

# Geodesic stability, the space of rays and uniform convexity in Mabuchi geometry

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We establish the essentially optimal form of Donaldson’s geodesic stability conjecture regarding existence of constant scalar curvature Kähler metrics. We carry this out by exploring in detail the metric geometry of Mabuchi geodesic rays, and the uniform convexity properties of the space of Kähler metrics.

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## 1 Introduction

In this paper we prove the essentially optimal form of S Donaldson’s geodesic stability conjecture [51]. The main result is obtained via a detailed analysis of the rays associated to the space of Kähler metrics.

Suppose  $(X, \omega)$  is a compact Kähler manifold with  $\dim X = n$ . We consider  $\mathcal{H}$ , the space of Kähler metrics cohomologous to  $\omega$ , with its  $L^p$ -type Mabuchi metric structures  $(\mathcal{H}, d_p)$  for  $p \geq 1$ ; see Darvas [32]. For simplicity, to describe our motivation, let us momentarily assume that  $X$  has no nontrivial holomorphic vector fields. In the recent breakthrough papers [25; 26; 27], Chen and Cheng provided the first existence theorems of constant scalar curvature Kähler (cscK) metrics inside the class  $\mathcal{H}$ . Such metrics

are minimizers of Mabuchi's K-energy functional  $\mathcal{K}: \mathcal{H} \rightarrow \mathbb{R}$  [67]. Together with Berman, Darvas and Lu [9], the Chen–Cheng results provided a full characterization of existence of cscK metrics in terms of  $d_1$ -properness of  $\mathcal{K}$ . As  $d_1$ -properness is actually equivalent with properness in terms of Aubin's  $J$ -functional [32], this also verified an old conjecture of Tian [77; 79, Conjecture 7.12], with the precise statement appearing in Darvas and Rubinstein [44, Conjecture 2.8].

Energy properness is the strongest form of stability. Contrasting this is uniform K-stability, one of the weakest such conditions. When the Kähler structure is induced by an ample line bundle, this criterion was first considered by G Székelyhidi [74], and was further studied by Dervan [47], Berman, Boucksom and Jonsson [7], Boucksom, Hisamoto and Jonsson [18; 17] and many others. The ultimate hope is that (uniform) K-stability is weak enough to be verified using computational techniques of algebraic geometry, this being the main motivation behind the Yau–Tian–Donaldson (YTD) conjecture, seeking to show that some form of K-stability is equivalent with existence of cscK metrics.

In this paper we focus on Donaldson's geodesic stability conjecture [51, Conjecture 12], lying between energy properness and uniform K-stability (see Conjecture 1.7 below). This conjecture predicts that it is enough to check properness of the K-energy along the geodesic rays of  $\mathcal{H}$  to ensure existence of cscK metrics. Initially, the predictions of Donaldson advocated for the use of smooth geodesic rays [51]. As we know now, the typical regularity of geodesics is merely  $C^{1,1}$  — see Chen [24], Błocki [11], Darvas and Lempert [42] and Chu, Tosatti and Weinkove [30] — even when connecting smooth endpoints. Hence, the present expectation is that (in its optimal form) Donaldson's geodesic stability conjecture should hold for rays that have at most two bounded derivatives. In Theorem 1.8 we essentially verify this form of the conjecture.

To carry out our plan, we first explore in depth the metric geometry of  $L^p$  geodesic rays (ie rays running inside the  $d_p$ -completions of  $\mathcal{H}$ ), a topic of independent interest. To do this, perhaps surprisingly, we need to understand uniform convexity of the  $L^p$  Mabuchi geometry when  $p > 1$ , extending work of Calabi and Chen in the particular case  $p = 2$  [22]. After exploring the metric space of  $L^p$  geodesic rays, we show that such rays can always be approximated via rays of  $C^{1,1}$  potentials, *with* converging radial K-energy. With slightly different formulation, the uniform  $L^1$  geodesic stability conjecture was verified in [26; 27], pointing out that it is enough to test energy properness along  $L^1$  geodesic rays to guarantee existence of cscK metrics. This result, together with our approximation theorems just mentioned, will yield the

geodesic stability theorem for rays of  $C^{1,\bar{1}}$  potentials, ie potentials with bounded complex Hessian.

In addition to the above, our results resolve a number of related open questions in Kähler geometry, specified in the paragraphs below. Also, in the particular case when the Kähler structure is induced by an ample line bundle, our theorems also make connection with the variational program designed to attack the uniform YTD conjecture (see Boucksom [13] and Chen and Cheng [27]). Roughly speaking, to verify the uniform YTD conjecture using our results, it is now enough to show that specific  $C^{1,\bar{1}}$  geodesic rays can be approximated by geodesic rays induced by the so-called test configurations of algebraic geometry; see Tian [78] and Donaldson [52] (with converging radial K-energy). On the surface this sounds simpler than approximating  $L^1$  metric geodesic rays [13, page 2], and time will tell what role our results will play in the solution of this problem.

**Uniform convexity and uniqueness of geodesic segments** By  $\mathcal{H}_\omega$  we denote the space of Kähler potentials associated to  $\mathcal{H}$ . The metric completions of  $(\mathcal{H}_\omega, d_p)$  are  $(\mathcal{E}_\omega^p, d_p)$ , and the latter spaces are complete geodesic metric spaces for any  $p \geq 1$  [32]. The distinguished  $d_p$ -geodesics running between the points of  $\mathcal{E}_\omega^p$  are called  $L^p$  finite-energy geodesics (or simply finite-energy geodesics, or  $L^p$  geodesics, if no confusion arises). These curves arise as limits of solutions to degenerate equations of complex Monge–Ampère type. We recall the basic properties of these spaces in Section 2.1.

For any  $p \in [1, \infty)$  it was shown in [27, Theorem 1.5] that the metrics  $d_p$  are “convex”: if  $[0, 1] \ni t \rightarrow u_t, v_t \in \mathcal{E}^p$  are two finite-energy geodesic segments then

$$(1) \quad d_p(u_\lambda, v_\lambda) \leq (1 - \lambda)d_p(u_0, v_0) + \lambda d_p(u_1, v_1), \quad \lambda \in [0, 1].$$

This property is called Busemann convexity in the metric geometry literature; see Jost [60, Section 2.2], going back to Busemann [21]. In the particular case  $p = 1$ , (1) was established in Berman, Darvas and Lu [8, Proposition 5.1], having applications to the convergence of the weak Calabi flow. When  $p = 2$ , (1) follows from the fact that  $(\mathcal{E}_\omega^2, d_2)$  is a complete CAT(0) metric space, as shown in Darvas [33, Theorem 1], building on estimates of [22, Theorem 1.1].

The CAT(0) property consists of the following estimate: if  $u \in \mathcal{E}_\omega^2$  and  $[0, 1] \ni t \rightarrow v_t \in \mathcal{E}_\omega^2$  is a finite-energy geodesic segment then

$$(2) \quad d_2(u, v_\lambda)^2 \leq (1 - \lambda)d_2(u, v_0)^2 + \lambda d_2(u, v_1)^2 - \lambda(1 - \lambda)d_2(v_0, v_1)^2, \quad \lambda \in [0, 1].$$

As is well known, (2) implies (1) [60, Proposition 2.3.2]. Unfortunately, there is very strong evidence that (2) cannot hold for the  $d_p$  metrics when  $p \neq 2$ . Indeed, when restricting to a toric Kähler manifold and toric Kähler metrics, the spaces  $(\mathcal{E}_\omega^p, d_p)$  are isometric to the flat  $L^p$  metric spaces of convex functions defined on a convex polytope of  $\mathbb{R}^n$ ; see Di Nezza and Guedj [49, Section 6]. It is well known however that CAT(0) Banach spaces are in fact Hilbert spaces — see Bridson and Haefliger [20] — evidencing that only  $(\mathcal{E}^2, d_2)$  can be CAT(0).

Despite this, in the first main result of this paper we show that adequate generalizations of the CAT(0) inequality (2) do hold for the  $d_p$  metrics, when  $p > 1$ . These can be viewed as the Kähler analogs of classical inequalities of Clarkson [31] and Ball, Carlen and Lieb [3], regarding the uniform convexity of  $L^p$  spaces. Consequently, the metric spaces  $(\mathcal{E}_\omega^p, d_p)$  are *uniformly convex* for  $p > 1$ , giving them extra structure that will be explored in the latter parts of the paper:

**Theorem 1.1** *Let  $p \in (1, \infty)$ . Suppose that  $u \in \mathcal{E}_\omega^p$ ,  $\lambda \in [0, 1]$  and  $[0, 1] \ni t \rightarrow v_t \in \mathcal{E}_\omega^p$  is a finite-energy geodesic segment. Then the following hold:*

- (i)  $d_p(u, v_\lambda)^2 \leq (1 - \lambda)d_p(u, v_0)^2 + \lambda d_p(u, v_1)^2 - (p - 1)\lambda(1 - \lambda)d_p(v_0, v_1)^2$  if  $1 < p \leq 2$ .
- (ii)  $d_p(u, v_\lambda)^p \leq (1 - \lambda)d_p(u, v_0)^p + \lambda d_p(u, v_1)^p - \lambda^{p/2}(1 - \lambda)^{p/2}d_p(v_0, v_1)^p$  if  $2 \leq p$ .

In the particular case  $p = 2$  this result recovers the inequalities of Calabi and Chen [22], however our proof of Theorem 1.1 is very different from the argument in [22], as the differentiation of  $d_p$  metrics is problematic for  $p \neq 2$ .

It was pointed out in the comments following [32, Theorem 4.17] that  $d_1$ -geodesic segments connecting the different points of  $(\mathcal{E}_\omega^1, d_1)$  are not unique. However, as a consequence of the above result it follows that uniqueness of  $d_p$ -geodesic segments does hold when  $p > 1$ :

**Theorem 1.2** *Let  $p \in (1, \infty)$ , and suppose that  $[0, 1] \ni t \rightarrow v_t \in \mathcal{E}_\omega^p$  is the  $L^p$  finite-energy geodesic connecting  $v_0, v_1 \in \mathcal{E}_\omega^p$ . Then  $t \rightarrow v_t$  is the only  $d_p$ -geodesic connecting  $v_0$  and  $v_1$ , ie  $(\mathcal{E}_\omega^p, d_p)$  is a uniquely geodesic metric space.*

**The metric geometry of geodesic rays** Next we explore the metric geometry of  $\mathcal{R}_u^p$ , the space of finite-energy  $L^p$  geodesic rays emanating from a fixed potential  $u \in \mathcal{E}_\omega^p$ . As a convention, given  $p \in [1, \infty)$ , a finite-energy geodesic ray  $[0, \infty) \ni t \rightarrow u_t \in \mathcal{E}_\omega^p$  with  $u_0 = u$  will be simply denoted by  $\{u_t\}_t \in \mathcal{R}_u^p$ .

In accordance with the metric space literature, two  $d_p$ -rays  $[0, \infty) \ni t \rightarrow u_t, v_t \in \mathcal{E}_\omega^p$  are parallel/asymptotic if  $d_p(u_t, v_t)$  is uniformly bounded for  $t \geq 0$  [20, Chapter II.8]. To start, we point out in Proposition 4.1 that for any  $v \in \mathcal{E}_\omega^p$  and  $\{u_t\}_t \in \mathcal{R}_u^p$  it is possible to find a unique  $\{v_t\}_t \in \mathcal{R}_v^p$  such that  $\{u_t\}_t$  and  $\{v_t\}_t$  are parallel. Consequently, the  $d_p$ -geometries verify Euclid’s 5<sup>th</sup> postulate for half-lines, answering an open question of Chen and Cheng [27, Remark 1.6], who proved this for  $p = 1$  under restrictive conditions on the slope of the K-energy along  $\{u_t\}_t$ . Thus, we can introduce a natural parallelism operator  $\mathcal{P}_{uv}: \mathcal{R}_u^p \rightarrow \mathcal{R}_v^p$  for any  $u, v \in \mathcal{E}_\omega^p$ . Moreover it is possible to introduce natural metric structures on  $\mathcal{R}_u^p$  and  $\mathcal{R}_v^p$  making this map an isometry:

**Theorem 1.3** *Let  $p \in [1, \infty)$ . For any  $u \in \mathcal{E}_\omega^p$ ,  $(\mathcal{R}_u^p, d_{u,p}^c)$  is a complete metric space. For any  $v \in \mathcal{E}_\omega^p$  the parallelism operator  $\mathcal{P}_{uv}: (\mathcal{R}_u^p, d_{u,p}^c) \rightarrow (\mathcal{R}_v^p, d_{v,p}^c)$  is an isometry.*

In this result, the  $d_{u,p}^c$  metric is called the *chordal  $L^p$  metric* between two rays, defined by the expression

$$(3) \quad d_{u,p}^c(\{u_t\}_t, \{v_t\}_t) := \lim_{t \rightarrow \infty} \frac{d_p(u_t, v_t)}{t}, \quad \{u_t\}_t \in \mathcal{R}_u^p, \{v_t\}_t \in \mathcal{R}_u^p.$$

That this limit exists and is finite follows from (1). Though not necessarily treated as a metric in other works, Chen and Cheng [27, Corollary 5.6] and Boucksom [13, Formula 1.2] also consider the expression on the right-hand side of (3), in the slightly restrictive cases of unit-speed geodesic rays and non-Archimedean metrics, respectively (see also [9, Lemma 3.1]). Moreover, one would think that the metrics of the graded filtrations defined in Boucksom and Jonsson [19, Section 3] should be related to the above concept as well.

It was pointed out recently that  $L^1$  Mabuchi geometry can be defined for big classes as well; see Darvas, Di Nezza and Lu [36]. Using this, it is possible to introduce the metric space of weak  $L^1$  rays in the big context (see Darvas, Di Nezza and Lu [40], where we embed singularity types into the space of  $L^1$  rays).

By the last part of the above theorem, there is no new information gained by considering different starting points for rays, hence it makes sense to restrict attention to the space  $(\mathcal{R}_\omega^p, d_p^c)$ , representing the space of rays emanating from  $0 \in \mathcal{H}_\omega$ . The above theorem points out that  $d_p^c$  thus defined gives a complete metric on the space of *all*  $L^p$  rays emanating from a fixed starting point, which includes the constant ray. In our next main result we point out that the resulting metric spaces have rich geometry:

**Theorem 1.4**  *$(\mathcal{R}_\omega^p, d_p^c)$  is a geodesic metric space for any  $p \in [1, \infty)$ . Additionally, the radial K-energy is convex along  $d_p^c$ -geodesic segments.*

The *radial K–energy* is defined for any  $\{u_t\}_t \in \mathcal{R}_\omega^p$ , and is given by the expression

$$\mathcal{K}\{u_t\} := \lim_{t \rightarrow \infty} \frac{\mathcal{K}(u_t)}{t},$$

where  $\mathcal{K}: \mathcal{E}_\omega^p \rightarrow (-\infty, \infty]$  is the extended K–energy of Mabuchi from Berman, Boucksom, Eyssidieux, Guedj and Zeriahi [5] and Berman, Darvas and Lu [8]. The radial K–energy is  $d_p^c$ –lsc, possibly equal to  $\infty$ , and in the setting of unit-speed geodesics, its definition agrees with the  $\mathbb{Y}$  invariant of [27]. Also, there is a clear parallel with the non-Archimedean K–energy (see [13]).

This theorem represents the radial version of [32, Theorem 2] and [8, Theorem 1.2] (building on Berman and Berndtsson [4]). In slight contrast with previous speculations in the literature (see for example [17] or [27, Definition 1.8]) it seems more natural to consider the space of *all*  $d_p$ –rays, not just the ones that have  $d_p$ –unit speed. Allowing for a bigger class of rays makes possible the construction of  $d_p^c$ –geodesic segments running between any two points of  $\mathcal{R}_\omega^p$ , with good convexity properties. Moreover, the convexity of the radial K–energy on  $\mathcal{R}_\omega^p$  could potentially be used to set up the study of optimal degenerations as a convex optimization problem (see Dervan and Székelyhidi [48]).

The  $d_p^c$ –geodesic segments constructed in the proof of the above theorem are called  *$d_p^c$ –chords*, as they are reminiscent of the classical chords in the chordal geometry of the unit sphere of  $\mathbb{R}^n$  (at least when restricting to  $d_p$ –unit-speed rays). When  $p > 1$ , due to uniform convexity (Theorem 1.1), we will construct the  $d_p^c$ –chords directly. When  $p = 1$ , in the absence of uniform convexity, the construction of  $d_1^c$ –chords is done using an approximation procedure, via our next main theorem.

We have  $\mathcal{R}_\omega^p \subset \mathcal{R}_\omega^{p'}$  for any  $p' \leq p$ . More importantly, by the proof of Theorem 1.4,  $d_p^c$ –chords are automatically  $d_{p'}^c$ –chords as well, giving further evidence that it is more advantageous to consider the space of all rays, not just the ones with  $d_p$ –unit speed. This latter fact again represents the radial version of a well-known phenomenon for the family of metric spaces  $(\mathcal{E}_\omega^p, d_p)$  for  $p \geq 1$ , according to which geodesics are “shared” when comparing different classes. Though the space of  $d_p$ –unit-speed rays seems to exhibit a metric structure reminiscent of the Tits geometry attached to CAT(0) spaces [20], none of the above properties hold for these structures.

Next we turn to approximation. The collection of geodesic rays  $\{u_t\}_t \in \mathcal{R}_\omega^1$  with  $u_t \in L^\infty$  for  $t \geq 0$  will be denoted by  $\mathcal{R}_\omega^\infty$ , and will be referred to as the set of *geodesic rays with bounded potentials*. In addition to having bounded potentials, the

rays of  $\mathcal{R}_\omega^\infty$  are actually  $t$ -Lipschitz, and they solve the geodesic equation of  $L^p$  Mabuchi geometry in the weak Bedford–Taylor sense, as opposed to the rays of  $\mathcal{R}_\omega^p$  for  $p \in [1, \infty)$ , that are only limits of solutions to such equations (See Section 2.1).

By  $\mathcal{H}_\omega^{1,\bar{1}}$  we will denote the set of potentials in  $\text{PSH}(X, \omega)$  whose Laplacian (or whose complex Hessian) is bounded. Analogously, the collection of geodesic rays  $\{u_t\}_t \in \mathcal{R}_\omega^1$  with  $u_t \in \mathcal{H}_\omega^{1,\bar{1}}$  for  $t \geq 0$  will be denoted by  $\mathcal{R}_\omega^{1,\bar{1}}$ , and will be referred to as the set of *geodesic rays with  $C^{1,\bar{1}}$  potentials*. The space of rays with bounded Hessian, denoted by  $\mathcal{R}_\omega^{1,1}$ , is defined similarly.

The next result points out that  $\mathcal{R}_\omega^\infty$  is  $d_p^c$ -dense in  $\mathcal{R}_\omega^p$  for any  $p \in [1, \infty)$ . Also, we show that  $\mathcal{R}_\omega^{1,\bar{1}}$  dense among rays with finite radial K-energy. In both cases one can approximate with converging radial K-energy:

**Theorem 1.5** *Let  $\{u_t\}_t \in \mathcal{R}_\omega^p$  with  $p \in [1, \infty)$ . The following hold:*

- (i) *There exists a sequence  $\{u_t^j\}_t \in \mathcal{R}_\omega^\infty$  such that  $u_t^j \searrow u_t$  for  $t \geq 0$ ,*

$$d_p^c(\{u_t^j\}_t, \{u_t\}_t) \rightarrow 0$$

*and  $\mathcal{K}\{u_t^j\} \rightarrow \mathcal{K}\{u_t\}$ .*

- (ii) *If  $\mathcal{K}\{u_t\} < \infty$ , then there exists a sequence  $\{v_t^j\}_t \in \mathcal{R}_\omega^{1,\bar{1}}$  such that  $v_t^j \searrow u_t$  for  $t \geq 0$ ,  $d_p^c(\{v_t^j\}_t, \{u_t\}_t) \rightarrow 0$  and  $\mathcal{K}\{v_t^j\} \rightarrow \mathcal{K}\{u_t\}$ .*

It remains to be seen if the condition  $\mathcal{K}\{u_t\} < \infty$  can be omitted in (ii). This theorem can be seen as a radial analog of [8, Theorem 1.3], perhaps also making progress on the variational program designed to attack the uniform YTD conjecture (see step (4) in [13, page 2]; compare [19, Conjecture 2.5]). Time will tell exactly how our results will fit into this program, but now it is enough to show that some  $C^{1,\bar{1}}$  rays can be approximated by rays induced by test configurations (with converging K-energy) to prove the uniform YTD conjecture.

As a first step in obtaining Theorem 1.5(i), in Theorem 4.5 we show that one can approximate by bounded geodesic rays with possibly diverging radial K-energy. The argument uses Ross and Witt Nyström [71], and this will suffice in the case  $\mathcal{K}\{u_t\} = \infty$ , since  $\mathcal{K}\{\cdot\}$  is  $d_p^c$ -lsc. However, to obtain (i) when  $\mathcal{K}\{u_t\}$  is finite, a much more delicate construction will be needed, building on the relative Kołodziej-type estimate of Darvas, Di Nezza and Lu [39]. The proof of (ii) builds on (i), and novel a priori  $C^{1,\bar{1}}$  estimates along geodesic segments that are “scalable” along rays. These will be obtained using the framework of He [58] and Guedj and Zeriahi [57].

**Applications to geodesic stability** We point out applications to existence of constant scalar curvature Kähler (csck) metrics in terms of geodesic stability, going back to Donaldson’s early conjectures in [51].

To start, we say that  $(X, \omega)$  is *geodesically  $L^p$ - or  $C^{1,\bar{1}}$ -semistable* if for any  $\{u_t\}_t \in \mathcal{R}_\omega^p / \mathcal{R}_\omega^{1,\bar{1}}$  we have that  $\mathcal{K}\{u_t\} \geq 0$  for  $p \in [1, \infty]$ . Regarding the relevance of semi-stability for the csck continuity method, we refer to [27]. As an immediate consequence of Theorem 1.5 we obtain the following:

**Theorem 1.6**  $(X, \omega)$  is geodesically  $L^1$ -semistable if and only if it is geodesically  $C^{1,\bar{1}}$ -semistable.

Let  $G := \text{Aut}_0(X)$  be the identity component of the group of holomorphic automorphisms of  $X$ . By  $I: \mathcal{E}_\omega^1 \rightarrow \mathbb{R}$  we denote the Monge–Ampère energy functional (sometimes called Aubin–Yau or Aubin–Mabuchi energy). Then, as explained in [44],  $G$  induces an isometry on  $\mathcal{E}_0^1 = \mathcal{E}_\omega^1 \cap I^{-1}(0)$ , and one can introduce the following pseudometric on the orbits  $\mathcal{E}_0^1/G$ :

$$d_{1,G}(Gu_0, Gu_1) := \inf_{g \in G} d_1(u_0, g.u_1).$$

Moreover, one can analogously define the space of *normalized rays*  $\mathcal{R}^p / \mathcal{R}^{1,\bar{1}} / \mathcal{R}^{1,1}$  for  $p \in [1, \infty]$ , where we restrict to rays  $\{u_t\}_t \in \mathcal{R}_\omega^p / \mathcal{R}_\omega^{1,\bar{1}} / \mathcal{R}_\omega^{1,1}$  with  $I(u_t) = 0$  for  $t \geq 0$ .

By showing that minimizers of the K-energy on  $\mathcal{E}_\omega^1$  are actually smooth csck potentials [26, Theorem 1.5], Chen and Cheng have verified the last remaining condition of the existence/properness principle of [44], applied to the case of csck metrics. Together with the necessity result [9, Theorem 1.5], their theorem showed that existence of csck metrics is equivalent with properness of  $\mathcal{K}$  in the following sense: there exists  $\delta, \gamma > 0$  such that

$$(4) \quad \mathcal{K}(u) \geq \delta d_{1,G}(G0, Gu) - \gamma, \quad u \in \mathcal{E}_\omega^1.$$

Clearly,  $d_{1,G}(Gv_0, Gv_1) \leq d_1(v_0, v_1)$  for  $v_0, v_1 \in \mathcal{E}_\omega^1$ , and we say that  $\{u_t\}_t \in \mathcal{R}^1$  is  *$G$ -calibrated* if the curve  $t \rightarrow Gu_t$  is a  $d_{1,G}$ -geodesic with the same speed as  $\{u_t\}_t$ , ie

$$d_{1,G}(Gu_0, Gu_t) = d_1(u_0, u_t), \quad t \geq 0.$$

Geometrically,  $\{u_t\}_t$  is  $G$ -calibrated if it cuts each  $G$ -orbit inside  $\mathcal{E}_\omega^1$  “perpendicularly”. When  $G = \{\text{Id}\}$ , every ray is  $G$ -calibrated.

Building on these concepts, it is natural to state the  $C^{1,1}$  uniform analog to Donaldson’s geodesic stability conjecture, with the original formulation of [51, Conjecture 12] more closely related to the language of “polystability”:

**Conjecture 1.7** ( $C^{1,1}$  uniform geodesic stability) *Let  $(X, \omega)$  be a compact Kähler manifold. Then the following are equivalent:*

- (i) *There exists a cscK metric in  $\mathcal{H}$ .*
- (ii) *There exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta \limsup_t \frac{1}{t} d_{1,G}(G0, Gu_t)$  for all geodesic rays  $\{u_t\}_t \in \mathcal{R}^{1,1}$ .*
- (iii)  *$\mathcal{K}$  is  $G$ -invariant and there exists  $\delta > 0$  such that for all  $G$ -calibrated geodesic rays  $\{u_t\}_t \in \mathcal{R}^{1,1}$  we have that  $\mathcal{K}\{u_t\} \geq \delta d_1(0, u_1)$ .*

The statement of (ii) clearly points out that uniform geodesic stability is simply the condition that tests energy properness (expressed in (4)) along a class of geodesic rays. As the notion of  $G$ -calibrated rays has an obvious analog in the case of the space of finite-dimensional rays as well (within the context of Kähler quantization), we included this condition here to perhaps facilitate in the future an alternative definition for uniform K-stability in the presence of vector fields.

As explained in [44, Proposition 5.5], in the above conjecture the  $d_1$  distance is interchangeable with Aubin’s  $J$ -functional. Lastly, given that rays induced by 1-parameter actions of  $G$  are never  $G$ -calibrated, the condition that  $\mathcal{K}$  is  $G$ -invariant (equivalent to vanishing Futaki invariant [54]) is necessary in the statement of (iii).

Using our above theorems, we prove in Theorems 6.2 and 6.3 that the  $C^{1,\bar{1}}$  and  $L^1$  versions of the uniform geodesic stability conjecture are equivalent. As alluded to previously, the breakthrough of Chen and Cheng [26; 27] together with Darvas [35, Theorem 4.7] essentially yielded the  $L^1$  version of this conjecture (see Theorem 6.1 below). Putting things together, we arrive at our most important main result, essentially settling Conjecture 1.7:

**Theorem 1.8** ( $C^{1,\bar{1}}$  uniform geodesic stability) *Let  $(X, \omega)$  be a compact Kähler manifold. Then the following are equivalent:*

- (i) *There exists a cscK metric in  $\mathcal{H}$ .*
- (ii) *There exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta \limsup_t \frac{1}{t} d_{1,G}(G0, Gu_t)$  for all  $\{u_t\}_t \in \mathcal{R}^{1,\bar{1}}$ .*
- (iii)  *$\mathcal{K}$  is  $G$ -invariant and there exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta d_1(0, u_1)$  for all  $G$ -calibrated geodesic rays  $\{u_t\}_t \in \mathcal{R}^{1,\bar{1}}$ .*

Clearly, given the obvious inclusions among classes of geodesic rays, the  $L^p$  versions of Conjecture 1.7 follow as well (with  $\mathcal{R}^p$  replacing  $\mathcal{R}^{1,\bar{1}}$  in the statement). Though slightly different in formulation, the  $L^\infty$  version of this result essentially confirms the equivalences between the conditions (3), (4) and (5) in [27, Question 1.12] (see also the closely related questions of [26, Remark 1.3]). When  $G = \{\text{Id}\}$ , the statement of the theorem can be made especially simple:

**Theorem 1.9** *Let  $(X, \omega)$  be a compact Kähler manifold without nontrivial holomorphic vector fields. Then the following are equivalent:*

- (i) *There exists a cscK metric in  $\mathcal{H}$ .*
- (ii) *There exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta d_1(0, u_1)$  for all  $\{u_t\}_t \in \mathcal{R}^{1,\bar{1}}$ .*

It remains to be seen if in the above stability results one can use rays that have potentials with fully bounded Hessian, not just bounded complex Hessian. Even if possible, this small improvement seems to require substantial amount of new work. Further optimizations are extremely unlikely, given that the typical regularity of geodesics breaks down beyond  $C^2$  estimates. One would think that generalizations to the context of extremal and conical-type cscK metrics should be possible, using our results together with He [59] and Zheng [82].

**Connections with the literature** Uniform convexity of metric spaces is an active area of research (see Ohta [69], Kell [61], Kuwae [65] and Naor and Silberman [68]). In particular, by [65, Proposition 2.5] the inequalities of Theorem 1.1 are essentially optimal.

The notion of K-stability goes back to work of Tian [78], with generalizations and refinements made along the way by Donaldson [52], Li and Xu [66], Székelyhidi [74] and many others. Though the precise form of K-stability is still not fully clarified for general Kähler manifolds — see Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman [1] — at least in the absence of nontrivial holomorphic vector fields, it is widely expected that uniform K-stability will be equivalent with existence of cscK metrics (see [27, Question 1.12; 13, Conjecture 4.9]). Informally, uniform K-stability simply says that Conjecture 1.7 holds for  $C^{1,\bar{1}}$  rays that are induced by the so-called test configurations of  $(X, \omega)$ .

Closing the gap between  $L^1$  uniform geodesic stability and uniform K-stability is the last remaining step in the variational program designed to attack the uniform YTD conjecture (see [13, page 2]), with our Theorem 1.8 representing an intermediate step. To

facilitate further progress in this direction, based on the findings of Theorem 1.4, one possible approach would be to develop the radial analog of the Kähler quantization scheme, recently extended to the  $d_p$  metric completions in Darvas, Lu and Rubinstein [43] (building on prior work by Berndtsson [10], Chen and Sun [29], Donaldson [52; 53], Phong and Sturm [70], Song and Zelditch [73], Tian [76] and others). Indeed, when the Kähler structure  $(X, \omega)$  is induced by an ample Hermitian line bundle  $(L, h)$ , it is pointed out by Boucksom, Eriksson and Jonsson [14; 19; 13] that  $\mathcal{R}_\omega^k$ , the space of finite-dimensional geodesic rays associated to the space of Hermitian metrics  $\mathcal{H}_\omega^k$  on  $H^0(X, L^k)$ , admits a natural metric  $d_{p,k}^c$ , likely representing the finite-dimensional analog of our  $d_p^c$  metrics. If one could show in the spirit of [43, Theorem 1.1] that the metric spaces  $(\mathcal{R}_\omega^k, d_{p,k}^c)$  approximate  $(\mathcal{R}_\omega^p, d_p^c)$  (or relevant parts of it) in the large  $k$ -limit, then that would open the door for a version of Theorem 1.5, where the rays from  $\mathcal{R}_\omega^{1,\bar{1}}$  are replaced by rays induced by test configurations. Even if successful, it is not clear how convergence of the radial K-energy can be achieved (see [19, Conjecture 2.5]), and for the difficulties that need to be overcome in this approach we refer to the comments following [13, Conjecture 4.9].

Further connections with geodesic rays are explored in [40], related to the metric geometry of the space of singularity types, and complex Monge–Ampère equations with prescribed singularity.

**Organization of the paper** In Section 2 we recall basic facts about the  $L^p$  Mabuchi geometry of the space of Kähler metrics, the relative Kołodziej-type estimate of [39], and we prove weighted versions of the classical inequalities of Clarkson and Ball, Carlen and Lieb that will be needed later. In Section 3 we prove Theorems 1.1 and 1.2 regarding uniform convexity, and uniqueness of geodesics in  $L^p$  Mabuchi geometry when  $p > 1$ . In Section 4 we study the chordal  $L^p$  metric structures on the space of geodesic rays and prove Theorem 1.4. In Section 5 we prove Theorem 1.5, our main approximation result, and in Section 6 we show that the  $C^{1,\bar{1}}$  version of the uniform geodesic stability conjecture holds.

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## 2 Preliminaries

### 2.1 The $L^p$ Finsler geometry of the space of Kähler potentials

In this short section we recall the basics of finite-energy pluripotential theory, as introduced by Guedj and Zeriahi [56], and the Finsler geometry of the space of Kähler potentials, as introduced by the first author [32]. For a detailed account on these matters we refer to the recent textbook [57] and lecture notes [35].

As a matter of convention for the duration of the paper we denote by  $V$  the total volume of the Kähler class  $[\omega]$ :

$$V := \int_X \omega^n.$$

By  $\text{PSH}(X, \omega)$  we denote the space of  $\omega$ -plurisubharmonic ( $\omega$ -psh) functions. Extending the ideas of Bedford and Taylor, Guedj and Zeriahi introduced the nonpluripolar Monge–Ampère mass for a general potential  $u \in \text{PSH}(X, \omega)$  as the following limit [56]:

$$\omega_u^n := \lim_{k \rightarrow \infty} \mathbb{1}_{\{u > -k\}} (\omega + \sqrt{-1} \partial \bar{\partial} \max(u, -k))^n.$$

For such measures one has an estimate on the total mass  $\int_X \omega_u^n \leq \int_X \omega^n = V$ , and  $\mathcal{E}_\omega := \{u \in \text{PSH}(X, \omega) \mid \int_X \omega_u^n = \int_X \omega^n = V\}$  is the set of potentials with full/maximum mass. Furthermore, potentials  $u \in \mathcal{E}_\omega$  that satisfy an  $L^p$ -type integral condition are members of the so-called *finite-energy spaces* of [56],

$$\mathcal{E}_\omega^p = \left\{ u \in \mathcal{E}_\omega \mid \int_X |u|^p \omega_u^n < +\infty \right\}.$$

Now we recall some of the main points on the  $L^p$  Finsler geometry of the space of Kähler potentials. By definition, the space of Kähler potentials  $\mathcal{H}_\omega$  is an open convex subset of  $C^\infty(X)$ , hence one can think of it as a trivial Fréchet manifold. As a result, one can introduce on  $\mathcal{H}_\omega$  a collection of  $L^p$ -type Finsler metrics. If  $u \in \mathcal{H}_\omega$  and  $\xi \in T_u \mathcal{H}_\omega \simeq C^\infty(X)$ , then the  $L^p$  norm of  $\xi$  is given by the expression

$$\|\xi\|_{p,u} = \left( \frac{1}{V} \int_X |\xi|^p \omega_u^n \right)^{1/p}.$$

When  $p = 2$ , this construction reduces to the Riemannian geometry of Mabuchi [67] (independently discovered by Semmes [72] and Donaldson [51]).

Using these Finsler metrics, one can introduce path length metric structures  $(\mathcal{H}_\omega, d_p)$ . In [32, Theorem 2], the first author identified the completion of these spaces with

$\mathcal{E}_\omega^p \subset \text{PSH}(X, \omega)$  from above, and it turns out that  $(\mathcal{E}_\omega^p, d_p)$  is a complete geodesic metric space.

The distinguished  $d_p$ -geodesic segments of the completion  $(\mathcal{E}_\omega^p, d_p)$  are constructed as upper envelopes of quasi-psh functions, as we now elaborate. Let  $S = \{0 < \text{Re } s < 1\} \subset \mathbb{C}$  be the unit strip, and  $\pi_{S \times X}: S \times X \rightarrow X$  denotes projection to the second component.

We consider  $u_0, u_1 \in \mathcal{E}_\omega^p$ . We say that the curve  $[0, 1] \ni t \rightarrow v_t \in \mathcal{E}_\omega^p$  is a *weak subgeodesic* connecting  $u_0$  and  $u_1$  if  $d_p(v_t, u_{0,1}) \rightarrow 0$  as  $t \rightarrow 0, 1$ , and the extension  $v(s, x) = v_{\text{Re } s}(x)$  is  $\pi^*\omega$ -psh on  $S \times X$ , ie

$$\pi^*\omega + i\partial_{S \times X}\bar{\partial}_{S \times X}v \geq 0, \quad \text{as currents on } S \times X.$$

As shown in [33; 32], a distinguished  $d_p$ -geodesic  $[0, 1] \ni t \rightarrow u_t \in \mathcal{E}_\omega^p$  connecting  $u_0$  and  $u_1$  can be obtained as the supremum of all weak subgeodesics:

$$(5) \quad u_t := \sup\{v_t \mid t \rightarrow v_t \text{ is a subgeodesic connecting } u_0 \text{ and } u_1\}, \quad t \in [0, 1].$$

Given  $u_0, u_1 \in \mathcal{E}_\omega^p$ , we call (5) the  $L^p$  *finite-energy geodesic* (or simply *finite-energy geodesic*) connecting  $u_0$  and  $u_1$ .

When the endpoints  $u_0$  and  $u_1$  are from  $\mathcal{H}_\omega$ , the finite-energy geodesic connecting them is actually  $C^{1,1}$  on  $\bar{S} \times X$ , as shown by Chu, Tosatti and Weinkove [30]. In this paper we only need the  $C^{1,\bar{1}}$  regularity of geodesics connecting smooth endpoints which had been previously proved by Chen [24] (for a survey see Błocki [11]). His proof relies on the so-called  $\varepsilon$ -geodesic that we now recall (for a survey see [35, Section 3.1]). Given  $u_0, u_1 \in \mathcal{H}_\omega$ , by  $u^\varepsilon \in C^\infty([0, 1] \times X)$  we denote the smooth  $\varepsilon$ -geodesic connecting  $u_0, u_1$ , ie  $[0, 1] \ni t \rightarrow u_t^\varepsilon \in \mathcal{H}_\omega$  solving the elliptic PDE, on  $[0, 1] \times X$ ,

$$(6) \quad (\ddot{u}_t^\varepsilon - |\nabla \dot{u}_t^\varepsilon|_{\omega_{u_t^\varepsilon}}^2) \frac{\omega_{u_t^\varepsilon}^n}{\omega^n} = \varepsilon, \quad u_0^\varepsilon := u_0, u_1^\varepsilon := u_1.$$

Note that  $t \mapsto u_t^\varepsilon$  is a subgeodesic connecting  $u_0$  and  $u_1$ . Given that the complex Hessian of  $u^\varepsilon$  is bounded on  $[0, 1] \times X$  [24], one can take the limit  $\varepsilon \rightarrow 0$ , to obtain  $u \in C^{1,\bar{1}}([0, 1] \times X)$ , the  $C^{1,\bar{1}}$ -geodesic connecting  $u_0$  and  $u_1$ :

$$(7) \quad [0, 1] \ni t \rightarrow u_t \in \mathcal{H}_\omega^{1,\bar{1}}.$$

As shown in [58, Theorem 1.1], if one merely has  $u_0, u_1 \in \mathcal{H}_\omega^{1,\bar{1}}$ , the curve in (7) still exists,  $u_t \in \mathcal{H}_\omega^{1,\bar{1}}$  for all  $t \in [0, 1]$ , however it is not known if the total Laplacian of  $u$  on  $[0, 1] \times X$  is bounded.

Due to the ‘‘Perron-type’’ definition of geodesics (5), finite-energy geodesic segments satisfy a comparison principle. In particular, we note the following simple consequence of the comparison principle for geodesics:

**Lemma 2.1** *Let  $u^\varepsilon$  be the smooth  $\varepsilon$ -geodesic connecting  $u_0, u_1 \in \mathcal{H}_\omega$  and  $u_t$  be the  $C^{1,1}$  geodesic connecting  $u_0$  and  $u_1$ . Then  $u^\varepsilon \leq u$ .*

Regarding the metric  $d_p$  the following double estimate holds for some dimensional constant  $C > 1$  and all  $p \geq 1$  [32, Theorem 3]:

$$\begin{aligned} \frac{1}{C}d_p(u_0, u_1)^p &\leq \frac{1}{V} \int_X |u_0 - u_1|^p \omega_{u_0}^n + \frac{1}{V} \int_X |u_0 - u_1|^p \omega_{u_1}^n \\ &\leq C d_p(u_0, u_1)^p, \quad u_0, u_1 \in \mathcal{E}_\omega^p. \end{aligned}$$

We recall that for any  $u \in \text{PSH}(X, \omega)$  there exists  $u_j \in \mathcal{H}_\omega$  such that  $u_j$  decreases to  $u$ . This is a result due to Demailly [45] with a simpler proof due to Błocki and Kołodziej [12]. It is well known that the Monge–Ampère energy  $I: \mathcal{E}_\omega^1 \rightarrow \mathbb{R}$  defined by

$$I(u) = \frac{1}{V(n+1)} \sum_{j=0}^n \int_X u \omega^{n-j} \wedge \omega_u^j$$

is affine along finite-energy geodesics [32]. Moreover, the same is true for  $\sup_X u_t$  when  $u_0 = 0$ :

**Lemma 2.2** *Let  $[0, 1] \ni t \rightarrow u_t \in \mathcal{E}_\omega^1$  be a finite-energy geodesic with  $u_0 = 0$ . Then  $t \rightarrow \sup_X u_t$  is affine.*

As  $\mathcal{E}_\omega^p \subset \mathcal{E}_\omega^1$  for all  $p \geq 1$ , Lemma 2.2 is true in  $\mathcal{E}_\omega^p$  for  $p \geq 1$ , as well.

This is essentially [34, Theorem 1(ii)], which is stated for bounded geodesics. Since finite-energy geodesic segments can be approximated decreasingly by bounded geodesic segments, the above result follows as a consequence of Hartogs’ lemma [57, Proposition 8.4]. For more on  $L^p$  Mabuchi geometry we refer to [35, Chapter 3].

## 2.2 The K-energy and constant scalar curvature metrics

Given a smooth closed  $(1, 1)$ -form  $\chi$  the  $\chi$ -contracted version of the Monge–Ampère energy is defined as

$$I_\chi(u) = \frac{1}{nV} \sum_{j=0}^{n-1} \int_X u \omega_u^j \wedge \omega^{n-1-j} \wedge \chi, \quad u \in \mathcal{E}_\omega^1.$$

It was shown in [32, Lemma 4.15; 8, Section 4] that the energy functionals  $I_X$  can be extended to  $(\mathcal{E}_\omega^1, d_1)$  as  $d_1$ -Lipschitz functionals.

We recall the definition of the  $\mathcal{I}$ -functional introduced by Aubin [2] (and extended to  $\mathcal{E}_\omega^1$  in [5]):

$$\mathcal{I}(u_0, u_1) = \frac{1}{V} \int_X (u_0 - u_1)(\omega_{u_1}^n - \omega_{u_0}^n), \quad u_0, u_1 \in \mathcal{E}_\omega^1.$$

From the definition it is clear that  $\mathcal{I}$  is symmetric and invariant under adding constants. By an integration by parts we see that  $\mathcal{I}$  is nonnegative.

The (extended) K-energy  $\mathcal{K}: \mathcal{E}^1 \rightarrow (-\infty, \infty]$  is defined as

$$(8) \quad \mathcal{K}(u) = \text{Ent}(\omega^n, \omega_u^n) + \bar{S}I(u) - nI_{\text{Ric}(\omega)}(u), \quad u \in \mathcal{E}_\omega^1,$$

where  $\text{Ent}(\omega^n, \omega_u^n)$  is the entropy of the measure  $\omega_u^n$  with respect to  $\omega^n$ ,

$$\text{Ent}(\omega^n, \omega_u^n) = V^{-1} \int_X \log(\omega_u^n / \omega^n) \omega_u^n,$$

and  $\bar{S} = \frac{1}{V} \int_X S_\omega \omega^n$  is the average scalar curvature, which can be seen to be independent of the choice of background metric. When restricted to  $\mathcal{H}_\omega$ , the above formula for the K-energy was originally introduced by Chen and Tian [23], with a similar formula already appearing in [77].

The first-order variation of  $\mathcal{K}$  is given by

$$\langle D\mathcal{K}(u), \delta v \rangle = V^{-1} \int_X \delta v (\bar{S} - S_{\omega_u}) \omega_u^n,$$

where  $\bar{S} = nV^{-1} \int_X \text{Ric}(\omega) \wedge \omega^{n-1}$ . Hence, the critical points of  $\mathcal{K}$  are the constant scalar curvature potentials, as these satisfy  $\bar{S} - S_{\omega_u} = 0$ . It was proved in [4] that the K-energy  $\mathcal{K}$  is convex along  $C^{1,1}$  geodesics. As a consequence, cscK potentials are minimizers of  $\mathcal{K}$ .

### 2.3 The relative Kołodziej-type estimate

In this short subsection we recall the basics of relative pluripotential theory that are needed to state the relative Kołodziej-type estimates of [39]. For more details we refer to the sequence of papers [38; 37; 36; 39].

Let  $E$  be a Borel subset of  $X$ . Given  $\chi \in \text{PSH}(X, \omega)$ , we define the  $\chi$ -relative capacity of  $E$  as

$$(9) \quad \text{Cap}_\chi(E) := \sup \left\{ \int_E \omega_u^n \mid u \in \text{PSH}(X, \omega), \chi - 1 \leq u \leq \chi \right\}.$$

When  $\chi = 0$ , we recover the classical Monge–Ampère capacity  $\text{Cap}_\omega$  (see eg [55]). For more on this concept we refer to [39, Section 4].

Given  $u \in \text{PSH}(X, \omega)$ , we recall the definition of envelopes with respect to singularity type, introduced by Ross and Witt Nyström [71]:

$$P[u] := \text{usc}\left(\lim_{C \rightarrow +\infty} P(0, u + C)\right) \in \text{PSH}(X, \omega),$$

where  $P(\phi, \psi) := \sup\{v \in \text{PSH}(X, \omega) \mid v \leq \phi \text{ and } v \leq \psi\}$ . In addition to appearing in the statement of the relative Kołodziej-type estimate below, this concept also plays a role in Theorem 4.5, where it is used to approximate geodesic rays, via [71].

Finally we recall the following  $L^\infty$  estimate from [39]:

**Theorem 2.3** [39, Theorem 3.3] *Let  $a \in [0, 1)$ ,  $A > 0$ ,  $\chi \in \text{PSH}(X, \theta)$  and  $0 \leq f \in L^p(X, \omega^n)$  for some  $p > 1$ . Assume that  $u \in \text{PSH}(X, \theta)$ , normalized by  $\sup_X u = 0$ , satisfies*

$$(10) \quad \theta_u^n \leq f\omega^n + a\theta_\chi^n.$$

Assume also that

$$(11) \quad \int_E f\omega^n \leq A[\text{Cap}_\chi(E)]^2$$

for every Borel subset  $E \subset X$ . If  $P[u]$  is less singular than  $\chi$  then

$$\chi - \sup_X \chi - C(\|f\|_{L^p}, p, (1-a)^{-1}, A) \leq u.$$

Here, given two potentials  $u, v \in \text{PSH}(X, \omega)$ , we say that  $u$  is less singular than  $v$  if  $u \geq v - C$  for some constant  $C$ .

This theorem generalizes the classical estimates of Kołodziej from [64], and it is used in [39] to solve complex Monge–Ampère equations with prescribed singularity type, and to resolve the log-concavity conjecture of the volume in pluripotential theory. Here we will use it in Section 5 to show that it is possible to approximate  $L^p$  geodesic rays with bounded ones that have converging radial K–energy.

## 2.4 Weighted Clarkson and Ball–Carlen–Lieb-type inequalities

In this short preliminary section we point out relevant extensions of well-known inequalities due to Clarkson [31] and Ball, Carlen and Lieb [3] for  $L^p$  spaces, introducing a

weight  $\lambda \in [0, 1]$  into these results. These theorems are almost certainly well known to experts in analysis, but we could not find the versions below in the literature.

**Theorem 2.4** *Suppose that  $p \geq 2$ ,  $\lambda \in [0, 1]$  and  $f, g \in L^p(\nu)$ , where  $\nu$  is a measure on the set  $X$ . Then*

$$(12) \quad \lambda \|f\|_p^p + (1 - \lambda) \|g\|_p^p \geq \|\lambda f + (1 - \lambda)g\|_p^p + \lambda^{p/2}(1 - \lambda)^{p/2} \|f - g\|_p^p.$$

**Proof** Since  $t \rightarrow |t|^{p/2}$  is a convex function, we can write the estimates

$$\begin{aligned} \lambda \|f\|_p^p + (1 - \lambda) \|g\|_p^p &\geq \int_X (\lambda f^2 + (1 - \lambda)g^2)^{p/2} d\nu \\ &= \int_X ((\lambda f + (1 - \lambda)g)^2 + \lambda(1 - \lambda)(f - g)^2)^{p/2} d\nu \\ &\geq \int_X (|\lambda f + (1 - \lambda)g|^p + \lambda^{p/2}(1 - \lambda)^{p/2}|f - g|^p) d\nu, \end{aligned}$$

where in the last step we have used that  $(a^2 + b^2)^{1/2} \geq (a^p + b^p)^{1/p}$  for  $a, b \geq 0$ .  $\square$

**Theorem 2.5** *Suppose that  $1 < p \leq 2$ ,  $\lambda \in [0, 1]$  and  $f, g \in L^p(\nu)$ , where  $\nu$  is a measure on the set  $X$ . Then*

$$(13) \quad \lambda \|f\|_p^2 + (1 - \lambda) \|g\|_p^2 \geq \|\lambda f + (1 - \lambda)g\|_p^2 + (p - 1)\lambda(1 - \lambda) \|f - g\|_p^2.$$

**Proof** The proof will be given using dyadic approximation. Indeed, it is enough to prove (13) for  $\lambda = k/2^m$ , where  $k, m \in \mathbb{N}$  with  $1 \leq k \leq 2^m$ . We will argue by induction on  $m$ . For  $m = 1$  and  $k = 0, 1, 2$ , the statement of (13) is either a triviality or reduces to [3, Proposition 3]. Let us assume that  $m > 1$  and the statement holds for  $m - 1$ . We can assume that  $k$  is odd, as otherwise the inequality reduces to the case  $m - 1$ . Using [3, Proposition 3], we start with the estimate

$$\begin{aligned} (14) \quad \frac{1}{2} \left\| \frac{k-1}{2^m} f + \left(1 - \frac{k-1}{2^m}\right) g \right\|_p^2 + \frac{1}{2} \left\| \frac{k+1}{2^m} f + \left(1 - \frac{k+1}{2^m}\right) g \right\|_p^2 \\ \geq \left\| \frac{k}{2^m} f + \left(1 - \frac{k}{2^m}\right) g \right\|_p^2 + (p-1) \left\| \frac{f}{2^m} - \frac{g}{2^m} \right\|_p^2. \end{aligned}$$

Since both  $k + 1$  and  $k - 1$  are even, by the inductive step we also have that

$$\begin{aligned} (15) \quad \frac{k+1}{2^m} \|f\|_p^2 + \left(1 - \frac{k+1}{2^m}\right) \|g\|_p^2 \\ \geq \left\| \frac{k+1}{2^m} f + \left(1 - \frac{k+1}{2^m}\right) g \right\|_p^2 + (p-1) \frac{k+1}{2^m} \left(1 - \frac{k+1}{2^m}\right) \|f - g\|_p^2, \end{aligned}$$

$$(16) \quad \frac{k-1}{2^m} \|f\|_p^2 + \left(1 - \frac{k-1}{2^m}\right) \|g\|_p^2 \geq \left\| \frac{k-1}{2^m} f + \left(1 - \frac{k-1}{2^m}\right) g \right\|_p^2 + (p-1) \frac{k-1}{2^m} \left(1 - \frac{k-1}{2^m}\right) \|f-g\|_p^2.$$

Adding (15) and (16) and then using (14), we arrive at

$$\frac{k}{2^m} \|f\|_p^2 + \left(1 - \frac{k}{2^m}\right) \|g\|_p^2 \geq \left\| \frac{k}{2^m} f + \left(1 - \frac{k}{2^m}\right) g \right\|_p^2 + (p-1) \frac{k}{2^m} \left(1 - \frac{k}{2^m}\right) \|f-g\|_p^2,$$

what we desired to prove. □

**Remark 2.6** As alluded to at the beginning of the subsection, when  $\lambda = \frac{1}{2}$ , Theorems 2.4 and 2.5 recover the well-known inequalities of Clarkson [31] and Ball, Carlen and Lieb [3, Proposition 3], respectively.

### 3 Uniform convexity and uniqueness of geodesics

Before proving the main result of this section, we first point out the following result about the “spread” of geodesic segments in  $\mathcal{E}_\omega^p$  sharing a common smooth endpoint:

**Theorem 3.1** *Suppose that  $p \geq 1$ ,  $u \in \mathcal{H}_\omega$  and  $[0, l] \ni t \rightarrow u_t, v_t \in \mathcal{E}_\omega^p$  are two finite-energy geodesic segments with  $u = u_0 = v_0$ , and  $l \in \mathbb{R}^+$ . Then*

$$(17) \quad \left[ \int_X |\dot{u}_0 - \dot{v}_0|^p \omega_u^n \right]^{1/p} \leq \frac{d_p(u_t, v_t)}{t}, \quad t \in [0, l].$$

**Proof** We first assume that  $u_l \geq v_l$ . Then we note that (thanks to convexity)  $u_t \geq v_t$  for all  $t \in [0, l]$ . Furthermore, using  $d_p$ -approximation of the endpoints  $u_l, v_l \in \mathcal{E}_\omega^p$  by decreasing sequences of potentials in  $\mathcal{H}_\omega$ , it is enough to prove (17) for  $C^{1,1}$ -geodesics  $t \rightarrow u_t, v_t$  with  $u_l, v_l \in \mathcal{H}_\omega$  (see [8, Proposition 4.3]).

Using the convexity condition (1) and [32, Lemma 5.1] for  $0 \leq s \leq t \leq l$  we have

$$\frac{d_p(u_t, v_t)^p}{t^p} \geq \frac{d_p(u_s, v_s)^p}{s^p} \geq \int_X \frac{(u_s - v_s)^p}{s^p} \omega_{u_s}^n.$$

As  $s \rightarrow 0^+$ , using the fact that the geodesics are  $C^{1,1}$ , we get that  $(u_s - v_s)^p/s^p$  uniformly converges to  $(\dot{u}_0 - \dot{v}_0)^p$ , which is a continuous function on  $X$ . Since  $\omega_{u_s}^n \rightarrow \omega_u^n$  weakly (see [32, Theorem 5(i)]) it follows that

$$\frac{d_p(u_t, v_t)^p}{t^p} \geq \int_X (\dot{u}_0 - \dot{v}_0)^p \omega_u^n.$$

We now treat the general case, when  $u_l$  and  $v_l$  may not be comparable. By the previous step, for  $t \in [0, l]$  we have

$$\frac{d_p(u_t, P(u_t, v_t))^p}{t^p} \geq \int_X |\dot{u}_0 - \dot{w}_0|^p \omega_u^n, \quad \frac{d_p(v_t, P(u_t, v_t))^p}{t^p} \geq \int_X |\dot{v}_0 - \dot{w}_0|^p \omega_u^n,$$

where  $[0, t] \ni s \mapsto w_s^t \in \mathcal{E}_\omega^p$  is the finite-energy geodesic connecting  $w_0 := u_0$  and  $w_t := P(u_t, v_t)$ .

Due to the comparison principle for geodesics, we note that  $\dot{w}_0^t \leq \dot{u}_0, \dot{v}_0$ . Using the Pythagorean formula [32, Corollary 4.14] and the inequality  $a^p + b^p \geq \max(a^p, b^p) \geq |a - b|^p$  for  $a, b \geq 0$ , we can sum up the above inequalities to arrive at the conclusion

$$\begin{aligned} \frac{d_p(u_t, v_t)^p}{t^p} &= \frac{d_p(u_t, P(u_t, v_t))^p}{t^p} + \frac{d_p(v_t, P(u_t, v_t))^p}{t^p} \\ &\geq \int_X |\dot{u}_0 - \dot{v}_0|^p \omega_u^n, \quad t \in [0, l]. \end{aligned} \quad \square$$

Before proceeding we note that Theorem 3.1 implies the following Lidskii-type inequality, proved in the case of Hodge-type Kähler metrics in [43]:

**Corollary 3.2** *If  $\alpha, \beta, \gamma \in \mathcal{E}_\omega^p$  with  $p \geq 1$  and  $\alpha \geq \beta \geq \gamma$ , then*

$$d_p(\beta, \gamma)^p \leq d_p(\alpha, \gamma)^p - d_p(\alpha, \beta)^p.$$

**Proof** By density it is enough to show this estimate for  $\alpha, \beta, \gamma \in \mathcal{H}_\omega$ . Let  $[0, 1] \ni t \rightarrow u_t, v_t \in \mathcal{E}_\omega^p$  be the increasing/decreasing  $C^{1,1}$ -geodesics joining  $u_0 := \beta$  and  $u_1 := \alpha$ , and  $v_0 := \beta$  and  $v_1 := \gamma$ , respectively. Then, due to  $t$ -monotonicity,  $\dot{v}_0 \leq 0 \leq \dot{u}_0$ . Then, by Theorem 3.1 and [32, Theorem 1],

$$\begin{aligned} d_p(\alpha, \gamma)^p &= d_p(u_1, v_1)^p \geq \int_X |\dot{u}_0 - \dot{v}_0|^p \omega_\beta^n \\ &\geq \int_X (|\dot{u}_0|^p + |\dot{v}_0|^p) \omega_\beta^n = d_p(\alpha, \beta)^p + d_p(\gamma, \beta)^p. \end{aligned} \quad \square$$

Next we prove the main result of this section about the uniform convexity of the spaces  $(\mathcal{E}_\omega^p, d_p)$  for  $p > 1$ . This will follow after an adequate combination of Theorem 3.1 and the extension of the inequalities of Clarkson and Ball, Carlen and Lieb, obtained in the previous section.

**Theorem 3.3** *Suppose that  $u \in \mathcal{E}_\omega^p$  for  $\lambda \in [0, 1]$  and  $[0, 1] \ni t \rightarrow v_t \in \mathcal{E}_\omega^p$  is a finite-energy geodesic segment. Then the following hold:*

- (i)  $d_p(u, v_\lambda)^2 \leq (1 - \lambda)d_p(u, v_0)^2 + \lambda d_p(u, v_1)^2 - (p - 1)\lambda(1 - \lambda)d_p(v_0, v_1)^2$  if  $1 < p \leq 2$ .
- (ii)  $d_p(u, v_\lambda)^p \leq (1 - \lambda)d_p(u, v_0)^p + \lambda d_p(u, v_1)^p - \lambda^{p/2}(1 - \lambda)^{p/2}d_p(v_0, v_1)^p$  if  $2 \leq p$ .

**Proof** To begin, let  $p \geq 1$  and  $\lambda \in [0, 1]$ . By density (and [8, Proposition 4.3]) we can assume that  $u, v_0, v_1 \in \mathcal{H}_\omega$  and hence  $t \rightarrow v_t$  is  $C^{1, \bar{1}}$ .

Fixing  $\varepsilon > 0$  momentarily, let  $[0, 1] \ni t \rightarrow v_t^\varepsilon \in \mathcal{H}_\omega$  be Chen’s smooth  $\varepsilon$ -geodesic connecting  $v_0, v_1 \in \mathcal{H}_\omega$ , see Section 2.1. Moreover, let  $[0, 1] \ni t \rightarrow \alpha_t^{\lambda, \varepsilon} \in \mathcal{E}_\omega^p$  be the  $C^{1, \bar{1}}$  geodesic connecting  $u$  and  $v_\lambda^\varepsilon$ . Let  $[0, \lambda] \ni t \rightarrow h_t^\varepsilon \in \mathcal{E}_\omega^p$  be the  $C^{1, \bar{1}}$  geodesic connecting  $v_0$  and  $v_\lambda^\varepsilon$ . Similarly, let  $[\lambda, 1] \ni t \rightarrow k_t^\varepsilon \in \mathcal{E}_\omega^p$  be the  $C^{1, \bar{1}}$  geodesic connecting  $v_\lambda^\varepsilon$  and  $v_1$ .

We now assume that  $2 \leq p$  to address (ii). Using Theorem 3.1 twice, for pairs of geodesics emanating from  $v_\lambda^\varepsilon$ , we conclude that

$$\int_X |\dot{\alpha}_1^{\lambda, \varepsilon} - \lambda \dot{h}_\lambda^\varepsilon|^p \omega_{v_\lambda^\varepsilon}^n \leq d_p(u, v_0)^p, \quad \int_X |\dot{\alpha}_1^{\lambda, \varepsilon} + (1 - \lambda) \dot{k}_\lambda^\varepsilon|^p \omega_{v_\lambda^\varepsilon}^n \leq d_p(u, v_1)^p.$$

In the first inequality, we have applied Theorem 3.1 for  $u_0 = v_0 = v_\lambda^\varepsilon$  (hence the geodesic segments  $h_t^\varepsilon$  and  $\alpha_t^{\lambda, \varepsilon}$  are both parametrized in the opposite directions). In the second inequality we have applied Theorem 3.1 for  $u_0 = v_0 = v_\lambda^\varepsilon$  (hence the geodesic segment  $k_t^\varepsilon$  is parametrized in the positive direction while  $\alpha_t^{\lambda, \varepsilon}$  is parametrized in the opposite direction; that’s why we have plus signs).

By the comparison principle for geodesics, we have that  $v_t^\varepsilon \leq h_t^\varepsilon \leq v_t$  for  $t \in [0, \lambda]$  and  $v_t^\varepsilon \leq k_t^\varepsilon \leq v_t$  for  $t \in [\lambda, 1]$  (see Lemma 2.1). Again, by the comparison principle, the concatenation of  $t \rightarrow h_t^\varepsilon$  and  $t \rightarrow k_t^\varepsilon$  is  $t$ -convex and we obtain that  $\dot{h}_\lambda^\varepsilon \rightarrow \dot{v}_\lambda$  and  $\dot{k}_\lambda^\varepsilon \rightarrow \dot{v}_\lambda$  uniformly on  $X$ . Using this and the above two estimates we can write

$$\begin{aligned} (18) \quad & (1 - \lambda)d_p(u, v_0)^p + \lambda d_p(u, v_1)^p \\ & \geq \int_X (1 - \lambda)|\dot{\alpha}_1^{\lambda, \varepsilon} - \lambda \dot{h}_\lambda^\varepsilon|^p + \lambda |\dot{\alpha}_1^{\lambda, \varepsilon} + (1 - \lambda) \dot{k}_\lambda^\varepsilon|^p \omega_{v_\lambda^\varepsilon}^n \\ & \geq (1 - \lambda) \int_X |\dot{\alpha}_1^{\lambda, \varepsilon} - \lambda \dot{v}_\lambda|^p \omega_{v_\lambda^\varepsilon}^n + \lambda \int_X |\dot{\alpha}_1^{\lambda, \varepsilon} + (1 - \lambda) \dot{v}_\lambda|^p \omega_{v_\lambda^\varepsilon}^n - O(\varepsilon) \\ & \geq \int_X |\dot{\alpha}_1^{\lambda, \varepsilon}|^p \omega_{v_\lambda^\varepsilon}^n + \lambda^{p/2}(1 - \lambda)^{p/2} \int_X |\dot{v}_\lambda|^p \omega_{v_\lambda^\varepsilon}^n - O(\varepsilon) \\ & = d_p(u, v_\lambda^\varepsilon)^p + \lambda^{p/2}(1 - \lambda)^{p/2} \int_X |\dot{v}_\lambda|^p \omega_{v_\lambda^\varepsilon}^n - O(\varepsilon), \end{aligned}$$

where in the third line we have used Theorem 2.4, and in the last line we have used [32, Theorem 1]. Letting  $\varepsilon \rightarrow 0$ , since  $\omega_{v_\varepsilon}^n \rightarrow \omega_{v_\lambda}^n$  and  $O(\varepsilon) \rightarrow 0$ , another application of [32, Theorem 1] gives (ii).

Now we assume that  $1 < p \leq 2$  and we address the inequality of (i). The proof is exactly the same, except for (18), where we use the estimate of Theorem 2.5 instead of Theorem 2.4. □

**Remark 3.4** Suppose that  $\omega$  is the curvature of a Hermitian line bundle  $(L, h)$ . By exactly the same arguments, one can show that the inequalities of Theorem 3.3 also hold for the finite-dimensional  $L^p$ -type metric spaces  $(\mathcal{H}_\omega^k, d_{p,k})$ , as considered in [43]. Using the quantization scheme of this paper [43, Theorem 1.2], an alternative proof of Theorem 3.3 can be thus given when  $[\omega]$  is integral.

Finally we point out that using the above result one can show that the finite-energy geodesic segments of  $\mathcal{E}_\omega^p$  are the only metric geodesics when  $p > 1$ :

**Theorem 3.5** *Let  $p \in (1, \infty)$ , and suppose that  $[0, 1] \ni t \rightarrow v_t \in \mathcal{E}_\omega^p$  is the finite-energy geodesic connecting  $v_0, v_1 \in \mathcal{E}_\omega^p$ . Then  $t \rightarrow v_t$  is the only  $d_p$ -geodesic connecting  $v_0$  and  $v_1$ .*

**Proof** Suppose that  $[0, 1] \ni t \rightarrow u_t \in \mathcal{E}_\omega^p$  is a  $d_p$ -geodesic connecting  $v_0$  and  $v_1$ , and let  $h_t \in \mathcal{E}_\omega^p$  be the  $d_p$ -midpoint of the finite-energy geodesic connecting  $u_t$  and  $v_t$  for  $t \in [0, 1]$ . Assuming that  $u_t \neq v_t$ , Theorem 3.3 implies that  $d_p(v_0, h_t) < \max\{d_p(v_0, u_t), d_p(v_0, v_t)\} = td_p(v_0, v_1)$ . Similarly,

$$d_p(v_1, h_t) < \max\{d_p(v_1, u_t), d_p(v_1, v_t)\} = (1 - t)d_p(v_0, v_1).$$

The triangle inequality now gives a contradiction, implying that  $u_t = v_t$  for  $t \in [0, 1]$ . □

A more careful analysis of the above proof yields the following:

**Proposition 3.6** *Suppose that  $p > 1$  and  $[0, 1] \ni l \rightarrow u_l \in \mathcal{E}_\omega^p$  is a finite-energy geodesic. Let  $v \in \mathcal{E}_\omega^p$  be such that  $d_p(v, u_0) \leq (t + \varepsilon)d_p(u_0, u_1)$  and  $d_p(v, u_1) \leq (1 - t + \varepsilon)d_p(u_0, u_1)$  for some  $\varepsilon > 0$  and  $t \in [0, 1]$ . Then there exists  $C(p) > 0$  such that*

$$d_p(v, u_t) \leq \varepsilon^{1/r} C d_p(u_0, u_1),$$

where  $r := \max(2, p)$ .

**Proof** Let  $h$  be the  $d_p$ -midpoint of the finite-energy geodesic connecting  $v$  and  $u_t$ . Then Theorem 3.3 implies that

$$\begin{aligned} d_p(u_0, h) &\leq \left[ \frac{1}{2} d_p(u_0, v)^r + \frac{1}{2} d_p(u_0, u_t)^r - c d_p(v, u_t)^r \right]^{1/r}, \\ d_p(u_1, h) &\leq \left[ \frac{1}{2} d_p(u_1, v)^r + \frac{1}{2} d_p(u_1, u_t)^r - c d_p(v, u_t)^r \right]^{1/r} \end{aligned}$$

for  $r := \max(p, 2)$ , and  $c := c(p) \in (0, 1)$ . Adding these estimates and using the triangle inequality we arrive at

$$\begin{aligned} d_p(u_0, u_1) &\leq [(t + \varepsilon)^r d_p(u_0, u_1)^r - c d_p(v, u_t)^r]^{1/r} \\ &\quad + [(1 - t + \varepsilon)^r d_p(u_0, u_1)^r - c d_p(v, u_t)^r]^{1/r}. \end{aligned}$$

After dividing by  $d_p(u_0, u_1)$ , basic calculus yields that

$$\frac{d_p(v, u_t)^r}{d_p(u_0, u_1)^r} \leq \max\left( \frac{(t + \varepsilon)^r - t^r}{c}, \frac{(1 - t + \varepsilon)^r - (1 - t)^r}{c} \right),$$

implying that  $d_p(v, u_t) \leq \varepsilon^{1/r} C d_p(u_0, u_1)$ , as desired.  $\square$

## 4 The metric geometry of weak $L^p$ geodesic rays

For  $u \in \mathcal{E}_\omega^p$  let  $\mathcal{R}_u^p$  denote the space of finite-energy  $L^p$  geodesic rays emanating from  $u$ . Note that we don't assume that the rays are unit-speed, or even nonconstant.

Following terminology from metric space theory [20], two rays  $\{u_t\}_t$  and  $\{v_t\}_t$  are parallel if  $d_p(u_t, v_t)$  is uniformly bounded. Given the characteristics of the finite-energy spaces, any ray admits a unique parallel ray emanating from an outside point, thus the  $d_p$ -geometries verify Euclid's 5<sup>th</sup> postulate for half-lines, answering an open question raised in [27, Remark 1.6]:

**Proposition 4.1** *Let  $u, v \in \mathcal{E}_\omega^p$ ; then for any  $\{u_t\}_t \in \mathcal{R}_u^p$  there exists a unique  $\{v_t\}_t \in \mathcal{R}_v^p$  such that  $\{u_t\}_t$  is parallel to  $\{v_t\}_t$ , giving a bijection  $\mathcal{P}_{uv}: \mathcal{R}_u^p \rightarrow \mathcal{R}_v^p$ . Moreover,  $d_p(u_t, v_t) \leq d_p(u, v)$  for  $t \geq 0$ .*

**Proof** We first observe that the inequality  $d_p(u_t, v_t) \leq d_p(u, v)$  holds for two parallel rays  $\{u_t\}_t$  and  $\{v_t\}_t$ . Indeed, if  $\{u_t\}_t$  is parallel to  $\{v_t\}_t$ , then by (1) we have, for  $0 < t < s$ ,

$$d_p(u_t, v_t) \leq \left(1 - \frac{t}{s}\right) d_p(u_0, v_0) + \frac{t}{s} d_p(u_s, v_s),$$

with the last term converging to 0 as  $s \rightarrow +\infty$ . Hence  $d_p(u_t, v_t) \leq d_p(u, v)$  for  $t \geq 0$ . If  $\{w_t\}_t \in \mathcal{R}_u^p$  is another ray which is parallel to  $\{u_t\}_t$ , then it is also parallel to  $\{v_t\}_t$ , hence  $d_p(w_t, v_t) \leq d_p(w_0, v_0) = 0$ , giving the uniqueness.

We first argue the existence part of the proposition for  $u \geq v$ , using the maximum principle. Consider the finite-energy geodesic segments  $[0, t] \ni l \rightarrow v_l^t \in \mathcal{E}_\omega^p$ , with  $v_0^t = v$  and  $v_t^t = u_t$ . Then by the comparison principle for geodesics, for  $0 \leq l \leq t \leq t'$  we get that  $v_l^{t'} \leq u_t = v_t^t$ , hence  $v_l^{t'} \leq v_l^t$ . Also, (1) implies

$$\frac{d_p(v_l^{t'}, u_l)}{t-l} \leq \frac{d_p(v, u)}{t}.$$

Putting the last two sentences together, Proposition 4.3 of [8] implies that  $l \rightarrow v_l := \lim_{t \rightarrow \infty} v_l^t \in \mathcal{E}_\omega^p$  is a finite-energy geodesic ray such that  $d_p(u_l, v_l) \leq d_p(u, v)$  for  $l \geq 0$ .

If  $u \leq v$ , the proposition holds by the same argument (the inequality  $v_l^t \leq v_l^{t'}$  being the only difference).

To treat the general case, we simply notice that  $h := \max(\sup_X u, \sup_X v) \in \mathcal{H}_\omega \subset \mathcal{E}_\omega^p$  and  $h \geq u, v$ . By the above arguments, we can introduce a ray  $\{h_t\}_t \in \mathcal{R}_h^p$  such that  $d_p(u_t, h_t) \leq d_p(u, h)$ . Since  $h \geq v$ , it is now possible to introduce another ray  $\{v_t\}_t \in \mathcal{R}_v^p$  with  $d_p(v_t, h_t) \leq d_p(v, h)$ . The estimate  $d_p(u_t, v_t) \leq d_p(u, h) + d_p(v, h)$  now follows from the triangle inequality, hence  $\{v_t\}_t$  is parallel to  $\{u_t\}$ .  $\square$

Next we introduce the chordal metric on  $\mathcal{R}_u^p$ ,

$$(19) \quad d_{u,p}^c(\{u_t\}_t, \{v_t\}_t) := \lim_{t \rightarrow \infty} \frac{d_p(u_t, v_t)}{t}, \quad \{u_t\}_t, \{v_t\}_t \in \mathcal{R}_u^p.$$

That the above increasing limit exists and is finite follows again from (1) and the triangle inequality. As we now clarify,  $(\mathcal{R}_\omega^p, d_p^c)$  is in fact a complete geodesic metric space.

**Theorem 4.2** *For any  $u \in \mathcal{E}_\omega^p$  with  $p \geq 1$ ,  $(\mathcal{R}_u^p, d_{u,p}^c)$  is a complete metric space. Moreover, for any  $v \in \mathcal{E}_\omega^p$  the map  $\mathcal{P}_{uv}: (\mathcal{R}_u^p, d_{u,p}^c) \rightarrow (\mathcal{R}_v^p, d_{v,p}^c)$  is an isometry.*

Some aspects of the proof below can be traced back to [9, Lemma 3.1].

**Proof** That  $d_{p,u}^c$  satisfies the triangle inequality follows from the triangle inequality of  $d_p$ . To argue nondegeneracy, suppose that  $d_{p,u}^c(\{u_t\}_t, \{v_t\}_t) = 0$ . This implies that

the increasing function  $f(t) = d_p(u_t, v_t)/t$  satisfies  $f(0) = 0$  and  $\lim_{t \rightarrow \infty} f(t) = 0$ . Consequently  $f(t) = 0$  for  $t \geq 0$ , implying that  $u_t = v_t$  for  $t \geq 0$ .

Now suppose that  $\{u_t^j\}_t \subset \mathcal{R}_u^p$  is a  $d_{u,p}^c$ -Cauchy sequence. Fixing  $l > 0$  we have that

$$(20) \quad \frac{d_p(u_l^j, u_l^k)}{l} \leq d_{u,p}^c(\{u_t^j\}_t, \{u_t^k\}_t).$$

Consequently  $\{u_l^j\}_j \subset \mathcal{E}_\omega^p$  is a  $d_p$ -Cauchy sequence with limit  $u_l \in \mathcal{E}_\omega^p$ . By the endpoint stability of geodesic segments in  $\mathcal{E}_\omega^p$  [8, Proposition 4.3] it follows that  $t \rightarrow u_t$  is a geodesic ray. More importantly, letting  $k \rightarrow \infty$  in (20) it follows that  $\frac{1}{l}d_p(u_l^j, u_l)$  is arbitrarily small for high enough  $j$  and any  $l > 0$ . This in turn implies that  $d_{u,p}^c(\{u_t^j\}_t, \{u_t\}_t) \rightarrow 0$ , giving completeness.

Let  $\{u_t\}_t, \{h_t\}_t \in \mathcal{R}_u^p$  and let  $\{v_t\}_t := \mathcal{P}_{uv}(\{u_t\}_t)$  and  $\{k_t\}_t := \mathcal{P}_{uv}(\{h_t\}_t)$ . By the triangle inequality and Proposition 4.1 we have

$$\begin{aligned} d_{u,p}(\{u_t\}_t, \{h_t\}_t) &= \lim_{t \rightarrow +\infty} \frac{d_p(u_t, h_t)}{t} \leq \lim_{t \rightarrow +\infty} \frac{d_p(u_t, v_t) + d_p(v_t, k_t) + d_p(k_t, h_t)}{t} \\ &\leq \lim_{t \rightarrow +\infty} \frac{d_p(u, v) + d_p(v, k_t) + d_p(v, u)}{t} = d_{u,p}(\{v_t\}_t, \{k_t\}_t). \end{aligned}$$

The reverse inequality also holds, due to symmetry, showing that  $\mathcal{P}_{uv}$  is an isometry.  $\square$

By this theorem, no extra information is gained by choice of initial metric; hence, going forward, we will only consider the space  $(\mathcal{R}_\omega^p, d_p^c)$ , the collection of rays emanating from  $0 \in \mathcal{H}_\omega \subset \mathcal{E}_\omega^p$ .

**Approximation of finite-energy rays** We now point out that bounded geodesic rays (running inside  $\text{PSH}(X, \omega) \cap L^\infty$ ) are dense among the rays of  $\mathcal{R}_\omega^p$ . Later, in the presence of finite radial K-energy we will sharpen this result further.

First we start with an auxiliary result, which is a consequence of Corollary 3.2, and it is the radial analog of [32, Lemma 4.16]:

**Lemma 4.3** *Let  $\{u_t\}_t, \{u_t^j\} \in \mathcal{R}_\omega^p$  be such that  $u_t^j$  is decreasing (increasing a.e.) to  $u_t$  as  $j \rightarrow \infty$  for all  $t \geq 0$ . Then  $d_p^c(\{u_t^j\}_t, \{u_t\}_t) \rightarrow 0$ .*

**Proof** We start by noticing that  $t \rightarrow \sup_X u_t$  and  $t \rightarrow \sup_X u_t^j$  are linear (Lemma 2.2). By our assumption we have that  $\sup_X u_1^j \rightarrow \sup_X u_1$  [57, Proposition 8.4]; hence, after possibly subtracting the same  $t$ -linear term from all our rays, without loss of

generality we can assume that  $\sup_X u_t, \sup_X u_t^j \leq 0$ . By convexity we will obtain that  $0 \geq u_t^j \geq u_t$  ( $0 \geq u_t \geq u_t^j$ ) for all  $j$  and  $t \geq 0$ . Consequently, Corollary 3.2 is applicable to yield that

$$(21) \quad \frac{d_p(u_t^j, u_t)^p}{t^p} \leq \frac{|d_p(0, u_t)^p - d_p(0, u_t^j)^p|}{t^p} = |d_p(0, u_1)^p - d_p(0, u_1^j)^p|, \quad t \geq 0,$$

where we have used that  $t \rightarrow d_p(0, u_t^j)$  and  $t \rightarrow d_p(0, u_t)$  are linear. This is because  $u_0^j = u_0 = 0$  and hence  $d_p(u_t^j, 0) = t d_p(u_1^j, 0)$  as  $u_t^j$  is a geodesic ray. Now Lemma 4.16 of [32] gives that  $d_p(u_1^j, u_1) \rightarrow 0$ , in particular  $d_p(0, u_1^j) \rightarrow d_p(0, u_1)$ , finishing the proof.  $\square$

**Remark 4.4** Analyzing the above argument we see that in Lemma 4.3 the conditions can be significantly weakened in some cases. For example, it is enough to assume that  $u_t \leq u_t^j$  for  $t \geq 0$  and  $j \geq 0$ , there exists  $C > 0$  such that  $u_1^j \leq C$  for  $j \geq 0$ , and that  $u_1^j$  converges to  $u_1$  pointwise on  $X$ , with the exception of a pluripolar set. Using [32, Lemma 5.1] we obtain that  $d_p(u_1, u_1^j)^p \leq \int_X |u_1 - u_1^j|^p \omega_{u_1}^n$ , and the dominated convergence theorem allows to conclude that the right-hand side of (21) still converges to zero.

**Theorem 4.5** Let  $\{u_t\}_t \in \mathcal{R}_\omega^p$ . Then there exists a sequence  $\{u_t^j\}_t \in \mathcal{R}_\omega^p$  such that  $u_t^j \in \text{PSH}(X, \omega) \cap L^\infty$  and  $u_t^j \searrow u_t$  as  $j \rightarrow \infty$  for all  $t \geq 0$ . In particular,  $d_p^c(\{u_t^j\}_t, \{u_t\}_t) \rightarrow 0$ , and we can choose  $\{u_t^j\}_t$  such that

$$(22) \quad \max(u_t, (\sup_X u_1 - j)t) \leq u_t^j \leq t \sup_X u_1.$$

**Proof** It follows from Lemma 2.2 that  $t \rightarrow \sup_X u_t/t$  for  $t > 0$  is constant, hence we can assume (by adding  $Ct$  to  $u_t$ ) that  $\sup_X u_t = 0$  for  $t \geq 0$ . By convexity of  $t \mapsto u_t(x)$ , for  $x \in X$  fixed, we have

$$u_t(x) \leq \frac{l-t}{l-s} u_s(x) + \frac{t-s}{l-s} u_l(x) \leq \frac{l-t}{l-s} u_s(x), \quad 0 < s < t < l.$$

Letting  $l \rightarrow +\infty$ , we see that  $t \rightarrow u_t$  is  $t$ -decreasing. For  $\tau \in \mathbb{R}$  and  $x \in X$ , we introduce

$$(23) \quad \psi_\tau(x) := \inf_{t>0} (u_t(x) - t\tau).$$

From Kiselman’s minimum principle [62] we have that  $\psi_\tau \equiv -\infty$  or  $\psi_\tau \in \text{PSH}(X, \omega)$ . More precisely, since  $\sup_X u_t = 0$  we have that  $\psi_\tau \in \text{PSH}(X, \omega)$  for  $\tau \leq 0$ , and  $\psi_\tau \equiv -\infty$  for all  $\tau > 0$ . Observe also that  $\tau \rightarrow \psi_\tau$  is  $\tau$ -decreasing and  $\tau$ -concave.

For all  $x \in X$  with  $\psi_0(x) > -\infty$ , the curve  $t \rightarrow u_t(x)$  is continuous in  $(0, +\infty)$ . Hence, by the involution property of the Legendre transform, for such  $x$  we have

$$(24) \quad u_t(x) = \sup_{\tau < 0} (\psi_\tau(x) + t\tau) = \sup_{\tau \in \mathbb{R}} (\psi_\tau(x) + t\tau), \quad t > 0.$$

For  $\varepsilon > 0, \tau < 0$ , set

$$\psi_\tau^\varepsilon(x) := \max(0, 1 + \varepsilon\tau)\psi_\tau \quad \text{and} \quad \phi_\tau^\varepsilon := P[\psi_\tau^\varepsilon].$$

We define  $\phi_0^\varepsilon := \lim_{\tau \rightarrow 0^-} \phi_\tau^\varepsilon$ .

Since  $\tau \rightarrow \psi_\tau$  is  $\tau$ -concave and  $\tau$ -decreasing with  $\psi_\tau \leq 0$ , it is elementary to see that  $\tau \rightarrow \psi_\tau^\varepsilon$  is also  $\tau$ -concave and  $\tau$ -decreasing. By elementary properties of  $P[\cdot]$  we get that  $\tau \rightarrow \phi_\tau^\varepsilon$  is also  $\tau$ -concave and  $\tau$ -decreasing (see the proof of [36, Proposition 4.6]). As a consequence of a result due to Ross and Witt Nyström [71] (further elaborated in [36, Corollary 1.3]) the curve

$$(25) \quad [0, \infty) \ni t \rightarrow u_t^\varepsilon(x) := \sup_{\tau < 0} (\phi_\tau^\varepsilon(x) + t\tau) \in \text{PSH}(X, \omega) \cap L^\infty$$

is a (bounded) geodesic ray emanating from 0.

We now prove that  $u_t^\varepsilon \searrow u_t$  as  $\varepsilon \searrow 0$  for any  $t \geq 0$ . For  $t = 0$  there is nothing to prove since  $u_0^\varepsilon = u_0 = 0$  on  $X$ . Fix now  $t > 0$  and  $x \in X$  with  $\psi_0(x) > -\infty$ . Then, using  $\tau$ -concavity, there exists  $C > 0$  depending on  $\psi_0(x)$  and  $t$  (but not on  $\varepsilon$ ) such that

$$u_t^\varepsilon(x) = \sup_{-C \leq \tau \leq 0} (\phi_\tau^\varepsilon(x) + t\tau) \quad \text{and} \quad u_t(x) = \sup_{-C \leq \tau \leq 0} (\psi_\tau(x) + t\tau).$$

By Lemma 4.6 below, the family of functions  $\tau \mapsto \phi_\tau^\varepsilon(x)$  decreases pointwise to the function  $\tau \mapsto \psi_\tau(x)$  as  $\varepsilon \rightarrow 0^+$  for  $\tau < 0$ . Using  $\tau$ -concavity and the fact that  $\psi_0(x) > -\infty$ , one can extend this convergence to  $\tau = 0$  as well. Hence, by Dini's theorem the convergence is uniform on  $[-C, 0]$ . It thus follows that  $u_t^\varepsilon(x) \searrow u_t(x)$  as  $\varepsilon \rightarrow 0^+$ . We conclude that  $u_t^\varepsilon$  decreases to  $u_t$  a.e. on  $X$ . But these are  $\omega$ -psh functions, so the convergence holds everywhere on  $X$ .

That  $d_p^c(\{u_t^\varepsilon\}_t, \{u_t\}_t) \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$  simply follows from Lemma 4.3.

Since  $\phi_\tau^\varepsilon = 0$  for  $\tau \leq -1/\varepsilon$  and  $\psi_\tau \leq \phi_\tau^\varepsilon$ , basic properties of Legendre transforms imply that  $u_t \leq u_t^\varepsilon \leq 0$  and  $-\frac{t}{\varepsilon} \leq u_t^\varepsilon \leq 0$ , since  $\psi_\tau \leq \phi_\tau^\varepsilon$  for all  $\tau$  and  $\phi_\tau^\varepsilon = 0$  for  $\tau < -\frac{1}{\varepsilon}$ . This immediately yields (22) with  $\varepsilon = \frac{1}{j}$ . □

**Lemma 4.6** Assume that  $\{u_t\}_t \in \mathcal{R}_\omega^1$  satisfies  $\sup_X u_t = 0$  for all  $t \geq 0$ . Then, for  $\psi_\tau$  as defined in (23), we have that  $\int_X \omega_{\psi_\tau}^n > 0$  for all  $\tau < 0$ . Additionally, for any  $\tau < 0$ ,

$$(26) \quad \lim_{\varepsilon \rightarrow 0} \phi_\tau^\varepsilon = \lim_{\varepsilon \rightarrow 0} P[(1 + \varepsilon\tau)\psi_\tau] = \psi_\tau.$$

**Proof** By the involution property, application of the Legendre transform twice gives back the original convex function. In particular,  $\sup_\tau \psi_\tau(x) = \lim_{\tau \rightarrow -\infty} \psi_\tau(x) = u_0(x)$  for all  $x \in X$  such that  $\lim_{t \rightarrow 0} u_t(x) = 0$ . In particular, we get that  $\psi_\tau$  increases a.e. to 0 as  $\tau \rightarrow -\infty$ . According to [37, Remark 2.5] we obtain that  $\lim_{\tau \rightarrow -\infty} \int_X \omega_{\psi_\tau}^n = \int_X \omega^n > 0$ .

Fixing  $\tau < 0$ , this last identity implies the existence of  $\tau_0 < \tau$  such that  $\int_X \omega_{\psi_{\tau_0}}^n > 0$ . By  $\tau$ -concavity of  $\tau \rightarrow \psi_\tau$  we get that

$$\psi_\tau \geq \frac{\tau}{\tau_0} \psi_{\tau_0} + \left(1 - \frac{\tau}{\tau_0}\right) \psi_0.$$

Finally, by monotonicity [80, Theorem 1.2] and the multilinearity of the nonpluripolar mass we obtain that  $\int_X \omega_{\psi_\tau}^n > 0$ , as desired.

To argue (26), we start by noting that  $\lim_{\varepsilon \rightarrow 0} P[(1 + \varepsilon\tau)\psi_\tau] \geq \psi_\tau$ , and according to [80, Theorem 1.2; 37, Theorem 2.3 and Remark 2.5] we get that

$$\int_X \omega_{\psi_\tau}^n \leq \int_X \omega_{\lim_\varepsilon P[(1 + \varepsilon\tau)\psi_\tau]}^n \leq \lim_\varepsilon \int_X \omega_{P[(1 + \varepsilon\tau)\psi_\tau]}^n = \lim_{\varepsilon \rightarrow 0} \int_X \omega_{(1 + \varepsilon\tau)\psi_\tau}^n.$$

By multilinearity of the nonpluripolar Monge–Ampère product [15, Proposition 1.4] the last limit is  $\int_X \omega_{\psi_\tau}^n$ . Hence we have equality everywhere, and all the integrals are positive. Consequently,  $\lim_{\varepsilon \rightarrow 0} P[(1 + \varepsilon\tau)\psi_\tau] \in F_{\psi_\tau}$  with the notation of [37, Theorem 3.12].

It follows from [34, Proposition 5.1] (or [38, Lemma 3.17]) that  $P[\psi_\tau] = \psi_\tau$  for all  $\tau \leq 0$  (the result in these works is only stated for rays of bounded potentials, however the proof only uses the comparison principle that holds for finite-energy rays as well, implying the result for these more general rays). Putting everything together, Theorem 3.12 of [37] implies that  $\lim_{\varepsilon \rightarrow 0} P[(1 + \varepsilon\tau)\psi_\tau] \leq P[\psi_\tau] = \psi_\tau$ , as desired.  $\square$

**The construction of geodesic segments in  $\mathcal{R}_\omega^p$**  Next we show that points of  $(\mathcal{R}_\omega^p, d_p^c)$  can be connected by geodesic segments. We first treat the case  $p > 1$ , where, due to uniform convexity, the construction can be carried out directly. The case  $p = 1$  will be treated using approximation, via Theorem 4.5.

**Theorem 4.7** If  $p > 1$ , then  $(\mathcal{R}_\omega^p, d_p^c)$  is a complete geodesic metric space.

The proof below shares similarities with the angle bisection techniques of [63].

**Proof** By Theorem 4.2, we only have to show that any two rays  $\{u_t\}_t, \{v_t\}_t \in \mathcal{R}_\omega^p$  can be joined by a distinguished  $d_p^c$ -geodesic when  $p > 1$ .

For any  $t \geq 0$ , we denote by  $[0, 1] \ni \alpha \rightarrow h_{t,\alpha} \in \mathcal{E}_\omega^p$  the finite-energy geodesic connecting  $u_t$  and  $v_t$ . Then  $d_p(u_t, h_{t,\alpha}) = d_p(h_{t,0}, h_{t,\alpha}) = d_p(h_{t,0}, h_{t,1}) = \alpha d_p(u_t, v_t)$ . To avoid introducing further variables, by  $[0, t] \ni s \rightarrow \frac{s}{t} h_{t,\alpha} \in \mathcal{E}_\omega^p$  we denote the finite-energy geodesic connecting 0 and  $h_{t,\alpha}$ . Finally, we can assume that  $u_t \neq v_t$  for  $t$  large enough. Indeed, if this does not hold, then (1) would give that  $\{u_t\}_t = \{v_t\}_t$  and the geodesic connecting the two rays is the constant one.

First we show that for any  $\alpha \in [0, 1]$  and  $l \geq 0$  there exists  $w_{l,\alpha} \in \mathcal{E}_\omega^p$  such that  $\lim_{t \rightarrow \infty} \frac{l}{t} h_{t,\alpha} = w_{l,\alpha}$ . By endpoint stability of geodesic segments [8, Proposition 4.3], this will automatically imply that  $\{w_{t,\alpha}\}_t \in \mathcal{R}_\omega^p$ . As we will see,  $\alpha \rightarrow \{w_{t,\alpha}\}_t$  will represent the  $d_p^c$ -geodesic connecting  $\{u_t\}_t$  and  $\{v_t\}_t$ .

Again, from (1) it follows that for any  $\alpha \in [0, 1]$  and  $0 < s \leq t$  we have

$$(27) \quad \frac{d_p(u_s, \frac{s}{t} h_{t,\alpha})}{s} \leq \frac{d_p(u_t, h_{t,\alpha})}{t} = \frac{\alpha d_p(u_t, v_t)}{t} \leq \alpha d_p^c(\{u_t\}_t, \{v_t\}_t),$$

$$(28) \quad \frac{d_p(v_s, \frac{s}{t} h_{t,\alpha})}{s} \leq \frac{d_p(v_t, h_{t,\alpha})}{t} = \frac{(1-\alpha) d_p(u_t, v_t)}{t} \leq (1-\alpha) d_p^c(\{u_t\}_t, \{v_t\}_t),$$

where the identities in the middle follow from the fact that  $t \rightarrow h_{t,\alpha}$  is a geodesic. We fix  $\varepsilon > 0$ . Since  $\frac{1}{s} d_p(u_s, v_s) \nearrow d_p^c(\{u_t\}_t, \{v_t\}_t)$ , (27) and (28) imply existence of  $s_{\alpha,\varepsilon} > 0$  such that for any  $s_{\alpha,\varepsilon} \leq s \leq t$  we have

$$\frac{d_p(u_s, \frac{s}{t} h_{t,\alpha})}{s} \leq (\alpha + \varepsilon) \frac{d_p(u_s, v_s)}{s} \quad \text{and} \quad \frac{d_p(v_s, \frac{s}{t} h_{t,\alpha})}{s} \leq (1 - \alpha + \varepsilon) \frac{d_p(u_s, v_s)}{s}.$$

Now Proposition 3.6 implies that  $d_p(h_{s,\alpha}, \frac{s}{t} h_{t,\alpha}) \leq \varepsilon^{1/r} C d_p(u_s, v_s)$  for any  $s_{\alpha,\varepsilon} \leq s \leq t$ . In particular, using (1), for any fixed  $l > 0$  such that  $\max(l, s_{\alpha,\varepsilon}) \leq s \leq t$  we have

$$(29) \quad \frac{d_p(\frac{l}{s} h_{s,\alpha}, \frac{l}{t} h_{t,\alpha})}{l} \leq \frac{d_p(h_{s,\alpha}, \frac{s}{t} h_{t,\alpha})}{s} \leq \varepsilon^{1/r} C \cdot \frac{d_p(u_s, v_s)}{s} \leq \varepsilon^{1/r} C d_p^c(\{u_t\}_t, \{v_t\}_t).$$

By shrinking  $\varepsilon$ , the expression on the right can be chosen to be as small as we want, implying that the sequence  $\{\frac{l}{t} h_{t,\alpha}\}_t \in \mathcal{E}_\omega^p$  is  $d_p$ -Cauchy. This is the crucial step! By [32, Theorem 2],  $(\mathcal{E}_\omega^p, d_p)$  is complete, hence  $\lim_t \frac{l}{t} h_{t,\alpha} =: w_{l,\alpha} \in \mathcal{E}_\omega^p$ , as proposed.

Moreover, letting  $t \rightarrow \infty$  on the left-hand side of (27) and (28), we obtain that

$$\frac{d_p(u_s, w_{s,\alpha})}{s} \leq \alpha d_p^c(\{u_t\}_t, \{v_t\}_t), \quad \frac{d_p(v_s, w_{s,\alpha})}{s} \leq (1-\alpha) d_p^c(\{u_t\}_t, \{v_t\}_t), \quad s > 0.$$

Letting  $s \rightarrow \infty$ , together with the triangle inequality we obtain

$$\begin{aligned} d_p^c(\{u_t\}_t, \{v_t\}_t) &\leq d_p^c(\{u_t\}_t, \{w_{t,\alpha}\}_t) + d_p^c(\{w_{t,\alpha}\}_t, \{v_t\}_t) \\ &\leq \alpha d_p^c(\{u_t\}_t, \{v_t\}_t) + (1-\alpha) \alpha d_p^c(\{u_t\}_t, \{v_t\}_t) \\ &= d_p^c(\{u_t\}_t, \{v_t\}_t). \end{aligned}$$

Thus everything is equal. In particular, we have  $d_p^c(\{u_t\}_t, \{w_{t,\alpha}\}_t) = \alpha d_p^c(\{u_t\}_t, \{v_t\}_t)$  and  $d_p^c(\{w_{t,\alpha}\}_t, \{v_t\}_t) = (1-\alpha) d_p^c(\{u_t\}_t, \{v_t\}_t)$ .

Suppose now that  $0 \leq \alpha \leq \beta \leq 1$ . These last two identities together with the triangle inequality give that

$$(30) \quad (\beta - \alpha) d_p^c(\{u_t\}_t, \{v_t\}_t) \leq d_p^c(\{w_{t,\beta}\}_t, \{w_{t,\alpha}\}_t).$$

To finish the proof we show that equality holds in this estimate. Indeed, another application of (1) gives that

$$\frac{d_p(\frac{l}{s} h_{s,\alpha}, \frac{l}{s} h_{s,\beta})}{l} \leq \frac{d_p(h_{s,\alpha}, h_{s,\beta})}{s} = \frac{(\beta - \alpha) d_p(u_s, v_s)}{s}, \quad s > 0.$$

Letting  $s \rightarrow \infty$  in this estimate, and after that  $l \rightarrow \infty$ , the reverse inequality in (30) follows, finishing the proof. □

The  $d_p^c$ -geodesic segment  $[0, 1] \ni \alpha \rightarrow \{w_{t,\alpha}\}_t \in \mathcal{R}_\omega^p$  constructed in the above theorem will be called the  $d_p^c$ -chord joining  $\{w_{t,0}\}$  and  $\{w_{t,1}\}$ , as this curve is reminiscent of the chords joining the different points in the unit sphere of  $\mathbb{R}^n$ .

Finally, using approximation, we point out that the same result holds for  $p = 1$  as well. First we remark that  $d_p^c$ -chords are automatically  $d_{p'}^c$ -chords for any  $p' \leq p$ . This observation is of independent interest, and is the “radial version” of a well-known phenomenon for the family of metric spaces  $(\mathcal{E}_\omega^p, d_p)$  for  $p \geq 1$ :

**Proposition 4.8** *Let  $1 \leq p' < p$  and  $\{u_t\}_t, \{v_t\}_t \in \mathcal{R}_\omega^p$ . Trivially  $\{u_t\}_t, \{v_t\}_t \in \mathcal{R}_\omega^{p'}$ , and the  $d_p^c$ -chord  $[0, 1] \ni \alpha \rightarrow \{w_{t,\alpha}\}_t \in \mathcal{R}_\omega^p$  connecting  $\{u_t\}_t, \{v_t\}_t$  is also a  $d_{p'}^c$ -chord.*

**Proof** To start, we trace the steps in the proof of Theorem 4.7 and notice that the curves  $\alpha \rightarrow h_{t,\alpha}$ , introduced in the argument, did not depend on the particular choice of  $p$ .

Fixing  $l \geq 0$  and  $\alpha \in [0, 1]$ , the crux of the proof is the fact that  $d_p(\frac{l}{s}h_{s,\alpha}, \frac{l}{s}h_{s,\alpha}) \rightarrow 0$  as  $s, t \rightarrow \infty$ , which follows from uniform convexity (when  $p > 1$ ), as elaborated in (29). Since  $1 \leq p' < p$ , we have that  $d_{p'}(\cdot, \cdot) \leq d_p(\cdot, \cdot)$  and  $\mathcal{E}_\omega^p \subset \mathcal{E}_\omega^{p'}$ , hence the same conclusion holds for  $p'$  as well:

$$d_{p'}\left(\frac{l}{s}h_{s,\alpha}, \frac{l}{t}h_{t,\alpha}\right) \leq d_p\left(\frac{l}{s}h_{s,\alpha}, \frac{l}{t}h_{t,\alpha}\right) \rightarrow 0 \quad \text{as } s, t \rightarrow 0.$$

The rest of the proof does not use uniform convexity, and goes through without any difficulties for  $p'$  in place of  $p$ , arriving at the conclusion that the chord  $[0, 1] \ni \alpha \rightarrow \{w_{t,\alpha}\}_t \in \mathcal{R}_\omega^p \subset \mathcal{R}_\omega^{p'}$  is a  $d_{p'}$ -chord as well.  $\square$

**Theorem 4.9**  $(\mathcal{R}_\omega^1, d_1^c)$  is a complete geodesic metric space. Moreover, the  $d_1^c$ -chords of this space can be constructed by the method of Theorem 4.7.

**Proof** Given  $\{w_{t,0}\}_t, \{w_{t,1}\}_t \in \mathcal{R}_\omega^1$ , we will show that there exists a  $d_1^c$ -chord  $[0, 1] \ni \alpha \mapsto \{w_{t,\alpha}\}_t \in \mathcal{R}_\omega^1$  joining  $\{w_{t,0}\}_t$  and  $\{w_{t,1}\}_t$ .

Fix any  $p > 1$ . Using Theorem 4.5 we can find  $\{w_{t,0}^k\}_t, \{w_{t,1}^k\}_t \in \mathcal{R}_\omega^p \subset \mathcal{R}_\omega^1$  such that  $w_{t,0}^k \searrow w_{t,0}$  and  $w_{t,1}^k \searrow w_{t,1}$  for all  $t \geq 0$ . Let  $[0, 1] \ni \alpha \rightarrow \{w_{t,\alpha}^k\}_t \in \mathcal{R}_\omega^p \subset \mathcal{R}_\omega^1$  be the  $d_1^c$ -geodesic joining  $\{w_{t,0}^k\}_t, \{w_{t,1}^k\}_t$ , which exists by Proposition 4.8.

We look at the construction of the curves  $\alpha \rightarrow \{w_{t,\alpha}^k\}$  in the proof of Theorem 4.7 and attempt to construct  $\alpha \rightarrow \{w_{t,\alpha}\}$  using the same method.

Using the fact that  $d_1(u, v) = I(u) - I(v)$  for  $u \geq v$ , and affinity of  $I$  along finite-energy geodesics, one deduces that for any  $\alpha \in [0, 1]$  and  $0 \leq s < t$  we have

$$\begin{aligned} (31) \quad d_1\left(\frac{s}{t}h_{t,\alpha}^k, \frac{s}{t}h_{t,\alpha}\right) &= I\left(\frac{s}{t}h_{t,\alpha}^k\right) - I\left(\frac{s}{t}h_{t,\alpha}\right) \\ &= \frac{s}{t}I(h_{t,\alpha}^k) - \frac{s}{t}I(h_{t,\alpha}) \\ &= \frac{s(1-\alpha)}{t}(I(w_{t,0}^k) - I(w_{t,0})) + \frac{s\alpha}{t}(I(w_{t,1}^k) - I(w_{t,1})) \\ &= s(1-\alpha)(I(w_{1,0}^k) - I(w_{1,0})) + s\alpha(I(w_{1,1}^k) - I(w_{1,1})), \end{aligned}$$

with the last expression converging to zero regardless of the values of  $t > 0$ . From here we get that  $d_1(\frac{s}{t}h_{t,\alpha}^k, \frac{s}{t}h_{t,\alpha}) \rightarrow 0$  as  $k \rightarrow \infty$ , uniformly with respect to  $t$ .

On the other hand, by Proposition 4.8 (and its proof) we get that  $d_1^c(\frac{s}{t}h_{t,\alpha}^k, w_{s,\alpha}^k) \rightarrow 0$  as  $t \rightarrow \infty$  for any fixed  $k \geq 0$ .

By construction, each sequence  $\{w_{s,\alpha}^k\}_k \in \mathcal{E}_\omega^1$  is decreasing and  $d_1$ -bounded; hence, by [32, Lemma 4.16], there exists  $\{w_{s,\alpha}\} \in \mathcal{E}_\omega^1$  such that  $d_1(w_{s,\alpha}^k, w_{s,\alpha}) \rightarrow 0$  as  $k \rightarrow \infty$ .

Lastly, the triangle inequality gives

$$d_1\left(\frac{s}{t}h_{t,\alpha}, w_{s,\alpha}\right) \leq d_1\left(\frac{s}{t}h_{t,\alpha}^k, \frac{s}{t}h_{t,\alpha}\right) + d_1^c\left(\frac{s}{t}h_{t,\alpha}^k, w_{s,\alpha}^k\right) + d_1(w_{s,\alpha}^k, w_{s,\alpha}).$$

Putting everything together, for  $s \geq 0$  fixed the first and last terms on the right-hand side can be made arbitrarily small for big  $k$ . Next, with  $k$  fixed, the same is true for the middle term for big  $t$ , ie  $d_1\left(\frac{s}{t}h_{t,\alpha}, w_{s,\alpha}\right) \rightarrow 0$  as  $t \rightarrow \infty$ .

As pointed out in the proof of Proposition 4.8, with this last fact in hand the rest of the proof of Theorem 4.7 goes through without any issues for  $p = 1$ . □

**Convexity of the radial K–energy** Let  $p \geq 1$ . The *radial K–energy* is defined for any  $\{u_t\}_t \in \mathcal{R}_\omega^p$ , and is given by the expression

$$\mathcal{K}\{u_t\} := \lim_{t \rightarrow \infty} \frac{\mathcal{K}(u_t)}{t},$$

where  $\mathcal{K}: \mathcal{E}_\omega^p \rightarrow (-\infty, \infty]$  is the extended K–energy of Mabuchi from [5; 8]. In the setting of unit-speed geodesics, this definition agrees with the  $\mathbb{Y}$  invariant of [27]. Also, there is clear parallel with the non-Archimedean K–energy of [13].

**Lemma 4.10** *Let  $\{u_t\}_t \in \mathcal{R}_u^p$  and  $\{v_t\}_t \in \mathcal{R}_v^p$  be parallel, with  $u, v \in \mathcal{E}^p$  satisfying  $\mathcal{K}(u) < +\infty$  and  $\mathcal{K}(v) < +\infty$ . Then  $\mathcal{K}\{u_t\} = \mathcal{K}\{v_t\}$ .*

**Proof** By the proof of Proposition 4.1 we can assume that either  $u \leq v$  or  $v \leq u$ .

For each  $t > 0$  let  $[0, t] \ni l \mapsto v_l^t \in \mathcal{E}_\omega^p$  be the finite-energy geodesic connecting  $v_0^t := v$  and  $v_t^t := u_t$ . It follows from Proposition 4.1 (and its proof) that  $\lim_{t \rightarrow +\infty} d_p(v_l^t, v_l) = 0$  for each  $l$  fixed. By convexity of  $\mathcal{K}$  [8, Theorem 1.2], for any  $0 < l < t$  we have that

$$\mathcal{K}(v_l^t) \leq \left(1 - \frac{l}{t}\right)\mathcal{K}(v) + \frac{l}{t}\mathcal{K}(u_t).$$

Thus, letting  $t \rightarrow +\infty$  and using lower semicontinuity of  $\mathcal{K}$  with respect to  $d_p$  [8, Theorem 4.7], we obtain

$$\frac{\mathcal{K}(v_l)}{l} \leq \frac{\mathcal{K}(v)}{l} + \mathcal{K}\{u_t\}.$$

Letting  $l \rightarrow +\infty$  yields  $\mathcal{K}\{v_t\} \leq \mathcal{K}\{u_t\}$ . The reverse inequality is obtained by reversing the roles of  $u$  and  $v$ . □

By the above lemma it makes sense to restrict to  $\mathcal{R}_\omega^p$  when considering the radial K–energy. Since  $d_p^c$ –convergence implies  $d_1^c$ –convergence it follows from Proposition 5.9

of [27] that the resulting functional

$$\mathcal{K}\{\cdot\}: \mathcal{R}_\omega^p \rightarrow (-\infty, \infty]$$

is  $d_p^c$ -lsc. In the last result of this section we point out that  $\mathcal{K}\{\cdot\}$  is also convex along the chords of  $\mathcal{R}_\omega^p$  for any  $p \geq 1$ :

**Theorem 4.11** *Suppose that  $p \geq 1$  and  $[0, 1] \ni \alpha \rightarrow \{w_{t,\alpha}\}_t \in \mathcal{R}_\omega^p$  is a  $d_p$ -chord joining  $\{u_t\}_t, \{v_t\}_t \in \mathcal{R}_\omega^p$ . Then  $\alpha \rightarrow \mathcal{K}\{w_{t,\alpha}\}$  is convex.*

**Proof** We use the notation and terminology of the proof of Theorems 4.7 and 4.9, and normalize  $\mathcal{K}$  such that  $\mathcal{K}(0) = 0$ . Using convexity of  $\mathcal{K}$  along finite-energy geodesics [8, Theorem 1.2] we know that for any  $0 < s \leq t$  and  $\alpha \in [0, 1]$  we have

$$\frac{\mathcal{K}(\frac{s}{t}h_{t,\alpha})}{s} \leq \frac{\mathcal{K}(h_{t,\alpha})}{t} \leq (1-\alpha)\frac{\mathcal{K}(u_t)}{t} + \alpha\frac{\mathcal{K}(v_t)}{t}.$$

Since  $d_p(\frac{s}{t}h_{t,\alpha}, w_{s,\alpha}) \rightarrow 0$ , given that  $\mathcal{K}$  is  $d_p$ -lsc [8, Theorem 1.2] it follows that

$$\frac{\mathcal{K}(w_{s,\alpha})}{s} \leq \liminf_{t \rightarrow \infty} \frac{\mathcal{K}(\frac{s}{t}h_{t,\alpha})}{s} \leq (1-\alpha)\mathcal{K}\{u_t\} + \alpha\mathcal{K}\{v_t\}.$$

The result now follows after taking the limit  $s \rightarrow \infty$ . □

**Remark 4.12** Many theorems that hold for the finite-energy metric spaces  $(\mathcal{E}_\omega^p, d_p)$  admit a radial version for  $(\mathcal{R}_\omega^p, d_p)$ . As we already pointed out, Theorem 1.4, Lemma 4.3 and also Theorem 5.1 below are examples of this phenomenon. This does not seem to be limited to only these results either. Indeed, though we will not pursue this further here, one can introduce radial analogs of the operators  $\max(\cdot, \cdot)$  and  $P(\cdot, \cdot)$ , and similar identities/inequalities/results hold for these as the ones described in [32; 33].

## 5 Approximation with converging radial K-energy

### Approximation with rays of bounded potentials

The goal of this subsection is to strengthen the conclusion of Theorem 4.5 and obtain Theorem 1.5(i) in the process:

**Theorem 5.1** *Let  $\{u_t\}_t \in \mathcal{R}_\omega^p$  with  $p \geq 1$ . Then there exists a sequence  $\{u_t^j\}_t \in \mathcal{R}_\omega^\infty$  such that  $u_t^j$  decreases to  $u_t$ , for each  $t > 0$  fixed and  $\mathcal{K}\{u_t^j\} \rightarrow \mathcal{K}\{u_t\}$ . In particular,  $\lim_{j \rightarrow +\infty} d_p^c(\{u_t^j\}, \{u_t\}) = 0$ . In addition,  $\sup_X u_t^j = \sup_X u_t$  for all  $j, t > 0$ .*

When  $\mathcal{K}\{u_t\} = +\infty$ , by the fact that  $\mathcal{K}\{\cdot\}$  is  $d_p^c$ -lsc [27, Proposition 5.9], we will simply invoke Theorem 4.5 for the existence of the sequence of  $\{u_t^j\}_t$ . If  $\mathcal{K}\{u_t\}$  is finite, we will need a much more delicate argument, resting on the relative Kołodziej-type estimate of [39, Theorem 3.3], as detailed in the argument below.

At places, the argument below shares some similarities with the proof of Theorem 3.2 of [41], with the relative Kołodziej-type estimate of [39] taking the place of Perelman’s estimates along the Kähler–Ricci flow on Fano manifolds. Before engaging in the proof of Theorem 5.1, we prove an auxiliary lemma:

**Lemma 5.2** *Let  $\{u_t\}_t \in \mathcal{R}_\omega^1$  with  $\sup_X u_t = 0$  for  $t \geq 0$ . Then*

$$(32) \quad \lim_{j \rightarrow \infty} \limsup_{t \rightarrow +\infty} \int_{\{u_t \leq -jt\}} \frac{(-u_t)}{t} \omega_{u_t}^n = 0.$$

**Proof** It follows from Theorem 4.5 that we can choose  $\{u_t^j\}_t \in \mathcal{R}_\omega^1$  such that  $u_t \leq \max(u_t, -jt) \leq u_t^j \leq 0$  and  $d_1^c(\{u_t^j\}, \{u_t\}) = 0$ . As  $u_t^j \geq u_t$ , by [32, Corollary 4.14] we have that  $d_1(u_t^j, u_t) = I(u_t^j) - I(u_t)$ . Hence  $\lim_{t \rightarrow +\infty} \frac{1}{t}(I(u_t^j) - I(u_t)) = 0$ .

From monotonicity and elementary properties of  $I(\cdot)$  we conclude that

$$\lim_{t \rightarrow \infty} \frac{I(\max(u_t, -jt)) - I(u_t)}{t} = 0,$$

ultimately implying

$$0 \leq \lim_j \lim_{t \rightarrow \infty} \int_X \frac{\max(u_t, -jt) - u_t}{t} \omega_{u_t}^n \leq (n+1) \lim_j \lim_{t \rightarrow \infty} \frac{I(\max(u_t, -jt)) - I(u_t)}{t} = 0.$$

Consequently both limits are equal to zero, and, on the set  $\{u_t \leq -2jt\}$ , we have  $0 \geq \max(u_t, -jt) - u_t \geq -\frac{1}{2}u_t$ . This and the above together yield (32).  $\square$

**Proof of Theorem 5.1** Using Theorem 4.5 and the fact that  $\mathcal{K}\{\cdot\}$  is  $d_p^c$ -lsc, we can assume that  $\mathcal{K}\{u_t\} < +\infty$ . Also, via Lemma 2.2, by possibly adding  $Ct$  to  $u_t$  we can additionally assume that  $\sup_X u_t = 0$ , ie  $t \rightarrow u_t$  is  $t$ -decreasing with  $u_\infty := \lim_{t \rightarrow \infty} u_t \in \text{PSH}(X, \omega)$ .

For each  $j > 1$  and  $l > 1$ , we let  $\varphi_l^j \in \mathcal{E}(X, \omega)$  be the unique  $\omega$ -psh function, whose existence is guaranteed by [56, Theorem A], such that

$$(33) \quad \omega_{\varphi_l^j}^n = \left(1 - \frac{1}{2j}\right) \mathbb{1}_{\{u_l > -jl\}} \omega_{u_l}^n + a_{j,l} \omega^n, \quad \sup_X \varphi_l^j = 0,$$

where  $0 \leq a_{j,l} \leq 1$  is a constant arranged so that the measure on the right-hand side has total mass equal to  $\int_X \omega^n$ .

Next we note that the conditions of Theorem 2.3 are satisfied with appropriate choice of data. Let  $a := (1 - 1/2^j)^{1/2} \in (0, 1)$ ,  $u := \varphi_l^j$ ,  $\chi := (1 - 1/2^j)^{1/2n} \max(u_l, -jl)$  and  $f := 1$ . Then, using locality of the nonpluripolar complex Monge–Ampère measure (see eg [56, Corollary 1.7]) we have that  $\omega_{\max(u_l, -jl)}^n \geq \mathbb{1}_{\{u_l > -jl\}} \omega_{u_l}^n$ ; hence,

$$\omega_u^n \leq a\omega_\chi^n + f\omega^n.$$

Moreover, due to [15, Proposition 4.3; 39, Lemma 4.2], there exists  $A(X, \omega) > 0$  such that for any Borel set  $E \subset X$  we have

$$\int_E f\omega^n = \int_E \omega^n \leq A \text{Cap}_\omega(E)^2 \leq A \left(1 - \left(1 - \frac{1}{2^j}\right)^{1/2n}\right)^{-2n} \text{Cap}_\chi(E)^2,$$

where  $\text{Cap}_\omega$  is the usual Monge–Ampère capacity and  $\text{Cap}_\chi$  is its relative version from [39, Section 3]. Lastly, we note that  $\chi \leq 0 = P[\varphi_l^j]$ , due to [34, Theorem 3], hence all the conditions of Theorem 2.3 are satisfied, to imply that

$$(34) \quad \varphi_l^j = u \geq \chi - C_j \geq \max(u_l, -jl) - C_j,$$

where  $C_j > 0$  is a constant depending on  $j$ , but not  $l > 1$ ! In particular,  $\varphi_l^j$  is bounded.

Moreover, for  $1 < j < k$  and  $l > 1$  we have

$$\omega_{\varphi_l^j}^n \leq \frac{1 - 2^{-j}}{1 - 2^{-k}} \omega_{\varphi_l^k}^n + \omega^n.$$

Similarly to (33), this allows for another application of Theorem 2.3, with the choice of data  $a := ((1 - 2^{-j}) / (1 - 2^{-k}))^{1/2} \in (0, 1)$ ,  $u := \varphi_l^j$ ,  $\chi := ((1 - 2^{-j}) / (1 - 2^{-k}))^{1/2n} \varphi_l^k$  and  $f := 1$ . Similarly to the above, the conditions of Theorem 2.3 are satisfied to yield that

$$(35) \quad \varphi_l^j = u \geq \chi - C_{j,k} \geq \varphi_l^k - C_{j,k},$$

where  $C_{j,k} > 0$  depends on  $j$  and  $k$ , but not on  $l > 1$ !

For each  $l > 1$  let  $[0, l] \ni t \mapsto u_t^{j,l}$  be the bounded geodesic segment joining 0 and  $\varphi_l^j + C_j$ . Then (34) and (35) together with the comparison principle for finite-energy geodesics imply that

$$(36) \quad \frac{C_j t}{l} \geq u_t^{j,l} \geq \max(u_t, -jt), \quad t \in [0, l],$$

and

$$(37) \quad \frac{C_j t}{l} \geq u_t^{j,l} \geq u_t^{k,l} - \frac{D_{j,k} t}{l}, \quad 0 < j < k, t \in [0, l],$$

where  $D_{j,k}$  depends on  $j$  and  $k$  but not on  $l > 1$ .

To show that the above geodesic sequences subconverge to appropriate geodesic rays, we first prove a number of estimates in the claims below.

**Claim 1** For any  $j > 1$  we have

$$\limsup_{t \rightarrow +\infty} \frac{\text{Ent}(\omega^n, \omega_{\varphi_t^j}^n)}{t} \leq \limsup_{t \rightarrow +\infty} \frac{\text{Ent}(\omega^n, \omega_{u_t}^n)}{t}.$$

Since  $\text{Ent}(\omega^n, \omega_{u_t}^n) < +\infty$  (because  $\mathcal{K}(u_t) < +\infty$ ), for any  $t \geq 0$ , we can write  $\omega_{\varphi_t^j}^n = f_{t,j} \omega^n$  and  $\omega_{u_t}^n = f_t \omega^n$ . Observe first that for any  $g_t \geq 0$  with  $\int_X g_t \omega^n = \int_X \omega^n$  we have that

$$\limsup_{t \rightarrow +\infty} \int_X \frac{g_t \log(g_t)}{t} \omega^n = \limsup_{t \rightarrow +\infty} \int_X \frac{(g_t + B) \log(g_t + B)}{t} \omega^n \quad \text{for all } B \geq 1.$$

This follows after splitting up both integrals using the partition  $\{0 \leq g_t \leq C\}$  and  $\{C < g_t\}$  for  $C > 0$  big and noticing that the lim sup of integrals on  $\{0 \leq g_t \leq C\}$  is always zero.

By construction, see (33),  $1 \leq f_{t,j} + 1 \leq f_t + 2$  and hence, since  $s \mapsto s \log(s)$  for  $s > 1$  is increasing,  $(f_{t,j} + 1) \log(f_{t,j} + 1) \leq (f_t + 2) \log(f_t + 2)$ . Using the above we then conclude

$$\begin{aligned} \limsup_{t \rightarrow +\infty} \int_X \frac{f_{t,j} \log(f_{t,j})}{t} \omega^n &= \limsup_{t \rightarrow +\infty} \int_X \frac{(f_{t,j} + 1) \log(f_{t,j} + 1)}{t} \omega^n \\ &\leq \limsup_{t \rightarrow +\infty} \int_X \frac{(f_t + 2) \log(f_t + 2)}{t} \omega^n \\ &= \limsup_{t \rightarrow +\infty} \int_X \frac{f_t \log f_t}{t} \omega^n. \end{aligned}$$

**Claim 2** We have

$$\lim_j \limsup_{t \rightarrow +\infty} \frac{\mathcal{I}(\varphi_t^j, u_t)}{t} = 0.$$

Before we start with the argument, we recall that  $\mathcal{I}(v, w) = \int_X (v - w)(\omega_w^n - \omega_v^n)$  for  $v, w \in \mathcal{E}_\omega^1$ . By (33) we have

$$\mathcal{I}(\varphi_t^j, u_t) \leq \frac{1}{2j} \int_X |\varphi_j^t - u_t| \omega_{u_t}^n + \int_{\{u_t \leq -jt\}} |\varphi_j^t - u_t| \omega_{u_t}^n + \int_X |\varphi_j^t - u_t| \omega^n,$$

and the claim follows from the following three estimates. First, the estimate of (34) and basic properties of  $\mathcal{I}(\cdot)$  give that

$$(38) \quad \lim_j \limsup_{t \rightarrow +\infty} \frac{1}{2j} \int_X \frac{|\varphi_j^t - u_t|}{t} \omega_{u_t}^n \leq \lim_j \limsup_{t \rightarrow +\infty} \frac{1}{2j} \frac{C_j}{t} + \lim_j \limsup_{t \rightarrow +\infty} \frac{1}{2j} \frac{|I(u_t)|}{t} = 0.$$

Second, by the dominated convergence theorem we have that

$$(39) \quad \lim_{t \rightarrow +\infty} \int_X \frac{|\varphi_t^j - u_t|}{t} \omega^n \leq \lim_{t \rightarrow +\infty} \int_X \frac{|C_j - u_\infty|}{t} \omega^n = 0.$$

Third, by Lemma 5.2 and (34),

$$(40) \quad \lim_j \limsup_{t \rightarrow +\infty} \int_{\{u_t \leq -jt\}} \frac{|\varphi_t^j - u_t|}{t} \omega_{u_t}^n \leq \lim_j \limsup_{t \rightarrow +\infty} \int_{\{u_t \leq -jt\}} \frac{|u_t| + C_j}{t} \omega_{u_t}^n = 0.$$

**Claim 3** We have

$$\lim_j \limsup_{t \rightarrow +\infty} \frac{|I(\varphi_t^j) - I(u_t)|}{t} = 0.$$

This claim follows from Claim 2 and Lemma 5.3 below, with  $\varphi_1 = \varphi_t^j$ ,  $\varphi_2 = u_t$  and  $\psi = 0$ . Indeed, given (34), we have that  $\max(-I(\varphi_t^j), -I(u_t)) \simeq Ct + C_j$  for a uniform constant  $C$ . Lemma 5.3 then gives

$$\left| \int_X (\varphi_t^j - u_t)(\omega_{u_t}^n - \omega^n) \right| \leq (Ct + C_j) f(\mathcal{I}(\varphi_t^j, u_t)/(Ct + C_j)).$$

Hence,

$$\lim_j \limsup_{t \rightarrow +\infty} \left| \int_X \frac{(\varphi_t^j - u_t)}{t} (\omega_{u_t}^n - \omega^n) \right| = 0.$$

Again, due to (34) and elementary properties of  $I(\cdot)$  we have that

$$0 \leq \limsup_{t \rightarrow +\infty} \frac{I(\varphi_t^j) - I(u_t)}{t} \leq \limsup_{t \rightarrow +\infty} \int_X \frac{\varphi_t^j - u_t}{t} \omega_{u_t}^n.$$

Putting these last two estimates together with (39), the claim follows.

**Claim 4** For any closed smooth real  $(1, 1)$ -form  $\alpha$  we have

$$\lim_j \limsup_{t \rightarrow +\infty} \frac{|I_\alpha(\varphi_t^j) - I_\alpha(u_t)|}{t} = 0.$$

Recall that  $I_\alpha(v) := \sum_{j=0}^{n-1} \int_X v \alpha \wedge \omega_v^j$ . Since  $\alpha$  can be written as the difference of two Kähler forms, and  $I_\alpha(\cdot)$  is monotone when  $\alpha \geq 0$ , notice that the claim follows if we can argue that

$$\lim_j \limsup_{t \rightarrow +\infty} \frac{|I_\omega(\varphi_t^j + C_j) - I_\omega(u_t)|}{t} = \lim_j \limsup_{t \rightarrow +\infty} \frac{I_\omega(\varphi_t^j + C_j) - I_\omega(u_t)}{t} = 0.$$

Using (34) we observe that this last identity is a consequence of

$$\lim_j \limsup_{t \rightarrow +\infty} \frac{I_\omega(\varphi_t^j + C_j) - I_\omega(u_t)}{t} = \lim_j \limsup_{t \rightarrow +\infty} \frac{I_\omega(\varphi_t^j) - I_\omega(u_t)}{t} = 0.$$

However we have that

$$\frac{I_\omega(\varphi_t^j) - I_\omega(u_t)}{t} = \frac{(n + 1)(I(\varphi_t^j) - I(u_t))}{t} - \frac{\int_X \varphi_t^j \omega_{\varphi_t^j}^n - \int_X u_t \omega_{u_t}^n}{t}.$$

So, by Claim 3, it is enough to show that

$$\lim_j \limsup_{t \rightarrow +\infty} \frac{|\int_X \varphi_t^j \omega_{\varphi_t^j}^n - \int_X u_t \omega_{u_t}^n|}{t} = 0.$$

Again, due to (34) and elementary properties of  $I(\cdot)$  we have that

$$\lim_j \limsup_{t \rightarrow +\infty} \frac{|\int_X (\varphi_t^j - u_t) \omega_{\varphi_t^j}^n|}{t} \leq \lim_j \limsup_{t \rightarrow +\infty} \frac{I(\varphi_t^j) - I(u_t)}{t} = 0,$$

where the last identity follows from Claim 3. Due to (33) and (39) we also have

$$\lim_j \limsup_{t \rightarrow +\infty} \frac{|\int_X u_t (\omega_{\varphi_t^j}^n - \omega_{u_t}^n)|}{t} \leq \lim_j \limsup_{t \rightarrow +\infty} \left( \frac{1}{2^j} \int_X \frac{|u_t|}{t} \omega_{u_t}^n + \int_{\{u_t \leq -jt\}} \frac{|u_t|}{t} \omega_{u_t}^n \right) = 0,$$

where the last equality follows from Lemma 5.2 and the fact that  $\int_X |u_t| \omega_{u_t}^n \leq d_1(0, u_t) = t d_1(0, u_1)$  [32, Theorem 3].

**Conclusion** There exists a sequence  $l_k \nearrow +\infty$  such that

$$\lim_{k \rightarrow +\infty} \frac{1}{l_k} \text{Ent}(\omega^n, \omega_{u_{l_k}}^n) = \limsup_{t \rightarrow +\infty} \frac{1}{t} \text{Ent}(\omega^n, \omega_{u_t}^n).$$

It then follows that

$$\begin{aligned} \limsup_{k \rightarrow +\infty} \frac{1}{l_k} (\text{Ent}(\omega^n, \omega_{\varphi_{l_k}^j}^n) - \text{Ent}(\omega^n, \omega_{u_{l_k}}^n)) & \leq \limsup_{k \rightarrow +\infty} \frac{1}{l_k} \text{Ent}(\omega^n, \omega_{\varphi_{l_k}^j}^n) - \lim_k \frac{1}{l_k} \text{Ent}(\omega^n, \omega_{u_{l_k}}^n) \\ & = \limsup_{k \rightarrow +\infty} \frac{1}{l_k} \text{Ent}(\omega^n, \omega_{\varphi_{l_k}^j}^n) - \limsup_{t \rightarrow +\infty} \frac{1}{t} \text{Ent}(\omega^n, \omega_{u_t}^n) \\ & \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \text{Ent}(\omega^n, \omega_{\varphi_t^j}^n) - \limsup_{t \rightarrow +\infty} \frac{1}{t} \text{Ent}(\omega^n, \omega_{u_t}^n). \end{aligned}$$

It thus follows from Claim 1 that

$$(41) \quad \lim_{j \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \frac{1}{l_k} (\text{Ent}(\omega^n, \omega_{\varphi_{l_k}^j}^n) - \text{Ent}(\omega^n, \omega_{u_{l_k}}^n)) \leq 0.$$

We continue with

$$\limsup_{k \rightarrow +\infty} \frac{\mathcal{K}(\varphi_{l_k}^j)}{l_k} \leq \limsup_{k \rightarrow +\infty} \frac{\mathcal{K}(\varphi_{l_k}^j) - \mathcal{K}(u_{l_k})}{l_k} + \limsup_{k \rightarrow +\infty} \frac{\mathcal{K}(u_{l_k})}{l_k}.$$

Thus, using the Chen–Tian formula (8) together with (41) and the estimates of Claims 3 and 4, we can continue to write

$$\lim_{j \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \frac{\mathcal{K}(\varphi_{l_k}^j)}{l_k} \leq \mathcal{K}\{u_t\}.$$

As a result, there exists an increasing sequence  $\{j_m\}_m \subset \mathbb{N}$  such that

$$\limsup_{k \rightarrow +\infty} \frac{\mathcal{K}(\varphi_{l_k}^{j_m})}{l_k} \leq \mathcal{K}\{u_t\} + \frac{1}{m}.$$

Hence, returning to the geodesic segments constructed at the beginning of the argument, by convexity of the K–energy we have, for all  $t \in [0, l_k]$ ,

$$(42) \quad \limsup_{k \rightarrow +\infty} \frac{\mathcal{K}(u_t^{j_m, l_k})}{t} \leq \limsup_{k \rightarrow +\infty} \frac{\mathcal{K}(\varphi_{l_k}^{j_m})}{l_k} \leq \mathcal{K}\{u_t\} + \frac{1}{m}.$$

Let us fix  $m \geq 1$  and  $t \in \mathbb{Q}^+$  momentarily. We use the compactness property of  $\mathcal{E}_\omega^1$  (see [8, Corollary 4.8]) to extract a subsequence (again denoted by  $l_k = l_k(m, t)$ ) such that  $d_1(u_t^{j_m, l_k}, u_t^m) \rightarrow 0$  as  $k \rightarrow \infty$  for some  $u_t^m \in \mathcal{E}_\omega^1$ . Using a diagonal Cantor process it is actually possible to pick the same subsequence of  $\{l_k\}_k$  for each  $m \geq 1$  and  $t \in \mathbb{Q}^+$ . Moreover, due to the endpoint stability of geodesic segments [8, Proposition 4.3], we get that the convergence extends for all  $t \geq 0$ : there exists  $u_t^m \in \mathcal{E}_\omega^1$  such that  $d_1(u_t^{j_m, l_k}, u_t^m) \rightarrow 0$  as  $k \rightarrow \infty$  for any  $t \geq 0$  and  $\{u_t^m\}_t \in \mathcal{R}_\omega^\infty$ .

Now we prove additional properties for our sequence  $\{u_t^m\}_t$ . By (36), we notice that  $\{u_t^m\}_t \in \mathcal{R}_\omega^\infty$ :

$$(43) \quad \max(u_t, -j_m t) \leq u_t^m \leq 0.$$

Moreover, by (37) we also have that  $\{u_t^m\}_t$  is  $m$ –decreasing!

Fixing  $t > 0$ , since  $d_1(u_t^{j_m, l_k}, u_t^m) \rightarrow 0$  as  $k \rightarrow \infty$ , due to  $d_1$ –lower semicontinuity of  $\mathcal{K}$ , from (42) we obtain that

$$(44) \quad \frac{\mathcal{K}(u_t^m)}{t} \leq \mathcal{K}\{u_t\} + \frac{1}{m} \quad \text{for all } t > 0;$$

hence,  $\mathcal{K}\{u_t^m\} \leq \mathcal{K}\{u_t\} + \frac{1}{m}$ , as desired.

Next, we argue that  $d_1(u_t^m, u_t) \rightarrow 0$  for any  $t \geq 0$ , as  $m \rightarrow \infty$ . But this is simply a consequence of Claim 3. Indeed, due to (43), we only need to argue that

$$(45) \quad \lim_{m \rightarrow +\infty} \frac{d_1(u_t^m, u_t)}{t} = \lim_{m \rightarrow +\infty} \frac{I(u_t^m) - I(u_t)}{t} = 0.$$

But, from  $I$ -linearity, for any  $t \in [0, l_k]$  we have that

$$\frac{I(u_t^m) - I(u_t)}{t} = \limsup_{k \rightarrow \infty} \frac{I(u_t^{j,l_m}) - I(u_t)}{t} = \limsup_{k \rightarrow \infty} \frac{I(\varphi_{l_k}^{j,m}) - I(u_{l_k})}{l_k},$$

and the right-hand side converges to zero as  $m \rightarrow +\infty$ , by Claim 3.

Since  $d_1(u_t^m, u_t) \rightarrow 0$  implies that  $u_t^m \searrow u_t$  [32, Theorem 5], hence we can invoke Lemma 4.3 to conclude that  $d_p^c(\{u_t^m\}_t, \{u_t\}_t) \rightarrow 0$  as  $m \rightarrow \infty$ .

Lastly, (43) together with  $u_t \leq u_t^m$  implies the normalization  $\sup_X u_t^m = \sup_X u_t = 0$  for  $t \in [0, 1]$ , as desired. □

In the above argument we have used the following lemma, whose proof goes along the same lines as [6, Theorem 5.8]:

**Lemma 5.3** *There exists a continuous nondecreasing function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $f(0) = 0$  such that, for all  $0 \geq \varphi_1, \varphi_2, \psi \in \mathcal{E}_\omega^1$ , we have*

$$\left| \int_X (\varphi_1 - \varphi_2)(\omega_{\varphi_2}^n - \omega_\psi^n) \right| \leq Af(\mathcal{I}(\varphi_1, \varphi_2)/A),$$

where  $A = \max(-I(\varphi_1), -I(\varphi_2), -I(\psi), 1)$ .

In the proof below we use  $C_n > 0$  to denote various numerical constants (only dependent on  $\dim X = n$ ) and  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  to denote a continuous nondecreasing function such that  $f(0) = 0$ . They may be different from place to place.

**Proof** By approximation of finite-energy potentials from above by smooth ones, we can assume that  $\varphi_1, \varphi_2$  and  $\psi$  are smooth (the convergence of the integrals is assured by the results of [56]; see for example [35, Proposition 2.11]). We set  $u = \varphi_1 - \varphi_2$  and  $v = \frac{1}{2}(\varphi_1 + \varphi_2)$ . For  $p = 0, \dots, n$ , let

$$a_p := \int_X u \omega_{\varphi_2}^p \wedge \omega_\psi^{n-p} \quad \text{and} \quad b_p := \int_X i \partial u \wedge \bar{\partial} u \wedge \omega_v^p \wedge \omega_\psi^{n-p-1}.$$

It follows from [56, Proposition 2.5] that

$$(46) \quad \mathcal{I}(\psi_1, \psi_2) \leq C_n (|I(\psi_1)| + |I(\psi_2)|) \quad \text{for all } 0 \geq \psi_1, \psi_2 \in \mathcal{E}_\omega^1.$$

In particular,  $\mathcal{I}(\psi, \varphi_j) \leq C_n A$  for  $j = 1, 2$ .

For  $p = 0, 1, \dots, n - 1$ , we have, using integration by parts and the Cauchy–Schwarz inequality,

$$\begin{aligned} &|a_p - a_{p+1}| \\ &= \left| \int_X i \partial u \wedge \bar{\partial}(\psi - \varphi_2) \wedge \omega_{\varphi_2}^p \wedge \omega_{\psi}^{n-p-1} \right| \\ &\leq \left( \int_X i \partial u \wedge \bar{\partial} u \wedge \omega_{\varphi_2}^p \wedge \omega_{\psi}^{n-p-1} \right)^{1/2} \left( \int_X i \partial(\psi - \varphi_2) \wedge \bar{\partial}(\psi - \varphi_2) \wedge \omega_{\varphi_2}^p \wedge \omega_{\psi}^{n-p-1} \right)^{1/2} \\ &\leq C_n b_p^{1/2} (\mathcal{I}(\varphi_2, \psi))^{1/2} \leq C_n b_p^{1/2} A^{1/2}. \end{aligned}$$

In the last line above we have used  $\omega_{\varphi_2} \leq 2\omega_v$  and the inequality

$$\int_X i \partial(\psi - \varphi_2) \wedge \bar{\partial}(\psi - \varphi_2) \wedge \omega_{\varphi_2}^p \wedge \omega_{\psi}^{n-p-1} \leq \int_X (\psi - \varphi_2)(\omega_{\varphi_2}^n - \omega_{\psi}^n).$$

It thus follows, by summing up the estimates of  $|a_p - a_{p+1}|$  above for  $p = 0, \dots, n - 1$ , that

$$(47) \quad |a_0 - a_n| \leq C_n A^{1/2} \sum_{p=0}^{n-1} b_p^{1/2}.$$

We claim that there is a nondecreasing continuous function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $f(0) = 0$  such that

$$b_p \leq A f(\mathcal{I}(\varphi_1, \varphi_2)/A), \quad 0 \leq p \leq n - 1.$$

We proceed by (backwards) induction. For  $p = n - 1$  we can simply take  $f(s) = C_n s$  for  $s \geq 0$ . By the same argument as above, using integration by parts and the Cauchy–Schwarz inequality we have, for  $0 \leq p \leq n - 2$ ,

$$\begin{aligned} &b_p - b_{p+1} \\ &= \int_X i \partial u \wedge \bar{\partial} u \wedge i \partial \bar{\partial}(\psi - v) \wedge \omega_v^p \wedge \omega_{\psi}^{n-p-2} \\ &\leq \left| \int_X i \partial u \wedge \bar{\partial}(\psi - v) \wedge i \partial \bar{\partial} u \wedge \omega_v^p \wedge \omega_{\psi}^{n-p-2} \right| \\ &\leq \left| \int_X i \partial u \wedge \bar{\partial}(\psi - v) \wedge (\omega_{\varphi_1} - \omega_{\varphi_2}) \wedge \omega_v^p \wedge \omega_{\psi}^{n-p-2} \right| \\ &\leq \left| \int_X i \partial u \wedge \bar{\partial}(\psi - v) \wedge \omega_{\varphi_1} \wedge \omega_v^p \wedge \omega_{\psi}^{n-p-2} \right| + \left| \int_X i \partial u \wedge \bar{\partial}(\psi - v) \wedge \omega_{\varphi_2} \wedge \omega_v^p \wedge \omega_{\psi}^{n-p-2} \right| \end{aligned}$$

$$\begin{aligned} &\leq C_n \left( \int_X i \partial u \wedge \bar{\partial} u \wedge \omega_v^{p+1} \wedge \omega_\psi^{n-p-2} \right)^{1/2} \\ &\quad \times \left( \int_X i \partial(\psi - v) \wedge \bar{\partial}(\psi - v) \wedge \omega_v^{p+1} \wedge \omega_\psi^{n-p-2} \right)^{1/2} \\ &\leq C_n \mathcal{I}(\psi, v)^{1/2} b_{p+1}^{1/2}, \end{aligned}$$

where we used several times that  $\omega_{\varphi_j} \leq 2\omega_v$ . Using (46), we thus have

$$b_p \leq b_{p+1} + AC_n(b_{p+1}/A)^{1/2} \leq Af(\mathcal{I}(\varphi_1, \varphi_2)/A) + AC_n f(\mathcal{I}(\varphi_1, \varphi_2)/A)^{1/2}.$$

Consequently, by possibly increasing  $f$ , we have that  $b_p \leq Af(\mathcal{I}(\varphi_1, \varphi_1)/A)$ , proving the claim. Comparing with (47), we thus have

$$|a_0 - a_n| \leq Af(\mathcal{I}(\varphi_1, \varphi_2)/A),$$

what we wanted to prove. □

### Approximation with rays of $C^{1,\bar{1}}$ potentials

The goal of this subsection is to prove Theorem 1.5(ii):

**Theorem 5.4** *Let  $p \geq 1$ . Suppose that  $\{u_t\}_t \in \mathcal{R}_\omega^p$  is such that  $\mathcal{K}\{u_t\} < \infty$ . Then there exists  $\{v_t^k\}_t \subset \mathcal{R}_\omega^{1,\bar{1}}$  such that  $v_t^k \searrow u_t$  for  $t \geq 0$ ,  $d_p^c(\{v_t^k\}_t, \{u_t\}_t) \rightarrow 0$  and  $\mathcal{K}\{v_t^k\} \rightarrow \mathcal{K}\{u_t\}$ . Additionally,  $\sup_X v_t^k = \sup_X u_t$  for all  $k, t > 0$ .*

To argue this result, we need two auxiliary theorems, whose proof will be given at the end of the section. First we will need the following theorem, which will allow us to obtain “scaled”  $C^{1,\bar{1}}$  estimates along geodesic rays, via convexity:

**Theorem 5.5** *Let  $[0, 1] \ni t \rightarrow u_t \in \mathcal{H}_\omega^{1,\bar{1}}$  be the  $C^{1,\bar{1}}$ -geodesic connecting  $u_0, u_1 \in \mathcal{H}_\omega^{1,\bar{1}}$ . Then there exists  $B > 0$ , only depending on  $(X, \omega)$ , such that*

$$[0, 1] \ni t \rightarrow \operatorname{ess\,sup}_X (\log(n + \Delta_\omega u_t) - Bu_t) \in \mathbb{R}$$

*is a convex function.*

The proof of this theorem is obtained using the estimates developed in [58]. We will also need the following smoothing argument along bounded geodesic rays, relying on the regularizing property of the weak Monge–Ampère flows, closely following the arguments of [57]:

**Theorem 5.6** Let  $B > 0$  be from Theorem 5.5, and  $\{u_t\}_t \in \mathcal{R}_\omega^\infty$  with  $\mathcal{K}\{u_t\} < \infty$  and  $\sup_X u_t = 0$  for  $t \geq 0$ . Then there exists  $\alpha > 0$ , depending on  $\{u_t\}_t$ , such that for all  $s > 0$  and  $j \in \mathbb{N}$  one can find  $u_s^j \in \mathcal{H}_\omega$  satisfying the following conditions:

- (i)  $\{u_s^j\}_j$  is decreasing and  $u_s \leq u_s^j \leq \alpha j 2^{-j}$ .
- (ii)  $\sup_X (\log(n + \Delta_\omega u_s^j) - B u_s^j) \leq \alpha 2^j (1 + s)$ .
- (iii)  $d_1(u_s^j, u_s) \leq \alpha 2^{-j} s + \alpha j 2^{-j}$ .
- (iv)  $\text{Ent}(\omega^n, \omega_{u_s^j}^n) \leq \text{Ent}(\omega^n, \omega_{u_s}^n)$ .

**Proof of Theorem 5.4** First we assume that  $\{u_t\}_t \in \mathcal{R}_\omega^\infty$  and  $\sup_X u_t = 0$  for  $t \geq 0$ . If  $\{u_t\}_t$  is the constant ray then we are done; hence, after rescaling, we can also assume that  $d_1(0, u_t) = t$  for  $t \geq 0$ . Let  $\{u_s^j\}_{s>0, j \in \mathbb{N}}$  be the potentials constructed as in Theorem 5.6.

Let us fix  $j \in \mathbb{N}$  momentarily. Given  $s > 0$ , by  $[0, s] \ni t \rightarrow u_t^{j,s} \in \mathcal{H}_\omega^{1, \bar{1}}$  we denote the  $C^{1, \bar{1}}$  geodesic connecting  $u_0^{j,s} := 0$  and  $u_s^{j,s} := u_s^j$ . Using condition (i) in Theorem 5.6 and the comparison principle for weak geodesics we get that

$$(48) \quad u_t \leq u_t^{j,s} \leq \frac{\alpha j 2^{-j} t}{s}, \quad t \in [0, s].$$

Since  $\{u_s^j\}_j$  is decreasing, by the comparison principle for weak geodesics,  $\{u_t^{j,s}\}_j$  is decreasing as well, for any  $t \in [0, s]$ .

Given  $t \in (0, s]$ , by Theorems 5.6(ii) and 5.5 we have that

$$(49) \quad \frac{\text{ess sup}_X (\log(1 + \frac{1}{n} \Delta_\omega u_t^{j,s}) - B u_t^{j,s})}{t} \leq \frac{\sup_X (\log(1 + \frac{1}{n} \Delta_\omega u_s^j) - B u_s^j)}{s} \leq \alpha 2^j \left(1 + \frac{1}{s}\right).$$

Finally, (1) and Theorem 5.6(iii) imply that

$$(50) \quad \frac{d_1(u_t^{j,s}, u_t)}{t} \leq \alpha 2^{-j} + \frac{\alpha}{s}, \quad t \in [0, s].$$

Fixing  $t > 0$ , (48) and (49) give that  $\{u_t^{j,s}\}_{\{s>t\}}$  is compact in the  $C^{1, \alpha}$  topology, implying existence of  $v_t^j \in \mathcal{H}_\omega^{1, \bar{1}}$  such that  $\|v_t^j - u_t^{j,s}\|_{C^{1, \alpha}} \rightarrow 0$  as  $s \rightarrow \infty$  (after passing to an  $s$ -subsequence). Moreover, letting  $s \rightarrow \infty$  in (48), (49) and (50), using Lemma A.1, we arrive at, for  $t \in (0, \infty)$ ,

$$(51) \quad u_t \leq v_t^j \leq 0, \quad \frac{1}{t} \left( \log \left( 1 + \frac{1}{n} \Delta_\omega v_t^j \right) - B v_t^j \right) \leq \alpha 2^j, \quad \frac{d_1(v_t^j, u_t)}{t} \leq \alpha 2^{-j}.$$

Using an Arzelà–Ascoli-type argument exactly the same way as in the proof of Theorem 5.1, after passing to another  $s$ -subsequence, we can assume that  $\|v_t^j - u_t^{j,s}\|_{C^{1,\alpha}} \rightarrow 0$  for all  $t > 0$  at the same time, implying existence of  $\{v_t^j\} \in \mathcal{R}^{1,\bar{1}}$  for any  $j \in \mathbb{N}$ . By (51) we get that  $d_1^c(\{v_t^j\}, \{u_t\}) \rightarrow 0$  as  $j \rightarrow \infty$ . Remark 4.4 implies that  $d_p^c(\{v_t^j\}, \{u_t\}) \rightarrow 0$ , as desired.

Finally, since  $\{u_t^{j,s}\}_j$  is decreasing for any  $t \in [0, s]$ , a diagonal Cantor process now implies that  $\{v_t^j\}_j$  can be chosen to be decreasing for any  $t > 0$ .

To show that  $\mathcal{K}\{v_t^j\} \rightarrow \mathcal{K}\{u_t\}$ , we first note that by [27, Proposition 5.9] we have  $\mathcal{K}\{u_t\} \leq \liminf_j \mathcal{K}\{v_t^j\}$ . Hence it is enough to show that  $\mathcal{K}\{v_t^j\} \leq \mathcal{K}\{u_t\} + f(\alpha 2^{-j})$  for any  $j \in \mathbb{N}$ , where  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is some continuous function with  $f(0) = 0$ .

Using conditions (iii) and (iv) in Theorem 5.6 we can start writing

$$(52) \quad \begin{aligned} \frac{\mathcal{K}(u_s^j) - \mathcal{K}(u_s)}{s} &\leq |\bar{S}| \frac{d_1(u_s^j, u_s)}{s} + n \frac{I_{\text{Ric}(\omega)}(u_s^j) - I_{\text{Ric}(\omega)}(u_s)}{s} \\ &\leq \frac{\alpha}{s} + \alpha 2^{-j} |\bar{S}| + n \frac{I_{\text{Ric}(\omega)}(u_s^j) - I_{\text{Ric}(\omega)}(u_s)}{s}. \end{aligned}$$

We can suppose that  $-C\omega \leq \text{Ric}(\omega) \leq C\omega$ , and for the rest of the proof  $C > 0$  will denote a constant only dependent on  $(X, \omega)$ . Using condition (i) in Theorem 5.6 multiple times, we can estimate

$$\begin{aligned} \frac{I_{\text{Ric}(\omega)}(u_s^j) - I_{\text{Ric}(\omega)}(u_s)}{s} &\leq C \frac{\sum_j \int_X (u_s^j - u_s) \omega \wedge \omega_{u_s^j}^{j-1} \wedge \omega_{u_s}^{n-j-1}}{s} \\ &\leq C \frac{\sum_j \int_X (u_s^j - u_s) \omega_{u_s^j/4+u_s/4}^{n_j}}{s} \\ &\leq C \frac{\sum_j \int_X (u_s^j - u_s) (\omega_{u_s^j/4+u_s/4}^{n_j} - \omega_{u_s}^n)}{s} + C \frac{I(u_s^j) - I(u_s)}{s} \\ &\leq f(\mathcal{I}(u_s^j, u_s)/s) + C\alpha 2^{-j} + \frac{C\alpha}{s}, \end{aligned}$$

where  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous function with  $f(0) = 0$ , and in the last line we used Lemma 5.3. Together with (52), this inequality implies that

$$(53) \quad \frac{\mathcal{K}(u_t^{j,s})}{t} \leq \frac{\mathcal{K}(u_s^j)}{s} \leq \mathcal{K}\{u_t\} + C\alpha 2^{-j} + f(\mathcal{I}(u_s^j, u_s)/s) + \frac{C\alpha}{s}.$$

Letting  $s \rightarrow \infty$ , since  $\mathcal{K}$  is convex and  $d_1$ -lsc, we obtain that

$$\mathcal{K}\{v_t^j\} \leq \mathcal{K}\{u_t\} + C\alpha 2^{-j} + f(\alpha 2^{-j}),$$

as desired, finishing the proof when  $\{u_t\}_t \in \mathcal{R}_\omega^\infty$ .

Now let  $\{u_t\}_t \in \mathcal{R}_\omega^p$  with  $\mathcal{K}\{u_t\} < \infty$ . We can still assume  $\sup_X u_t = 0$ . By Theorem 5.1, there exists  $\{u_t^k\} \in \mathcal{R}_\omega^\infty$  such that  $u_t^k \searrow u_t$  for  $t \geq 0$ ,  $d_p^c(\{u_t^k\}, \{u_t\}) \leq 1/2^k$ ,  $|\mathcal{K}\{u_t\} - \mathcal{K}\{u_t^k\}| \leq 1/2^k$  and  $\sup_X u_t^k = 0$ .

Let  $\{u_s^{k,j}\}_j$  be the potentials of Theorem 5.6 associated to the rays  $\{u_t^k\}_t$ . By the construction of these potentials (as elaborated in the proof of Theorem 5.6) and [57, Corollary 2.2], it follows that  $\{u_s^{k,j}\}_{k,j}$  is decreasing both in  $j$  and  $k$  for any fixed  $s > 0$ ! For each  $k$  fixed we choose  $j_k \in \mathbb{N}$  large enough such that, in view of Theorem 5.6 and (53),  $d_1(u_s^{k,j_k}, u_s^k) \leq 2^{-k}s + C_k$  and

$$\mathcal{K}(u_s^{k,j_k}) \leq s(\mathcal{K}\{u_t^k\} + 2^{-k}) + C_k, \quad s > 0.$$

Moreover, we can choose the sequence  $\{j_k\}_k$  such that  $j_{k+1} > j_k$ . Using this, a diagonal Cantor process applied to  $\{u_s^{k,j_k}\}_s$  yields rays  $\{v_t^k\}_t \in \mathcal{R}_\omega^{1,\bar{1}}$  such that  $u_t^k \leq v_t^k \leq 0$ ,  $d_p^c(\{v_t^k\}, \{u_t^k\}) \leq 1/2^k$  and  $\mathcal{K}\{v_t^k\} - \mathcal{K}\{u_t^k\} \leq 1/2^k$ ; moreover(!)  $\{v_t^k\}_k$  is decreasing for any fixed  $t > 0$ .

As  $(\mathcal{R}_\omega^p, d_p^c)$  is complete, we obtain that  $d_p^c(\{v_t^k\}_t, \{u_t\}_t) \rightarrow 0$  and  $\limsup_k \mathcal{K}\{v_t^k\} \leq \mathcal{K}\{u_t\}$ . Using again [27, Proposition 5.9] we have that  $\mathcal{K}\{u_t\} \leq \liminf_k \mathcal{K}\{v_t^k\}$ , hence we have equality as desired. □

**Remark 5.7** It follows from (51), that the approximating rays  $\{v_t^j\}_t \in \mathcal{R}^{1,\bar{1}}$  in the previous theorem additionally satisfy the estimate

$$\frac{1}{t} \operatorname{ess\,sup}_X \left( \log \left( 1 + \frac{1}{n} \Delta_\omega v_t^j \right) - Bv_t^j \right) \leq \alpha 2^j, \quad t > 0, \quad j \in \mathbb{N}.$$

**The proof of Theorem 5.5** Let us denote the log of the left-hand side of (6) by  $F(u^\varepsilon)$ . Given a smooth function  $h \in C^\infty([0, 1] \times X)$ , if  $h$  attains its maximum at  $(t, x) \in (0, 1) \times X$ , then ellipticity of (6) gives that

$$(54) \quad DF(u^\varepsilon)(h)(t, x) := \frac{d}{ds} \Big|_{s=0} F(u^\varepsilon + sh)(t, x) \leq 0.$$

**Proof of Theorem 5.5** Let us first assume that  $u_0, u_1 \in \mathcal{H}_\omega$  and let  $[0, 1] \ni t \mapsto u_t^\varepsilon$  be the smooth  $\varepsilon$ -geodesic connecting  $u_0$  and  $u_1$  (see Section 2.1). Fix  $(t, x) \in [0, 1] \times X$  and  $\varepsilon > 0$  momentarily. In [58, page 339] (after equation (2.19)) it is shown that for some constants  $B, C > 1$ , dependent only on  $(X, \omega)$ , we have that

$$(55) \quad DF(u^\varepsilon)(\log(n + \Delta_\omega u_t^\varepsilon) - Bu_t^\varepsilon)(t, x) \geq \sum_{j=1}^n \frac{1}{1 + (u_t^\varepsilon)_{jj}} - C,$$

where we have used normal coordinates of  $\omega$  at  $x$  and  $i\partial\bar{\partial}u_t^\varepsilon$  is assumed to be diagonal. Additionally, fix  $\delta > 0$  and  $g_\delta(t) := \frac{1}{2}\delta t^2$ . We also have

$$(56) \quad DF(u^\varepsilon)(g_\delta(t))(t, x) = \frac{\delta}{\ddot{u}_t - |\nabla u_t^\varepsilon|_{\omega_{u_t^\varepsilon}}^2}.$$

Assume that  $h_{\varepsilon,\delta}(t, x) := \log(n + \Delta_\omega u_t^\varepsilon) - Bu_t^\varepsilon + g_\delta(t)$  is maximized at  $(t, x) \in (0, 1) \times X$ . Then, by (54), (55) and (56), we obtain at  $(t, x)$  that

$$(57) \quad \begin{aligned} 0 \geq DF(u^\varepsilon)(h_{\varepsilon,\delta}) &\geq \sum_{j=1}^n \frac{1}{1 + (u_t^\varepsilon)_{j\bar{j}}} + \frac{\delta}{\ddot{u}_t - |\nabla u_t^\varepsilon|_{\omega_{u_t^\varepsilon}}^2} - C \\ &\geq (n + 1) \left[ \frac{\delta}{(1 + (u_t^\varepsilon)_{1\bar{1}}) \cdots (1 + (u_t^\varepsilon)_{n\bar{n}})(\ddot{u}_t - |\nabla u_t^\varepsilon|_{\omega_{u_t^\varepsilon}}^2)} \right]^{1/(n+1)} - C \\ &= (n + 1) \left[ \frac{\delta}{\varepsilon} \right]^{1/(n+1)} - C. \end{aligned}$$

In the second line we have used the inequality of arithmetic and geometric means while for the last identity we have used (6). Thus, for  $\varepsilon < \delta(n + 1)^{n+1}/C^{n+1}$  we get a contradiction in the above inequality, implying that the maximum of  $h_{\varepsilon,\delta}$  cannot be attained at  $(t, x)$ , an interior point of  $[0, 1] \times X$ . In particular, we have that

$$\sup_X h_{\varepsilon,\delta}(t, x) \leq \max\left(\sup_X h_{\varepsilon,\delta}(0, x), \sup_X h_{\varepsilon,\delta}(1, x)\right), \quad t \in [0, 1], \varepsilon < \delta(n + 1)^{n+1}/C^{n+1}.$$

Letting  $\varepsilon \searrow 0$  and  $\delta \searrow 0$  thereafter, via Lemma A.1 we arrive at

$$\operatorname{ess\,sup}_X h_{0,0}(t, x) \leq \max\left(\sup_X h_{0,0}(0, x), \sup_X h_{0,0}(1, x)\right), \quad t \in [0, 1],$$

motivating the introduction of  $M_{u_0,u_1}(t) := \operatorname{ess\,sup}_X (\log(n + \Delta_\omega(u_t)) - Bu_t)$ . Indeed, we can simply write

$$(58) \quad M_{u_0,u_1}(t) \leq \max(M_{u_0,u_1}(0), M_{u_0,u_1}(1)), \quad t \in [0, 1].$$

Next we observe that (58) also holds when we merely have  $u_0, u_1 \in \mathcal{H}_\omega^{1,\bar{1}}$ . Indeed, we pick sequences  $u_0^j \searrow u_0$  and  $u_1^j \searrow u_1$ , as in Proposition A.2. Then we apply (58) to  $M_{u_0^j,u_1^j}$  and the comparison principle [11, Theorem 21] together with Lemma A.1 gives (58) for  $u_0, u_1 \in \mathcal{H}_\omega^{1,\bar{1}}$ .

To finish, we show that  $M_{u_0,u_1}(t)$  is actually convex. Let  $a, b \in [0, 1]$ . Then  $t \rightarrow v_t := u_a + t(b - a) + \frac{1}{B}(M_{u_0,u_1}(a) + t(M_{u_0,u_1}(b) - M_{u_0,u_1}(a)))$  is the  $C^{1,\bar{1}}$  geodesic connecting  $v_0 := u_a + M_{u_0,u_1}(a)/B$  and  $v_1 := u_b + M_{u_0,u_1}(b)/B$ . Applying (58) to

$v_0$  and  $v_1$ , we arrive at

$$\begin{aligned} M_{u_0, u_1}(a + t(b - a)) - M_{u_0, u_1}(a) - t(M_{u_0, u_1}(b) - M_{u_0, u_1}(a)) \\ = M_{v_0, v_1}(t) \leq \max(M_{v_0, v_1}(0), M_{v_0, v_1}(1)) = 0, \end{aligned}$$

hence  $t \rightarrow M_{u_0, u_1}(t)$  is convex, as desired. □

**The proof of Theorem 5.6** In the proof of Theorem 5.6 we will use the formalism of [57] adapted to our context. Fixing  $\varphi_0 \in \mathcal{E}_\omega^1$  with  $\sup_X \varphi_0 = 0$ , we consider the parabolic PDE on  $[0, \infty) \times X$ , with initial data given by  $\varphi_0$ ,

$$(59) \quad \frac{d}{dt} \varphi_t = \log \left[ \frac{(\omega + i \partial \bar{\partial} \varphi_t)^n}{\omega^n} \right].$$

To avoid cumbersome notation, we will denote  $t$ -derivatives by dots in what follows. As shown in [57],  $(t, x) \rightarrow \varphi_t(x)$  is smooth on  $(0, \infty) \times X$ . The initial condition simply means that  $d_1(\varphi_t, \varphi_0) \rightarrow 0$ , as  $t \rightarrow 0$  [57, Section 5.2.2]. When  $\varphi_0 \in \mathcal{H}_\omega$ , we actually have that  $\|\varphi_t - \varphi_0\|_{C^\infty} \rightarrow 0$  as  $t \rightarrow 0$ . Moreover, it is shown in [50, Theorem B] that if  $\varphi_0 \in \mathcal{E}_\omega^1$  and  $\varphi_0^j \in \mathcal{H}_\omega$  converges in  $L^1(X, \omega^n)$  to  $\varphi_0$ , then for any  $t > 0$  we have that  $\|\varphi_t^j - \varphi_t\|_{C^\infty} \rightarrow 0$ , where  $\{\varphi_t^j\}_j$  are the smooth solutions to (59) with initial data  $\varphi_0^j$ . All this implies that the a priori estimates and maximum principles developed in [57, Section 2] for smooth initial data, also apply for initial data in  $\mathcal{E}_\omega^1$ , as above (for our applications  $\varphi_0$  will be actually bounded).

For the remainder of this section we pick a small constant  $\lambda > 0$  depending only on  $(X, \omega)$  such that  $\int_X e^{-2\lambda\phi} \omega^n$  is uniformly bounded for all  $\phi \in \text{PSH}(X, \omega)$  normalized by  $\sup_X \phi = 0$  (see [75, Proposition 2.1; 81]).

Let  $v$  be the unique continuous  $\omega$ -psh function such that

$$(60) \quad \omega_v^n := e^{\lambda v - \lambda \varphi_0 - n \log \lambda} \omega^n.$$

By our choice of  $\lambda$ , it follows from [64; 6] (or much more generally [39, Theorem 5.3]) that  $v$  is uniformly bounded by a constant depending only on  $(X, \omega)$ .

**Lemma 5.8** *With  $\lambda \in (0, 1)$  and  $v$  as above, we have that*

$$(61) \quad (1 - \lambda t)\varphi_0 + \lambda t v + n(t \log t - t) \leq \varphi_t \leq 0, \quad t \in [0, 1].$$

**Proof** Let  $\psi_t := (1 - \lambda t)\varphi_0 + \lambda t v + n(t \log t - t)$  for  $t \in [0, 1]$ . Then

$$\dot{\psi}_t = \lambda(v - \varphi_0) + n \log t = \log \left( \lambda^n t^n \cdot \frac{\omega_v^n}{\omega^n} \right) \leq \log \left( \frac{\omega_{\psi_t}^n}{\omega^n} \right).$$

This implies that  $l \rightarrow \psi_t$  is a subsolution to (59), and an application of the maximum principle [57, Corollary 2.2] yields the first inequality in (61).

The second inequality follows from [57, Corollary 2.2], after comparing  $t \rightarrow \varphi_t$  with the constant solution  $t \rightarrow \chi_t := \sup_X \varphi_0 = 0$  of (59). □

Simplifying (61), we actually obtain that

$$(62) \quad \varphi_0 \leq \varphi_t + Ct - Ct \log t, \quad t \in [0, 1],$$

for some constant  $C > 0$  dependent on  $(X, \omega)$ . This can be taken one step further, as we now describe:

**Corollary 5.9** *There exists a constant  $C > 1$  depending on  $(X, \omega)$  such that  $w_t \geq w_{t/2}$  for any  $t \in [0, 1]$ , where*

$$(63) \quad w_t := \varphi_t + Ct - Ct \log t.$$

**Proof** Fixing  $s \in (0, 1)$ , we apply (62) to the flow  $t \mapsto \varphi_{s/2+t}$ , starting from  $\varphi_{s/2}$ . By (61) and (62) we have that  $\|e^{-\lambda\varphi_{s/2}}\|_{L^2}$  is controlled by  $\|e^{-\lambda\varphi_0}\|_{L^2}$  which is uniformly bounded by a constant depending on  $(X, \omega)$ . Hence, for  $t := \frac{s}{2} \in [0, 1]$  in (62), we have

$$\varphi_s \geq \varphi_{s/2} - Cs/2 + C(s/2) \log(s/2),$$

where  $C$  only depends on  $(X, \omega)$ . Thus, after possibly increasing  $C > 0$ , the function  $w_t := \varphi_t + Ct - Ct \log t$  satisfies  $w_t \geq w_{t/2}$  for  $t \in [0, 1]$ . □

We also point out the following simple monotonicity result:

**Lemma 5.10** *The map  $[0, \infty) \ni t \rightarrow \text{Ent}(\omega^n, \omega_{\varphi_t}^n) \in \mathbb{R}$  is decreasing.*

**Proof** First let us assume that  $\varphi_0 \in \mathcal{H}_\omega$  in (59). For  $t \geq 0$ , we can start by computing

$$\begin{aligned} \frac{d}{dt} \text{Ent}(\omega^n, \omega_{\varphi_t}^n) &= \frac{d}{dt} \int_X \dot{\varphi}_t (\omega + i\partial\bar{\partial}\varphi_t)^n \\ &= \int_X \ddot{\varphi}_t (\omega + i\partial\bar{\partial}\varphi_t)^n - \int_X |\nabla \dot{\varphi}_t|_{\omega_{\varphi_t}}^2 (\omega + i\partial\bar{\partial}\varphi_t)^n \\ &= \int_X (\Delta_{\omega_{\varphi_t}} \dot{\varphi}_t) (\omega + i\partial\bar{\partial}\varphi_t)^n - \int_X |\nabla \dot{\varphi}_t|_{\omega_{\varphi_t}}^2 (\omega + i\partial\bar{\partial}\varphi_t)^n \\ &= - \int_X |\nabla \dot{\varphi}_t|_{\omega_{\varphi_t}}^2 (\omega + i\partial\bar{\partial}\varphi_t)^n \leq 0. \end{aligned}$$

In the second line above we have used  $\ddot{\varphi}_t = \Delta_{\omega_{\varphi_t}} \dot{\varphi}_t$ , which follows from the equation of the flow (59). Consequently,  $t \rightarrow \text{Ent}(\omega^n, \omega^n_{\varphi_t})$  is decreasing on  $[0, \infty)$ , when  $\varphi_0 \in \mathcal{H}_\omega$ .

For general  $\varphi_0 \in \mathcal{E}^1_\omega$ , let  $\varphi_0^j \in \mathcal{H}_\omega$  be such that  $d_1(\varphi_0^j, \varphi_0) \rightarrow 0$  and  $\text{Ent}(\omega^n, \omega^n_{\varphi_0^j}) \rightarrow \text{Ent}(\omega^n, \omega^n_{\varphi_0})$  (such a sequence exists by [8, Theorem 1.3]). Fixing  $t > 0$ , by Theorem B of [50] we have that  $\varphi_t^j \rightarrow_{C^\infty} \varphi_t$ , hence we can conclude that

$$\text{Ent}(\omega^n, \omega^n_{\varphi_t}) = \lim_j \text{Ent}(\omega^n, \omega^n_{\varphi_t^j}) \leq \lim_j \text{Ent}(\omega^n, \omega^n_{\varphi_0^j}) = \text{Ent}(\omega^n, \omega^n_{\varphi_0}),$$

finishing the proof. □

For the remainder of this section, let  $\{u_t\}_t \in \mathcal{R}^\infty$  with  $\sup_X u_t = 0$  for  $t \geq 0$  and  $\mathcal{K}\{u_t\} < +\infty$ , as in the statement of Theorem 5.6. Since  $\sup_X u_s = 0$  for  $s \geq 0$ , by the weak  $L^1$ -compactness of  $\text{PSH}(X, \omega)$  we have that  $u_s \searrow u_\infty \in \text{PSH}(X, \omega)$ .

We fix  $s > 0$  for the remainder of this section, and we construct the sequence  $u_s^j$  as follows. For each  $j$  we define

$$u_s^j := w_{s, 2^{-j}},$$

where  $w_{s,t} \in \mathcal{H}_\omega$  is constructed in (63) with respect to the flow  $t \rightarrow \varphi_{s,t}$ , starting from  $\varphi_{s,0} := u_s$ . The estimate of Corollary 5.9, together with (62) yields the condition (i) in Theorem 5.6 for  $\alpha := 3C$ . Condition (iv) follows automatically from Lemma 5.10.

Next we address condition (ii) in Theorem 5.6, which is closely related to Corollary 4.5 of [57]:

**Lemma 5.11** *We have that*

$$(64) \quad \sup_X (\log(n + \Delta_\omega u_s^j) - B u_s^j) \leq \alpha 2^j (1 + s), \quad j \in \mathbb{N}, s > 0.$$

**Proof** From [57, Corollary 4.5] we obtain that for any  $j \in \mathbb{N}$  and  $s > 0$  we have

$$(65) \quad \frac{1}{2^j} \log(n + \Delta_\omega u_s^j) = \frac{1}{2^j} \log(n + \Delta_\omega \varphi_{s, 2^{-j}}) \leq C(\text{osc}_X \varphi_{s, 2^{-j-1}} + 1),$$

where  $C > 0$  only depends on  $(X, \omega)$ . Using (62) we have that

$$\text{osc}_X \varphi_{s, 2^{-j-1}} \leq -\inf_X \varphi_{s,0} + \alpha = -\inf_X u_s + \alpha.$$

By [34, Theorem 1] we have that  $\inf_X u_s = m_{\{u_t\}}^s$  for some constant  $m_{\{u_t\}}^s \leq 0$ . Consequently (64) follows after putting the above together with condition (i) in Theorem 5.6 (and possibly increasing the value of  $\alpha > 0$ ). □

Next we address condition (iii) in Theorem 5.6:

**Lemma 5.12** We have that  $d_1(u_s^j, u_s) \leq \alpha 2^{-j} s + \alpha j 2^{-j}$  for any  $j \in \mathbb{N}$  and  $s > 0$ .

**Proof** For the flow  $t \mapsto \varphi_{s,t}$ , using (59), we can write

$$\begin{aligned} I(\varphi_{s,t}) - I(\varphi_{s,0}) &= \int_0^t \frac{d}{dl} I(\varphi_{s,l}) dl = \int_0^t \text{Ent}(\omega^n, \omega_{\varphi_{s,l}}^n) dl \\ &\leq \text{Ent}(\omega^n, \omega_{\varphi_{s,0}}^n) t = \text{Ent}(\omega^n, \omega_{u_s}^n) t, \end{aligned}$$

where we have used Lemma 5.10. Recall that for  $u_s^j := w_{s,2^{-j}}$ , due to property (i) we can continue:

$$d_1(u_s^j, u_s) = I(u_s^j) - I(u_s) \leq I(\varphi_{s,2^{-j}}) - I(\varphi_{s,0}) + \alpha j 2^{-j} \leq \text{Ent}(\omega^n, \omega_{u_s}^n) 2^{-j} + \alpha j 2^{-j}.$$

After invoking Lemma 5.13 below, and possibly adjusting  $\alpha > 0$  again, the proof is finished. □

As promised above, we argue that along  $\{u_t\}_t$  the entropy has sublinear growth:

**Lemma 5.13** There exists  $C > 0$  depending on  $d_1(0, u_1)$  and  $\mathcal{K}\{u_t\}$  such that  $\text{Ent}(\omega^n, \omega_{u_t}^n) \leq Ct$  for  $t \geq 0$ .

**Proof** Let  $D > \mathcal{K}\{u_t\}$ . By the Chen–Tian formula for the extended K–energy (8) we obtain that

$$\text{Ent}(\omega^n, \omega_{u_t}^n) \leq Dt - \bar{S} I(u_t) + n I_{\text{Ric}(\omega)}(u_t) \leq C_1 t + C_2 d_1(0, u_t) + C_3 d_1(0, u_t) \leq Ct,$$

where we have used [41, Proposition 2.5] in the second estimate. Here  $C_1$ ,  $C_2$  and  $C_3$  are uniform constants. Since  $u_t \leq 0$ , we have  $d_1(0, u_t) = |I(u_t)|$ , while [41, Proposition 2.5] gives the bound  $n |I_{\text{Ric}(\omega)}(u_t)| \leq C_2 d_1(0, u_t)$ . Since  $u_t$  is a geodesic ray starting from 0, we have that  $d_1(u_t, 0) = t d_1(0, u_1)$ . This gives the last estimate above. The constants  $C_1$ ,  $C_2$  and  $C_3$  depend only on an upper bound for  $\mathcal{K}\{u_t\}$ ,  $d_1(0, u_1)$  and  $(X, \omega)$ . □

## 6 Applications to geodesic stability

First we point out how the  $L^1$  version of Conjecture 1.7 can be derived from [26; 27; 35, Theorem 4.7]. As alluded to in the introduction, the argument is implicitly

contained in [26; 27], but we provide a short proof here as this result is not explicitly stated in that paper. Recall that  $G = \text{Aut}_0(X, J)$ , and for the definition of  $G$ -calibrated rays we refer back to the introduction.

**Theorem 6.1** ( $L^1$  uniform geodesic stability) *Let  $(X, \omega)$  be a compact Kähler manifold. Then the following are equivalent:*

- (i) *There exists a cscK metric in  $\mathcal{H}_\omega$ .*
- (ii) *There exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta \limsup_t \frac{1}{t} d_{1,G}(G0, Gu_t)$  for all  $\{u_t\}_t \in \mathcal{R}^1$ .*
- (iii)  *$\mathcal{K}$  is  $G$ -invariant and there exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta d_1(0, u_1)$  for all  $G$ -calibrated geodesic rays  $\{u_t\}_t \in \mathcal{R}^1$ .*

We recall that  $\mathcal{R}^p / \mathcal{R}^{1, \bar{1}}$  for  $p \in [1, \infty]$  is the set of rays  $\{u_t\}_t \in \mathcal{R}_\omega^p / \mathcal{R}_\omega^{1, \bar{1}}$  normalized by the condition  $I(u_t) = 0$  for  $t \geq 0$ .

**Proof** By [26, Theorem 1.5], the conditions of [35, Theorem 4.7] are satisfied. Indeed, it was pointed out in [44, Theorem 10.1] that all the conditions (A1)–(A4) and (P1)–(P6) hold with the exception of (P3), which is exactly the content of [26, Theorem 1.5].

After comparing with the conclusion of [35, Theorem 4.7], we only need to argue that condition (ii) implies that  $\mathcal{K}$  is  $G$ -invariant. However we notice that (ii) implies that  $(X, \omega)$  is  $L^1$ -geodesically semistable, in the sense that,  $\mathcal{K}\{u_t\} \geq 0$  for any  $\{u_t\}_t \in \mathcal{R}_\omega^1$ . Now Lemma 4.1 of [27] implies that  $\mathcal{K}$  is  $G$ -invariant, as desired.  $\square$

To show that Theorem 1.8 holds, we argue in the next two results that conditions (ii) and (iii) in the previous theorem are equivalent with their  $C^{1, \bar{1}}$  version:

**Theorem 6.2** *Let  $(X, \omega)$  be a compact Kähler manifold. Then the following are equivalent:*

- (i) *There exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta \limsup_t \frac{1}{t} d_{1,G}(G0, Gu_t)$  for all  $\{u_t\}_t \in \mathcal{R}^1$ .*
- (ii) *There exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta \limsup_t \frac{1}{t} d_{1,G}(G0, Gu_t)$  for all  $\{u_t\}_t \in \mathcal{R}^{1, \bar{1}}$ .*

**Proof** We only need to argue that (ii)  $\implies$  (i). Let  $\{u_t\}_t \in \mathcal{R}^1$ . We can assume that  $\mathcal{K}\{u_t\} < \infty$ , otherwise there is nothing to prove.

We pick  $\{u_t^k\}_t \in \mathcal{R}_\omega^{1,\bar{1}}$ , as in Theorem 5.4. We notice that  $|I(u_t) - I(u_t^k)| \rightarrow 0$  as  $k \rightarrow \infty$  for fixed  $t \geq 0$  (because  $d_1(u_t, u_t^k) \rightarrow 0$ ); hence, by subtracting a linear term from each  $\{u_t^k\}_t$ , we can assume that  $\{u_t^k\}_t \in \mathcal{R}^{1,\bar{1}}$  with  $\mathcal{K}\{u_t^k\} \rightarrow \mathcal{K}\{u_t\}$  and  $d_1^c(\{u_t^k\}_t, \{u_t\}_t) \rightarrow 0$  still holding. Moreover, we have the sequence of inequalities

$$\begin{aligned} \limsup_t \frac{|d_{1,G}(G0, Gu_t) - d_{1,G}(G0, Gu_t^k)|}{t} &\leq \limsup_t \frac{d_{1,G}(Gu_t, Gu_t^k)}{t} \\ &\leq \limsup_t \frac{d_1(u_t, u_t^k)}{t} \\ &= d_1^c(\{u_t\}_t, \{u_t^k\}_t). \end{aligned}$$

Since the last term converges to zero as  $k \rightarrow \infty$ , we obtain that  $\limsup_t \frac{1}{t} d_{1,G}(G0, Gu_t^k)$  converges to  $\limsup_t \frac{1}{t} d_{1,G}(G0, Gu_t)$ , as desired.  $\square$

**Theorem 6.3** *Let  $(X, \omega)$  be a compact Kähler manifold. Then the following are equivalent:*

- (i)  $\mathcal{K}$  is  $G$ -invariant and there exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta d_1(0, u_1)$  for all  $G$ -calibrated geodesic rays  $\{u_t\}_t \in \mathcal{R}^1$ .
- (ii)  $\mathcal{K}$  is  $G$ -invariant and there exists  $\delta > 0$  such that  $\mathcal{K}\{u_t\} \geq \delta d_1(0, u_1)$  for all  $G$ -calibrated geodesic rays  $\{u_t\}_t \in \mathcal{R}^{1,\bar{1}}$ .

**Proof** We only need to argue that (ii)  $\implies$  (i). Let  $\{u_t\}_t \in \mathcal{R}^1$  be  $G$ -calibrated and nonconstant. We can assume that  $\mathcal{K}\{u_t\} < \infty$ , otherwise there is nothing to prove.

Using Theorem 5.4, we pick  $\{u_t^k\}_t \in \mathcal{R}_\omega^{1,\bar{1}}$  such that  $d_1^c(\{u_t^k\}_t, \{u_t\}_t) \rightarrow 0$  and  $\mathcal{K}\{u_t^k\} \rightarrow \mathcal{K}\{u_t\}$ . By adjusting with small constants, we can assume that  $\{u_t^k\}_t \in \mathcal{R}^{1,\bar{1}}$ , and neither of these rays is constant. Unfortunately  $\{u_t^k\}_t$  may not be  $G$ -calibrated, and the bulk of the proof consists of finding a new sequence  $\{\tilde{u}_t^k\}_t \in \mathcal{R}^{1,\bar{1}}$  that satisfies this property.

For any  $k \geq 1$  and  $t \geq 1$ , let  $g_t^k \in G$  be such that

$$(66) \quad d_1(0, g_t^k \cdot u_t^k) \geq d_{1,G}(0, u_t^k) \geq d_1(0, g_t^k \cdot u_t^k) - \frac{1}{t}.$$

The following estimates will be used later:

$$\begin{aligned}
 (67) \quad d_1(0, g_t^k \cdot u_t^k) &\geq d_1(0, g_t^k \cdot u_t) - d_1(g_t^k \cdot u_t^k, g_t^k \cdot u_t) \\
 &\geq d_{1,G}(G \cdot 0, G \cdot u_t) - d_1(u_t, u_t^k) \\
 &\geq d_1(0, u_t^k) - 2d_1(u_t, u_t^k) \\
 &\geq d_1(0, u_t^k) - 2td_1^c(\{u_t\}_t, \{u_t^k\}_t),
 \end{aligned}$$

where in the first line we have used the triangle inequality; in the second line we have used the definition of  $d_{1,G}$  and the fact that  $G$  acts on  $\mathcal{E}_0^1$  by  $d_1$ -isometry (see [44, Lemma 5.9]); in the third line we have used the triangle inequality and that  $\{u_t\}_t$  is  $G$ -calibrated; in the last line we have used that  $(0, +\infty) \ni t \mapsto d_1(u_t^k, u_t)/t$  is increasing (see [9]).

Let  $[0, t] \ni l \rightarrow \rho_l^{k,t} \in \mathcal{E}_\omega^1$  be the finite-energy geodesic connecting  $0$  and  $g_t^k \cdot u_t^k$ . From (66) and [35, Lemma 4.9] it follows that

$$(68) \quad d_1(0, \rho_l^{k,t}) \geq d_{1,G}(G \cdot 0, G \cdot \rho_l^{k,t}) \geq d_1(0, \rho_l^{k,t}) - \frac{1}{t}, \quad l \in [0, t].$$

Using  $G$ -invariance, convexity of  $\mathcal{K}$  and that  $\mathcal{K}(0) = 0$ , for any  $l \in [0, t]$  we have that

$$(69) \quad \frac{\mathcal{K}(\rho_l^{k,t})}{l} \leq \frac{\mathcal{K}(g_t^k \cdot u_t^k)}{t} = \frac{\mathcal{K}(u_t^k)}{t} \leq \mathcal{K}\{u_t^k\}.$$

Due to [8, Corollary 4.8], after possibly selecting a subsequence  $t_j \rightarrow \infty$ , there exists  $\tilde{u}_l^k \in \mathcal{E}_\omega^1$  for any  $l > 0$  such that  $d_1(\tilde{u}_l^k, \rho_l^{k,t_j}) \rightarrow 0$ . After taking the limit in (68), due to [8, Proposition 4.3] we find that  $\{\tilde{u}_t^k\}_t \in \mathcal{R}^1$  is  $G$ -calibrated. Moreover, due to (67), there exists  $k_0$  such that  $\{\tilde{u}_t^k\}_t$  is not the constant ray for  $k \geq k_0$ .

Next we argue that  $\{\tilde{u}_t^k\}_t \in \mathcal{R}^{1,\bar{1}}$ . To start, for  $t \geq 1$  using (66) and the fact that  $G$  acts by  $d_1$ -isometries (see [44, Lemma 5.9]), we get that

$$\begin{aligned}
 (70) \quad d_1(0, g_t^k \cdot 0) &= d_1(0, (g_t^k)^{-1} \cdot 0) \leq d_1(u_t^k, (g_t^k)^{-1} \cdot 0) + d_1(u_t^k, 0) \\
 &= d_1(g_t^k \cdot u_t^k, 0) + d_1(u_t^k, 0) \leq 2d_1(u_t^k, 0) + \frac{1}{t} \leq 2d_1(u_t^k, 0) + 1.
 \end{aligned}$$

Next, let  $B > 0$  as in the statement of Theorem 5.5. Using Lemma 6.4 below, we have the estimates

$$\max\left(\sup_X |g_t^k \cdot 0|, \sup_X \log |\nabla g_t^k|_\omega, \sup_X (\log(n + \Delta_\omega(g_t^k \cdot 0)) - Bg_t^k \cdot 0)\right) \leq Cd_1(g_t^k \cdot 0, 0) + C.$$

Recall that  $g_t^k . u_t^k = (g_t^k)^* u_t^k + g_t^k . 0$  (see [44, Lemma 5.8]). In particular, Theorem 1 of [34], Remark 5.7 and (70) give that

$$\begin{aligned} \sup_X |g_t^k . u_t^k| &\leq \sup_X |u_t^k| + \sup_X |g_t^k . 0| \leq Ct + 2Cd_1(u_1^k, 0)t + C \leq Ct + C, \\ \sup_X (\log(n + \Delta_\omega(g_t^k . u_t^k)) - Bg_t^k . u_t^k) &\leq Ct + 2Cd_1(u_1^k, 0)t + C \leq Ct + C, \end{aligned}$$

where  $C$  depends on  $k$  but not on  $t \geq 1$ ! Using [34, Theorem 1] and Theorem 5.5, we find that  $\sup_X |\rho_l^{k,t}| \leq Cl + C$  and  $\sup_X (\log(n + \Delta_\omega \rho_l^{k,t}) - B\rho_l^{k,t}) \leq Cl + C$  for any  $l \in [0, t]$ . Lastly, letting  $t_j \rightarrow \infty$ , we arrive at  $\sup_X |\tilde{u}_l^k| \leq Cl + C$  and  $\sup_X (\log(n + \Delta_\omega \tilde{u}_l^k) - B\tilde{u}_l^k) \leq Cl + C$  for any  $l \geq 0$ , what we wanted to argue.

Due to the fact that  $\mathcal{K}$  is  $d_1$ -lsc,  $G$ -invariant and convex, similarly to (69), we find that for all  $l > 0$  and  $k \geq k_0$  we have

$$\begin{aligned} (71) \quad \frac{\mathcal{K}(\tilde{u}_l^k)}{d_1(0, \tilde{u}_l^k)} &\leq \liminf_{t_j \rightarrow \infty} \frac{\mathcal{K}(\rho_l^{k,t_j})}{d_1(0, \rho_l^{k,t_j})} \leq \liminf_{t_j \rightarrow \infty} \frac{\mathcal{K}(g_{t_j}^k . u_{t_j}^k)}{d_1(0, g_{t_j}^k . u_{t_j}^k)} \\ &= \liminf_{t_j \rightarrow \infty} \frac{\mathcal{K}(u_{t_j}^k)}{d_1(0, g_{t_j}^k . u_{t_j}^k)} \leq \liminf_{t_j \rightarrow \infty} \frac{\mathcal{K}(u_{t_j}^k)}{d_1(0, u_{t_j}^k) - 2t_j d_1^c(\{u_t^k\}_t, \{u_t\}_t)} \\ &= \liminf_{t_j \rightarrow \infty} \left[ \frac{\mathcal{K}(u_{t_j}^k)}{d_1(0, u_{t_j}^k)} \cdot \frac{d_1(0, u_{t_j}^k)}{d_1(0, u_{t_j}^k) - 2t_j d_1^c(\{u_t^k\}_t, \{u_t\}_t)} \right] \\ &\leq \frac{\mathcal{K}\{u_t^k\}}{d_1(0, u_1^k)} \cdot \frac{d_1(0, u_1^k)}{d_1(0, u_1^k) - 2d_1^c(\{u_t^k\}_t, \{u_t\}_t)}, \end{aligned}$$

where in the second line we have used (67), and all the denominators are nonzero since  $\{\tilde{u}_t^k\}_t$  and  $\{u_t^k\}_t$  are nonconstant for  $k \geq k_0$ .

Finally, we use that (ii) holds for  $\{\tilde{u}_t^k\}_t \in \mathcal{R}^{1,\bar{1}}$ . Consequently, after letting  $l, t \rightarrow \infty$  in (71), we arrive at

$$\delta \leq \frac{\mathcal{K}\{\tilde{u}_t^k\}}{d_1(0, \tilde{u}_1^k)} \leq \frac{\mathcal{K}\{u_t^k\}}{d_1(0, u_1^k)} \cdot \frac{d_1(0, u_1^k)}{d_1(0, u_1^k) - 2d_1^c(\{u_t^k\}_t, \{u_t\}_t)}.$$

Letting  $k \rightarrow \infty$ , we now obtain that  $\delta \leq \mathcal{K}\{u_t\}/d_1(0, u_1)$ , finishing the proof. □

**Lemma 6.4** *Let  $(X, \omega)$  be a compact Kähler manifold. There exists  $C := C(X, \omega) > 0$  such that for all  $g \in G$  we have that*

$$\sup_X |g.0| \leq Cd_1(0, g.0) + C \quad \text{and} \quad \sup_X \log(n + \Delta_\omega(g.0)) \leq Cd_1(0, g.0) + C.$$

The Laplacian estimate from this lemma is equivalent with the following estimate for the gradient  $\nabla g$ , as a self-map of  $X$ :

$$(72) \quad \sup_X |\nabla g|_\omega^2 \leq e^{Cd_1(0, g.0) + C}, \quad g \in G.$$

The desired Laplacian estimate of the lemma can be extracted from the arguments of [28], as we now elaborate.

**Proof** Fix  $g \in G$ . Using [44, Lemma 5.8], and the fact that  $(g^{-1})^*(g^*\omega) = \omega$ , we obtain that  $0 = g^{-1}.(g.0) = g^{-1}.0 + (g^{-1})^*(g.0)$ . In particular, we have that

$$(73) \quad -\inf_X g.0 = \sup_X g^{-1}.0.$$

Due to [32, Corollary 4; 35, Lemma 3.45] we have that

$$0 \leq \sup_X g.0 \leq \int_X g.0 \omega^n + C \leq Cd_1(0, g.0) + C,$$

and

$$0 \leq \sup_X g^{-1}.0 \leq \int_X g^{-1}.0 \omega^n \leq Cd_1(0, g^{-1}.0) + C.$$

Since  $d_1(0, g^{-1}.0) = d_1(g.0, 0)$ , putting the above together with (73), one of the desired estimates follows:

$$(74) \quad \sup_X |g.0| \leq Cd_1(0, g.0) + C.$$

Now we address the Laplacian estimate. To start, we note that there exists  $C := C(X, \omega) > 0$  such that  $-C\omega \leq \text{Ric}(\omega) \leq C\omega$ . Pulling back by  $g$  we obtain that  $\text{Ric}(\omega_{g.0}) \leq C\omega_{g.0}$ . We introduce  $F_g := \log(\omega_{g.0}^n / \omega^n)$ . We obtain that

$$i\partial\bar{\partial}F_g = \text{Ric}(\omega) - \text{Ric}(\omega_{g.0}) \geq -C\omega - C\omega_{g.0}.$$

In particular,  $\frac{1}{2}g.0 + \frac{1}{2C}F_g \in \text{PSH}(X, \omega)$ , implying that

$$\sup_X \left( \frac{1}{2}g.0 + \frac{1}{2C}F_g \right) \leq C + \int_X \left( \frac{1}{2}g.0 + \frac{1}{2C}F_g \right) \omega^n \leq \frac{1}{2}d_1(0, g.0) + C.$$

Here, we used Jensen’s inequality to obtain  $\int_X F_g \omega^n \leq 0$ . Using (74) we arrive at

$$(75) \quad \sup_X F_g \leq Cd_1(0, g.0) + C.$$

To obtain the Laplacian estimate, we start with Yau’s calculation (for a survey, see [16, Proposition 4.1.2]):

$$\text{Tr}_{\omega_{g,0}}[i \partial \bar{\partial} \log \text{Tr}_{\omega} \omega_{g,0}] \geq \frac{\text{Tr}_{\omega}[i \partial \bar{\partial} \log(\omega_{g,0}^n / \omega^n)]}{\text{Tr}_{\omega} \omega_{g,0}} - C \text{Tr}_{\omega_{g,0}} \omega,$$

where  $C > 0$  only depends on  $(X, \omega)$ . Let  $B := 2C + 1$ . Using the fact that  $\text{Ric}(\omega_{g,0}) \leq C \omega_{g,0}$ , we can continue:

$$\begin{aligned} \text{Tr}_{\omega_{g,0}}[i \partial \bar{\partial}(\log \text{Tr}_{\omega} \omega_{g,0} - Bg \cdot 0)] &\geq \frac{\text{Tr}_{\omega}[-C \omega_{g,0} - C \omega]}{\text{Tr}_{\omega} \omega_{g,0}} - C \text{Tr}_{\omega_{g,0}} \omega - B \text{Tr}_{\omega_{g,0}}[i \partial \bar{\partial} g \cdot 0] \\ &\geq -\frac{nC}{\text{Tr}_{\omega} \omega_{g,0}} + (B - C) \text{Tr}_{\omega_{g,0}} \omega - nB - C \\ &\geq (B - 2C) \text{Tr}_{\omega_{g,0}} \omega - nB - C \\ &\geq \text{Tr}_{\omega_{g,0}} \omega - C \geq \left(\frac{\omega_{g,0}^n}{\omega^n}\right)^{-1/(n-1)} (\text{Tr}_{\omega} \omega_{g,0})^{1/(n-1)} - C \\ &= F_g^{-1/(n-1)} (\text{Tr}_{\omega} \omega_{g,0})^{1/(n-1)} - C. \end{aligned}$$

Let  $x_0 \in X$  be the point where  $\log \text{Tr}_{\omega} \omega_{g,0} - Bg \cdot 0$  is maximized. Using the above estimate and (75) we obtain that  $\text{Tr}_{\omega} \omega_{g,0}(x_0) \leq C d_1(0, g \cdot 0) + C$ . Together with (74) we arrive at  $\sup_X \log(n + \Delta_{\omega}(g \cdot 0)) \leq C d_1(0, g \cdot 0) + C$ . □

## Appendix

Here we address two likely known facts about Kähler potentials with bounded Laplacian, whose proof we could not find in the literature.

**Lemma A.1** *Let  $u, u_j \in \mathcal{H}_{\omega}^{1,1}$  and  $B \in \mathbb{R}$ . If  $u_j \searrow u$  then*

$$(76) \quad \liminf_j \text{ess sup}_X (\log(n + \Delta_{\omega} u_j) - B u_j) \geq \text{ess sup}_X (\log(n + \Delta_{\omega} u) - B u).$$

**Proof** After picking a subsequence, we can assume without loss of generality that the  $\liminf$  on the left-hand side is actually a limit. Let  $\delta \in \mathbb{R}$  be such that

$$\log(n + \Delta_{\omega} u_j(x)) - B u_j(x) < \delta$$

for a.e.  $x \in X$  and  $j \in \mathbb{N}$ . To conclude, it is enough to show that

$$(77) \quad \log(n + \Delta_{\omega} u) - B u \leq \delta \quad \text{a.e. on } X.$$

By assumption,  $\Delta_\omega u_j + n \leq e^{Bu_j + \delta}$  in the weak sense of positive measures on  $X$ . By Dini’s lemma we have that  $\|u_j - u\|_{C^0} \rightarrow 0$ , hence passing to the weak limit we have that  $\Delta_\omega u + n \leq e^{Bu + \delta}$ , again in the weak sense of positive measures on  $X$ . Since all our measures have bounded densities, (77) follows.  $\square$

Complementing the above lemma, in the next result we point out that the quantity on the right-hand side of (76) can be realized with an appropriate decreasing sequence, constructed via the method of [46]. Let us recall some elements of this work. We denote by  $\text{exp}_x: T_x X \rightarrow X$  the “quasiholomorphic exponential map” of  $\omega$  (see [46, Section 2]). Let  $\chi: \mathbb{R} \rightarrow \mathbb{R}$  be an even nonnegative smooth function supported in  $[0, 1]$  such that  $\int_{\mathbb{C}^n} \chi(\|\xi\|^2) d\lambda(\xi) = 1$ . Given  $u \in \text{PSH}(X, \omega)$ , one can introduce  $u_\varepsilon \in C^\infty(X)$  by

$$u_\varepsilon(x) := \frac{1}{\varepsilon^{2n}} \int_{T_x X} u(\text{exp}_x(\xi)) \chi\left(\frac{|\xi|^2}{\varepsilon^2}\right) d\lambda(\xi),$$

where  $d\lambda$  is the Lebesgue measure on  $T_x X$  with respect to  $\omega$ .

**Proposition A.2** *Let  $u \in \mathcal{H}_\omega^{1, \bar{1}}$  and  $B \in \mathbb{R}$ . There exists  $u_j \in \mathcal{H}_\omega$  such that  $u_j$  converges to  $u$  decreasingly (and uniformly by Dini’s lemma) and*

$$(78) \quad \limsup_j \sup_X (\log(n + \Delta_\omega u_j) - Bu_j) = \text{ess sup}_X (\log(n + \Delta_\omega u) - Bu).$$

**Proof** By possibly rescaling  $u$  with a small constant, we can assume that there exists  $\delta > 0$  such that  $\omega_u \geq \delta\omega$ . In particular, it follows from the estimate of [46, Theorem 4.1] that for small enough  $\varepsilon > 0$  we actually have that  $u_\varepsilon \in \mathcal{H}_\omega$ . Moreover,  $\|u_\varepsilon - u\|_{C^0} \rightarrow 0$ . Also, it follows from [46, Theorem 3.8] that

$$i \partial \bar{\partial} u_\varepsilon(\zeta, \zeta) = \int_{T_x X} i \partial \bar{\partial} u|_{\text{exp}_x(\varepsilon \xi)}(\zeta, \zeta) \chi(|\xi|^2) d\lambda(\xi) + O(|\varepsilon|)(\zeta, \zeta), \quad \zeta \in T_x X, x \in X.$$

Consequently, by an elementary local calculation, we have that

$$\lim_{\varepsilon \rightarrow 0} \sup_X (\log(n + \Delta_\omega u_\varepsilon) - Bu_\varepsilon) = \text{ess sup}_X (\log(n + \Delta_\omega u) - Bu).$$

After possibly adding small constants to  $u_\varepsilon$ , we can construct the decreasing sequence desired.  $\square$

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