalg_{\mathfrak{C}^{\mathrm{sc}}[\mathrm{N}^{\mathrm{sc}}(\mathbb{C})]} \overline{\mathfrak{C}}^{R}_{\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}(\overline{X}).$$

The *contravariant scaled straightening functor* $\mathrm{St}_{\mathbb{C}}$ is the functor which sends $\overline{X} \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ to the functor

$$\mathbb{C}^{(\mathrm{op},\mathrm{op})} \to \mathrm{Set}^{+}_{\Delta}, \quad x \mapsto \mathrm{Map}_{\mathbb{C}(X)}(x, v)^{\mathrm{op}}.$$

It is straightforward to check that $\mathrm{St}_{\mathbb{C}}$ as defined in (1-1) is indeed naturally isomorphic to $\mathrm{St}_{\mathbb{C}}$ as defined in Definition 1.5.2.

2 The relative 2–nerve

In this section, we define the central notion of this work: the relative 2–nerve. We begin with the definition of the scaled nerve.

Definition 2.0.1 Let I be a linearly ordered finite set. We define a 2–category $\mathbb{O}^I$ as follows:[2]

- The objects of $\mathbb{O}^I$ are the elements of I.
- The category $\mathbb{O}^I(i,j)$ of morphisms between objects $i, j \in I$ is defined as the poset of finite sets $S \subseteq I$ such that $\min(S) = i$ and $\max(S) = j$ ordered by inclusion.
- The composition functors are given, for $i, j, l \in I$, by

$$\mathbb{O}^I(i,j) \times \mathbb{O}^I(j,l) \to \mathbb{O}^I(i,l), \quad (S,T) \mapsto S \cup T.$$

Remark 2.0.2 The association $[n] \mapsto \mathbb{O}^{[n]}$ extends to a cosimplicial 2–category, so that, for a 2–category $\mathbb{C}$, we obtain a simplicial set $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ with set of n–simplices

$$\mathrm{N}^{\mathrm{sc}}(\mathbb{C})_n = \mathrm{Fun}_{\mathrm{Cat}}(\mathbb{O}^{[n]}, \mathbb{C}).$$

The simplicial set $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ is precisely the scaled nerve of [6], where the scaled 2–simplices are those for which the 2–morphism $\{0,2\} \Rightarrow \{0,1,2\}$ in $\mathbb{O}^{[2]}$ is mapped to an invertible 2–morphism in $\mathbb{C}$.

For a linearly ordered set I, we consider the under 2–category

$$\widetilde{D}^I = ((\mathbb{O}^I)^{(-,\mathrm{op})})_{\min(I)/} \tag{2-1}$$

and denote its homotopy category by D^I. For the standard ordinal $[n] = \{0, 1, \ldots, n\}$, we will use the notation $D^n = D^{[n]}$.

Proposition 2.0.3 *The category D^I can be identified with the poset whose*

- *elements are those subsets $S \subseteq I$ such that $\min(S) = \min(I)$, and*
- *we have $S \leq T$ if and only if the following conditions hold*:
  (1) $\max(S) \leq \max(T)$.
  (2) $T \subset S \cup [\max(S), \max(T)]$.

[2] This notation is a deliberate nod to the *orientals* defined by Street in [7]. In the terminology of this work, the 2–category $\mathbb{O}^{[n]}$ is the free 2–category on the n–simplex.

Proof We note that, for $S, T \in \widetilde{D}^I$, a 1–morphism in $\widetilde{D}^I$ from S to T corresponds to a subset $U \subset I$ satisfying $\min(U) = \max(S)$, $\max(U) = \max(T)$ and $T \subseteq S \cup U$. In particular, we have

$$T \subseteq S \cup [\max(S), \max(T)],$$

so that S must contain all elements $t \in T$ with $t \leq \max(S)$, implying condition (2). Conversely, if a pair (S, T) satisfies the conditions (1) and (2), then the set $U_0 := [\max(S), \max(T)]$ defines an 1–morphism in $\widetilde{D}^I$ from S to T. Further, given any 1–morphism in $\widetilde{D}^I$ from S to T, represented by a subset U, we have an inclusion $U \subset U_0$, which represents a 2–morphism in $\widetilde{D}^I$ from U_0 to U. This shows that all nonempty morphism categories of $\widetilde{D}^I$ are connected (even contractible) and implies the claim. □

A systematic analysis of the poset D^n will be given in Section 3.2. Graphical representations for $n \leq 4$ are provided in Figure 1. Let I be a finite nonempty linearly ordered set and suppose $J \subset I$ is a nonempty subset. Then pullback along 1–morphisms in $\mathbb{O}^I$ determines a *pullback functor*

$$\rho_{J,I} \colon \mathbb{O}^I(\min(I), \min(J))^{\mathrm{op}} \times D^J \to D^I, \quad (S_1, S_2) \mapsto S_1 \cup S_2. \tag{2-2}$$

Definition 2.0.4 Let $\mathbb{C}$ be a 2–category and let

$$F \colon \mathbb{C}^{(\mathrm{op},\mathrm{op})} \to \mathrm{Set}_\Delta^+$$

be a Set_Δ^+–enriched functor. We define a marked simplicial set $\mathbb{X}_{\mathbb{C}}(F)$, called the *relative 2–nerve of* F, as follows. An n–simplex of $\mathbb{X}_{\mathbb{C}}(F)$ consists of

(1) an n–simplex $\sigma \colon \Delta^n \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C})$,

(2) for every nonempty subset $I \subset [n]$, a map of marked simplicial sets

$$\theta_I \colon \mathrm{N}(D^I)^\flat \to F(\sigma(\min(I)))$$

such that, for every $J \subset I \subset [n]$, the diagram

$$\begin{array}{ccc} \mathrm{N}\big(\mathbb{O}^I(\min(I), \min(J))^{\mathrm{op}}\big)^\flat \times \mathrm{N}(D^J)^\flat & \xrightarrow{\rho_{J,I}} & \mathrm{N}(D^I)^\flat \\ \downarrow{\scriptstyle \mathrm{N}(\sigma)\times\theta_J} & & \downarrow{\scriptstyle \theta_I} \\ \mathrm{N}\big(\mathbb{C}\big(\sigma(\min(I)), \sigma(\min(J))\big)^{\mathrm{op}}\big)^\flat \times F(\min(J)) & \xrightarrow{F(-)} & F(\min(I)) \end{array}$$

commutes. The marked edges of $\mathbb{X}_{\mathbb{C}}(F)$ are defined as follows: An edge e of $\mathbb{X}_{\mathbb{C}}(F)$ consists of a morphism $f \colon x \to y$ in $\mathbb{C}$, together with vertices $A_x \in F(x)$ and

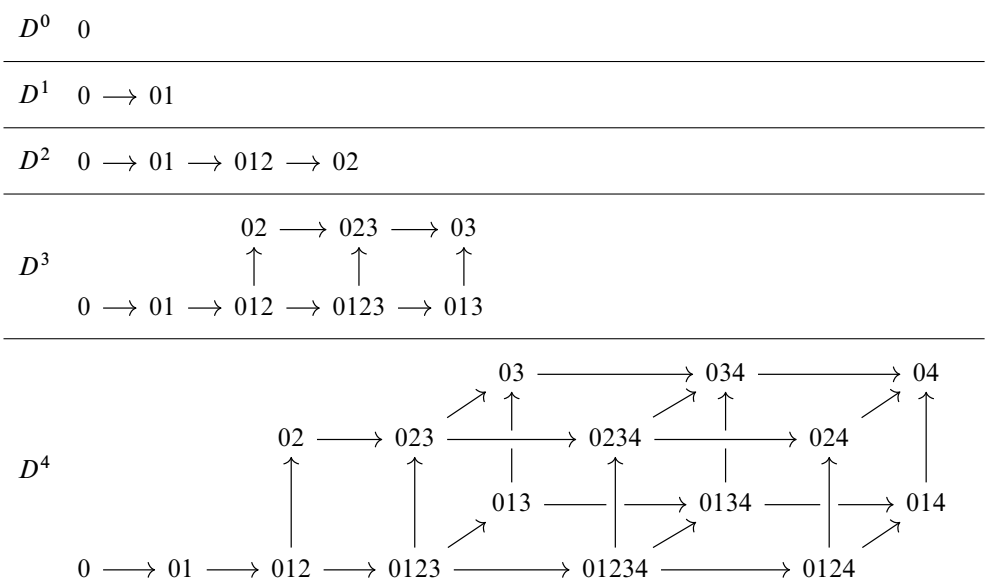


Figure 1: The posets D^n for $n \leq 4$.

$A_y \in F(y)$, and an edge $\widetilde{e}\colon A_x \to F(f)(A_y)$ in $F(x)$. We declare e to be marked if $\widetilde{e}$ is marked. Finally, we consider $\mathbb{X}_{\mathbb{C}}(F)$ as a simplicial set over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$ by means of the forgetful functor.

The construction $F \mapsto \mathbb{X}_{\mathbb{C}}(F)$ is functorial with respect to Set^+_Δ–enriched natural transformations and therefore defines a functor

$$\mathbb{X}_{\mathbb{C}}\colon \mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) \to (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}, \tag{2-3}$$

which we refer to as the *relative* 2*–nerve functor*.

To conclude this section we observe that $\mathbb{X}_{\mathbb{C}}$ preserves all limits and therefore, invoking the adjoint functor theorem, we obtain an adjunction

$$\mathbb{\Phi}_{\mathbb{C}}\colon (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \rightleftarrows \mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) :\mathbb{X}_{\mathbb{C}}. \tag{2-4}$$

with left adjoint $\mathbb{\Phi}_{\mathbb{C}}$.

3 Key propositions

We will show that the adjunction (2-4) is a Quillen adjunction by verifying that $\mathbb{X}_{\mathbb{C}}$ preserves trivial fibrations and that $\mathbb{\Phi}_{\mathbb{C}}$ preserves weak equivalences. To this end, we

will establish the two following results, whose proofs form the technical heart of this work.

Proposition 3.0.1 *Let $\mathbb{C}$ be a 2–category. Then $\mathbb{X}_{\mathbb{C}}$ maps trivial fibrations in the projective model structure to trivial fibrations in the scaled Cartesian model structure.*

Proposition 3.0.2 *Let $\mathbb{C} = \mathbb{O}^n$ with $n \geq 2$ and consider the inclusion morphism,*

$$(\Lambda^n_i)^\flat \to (\Delta^n)^\flat, \quad 0 < i < n,$$

as a morphism in $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)}$. Then the induced map,

$$\Phi_{\mathbb{O}^n}((\Lambda^n_i)^\flat) \to \Phi_{\mathbb{O}^n}((\Delta^n)^\flat),$$

is a trivial cofibration in the projective model structure.

Proposition 3.0.1 implies that $\Phi_{\mathbb{C}}$ preserves cofibrations. Proposition 3.0.2 will be the main technical ingredient to show that show that $\Phi_{\mathbb{C}}$ is weakly equivalent to $\mathrm{St}_{\mathbb{C}}$ and hence preserves weak equivalences. Moreover, $\Phi_{\mathbb{C}}$ will then induce an equivalence on derived categories (since $\mathrm{St}_{\mathbb{C}}$ does), and thus will be a Quillen equivalence.

3.1 Proposition 3.0.1

Throughout this section, we fix a 2–category $\mathbb{C}$. We want to show that $\mathbb{X}_{\mathbb{C}}$ preserves trivial fibrations. Therefore we need to consider lifting problems against arbitrary cofibrations of marked simplicial sets. These are generated by

$$\{(\Delta^1)^\flat \hookrightarrow (\Delta^1)^\sharp\} \quad \text{and} \quad \{(\partial\Delta^n)^\flat \hookrightarrow (\Delta^n)^\flat\}.$$

Consequently, we will confine ourselves to these cases and find solutions to lifting problems of the form

(3-1)
$$\begin{array}{ccc} (\Delta^1)^\flat & \longrightarrow & \mathbb{X}_{\mathbb{C}}(F) \\ \downarrow & \nearrow & \downarrow \\ (\Delta^1)^\sharp & \longrightarrow & \mathbb{X}_{\mathbb{C}}(G) \end{array}$$

(3-2)
$$\begin{array}{ccc} (\partial\Delta^n)^\flat & \xrightarrow{f} & \mathbb{X}_{\mathbb{C}}(F) \\ \downarrow & \nearrow & \downarrow \\ (\Delta^n)^\flat & \xrightarrow{g} & \mathbb{X}_{\mathbb{C}}(G) \end{array}$$

where $F \to G$ is a Set^+_Δ–enriched natural transformation of Set^+_Δ–enriched functors $\mathbb{C}^{(\mathrm{op},\mathrm{op})} \to \mathrm{Set}^+_\Delta$ which is a pointwise trivial fibration. Here, the map $\mathbb{X}_{\mathbb{C}}(F) \to \mathbb{X}_{\mathbb{C}}(G)$

is considered as a morphism in $\mathrm{Set}^{+}_{\Delta/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}$. A quick inspection shows that we can solve the lifting problem (3-1). The lifting problem given by (3-2) is slightly more involved and requires some preparation.

We begin by developing some terminology that will help unravel the essence of these kinds of lifting problems. In the following discussion, markings will not play any role, so we systematically ignore them and only refer to the underlying simplicial sets.

Let $n \geq 0$ and let $J \subset [n]$ be a nonempty subset. The pullback functor

$$\rho^n_J \colon \mathbb{O}^{[n]}(0, \min(J))^{\mathrm{op}} \times D^J \to D^n \tag{3-3}$$

is easily verified to be fully faithful; we denote its image by $A(J)$. We denote by $\mathcal{D}^n$ the nerve of the poset D^n and by $\mathcal{A}(J)$ the nerve of the poset $A(J)$. We can now define

$$\mathcal{S}^n := \bigcup_{\Delta^J \subset \partial\Delta^n} \mathcal{A}(J) \subset \mathcal{D}^n. \tag{3-4}$$

Suppose we are given a lifting problem of the form (3-2). According to Definition 2.0.4, the map g corresponds to a functor $\sigma \colon \mathbb{O}^{[n]} \to \mathbb{C}$ together with a compatible collection of maps

$$\{g_I \colon \mathcal{D}^I \to G(\sigma(\min(I)))\}$$

parametrized by the nonempty subsets $I \subset [n]$. In particular, choosing $I = [n]$, we obtain a functor

$$\overline{g} \colon \mathcal{D}^n \to G(\sigma(0)).$$

The additional data contained in the map f amounts to a compatible collection of functors

$$\{f_J \colon \mathcal{D}^J \to F(\sigma(\min(J)))\}$$

parametrized by the subsets J such that $\Delta^J \subset \partial\Delta^n$. We can use the functoriality of F to transfer the f_J from $\sigma(\min(J))$ to $\sigma(0)$, to obtain a functor as the composite

$$f^n_J \colon A(J) \xleftarrow{\rho^n_J} \mathbb{O}^{[n]}(0, \min(J))^{\mathrm{op}} \times \mathcal{D}^J \xrightarrow{\sigma \times f_J} \mathbb{C}\big(\sigma(0), \sigma(\min(J))\big)^{\mathrm{op}} \times F(\sigma(\min(J))) \xrightarrow{F} F(\sigma(0)).$$

The various functors $\{f^n_J\}$ assemble to define a functor

$$\overline{f} \colon \mathcal{S}^n \to F(\sigma(0))$$

making the solid part of the diagram

$$\begin{array}{ccc} \mathcal{S}^n & \xrightarrow{\bar{f}} & F(\sigma(0)) \\ \downarrow & \nearrow & \downarrow \\ \mathcal{D}^n & \xrightarrow{\bar{g}} & G(\sigma(0)) \end{array} \tag{3-5}$$

commute. We call (3-5) the *reduced lifting problem* associated to (3-2).

Proposition 3.1.1 *The solutions of a lifting problem of the form* (3-2) *are in bijection with the solutions of the associated reduced lifting problem* (3-5).

Proof Left to the reader. □

Since the morphism $\mathcal{S}^n \to \mathcal{D}^n$ is a cofibration, Proposition 3.0.1 follows.

3.2 Proposition 3.0.2

This section is devoted to the key technical claim of this paper. Namely we show that the images of inner horn inclusions under $\Phi_{\mathbb{C}}$ are objectwise marked weak equivalences. We begin by identifying the images of $(\Lambda^n_i)^\flat$ and $(\Delta^n)^\flat$ under $\Phi_{\mathbb{O}^n}$.

Definition 3.2.1 For $n \geq 0$, we define a marked simplicially enriched functor

$$\mathfrak{D}^n\colon (\mathbb{O}^n)^{(\mathrm{op},\mathrm{op})} \to \mathrm{Set}^+_\Delta, \quad j \mapsto (\mathcal{D}^{[j,n]})^\flat.$$

For $0 < i < n$, we define a second marked simplicially enriched functor

$$\mathfrak{L}^n_i\colon (\mathbb{O}^n)^{(\mathrm{op},\mathrm{op})} \to \mathrm{Set}^+_\Delta$$

given on objects by

$$j \mapsto \left(\bigcup_{\substack{I \subset [j,n] \\ \Delta^I \subset \Lambda^n_i}} \mathbb{O}^{[j,n]}(j, \min(I))^{\mathrm{op}} \times \mathcal{D}^I \right)^\flat \subset (\mathcal{D}^{[j,n]})^\flat$$

with enriched functoriality given by

$$\begin{aligned} \mathbb{O}^n(k,j)^{\mathrm{op}} \times \mathbb{O}^{[j,n]}(j, \min(I))^{\mathrm{op}} \times \mathcal{D}^I &\to \mathbb{O}^{[k,n]}(k, \min(I))^{\mathrm{op}} \times \mathcal{D}^I, \\ (S,T,U) &\mapsto (S \cup T, U). \end{aligned}$$

To ease writing, we refer to those subsets $I \subset [n]$ such that $\Delta^I \subset \Lambda^n_i$ as *i–admissible*.

Lemma 3.2.2 *Consider* $(\Delta^n)^\flat \to \Delta^n$ *and* $(\Lambda^n_i)^\flat \to \Delta^n$ *as objects in* $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)}$.

(1) *For* $n \geq 0$,
$$\Phi_{\mathbb{O}^n}((\Delta^n)^\flat) \cong \mathfrak{D}^n.$$

(2) *For* $0 < i < n$,
$$\Phi_{\mathbb{O}^n}((\Lambda^n_i)^\flat) \cong \mathfrak{L}^n_i.$$

Proof Follows from unravelling the definitions. □

It is worth noting that, for $i, j > 0$, $[j, n] \subset [n]$ is itself an i–admissible set. Consequently, we have that, for $j > 0$, $\mathfrak{L}^n_i(j) \cong \mathfrak{D}^n(j)$. This reduces the problem of showing that $\mathfrak{L}^n_i \Rightarrow \mathfrak{D}^n$ is an objectwise marked weak equivalence to the problem of showing that $\mathfrak{L}^n_i(0) \to \mathfrak{D}^n(0)$ is a marked weak equivalence. Moreover, since both marked simplicial sets carry the minimal marking, it will suffice to show that this is a weak equivalence in the Joyal model structure.

Definition 3.2.3 For ease of notation, we set
$$\mathcal{L}^n_i := \mathfrak{L}^n_i(0) = \bigcup_{\Delta^J \subset \Lambda^n_i} \mathcal{A}(J) \subset \mathcal{D}^n.$$

We now turn to showing that the map of simplicial sets
$$\mathcal{L}^n_i \hookrightarrow \mathcal{D}^n$$
is a trivial cofibration for the Joyal model structure. This in turn is equivalent to the statement that the induced functor of simplicial categories
$$\mathfrak{C}[\mathcal{L}^n_i] \hookrightarrow \mathfrak{C}[\mathcal{D}^n]$$
is a weak equivalence. This latter functor induces a bijection on objects. Furthermore, since $\mathcal{D}^n$ is the nerve of a poset, the simplicial set $\mathfrak{C}[\mathcal{D}^n](S, T)$ is contractible if $S \leq T$ and empty otherwise. Therefore, Proposition 3.0.2 is an immediate corollary of the following main result of this section:

Theorem 3.2.4 *Let* $0 < i < n$ *and let* $S \leq T$ *be elements of* D^n. *Then the simplicial set* $\mathfrak{C}[\mathcal{L}^n_i](S, T)$ *is contractible.*

Corollary 3.2.5 *For every $0 < i < n$, the map of simplicial sets*

$$\mathcal{L}^n_i \hookrightarrow \mathcal{D}^n$$

is a trivial cofibration for the Joyal model structure.

To prepare the proof of Theorem 3.2.4, we begin with an explicit description of the mapping spaces $\mathfrak{C}[\mathcal{K}](S,T)$, where $\mathcal{K} \subset \mathcal{D}^n$ is a simplicial subset. We define a *strict chain of length ℓ between S and T* to be a sequence

$$S = M_0 < M_1 < M_2 < \cdots < M_{\ell-1} < M_\ell = T$$

of objects in D^n. Let $P^n(S,T)$ denote the poset of strict chains between S and T in D^n with length $\ell > 0$, ordered by refinement $\underline{M} \subset \underline{M}'$. Furthermore, let $\mathcal{P}^n(S,T)$ denote the nerve of $P^n(S,T)$. We define

$$\mathcal{P}^n_{\mathcal{K}}(S,T) \subset \mathcal{P}^n(S,T)$$

to be the simplicial subset consisting of those simplices whose corresponding sequence of chain refinements

$$\underline{M}^{(0)} \subset \underline{M}^{(1)} \subset \cdots \subset \underline{M}^{(k)}$$

satisfies the following condition:

- For every consecutive pair of elements $M^{(0)}_r < M^{(0)}_{r+1}$ in the chain $\underline{M}^{(0)}$, the simplex of $\mathcal{D}^n$ corresponding to the totally ordered chain

  $$\{M' \in \underline{M}^{(k)} \mid M^{(0)}_r < M' < M^{(0)}_{r+1}\} \subset D^n$$

  is contained in $\mathcal{K}$.

With this terminology, we have the following:

Proposition 3.2.6 *Let $\mathcal{K} \subset \mathcal{D}^n$ be a simplicial subset containing the pair (S,T) of vertices of $\mathcal{D}^n$. Then there is an isomorphism of simplicial sets*

$$\mathfrak{C}[\mathcal{K}](S,T) \cong \mathcal{P}^n_{\mathcal{K}}(S,T).$$

Proof This follows by explicit computation of $\mathfrak{C}[\mathcal{K}](S,T)$, representing $\mathcal{K}$ as a colimit over its nondegenerate simplices based on [5, 1.1.5.9]. It can also be seen very nicely by using the Dugger–Spivak necklace model for ordered simplicial sets [1, Section 4.11]. □

For the remainder of this section, we will always identify mapping spaces of the form $\mathfrak{C}[\mathcal{K}](S,T)$ with the model $\mathcal{P}^n_{\mathcal{K}}(S,T)$ provided by Proposition 3.2.6. In particular, we will implicitly use this chain model to describe the mapping spaces of the simplicial category $\mathfrak{C}[\mathcal{L}^n_i]$.

We will now introduce some terminology which will allow us to get a handle on the combinatorics of the poset D^n, the simplicial set $\mathcal{L}^n_i$ and its associated simplicial category $\mathfrak{C}[\mathcal{L}^n_i]$. For the remainder of the section, we fix $n \geq 2$ and $0 < i < n$. We first note that the poset D^n can be realized as a full subposet of $\mathbb{Z}^{n-1}$. Namely, let e_j denote the j^{th} unit coordinate vector of $\mathbb{Z}^{n-1}$ and consider the embedding

$$x\colon D^n \hookrightarrow \mathbb{Z}^{n-1}, \quad J \mapsto \max(J)e_{n-1} + \sum_{\substack{0<j<\max(J)\\ j\notin J}} e_j.$$

In fact, the visualizations of the posets in Figure 1 are obtained precisely via this embedding. We further denote by x_i the postcomposition of x with projection to the i^{th} coordinate of $\mathbb{Z}^{n-1}$. Various useful notions arise from these geometric coordinates on D^n:

Definition 3.2.7 Let $S \leq T$ be a pair of comparable elements of D^n.

(1) The edge $S \leq T$ is called *atomic* if $x(T) - x(S)$ is a unit coordinate vector of $\mathbb{Z}^{n-1}$. Note that being atomic means that $S < T$ and there does not exist $M \in D^n$ with $S < M < T$.

(2) We define the *atomic distance* between S and T as the taxicab distance

$$d(S,T) := \sum_{1\leq i\leq n-1} x_i(T) - x_i(S).$$

Note that the atomic distance between S and T measures the length of any chain

$$S = M_0 < M_1 < \cdots < M_l = T$$

with $M_r < M_{r+1}$ atomic.

(3) We call S and T *close* if there exists a rectilinear unit cube in $\mathbb{Z}^{n-1}$ containing the points $x(S)$ and $x(T)$ among its vertices. Equivalently, S and T are close if $x_{n-1}(T) - x_{n-1}(S) \leq 1$.

(4) We call S and T *distant* if they are not close.

It will be convenient to introduce notation for a certain class of i–admissible subsets of $[n]$. A subset $J \subset [n]$ is called *i–superior* if it is maximal among all i–admissible subsets with the same minimal element. Note that, by maximality, the collection of simplicial sets $\{\mathcal{A}(J)\}$, where J runs through all i–superior subsets, covers $\mathcal{L}_i^n$:

$$\mathcal{L}_i^n = \bigcup_{\Delta^J \subset \Lambda_i^n} \mathcal{A}(J) = \bigcup_{\substack{\Delta^J \subset \Lambda_i^n \\ i\text{–superior}}} \mathcal{A}(J) \subset \mathcal{D}^n.$$

This covering indexed by i–superior sets will play a central role in the proof of Theorem 3.2.4.

Remark 3.2.8 The i–superior subsets of $[n]$ are precisely the subsets of the form

(1) $[n] \setminus \{j\}$ where $j \neq i$,

(2) $\{k, k+1, \dots, n\}$ where $k \geq 1$.

Proof of Theorem 3.2.4 The proof will be given by induction on the atomic distance between S and T.

For the base of the induction, suppose that S and T have atomic distance 1. Then they are close, so that, by Lemma 3.2.9, the edge $S < T$ is contained in $\mathcal{L}_i^n$. Therefore, $\mathfrak{C}[\mathcal{L}_i^n](S, T)$ consists of a single vertex representing this edge and is hence contractible.

Suppose now, for the inductive step, that S and T have atomic distance $d \geq 2$ and assume that, for every pair $S' < T'$ with atomic distance $d(S', T') < d$, the simplicial set $\mathfrak{C}[\mathcal{L}_i^n](S', T')$ is contractible. We distinguish three cases:

(I) *There does not exist an i–superior subset J such that $A(J)$ contains both S and T.*

This implies that every vertex of $\mathfrak{C}[\mathcal{L}_i^n](S, T)$ corresponds to a chain between S and T of length ≥ 2. Further, by Lemma 3.2.9, S and T must be distant, so that, by Lemma 3.2.10, the poset $P^n(S, T)_{\geq 2} \subset P^n(S, T)$ of chains between S and T of length ≥ 2 is contractible. The functor

$$p\colon (P^n(S, T)_{\geq 2})^{\mathrm{op}} \to \mathrm{Set}_\Delta,$$
$$\{S < M_1 < \cdots < M_k < T\} \mapsto \mathfrak{C}[\mathcal{L}_i^n](S, M_1) \times \mathfrak{C}[\mathcal{L}_i^n](M_1, M_2) \times \cdots \times \mathfrak{C}[\mathcal{L}_i^n](M_k, T),$$

satisfies

$$\operatorname{colim} p \cong \mathfrak{C}[\mathcal{L}_i^n](S, T).$$

By Lemma 3.2.13, the diagram p is projectively cofibrant, so that, by Remark 3.2.12, the latter colimit is a homotopy colimit. By induction hypothesis, the values of p are contractible simplicial sets, so that, since $P^n(S,T)_{\geq 2}$ is contractible as well, we deduce that $\mathfrak{C}[\mathcal{L}_i^n](S,T)$ is contractible.

(II) *There exists an i–superior subset J such that $A(J)$ contains every $M \in D^n$ with $S \leq M \leq T$.*

By Proposition 3.2.6, we have that

$$\mathfrak{C}[\mathcal{L}_i^n](S,T) \cong \mathcal{P}^n(S,T),$$

where the latter simplicial set is the nerve of the poset $P^n(S,T)$ with minimal element $S < T$ and hence contractible.

(III) *There exists an i–superior subset J such that $A(J)$ contains S and T, but there does not exist an i–superior subset satisfying the hypothesis of* (II)enumz.

This is the most subtle case, since both chains between S and T of length ≥ 2 which may not be contained completely in any $A(J)$ as well as chains between S and T of length 1 which are contained in some $A(J)$ contribute to $\mathfrak{C}[\mathcal{L}_i^n](S,T)$. In a sense, the argument in this case will be a hybrid of the arguments for the extreme cases (I)enumz and (II)enumz.

Denote by $\mathcal{U}_i(S,T)$ the poset whose elements are (possibly empty) sets

$$\underline{J} = \{J_1, \ldots, J_k\},$$

where each J_r is an i–superior subset of $[n]$ such that $A(J)$ contains S and T. Define the poset

$$\mathcal{V}_i(S,T) := (\mathcal{U}_i(S,T) \times \{1 < 2\}) \setminus \{(\varnothing, 1)\}$$

and further a functor

$$p\colon (\mathcal{V}_i(S,T))^{\mathrm{op}} \to \mathrm{Cat}, \quad (\underline{J}, m) \mapsto \begin{cases} P^n_{A(\underline{J})}(S,T) & \text{for } m = 1, \\ P^n_{A(\underline{J})}(S,T)_{\geq 2} & \text{for } m = 2, \end{cases}$$

where

$$P^n_{A(\underline{J})}(S,T) \subset P^n(S,T)$$

denotes the subposet consisting of those chains between S and T which are contained in every poset $A(J)$, $J \in \underline{J}$. Denote by χ the Grothendieck construction of p, as

defined in the introduction. The category χ is again a poset: an element of χ is given by a triple

$$(\underline{J}, l, \{S < M_1 < \cdots < M_k < T\})$$

consisting of a set $\underline{J} \in \mathcal{U}_i(S,T)$ of i–superior subsets, a number $l \in \{1,2\}$ and a chain between S and T of length $\geq l$ lying in all posets $A(J)$, $J \in \underline{J}$.

The poset $\mathcal{V}_i(S,T)$ has a maximal element $(\underline{J}, 2)$, where $\underline{J}$ is the set of all i–superior subsets J such that $A(J)$ contains S and T. Hence it is contractible. By Lemmas 3.2.10 and 3.2.11, for every $\underline{J} \in \mathcal{U}_i(S,T)$, the poset $P^n_{A(\underline{J})}(S,T)_{\geq 2}$ is contractible. The posets $P^n_{A(\underline{J})}(S,T)$ are contractible since they have a minimal element given by the chain $S < T$ of length 1. Therefore, by Lemma 3.2.14, the poset χ is contractible.

To conclude, define a functor $q\colon \chi^{\mathrm{op}} \to \mathrm{Set}_\Delta$ by assigning to the element

$$(\underline{J}, l, \{S < M_1 < \cdots < M_k < T\})$$

the simplicial set

$$\mathfrak{C}[\mathcal{A}(\underline{J})](S, M_1) \times \cdots \times \mathfrak{C}[\mathcal{A}(\underline{J})](M_k, T),$$

where

$$\mathcal{A}(\underline{J}) = \begin{cases} \mathcal{L}^n_i & \text{if } \underline{J} = \varnothing, \\ \bigcap_{J \in \underline{J}} \mathcal{A}(J) & \text{otherwise.} \end{cases}$$

Note that the values of q are contractible simplicial sets: for $\underline{J} \neq \varnothing$ this follows, since $\mathcal{A}(\underline{J})$ is the nerve of the poset $\bigcap_{J \in \underline{J}} A(J)$; for $\underline{J} = \varnothing$, we have $l = 2$, so that the contractibility of the involved mapping spaces follows from the induction hypothesis. By Remark 3.2.12, the diagram q is projectively cofibrant and we have, by construction,

$$\operatorname{colim} q \cong \mathfrak{C}[\mathcal{L}^n_i](S,T).$$

Therefore, by Lemma 3.2.13, this latter colimit is a homotopy colimit of contractible simplicial sets parametrized by the contractible poset χ. Consequently, $\mathfrak{C}[\mathcal{L}^n_i](S,T)$ is contractible. □

Lemma 3.2.9 *Suppose that $S \leq T$ in D^n are close. Then there exists an i–admissible subset $J \subset [n]$ such that $A(J)$ contains every $M \in D^n$ with $S \leq M \leq T$.*

Proof This is geometrically clear from the embedding $x\colon D^n \hookrightarrow \mathbb{Z}^{n-1}$: Every rectilinear unit k–cube in $x(D^n)$, where $1 \leq k \leq n-1$, is contained in one of the subsets

(1) $x\big(A([n] \setminus \{n\})\big)$,

(2) $x\big(A(\{n-1, n\})\big)$,

(3) $x\big(A(\{n\})\big)$.

If $x(S)$ and $x(T)$ are vertices of the same rectilinear unit cube, then the same must be true for $x(M)$, where $S < M < T$. Therefore, we may take J to be one of the subsets $[n]\setminus\{n\}$, $\{n-1,n\}$ or $\{n\}$, respectively. □

Lemma 3.2.10 *Let $S \leq T$ be distant elements of D^n. Then the subposet*

$$P^n(S,T)_{\geq 2} \subset P^n(S,T),$$

consisting of chains of length ≥ 2, is contractible.

Proof Let $G(S,T) \subset D^n$ denote the full subposet on those V such that $S < V < T$. By definition, $P^n(S,T)_{\geq 2}$ is the category of nondegenerate simplices of $G(S,T)$, and thus has the same homotopy type as $G(S,T)$. It will therefore suffice to show that $G(S,T)$ is contractible.

Denote by $G(S,T)^{>}$ the full subposet of $G(S,T)$ on those objects V for which $x_{n-1}(V) > x_{n-1}(S)$. We then make two observations:

(1) $G(S,T)^{>}$ has an initial element, given by the preimage of $x(S)+e_{n-1}$, and is thus contractible.

(2) The morphism $r\colon G(S,T) \to G(S,T)^{>}$ which maps

$$V \mapsto \begin{cases} x(V)+e_{n-1} & \text{if } x_{n-1}(V) = x_{n-1}(S), \\ x(V) & \text{otherwise,} \end{cases}$$

is well defined (since S and T are distant) and defines a homotopy inverse to the inclusion $G(S,T)^{>} \hookrightarrow G(S,T)$. Consequently, $G(S,T)$ is contractible. □

Lemma 3.2.11 *Let $S \leq T$ elements of D^n satisfying the hypothesis of* (III)enumz *in the proof of Theorem 3.2.4. Let $\underline{J} \in \mathcal{U}_i(S,T)$ be a set of i–superior subsets. Then the subposet*

$$P^n_{A(\underline{J})}(S,T)_{\geq 2} \subset P^n(S,T)_{\geq 2},$$

consisting of those chains which are contained in every $A(J)$, $J \in \underline{J}$, is contractible.

Proof First note that the poset $P^n_{A(\underline{J})}(S,T)_{\geq 2}$ can be identified with the category of simplices of the poset $G(S,T)_{\underline{J}}$ consisting of those objects V such that $S < V < T$ and such that, for every $J \in \underline{J}$, we have $V \in J$. In particular, it suffices to show that the posets $G(S,T)_{\underline{J}}$ are contractible.

The i–superior subsets $J \in \underline{J}$ must be of the form $[n]\setminus j$ with $j > \max(S)$ and $j \neq n, i$. Indeed, all other i–superior sets would contradict the hypothesis of (III)enumz. In

particular, this implies that $\max(T) = n$ and that $S \cap T = \{0\}$. To simplify notation, we will identify $\underline{J} = \{[n] \setminus j_\ell\}_{\ell=1}^{r}$ with the set $X := \{j_1, j_2, \dots, j_r\}$ and distinguish the following two cases:

(1) Suppose $S = \{0\}$. Let $k \in [n] \setminus \{0\}$ be the smallest element such that $k \notin X$. We observe that $\{0, k\} \in G(S,T)_{\underline{J}}$ and define $G(S,T)_{\underline{J}}^{\leq k}$ to be the full subposet on those objects V such that $V \leq \{0,k\}$. The inclusion functor

$$i_k \colon G(S,T)_{\underline{J}}^{\leq k} \subset G(S,T)_{\underline{J}},$$

admits a section

$$r_k \colon G(S,T)_{\underline{J}} \to G(S,T)_{\underline{J}}^{\leq k}, \quad V \mapsto \begin{cases} V & \text{if } \max(V) \leq k, \\ (V \cap [k]) \cup \{k\} & \text{otherwise.} \end{cases}$$

Furthermore, we have $i_k \circ r_k(V) \leq V$ thus inducing a natural transformation between $i_k \circ r_k$ and id. This shows that both posets have the same homotopy type. Since $G(S,T)_{\underline{J}}^{\leq k}$ has a final element $\{0,k\}$, we see that $|G(S,T)_{\underline{J}}|$ is contractible.

(2) Suppose $S \neq \{0\}$ and set $s = \max(S)$. We define $G(S,T)_{\underline{J}}^{s} \subset G(S,T)_{\underline{J}}$ to consist of those sets V which contain s. This poset has a final element, given by $T \cup \{s\}$. As above, the inclusion $i_s \colon G(S,T)_{\underline{J}}^{s} \subset G(S,T)_{\underline{J}}$ admits a section

$$r_s \colon G(S,T)_{\underline{J}} \to G(S,T)_{\underline{J}}^{s}, \quad V \mapsto \begin{cases} V & \text{if } s \in V, \\ V \cup \{s\} & \text{otherwise,} \end{cases}$$

which is further a homotopy inverse. As above, this implies the contractibility of $|G(S,T)_{\underline{J}}|$, concluding the proof. □

For $\mathcal{C}$ a small category and $F \colon \mathcal{C} \to \mathrm{Set}_\Delta$ a diagram, we call F *projectively cofibrant* if the natural transformation $\varnothing \to F$ from the constant diagram to F has the left lifting property with respect to all pointwise trivial fibrations in the Kan–Quillen model structure.

Remark 3.2.12 The projectively cofibrant functors are of use to us precisely because, for a projectively cofibrant functor $F \colon \mathcal{C} \to \mathrm{Set}_\Delta$, the canonical morphism

$$\mathrm{hocolim}_{\mathcal{C}} F \to \mathrm{colim}_{\mathcal{C}} F$$

is a weak equivalence. See [5, A.2.8].

Lemma 3.2.13 *Let X be a simplicial set, $\{U_i\}_{i\in I}$ a finite cover of X by nonempty simplicial subsets, and $P \subset \mathcal{P}(I) \setminus \varnothing$ a full subposet containing all $J \subset I$ such that $\bigcap_{j\in J} U_j \subset X$ is nonempty. Then the diagram*

$$F\colon P^{\mathrm{op}} \to \mathrm{Set}_\Delta, \quad J \mapsto \bigcap_{j\in J} U_j,$$

is projectively cofibrant.

Proof Since the source of F is a finite poset and every simplicial set is cofibrant, an argument similar to that of [5, Proposition A.2.9.19] shows that it is sufficient to check that $\mathrm{colim}_{P^{\mathrm{op}}_{/J}\setminus\{J\}} F \to F(J)$ is a cofibration. A quick calculation shows that

$$\mathrm{colim}_{P^{\mathrm{op}}_{/J}\setminus\{J\}} F \cong \bigcup_{K\supsetneq J} \Big(\bigcap_{k\in K} U_k \Big) \subset F(J),$$

completing the proof. □

Lemma 3.2.14 *Let P be a contractible poset, and let $F\colon P^{\mathrm{op}} \to \mathrm{Cat}$ be a diagram such that $F(p)$ is contractible for all $p \in P$. Then the Grothendieck construction $\chi(F)$ is contractible.*

Proof This follows from the well-known fact that the geometric realization of the Grothendieck construction computes the homotopy colimit of the geometric realizations of the various categories of the diagrams (as can be deduced from [5, 3.3.4.3], for example). A direct argument adapted to the given situation is the following: We consider the resulting Cartesian fibration $\pi\colon \chi(F) \to P$. For $p \in P$, we relate the fibre $F(p)$ to the overcategory $\chi(F)_{p/}$. Clearly there is an inclusion

$$i\colon F(p) \to \chi(F)_{p/}, \quad c \mapsto (p, c).$$

Setting f_q to be the unique morphism in P from p to q, we can also define a functor

$$r\colon \chi(F)_{p/} \to F(p), \quad (q, c) \mapsto F(f_q)(c).$$

It is immediate that $r \circ i = \mathrm{id}_{F(p)}$.

We then observe that the canonical morphisms $(p, F(f_q)(c)) \to (q, c)$ given by $(f_q, \mathrm{id}_{F(f_q)(c)})$ form a natural transformation $i \circ r \to \mathrm{id}_{\chi(F)_{p/}}$. Consequently, i is a homotopy equivalence, and so by assumption the overcategories $\chi(F)_{p/}$ are all contractible. By Quillen's Theorem A, π induces a homotopy equivalence of nerves, and so, since P is contractible, $\chi(F)$ is contractible. □

4 The Quillen equivalence

The goal of this section will be to show that the adjunction

$$\Phi_{\mathbb{C}}\colon (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \rightleftarrows \mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) :\mathbb{X}_{\mathbb{C}}$$

from (2-4) is in fact a Quillen equivalence. The proof strategy precisely parallels the argument of [5] to show that the ordinary relative nerve defines a Quillen equivalence: we first relate the values of $\Phi_{\mathbb{C}}$ and the contravariant scaled straightening functor $\mathrm{St}_{\mathbb{C}}$ from Definition 1.5.2 on simplices over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$. We then show that the comparison maps thus obtained glue to produce global natural comparison maps between the values of $\Phi_{\mathbb{C}}$ and $\mathrm{St}_{\mathbb{C}}$ on any marked simplicial set over $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$. We show that this natural transformation is an objectwise weak equivalence. Since $\Phi_{\mathbb{C}}$ preserves cofibrations, this comparison shows, by 2-out-of-3, that it also preserves trivial cofibrations and is thus a Quillen adjunction. In addition, this comparison with Lurie's straightening functor shows that the left derived functors of $\Phi_{\mathbb{C}}$ and $\mathrm{St}_{\mathbb{C}}$ are equivalent, allowing us to conclude the proof.

4.1 Base change

Given a 2–category $\mathbb{C}$, the contravariant scaled straightening functor $\mathrm{St}_{\mathbb{C}}$ associates to a marked simplicial set $X \in \mathrm{Set}^+_{\Delta/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}$ the Set^+_Δ–enriched functor

$$\mathrm{St}_{\mathbb{C}}(X)\colon \mathbb{C}^{(\mathrm{op},\mathrm{op})} \to \mathrm{Set}^+_\Delta$$

whose value at $c \in \mathbb{C}$ is the marked simplicial set

$$\mathrm{St}_{\mathbb{C}}(X)(c) = \mathrm{Map}_{\mathbb{C}(X)}(c, v)^{\mathrm{op}}$$

of maps in the Set^+_Δ–enriched category

$$\mathbb{C}(X) = \mathbb{C} \amalg_{\mathfrak{C}^{\mathrm{sc}}[\mathrm{N}^{\mathrm{sc}}(\mathbb{C})]} \mathfrak{C}^{\mathrm{sc}}[\mathrm{N}^{\mathrm{sc}}(\mathbb{C}) \amalg_{(X\times\{0\})_\flat} X \times \Delta^1 \amalg_{(X\times\{1\})_\flat} \{*\}]. \tag{4-1}$$

To relate the straightening functor $\mathrm{St}_{\mathbb{C}}$ from Definition 1.5.2 to the left adjoint $\Phi_{\mathbb{C}}$, we begin by analyzing its base change behaviour with respect to 2–functors $\mathbb{C} \to \mathbb{D}$.

Proposition 4.1.1 *Let $f\colon \mathbb{C} \to \mathbb{D}$ be a functor between 2–categories. Then the diagram*

$$\begin{array}{ccc} \mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{D}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) & \xrightarrow{f^*} & \mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) \\ \downarrow{\scriptstyle \mathbb{X}_{\mathbb{D}}} & & \downarrow{\scriptstyle \mathbb{X}_{\mathbb{C}}} \\ (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{D})} & \xrightarrow{\mathrm{N}^{\mathrm{sc}}(f)^*} & (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \end{array} \tag{4-2}$$

whose horizontal morphisms are given by pullback along f and $\mathrm{N}^{\mathrm{sc}}(f)$, respectively, commutes up to natural isomorphism.

Proof This is immediate from the definition of the relative 2–nerve. □

Corollary 4.1.2 *Let $f\colon \mathbb{C} \to \mathbb{D}$ be a functor between 2–categories. Then the diagram*

$$\begin{array}{ccc} \mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{D}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) & \xleftarrow{f_!} & \mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) \\ \Phi_{\mathbb{D}} \uparrow & & \Phi_{\mathbb{C}} \uparrow \\ (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{D})} & \xleftarrow{\mathrm{N}^{\mathrm{sc}}(f)_!} & (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \end{array}$$

obtained from (4-2) *by passing to left adjoints of the horizontal functors, commutes up to natural isomorphism.*

4.2 Comparison for simplices

We now provide a map

$$\eta_{\mathbb{C}}(X)\colon \mathrm{St}_{\mathbb{C}}(X) \to \Phi_{\mathbb{C}}(X) \tag{4-3}$$

for the following choices of $\mathbb{C}$ and X:

(1) $\mathbb{C} = \mathbb{O}^n$ and $X = (\Delta^n)^\flat$,

(2) $\mathbb{C} = \mathbb{O}^1$ and $X = (\Delta^1)^\sharp$.

Following [5, 3.2.5.10], it will then be shown that these choices canonically extend to determine maps $\eta_{\mathbb{C}}(X)\colon \mathrm{St}_{\mathbb{C}}(X) \to \Phi_{\mathbb{C}}(X)$ for all $\mathbb{C}$ and $X \in (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}$.

(1) $\mathbb{C} = \mathbb{O}^n$ and $X = (\Delta^n)^\flat$. Per Lemma 3.2.2, we have that $\Phi_{\mathbb{O}^n}((\Delta^n)^\flat) \cong \mathfrak{D}^n$. To determine the value of the straightening functor, note that formula (4-1) yields the Set^+_Δ–enriched category

$$\mathbb{O}^n((\Delta^n)^\flat) = \mathbb{O}^n \amalg_{\mathfrak{C}^{\mathrm{sc}}[\Delta^n_\flat \times \{0\}]} \mathfrak{C}^{\mathrm{sc}}[\Delta^n_\flat \times \Delta^1] \amalg_{\mathfrak{C}^{\mathrm{sc}}[\Delta^n_\flat \times \{1\}]} \mathfrak{C}^{\mathrm{sc}}[\{*\}].$$

For $i, j \in \mathbb{O}^n$, the marked simplicial set

$$\mathrm{Map}_{\mathfrak{C}^{\mathrm{sc}}[\Delta^n_\flat \times \Delta^1]}((i,0),(j,1))$$

can be identified with the nerve of the poset $P(i,j)$ of chains in the poset $[n]\times[1]$ between $(i,0)$ and $(j,1)$, ordered by refinement. We define the map

$$P(i,j)^{\mathrm{op}} \to D^{[i,n]}, \quad C \mapsto C_0 \cup \min(C_1),$$

where we set $C_i := C \cap ([n] \times \{i\})$. Passing to nerves, one verifies that this map factors to define a map

$$\mathrm{Map}_{\mathbb{O}^n((\Delta^n)^\flat)}(i, *)^{\mathrm{op}} \to \mathcal{D}^{[i,n]} \tag{4-4}$$

natural in i. This yields the desired map

$$\mathrm{St}_{\mathbb{O}^n}(\Delta^n_\flat) \to \Phi_{\mathbb{O}^n}(\Delta^n_\flat). \tag{4-5}$$

(2) $\mathbb{C} = \mathbb{O}^1$ and $X = (\Delta^1)^\sharp$. On underlying unmarked simplicial sets, we may use the maps (4-4) to define the desired map

$$\mathrm{St}_{\mathbb{O}^1}((\Delta^1)^\sharp) \to \Phi_{\mathbb{O}^1}((\Delta^1)^\sharp). \tag{4-6}$$

We then conclude by observing that the markings are compatible as well.

4.3 Comparison for simplicial sets

We now extend the comparison maps (4-3) from simplices to simplicial sets following a standard technique from [5].

Proposition 4.3.1 *There exists a unique family of natural transformations*

$$\eta_{\mathbb{C}}(X)\colon \mathrm{St}_{\mathbb{C}}(X) \to \Phi_{\mathbb{C}}(X) \tag{4-7}$$

indexed by pairs $(\mathbb{C}, X)$, *where* $\mathbb{C}$ *is a* 2*–category and* $X \in (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}$ *with the following properties*:

(1) *For every map* $g\colon X \to Y$ *of marked simplicial sets over* $\mathrm{N}^{\mathrm{sc}}(\mathbb{C})$, *the diagram*

$$\begin{array}{ccc} \mathrm{St}_{\mathbb{C}}(X) & \xrightarrow{\eta_{\mathbb{C}}(X)} & \Phi_{\mathbb{C}}(X) \\ {\scriptstyle \mathrm{St}_{\mathbb{C}}(g)}\downarrow & & \downarrow{\scriptstyle \Phi_{\mathbb{C}}(g)} \\ \mathrm{St}_{\mathbb{C}}(Y) & \xrightarrow{\eta_{\mathbb{C}}(Y)} & \Phi_{\mathbb{C}}(Y) \end{array}$$

commutes.

(2) *For every* 2*–functor* $f\colon \mathbb{C} \to \mathbb{D}$, *the diagram*

$$\begin{array}{ccc} f_! \,\mathrm{St}_{\mathbb{C}}(X) & \xrightarrow{f_! \circ \eta_{\mathbb{C}}(X)} & f_! \Phi_{\mathbb{C}}(X) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{St}_{\mathbb{D}}(\mathrm{N}^{\mathrm{sc}}(f)_! X) & \xrightarrow{\eta_{\mathbb{C}}(\mathrm{N}^{\mathrm{sc}}(f)_! X)} & \Phi_{\mathbb{D}}(\mathrm{N}^{\mathrm{sc}}(f)_! X) \end{array}$$

commutes.

(3) *For* $\mathbb{C} = \mathbb{O}^n$ *and* $X = (\Delta^n)^\flat$, *the map* $\eta_{\mathbb{C}}(X)$ *coincides with the map* (4-5).

(4) *For* $\mathbb{C} = \mathbb{O}^1$ *and* $X = (\Delta^1)^\sharp$, *the map* $\eta_{\mathbb{C}}(X)$ *coincides with the map* (4-6).

The proof of Proposition 4.3.1 is a routine application of arguments like those of [5, Remark 3.2.5.10]. One first uses base change to compute the value of $\eta_{\mathbb{O}^n}$ on any simplex of $\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)$, and then shows naturality on the full subcategory on those objects. The remainder of the argument is completely parallel to loc. cit.

Proposition 4.3.2 *For every* 2*–category* $\mathbb{C}$ *and every* $X \in (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}$, *the map*

$$\eta_{\mathbb{C}}(X)\colon \mathrm{St}_{\mathbb{C}}(X) \to \Phi_{\mathbb{C}}(X)$$

from (4-7) *is a weak equivalence with respect to the projective model structure.*

Proof We first show that $\eta_{\mathbb{C}}$ is a weak equivalence on objects of the form

$$(\Delta^n)^\flat \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C}), \quad n \leq 1, \qquad \text{and} \qquad (\Delta^n)^\sharp \to \mathrm{N}^{\mathrm{sc}}(\mathbb{C}), \quad n \leq 1.$$

The case $n = 0$ is trivial. For $n = 1$, since left Kan extension preserves weak equivalences between cofibrant objects, we reduce to the case of the identity map $\mathrm{id}\colon (\Delta^1)^\flat \to \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^1)$. Then we have

$$\mathrm{St}_{\mathbb{O}^1}((\Delta^1)^\flat)(1) \cong \Phi_{\mathbb{O}^1}((\Delta^1)^\flat)(1) \cong \Delta^0.$$

Similarly, one observes that

$$\mathrm{St}_{\mathbb{O}^1}((\Delta^1)^\flat)(0) \cong (\Lambda^2_2)^\dagger,$$

where $(\Lambda^2_2)^\dagger$ denotes the right horn with degenerate edges and $\{1, 2\}$ marked, and the map to $\mathcal{D}^{[0,1]}$ is given by collapsing the marked edge, hence a marked equivalence. The case of $(\Delta^1)^\sharp \to \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^1)$ follows analogously, keeping track of the new marked edges. The claim now follows from Proposition 3.0.2 and Lemma 4.3.3 below. □

Lemma 4.3.3 *Let* $\overline{K} = (K, K^{\mathrm{t}})$ *be a scaled simplicial set. Suppose we are given two left adjoint functors,*

$$L_1, L_2\colon (\mathrm{Set}^+_\Delta)_{/\overline{K}} \to C,$$

where C *is a left proper combinatorial model category and* L_1 *is a left Quillen functor. Suppose further that* L_2 *preserves cofibrations and that it maps the morphisms*

$$(\Lambda^n_i)^\flat \to (\Delta^n)^\flat, \quad n \geq 2,\ 0 < i < n, \tag{4-8}$$

to weak equivalences. Given a natural transformation $\eta\colon L_1 \Rightarrow L_2$ which is a weak equivalence on objects of the form

$$(\Delta^n)^\flat \to K, \quad n \leq 1, \qquad \textit{and} \qquad (\Delta^n)^\sharp \to K, \quad n \leq 1.$$

Then η is a levelwise weak equivalence.

Proof Every marked simplicial set over X can be expressed as a filtered colimit of its skeleta. Since C is a combinatorial model category, weak equivalences are stable under filtered colimits. This shows that we can reduce to the case of objects

$$X \to K, \quad X, \text{ finite-dimensional.}$$

In addition we know that every simplicial set can be expressed as a filtered colimit of simplicial sets containing only finitely many nondegenerate simplices, allowing us to further reduce to the latter case. Recall that given a pushout diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 \end{array}$$

in $(\mathrm{Set}^+_\Delta)_{/\overline{K}}$ with one of the morphisms $X_1 \to Y_1$ or $X_1 \to X_2$ a cofibration, then, since L_1 and L_2 preserve cofibrations and C is left proper, it follows that, if the components of the natural transformation associated to X_1, X_2 and Y_1 are weak equivalences, then so is the component at Y_2.

Any marked simplicial set X with finitely many nondegenerate simplices can by obtained from $X^\flat$ by a finite sequence of pushouts along the monomorphism $(\Delta^1)^\flat \to (\Delta^1)^\sharp$. This implies that we can reduce to the case of minimally marked simplicial sets.

We claim that if the natural transformation is a weak equivalence at $(\Delta^n)^\flat$ for $n \geq 0$, then the result follows in general. In order to see this, we proceed by induction on the dimension of our simplicial sets. The case $n = 0$ is clearly true. We prove now the case n, provided that the result holds for $n-1$. Since we can assume that our simplicial set X has finitely many nondegenerate simplices, we can express it as an iterated pushout of $\mathrm{sk}_{n-1}(X)$ along the cofibration $(\partial\Delta^n)^\flat \to (\Delta^n)^\flat$, and the claim holds.

We claim that the natural transformation is a weak equivalence for $(\Delta^n)^\flat$ with $n \geq 0$. We proceed by induction. The claim holds for $n \leq 1$ by hypothesis. Suppose that the claim holds for $n-1$. Then an iterated pushout along cofibrations shows that

the component of the natural transformation at $(\Lambda^n_i)^\flat$ is a weak equivalence for every $0 \leq i \leq n$. Observe that in the model structure associated to $\overline{K}$, minimally marked inner horn inclusions are trivial cofibrations. It follows that the image of those maps under L_1 is a trivial cofibration, and by assumption the same is true of L_2. Therefore the result follows from commutativity of the diagram

$$\begin{array}{ccc} L_1((\Lambda^n_i)^\flat) & \longrightarrow & L_1((\Delta^n)^\flat) \\ \downarrow & & \downarrow \\ L_2((\Lambda^n_i)^\flat) & \longrightarrow & L_2((\Delta^n)^\flat) \end{array}$$

and 2-out-of-3. □

Remark 4.3.4 Under the assumption that L_2 is also a left Quillen functor, preservation of trivial cofibrations implies that the morphisms (4-8) are mapped to weak equivalences. Consequently, Lemma 4.3.3 yields a criterion allowing us to quickly check when two left Quillen functors are equivalent. We will reapply the lemma in this situation when proving Theorem 4.4.1 below.

Corollary 4.3.5 *Let $\mathbb{C}$ be a 2–category. Then the adjunction*

$$\Phi_{\mathbb{C}} \colon (\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})} \rightleftarrows \mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta) : \mathcal{X}_{\mathbb{C}}$$

is a Quillen equivalence.

Proof By Proposition 3.0.1, $\mathcal{X}_{\mathbb{C}}$ preserves trivial fibrations, so that $\Phi_{\mathbb{C}}$ preserves cofibrations. As a left Quillen functor, the functor $\mathrm{St}_{\mathbb{C}}$ preserves weak equivalences. By 2-out-of-3, we deduce from Proposition 4.3.2 that $\Phi_{\mathbb{C}}$ preserves weak equivalences as well. In particular, $\Phi_{\mathbb{C}}$ preserves trivial cofibrations, so that $\Phi_{\mathbb{C}}$ is a left Quillen functor. The fact that $\Phi_{\mathbb{C}}$ is in fact a left Quillen equivalence also follows immediately from Proposition 4.3.2, since $\mathrm{St}_{\mathbb{C}}$ is a left Quillen equivalence by [6, Theorem 3.8.1]. □

It is immediate that $\mathcal{X}_{\mathbb{C}}$ defines an equivalence between the ∞–categories associated to the model categories $\mathrm{Fun}_{\mathrm{Set}^+_\Delta}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}^+_\Delta)$ and $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})}$, respectively, by means of ∞–categorical localization.

Since both model categories are in fact simplicial model categories, the ∞–categories thus obtained can be described explicitly as the coherent nerves of the simplicial subcategories of fibrant–cofibrant objects. In this context, it becomes desirable to lift the Quillen equivalence to a simplicially enriched Quillen equivalence so as to

obtain a description of the equivalences induced by $\mathbb{X}_{\mathbb{C}}$ and $\Phi_{\mathbb{C}}$ via coherent nerves. Unfortunately, this is not possible. However, we at least have the following result:

Proposition 4.3.6 *The functor $\mathbb{X}_{\mathbb{C}}$ can be naturally extended to a simplicially enriched functor $\widetilde{\mathbb{X}_{\mathbb{C}}}$. In particular, we obtain the explicit description*

$$\mathrm{N}(\widetilde{\mathbb{X}_{\mathbb{C}}})\colon \mathrm{N}(\mathrm{Fun}_{\mathrm{Set}_\Delta^+}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}_\Delta^+)^\circ) \to \mathrm{N}((\mathrm{Set}_\Delta^+)^\circ_{/\mathrm{N}^{\mathrm{sc}}(\mathbb{C})})$$

of the equivalence of ∞–categories induced by $\mathbb{X}_{\mathbb{C}}$.

Proof We show that the simplicial enrichment $\widetilde{\mathbb{X}_{\mathbb{C}}}$ exists. Since both model categories are simplicial model categories, the remaining statement follows from [5, Remark A.3.1.9 and Proposition A.3.10] and Corollary 4.3.5. Recall that the simplicial enrichment of the category

$$\mathrm{Fun}_{\mathrm{Set}_\Delta^+}(\mathbb{C}^{(\mathrm{op},\mathrm{op})}, \mathrm{Set}_\Delta^+)$$

is adjoint to the tensor structure given by the formula

$$(K, F) \mapsto K^\sharp \times F,$$

where the Cartesian product is taken pointwise. We then provide a map of simplicial sets

$$\widetilde{\mathbb{X}_{\mathbb{C}}}\colon \mathrm{Map}(F, G) \to \mathrm{Map}(\mathbb{X}_{\mathbb{C}}(F), \mathbb{X}_{\mathbb{C}}(G))$$

as follows: Given an n–simplex $\sigma\colon \Delta^n \to \mathrm{Map}(F, G)$, we apply $\mathbb{X}_{\mathbb{C}}$ to the adjoint map

$$(\Delta^n)^\sharp \times F \to G$$

to obtain

$$\mathbb{X}_*((\Delta^n)^\sharp) \times \mathbb{X}_{\mathbb{C}}(F) \mapsto \mathbb{X}_{\mathbb{C}}(G),$$

where $\mathbb{X}_*$ denotes the 2–nerve relative to the final 2–category $*$. We then precompose with the map $(\Delta^n)^\sharp \to \mathbb{X}_*((\Delta^n)^\sharp)$ induced by pullback along the morphisms p_I of posets from (4-9) below to obtain a map

$$(\Delta^n)^\sharp \times \mathbb{X}_{\mathbb{C}}(F) \mapsto \mathbb{X}_{\mathbb{C}}(G)$$

and finally define $\widetilde{\mathbb{X}_{\mathbb{C}}}(\sigma)$ to be the n–simplex adjoint to this latter map. This construction defines a map $\widetilde{\mathbb{X}_{\mathbb{C}}}$ of simplicial sets and provides the desired Set_Δ–enrichment of $\mathbb{X}_{\mathbb{C}}$. □

4.4 Comparison of the 1–categorical and 2–categorical relative nerves

We conclude with a comparison between the relative 1–nerve and the relative 2–nerve in the case when $\mathbb{C} = C$ is a 1–category. To relate the two nerves, consider, for an ordinal $I \subset [n]$, the map of posets

$$p_I \colon D^I \to I, \quad S \mapsto \max(S), \tag{4-9}$$

and denote its nerve by π_I. Given an n–simplex of $\chi(F)$, comprising a collection of compatible maps

$$\Delta^I \mapsto F(\sigma(\min(I)))$$

parametrized by all nonempty subsets $I \subset [n]$, we produce, by pulling back along the various maps π_I, a collection of maps

$$\mathcal{D}^I \mapsto F(\sigma(\min(I))).$$

Unravelling the definitions, using that C is a 1–category, it follows that this collection is in fact compatible and defines an n–simplex of $\chi(F)$. Compatibility with the simplicial structures yields a comparison map

$$\pi^* \colon \chi_C(F) \to \mathbb{X}_C(F) \tag{4-10}$$

of marked simplicial sets over $\mathrm{N}(C)$, natural in F.

Theorem 4.4.1 *Let C be a 1–category. Then, for every functor $F \colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$, the morphism*

$$\pi^* \colon \chi_C(F) \to \mathbb{X}_C(F)$$

from (4-10) *is an equivalence of Cartesian fibrations over* $\mathrm{N}(C)$.

Proof Since both $\mathbb{X}_C(F)$ and $\chi_C(F)$ are Cartesian fibrations over $\mathrm{N}(C)$, it suffices to show that the morphism π^* induces a marked equivalence of each fibre. By base change, we thus reduce to the case when C is the final 1–category. In this situation, the relative 1–nerve is part of the Quillen equivalence

$$\phi_* \colon \mathrm{Set}_\Delta^+ \rightleftarrows \mathrm{Set}_\Delta^+ \colon \chi_*,$$

where both ϕ and χ are the identity functors. The relative 2–nerve is the right Quillen functor of the Quillen equivalence

$$\Phi_* \colon \mathrm{Set}_\Delta^+ \rightleftarrows \mathrm{Set}_\Delta^+ \colon \mathbb{X}_*$$

from Corollary 4.3.5. To show that the natural transformation $\pi^*\colon \chi_* \Rightarrow \mathbb{X}_*$ is a weak equivalence on fibrant objects, it suffices to show that the adjoint transformation $\pi_!\colon \Phi_* \Rightarrow \phi_*$ is a weak equivalence. By Lemma 4.3.3, it suffices to verify that $\pi_!$ is a weak equivalence on the marked simplicial sets $(\Delta^0)^\flat$, $(\Delta^1)^\flat$ and $(\Delta^1)^\sharp$. In these cases, $\pi_!$ induces an isomorphism of marked simplicial sets, concluding the argument. □

List of symbols

For ease of reading, we provide a (noncomprehensive) list of notation appearing in this paper, as well as references to the relevant pages.

$A(J)$, $\mathcal{A}(J)$	Subposets of D^I corresponding to faces of simplices	3162
D^I, $\mathcal{D}^I$	Poset used in defining the $(\infty,2)$ relative nerve	3158
$P^n(S,T)$, $\mathcal{P}^n(S,T)$	Poset computing the homotopy type of mapping spaces in $\mathcal{D}^n$	3165
$\mathcal{L}^n_i$	Simplicial subset of $\mathcal{D}^n$ obtained from the horn Λ^n_i under $\Phi_{\mathbb{O}^n}$	3167
$\mathbb{O}^I$	The free 2–category on the I–simplex	3158
$\mathrm{St}_{\mathbb{C}}$	Scaled straightening functor	3157
$\mathbb{X}_{\mathbb{C}}$	The relative 2–nerve	3159
χ_C	The relative 1–nerve	3148
$\eta_{\mathbb{C}}$	Comparison map between scaled straightening and the left adjoint to the relative nerve	3174
$\mathfrak{D}^n$	The image of $(\Delta^n)^\flat \to \Delta^n$ under $\Phi_{\mathbb{O}^n}$	3163
$\mathfrak{L}^n_i$	The image of $(\Lambda^n_i)^\flat \to \Delta^n$ under $\Phi_{\mathbb{O}^n}$	3163
$\rho_{I,J}$	Pullback functors, which relate $\mathcal{D}^I$ to $\mathcal{D}^J$	3159
p_I, π_I	Functors comparing $\mathcal{D}^I$ to Δ^I	3180
x	Embedding of $\mathcal{D}^n$ into the lattice $\mathbb{Z}^{n-1}$	3166

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