msp

Algebraic & Geometric Topology 20 (2020) 2413–2450

Amenable signatures, algebraic solutions and filtrations of the knot concordance group

TAEHEE KIM

It is known that each of the successive quotient groups of the grope and solvable filtrations of the knot concordance group has an infinite-rank subgroup. The generating knots of these subgroups are constructed using iterated doubling operators. In this paper, for each of the successive quotients of the filtrations we give a new infinite-rank subgroup which trivially intersects the previously known infinite-rank subgroups. Instead of iterated doubling operators, the generating knots of these new subgroups are constructed using the notion of algebraic n–solutions, which was introduced by Cochran and Teichner. Moreover, for any slice knot K whose Alexander polynomial has degree greater than 2, we construct the generating knots so that they have the same derived quotients and higher-order Alexander invariants up to a certain order as the knot K.

In the proof, we use an L^2–theoretic obstruction for a knot to being $n.5$–solvable given by Cha, which is based on L^2–theoretic techniques developed by Cha and Orr. We also generalize and use the notion of algebraic n–solutions to the notion of R–algebraic n–solutions, where R is either the rationals or the field of p elements for a prime p.

57M25; 57N70

Dedicated to the memory of Tim D Cochran

1 Introduction

We address the structure of the grope and solvable filtrations of the knot concordance group. Two oriented knots K_0 and K_1 in the 3–sphere S^3 are *concordant* if there is a topologically locally flat and properly embedded annulus in $S^3 \times [0, 1]$ whose boundary is the union of $K_0 \times \{0\}$ and $-K_1 \times \{1\}$. It is known that K_0 and K_1 are concordant if and only if the connected sum $K_0 \# (-K_1)$ bounds a locally flat and properly embedded disk in the 4–ball D^4, namely, $K_0 \# (-K_1)$ is a *slice knot*. Concordance classes form an abelian group under connected sum, which is called *the knot concordance group* and denoted by $\mathcal{C}$.

Published: 4 November 2020 DOI: 10.2140/agt.2020.20.2413

In the late 1990s, Cochran, Orr and Teichner [15] introduced the grope and solvable filtrations of the knot concordance group, denoted by $\{\mathcal{G}_n\}$ and $\{\mathcal{F}_n\}$, respectively, which are indexed by nonnegative half-integers. The subgroup $\mathcal{G}_n$ consists of knots bounding a *grope* of height n in D^4, where a grope is a certain 2–complex constructed by attaching surfaces along their boundaries. See Definition 2.1 for a precise definition of a grope. Similarly, $\mathcal{F}_n$ is the subgroup of knots such that the zero-framed surgery on the knot bounds an *n–solution*, where an n–solution is a 4–manifold satisfying certain conditions on the (equivariant) intersection form with twisted coefficients (see Definition 2.3). For all n it is known that $\mathcal{G}_{n+2} \subset \mathcal{F}_n$ [15, Theorem 8.11], and one may consider an n–solution as an order n approximation of the exterior of a slice disk in D^4. These filtrations reflect classical invariants at low levels. For instance, a knot K has vanishing Arf invariant if and only if $K \in \mathcal{F}_0$, and K is algebraically slice if and only if $K \in \mathcal{F}_{0.5}$ [15]. Furthermore, a knot in $\mathcal{F}_{1.5}$ has vanishing Casson–Gordon invariants [15], but it is known that there exists a knot with vanishing Casson–Gordon invariants which is not in $\mathcal{F}_{1.5}$; see Kim [31].

We review results on finding infinite-rank subgroups of the successive quotients $\mathcal{F}_n/\mathcal{F}_{n.5}$ and $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ for $n \geq 2$. We are interested in the cases for $n \geq 2$ since the cases for $n < 2$ were well known from classical invariants. For finding infinite-rank subgroups of the quotients, there are three different approaches, using

(1) rationally universal solvable representations;

(2) algebraic n–solutions;

(3) iterated doubling operators.

In this paper, by an iterated doubling operator we mean any of the iterated (generalized) doubling operators used in Cochran, Harvey and Leidy [11; 13], Cha [2] and Horn [27]. For a more precise definition of an iterated doubling operator, see the proof of Theorem 4.3.

First, in their seminal papers [15; 16], using the von Neumann–Cheeger–Gromov $\rho^{(2)}$–invariants associated to *rationally universal solvable representations*, Cochran, Orr and Teichner showed that $\mathcal{F}_2/\mathcal{F}_{2.5}$ has an infinite-rank subgroup, giving the first example of nonslice knots with vanishing Casson–Gordon invariants.

For each integer $n \geq 2$, Cochran and Teichner showed that $\mathcal{F}_n/\mathcal{F}_{n.5}$ and $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ are infinite and have positive rank using Cheeger–Gromov's universal bound on $\rho^{(2)}$–invariants and their new notion of an *algebraic n–solution* [17]. Later, Cochran and

Teichner's work was refined further by Cochran and the author [14]. Also in [14], the notion of an algebraic n–solution was generalized.

Using *iterated doubling operators* and unlocalized higher-order Blanchfield linking forms, Cochran, Harvey, and Leidy obtained the first example of an infinite-rank subgroup of $\mathcal{F}_n/\mathcal{F}_{n.5}$ for each integer $n \geq 2$ [11]. They extended their work further and showed that the solvable filtration $\{\mathcal{F}_n\}$ has refined filtrations related to primary decomposition whose successive quotient groups have infinite rank [13]. For the grope filtration, Horn [27] proved that $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ has an infinite-rank subgroup for each integer $n \geq 2$, where the generating knots of the subgroups were constructed using iterated doubling operators.

In all of the above work, knots were obstructed to being in $\mathcal{F}_{n.5}$ (hence not in $\mathcal{G}_{n+2.5}$) using the $\rho^{(2)}$–invariants associated to a representation mapping to a poly-torsion-free-abelian (henceforth PTFA) group. By a PTFA group, we mean a group which allows a subnormal series whose successive quotient groups are torsion-free and abelian. These obstructions are essentially based on the vanishing criterion of the $\rho^{(2)}$–invariants associated to a PTFA representation given by Cochran, Orr and Teichner [15, Theorem 4.2]. For the reader's convenience, we review the $\rho^{(2)}$–invariant in Section 2.2, where the $\rho^{(2)}$–invariant is defined to be an L^2–signature defect.

In [7], Cha and Orr developed L^2–theoretic methods: for a representation to a group which is amenable and lies in Strebel's class $D(R)$ for some commutative ring R, they proved the homology cobordism invariance of the L^2–Betti numbers and the $\rho^{(2)}$–invariants, and presented a method for controlling the L^2–dimension of homology with L^2–coefficients. (The reader may refer to [7] for definitions of amenable groups and Strebel's class $D(R)$, but the definitions will not be needed in this paper.)

Based on the work in [7], Cha found a vanishing criterion of $\rho^{(2)}$–invariants associated to a representation to a group which is amenable and lies in Strebel's class $D(R)$, where $R=\mathbb{Q}$ or $\mathbb{Z}_p$ for a prime p [2, Therorem 1.3], which extends [15, Theorem 4.2]. In this paper, we call this extended vanishing criterion the *amenable signature theorem* (*for $n.5$–solvability*); see Theorem 2.5. We note that the above class of groups includes PTFA groups and some groups with torsion elements (see Lemma 2.7). Using the amenable signature theorem, Theorem 2.5, Cha constructed an infinite-rank subgroup of $\mathcal{F}_n/\mathcal{F}_{n.5}$ for each integer $n \geq 2$ for which the $\rho^{(2)}$–invariants associated to a PTFA representation vanish [2, Theorem 1.4]. The generating knots of the infinite-rank subgroups in [2] were constructed also using iterated doubling operators.

In this paper, we combine algebraic n–solutions and the amenable signature theorem to find new infinite-rank subgroups of the grope and solvable filtrations, and give Theorem 1.1 below. In the following, $\Delta_K(t)$ denotes the Alexander polynomial of a knot K. For a group G, let $G^{(0)} := G$, and inductively for $n \geq 0$ define $G^{(n+1)} := [G^{(n)}, G^{(n)}]$. The group $G^{(n)}$ is called the n^{th} *derived subgroup* of G. Also recall that $\mathcal{G}_{n+2} \subset \mathcal{F}_n$ for all n.

Theorem 1.1 *Let $n \geq 2$ be an integer. Let K be a slice knot with* $\deg \Delta_K(t) > 2$. *Then there is a sequence of knots $K_1, K_2, \ldots$ satisfying the following*:

(1) *For each i, there is an isomorphism*

$$\pi_1(S^3 \smallsetminus K_i)/\pi_1(S^3 \smallsetminus K_i)^{(n+1)} \to \pi_1(S^3 \smallsetminus K)/\pi_1(S^3 \smallsetminus K)^{(n+1)}$$

which preserves the peripheral structures.

(2) *The K_i are in $\mathcal{G}_{n+2}$ and they are linearly independent modulo $\mathcal{F}_{n.5}$. In particular, the knots K_i generate an infinite-rank subgroup in each of $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ and $\mathcal{F}_n/\mathcal{F}_{n.5}$.*

(3) *In $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$, the infinite-rank subgroup generated by the K_i trivially intersects the infinite-rank subgroup in* [27].

(4) *In $\mathcal{F}_n/\mathcal{F}_{n.5}$, the infinite-rank subgroup generated by the K_i trivially intersects the infinite-rank subgroups in* [11; 13; 2].

In Theorem 1.1, the condition $\deg \Delta_K(t) > 2$ is the best possible; by the work of Friedl and Teichner [24], Theorem 1.1 does not hold for any $n \geq 2$ for a certain slice knot K with $\deg \Delta_K(t) = 2$ (also refer to [14, Proposition 5.10]). Properties (1) and (2) of Theorem 1.1 are proven in Theorem 4.2. Theorem 4.3 gives properties (3) and (4).

An advantage of constructing the K_i using not iterated doubling operators but algebraic n–solutions is that Theorem 1.1(1) is obtained for *any* slice knot K with $\deg \Delta_K(t) > 2$. We remark that Theorem 1.1(1) is related to studying the structure on knot concordance under a fixed Seifert form. It is well known by the work of Freedman and Quinn [20; 21] that a knot with trivial Alexander polynomial is slice, and hence determines a unique concordance class, namely the class of slice knots. It was asked if there is any other Alexander polynomial or a Seifert form which determines a unique concordance class. This question was answered in the negative that for each Seifert form of a knot K with nontrivial Alexander polynomial, there exist infinitely many mutually nonconcordant knots K_i having the Seifert form; see Livingston [35] and Kim [32] (also refer to

Kim [34]). This result was refined further: in [14, Theorem 5.1] it was shown under the condition deg $\Delta_K(t) > 2$ that for each $n \geq 2$ those mutually nonconcordant knots K_i can be constructed using algebraic n–solutions such that the K_i are mutually distinct in $\mathcal{F}_n/\mathcal{F}_{n.5}$ and satisfy property (1) in Theorem 1.1. Furthermore, in [14] it was shown that the K_i have the same m^{th} *order Seifert presentation* (see [14, Definitions 5.5 and 5.9]) as K for $m = 0, 1, \dots, n-1$, and hence have the same Seifert form. In this paper, in Theorem 1.1, we refine it further and construct the K_i so that they are linearly independent in $\mathcal{F}_n/\mathcal{F}_{n.5}$, still satisfying property (1) in Theorem 1.1. We also note that the K_i in Theorem 1.1 have the same m^{th} order Seifert presentation for $m = 0, 1, \dots, n-1$. (This can be easily shown using the same arguments in the proof of [14, Theorem 5.1] and will not be discussed in this paper.)

We also refer the reader to Cha and Kim [33; 5] for more applications of algebraic n–solutions to doubly slice knots and the solvable filtration of the *rational* knot concordance group.

We construct the K_i in Theorem 1.1 using a well-known process called *infection* or *satellite construction*, which involves a *seed knot*, *axes* (*knots*) and *infection knots* (see Section 3). Then, we show that the K_i are linearly independent modulo $\mathcal{F}_{n.5}$ using the amenable signature theorem, Theorem 2.5, following the ideas in [2]. We give more details: Let J be a nontrivial linear combination of the K_i. Given any n–solution V for J, to obstruct V from being an $n.5$–solution, we use the $\rho^{(2)}$–invariants associated to a representation factoring through $\pi_1 W \to \pi_1 W/\mathcal{P}^{n+1}\pi_1 W$, where W is a certain 4–manifold containing V. Here, $\mathcal{P}^{n+1}\pi_1 W$ is a subgroup of $\pi_1 W$ obtained from a certain *mixed-coefficient commutator series* of $\pi_1 W$ which is defined depending on the choice of a prime p. See Definition 2.6 for a precise definition of $\mathcal{P}^{n+1}\pi_1 W$. We remark that, as can be seen in the proof of Theorem 4.2, the choice of a prime p is specific to the linear combination J. Namely, we can choose a representation specific to J and use the corresponding $\rho^{(2)}$–invariant, and this makes it easier to show that J is not trivial modulo $\mathcal{F}_{n.5}$. In the representation, the quotient group $\pi_1 W/\mathcal{P}^{n+1}\pi_1 W$ has torsion elements, hence is not a PTFA group. But it is amenable and lies in Strebel's class $D(\mathbb{Z}_p)$, and this fact enables us to use the amenable signature theorem, Theorem 2.5, and obtain desired computations of $\rho^{(2)}$–invariants.

When we construct the K_i using infection, we need to choose the axes in a subtle way: for a given slice knot K with deg $\Delta_K(t) > 2$, we use K as a seed knot and choose axes η_i for $1 \leq i \leq m$ such that for each homomorphism on π_1 induced from the inclusion $M(K) \to W$, where $M(K)$ is the 0–surgery on K and W is an n–solution for K, the axes η_i should satisfy a certain nontriviality property under the homomorphism. The desired nontriviality property and existence of such axes are given in Theorem 3.4, which is a key technical theorem in this paper. Roughly speaking, the nontriviality property requires that for a given n–solution W and $\mathcal{P}^{n+1}\pi_1 W$, there exists some η_i such that $\eta_i \notin \mathcal{P}^{n+1}\pi_1 W$. To prove Theorem 3.4 we generalize the notion of an algebraic n–solution in [17; 14] and define the notion of an *R–algebraic n–solution*, where $R = \mathbb{Q}$ or $\mathbb{Z}_p$ for a prime p (see Definition 5.8). We also use a nontriviality theorem on homology with twisted coefficients which is obtained using higher-order Blanchfield linking forms (see Theorem 5.2 and the proof of Theorem 3.4).

We prove Theorem 1.1(3)–(4) in Theorem 4.3, following the ideas in [13, Section 9]. For each positive half-integer n, a prime p and a group G, we define a subgroup $G^{(n)}_{\mathrm{cot},p}$ of G using a localization of a group ring (see Section 4.2). Then, using it we define the notion of *(n,p)–solvable knots* (Definition 4.4). We note that $G^{(n)} \subset G^{(n)}_{\mathrm{cot},p}$ for all prime p, and an n–solvable knot is (n,p)–solvable for all prime p. Finally, we show that a nontrivial linear combination of the K_i in Theorem 1.1 is not $(n.5,p)$–solvable for some prime p, but a knot concordant to a linear combination of the knots in [27; 11; 13; 2] is $(n.5,p)$–solvable for all prime p.

Other than finding infinite-rank subgroups of $\mathcal{F}_n/\mathcal{F}_{n.5}$ and $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$, there are many interesting results on the grope and solvable filtrations; see Kim and Kim [29; 30], Cochran, Harvey and Leidy [13; 12], Burke [1], Davis [18] and Jang [28]. For instance, it is known that for each integer $n \geq 2$ and for each of $\mathcal{F}_n/\mathcal{F}_{n.5}$ and $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$, there exists a subgroup infinitely generated by knots of order 2 [12; 28]. These knots are constructed using iterated doubling operators, and it is unknown whether or not one can construct a subgroup infinitely generated by knots of order 2 using algebraic n–solutions.

We can also define the grope and solvable filtrations $\{\mathcal{G}^s_n\}$ and $\{\mathcal{F}^s_n\}$ of the smooth knot concordance group. We note that all the results in this paper also hold under this smooth setting: all our examples of manifolds and gropes are constructed smoothly, and the obstructions obtained from the amenable signature theorem, Theorem 2.5, are topological obstructions. Moreover, by the work of Freedman and Quinn [21], it is known that in fact the smooth and topological filtrations are equivalent, that is, $K \in \mathcal{G}_n$ (resp. $K \in \mathcal{F}_n$) if and only $K \in \mathcal{G}^s_n$ (resp. $K \in \mathcal{F}^s_n$); see [3, Remark 2.19; 13, page 454].

This paper is organized as follows. In Section 2, we introduce the grope and solvable filtrations of the knot concordance group and give the amenable signature theorem,

Theorem 2.5. In Section 3, we discuss the construction of examples using infection (or satellite construction). We prove Theorem 1.1 in Section 4, and discuss higher-order Blanchfield linking forms and the notion of R–algebraic n–solutions in Section 5.

In this paper, manifolds are assumed to be compact, and oriented, and $\mathbb{Z}_p$ denotes the field of p elements for a prime p. By abuse of notation we use the same symbol for a knot and its homotopy class and homology classes. Homology groups come with integer coefficients unless specified otherwise. For a knot K, we denote by $E(K)$ and $M(K)$ the exterior of K in S^3 and the 0–surgery on K in S^3, respectively.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (no. 2011-0030044(SRC-GAIA) and no. 2015R1D1A1A01056634).

2 Preliminaries

In this section, we review the grope and solvable filtrations of the knot concordance group, and recall necessary results on amenable signatures and mixed-coefficient commutator series of a group in [2].

2.1 The grope and solvable filtrations

In this subsection we review the notions of a grope, an n–solution, an n–cylinder and the grope and solvable filtrations of the knot concordance group.

Definition 2.1 [22; 15, Definition 7.9] A *grope of height* 1 is a compact connected surface with a single boundary component. This boundary component is called *the base circle*. Let Σ be a grope of height 1 of genus g. Let $\{\alpha_i, \beta_i\}_{1\le i\le g}$ be a standard symplectic basis of circles on Σ such that α_i and β_i are dual to each other. For an integer $n \geq 1$, a *grope of height* $n+1$ is a 2–complex obtained by attaching gropes of height n to each α_i and β_i along the base circles. A *grope of height* $n.5$ is a 2–complex obtained by attaching gropes of height n to each α_i and gropes of height $n-1$ to each β_i along the base circles. Here, the surface Σ is called *the bottom stage* of the grope, and a grope of height 0 is understood to be the empty set. For a grope embedded in a 4–manifold, we require that the grope has a neighborhood which is diffeomorphic to the product of $\mathbb{R}$ and the standard neighborhood of the (abstract) grope in $\mathbb{R}^3$. In this paper, a grope in a 4–manifold means a grope smoothly embedded in the 4–manifold in this way. We also note that in the literature a grope defined in this way is called a *symmetric grope*.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $n \in \frac{1}{2}\mathbb{N}_0$, we denote by $\mathcal{G}_n$ the subset of $\mathcal{C}$ which consists of knots bounding a grope of height n in D^4. It is known that each $\mathcal{G}_n$ is a subgroup of $\mathcal{C}$ and $\mathcal{G}_m \subset \mathcal{G}_n$ for $m \geq n$, and hence $\{\mathcal{G}_n\}$ is a filtration of $\mathcal{C}$. We call it the *grope filtration* of $\mathcal{C}$.

For a group G, recall that $G^{(n)}$ denotes the n^{th} derived subgroup of G. Now, for a 4–manifold W, let $\pi := \pi_1 W$ and let $R := \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{Z}_p$. Then, for each $n \geq 0$, there is the equivariant intersection form

$$\lambda_n^R \colon H_2(W; R[\pi/\pi^{(n)}]) \times H_2(W; R[\pi/\pi^{(n)}]) \to R[\pi/\pi^{(n)}].$$

We drop the decoration R from λ_n^R when it is understood from the context. Below, we generalize the notions of an n–cylinder and a rational n–cylinder in [14].

Definition 2.2 (Section 2 of [14] for $R = \mathbb{Z}$ and $\mathbb{Q}$) Let $R = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{Z}_p$. Let n be a nonnegative integer. Let W be a compact connected 4–manifold with $\partial W = \coprod_{i=1}^{\ell} M_i$, where each M_i is a connected component with $H_1(M_i) \cong R$. Let $\pi := \pi_1 W$ and $r = \frac{1}{2}\operatorname{rank}_R \operatorname{Coker}\{H_2(\partial W; R) \to H_2(W; R)\}$.

(1) W is an *R–coefficient n–cylinder* if each inclusion from M_i to W induces an isomorphism on $H_1(M_i; R)$ and there exist $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_r$ in $H_2(W; R[\pi/\pi^{(n)}])$ such that $\lambda_n(x_i, x_j) = 0$ and $\lambda_n(x_i, y_j) = \delta_{ij}$ for $1 \leq i, j \leq r$.

(2) W is an *R–coefficient $n.5$–cylinder* if W satisfies (1), and furthermore there exist lifts $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_r$ of $x_1, x_2, \ldots, x_r$ in $H_2(W; R[\pi/\pi^{(n+1)}])$ such that $\lambda_{n+1}(\widetilde{x}_i, \widetilde{x}_j) = 0$ for $1 \leq i, j \leq r$.

We also require W to be spin when $R = \mathbb{Z}$. A $\mathbb{Z}$–coefficient n–cylinder is also called an *n–cylinder*. The submodule generated by $x_1, x_2, \ldots, x_r$ (resp. $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_r$) is called an *$n$–Lagrangian* (resp. *$(n+1)$–Lagrangian*), and the submodule generated by $y_1, y_2, \ldots, y_r$ is called its *n–dual*.

It is obvious that an n–cylinder is a $\mathbb{Q}$–coefficient n–cylinder. An n–cylinder in [14] appeared as a generalization of an n–solution in [15]. Below we generalize the notion of an n–solution to the notion of an R–coefficient n–solution.

Definition 2.3 (Section 8 of [15] for $R=\mathbb{Z}$ and $\mathbb{Q}$) An R–coefficient n–cylinder with a single boundary component is also called an *R–coefficient n–solution.* A $\mathbb{Z}$–coefficient n–solution is called an *n–solution.* A closed 3–manifold M with $H_1(M)\cong\mathbb{Z}$ is *n–solvable via W* if there exists an n–solution W with boundary M. A knot K in S^3 is *n–solvable via W* and W is an *n–solution for K* if $M(K)$ is n–solvable via W.

For each $n\in\frac{1}{2}\mathbb{N}_0$, we denote by $\mathcal{F}_n$ the subset of n–solvable knots in $\mathcal{C}$. It is known that $\mathcal{F}_n$ is a subgroup of $\mathcal{C}$ and $\mathcal{F}_m\subset\mathcal{F}_n$ if $m\geq n$ [15]. Therefore, $\{\mathcal{F}_n\}$ is a filtration of $\mathcal{C}$, and we call it the *solvable filtration* of $\mathcal{C}$. The filtrations $\{\mathcal{G}_n\}$ and $\{\mathcal{F}_n\}$ have the following relationship.

Theorem 2.4 [15, Theorem 8.11] *Let $n\in\frac{1}{2}\mathbb{N}_0$. If a knot K bounds a grope of height $n+2$ in D^4, then K is n–solvable. That is, $\mathcal{G}_{n+2}\subset\mathcal{F}_n$.*

It is unknown whether or not the converse $\mathcal{F}_n\subset\mathcal{G}_{n+2}$ holds in general, but it is known that $\mathcal{F}_n=\mathcal{G}_{n+2}$ when $n=0,0.5$ [15, Theorem 8.13 and Remark 8.14].

2.2 Amenable signatures

In this subsection, we briefly review the von Neumann–Cheeger–Gromov $\rho^{(2)}$–invariants [9], and introduce the amenable signature theorem, Theorem 2.5. For a closed 3–manifold M, let Γ be a countable group and $\psi\colon\pi_1M\to\Gamma$ a group homomorphism. Then Chang and Weinberger [8] showed that there exists a group G containing Γ and a 4–manifold W with $\partial W=M$ such that $\phi\colon\pi_1M\xrightarrow{\psi}\Gamma\hookrightarrow G$ extends to $\pi_1W\to G$. Then, for $\mathcal{N}G$, which is the group von Neumann algebra of G, using ϕ we obtain a homomorphism $\mathbb{Z}[\pi_1W]\to\mathcal{N}G$ and the corresponding equivariant hermitian intersection form

$$\lambda\colon H_2(W;\mathcal{N}G)\times H_2(W;\mathcal{N}G)\to\mathcal{N}G$$

and the L^2–signature $\operatorname{sign}^{(2)}_G(W)\in\mathbb{R}$. Let $\operatorname{sign}(W)$ be the ordinary signature of W and let $S_G(W):=\operatorname{sign}^{(2)}_G(W)-\operatorname{sign}(W)$, an L^2–signature defect. Then the *von Neumann–Cheeger–Gromov $\rho^{(2)}$–invariant associated to* (M,ψ) is defined to be

$$\rho^{(2)}(M,\psi):=S_G(W).$$

It is known that it is independent of the choices of a group G and a 4–manifold W. For more details on the von Neumann–Cheeger–Gromov $\rho^{(2)}$–invariants, refer to [17; 2].

Using L^2–theoretic techniques and the amenable signature theorem on homology bordism developed in [7], Cha obtained the following obstruction for a knot to be $n.5$–solvable, which generalizes the obstruction from the $\rho^{(2)}$–invariant associated with a PTFA representation in [15]:

Theorem 2.5 (amenable signature theorem for $n.5$–solvability [2, Theorem 1.3]) *Let K be an $n.5$–solvable knot. Let G be an amenable group lying in Strebel's class $D(R)$ for $R=\mathbb{Q}$ or $\mathbb{Z}_p$ such that $G^{(n+1)}=\{e\}$. Let $\phi\colon \pi_1 M(K)\to G$ be a homomorphism which sends the meridian of K to an infinite-order element in G. Suppose ϕ extends to an $n.5$–solution for $M(K)$. Then $\rho^{(2)}(M(K),\phi)=0$.*

The only amenable groups in Strebel's class $D(R)$ which we will use in this paper are the groups given in Lemma 2.7 below, and we will not need the definitions of amenable group and Strebel's class $D(R)$. (One may find the definitions in [7].)

2.3 Mixed-coefficient commutator series

In this subsection, we give examples of groups which are amenable and in $D(R)$. Namely, for a group G, we will construct a certain subnormal series $\{\mathcal{P}^k G\}$ of G such that $G/\mathcal{P}^k G$ are amenable and in $D(R)$.

Definition 2.6 [2, Definition 4.1] Let G be a group and $\mathcal{P}=(R_0,R_1,\dots)$ be a sequence of rings with unity. The *$\mathcal{P}$–mixed-coefficient commutator series* $\{\mathcal{P}^k G\}$ of G is defined inductively as follows: Let $\mathcal{P}^0 G := G$. For a nonnegative integer k, we define

$$\mathcal{P}^{k+1}G := \operatorname{Ker}\left\{\mathcal{P}^k G \to \frac{\mathcal{P}^k G}{[\mathcal{P}^k G,\mathcal{P}^k G]} \to \frac{\mathcal{P}^k G}{[\mathcal{P}^k G,\mathcal{P}^k G]}\otimes_{\mathbb{Z}} R_k\right\}.$$

Note that $(\mathcal{P}^k G/[\mathcal{P}^k G,\mathcal{P}^k G])\otimes_{\mathbb{Z}} R_k \cong H_1(G;R_k[G/\mathcal{P}^k G])$. Also, $G^{(k)}\subset \mathcal{P}^k G$ for all $\mathcal{P}$ and k, and the group $\mathcal{P}^k G/\mathcal{P}^{k+1}G$ injects into $H_1(G;R_k[G/\mathcal{P}^k G])$.

Using $\mathcal{P}$–mixed-coefficient commutator series, we obtain groups which are amenable and in $D(R)$ as below.

Lemma 2.7 [2, Lemma 4.3] *Let G be a group and n be a nonnegative integer. Let $\mathcal{P}=(R_0,R_1,\dots)$ be a sequence of rings with unity such that for each $k<n$, every integer relatively prime to p is invertible in R_k. Then, for each $k\le n$, the group $G/\mathcal{P}^k G$ is amenable and lies in $D(\mathbb{Z}_p)$.*

Later, in the proof of Theorem 1.1, we will use a $\mathcal{P}$–mixed-coefficient commutator series where $\mathcal{P} = (R_0, R_1, \dots, R_n)$ with $R_i = \mathbb{Q}$ for $0 \le i \le n-1$ and $R_n = \mathbb{Z}_p$ for some prime p and a representation to the group $G/\mathcal{P}^{n+1}G$ for some group G. In this case, $G/\mathcal{P}^{n+1}G$ has a subgroup $\mathcal{P}^n G/\mathcal{P}^{n+1}G$ which injects into a $\mathbb{Z}_p$–vector space $H_1(G; \mathbb{Z}_p[G/\mathcal{P}^n G])$. (Therefore $G/\mathcal{P}^{n+1}G$ has p–torsion elements and it is not a PTFA group.)

3 Construction of knots bounding a grope of height $n+2$

In this section, we discuss how to construct a knot bounding a grope of height $n+2$ in D^4. We will construct such a knot using a process called *infection* or *satellite construction*. Let K be a knot in S^3 and n a positive integer. Let $\eta_1, \dots, \eta_m$ be disjoint simple closed curves in $E(K)$ such that $\eta_i \in \pi_1 E(K)^{(n)}$ for all i. Suppose the η_i form an unlink in S^3. Let $J_1, \dots, J_m$ be knots. For each $1 \le i \le m$, remove the open tubular neighborhood of η_i in S^3 and glue in the exterior of J_i by identifying their common boundaries using an orientation-reversing homeomorphism in such a way that a meridian (resp. a 0–linking longitude) of η_i is identified with a 0–linking longitude (resp. a meridian) of J_i. The resulting 3–manifold is homeomorphic to S^3, and now the knot K becomes a new knot in this S^3. We denote this knot by $K(\eta_i; J_i)$ or $K(\eta_1, \dots, \eta_m; J_1, \dots, J_m)$ and say that it is obtained by *infecting K by J_i along η_i*. In the case that $J_i = J$ for some knot J for all i, we simply write $K(\eta_1, \dots, \eta_m; J)$ or $K(\eta_i; J)$. We call K, η_i and J_i the *seed knot*, the *axes* and the *auxiliary knots* (or *infection knots*), respectively. For more details, we refer the reader to [16].

Roughly speaking, a grope of height $n+2$ bounded by $K(\eta_i; J_i)$ is constructed by "stacking" gropes of height 2 bounded by J_i and gropes of height n bounded by η_i. Construction of a grope bounding the knot resulting from infection was investigated in [17] and later in [27]. Afterwards, a more systematic way of construction was given in [3; 6]. For instance, the following theorem is implicitly proved in [6, Definition 4.4]. In the following, a *capped grope* is a grope with disks attached along symplectic basis curves on the top stage surfaces of the grope, where the disks are called *caps*.

Theorem 3.1 [6] *Let K be a knot and let η_i for $1 \le i \le m$ be curves in $S^3 \smallsetminus K$ which form an unlink in S^3. Suppose that η_i bound disjoint capped gropes of height n in S^3 which do not meet K except for the caps. For each i with $1 \le i \le m$, suppose that J_i is a knot which bounds a grope of height 2 in D^4. Then the knot $K(\eta_i; J_i)$ bounds a grope of height $n+2$ in D^4.*

The following lemma shows how to find axes η_i bounding gropes of height n:

Lemma 3.2 [17, Lemma 3.9] *Let K be a knot and let η_i for $1 \le i \le m$ be curves in $S^3 \smallsetminus K$ which form an unlink in S^3. Suppose $\eta_i \in \pi_1(S^3 \smallsetminus K)^{(n)}$. Then there exist capped gropes G_i of height n disjointly embedded in S^3 such that G_i do not meet K except for the caps and for each i the grope G_i is bounded by a knot in the homotopy class of η_i.*

Note that in the above lemma, the base circles of G_i form an unlink in S^3 since G_i are disjointly embedded capped gropes. Theorem 3.1 and Lemma 3.2 yield the following corollary immediately:

Corollary 3.3 *Let K be a knot and let η_i for $1 \le i \le m$ be curves in $S^3 \smallsetminus K$ which form an unlink in S^3. Suppose $\eta_i \in \pi_1(S^3 \smallsetminus K)^{(n)}$. For each i with $1 \le i \le m$, suppose J_i is a knot bounding a grope of height 2 in D^4. Then we can homotope η_i so that the knot $K(\eta_i; J_i)$ bounds a grope of height $n+2$ in D^4.*

We discuss further how to choose η_i and J_i. Namely, to prove Theorem 1.1 we will need to construct infinitely many knots bounding a grope of height $n+2$ in D^4 which are linearly independent modulo $\mathcal{F}_{n.5}$. For that purpose, we need to make specific choices for η_i and J_i.

The following theorem is a generalization of [14, Theorem 5.13], and in Section 4 it will be used for the choice of axes η_i in the construction of the generating knots of the subgroups in Theorem 1.1.

Theorem 3.4 *Let $n \ge 1$ be an integer. Let K be a knot with nontrivial Alexander polynomial $\Delta_K(t)$. Suppose $\deg \Delta_K(t) > 2$ if $n > 1$. Let Σ be a Seifert surface for K. Then there exists an unlink $\{\eta_1, \dots, \eta_m\}$ in S^3 which does not meet Σ and satisfies the following:*

(1) *For all i, $\eta_i \in \pi_1 M(K)^{(n)}$ and the η_i bound capped gropes of height n which are disjointly embedded in $S^3 \smallsetminus K$. Here, the caps are allowed to intersect K.*

(2) *Let $\mathcal{P} = (R_0, R_1, \dots, R_n)$, where $R_i = \mathbb{Q}$ for $0 \le i \le n-1$ and $R_n = \mathbb{Z}_p$, where p is a prime greater than the top coefficient of $\Delta_K(t)$. Then, for each n–cylinder W with $M(K)$ as one of its boundary components, there exists some η_i such that $j_*(\eta_i) \notin \mathcal{P}^{n+1}\pi_1 W$, where $j_*\colon \pi_1 M(K) \to \pi_1 W$ is the inclusion-induced homomorphism.*

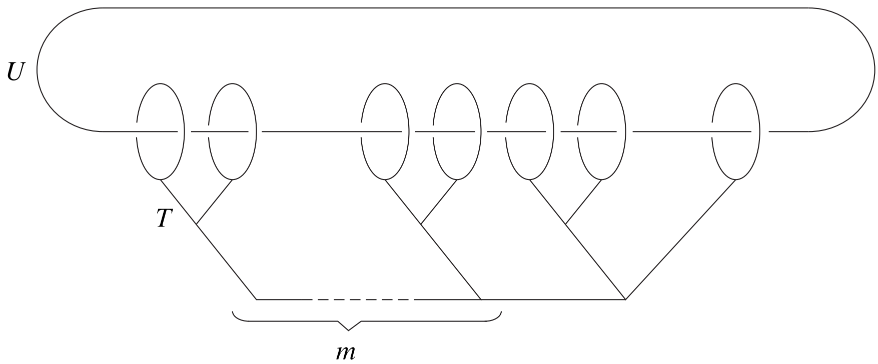

Figure 1: A clasper surgery description of the knot P_m.

We note that the $\{\eta_1, \dots, \eta_m\}$ is independent from the choice of $\mathcal{P}$ and W. The above theorem is the most technical part of the paper, and its proof is postponed to the end of Section 5.2, where we give a proof using the notion of algebraic n–solutions. Theorem 3.4 is significant since it shows that there is a *finite* set of η_i in $S^3 \smallsetminus K$ (in fact, in $S^3 \smallsetminus \Sigma$) which satisfies the nontriviality property in Theorem 3.4(2) for *every n–cylinder* for $M(K)$. (If one needs to find a finite set of η_i satisfying the nontriviality property for *a specific choice of an n–cylinder*, it can be more easily done using Theorem 5.2.) We also note that the η_i are explicitly constructed by taking conjugates and commutators of the curves dual to the bands of Σ; see the proofs of Theorems 3.4 and 5.11.

We will need the following lemma for the choice of infection knots J_i when we show linear independence of knots $K(\eta_1, \dots, \eta_m; J_i)$ for $i \geq 1$ in the proof of Theorem 1.1. In the following, σ_{J_i} denotes the Levine–Tristram signature function for J_i.

Lemma 3.5 [28, Proposition 3.4] *For any positive constant C, there exists a sequence of knots $J_1, J_2, \dots$ and an increasing sequence of odd primes $p_1, p_2, \dots$ which satisfy the following; let $\omega_i := e^{2\pi\sqrt{-1}/p_i}$.*

(1) *Each J_i bounds a grope of height 2 in D^4.*
(2) $\sum_{r=0}^{p_i-1} \sigma_{J_i}(\omega_i^r) > p_i \cdot C$.
(3) $\sum_{r=0}^{p_j-1} \sigma_{J_i}(\omega_j^r) = 0$ *for $j < i$.*

In [28, Proposition 3.4], the J_i were constructed to satisfy more conditions, such as $\int_{S^1} \sigma_{J_i}(\omega)\, d\omega = 0$, to obtain the knots $K(\eta_1, \dots, \eta_m; J_i)$ for which the $\rho^{(2)}$–invariants associated to a PTFA representation vanish. We do not need this property in this paper, and for our purpose we can use simpler J_i. That is, the proof of [28, Proposition 3.4] tells us that, in Lemma 3.5, for each i we can take J_i to be a connected sum of N copies of $P_{m_{i+1}} \# (-P_{m_i})$, where N is an integer bigger than $\frac{1}{2}C$, P_m is a knot whose clasper surgery description is as given in Figure 1, and $m_1, m_2, \dots$ is a certain increasing sequence of positive integers. In Figure 1, the knot P_m is obtained from the unknot U by performing clasper surgery along the tree T. The knot P_m for $m = 1$ was given in [17, Figure 3.7], and the knots P_m for $m > 1$ were given in [27]. The surgery descriptions of P_m are given in [17, Figure 3.6; 27, Figure 3].

4 Infinite-rank subgroups of filtrations

In this section, we prove Theorem 1.1, which will be a direct consequence of Theorems 4.2 and 4.3. First, we construct the generating knots of the desired subgroups in Theorem 1.1. Let $n > 1$. Let K be a slice knot with $\deg \Delta_K(t) > 2$. Let $\{\eta_1, \dots, \eta_m\}$ be an unlink in S^3 lying in $E(K)$ given by Theorem 3.4. By Corollary 3.3, we can homotope η_i so that $K(\eta_i; J)$ bounds a grope of height $n+2$ in D^4 for every knot J which bounds a grope of height 2 in D^4. By the work in [9; 36], there exists a constant C such that $|\rho^{(2)}(M(K), \phi)| < C$ for all homomorphisms ϕ: $\pi_1 M(K) \to G$ for every group G. (Moreover, in [4] it was shown that we can take $C = 69713280 \cdot$ (crossing number of K).) Using Lemma 3.5 with this constant C, choose a sequence of knots $J_1, J_2, \dots$ such that the prime p_1 is greater than the top coefficient of $\Delta_K(t)$. Finally, for each $i \ge 1$ define $K_i := K(\eta_1, \dots, \eta_m; J_i)$. (Therefore, the choice of $\eta_1, \dots, \eta_m$ is independent of K_i.)

4.1 Linear independence of examples

In this subsection, in Theorem 4.2 we show that the knots K_i defined as above satisfy Theorem 1.1(1)–(2). To prove Theorem 4.2, we will need the following lemma. Recall that for a 4–manifold W with a homomorphism ϕ: $\pi_1 W \to G$, we let $S_G(W) := \operatorname{sign}^{(2)}(W) - \operatorname{sign}(W)$.

Lemma 4.1 *Let K be a slice knot. Let $\{\eta_1, \eta_2, \dots, \eta_m\}$ be an unlink in $E(K)$ such that $\eta_i \in \pi_1 E(K)^{(n)}$ for all $1 \le i \le m$. Let $M := M(K(\eta_i; J_i))$. Suppose J_i is a knot with vanishing Arf invariant for $1 \le i \le m$. Then, letting $\mathbb{Z}_\infty := \mathbb{Z}$, we have the following:*

(1) *There exists an n–solution W for M satisfying the following: suppose that ϕ: $\pi_1 W \to G$ is a homomorphism, where G is an amenable group lying in Strebel's class $D(R)$ for some ring R. Then $S_G(W) = \sum_{i=1}^m \rho^{(2)}(M(J_i), \phi_i)$,*

where for each i the map $\phi_i\colon \pi_1 M(J_i) \to \mathbb{Z}_{d_i}$ is a surjective homomorphism sending the meridian of J_i to $1 \in \mathbb{Z}_{d_i}$ with d_i the order of $\phi(\eta_i)$ in G.

(2) *There exists an n–cylinder V with $\partial V = M(K) \amalg (-M)$ satisfying the following: suppose that $\phi\colon \pi_1 V \to G$ is a homomorphism, where G is an amenable group lying in Strebel's class $D(R)$ for some ring R. Then $S_G(V) = -\sum_{i=1}^m \rho^{(2)}(M(J_i), \phi_i)$, where for each i the map $\phi_i\colon \pi_1 M(J_i) \to \mathbb{Z}_{d_i}$ is a surjective homomorphism sending the meridian of J_i to $1 \in \mathbb{Z}_{d_i}$ with d_i the order of $\phi(\eta_i)$ in G.*

Proof Part (1) is essentially due to [2, Proposition 4.4] and its proof, noticing that

$$\rho^{(2)}(M,\phi) = \rho^{(2)}(M(K),\phi) + \sum_{i=1}^m \rho^{(2)}(M(J_i),\phi_i),$$

where, by abuse of notation, ϕ also denotes the restriction of the map $\phi\colon \pi_1 W \to G$ to the corresponding subspace.

We prove part (2). Since each J_i has vanishing Arf invariant, it is 0–solvable. Let W_i be a 0–solution for J_i. By doing surgery along $\pi_1 W_i^{(1)}$ if necessary, we may assume that $\pi_1 W_i \cong \mathbb{Z}$, generated by the meridian of J_i.

Let $V = M(K) \times [0,1] \cup \big(\coprod_{i=1}^m -W_i\big)$, where each $-W_i$ is attached to $M(K) \times [0,1]$ by identifying the solid torus $M(J_i) \smallsetminus E(J_i) \subset \partial W_i$ with the tubular neighborhood of $\eta_i \times 0 \subset M(K) \times 0$ in such a way that a 0–linking longitude of η_i is identified with a meridian of J_i and a meridian of η_i is identified with a 0–linking longitude of J_i. Then $\partial V = M(K) \amalg (-M)$ and an inclusion from a boundary component $M(K)$ or M to V induces an isomorphism on first homology. Using Mayer–Vietoris sequences, one can also show that 0–Lagrangians and 0–duals of W_i give rise to an n–Lagrangian and its n–dual of V. This shows that V is an n–cylinder.

As in [11, Lemma 2.3], one can obtain that

$$\rho^{(2)}(M(K),\phi_K) - \rho^{(2)}(M,\phi_M) = -\sum_{i=1}^m \rho^{(2)}(M(J_i),\phi_i),$$

where ϕ_K, ϕ_M and ϕ_i are the restrictions of ϕ to the corresponding subspaces. By the definition of $\rho^{(2)}$–invariants, we have $\rho^{(2)}(M(K),\phi_K) - \rho^{(2)}(M,\phi_M) = S_G(V)$. Note that since ϕ_i factors through $\pi_1 W_i \cong \mathbb{Z}$, the image of ϕ_i is the abelian subgroup in G generated by the image of the meridian of J_i. Since the meridian of J_i is identified with the 0–linking longitude of η_i in $M(K)$, by the subgroup property of

$\rho^{(2)}$–invariants (see [16, page 108]) we may assume that the map $\phi_i\colon \pi_1 M(J_i) \to \mathbb{Z}_{d_i}$ is a surjective homomorphism sending the meridian of J_i to $1 \in \mathbb{Z}_{d_i}$ with d_i the order of $\phi(\eta_i)$ in G. □

Theorem 4.2 *Let $n \geq 2$ be an integer and let K be a slice knot with* $\deg \Delta_K(t) > 2$. *Let K_i be the knots obtained from K as in the beginning of Section 4. Then $K_i \in \mathcal{G}_{n+2}$ for all i, and K_i are linearly independent modulo $\mathcal{F}_{n.5}$. Moreover, letting $G := \pi_1(S^3 \smallsetminus K)$ and $G_i := \pi_1(S^3 \smallsetminus K_i)$, we can construct K_i such that for each i there is an isomorphism $G_i/G_i^{(n+1)} \to G/G^{(n+1)}$ preserving peripheral structures.*

Proof Since there is a degree 1 map from $E(J_i)$ to $E(\text{unknot})$, for each i, there is a degree 1 map $f_i\colon E(K_i) \to E(K)$ relative to the boundary such that f_i is the identity outside the copies of $E(J_i)$. Since $\eta_\ell \in \pi_1 M(K)^{(n)}$ for all $\ell = 1, \dots, m$, by [10, Theorem 8.1] the f_i induces an isomorphism $G_i/G_i^{(n+1)} \to G/G^{(n+1)}$ which preserves peripheral structures. This proves the last part of Theorem 4.2.

The K_i bound a grope of height $n+2$ in D^4 by Corollary 3.3 and Lemma 3.5, hence $K_i \in \mathcal{G}_{n+2}$. Let $J := \#_i\, a_i K_i$, where $a_i \in \mathbb{Z}$, a nontrivial connected sum of finitely many copies of K_i and their inverses $-K_i$. To prove the theorem it suffices to show that J is not $n.5$–solvable.

Suppose J is $n.5$–solvable. We may assume $a_1 \neq 0$ while preserving the properties given by Lemma 3.5, and, furthermore, by taking the inverse of K_1 if necessary, we may assume $a_1 > 0$. Note that the J_i have vanishing Arf invariant since they bound a grope of height 2 in D^4 and hence are 0–solvable. We construct building blocks for a certain 4–manifold as follows:

(1) Let V be an $n.5$–solution for $M(J)$.

(2) Let E be the standard cobordism between $M(J)$ and $\coprod_i a_i M(K_i)$ as constructed (with the name C) in [16, page 113]. We may assume that $\partial E = \big(\coprod_i a_i M(K_i)\big) \amalg (-M(J))$.

(3) Let V_1 be an n–cylinder with $\partial V = M(K) \amalg (-M(K_1))$ as given in Lemma 4.1(2).

(4) Let W_i be an n–solution for $M(K_i)$ as given in Lemma 4.1(1).

Let $b_1 := a_1 - 1$ and $b_i = |a_i|$ for $i \geq 2$. For $1 \leq s \leq b_i$, let W_i^s be a copy of $-\epsilon_i W_i$, where $\epsilon_i = a_i/|a_i|$. Now we define

$$W = V \bigcup_{\partial_- E} E \bigcup_{\partial_+ E} \left(V_1 \amalg \left(\coprod_i \coprod_{s=1}^{b_i} W_i^s \right) \right),$$

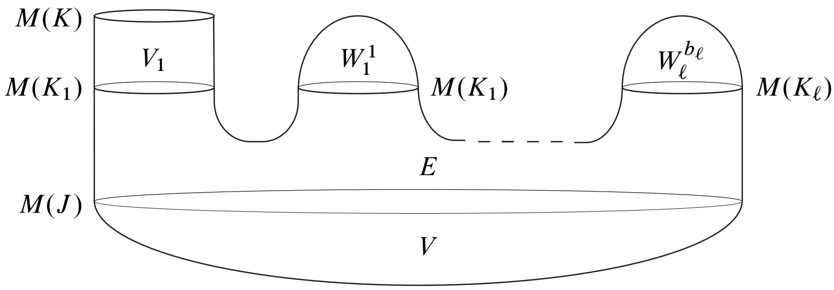

Figure 2: Cobordism W.

where $\partial_+ E = \coprod_i a_i M(K_i)$ and $\partial_- E = M(J)$. See Figure 2. Note that $\partial W = M(K)$. Using Mayer–Vietoris sequences, one can easily show that the n–Lagrangians and n–duals of V_1, W_i^s and V form an n–Lagrangian and its n–dual for W, and therefore W is an n–solution for $M(K)$.

Let $\mathcal{P} := (R_0, R_1, \ldots, R_n)$, where $R_i = \mathbb{Q}$ for $i \le n-1$ and $R_n = \mathbb{Z}_{p_1}$. Using Definition 2.6, we obtain $\mathcal{P}^j \pi_1 W$ for $1 \le j \le n+1$, which are subgroups of $\pi_1 W$. Let $G := \pi_1 W/\mathcal{P}^{n+1}\pi_1 W$, which is amenable and lies in $D(\mathbb{Z}_{p_1})$ by Lemma 2.7, and let $\phi\colon \pi_1 W \to G$ be the quotient map. For convenience, we denote a restriction of ϕ by ϕ as well.

Since $\partial W = M(K)$, we have $S_G(W) = \rho^{(2)}(M(K), \phi)$. On the other hand, by Novikov additivity, we have

$$S_G(W) = S_G(V) + S_G(E) + S_G(V_1) + \sum_i \sum_{s=1}^{b_i} S_G(W_i^s).$$

Here, $S_G(V) = 0$ by the amenable signature theorem, Theorem 2.5.

Also, $S_G(E) = 0$ since $\operatorname{Coker}\{H_2(\partial_- E) \to H_2(E)\} = 0$ (see [16, Lemma 4.2; 11, Lemma 2.4]).

By Lemma 4.1(2),

$$S_G(V_1) = -\sum_{i=1}^{m} \rho^{(2)}(M(J_1), \phi_i),$$

where $\phi_i\colon \pi_1 M(J_1) \to \mathbb{Z}_{d_i}$ is a surjective homomorphism such that d_i is the order of $\phi(\eta_i)$ in G and ϕ_i sends the meridian of J_i, which is the 0–linking longitude of η_i, to $1 \in \mathbb{Z}_{d_i}$. Since $\eta_i \in \pi_1 M(K)^{(n)}$, one can see that $\phi(\eta_i)$ lies in $\mathcal{P}^n\pi_1 W/\mathcal{P}^{n+1}\pi_1 W$. Since $\mathcal{P}^n\pi_1 W/\mathcal{P}^{n+1}\pi_1 W$ injects into $H_1(\pi_1 W; \mathbb{Z}_{p_1}[\pi_1 W/\mathcal{P}^n\pi_1 W])$, which is a $\mathbb{Z}_{p_1}$–vector space, we have $d_i = 1$ or p_1.

Then, by [16, Proposition 5.1] (or by [23, Corollary 4.3]), $\rho^{(2)}(M(J_1),\phi_i)=0$ if $d_i=1$, and $\rho^{(2)}(M(J_1),\phi_i)=(1/p_1)\sum_{r=0}^{p_1-1}\sigma_{J_1}(e^{2\pi r\sqrt{-1}/p_1})$ if $d_i=p_1$, where σ_{J_1} denotes the Levine–Tristram signature function for J_1. Furthermore, since the η_ℓ for $1\le\ell\le m$ were chosen using Theorem 3.4, there exists some η_i such that $\phi(\eta_i)\neq e$ in G. Therefore $d_i=p_1$ for some i and we have

$$S_G(V_1)\le -\frac{1}{p_1}\sum_{r=0}^{p_1-1}\sigma_{J_1}(e^{2\pi r\sqrt{-1}/p_1}).$$

We compute $S_G(W_i^r)$. By Lemma 4.1(1), we have

$$S_G(W_i)=\sum_{j=1}^{m}\rho^{(2)}(M(J_i),\phi_j).$$

When $i=1$, $W_1^s=-W_1$ and hence

$$S_G(W_1^s)=-\sum_{r=0}^{p_1-1}\sigma_{J_1}(e^{2\pi r\sqrt{-1}/p_1})\ \text{ or } 0,$$

computed as above. When $i\ge 2$, similarly to the case $i=1$, we have

$$S_G(W_i^s)=\pm\frac{1}{p_1}\sum_{r=0}^{p_1-1}\sigma_{J_i}(e^{2\pi r\sqrt{-1}/p_1})\ \text{ or } 0.$$

But, by Lemma 3.5(3), we have $S_G(W_i^s)=0$ for $i\ge 2$.

Summing up the above computations, we obtain that

$$S_G(W)\le -\frac{1}{p_1}\sum_{r=0}^{p_1-1}\sigma_{J_1}(e^{2\pi r\sqrt{-1}/p_1})$$

and therefore we have $|S_G(W)|>C$ by our choice of J_1 and Lemma 3.5(2). But, from $S_G(W)=\rho^{(2)}(M(K),\phi)$ and our choice of C, we have $|S_G(W)|<C$, which is a contradiction. □

4.2 Distinction from the knots constructed via iterated doubling operators

The purpose of this subsection is to prove Theorem 4.3 below. We note that in [27] it was shown that a nontrivial combination of the knots generating the infinite-rank subgroup of $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ in [27] is nontrivial modulo $\mathcal{F}_{n.5}$. That is, the infinite-rank subgroup of $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ in [27] injects into $\mathcal{F}_n/\mathcal{F}_{n.5}$ under the quotient map $\mathcal{C}/\mathcal{G}_{n+2.5} \to \mathcal{C}/\mathcal{F}_{n.5}$.

Theorem 4.3 *Let $n \geq 2$ be an integer and let K_i be the knots in Theorem 4.2. In $\mathcal{F}_n/\mathcal{F}_{n.5}$, the infinite-rank subgroup generated by the K_i trivially intersects the infinite-rank subgroups of $\mathcal{F}_n/\mathcal{F}_{n.5}$ in* [11; 27; 13; 2]*. In particular, in $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$, the infinite-rank subgroup generated by K_i trivially intersects the infinite-rank subgroup of $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ in* [27].

We give a proof of Theorem 4.3 at the end of this subsection. The proof of the above theorem is based on ideas in [13, Section 9]. Using the terms in [13, Section 9], our knots K_i can be considered as *generalized COT knots*, and on the other hand the knots in [11; 27; 13; 2] are called *CHL knots*. Therefore, to prove Theorem 4.3, we need to extend the results in [13, Section 9], which are about distinguishing COT knots from CHL knots, to the case of generalized COT knots. We will do this by showing the following:

(1) For each $n > 0$ and a prime p, we define a subset $\mathcal{F}_n^{\mathrm{cot},p}$ of the set of (isotopy classes of) knots in S^3.
(2) We show that if K is a nontrivial linear combination of the K_i in Theorem 4.2, then we have $K \notin \mathcal{F}_{n.5}^{\mathrm{cot},p}$ for some prime p.
(3) We show that if a knot K is concordant to a nontrivial linear combination of CHL knots, then $K \in \mathcal{F}_{n.5}^{\mathrm{cot},p}$ for all prime p.

First, for each $n > 0$, we define a subset $\mathcal{F}_n^{\mathrm{cot},p}$, which generalizes the subset $\mathcal{F}_n^{\mathrm{cot}}$ defined in [13]. Let G be a group such that $H_1(G) \cong \mathbb{Z}$. For each $k \geq 0$, let $G_r^{(k)}$ be the k^{th} rational derived subgroup of G. That is, $G_r^{(0)} := G$ and

$$G_r^{(k+1)} := \operatorname{Ker}\left\{ G_r^{(k)} \to \frac{G_r^{(k)}}{[G_r^{(k)}, G_r^{(k)}]} \to \frac{G_r^{(k)}}{[G_r^{(k)}, G_r^{(k)}]} \otimes_{\mathbb{Z}} \mathbb{Q} \right\}.$$

Now fix an integer $n \geq 0$ and a prime p. Since $G^{(1)}/G_r^{(n)}$ is a PTFA group [26, Proposition 2.1], $\mathbb{Z}_p[G^{(1)}/G_r^{(n)}]$ embeds into its skew quotient field, say $\mathbb{K}$ (see Lemma 5.1). Since $H_1(G) = G/G^{(1)} \cong \mathbb{Z}$, the group ring $\mathbb{Z}_p[G/G_r^{(n)}]$ can be embedded into the noncommutative Laurent polynomial ring $\mathbb{K}[t^{\pm 1}]$, which is a noncommutative PID. Now we define

$$G_{\mathrm{cot},p}^{(n+1)} := \operatorname{Ker}\left\{ G_r^{(n)} \to \frac{G_r^{(n)}}{[G_r^{(n)}, G_r^{(n)}]} \to \frac{G_r^{(n)}}{[G_r^{(n)}, G_r^{(n)}]} \otimes_{\mathbb{Z}[G/G_r^{(n)}]} \mathbb{K}[t^{\pm 1}] \right\}.$$

Note that $G^{(n+1)} \subset G^{(n+1)}_{\mathrm{cot},p}$ for all prime p and

$$\frac{G_r^{(n)}}{[G_r^{(n)}, G_r^{(n)}]} \otimes_{\mathbb{Z}[G/G_r^{(n)}]} \mathbb{K}[t^{\pm 1}] \cong H_1(G; \mathbb{K}[t^{\pm 1}])$$

since $\mathbb{K}[t^{\pm 1}]$ is a flat left module over $\mathbb{Z}[G/G_r^{(n)}]$ by [37, Proposition II.3.5].

For a 4–manifold W with $H_1(W) \cong \mathbb{Z}$, let $\pi := \pi_1 W$. For each $n > 0$ and a prime p, there is the equivariant intersection form

$$\lambda_n^p \colon H_2(W; \mathbb{Z}[\pi/\pi^{(n)}_{\mathrm{cot},p}]) \times H_2(W; \mathbb{Z}[\pi/\pi^{(n)}_{\mathrm{cot},p}]) \to \mathbb{Z}[\pi/\pi^{(n)}_{\mathrm{cot},p}].$$

Definition 4.4 Let n be a positive integer and p a prime. A knot K is *(n,p)–solvable via W* if there exists a compact connected 4–manifold W with boundary $M(K)$ satisfying the following; let $\pi := \pi_1 W$ and $r := \frac{1}{2} \operatorname{rank}_{\mathbb{Z}} H_2(W)$.

(1) The inclusion $M \to W$ induces an isomorphism $H_1(M) \to H_1(W)$.

(2) There exist $x_1, x_2, \dots, x_r$ and $y_1, y_2, \dots, y_r$ in $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}_{\mathrm{cot},p}])$ such that $\lambda_n^p(x_i, x_j) = 0$ and $\lambda_n^p(x_i, y_j) = \delta_{ij}$ for $1 \le i, j \le r$.

We say K is *$(n.5,p)$–solvable via W* if the following additional condition is satisfied:

(3) There exist $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_r$ in $H_2(W; \mathbb{Z}[\pi/\pi^{(n+1)}_{\mathrm{cot},p}])$ such that $\lambda_{n+1}^p(\widetilde{x}_i, \widetilde{x}_j) = 0$ for $1 \le i, j \le r$ and $\widetilde{x}_i$ and x_i are represented by the same surface for each i.

We denote the set of (n,p)–solvable knots and $(n.5,p)$–solvable knots by $\mathcal{F}_n^{\mathrm{cot},p}$ and $\mathcal{F}_{n.5}^{\mathrm{cot},p}$, respectively. The submodules generated by x_i and y_i are called an *(n,p)–Lagrangian* and an *(n,p)–dual*, respectively. The submodule generated by $\widetilde{x}_i$ is called an *$(n+1,p)$–Lagrangian.*

Since $\pi^{(n)} \subset \pi^{(n)}_{\mathrm{cot},p}$, if a knot K is n–solvable, then it is (n,p)–solvable for all prime p. We note that the notion of $G^{(n)}_{\mathrm{cot},p}$ is not functorial. That is, for a group homomorphism $\phi \colon G \to H$, in general, $\phi(G^{(n)}_{\mathrm{cot},p}) \not\subseteq H^{(n)}_{\mathrm{cot},p}$. Due to this fact, the subset $\mathcal{F}_n^{\mathrm{cot},p}$ does not descend to a subgroup of $\mathcal{C}$.

Theorem 4.5 below generalizes [13, Theorem 5.2; 2, Theorem 3.2]. Theorem 4.5 can be proved using the arguments in the proof of [2, Theorem 3.2], and therefore we give only a sketch of the proof.

Theorem 4.5 *Let p be a prime and n a positive integer. Let K be a knot which is $(n.5, p)$–solvable via W. Let G be an amenable group lying in Strebel's class $D(\mathbb{Z}_p)$. Let $\pi := \pi_1 W$ and suppose we are given a homomorphism*

$$\phi\colon \pi_1 M(K) \to \pi \to \pi/\pi^{(n+1)}_{\mathrm{cot},p} \to G$$

which sends the meridian of K to an infinite-order element in G. Then

$$\rho^{(2)}(M(K), \phi) = 0.$$

Proof sketch We have $\rho^{(2)}(M(K),\phi) = S_G(W) = \mathrm{sign}^{(2)}_G(W) - \mathrm{sign}(W)$, and we will show that $S_G(W) = 0$. From Definition 4.4, we have an $(n+1, p)$–Lagrangian in $H_2(W; \mathbb{Z}[\pi/\pi^{(n+1)}_{\mathrm{cot},p}])$ generated by $\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_r$ and we have its n–dual in $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}_{\mathrm{cot},p}])$ generated by $y_1, y_2, \dots, y_r$, where $r = \frac{1}{2}\operatorname{rank}_{\mathbb{Z}} H_2(W)$. The images of $\widetilde{x}_i$ and y_i consist of a 0–Lagrangian and its 0–dual in $H_2(W;\mathbb{Q})$, and therefore $\mathrm{sign}(W) = 0$.

We show that $\mathrm{sign}^{(2)}_G(W) = 0$. Let $\mathcal{N}G$ be the group von Neumann algebra of G and

$$\lambda\colon H_2(W;\mathcal{N}G) \times H_2(W;\mathcal{N}G) \to \mathcal{N}G$$

be the corresponding intersection form, where the coefficients of the homology groups are twisted via ϕ. Since $\phi\colon \pi_1 M(K) \to G$ factors through $\pi/\pi^{(n+1)}_{\mathrm{cot},p}$, we have the induced map $H_2(W; \mathbb{Z}[\pi/\pi^{(n+1)}_{\mathrm{cot},p}]) \to H_2(W;\mathcal{N}G)$.

Let H be the submodule of $H_2(W;\mathcal{N}G)$ generated by the images of $\widetilde{x}_i$ under this map, and $\overline{H}$ be the submodule of $H_2(W;\mathbb{Z}_p)$ generated by the images of $\widetilde{x}_i$. Since the image of $\widetilde{x}_i$ in $H_2(W;\mathbb{Z}_p)$ has a dual, which is the image of y_i, we have $\dim_{\mathbb{Z}_p} \overline{H} = r$. By Lemmas 3.13 and 3.14 in [2], we have $\dim^{(2)} H_2(W;\mathcal{N}G) = \dim_{\mathbb{Z}_p} H_2(W;\mathbb{Z}_p) = 2r$, where $\dim^{(2)}$ denotes the L^2–dimension function on $\mathcal{N}G$–modules. By Theorem 3.11 of [2], we have

$$\dim^{(2)} H \geq \dim^{(2)} H_2(W;\mathcal{N}G) - \dim_{\mathbb{Z}_p} H_2(W;\mathbb{Z}_p) + \dim_{\mathbb{Z}_p} \overline{H} = r.$$

Now, by [2, Proposition 3.7], we have $\mathrm{sign}^{(2)}_G(W) = 0$. □

Now we give a proof of Theorem 4.3.

Proof of Theorem 4.3 Since $\mathcal{G}_{n+2.5} \subset \mathcal{F}_{n.5}$, the last part follows from the fact that the basis knots of the infinite-rank subgroup $\mathcal{G}_{n+2}/\mathcal{G}_{n+2.5}$ in [27] are linearly independent modulo $\mathcal{F}_{n.5}$, which is shown in the proof of [27, Theorem 5.2].

Let J be a nontrivial linear combination of the K_i and let L be a knot which is concordant to a linear combination of the n–solvable knots in [11; 27; 13; 2], which are constructed using iterated doubling operators and generate an infinite-rank subgroup of $\mathcal{F}_n/\mathcal{F}_{n.5}$. We will show $J \notin \mathcal{F}_{n.5}^{\mathrm{cot},p}$ for some prime p, but $L \in \mathcal{F}_{n.5}^{\mathrm{cot},p}$ for all prime p, and this will prove the theorem.

First, suppose $J = \#_i\, a_i K_i$ with $a_i \in \mathbb{Z}$. As in the proof of Theorem 4.2, we may assume $a_1 > 0$. We will show $J \notin \mathcal{F}_{n.5}^{\mathrm{cot},p_1}$. Suppose to the contrary that J is $(n.5, p_1)$–solvable via V. We follow the proof of Theorem 4.2 with the changes and observations below:

(1) Now V is not an $n.5$–solution; it is only an $(n.5, p_1)$–solution.

(2) Use $\pi_1 W_{\mathrm{cot},p_1}^{(n+1)}$ instead of $\mathcal{P}^{n+1}\pi_1 W$. We keep $\mathcal{P}^n\pi_1 W$ as it is, noting that $\mathcal{P}^n\pi_1 W = \pi_1 W_r^{(n)}$.

(3) $\mathcal{P}^n\pi_1 W/\pi_1 W_{\mathrm{cot},p_1}^{(n+1)}$ is a $\mathbb{Z}_{p_1}$–vector space: from the definitions, one can see that $\mathcal{P}^{n+1}\pi_1 W \subset \pi_1 W_{\mathrm{cot},p_1}^{(n+1)}$, and therefore $\mathcal{P}^n\pi_1 W/\pi_1 W_{\mathrm{cot},p_1}^{(n+1)}$ is a quotient group of $\mathcal{P}^n\pi_1 W/\mathcal{P}^{n+1}\pi_1 W$, which is a $\mathbb{Z}_{p_1}$–vector space.

(4) We change the definition of G: let $G := \pi_1 W/\pi_1 W_{\mathrm{cot},p_1}^{(n+1)}$. Then G admits a subnormal series

$$\{e\} \subset G_r^{(n)} \subset G_r^{(n-1)} \subset \cdots \subset G_r^{(1)} \subset G$$

whose successive quotients are abelian and have no torsion coprime to p_1. Therefore, the group G is amenable and lies in Strebel's class $D(\mathbb{Z}_{p_1})$ (see [7, Lemma 6.8]).

(5) Use Theorem 4.5 instead of Theorem 2.5 to show $S_G(V) = 0$.

(6) The crucial part is to show that there is some η_i such that $\phi(\eta_i) \neq e$ for the homomorphism $\phi\colon \pi_1 W \to G$. In the proof of Theorem 4.2, we use Theorem 3.4 to show this when $G = \pi_1 W/\mathcal{P}^{n+1}\pi_1 W$. Theorem 3.4 is proved in the end of Section 5.2, in which we show that there exists some η_i such that $\eta_i \notin \mathcal{P}^{n+1}\pi_1 W$. But in that proof a stronger fact is proved for η_i: this η_i maps to a nontrivial element in $H_1(W; \mathbb{K}[t^{\pm 1}])$, where $\mathbb{K}$ is the skew quotient field of $\mathbb{Z}_p[\pi_1 W^{(1)}/\pi_1 W_r^{(n)}]$ (in our case $p = p_1$). This implies that this η_i maps to a nontrivial element even in $\mathcal{P}^n\pi_1 W/\pi_1 W_{\mathrm{cot},p_1}^{(n+1)}$. Therefore, for this η_i, we have $\phi(\eta_i) \neq e$.

Then, we obtain $|S_G(W)| > C$ as in the proof of Theorem 4.2, which is a contradiction. This completes the proof that $J \notin \mathcal{F}_{n.5}^{\mathrm{cot},p_1}$.

Next, we show $L \in \mathcal{F}_{n.5}^{\mathrm{cot},p}$ for all prime p. The proof is similar to that of Proposition 9.2 in [13]. Suppose L is concordant to a knot $J := \#_i\, a_i J_i$, where $a_i \in \mathbb{Z}$ and J_i are

n–solvable knots in [11; 27; 13; 2], which are constructed using iterated doubling operators and generate an infinite-rank subgroup of $\mathcal{F}_n/\mathcal{F}_{n.5}$.

We explain iterated doubling operators in more detail. Let R be a slice knot and α be a finite set of simple closed curves $\{\eta_1, \eta_2, \dots, \eta_m\}$ in $S^3 \smallsetminus R$ such that the η_i form an unlink in S^3 and $\eta_i \in \pi_1(S^3 \smallsetminus R)^{(1)}$ for all i. For a given knot J, we write $R_\alpha(J)$ for the knot $R(\eta_1, \dots, \eta_m; J)$, the knot obtained by infecting R by J along η_ℓ for $1 \le \ell \le m$ as defined in Section 3, and we say R_α is a *doubling operator*. Then, an *iterated doubling operator* (at level n) is obtained by applying doubling operators n times: $R^n_{\alpha_n} \circ R^{n-1}_{\alpha_{n-1}} \circ \dots \circ R^1_{\alpha_1}$. In this paper, by a knot constructed using iterated doubling operators we mean a knot $(R^n_{\alpha_n} \circ R^{n-1}_{\alpha_{n-1}} \circ \dots \circ R^1_{\alpha_1})(J)$ for some knot J with vanishing Arf invariant. Here, we need J to have vanishing Arf invariant to make the resulting knot n–solvable (see [16, Proposition 3.1]). Furthermore, to simplify notation we consider only the iterated doubling operators where each α_i for R^i is a single curve. Our proof can be easily and straightforwardly adapted to the general case of iterated doubling operators with multiple curves at each level.

Now, since L is concordant to $J = \#\, a_i J_i$, there exists a homology cobordism V between $M(L)$ and $M(J)$. Let C be the standard cobordism between $M(J)$ and $\coprod a_i M(J_i)$. Then $V \cup C$, the union along the common boundary $M(J)$, has boundary $\partial(V \cup C) = \coprod(-a_i M(J_i)) \amalg M(L)$.

Let us write $J_i = (R^n_{\alpha_n} \circ R^{n-1}_{\alpha_{n-1}} \circ \dots \circ R^1_{\alpha_1})(J_i^0)$ for some knot J_i^0. Let μ_0 be the meridian of J_i^0. Let W_i be an n–solution for $M(J_i)$ such that $H_2(W_i) \cong H_2(V_i)$ for a 0–solution V_i for J_i^0. We may assume $V_i \subset W_i$ and $\pi_1 V_i \cong \mathbb{Z}$, which is generated by the meridian μ_0 of J_i^0. The existence of such W_i is well known in the literature (see the proofs of [13, Theorem 6.2; 2, Proposition 4.4]).

Let W be the union of $V \cup C$ and $\coprod a_i W_i$ along their common boundary $\coprod a_i M(J_i)$. Then $\partial W = M(L)$. We will show that W is an $(n.5, p)$–solution for L for all prime p.

Using Mayer–Vietoris sequences, one can see that $H_2(W) \cong \bigoplus H_2(W_i)^{|a_i|}$, and therefore $H_2(W) \cong \bigoplus H_2(V_i)^{|a_i|}$. Since $\alpha_j \in \pi_1(S^3 \smallsetminus R^j)^{(1)}$ for $j = 1, 2, \dots, n$, one can easily see that $\mu_0 \in \pi_1 M(J_i)^{(n)}$, and hence $\mu_0 \in \pi_1 W^{(n)}$. Since $\pi_1 V_i \cong \mathbb{Z}$ is generated by the meridian μ_0, the 0–Lagrangians and 0–duals of V_i for all i form an n–Lagrangian, say $[L_j]$, and n–dual, say $[D_j]$, of W. Since $\pi^{(n)} \subset \pi^{(n)}_{\mathrm{cot},p}$, $[L_j]$ and $[D_j]$ are an (n, p)–Lagrangian and its (n, p)–dual, respectively, for all prime p.

We will show that the $[L_j]$ form an $(n+1, p)$–Lagrangian of W for all prime p, which implies that W is an $(n.5, p)$–solution for all prime p. Since each of the L_j is a surface in V_i for some i and $V_i \subset W$ for all i, it suffices to show that $\pi_1 V_i$ maps to $\pi_1 W^{(n+1)}_{\mathrm{cot},p}$ by the inclusion-induced homomorphism. Since $\pi_1 V_i \cong \mathbb{Z}$ is generated by the meridian μ_0, we only need to show $\mu_0 \in \pi_1 W^{(n+1)}_{\mathrm{cot},p}$.

Fix a prime p and let $G := \pi_1 W$. Since $\mu_0 \in G^{(n)}$, by the definition of $G^{(n+1)}_{\mathrm{cot},p}$ we only need to show $\mu_0 \in G^{(n)}_r/[G^{(n)}_r, G^{(n)}_r]$ is $\mathbb{K}[t^{\pm 1}]$–torsion, where $\mathbb{K}$ is the skew quotient field of the group ring $\mathbb{Z}_p[G^{(1)}/G^{(n)}_r]$. From the iterated doubling operator construction of J_i, the meridian μ_0 is identified with the curve α_1. Let $J^1_i := R^1_{\alpha_1}(J^0_i)$ and let μ_1 be the meridian of J^1_i. Again since $\alpha_j \in \pi_1(S^3 \smallsetminus R^j)^{(1)}$ for all j, one can easily prove $\mu_1 \in G^{(n-1)}$. Therefore $\pi_1 M(J^1_i) \subset G^{(n-1)}$, and hence $\pi_1 M(J^1_i)^{(1)} \subset G^{(n)}_r$ and $\pi_1 M(J^1_i)^{(2)} \subset [G^{(n)}_r, G^{(n)}_r]$. Now let $\Delta(t)$ be the Alexander polynomial of J^1_i, which is the same as that of R^1. Since $\alpha_1 \in \pi_1(S^3 \smallsetminus R^1)^{(1)}$, the polynomial $\Delta(t)$ annihilates α_1 in the module $\pi_1 M(J^1_i)^{(1)}/\pi_1 M(J^1_i)^{(2)}$, and therefore $\Delta(\mu_1)$ annihilates α_1 in the module $G^{(n)}_r/[G^{(n)}_r, G^{(n)}_r]$. Since $n \geq 2$ and $\mu_1 \in G^{(n-1)}$, we have $\mu_1 \in G^{(1)}$ and $\Delta(\mu_1) \in \mathbb{Z}_p[G^{(1)}/G^{(n)}_r] \subset \mathbb{K} \subset \mathbb{K}[t^{\pm 1}]$. Furthermore, since $\Delta(\mu_1)$ augments to 1, ie $\Delta(1) = 1$, $\Delta(\mu_1) \neq 0$ in $\mathbb{K}[t^{\pm 1}]$. This implies that α_1 is $\mathbb{K}[t^{\pm 1}]$–torsion. □

5 Modulo p Blanchfield linking forms and algebraic n–solutions

This section is devoted to proving Theorem 3.4. To this end, in Section 5.1 we give Theorem 5.2, which asserts the nontriviality of some homomorphisms on first homology with twisted coefficients induced from the inclusion from $M(K)$ to an n–cylinder one of whose boundary components is $M(K)$. Then, in Section 5.2 we introduce the notion of $\mathbb{Z}_p$–coefficient algebraic n–solutions, which generalizes the notion of algebraic n–solutions in [17; 14], and relevant theorems. Finally, we give a proof of Theorem 3.4 at the end of Section 5.

To have a quick look at the proof of Theorem 3.4, one may read Theorem 5.2, Proposition 5.10, Theorem 5.11 and the paragraph following Theorem 5.11, and then skip to the proof of Theorem 3.4.

5.1 Modulo p Blanchfield linking forms

The aim of this subsection is to prove Theorem 5.2 below, which plays a key role in the proof of Theorem 3.4. Theorem 5.2 generalizes [14, Theorem 3.8] to the case of a homomorphism on first homology whose coefficient group is a localization of a group ring with $\mathbb{Z}_p$–coefficients.

First, we need the following lemma:

Lemma 5.1 (Proposition 2.5 of [15] for $R = \mathbb{Q}$ and Lemma 5.2 of [2] for $R = \mathbb{Z}_p$) *Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$. If Γ is a PTFA group, then $R\Gamma$ is an Ore domain. That is, $R\Gamma$ embeds in the (skew) quotient field $\mathcal{K} = R\Gamma(R\Gamma \smallsetminus \{0\})^{-1}$.*

Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$. Let Γ be a PTFA group such that $H_1(\Gamma) \cong \Gamma/[\Gamma,\Gamma] \cong \mathbb{Z}$. Let $\mathcal{K}$ be the (skew) quotient field of $R\Gamma$ obtained by Lemma 5.1. Since a subgroup of a PTFA group is also PTFA, the group $[\Gamma,\Gamma]$ is PTFA. By Lemma 5.1, the group ring $R[\Gamma,\Gamma]$ embeds into the (skew) quotient field, say $\mathbb{K}$. Since $H_1(\Gamma) \cong \mathbb{Z} = \langle t \rangle$, we have a (noncommutative) PID $\mathbb{K}[t^{\pm 1}]$ such that $R\Gamma \subset \mathbb{K}[t^{\pm 1}] \subset \mathcal{K}$.

Theorem 5.2 (Theorem 3.8 of [14] for $R = \mathbb{Q}$) *Let $n \geq 1$ be an integer. Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$. For a knot K, let W be an n–cylinder with $M(K)$ as one of its boundary components. Let Γ be an PTFA group such that $H_1(\Gamma) \cong \mathbb{Z} = \langle t \rangle$ and $\Gamma^{(n)} = \{e\}$. Let $\phi\colon \pi_1 W \to \Gamma$ be a homomorphism which induces an isomorphism $H_1(W) \to H_1(\Gamma)$. Let $d := \operatorname{rank}_R H_1(M_\infty; R)$, where M_∞ is the infinite cyclic cover of $M(K)$. Then we have*

$$\operatorname{rank}_{\mathbb{K}} \operatorname{Im}\{i_*\colon H_1(M(K);\mathbb{K}[t^{\pm 1}]) \to H_1(W;\mathbb{K}[t^{\pm 1}])\} \geq \begin{cases} \frac{1}{2}(d-2) & \text{if } n > 1, \\ \frac{1}{2}d & \text{if } n = 1, \end{cases}$$

where i_* is the inclusion-induced homomorphism.

We give a proof of Theorem 5.2 at the end of this subsection, after showing needed materials.

We note that the image of i_* is nontrivial if $d > 2$ when $n > 1$ and if $d > 0$ when $n = 1$. In the above setting, when $R = \mathbb{Q}$, the rank d is equal to the degree of $\Delta_K(t)$. When $R = \mathbb{Z}_p$, the rank d is still equal to the degree of $\Delta_K(t)$ if the prime p is bigger than the top coefficient of $\Delta_K(t)$.

To prove Theorem 5.2, we need to generalize the various results which were used for the proof of [14, Theorem 3.8] to the case of homology with twisted coefficients, which are obtained as a localization of a group ring with $\mathbb{Z}_p$–coefficients. A key ingredient of the proof of Theorem 5.2 is higher-order Blanchfield linking forms, which are adapted to homology with such coefficients.

We briefly review higher-order Blanchfield linking forms. The Blanchfield linking form in a noncommutative setting was first defined by Duval [19] on boundary links over a group ring of a free group. Then Cochran, Orr and Teichner introduced the noncommutative (higher-order) Blanchfield linking form for a knot over a group ring $\mathbb{Z}\Gamma$ of a PTFA group Γ and its localizations [15]. This was generalized to the Blanchfield linking form for a knot over a group ring $\mathbb{Z}_p\Gamma$ for a PTFA group Γ by Cha [2]. In this paper, we need the Blanchfield linking form for a knot over a (PID) localization of a group ring $\mathbb{Z}_p\Gamma$ for a PTFA group Γ. It will be defined in Theorem 5.3, and will be used to prove Theorem 5.2.

Let Γ be a PTFA group and $R=\mathbb{Q}$ or $\mathbb{Z}_p$. Then, by Lemma 5.1, $R\Gamma$ has the (skew) quotient field of $\mathcal{K}$. If $\mathcal{R}$ is a ring such that $R\Gamma\subset\mathcal{R}\subset\mathcal{K}$ and M is a closed 3–manifold such that $H_1(M)\cong\mathbb{Z}$ and $\pi_1 M\to\Gamma$ is a nontrivial representation, then we have the composition of maps

$$H_1(M;\mathcal{R})\to\overline{H^2(M;\mathcal{R})}\to\overline{H^1(M;\mathcal{K}/\mathcal{R})}\to\overline{\mathrm{Hom}_R(H_1(M;\mathcal{R}),\mathcal{K}/\mathcal{R})},$$

where the maps are Poincaré duality, the inverse of a Bockstein homomorphism and the Kronecker evaluation map. Here, the inverse of a Bockstein homomorphism exists since $H^1(M;\mathcal{K})=H^2(M;\mathcal{K})=0$ and hence the Bockstein homomorphism is in fact an isomorphism (see [15, Proposition 2.11]). Also, $\overline{H^*(M;\mathcal{R})}$ are made into right $\mathcal{R}$–modules using the involution of $\mathcal{R}$.

The following theorem shows that under this setting, if $\mathcal{R}$ is a PID, then the composition gives rise to a nonsingular symmetric linking form.

Theorem 5.3 (Theorem 2.13 of [15] for $R=\mathbb{Q}$ and essentially due to [2, Section 5] for $R=\mathbb{Z}_p$) *Let $R=\mathbb{Q}$ or $\mathbb{Z}_p$. Let M be a closed 3–manifold with $H_1(M)\cong\mathbb{Z}$. Let $\phi\colon\pi_1 M\to\Gamma$ be a nontrivial PTFA coefficient system. Let $\mathcal{R}$ be a* (*noncommutative*) *PID such that $R\Gamma\subset\mathcal{R}\subset\mathcal{K}$ where $\mathcal{K}$ is the* (*skew*) *quotient field of $R\Gamma$. Then there exists a nonsingular symmetric linking form* (called the *Blanchfield linking form*)

$$\mathrm{Bl}\colon H_1(M;\mathcal{R})\to\overline{H^2(M;\mathcal{R})}\to\overline{H^1(M;\mathcal{K}/\mathcal{R})}\to\overline{\mathrm{Hom}_R(H_1(M;\mathcal{R}),\mathcal{K}/\mathcal{R})}.$$

Proof When $R=\mathbb{Z}_p$, the proof is identical to that of [15, Theorem 2.13] except for the following change: in the proof, replace [15, Proposition 2.11] by [2, Lemma 5.3]. □

Lemma 5.4 below is needed to prove Proposition 5.5. When $R=\mathbb{Q}$, it was proved in [14]. The proof for the case $R=\mathbb{Z}_p$ is essentially the same as that for the case $R=\mathbb{Q}$, and hence it is omitted.

Lemma 5.4 (Lemma 3.5 of [14] for $R=\mathbb{Q}$) *Let $R=\mathbb{Q}$ or $\mathbb{Z}_p$. Let W be an R–coefficient n–cylinder with M as one of its boundary components. Let Γ be a PTFA group such that $\Gamma^{(n)}=\{e\}$, and let $\phi\colon \pi_1 M\to\Gamma$ be a nontrivial coefficient system which extends to $\pi_1 W$. Let $\mathcal{R}$ be a (noncommutative) PID such that $R\Gamma\subset\mathcal{R}\subset\mathcal{K}$, where $\mathcal{K}$ is the (skew) quotient field of $R\Gamma$. Then the sequence of maps*

$$TH_2(W,M;\mathcal{R})\to H_1(M;\mathcal{R})\to H_1(W;\mathcal{R})$$

is exact. (Here for an $\mathcal{R}$–module N, TN denotes the $\mathcal{R}$–torsion submodule of N.)

For an $\mathcal{R}$–submodule P of $H_1(M;\mathcal{R})$, we define

$$P^{\perp}:=\{x\in H_1(M;\mathcal{R})\mid \mathrm{Bl}(x)(y)=0 \text{ for all } y\in P\}.$$

We say that a submodule P of $H_1(M;\mathcal{R})$ is *self-annihilating with respect to* Bl if $P=P^{\perp}$. The following proposition generalizes [15, Theorem 4.4; 14, Proposition 3.6]:

Proposition 5.5 (Proposition 3.6 of [14] for $R=\mathbb{Q}$) *Suppose the same hypotheses as in Lemma 5.4. If $P=\mathrm{Ker}\{i_*\colon H_1(M;\mathcal{R})\to H_1(W;\mathcal{R})\}$, where i_* is the inclusion-induced homomorphism, then $P\subset P^{\perp}$. Moreover, if W is an R–coefficient n–solution for M, then $P=P^{\perp}$.*

Proof For $R=\mathbb{Z}_p$, make the following changes in the proof of [14, Proposition 3.6]: replace Corollary 3.3 and Lemma 3.5 in [14] by [2, Lemma 5.3] and Lemma 5.4 above, respectively. □

The proof of the following lemma is based on the ideas in [10, Sections 3 and 4]. In particular, Lemma 5.6(1) is a direct extension of [10, Lemma 3.9].

Lemma 5.6 (1) *Let G be a PTFA group and $R=\mathbb{Z}_p$. Let A be a wedge of m circles. Suppose we have a nontrivial coefficient system $\pi_1 A\to G$. Then* $\mathrm{rank}_{RG}\, H_1(A;RG)=m-1$.

(2) *Suppose the same hypotheses as in Theorem 5.2. Suppose $n>1$ and $R=\mathbb{Z}_p$. Then* $\mathrm{rank}_{\mathbb{K}}\, H_1(E(K);\mathbb{K}[t^{\pm1}])\geq d-1$.

Proof Note that A is a finite 1–complex with Euler characteristic $1-m$. By the invariance of the Euler characteristic, we have $\mathrm{rank}_{RG}\, H_0(A;RG)-\mathrm{rank}_{RG}\, H_1(A;RG)=1-m$. Since the homomorphism $\pi_1 A\to G$ is nontrivial, for the (skew) quotient field $\mathbb{K}$ of RG and the map $\phi\colon \pi_1 A \to G \to \mathbb{K}$ we have

$$H_0(A;\mathbb{K}) = \mathbb{K}/\{\phi(g)a - a \mid g \in \pi_1 A,\ a \in \mathbb{K}\} = 0.$$

Now we have $\operatorname{rank}_{RG} H_0(A;RG) = 0$ and $\operatorname{rank}_{RG} H_1(A;RG) = m-1$. This proves (1).

Let $\widetilde{E}$ be the infinite cyclic cover of $E(K)$ and let $G := [\Gamma,\Gamma]$ (so $\mathbb{K}$ is the quotient field of RG). Then $H_1(E(K);\mathbb{K}[t^{\pm1}]) \cong H_1(\widetilde{E};\mathbb{K})$ and $d = \operatorname{rank}_R H_1(\widetilde{E};R)$. Since $\mathbb{K}$ is the (skew) quotient field of RG, we have $\operatorname{rank}_{\mathbb{K}} H_1(\widetilde{E};\mathbb{K}) = \operatorname{rank}_{RG} H_1(\widetilde{E};RG)$, and it suffices to show that $\operatorname{rank}_{RG} H_1(\widetilde{E};RG) \geq d-1$.

Let A be a wedge of d circles and let $j\colon A \to \widetilde{E}$ be a map which induces a monomorphism on $H_1(-;R)$. For convenience we identify A with $j(A)$. Since $\widetilde{E}$ has the homotopy type of a 2–complex, we may assume $(\widetilde{E},A)$ is a 2–complex. Let $C_2 \to C_1 \to C_0$ be the free RG–chain complex obtained from the cell structure of the G–cover of $(\widetilde{E},A)$.

Since $H_2(\widetilde{E};R) = 0$ and $H_1(A;R) \to H_1(\widetilde{E};R)$ is injective, $H_2(\widetilde{E},A;R) = 0$. Therefore, $C_2 \otimes_{RG} R \to C_1 \otimes_{RG} R$ is injective. Since PTFA groups lie in Strebel's class $D(R)$ [38], the group G is in $D(R)$. Therefore, $C_2 \to C_1$ is also injective, and hence $H_2(\widetilde{E},A;RG) = 0$. Then it follows that the map $H_1(A;RG) \to H_1(\widetilde{E};RG)$ is injective. Now $\operatorname{rank}_{RG} H_1(\widetilde{E};RG) \geq \operatorname{rank}_{RG} H_1(A;RG) = d-1$, where the equality holds by (1). □

Now we are ready to give a proof of Theorem 5.2.

Proof of Theorem 5.2 For $R = \mathbb{Z}_p$, the proof is the same as that of [14, Theorem 3.8] (for the case $R = \mathbb{Q}$) except for the following changes: noticing that an n–cylinder is a $\mathbb{Z}_p$–coefficient n–cylinder by Proposition 5.7 below, replace [15, Theorem 2.13], [14, Proposition 3.6] and the coefficients $\mathbb{Q}$ by Theorem 5.3, Proposition 5.5 and the coefficients $\mathbb{Z}_p$, respectively. One also needs to replace [10, Corollary 4.8] by Lemma 5.6(2). □

Proposition 5.7 *An n–cylinder is a $\mathbb{Z}_p$–coefficient n–cylinder.*

Proof Let W be an n–cylinder with connected boundary components M_i for $1 \leq i \leq \ell$. Since $H_1(M_i) \cong H_1(W) \cong \mathbb{Z}$, the map $H_1(M_i;\mathbb{Z}_p) \to H_1(W;\mathbb{Z}_p)$ is an isomorphism for each i. For $R = \mathbb{Z}$ or $\mathbb{Z}_p$, let $r(R) = \operatorname{rank}_R \operatorname{Coker}\{H_2(\partial W;R) \to H_2(W;R)\}$.

By naturality of intersection forms, it suffices to show that $r(\mathbb{Z}) = r(\mathbb{Z}_p)$. From the long exact sequence

$$0 \to \mathrm{Coker}\{H_2(\partial W; R) \to H_2(W; R)\} \to H_2(W, \partial W; R) \\ \to H_1(\partial W; R) \to H_1(W; R) \to 0$$

and the fact that $\mathrm{rank}_R\, H_2(W, \partial W; R) = \mathrm{rank}_R\, H_2(W; R)$, we obtain that

$$r(R) = \mathrm{rank}_R\, H_2(W; R) - (\ell - 1).$$

Since $H_1(M_i) \cong H_1(W) \cong \mathbb{Z}$, the groups $H_1(W)$ and $\mathrm{Coker}\{H_1(\partial W) \to H_1(W)\}$ have no p–torsion. Therefore, by [2, Lemma 3.14], we have

$$\mathrm{rank}_{\mathbb{Z}}\, H_2(W) = \mathrm{rank}_{\mathbb{Z}_p}\, H_2(W; \mathbb{Z}_p).$$

Therefore, $r(\mathbb{Z}) = r(\mathbb{Z}_p)$. □

5.2 Algebraic n–solutions

In this section, we introduce the notion of *R–algebraic n–solution*, where $R = \mathbb{Q}$ or $\mathbb{Z}_p$, and use it we prove Theorem 3.4. The notion of an algebraic n–solution was introduced by Cochran and Teichner in [17] and later generalized by Cochran and the author in [14, Section 6]. The new notion of an R–algebraic n–solution generalizes an algebraic n–solution to the case of $\mathbb{Z}_p$–coefficients. An R–algebraic n–solution may be considered as an algebraic abstraction of an n–cylinder.

All the results in this section are based on the ideas and results in [17; 14], which deal with the case $R = \mathbb{Q}$. In fact, the results in this section are mainly a $\mathbb{Z}_p$–coefficient version of the corresponding result with $\mathbb{Q}$–coefficients in [14, Section 6]. But, since Section 6 in [14] is quite technical, we rewrite the theorems with $\mathbb{Z}_p$–coefficients carefully and completely, and clarify how one should modify the proofs of the corresponding theorems in [14].

For a group G, let $G_k := G/G_r^{(k)}$, where $G_r^{(k)}$ is the k^{th} rational derived subgroup of G. Since G_k is a PTFA group, $\mathbb{Z}G_k$ (and $\mathbb{Q}G_k$) embeds into the (skew) quotient field, which we denote by $\mathbb{K}(G_k)$. By Lemma 5.1, the group ring $\mathbb{Z}_pG_k$ also embeds into the (skew) quotient field, and we also denote it by $\mathbb{K}(G_k)$, or $\mathbb{K}_p(G_K)$ to emphasize the coefficients $\mathbb{Z}_p$. We write $\mathbb{K}$ for $\mathbb{K}(G_k)$ when it can be understood from the context.

The following is a generalization of an algebraic n–solution [17, Definition 6.1; 14, Definition 6.1]:

Definition 5.8 (Definition 6.1 of [14] for $R=\mathbb{Q}$) Let $R=\mathbb{Q}$ or $\mathbb{Z}_p$ and let $\mathbb{K}(G_k)$ be the (skew) quotient field of RG_k. Let S be a group with $H_1(S;R)\neq 0$. Let F be a free group of rank $2g$ and let $i\colon F\to S$ be a homomorphism. A nontrivial homomorphism $r\colon S\to G$ is an *R–algebraic n–solution* for $i\colon F\to S$ if the following hold:

(1) For each $0\le k\le n-1$, the map $r_*\colon H_1(S;\mathbb{K}(G_k))\to H_1(G;\mathbb{K}(G_k))$ is nontrivial.

(2) For each $0\le k\le n-1$, the map $i_*\colon H_1(F;\mathbb{K}(G_k))\to H_1(S;\mathbb{K}(G_k))$ is surjective.

A $\mathbb{Q}$–algebraic n–solution is also called an *algebraic n–solution*.

Remark 5.9 (1) For $k<n$, an R–algebraic n–solution is an R–algebraic k–solution.

(2) Since $\mathbb{K}(G_k)$ is a flat RG_k–module [37, Proposition II.3.5], $H_1(-;\mathbb{K}(G_k))\cong H_1(-;RG_k)\otimes_{RG_k}\mathbb{K}(G_k)$.

(3) We have $H_1(G;RG_k)\cong H_1(G;\mathbb{Z}G_k)\otimes_{\mathbb{Z}} R\cong G_r^{(k)}/[G_r^{(k)},G_r^{(k)}]\otimes_{\mathbb{Z}} R$.

(4) The case $k=0$ in the condition (2) implies that $H_1(F;R)\to H_1(S;R)$ is surjective.

The following proposition shows the relationship between n–cylinders and R–algebraic n–solutions. Roughly speaking, for a knot K and an n–cylinder W for $M(K)$, the homomorphism on the fundamental groups of the infinite cyclic covers of $M(K)$ and W induced from the inclusion gives rise to an R–algebraic n–solution.

Proposition 5.10 (Proposition 6.3 of [14] for $R=\mathbb{Q}$) *Let $n\ge 1$. Let K be a knot with nontrivial Alexander polynomial $\Delta_K(t)$. Suppose the degree of $\Delta_K(t)$ is greater than 2 if $n>1$. Let W be an n–cylinder with $M(K)$ as one of its boundary components. Let Σ be a capped-off Seifert surface for K. Let $S:=\pi_1M(K)^{(1)}$ and $G:=\pi_1W^{(1)}$. Let $R:=\mathbb{Q}$ or $\mathbb{Z}_p$, where p is a prime greater than the top coefficient of $\Delta_K(t)$. Let F be a free group of rank $2g$, where g is the genus of Σ and let $F\to\pi_1(M(K)\smallsetminus\Sigma)$ be a homomorphism inducing an isomorphism on $H_1(-;R)$. Let i be the composition $F\to\pi_1(M(K)\smallsetminus\Sigma)\to S$. Then the inclusion-induced map $j\colon S\to G$ is an R–algebraic n–solution for $i\colon F\to S$.*

Proof For $R=\mathbb{Z}_p$, modify the proof of Proposition 6.3 in [14] (for the case $R=\mathbb{Q}$) as follows:

(1) Replace the coefficients $\mathbb{Z}$ by $\mathbb{Z}_p$ and let $\mathbb{K}$ be the (skew) quotient field of $\mathbb{Z}_p\Gamma^{(1)} = \mathbb{Z}_p G_k$.

(2) Replace [14, Theorem 3.8] by Theorem 5.2 in this paper.

(3) In the proof of [14, Proposition 6.3], it was given that $d := \operatorname{rank}_{\mathbb{Q}} H_1(M_\infty;\mathbb{Q})$, where M_∞ is the infinite cyclic cover of $M(K)$, and it was used that d is equal to the degree of $\Delta_K(t)$. In our case with $\mathbb{Z}_p$–coefficients, we set $d := \operatorname{rank}_{\mathbb{Z}_p} H_1(M_\infty;\mathbb{Z}_p)$. Note that d is still equal to the degree of $\Delta_K(t)$ since p is greater than the top coefficient of $\Delta_K(t)$ by our hypothesis.

(4) To establish property (2) in Definition 5.8, letting A be a wedge of $2g$ circles, choose a map $A \to Y = M(K) \smallsetminus \Sigma$ inducing $F \to \pi_1 Y$ that is 1–connected on homology with $\mathbb{Z}_p$–coefficients.

(5) Use the $\mathbb{Z}_p$–coefficient version of [15, Proposition 2.10] to show that the map $F \to \pi_1 Y$ induces a 1–connected map on homology with $\mathbb{K}(G_k)$–coefficients: Proposition 2.10 in [15] holds with $\mathbb{Z}_p$–coefficients since G_k lies in Strebel's class $D(\mathbb{Z}_p)$ (see Lemma 2.7).

(6) The proof of [14, Proposition 6.3] uses Harvey's work in [25] with the (skew) quotient field $\mathbb{K}$ of $\mathbb{Z}G_k$. We can still use Harvey's work in the same way for the case of $\mathbb{Z}_p$–coefficients with our $\mathbb{K} = \mathbb{K}_p(G_k)$, the (skew) quotient field of $\mathbb{Z}_p G_k$. □

We note that in the proofs of Proposition 5.10 and [14, Proposition 6.3], it is implicitly proved that the map $i_*\colon H_1(F;\mathbb{K}(G_k)) \to H_1(S;\mathbb{K}(G_k))$ is surjective for all $k \ge 0$.

By Remark 5.9(3), for a group S and $R = \mathbb{Q}$ or $\mathbb{Z}_p$, in an abuse of notation we will regard an element of $S^{(n)}$ as an element of $H_1(S;RS_n)$.

Theorem 5.11 (Theorem 6.4 of [14] for $R = \mathbb{Q}$ only) *Let F be a free group of rank $2g$ and let $i\colon F \to S$ be a group homomorphism. For each $n \ge 0$, there exists a finite collection $\mathcal{P}_n$ of sets of $2g-1$ (if $n > 0$) or $2g$ (if $n = 0$) elements of $F^{(n)}$ which satisfies the following:*

If $r\colon S \to G$ is an R–algebraic n–solution for $i\colon F \to S$ for both $R = \mathbb{Q}$ and $R = \mathbb{Z}_p$, then there is an element in $\mathcal{P}_n$ which maps to a generating set of $H_1(S;\mathbb{K}(G_n))$ under the composition

$$F^{(n)} \to S^{(n)} \to H_1(S;RS_n) \xrightarrow{r_*} H_1(S;RG_n) \to H_1(S;\mathbb{K}(G_n))$$

for both $R = \mathbb{Q}$ and $R = \mathbb{Z}_p$.

An element of $\mathcal{P}_n$ is called an (unordered) *tuple*, and a tuple mapped to a generating set of $H_1(S;\mathbb{K}(G_n))$ in the above theorem is called a *special tuple*. In the proof of Theorem 3.4, the η_i will be defined as the image of the elements of the tuples in $\mathcal{P}_{n-1}$ under the homomorphism $i\colon F\to S$, which is a finite set, and the existence of a special tuple will give us the existence of the desired curve η_i.

Proof Suppose that $x_1,x_2,\dots,x_{2g}$ generate F. Let $\mathcal{P}_0=\{\{x_1,x_2,\dots,x_{2g}\}\}$, a set consisted of a single $2g$–tuple. When $n=0$, the above composition becomes $F\to H_1(S;R)$ and the theorem follows since $H_1(F;R)\to H_1(S;R)$ is surjective by Remark 5.9(4) and x_i generate $H_1(F;R)\cong R^{2g}$.

Now assume $n\ge 1$. We define $\mathcal{P}_n$ inductively as in the proof of [14, Theorem 6.4]. Define

$$\mathcal{P}_1:=\bigl\{\{[x_i,x_1],\dots,[x_i,x_{i-1}],[x_i,x_{i+1}],\dots,[x_i,x_{2g}]\}\mid 1\le i\le 2g\bigr\},$$

which is a set of $(2g{-}1)$–tuples. Suppose $\mathcal{P}_k$ has been constructed for $k\ge 1$. We define $\mathcal{P}_{k+1}$ as follows: a $(2g{-}1)$–tuple $\{z_1,\dots,z_{2g-1}\}$ is in $\mathcal{P}_{k+1}$ if and only if there is $\{w_1,\dots,w_{2g-1}\}\in\mathcal{P}_k$ such that for each $1\le i\le 2g-1$, $z_i=[w_i,w_i^{x_j}]$ for some j with $1\le j\ne i\le 2g$ or $z_i=[w_i,w_k]$ for some k with $1\le k\ne i\le 2g-1$. Here $w_i^{x_j}$ denotes $x_j^{-1}w_ix_j$.

By [14, Theorem 6.4] the conclusion of the theorem holds when $R=\mathbb{Q}$. That is, there is an element in $\mathcal{P}_n$ mapping to a generating set of $H_1(S;\mathbb{K}(G_n))$, where $\mathbb{K}(G_n)$ is the (skew) quotient field of $\mathbb{Q}G_n$.

We assert that the above element in $\mathcal{P}_n$ also maps to a generating set of $H_1(S;\mathbb{K}(G_n))$ when $\mathbb{K}(G_n)=\mathbb{K}_p(G_n)$, the (skew) quotient field of $\mathbb{Z}_pG_n$. This is proved by modifying the proof of Theorem 6.4 in [14] as follows:

(1) Replace the group rings over $\mathbb{Z}$–coefficients by the corresponding group rings over $\mathbb{Z}_p$–coefficients. For example, change $\mathbb{Z}F_n$ to $\mathbb{Z}_pF_n$.

(2) Let $\mathbb{K}(G_n)$ denote the (skew) quotient field of $\mathbb{Z}_pG_n$.

(3) Use Lemma 5.6(1) in this paper instead of [10, Lemma 3.9].

(4) Use Lemma 5.12 below instead of [14, Lemma 6.5]. (Note that we need the hypothesis that $r\colon S\to G$ is a ($\mathbb{Q}$–)algebraic n–solution in Lemma 5.12.)

(5) In the last part of the proof, one needs to use that $H_1(F;\mathbb{Z}_pG_n)\to H_1(S;\mathbb{Z}_pG_n)$ is surjective after tensoring with $\mathbb{K}_p(G_n)$. Note that this is where we use the assumption that $r\colon S\to G$ is a $\mathbb{Z}_p$–algebraic n–solution. □

Let R be a ring. For each $1 \le i \le 2g$, let $\partial_i; F \to \mathbb{Z}F \to RF$ be the Fox free differential calculus defined by $\partial_i(x_j) = \delta_{ij}$ and $\partial_i(gh) = \partial_i g + (\partial_i h)g^{-1}$. For $k \ge 0$, let $\pi_k\colon RF \to RF_k$ be the quotient map. By abuse of notation, denote by r the map $RF_k \to RG_k$ induced from $r \circ i\colon F \to S \to G$. We denote by d_i^k the composition $r \circ \pi_k \circ \partial_i\colon F \to RG_k$ for $1 \le i \le 2g$. We simply write d_i for d_i^k when the superscript is understood from the context.

Lemma 5.12 (Lemma 6.5 of [14] for $R = \mathbb{Q}$) *Let $n \ge 1$. Suppose we are given an algebraic n–solution $r\colon S \to G$ for $i\colon F \to S$. Then there exists a $(2g-1)$–tuple $\{w_1, \dots, w_{2g-1}\} \in \mathcal{P}_n$ such that the $2g-1$ vectors*

$$\{(d_1(w_i), d_2(w_i), \dots, d_{2g-1}(w_i)) \mid 1 \le i \le 2g-1\}$$

in $(RG_n)^{2g-1}$ are right linearly independent over RG_n for $R = \mathbb{Q}$ and $R = \mathbb{Z}_p$ for all prime p.

Proof In the proof of [14, Lemma 6.5], for each $1 \le k \le n$ a tuple $\{w_1^k, \dots, w_{2g-1}^k\} \in \mathcal{P}_k$ was constructed inductively such that

$$\{(d_1(w_i^k), d_2(w_i^k), \dots, d_{2g-1}(w_i^k)) \mid 1 \le i \le 2g-1\}$$

in $(\mathbb{Q}G_k)^{2g-1}$ are right linearly independent over $\mathbb{Q}G_k$ using the fact that $r\colon S \to G$ is a ($\mathbb{Q}$–)algebraic n–solution.

We will show that for each k, the vectors

$$\{(d_1(w_i^k), d_2(w_i^k), \dots, d_{2g-1}(w_i^k)) \mid 1 \le i \le 2g-1\}$$

are also right linearly independent over $\mathbb{Z}_pG_k$, and it will complete the proof by taking $\{w_1, \dots, w_{2g-1}\} = \{w_1^n, \dots, w_{2g-1}^n\}$. The proof follows the lines of the proof of [14, Lemma 6.5], and we explain the needed modifications below.

For $k = 1$, in the proof of [14, Lemma 6.5] we assume that $(r \circ \pi_1)(x_{2g})$ is nontrivial in G_1 and take $\{w_1^1, \dots, w_{2g-1}^1\} = \{[x_{2g}, x_1], \dots, [x_{2g}, x_{2g-1}]\}$. Then it was shown that the vectors $\{(d_1(w_i^1), d_2(w_i^1), \dots, d_{2g-1}(w_i^1)) \mid 1 \le i \le 2g-1\}$ are linearly independent over $\mathbb{Q}G_1$ (in the first paragraph of [14, page 1428]). Using the same argument, it can be seen that $\{w_1^1, \dots, w_{2g-1}^1\}$ are also linearly independent over $\mathbb{Z}_pG_1$.

We use an induction argument. Suppose that for $k < n$ it has been shown that the vectors $\{(d_1(w_i^k), d_2(w_i^k), \dots, d_{2g-1}(w_i^k)) \mid 1 \le i \le 2g-1\}$ are linearly independent over $\mathbb{Z}_pG_k$. In [14] it was shown that we may assume $(r \circ \pi_{k+1})(w_1^k) \ne e$ in G_{k+1}.

Then $\{w_1^{k+1}, \dots, w_{2g-1}^{k+1}\}$ was defined in [14] as follows: we take $w_i^{k+1} = [w_i^k, w_i^{x_{2g}}]$ if $(r \circ \pi_{k+1})(w_i^k) \neq e$ in G_{k+1} and $w_i^{k+1} = [w_i^k, w_1^k]$ otherwise.

Then it was shown in [14] that in $\mathbb{Z}G_{k+1}$, for some $t_i \in \mathbb{Z}F$,

$$\begin{aligned}(d_1^{k+1}(w_i^{k+1}), d_2^{k+1}(w_i^{k+1}), \dots, d_{2g-1}^{k+1}(w_i^{k+1}))\\ = (d_1^{k+1}(w_i^k), d_2^{k+1}(w_i^k), \dots, d_{2g-1}^{k+1}(w_i^k)) \cdot (r \circ \pi_{k+1})(t_i),\end{aligned}$$

where $(r \circ \pi_{k+1})(t_i) \neq e$ in $\mathbb{Z}G_{k+1}$. We note that, using the same argument in [14] one can show that $(r \circ \pi_{k+1})(t_i) \neq e$ in $\mathbb{Z}_p G_{k+1}$ as well. This implies that it suffices to show that the vectors $\{(d_1^{k+1}(w_i^k), d_2^{k+1}(w_i^k), \dots, d_{2g-1}^{k+1}(w_i^k)) \mid 1 \leq i \leq 2g-1\}$ are linearly independent over $\mathbb{Z}_p G_{k+1}$.

For simplicity, for $1 \leq i \leq 2g-1$, let $\boldsymbol{v}_i^{k+1} = (d_1^{k+1}(w_i^k), d_2^{k+1}(w_i^k), \dots, d_{2g-1}^{k+1}(w_i^k))$ and $\boldsymbol{v}_i^k = (d_1^k(w_i^k), d_2^k(w_i^k), \dots, d_{2g-1}^k(w_i^k))$. By the induction hypothesis, the vectors $\boldsymbol{v}_i^k$ for $1 \leq i \leq 2g-1$ are linearly independent over $\mathbb{Z}_p G_{k+1}$ (and over $\mathbb{Q}G_{k+1}$). Let $H = G_r^{(k)}/G_{r+1}^{(k)} = \operatorname{Ker}\{G_{k+1} \to G_k\}$. Then H is a torsion-free abelian group, and hence lies in Strebel's class $D(\mathbb{Z}_p)$ [38]. In the last paragraph of the proof of [14, Lemma 6.5] it was shown that linear independence of the vectors $\boldsymbol{v}_i^k$ over $\mathbb{Z}G_k$ implies linear independence of the vectors $\boldsymbol{v}_i^{k+1}$ over $\mathbb{Z}G_{k+1}$. Using the same argument and the fact that the group H lies in $D(\mathbb{Z}_p)$, changing the coefficient group for group rings from $\mathbb{Z}$ to $\mathbb{Z}_p$ in the proof, one can show that linear independence of the vectors $\boldsymbol{v}_i^k$ over $\mathbb{Z}_p G_k$ implies linear independence of the vectors $\boldsymbol{v}_i^{k+1}$ over $\mathbb{Z}_p G_{k+1}$. This completes the proof. □

Now we give a proof of Theorem 3.4.

Proof of Theorem 3.4 Let p be a prime greater than the top coefficient of $\Delta_K(t)$ and W an n–cylinder one of whose boundary components is $M(K)$. Let $\mathcal{P} = (R_0, \dots, R_n)$ where $R_i = \mathbb{Q}$ for $i \leq n-1$ and $R_n = \mathbb{Z}_p$. With this $\mathcal{P}$, recall that for a group G and $k \leq n$, $\mathcal{P}^k G = G_r^{(k)}$, the k^{th} rational derived subgroup of G. Recall that for a group G, we denote $G/G_r^{(k)}$ by G_k.

For convenience, we also denote by Σ the capped-off Seifert surface for K. Suppose Σ has genus g. Let $M := M(K)$, $S := \pi_1 M^{(1)}$, and $G := \pi_1 W^{(1)}$. Let F be a free group of rank $2g$ and $F \to \pi_1(M \smallsetminus \Sigma)$ a homomorphism inducing an isomorphism on $H_1(-;R)$ for $R = \mathbb{Q}$ and $\mathbb{Z}_p$. Let $i\colon F \to \pi_1(M \smallsetminus \Sigma) \to S$ be a map induced from this map. Since G_{n-1} is PTFA [26, Proposition 2.1], $\mathbb{Z}_p G_{n-1}$ embeds into the (skew) quotient field, say $\mathbb{K}$, by Lemma 5.1.

Now, by Proposition 5.10, the inclusion-induced homomorphism $j\colon S \to G$ is an R–algebraic n–solution for $i\colon F \to S$ for $R = \mathbb{Q}$ and $\mathbb{Z}_p$, and hence an R–algebraic $(n-1)$–solution for both $R = \mathbb{Q}$ and $\mathbb{Z}_p$ (see Remark 5.9(1)). Let $\mathcal{P}_{n-1}$ be the finite collection of tuples of $2g-1$ (if $n > 1$) or $2g$ (if $n = 1$) elements of $F^{(n-1)}$ obtained using Theorem 5.11 with the map $i\colon F \to S$ above (note that $\mathcal{P}_{n-1}$ is independent of the choice of $\mathcal{P}$). The image of the elements of the tuples in $\mathcal{P}_{n-1}$ under the homomorphism $i\colon F \to S$ is a finite subset of $S^{(n-1)} = \pi_1 M^{(n)}$, which we denote by $\eta_1, \eta_2, \dots \eta_m$. Since the map i factors through $\pi_1(M \smallsetminus \Sigma)$, we can find the representatives η_i in $S^3 \smallsetminus \Sigma$, and by crossing change we can make η_i form an unlink. Now since $j\colon S \to G$ is an R–algebraic $(n-1)$–solution for $i\colon F \to S$, by Theorem 5.11 there exists a $(2g-1)$–tuple $\{w_1, w_2, \dots, w_{2g-1}\} \subset F^{(n-1)}$ (or a $2g$–tuple $\{w_1, \dots, w_{2g}\} \subset F$ if $n = 1$) which maps to a generating set of $H_1(S; \mathbb{K})$ as a $\mathbb{K}$–module. Therefore, the η_i for $1 \le i \le m$ form a generating set of $H_1(S; \mathbb{K})$ as a $\mathbb{K}$–module.

Let M_∞ and W_∞ denote the infinite cyclic covers of M and W, respectively. Let $\Gamma := \pi_1 W / \mathcal{P}^n \pi_1 W = \pi_1 W / \pi_1 W_r^{(n)}$. Then Γ is a PTFA group such that $\Gamma^{(n)} = \{e\}$. Since $H_1(\Gamma) \cong \mathbb{Z} = \langle t \rangle$ and

$$[\Gamma, \Gamma] = \pi_1(W)^{(1)} / \pi_1 W_r^{(n)} = G / G_r^{(n-1)} = G_{n-1},$$

we have $\Gamma \cong G_{n-1} \rtimes \langle t \rangle$. Therefore, we have

$$\begin{aligned} H_1(W_\infty; \mathbb{K}) &\cong H_1(W_\infty; \mathbb{Z}_p G_{n-1}) \otimes_{\mathbb{Z}_p G_{n-1}} \mathbb{K} \\ &\cong H_1(W_\infty; \mathbb{Z}_p[\Gamma, \Gamma]) \otimes_{\mathbb{Z}_p G_{n-1}} \mathbb{K} \\ &\cong H_1(W; \mathbb{Z}_p \Gamma) \otimes_{\mathbb{Z}_p G_{n-1}} \mathbb{K} \\ &\cong H_1\big(W; \mathbb{Z}_p[G_{n-1} \rtimes \langle t \rangle]\big) \otimes_{\mathbb{Z}_p G_{n-1}} \mathbb{K} \\ &\cong H_1(W; \mathbb{K}[t^{\pm 1}]). \end{aligned}$$

Similarly, we have $H_1(M_\infty; \mathbb{K}) \cong H_1(M; \mathbb{K}[t^{\pm 1}])$.

Therefore, since $H_1(S; \mathbb{K}) \cong H_1(M_\infty; \mathbb{K}) \cong H_1(M; \mathbb{K}[t^{\pm 1}])$, the η_i for $1 \le i \le m$ can be regarded as a generating set of $H_1(M; \mathbb{K}[t^{\pm 1}])$ as a $\mathbb{K}$–module.

Let $d := \deg \Delta_K(t) = \operatorname{rank}_{\mathbb{Q}} H_1(M_\infty; \mathbb{Q})$. Since p is greater than the top coefficient of $\Delta_K(t)$, we have $d = \operatorname{rank}_{\mathbb{Z}_p} H_1(M_\infty; \mathbb{Z}_p)$. Note that $d \ge 4$ if $n \ge 2$ and $d \ge 2$ if $n = 1$ by assumption. Since Γ is a PTFA group such that $\Gamma^{(n)} = \{e\}$, by Theorem 5.2 this implies that the map $H_1(M; \mathbb{K}[t^{\pm 1}]) \to H_1(W; \mathbb{K}[t^{\pm 1}])$ is nontrivial. Therefore, since the η_i generate $H_1(M;\mathbb{K}[t^{\pm 1}])$ as a $\mathbb{K}$–module, there exists some element η_i which maps to a nontrivial element in $H_1(W;\mathbb{K}[t^{\pm 1}])$.

Note that $H_1(W;\mathbb{K}[t^{\pm 1}])\cong H_1(W;\mathbb{Z}_p\Gamma)\otimes_{\mathbb{Z}_pG_{n-1}}\mathbb{K}$. By the definition of $\mathcal{P}^{n+1}\pi_1 W$, the group $\mathcal{P}^n\pi_1 W/\mathcal{P}^{n+1}\pi_1 W$ injects to $H_1(\pi_1 W;\mathbb{Z}_p\Gamma)\cong H_1(W;\mathbb{Z}_p\Gamma)$. Since $\eta_i\in\pi_1 M^{(n)}$, it maps into $\pi_1 W^{(n)}$, hence into $\mathcal{P}^n\pi_1 W$. From these observations, one can deduce that the element η_i, which maps to a nontrivial element in $H_1(W;\mathbb{K}[t^{\pm 1}])$, also maps nontrivially to $\mathcal{P}^n\pi_1 W/\mathcal{P}^{n+1}\pi_1 W$. In particular, we see that $\eta_i\notin\mathcal{P}^{n+1}\pi_1 W$.

Finally, by Lemma 3.2 we can homotope all η_i for $1\le i\le m$ so that all of the η_i bound capped gropes of height n which are disjointly embedded in $S^3\smallsetminus K$, as desired. □

References

[1] **J R Burke**, *Infection by string links and new structure in the knot concordance group*, Algebr. Geom. Topol. 14 (2014) 1577–1626 MR

[2] **J C Cha**, *Amenable L^2–theoretic methods and knot concordance*, Int. Math. Res. Not. 2014 (2014) 4768–4803 MR

[3] **J C Cha**, *Symmetric Whitney tower cobordism for bordered 3–manifolds and links*, Trans. Amer. Math. Soc. 366 (2014) 3241–3273 MR

[4] **J C Cha**, *A topological approach to Cheeger–Gromov universal bounds for von Neumann ρ–invariants*, Comm. Pure Appl. Math. 69 (2016) 1154–1209 MR

[5] **J C Cha**, **T Kim**, *Covering link calculus and iterated Bing doubles*, Geom. Topol. 12 (2008) 2173–2201 MR

[6] **J C Cha**, **T Kim**, *Unknotted gropes, Whitney towers, and doubly slicing knots*, Trans. Amer. Math. Soc. 371 (2019) 2383–2429 MR

[7] **J C Cha**, **K E Orr**, *L^2–signatures, homology localization, and amenable groups*, Comm. Pure Appl. Math. 65 (2012) 790–832 MR

[8] **S Chang**, **S Weinberger**, *On invariants of Hirzebruch and Cheeger–Gromov*, Geom. Topol. 7 (2003) 311–319 MR

[9] **J Cheeger**, **M Gromov**, *Bounds on the von Neumann dimension of L^2–cohomology and the Gauss–Bonnet theorem for open manifolds*, J. Differential Geom. 21 (1985) 1–34 MR

[10] **T D Cochran**, *Noncommutative knot theory*, Algebr. Geom. Topol. 4 (2004) 347–398 MR

[11] **T D Cochran**, **S Harvey**, **C Leidy**, *Knot concordance and higher-order Blanchfield duality*, Geom. Topol. 13 (2009) 1419–1482 MR

[12] **T D Cochran**, **S Harvey**, **C Leidy**, *2–Torsion in the n–solvable filtration of the knot concordance group*, Proc. Lond. Math. Soc. 102 (2011) 257–290 MR

[13] **T D Cochran**, **S Harvey**, **C Leidy**, *Primary decomposition and the fractal nature of knot concordance*, Math. Ann. 351 (2011) 443–508 MR

[14] **T D Cochran**, **T Kim**, *Higher-order Alexander invariants and filtrations of the knot concordance group*, Trans. Amer. Math. Soc. 360 (2008) 1407–1441 MR

[15] **T D Cochran**, **K E Orr**, **P Teichner**, *Knot concordance, Whitney towers and L^2–signatures*, Ann. of Math. 157 (2003) 433–519 MR

[16] **T D Cochran**, **K E Orr**, **P Teichner**, *Structure in the classical knot concordance group*, Comment. Math. Helv. 79 (2004) 105–123 MR

[17] **T D Cochran**, **P Teichner**, *Knot concordance and von Neumann ρ–invariants*, Duke Math. J. 137 (2007) 337–379 MR

[18] **C W Davis**, *Linear independence of knots arising from iterated infection without the use of Tristram–Levine signature*, Int. Math. Res. Not. 2014 (2014) 1973–2005 MR

[19] **J Duval**, *Forme de Blanchfield et cobordisme d'entrelacs bords*, Comment. Math. Helv. 61 (1986) 617–635 MR

[20] **M H Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 MR

[21] **M H Freedman**, **F Quinn**, *Topology of 4–manifolds*, Princeton Math. Ser. 39, Princeton Univ. Press (1990) MR

[22] **M H Freedman**, **P Teichner**, *4–manifold topology, I: Subexponential groups*, Invent. Math. 122 (1995) 509–529 MR

[23] **S Friedl**, *L^2–eta-invariants and their approximation by unitary eta-invariants*, Math. Proc. Cambridge Philos. Soc. 138 (2005) 327–338 MR

[24] **S Friedl**, **P Teichner**, *New topologically slice knots*, Geom. Topol. 9 (2005) 2129–2158 MR

[25] **S L Harvey**, *Higher-order polynomial invariants of 3–manifolds giving lower bounds for the Thurston norm*, Topology 44 (2005) 895–945 MR

[26] **S L Harvey**, *Homology cobordism invariants and the Cochran–Orr–Teichner filtration of the link concordance group*, Geom. Topol. 12 (2008) 387–430 MR

[27] **P D Horn**, *The non-triviality of the grope filtrations of the knot and link concordance groups*, Comment. Math. Helv. 85 (2010) 751–773 MR

[28] **H J Jang**, *Two-torsion in the grope and solvable filtrations of knots*, Int. J. Math. 28 (2017) art. id. 1750023 MR

[29] **S-G Kim**, **T Kim**, *Polynomial splittings of metabelian von Neumann rho-invariants of knots*, Proc. Amer. Math. Soc. 136 (2008) 4079–4087 MR

[30] **S-G Kim**, **T Kim**, *Splittings of von Neumann rho-invariants of knots*, J. Lond. Math. Soc. 89 (2014) 797–816 MR

[31] **T Kim**, *Filtration of the classical knot concordance group and Casson–Gordon invariants*, Math. Proc. Cambridge Philos. Soc. 137 (2004) 293–306 MR

[32] **T Kim**, *An infinite family of non-concordant knots having the same Seifert form*, Comment. Math. Helv. 80 (2005) 147–155 MR

[33] **T Kim**, *New obstructions to doubly slicing knots*, Topology 45 (2006) 543–566 MR

[34] **T Kim**, *Knots having the same Seifert form and primary decomposition of knot concordance*, J. Knot Theory Ramifications 26 (2017) art. id. 1750103 MR

[35] **C Livingston**, *Seifert forms and concordance*, Geom. Topol. 6 (2002) 403–408 MR

[36] **M Ramachandran**, *Von Neumann index theorems for manifolds with boundary*, J. Differential Geom. 38 (1993) 315–349 MR

[37] **B Stenström**, *Rings of quotients: an introduction to methods of ring theory*, Grundl. Math. Wissen. 217, Springer (1975) MR

[38] **R Strebel**, *Homological methods applied to the derived series of groups*, Comment. Math. Helv. 49 (1974) 302–332 MR

Department of Mathematics, Konkuk University
Seoul, South Korea

tkim@konkuk.ac.kr

Received: 15 June 2018 Revised: 11 September 2019