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Euler characteristics of Gothic Teichmüller curves

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We compute the Euler characteristics of the recently discovered series of Gothic Teichmüller curves. The main tool is the construction of "Gothic" Hilbert modular forms vanishing at the images of these Teichmüller curves.

Contrary to all previously known examples, the Euler characteristic is not proportional to the Euler characteristic of the ambient Hilbert modular surfaces. This results in interesting "varying" phenomena for Lyapunov exponents.

32G15; 11F27, 11F46, 37D40

1 Introduction

Teichmüller curves are complex geodesics in the moduli space of curves $\mathcal{M}_g$. They arise as the $\mathrm{SL}_2(\mathbb{R})$–orbits of flat surfaces with optimal dynamics, called Veech surfaces. If the Veech surface is not obtained by a covering construction from a lower-genus surface, it is called primitive and the resulting Teichmüller curve is called primitive too.

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There are very few constructions of primitive Teichmüller curves (see Möller [20] for a list of known examples). Each infinite collection of primitive Teichmüller curves in a fixed genus stems from an invariant submanifold "like the minimal stratum $\Omega\mathcal{M}_2(2)$" in genus two (see Section 2.2), by the finiteness results from Eskin, Filip and Wright [6]. While the geometry of $\Omega\mathcal{M}_2(2)$ and of the Prym loci is well understood now, the geometry of the two invariant submanifolds "like $\Omega\mathcal{M}_2(2)$" recently discovered by Eskin, McMullen, Mukamel and Wright in [8] is basically unexplored. Here we focus on the Gothic locus $\Omega G \subset \Omega\mathcal{M}_4(2,2,2)$ of flat genus four surfaces, introduced already in McMullen, Mukamel and Wright [16].

While interest in Teichmüller curves originates from dynamics, their geometry is strongly determined by modular forms. Teichmüller curves in an infinite series of fixed genus always map via the Torelli map to the locus of real multiplication, ie to a Hilbert modular surface (see Möller [17] together with [6]). Conversely, the intersection of ΩG with the locus of real multiplication by the order $\mathcal{O}_D$ is a union of Teichmüller curves G_D. These Teichmüller curves are primitive if and only if D is not a square, which we assume in the rest of this paper. The modular forms in question are thus Hilbert modular forms, supposed to cut out the Teichmüller curves G_D inside the Hilbert modular surface.

Contrary to the expectation from the situation in genus two and in the Prym loci of genus three and four, there is no Hilbert modular form whose vanishing locus is precisely equal to the Gothic Teichmüller curves G_D! Yet, there is a "Gothic" Hilbert modular form $\mathcal{G}_D$ whose vanishing locus is only slightly larger than G_D, the difference being a collection of modular curves, whose parameters can all be computed.

In order to state the results, we roughly recall the definition of ΩG; see Section 2.1 for more details. A flat surface (X,ω) in the stratum $\Omega\mathcal{M}_4(2,2,2)$ is Gothic if it admits an involution J leaving ω antiinvariant and fixing the zeros of ω and an "odd" degree three map $X \to B$ to an elliptic curve B mapping all the zeros to a single point. The involution J induces a degree two map $X \to A$ to another elliptic curve A. The complement of both A and B in the Jacobian of X inherits a polarization of type $(1,6)$, as we show in Section 3. Consequently, the number 6 plays a prominent role in the paper: the Gothic Teichmüller curves G_D naturally live on Hilbert modular surfaces $X_D(\mathfrak{b})$, where $\mathfrak{b}$ is an $\mathcal{O}_D$–ideal of norm 6.

Our first goal is to give a natural decomposition of G_D into (perhaps still reducible) components and to compute explicitly their Euler characteristics. They can be written in terms of Euler characteristics of those Hilbert modular surfaces and of the reducible locus Red_{23}, parametrizing $(2,3)$–polarized products of elliptic curves with real multiplication by $\mathcal{O}_D$.

Theorem 1.1 *Let D be a nonsquare discriminant. The Gothic Teichmüller curve G_D is nonempty if and only if $D \equiv 0, 1, 4, 9, 12, 16 \bmod 24$.*

In this case, G_D consists of different subcurves $G_D(\mathfrak{b})$ corresponding to different $\mathcal{O}_D$–ideals $\mathfrak{b}$ of norm 6. The Euler characteristics of all these subcurves agree and are equal to

$$-\chi(G_D(\mathfrak{b})) = \tfrac{3}{2}\chi(X_D(\mathfrak{b})) + 2\chi(\mathrm{Red}_{23}(\mathfrak{b})).$$

We give a completely explicit formula in Theorem 11.1, and a table for small discriminants (Table 1) can be found at the end of the paper. The Euler characteristic of the Hilbert modular surface $X_D(\mathfrak{b})$ is equal to the Euler characteristic of a standard Hilbert modular surface if D is fundamental, and differs by a simple factor in general; see Proposition 4.3 for the complete formula. In any case, $\chi(X_D(\mathfrak{b}))$ is independent of the choice of the ideal of norm 6. We strongly suspect the subcurves $G_D(\mathfrak{b})$ defined in the theorem to be irreducible but we do not attempt to prove this here.

The presence of modular curves in the vanishing locus of the Hilbert modular form $\mathcal{G}_D$ has another consequence that makes characteristic invariants of the Gothic Teichmüller curves behave differently than all the examples known so far. We phrase this in terms of Lyapunov exponents in Section 11 and restate it geometrically here.

Teichmüller curves are Kobayashi geodesic algebraic curves C in Hilbert modular surfaces. If $z \mapsto (z, \varphi(z))$ is the universal covering map of a Kobayashi geodesic, then for any $M \in \mathrm{GL}_2(\mathbb{Q}(\sqrt{D}))$ the map $z \mapsto (Mz, M^\sigma \varphi(z))$ descends to another Kobayashi geodesic. All modular curves arise by this twisting procedure from the diagonal and the twists of Teichmüller curves are interesting special curves on Hilbert modular surfaces. However, this twisting does not change the most basic algebraic invariant,

$$\lambda_2(C) = \frac{C \cdot [\omega_2]}{C \cdot [\omega_1]},$$

where $[\omega_i]$ are the foliation classes on the Hilbert modular surface. For modular curves $\lambda_2 = 1$ and, in general, the list of known $\lambda_2(C)$ of Kobayashi geodesics C was a rather short (and finite) list (see the summary in Möller and Zagier [21, Section 1]). As a consequence of the decomposition of $\{\mathcal{G}_D = 0\}$ into several components we obtain:

Corollary 1.2 *The sequence of invariants $\lambda_2(G_D)$ is infinite and tends to $\frac{3}{13}$ for $D \to \infty$.*

This corollary is proved in the equivalent formulation of Theorem 11.2; see also Proposition 11.3. It is an open question whether for a *fixed* Hilbert modular surface the set of $\lambda_2(C)$ for all its Kobayashi geodesics C is finite or infinite.

We next summarize the main steps in the proof of Theorem 1.1 and explain the origin of the Gothic modular form $\mathcal{G}_D$. Analyzing the definition of the Gothic locus (Section 2), we obtain that the image of a Gothic Veech surface in its $(1,6)$–polarized Prym abelian surface is a curve with a triple point at the origin and horizontal tangents at three nonzero 2–torsion points (Section 6). To construct these images as the vanishing locus of a theta function, we need to impose five conditions, two stemming from the multiplicity at the origin and the rest from the behavior at the 2–torsion points. The (odd) theta functions vary in a 3–dimensional projective space, so we can impose the first three conditions and, by restricting to a divisor $\{\mathcal{G}_D = 0\}$ in the Hilbert modular surface, we can also satisfy the last two conditions. Teichmüller curves exist due to dimension miracles. From our point of view this is manifested by the last two conditions holding simultaneously along $\{\mathcal{G}_D = 0\}$, due to theta value relations at 2–torsion points (Section 5).

Contrary to the previous known cases in $\Omega\mathcal{M}_2(2)$ and the Prym locus, the vanishing locus of the Gothic modular form $\mathcal{G}_D$ contains some "spurious" components apart from the Gothic Teichmüller curves. These components form the $(2,3)$–reducible locus, points in the Hilbert modular surface corresponding to products of elliptic curves with the natural $(2,3)$–product polarization (Section 7). By studying the vanishing order of the modular form along both the Gothic Teichmüller curve and the reducible locus (Section 8), we can finally relate their Euler characteristics with the Euler characteristic of the Hilbert modular surface in which they live (Section 11). This also allows us to give a formula for the Lyapunov exponents of the individual Gothic Teichmüller curves and to compute those of the Gothic locus.

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2 Examples of Gothic Teichmüller curves

In this section we introduce the Gothic locus and the Gothic Teichmüller curves, following [16]. Not all Gothic Teichmüller curves can be presented in the shape of a Gothic cathedral. In fact, the simplest example of a Gothic Teichmüller curve already appeared in work of Ward [24] on triangular billiards.

2.1 The Gothic locus

Given a Riemann surface X with an involution J we say that a map $\pi_B\colon X \to B$ is *odd* if there exists an involution $j\colon B \to B$ such that $\pi_B \circ J = j \circ \pi_B$. Following [16] we define the *Gothic locus* ΩG to be the set of Riemann surfaces $(X,\omega) \in \Omega\mathcal{M}_4(2^3, 0^3)$ such that

- (i) there exists an involution $J \in \mathrm{Aut}(X)$ whose fixed points are the six marked points, the zeros $Z = \mathcal{Z}(\omega) = \{z_1, z_2, z_3\}$ and the marked regular points $P = \{p_1, p_2, p_3\}$,
- (ii) the one-form ω is J–antiinvariant, that is, $J^*\omega = -\omega$, and
- (iii) there exists a genus one curve B and an odd map $\pi_B\colon X \to B$ of degree 3 such that $|\pi_B(Z)| = 1$.

Every flat surface $(X,\omega) \in \Omega G$ in the Gothic locus thus comes with maps

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\pi_A} & \downarrow h & {}^{\pi_B}\searrow & \\ A & & & & B \\ & {}_{p}\searrow & & \swarrow_{r} & \\ & & \mathbb{P}^1 & & \end{array} \tag{1}$$

where

- $\pi_A\colon X \to X/J \cong A$ is of degree 2,
- $\pi_B\colon X \to B$ is an odd, degree 3 ramified covering such that $|\pi_B(Z)| = 1$,
- $r\colon B \to B/j \cong \mathbb{P}^1$ is the quotient map, and
- $p\colon A \to \mathbb{P}^1$ is the degree 3 ramified covering that makes the diagram commutative.

These maps can be illustrated on the hexagon form in Figure 1. It admits an automorphism R of order 6 with $R^*\omega = \zeta_6\omega$. Then $J = R^3$ and π_A and π_B are the quotients

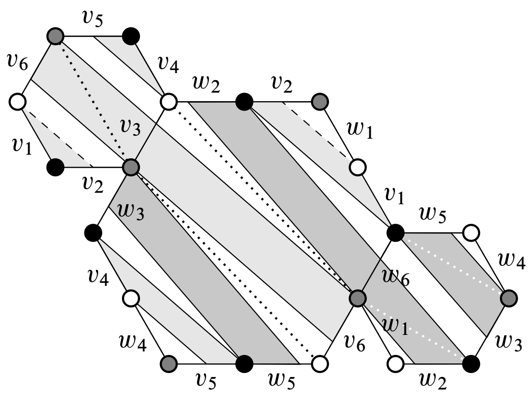


Figure 1: The hexagon form in the Gothic locus (from [16]).

by R^3 and R^2, respectively. Note, however, that the map π_B will not be Galois in general. The reason for the definition is that ΩG turns out to be an unexpected $SL_2(\mathbb{R})$–orbit closure.

Theorem 2.1 [16] *The Gothic locus is a closed irreducible variety of dimension* 4*, locally defined by linear equations in period coordinates.*

In fact, $v_1, \dots, v_6, w_1, \dots, w_6$ are periods on the 10–dimensional space $\Omega\mathcal{M}_4(2,2,2)$, and $\sum_{i=1}^6 v_i = 0 = \sum_{i=1}^6 w_i$ by construction. In fact, $v_1, \dots, v_5, w_1, \dots, w_5$ form a coordinate system. In this coordinates ΩG is cut out by the conditions

$$(2) \qquad v_{i+3} = -v_i, \quad w_{i+3} = -w_i, \quad v_1 + v_3 + v_5 = 0, \quad w_1 + w_3 + w_5 = 0$$

for $i = 1, 2, 3$.

The branch points of the maps in the diagram (1) give a collection of special points. We introduce notation for later use. Given a point $x \in X$ we will denote the other points in the same π_B–fiber by $\pi_B^{-1}(\pi_B(x)) = \{x =: x^{(1)}, x^{(2)}, x^{(3)}\}$ and $h = r \circ \pi_B = p \circ \pi_A$. The preimages of the ramification points of h and their behavior under the maps π_A and π_B can be described in the following way:

- The image point $e_4' = \pi_B(Z)$ is fixed by j, since each z_i is fixed by J. This point is therefore sent by r to a ramification point $e_4 = r(e_4')$ of h. In particular we can choose the group law on B so that e_4' agrees with the origin O.
- The image points $e_i' = \pi_B(p_i)$ for $i = 1, 2, 3$ are also fixed by j, giving rise to the other three points of order 2 in B. Their preimages under π_B are given by $\pi_B^{-1}(e_i') = \{p_i, q_i, J(q_i)\}$.

- There exist three other ramification points of the map h, among the preimages of which there exist points $\{y_i, J(y_i)\}$ for $i = 1, 2, 3$ with ramification index 2 with respect to h each.

In summary, we have

$$\begin{array}{ccccc} & & \{p_i, q_i, J(q_i)\} & & \\ & \overset{\pi_A}{\swarrow} & \downarrow h & \overset{\pi_B}{\searrow} & \\ \{\overline{p}_i, \overline{q}_i\} & & & & e_i' \\ & \underset{p}{\searrow} & & \underset{r}{\swarrow} & \\ & & e_i & & \end{array} \qquad \begin{array}{ccccc} & & \{z_1, z_2, z_3\} & & \\ & \overset{\pi_A}{\swarrow} & \downarrow h & \overset{\pi_B}{\searrow} & \\ \{\overline{z}_1, \overline{z}_2, \overline{z}_3\} & & & & e_4' \\ & \underset{p}{\searrow} & & \underset{r}{\swarrow} & \\ & & e_4 & & \end{array} \tag{3}$$

Recall that the stratum $\Omega\mathcal{M}_g(2,2,2)$ has two connected components, distinguished by the parity of the spin structure. One can take a flat surface in ΩG (eg the hexagon form) and compute the winding numbers of a symplectic basis to prove that the Gothic locus lies in the component $\Omega\mathcal{M}_g^{\text{even}}(2,2,2)$ with even spin structure (see also the argument using θ_{null} in [16, Section 4]). We will however not use this fact when cutting out, in Section 6, the image of the Veech surfaces in their Prym varieties with theta functions.

The one-form ω obviously belongs to the tangent space to a three-dimensional subvariety of $\mathrm{Jac}(X)$, the complement of A, since ω is J–invariant. We can reduce the considerations to abelian surfaces, the complement of both A and B, thanks to the following observation:

Lemma 2.2 [16] *For $(X, \omega) \in \Omega G$ the π_B–pushforward is zero.*

Proof The differential $(\pi_B)_*(\omega)$ vanishes at e_4', since all the π_B–preimages of that point are zeros of ω, and this pushforward differential is holomorphic. On the elliptic curve B this implies $(\pi_B)_*(\omega) = 0$. □

2.2 Gothic Teichmüller curves: cathedrals and semiregular hexagons

The Gothic locus ΩG is "like $\Omega\mathcal{M}_2(2)$" in a precise sense: it is an affine invariant manifold of dimension four and rank two (in the sense of [26]). In this situation, the intersection with the locus where the Prym variety (as defined in detail in Section 3) has real multiplication by a quadratic field is a union of Teichmüller curves. That is, if we let

$$\Omega G_D = \{(X, \omega) \in \Omega G : \omega \text{ is an eigenform for real multiplication by } \mathcal{O}_D \text{ on } \mathrm{Prym}\, X\},$$

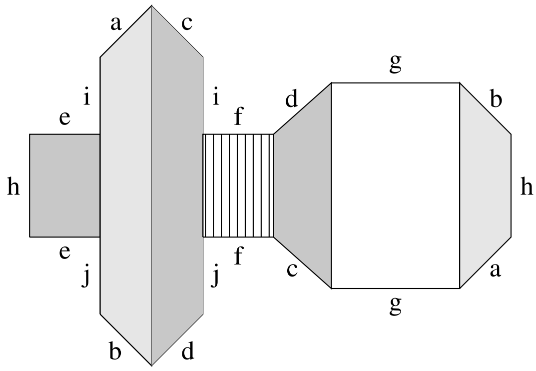


Figure 2: A cathedral-shaped surface in the Gothic locus.

the image $G_D \subset \mathcal{M}_4$ is a finite union of Teichmüller curves by [16, Theorem 1.7]. We give flat pictures of some of these Teichmüller curves.

The first flat picture is the Gothic cathedral, Figure 2. It was obtained in [16] by shearing jointly the light gray cylinders in Figure 1 (which preserves membership in ΩG) and a cut-and-paste operation until these light gray cylinders become the ones containing the sides e, f and g and the dark gray cylinders transform into the cylinders containing

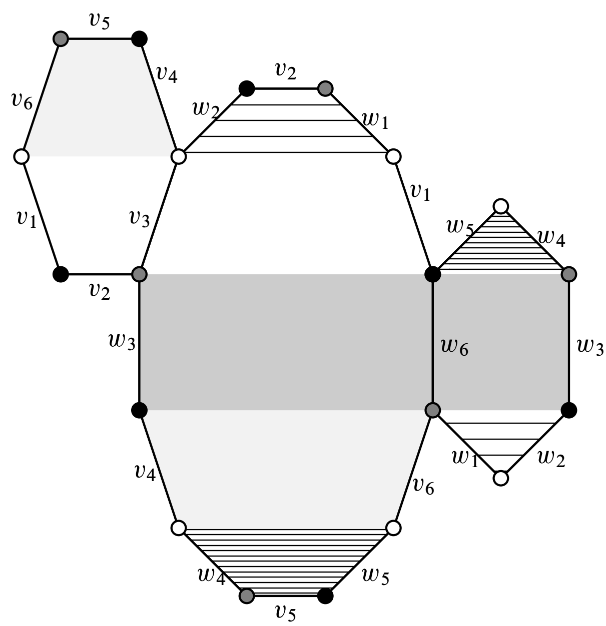


Figure 3: A duck-shaped surface in the Gothic locus

the sides i and j. After normalizing in the horizontal and vertical directions, one can furthermore assume that the periods have the form

$$\mathrm{a}=\mathrm{d}=\tfrac{1}{2}(1+i),\quad \mathrm{b}=\mathrm{c}=\tfrac{1}{2}(1-i),\quad \mathrm{h}=i,\quad \mathrm{e}=\mathrm{f}=\tfrac{1}{2}\mathrm{g}=\alpha,\quad \mathrm{i}=\mathrm{j}=\beta i$$

for some $\alpha,\beta\in\mathbb{R}$. For appropriate values of α and β the ratios of the moduli of all vertical and of all horizontal cylinders are commensurable and, therefore, the Veech group of the cathedral contains parabolic elements fixing the vertical and horizontal direction. In fact, this happens whenever $\alpha=x+y\sqrt{d}$ and $\beta=-3x-\frac{3}{2}+3y\sqrt{d}$ for $d>0$ and $x,y\in\mathbb{Q}$. The product of such parabolic elements is then hyperbolic and has quadratic trace field $\mathbb{Q}(\sqrt{d})$ and, consequently, Figure 2 generates a Teichmüller curve [16, Section 9].

A more precise computation shows that eg for $x=0$, $y=\frac{1}{2}$ and $d=2$ the period matrix of $\mathrm{Prym}(X,\pi_{\boldsymbol{A}},\pi_{\boldsymbol{B}})$ (see Section 3) is equivalent to

$$\Pi=\begin{pmatrix}-3\sqrt{2} & -3\sqrt{2}-1+3i & -3+3i & 3i\sqrt{2}-3i\\ 3\sqrt{2} & 3\sqrt{2}-1+3i & -3+3i & -3i\sqrt{2}-3i\end{pmatrix}.$$

This abelian variety admits real multiplication by $\mathcal{O}_{288}$, as can by seen by the analytic and rational representations

$$A_{\sqrt{288}/2}=\begin{pmatrix}\frac{\sqrt{288}}{2} & 0\\ 0 & -\frac{\sqrt{288}}{2}\end{pmatrix}\quad\text{and}\quad R_{\sqrt{288}/2}=\begin{pmatrix}18 & 11 & -3 & -9\\ -18 & -9 & 9 & 9\\ 18 & 15 & -3 & -3\\ 0 & 6 & 6 & -6\end{pmatrix},$$

ie the identity $A_{\sqrt{288}/2}\Pi=\Pi R_{\sqrt{288}/2}$ holds.

Alternatively, one can move the corners of the hexagon while maintaining the relations (2) and the surface becomes horizontally and vertically periodic with cylinders as in Figure 3. Concretely, we may take

$$\begin{aligned} &v_1=x+yi, \quad v_2=2x, \quad v_3=x-yi,\\ &w_1=1-i, \quad w_2=1+i, \quad w_3=2i \end{aligned} \tag{4}$$

for $x,y\in\mathbb{R}$.

Proposition 2.3 *Let $x,y\in\mathbb{Q}(\sqrt{d})$ be such that*

$$\frac{1+3x}{y(1+x)},\ \frac{x(y+3)}{y+1}\in\mathbb{Q}.$$

Then the flat surface in Figure 3 generates a Teichmüller curve in $G_{\boldsymbol{D}}$ for some D such that $\mathbb{Q}(\sqrt{D})=\mathbb{Q}(\sqrt{d})$.

Proof The coordinates in (4) are chosen so that the flat surface admits a horizontal and a vertical cylinder decomposition. The moduli of the horizontal cylinders are given by $(m_1, m_2) = (2(1+3x)/y, 2(1+x))$ while the moduli of vertical cylinders are $(m'_1, m'_2) = (2(y+1)/x, 2(y+3))$, the commensurability of which is given by the above conditions on x and y. □

It is amusing to note that a curve in this series of Teichmüller curves in G_D was in the literature long before the discovery of the whole series. The (irreducible) curve G_{12} will be our second running example. We recall the notation $\mathcal{T}(m,n)$ of Wright [25] for the Veech–Ward–Bouw–Möller curve generated by the unfolding of the (m,n,∞)–triangle (see also [24]), and the "semiregular polygons" decomposition of the corresponding Veech surface $(Y_{m,n}, \eta_{m,n})$ of Hooper [13].

Proposition 2.4 *The Teichmüller curve G_{12} agrees with the Veech–Ward–Bouw–Möller curve $\mathcal{T}(3,6)$. It is generated by the flat surface in Figure 3 with $x = \frac{\sqrt{3}}{3}$ and $y = -\sqrt{3}$, which agrees with the semiregular polygon decomposition of $(Y_{3,6}, \eta_{3,6})$ after scaling the axes by $\frac{3}{4}$ and $\frac{4}{\sqrt{3}}$. The Veech group of G_{12} is the triangle group $\Delta(3,6,\infty)$, hence $\chi(G_{12}) = -\frac{1}{2}$.*

Proof The equivalence of the flat presentation is a straightforward check using the notation conventions given in the references. To see that this example corresponds to discriminant $D = 12$ in the Gothic series it is enough to check that

$$\Pi = \begin{pmatrix} (18i-6)(\sqrt{3}+1)/(\sqrt{3}+3) & 12i(\sqrt{3}+1)/(\sqrt{3}+3) & 6i-6 & 4 \\ (18i-6)(\sqrt{3}-1)/(\sqrt{3}-3) & 12i(\sqrt{3}-1)/(\sqrt{3}-3) & 6i-6 & 4 \end{pmatrix}$$

gives the period matrix of the corresponding Prym variety $\operatorname{Prym}(X, \pi_A, \pi_B)$ (see Section 3) and it admits real multiplication by $\mathcal{O}_{12}$ defined by the analytic and rational representation

$$A_{\sqrt{12}/2} = \begin{pmatrix} \frac{\sqrt{12}}{2} & 0 \\ 0 & -\frac{\sqrt{12}}{2} \end{pmatrix} \quad \text{and} \quad R_{\sqrt{12}/2} = \begin{pmatrix} 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 3 \\ 3 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix}.$$ □

3 Prym varieties for two maps

Given a finite collection of maps $\pi_i \colon X \to Y_i$ between curves, the Prym variety $\operatorname{Prym}(X, \pi_1, \ldots, \pi_n)$ (in a generalized sense) is the complementary abelian variety to the image of the maps π_i^*: Jac $Y_i \to$ Jac X, that is, the perpendicular space to the tangent spaces $\bigoplus \Omega_{Y_i}^\vee$ divided by its intersection with the period lattice. The main goal of this section is to determine the signature of the polarization on this Prym variety $\mathrm{Prym}(X) = \mathrm{Prym}(X, \pi_A, \pi_B)$ in the case of a Gothic flat surface.

Proposition 3.1 *The restriction of the principal polarization on* $\mathrm{Jac}(X)$ *is a polarization on* $\mathrm{Prym}\, X$ *of type* $(1,6)$. *Consequently, the dual Prym variety* $\mathrm{Prym}^\vee X$ *has a natural polarization of type* $(1,6)$, *too.*

We first recall an equivalent definition of complementary abelian subvarieties in terms of endomorphisms. Let $(T, \mathcal{L})$ be an abelian variety, that is, a complex torus $T = V/\Lambda$ together with a positive-definite line bundle $\mathcal{L}$. Given an abelian subvariety ι: $Y \to T$, one can define its exponent e_Y as the exponent $e(\mathcal{L})$ of the induced polarization $\iota^*\mathcal{L}$, and its norm endomorphism $N_Y \in \mathrm{End}(T)$ and symmetric idempotent $\varepsilon_Y \in \mathrm{End}_{\mathbb{Q}}(T)$ as

$$N_Y := \iota\psi_{\iota^*\mathcal{L}}\check{\iota}\phi_{\mathcal{L}} \quad \text{and} \quad \varepsilon_Y := \frac{1}{e_Y} N_Y, \tag{5}$$

respectively, where $\phi_{\mathcal{L}}$: $T \to T^\vee$ is the isogeny associated to a line bundle $\mathcal{L}$ and $\psi_{\mathcal{L}} = e(\mathcal{L})\phi_{\mathcal{L}}^{-1}$ (see [3, Section 5.3]). In the case of $\mathcal{L}$ being a principal polarization, the exponent of Y is precisely $e_Y = \min\{n > 0 : n\varepsilon_Y \in \mathrm{End}(T)\}$ (see Proposition 12.1.1 of [3]).

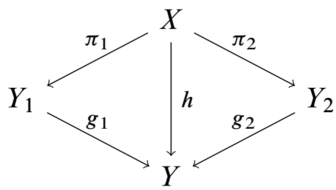
The assignment $Y \mapsto \varepsilon_Y$ and its inverse $\varepsilon \mapsto X^\varepsilon := \mathrm{im}(n\epsilon)$, for any $n > 0$ such that $n\varepsilon \in \mathrm{End}(T)$, induce a bijection between the set of abelian subvarieties of T and the set of symmetric (with respect to the Rosati involution $f \mapsto f' = \phi_{\mathcal{L}}^{-1} \hat{f} \phi_{\mathcal{L}}$) idempotents in $\mathrm{End}_{\mathbb{Q}}(T)$. Accordingly, the canonical involution $\varepsilon \mapsto 1 - \varepsilon$ on the set of symmetric idempotents induces an involution $Y \mapsto Z := X^{1-\varepsilon_Y}$ on the set of abelian subvarieties. The abelian subvariety Z is called the *complementary abelian subvariety* of Y, and the exponent e_Z agrees with e_Y in the case of $\mathcal{L}$ being a principal polarization. The map (N_Y, N_Z): $X \to Y \times Z$ is an isogeny and the identities

$$N_Y|_Y = e_Y \,\mathrm{Id}, \quad N_Y|_Z = 0 = N_Y N_Z = 0, \quad e_Y N_Z + e_Z N_Y = e_Y e_Z \,\mathrm{Id}$$

hold [3, Section 5.3].

Now let π: $X \to Y$ be a morphism between curves. The pullback map defines a homomorphism π^*: Jac $Y \to$ Jac X. This map is, moreover, injective whenever π does not factor through a cyclic étale cover of degree ≥ 2. Under these conditions, the Prym variety $\mathrm{Prym}(X, \pi)$ of the map π is defined as the complementary abelian variety of $\pi^*(\operatorname{Jac} Y)$ (or, equivalently, as the connected component of the identity of the kernel $\ker N_{\pi^*(\operatorname{Jac} Y)}$). The Jacobian of X decomposes, up to isogeny, as $\operatorname{Jac} X \sim \pi^*(\operatorname{Jac} Y) \times \operatorname{Prym}(X, \pi)$. Note that, in general, $\operatorname{Prym}(X, \pi)$ is not a Prym variety in the classical sense (see [3, Section 12]), as the induced polarization will not be a multiple of the principal polarization.

Consider now a pair of morphisms of curves $\pi_1\colon X \to Y_1$ and $\pi_2\colon X \to Y_2$, together with the corresponding homomorphisms $\pi_1^*\colon \operatorname{Jac} Y_1 \to \operatorname{Jac} X$ and $\pi_2^*\colon \operatorname{Jac} Y_2 \to \operatorname{Jac} X$. Assume moreover that there exist morphisms $g_1\colon Y_1 \to Y$ and $g_2\colon Y_2 \to Y$ to some curve Y such that the diagram



commutes. Under a mild nonfactorization condition, one can decompose $\operatorname{Jac} X$ further in terms of Jacobians.

Proposition 3.2 [14] *Suppose g_1 and g_2 do not both factorize via the same morphism $Y_0 \to Y$ of degree ≥ 2. Then $\pi_2^* \operatorname{Prym}(Y_2, g_2)$ is an abelian subvariety of* $\operatorname{Prym}(X, \pi_1)$. *In particular,* $\operatorname{Jac} X$ *decomposes, up to isogeny, as*

$$\operatorname{Jac} X \sim h^*(\operatorname{Jac} Y) \times \pi_1^* \operatorname{Prym}(Y_1, g_1) \times \pi_2^* \operatorname{Prym}(Y_2, g_2) \times P$$

for some subvariety P of $\operatorname{Jac} X$.

The subvariety P is called the *Prym variety* $\operatorname{Prym}(X, \pi_1, \pi_2)$ of the pair of coverings (π_1, π_2). In the case that $Y = \mathbb{P}^1$, the summand $h^*(\operatorname{Jac} Y)$ is of course trivial and $\operatorname{Prym}(Y_j, g_j) = \operatorname{Jac} Y_j$.

We now specialize to the Gothic situation and give explicitly the various norm endomorphisms for later use. We write $A^\vee$ and $B^\vee$ for the image of π_A^* and π_B^*, respectively.

Proposition 3.3 *Let T be a principally polarized abelian variety and $A^\vee, B^\vee \subset T$ be abelian subvarieties with coprime exponents e_A and e_B and such that $N_A N_B = 0$. Then $Y = A^\vee \times B^\vee$ is a subvariety of T. Moreover, the norm endomorphisms of Y and its complementary abelian variety P satisfy*

$$N_Y = e_B N_A + e_A N_B \quad \text{and} \quad N_P = e_A e_B \operatorname{Id} - N_Y.$$

Proof Injectivity of $Y \to T$ follows from coprimality. Writing $N = e_B N_A + e_A N_B$, one has $N^2 = e_A e_B N$ and $N|_Y = e_A e_B \operatorname{Id}_Y$. The idempotent $\varepsilon = (1/e_A e_B) N$ corresponds to the abelian subvariety Y and it is of exponent $e_A e_B$ since $e_Y = \min\{n > 0 : n\varepsilon_Y \in \operatorname{End}(T)\}$ and $(e_A, e_B) = 1$. The rest of the claims follow. □

Proof of Proposition 3.1 Thanks to diagram (1) and since $\gcd(2,3) = 1$ the hypothesis of Proposition 3.2 is met. Moreover, $A^\vee \times B^\vee$ has a polarization of type $(1,6)$ and, by [3, Corollary 12.1.5], the same holds for the complementary abelian variety. □

4 Hilbert modular surfaces and modular embeddings

The Prym–Torelli map t associates with a flat surfaces in the Gothic locus, or more generally with any genus four surface admitting maps π_A and π_B that fit into the diagram (1), the dual Prym variety $\operatorname{Prym}^\vee(X, \pi_A, \pi_B)$. (The reason for dualizing will become apparent in Section 6.) By Proposition 3.1 this gives a map $t\colon \Omega G \to \mathcal{A}_{2,(1,6)}$ to the moduli space of $(1,6)$–polarized abelian surfaces. The goal of this section is to recall some basic properties of Hilbert modular surfaces that arise from the following observation:

Proposition 4.1 *The Prym–Torelli image $t(G_D)$ of the Gothic locus is contained in the image of a Hilbert modular surface $X_D(\mathfrak{b})$ inside the moduli space of $(1,6)$–polarized abelian surfaces, where $\mathfrak{b}$ is an $\mathcal{O}_D$–ideal of norm 6.*

We compute here the Euler characteristics of these Hilbert modular surfaces $X_D(\mathfrak{b})$ and discuss the modular embeddings that induce the map $X_D(\mathfrak{b}) \to \mathcal{A}_{2,(1,6)}$.

4.1 Hilbert modular surfaces

For any positive discriminant $D \equiv 0, 1 \bmod 4$, write $D = b^2 - 4ac$ for suitable $a, b, c \in \mathbb{Z}$. The (unique) *quadratic order of discriminant* D is defined as $\mathcal{O}_D = \mathbb{Z}[T]/(aT^2 + bT + c)$. This order agrees with $\mathcal{O}_D = \mathbb{Z} \oplus \gamma_D \mathbb{Z}$ inside the quadratic field $K = \mathbb{Q}(\sqrt{D})$, where $\gamma := \gamma_D = \frac{1}{2}(D + \sqrt{D})$, provided that D is not a square.

For any fractional ideal $\mathfrak{c} \subset K$, we denote by $\mathfrak{c}^\vee$ the dual with respect to the trace pairing, ie $\mathfrak{c}^\vee = \{x \in K : \operatorname{tr}_{\mathbb{Q}}^K(x\mathfrak{c}) \subset \mathbb{Z}\}$. In particular, $\mathcal{O}_D^\vee = \frac{1}{\sqrt{D}} \mathcal{O}_D$.

Let $\mathfrak{b}$ be an $\mathcal{O}_D$–ideal. The $\mathcal{O}_D$–module $\mathfrak{b} \oplus \mathcal{O}_D^\vee$ is preserved by the *Hilbert modular group*

$$\operatorname{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee) = \begin{pmatrix} \mathcal{O}_D & \sqrt{D}\,\mathfrak{b} \\ \frac{1}{\sqrt{D}} \mathfrak{b}^{-1} & \mathcal{O}_D \end{pmatrix} \cap \operatorname{SL}_2(K).$$

Associated with $\mathfrak{b}$ we can construct the *Hilbert modular surface*

$$X_D(\mathfrak{b}) = \mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee)\backslash\mathbb{H}^2.$$

4.2 Abelian surfaces with real multiplication and a $(1, n)$–polarization

An abelian surface T admits *real multiplication* by $\mathcal{O}_D$ if there exists an embedding $\mathcal{O}_D \hookrightarrow \mathrm{End}(T)$ by self-adjoint endomorphisms. We will always assume that the action is *proper*, in the sense that it cannot be extended to an action of a larger quadratic discriminant $\mathcal{O}_E \supset \mathcal{O}_D$.

The different components of the moduli space of $(1, n)$–polarized abelian varieties with a choice of real multiplication by $\mathcal{O}_D$ are parametrized by certain Hilbert modular surfaces (see [12, Chapter 7]).

More precisely, suppose that $(T = \mathbb{C}^2/\Lambda, \mathcal{L})$ is an abelian variety with a $(1, n)$–polarization $\mathcal{L}$ and a choice of real multiplication by $\mathcal{O}_D$. Then Λ is a rank-two $\mathcal{O}_D$–module with symplectic pairing of signature $(1, n)$. By [2] such a lattice splits as a direct sum of $\mathcal{O}_D$–modules. Moreover, although $\mathcal{O}_D$ is not a Dedekind domain for nonfundamental discriminants D, any rank-two $\mathcal{O}_D$–module is isomorphic to $\mathfrak{b} \oplus \mathcal{O}_D^\vee$ for some $\mathcal{O}_D$–ideal $\mathfrak{b}$. The isomorphism can moreover be chosen so that the symplectic form is mapped to the trace pairing $\langle (a, b)^T, (\tilde{a}, \tilde{b})^T \rangle = \mathrm{tr}_{\mathbb{Q}}^K(a\tilde{b} - \tilde{a}b)$. The type of such a polarization is (d_1, d_2), where $d_i \in \mathbb{N}$ are uniquely determined by $d_1 \mid d_2$ and $\mathcal{O}_D/\mathfrak{b} \cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z}$.

In the case of a polarization of type $(1, n)$, it follows (see for example Proposition 5.2.1 of [5]) that the ideal $\mathfrak{b}$ can be generated as a $\mathbb{Z}$–module by $\left(\frac{1}{2}(r + \sqrt{D}), n\right)$ for some $0 \le r < 2n$. In particular, $N_{\mathbb{Q}}^K(\mathfrak{b}) = n$.

Conversely, for any ideal $\mathfrak{b}$ of norm n and $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathbb{H}^2$, we define the lattice

$$\Lambda_{\mathfrak{b},\boldsymbol{\tau}} = \{(a + b\tau_1, a^\sigma + b^\sigma \tau_2)^T \mid a \in \mathfrak{b},\ b \in \mathcal{O}^\vee\}.$$

The quotient $T_{\boldsymbol{\tau}} = \mathbb{C}^2/\Lambda_{\mathfrak{b},\boldsymbol{\tau}}$ is an abelian surface with a $(1, n)$–polarization (given by the trace pairing) and real multiplication by $\mathcal{O}_D$. The isomorphism class of $T_{\boldsymbol{\tau}}$ depends only on the image of $\boldsymbol{\tau}$ in $X_D(\mathfrak{b})$.

The proof of Proposition 4.1 follows from this observation and the real multiplication built into the definition of the Gothic curves G_D.

It also follows that the locus of $(1, n)$–polarized abelian varieties with a choice of real multiplication by $\mathcal{O}_D$ has as many components as ideals $\mathfrak{b}$ of norm n in $\mathcal{O}_D$, each of these components being parametrized by the Hilbert modular surface $X_D(\mathfrak{b})$. Concretely, for the case we are interested in:

Proposition 4.2 *The moduli space of $(1,6)$–polarized abelian surfaces with a choice of real multiplication by $\mathcal{O}_D$ is empty for $D \equiv 5 \bmod 8$ or $D \equiv 2 \bmod 3$.*

It is nonempty and irreducible for $D \equiv 0, 12 \bmod 24$, it has two irreducible components for $D \equiv 4, 9, 16 \bmod 24$ and four for $D \equiv 1 \bmod 24$.

Proof By the preceding discussion, the locus of $(1,6)$–polarized abelian varieties with a choice of real multiplication by $\mathcal{O}_D$ is nonempty if and only if there is an $\mathcal{O}_D$–ideal $\mathfrak{b}$ with $N_{\mathbb{Q}}^{K}(\mathfrak{b}) = 6$, ie if and only if $D \equiv 0, 1, 4, 9, 12, 16 \bmod 24$.

Each connected component of this locus is parametrized by a Hilbert modular surface $X_D(\mathfrak{b})$ for an $\mathcal{O}_D$–ideal $\mathfrak{b}$ of norm 6. For $D \equiv 0, 12 \bmod 8$ there is exactly one prime ideal $\mathfrak{b}_2$ of norm 2 and one prime ideal $\mathfrak{b}_3$ of norm 3, so that the locus is connected. For $D \equiv 9 \bmod 24$ the prime 2 splits (but 3 is ramified) and for $D \equiv 4, 16 \bmod 24$ the prime 3 splits (but 2 is ramified), resulting in two connected components. For $D \equiv 1 \bmod 24$ both primes split. □

Note, however, that the locus of real multiplication in $\mathcal{A}_{2,(1,6)}$ has in general fewer components than the moduli space of abelian surfaces with a chosen real multiplication by $\mathcal{O}_D$. In fact, the abelian varieties parametrized by $X_D(\mathfrak{b})$ and by $X_D(\mathfrak{b}^\sigma)$ map to the same subsurface in $\mathcal{A}_{2,(1,6)}$.

4.3 Euler characteristics

The notion of Euler characteristic (of curves and of Hilbert modular surfaces) refers throughout to orbifold Euler characteristics. Let $D = f^2 D_0$ be the factorization of the discriminant into a fundamental discriminant D_0 and a square of $f \in \mathbb{N}$. The Euler characteristic of Hilbert modular surfaces has been computed by Siegel [23] for the more usual Hilbert modular surface $X_D = X_D(\mathcal{O}_D)$. A reference including also the case of nonfundamental discriminants is [1, Theorem 2.12]. Altogether,

$$\chi(X_D) = 2 f^3 \zeta_{\mathbb{Q}(\sqrt{D})}(-1) \biggl(\sum_{r \mid f} \left(\frac{D_0}{r} \right) \frac{\mu(r)}{r^2} \biggr),$$

where μ is the Möbius function and $\left(\frac{a}{b}\right)$ is the Jacobi symbol. The case we are interested in can be deduced from this formula.

Proposition 4.3 *The Euler characteristics of $X_D(\mathfrak{b})$ for $\mathfrak{b}$ of norm 6 and of X_D are related by*

$$\kappa_D := \frac{\chi(X_D(\mathfrak{b}))}{\chi(X_D)} = \begin{cases} 1 & \text{if } \gcd(6,f)=1, \\ \frac{3}{2} & \text{if } \gcd(6,f)=2, \\ \frac{4}{3} & \text{if } \gcd(6,f)=3, \\ 2 & \text{if } \gcd(6,f)=6. \end{cases}$$

Proof The groups $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ and $\mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee)$ are commensurable. To determine the indices in their intersection, we conjugate both groups by $\left(\begin{smallmatrix}\sqrt{D} & 0 \\ 0 & 1\end{smallmatrix}\right)$. This takes the first group into $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D)$ and the second group into $\mathrm{SL}_2(\mathfrak{b} \oplus \mathcal{O}_D)$. The two images under conjugation contain

$$\Gamma_{\mathfrak{b}} = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(K) : a,d \in \mathcal{O}_D,\ b \in \mathfrak{b},\ c \in \mathcal{O}_D\right\},$$

with a finite index that we now calculate. We factorize $\mathfrak{b} = \mathfrak{p}_2\mathfrak{p}_3$ into the primes of norm 2 and 3 and consider the action of $\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D)$ on $\mathbb{P}^1(\mathcal{O}_D/\mathfrak{p}_2) \times \mathbb{P}^1(\mathcal{O}_D/\mathfrak{p}_3)$. This action is transitive, in fact elementary matrices in $\mathrm{SL}_2(\mathcal{O}_D/\mathfrak{p}_2)$ and $\mathrm{SL}_2(\mathcal{O}_D/\mathfrak{p}_3)$ generate a transitive group and elementary matrices can obviously be lifted. Since $\Gamma_{\mathfrak{b}}$ is precisely the stabilizer of $((0:1),(0:1))$, we conclude

$$[\mathrm{SL}(\mathcal{O}_D \oplus \mathcal{O}_D) : \Gamma_{\mathfrak{b}}] = |\mathbb{P}^1(\mathcal{O}_D/\mathfrak{p}_2) \times \mathbb{P}^1(\mathcal{O}_D/\mathfrak{p}_3)| = 12.$$

If $\mathfrak{b}$ is an invertible ideal, we use $\mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D) = \mathrm{SL}(\mathcal{O}_D \oplus \mathfrak{b}^{-1})$ and consider the projection

$$\mathrm{pr}\colon \mathrm{SL}(\mathcal{O}_D \oplus \mathfrak{b}^{-1}) \to \mathrm{SL}(\mathcal{O}_D/\mathfrak{p}_3 \oplus \mathfrak{b}^{-1}/\mathfrak{p}_2^{-1}) \times \mathrm{SL}(\mathcal{O}_D/\mathfrak{p}_2 \oplus \mathfrak{b}^{-1}/\mathfrak{p}_3^{-1}),$$

where in the range the modules are considered as an $\mathcal{O}_D/\mathfrak{p}_3$–module and an $\mathcal{O}_D/\mathfrak{p}_2$–module (ie as vector spaces), respectively. Even the smaller group $\Gamma_{\mathfrak{b}}$ contains the kernel of pr, and in fact $\Gamma_{\mathfrak{b}}$ is precisely the stabilizer of $((0:1),(0:1))$. We conclude that its index is 12 in $\mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D)$ and this completes the case $\gcd(6,f)=1$.

If $\gcd(6,f)=2$, we use that $\mathfrak{p}_2^{-1} = \mathcal{O}_{D/4}$ and consider $\mathrm{SL}(\mathcal{O}_D \oplus \mathfrak{b}^{-1})$ as a subgroup of $\mathrm{SL}(\mathcal{O}_{D/4} \oplus \widetilde{\mathfrak{p}}_3^{-1})$, where the tilde indicates that we now extended scalars of $\mathfrak{p}_3^{-1}$ to form an $\mathcal{O}_{D/4}$–module. We consider the projection

$$\mathrm{pr}\colon \mathrm{SL}(\mathcal{O}_{D/4} \oplus \widetilde{\mathfrak{p}}_3^{-1}) \to \mathrm{SL}(\mathcal{O}_{D/4}/\widetilde{\mathfrak{p}}_3 \oplus \widetilde{\mathfrak{p}}_3^{-1}/\mathcal{O}_{D/4}) \times \mathrm{SL}(\mathcal{O}_{D/4}/\mathfrak{p}_2 \oplus \widetilde{\mathfrak{p}}_3^{-1}/\mathfrak{p}_2\widetilde{\mathfrak{p}}_3^{-1}).$$

Again, even the smaller group $\Gamma_{\mathfrak{b}}$ contains the kernel of pr. The image of $\mathrm{SL}(\mathcal{O}_D \oplus \mathfrak{b}^{-1})$ under pr is contained in the full first factor times the lower-triangular matrices in the second factor, as can be checked using a set of generators for these groups consisting of elementary matrices. The image of $\Gamma_{\mathfrak{b}}$ under pr is the stabilizer of $(0:1)$ in the first factor times the lower-triangular matrices with $\mathcal{O}_D/\mathfrak{p}_2 \subset \mathcal{O}_{D/4}/\mathfrak{p}_2 \cong \tilde{\mathfrak{p}}_3^{-1}/\mathfrak{p}_2\tilde{\mathfrak{p}}_3^{-1}$ in the lower-left corner in the second factor. This subgroup is of index $4 \cdot 2 = 8$ and this concludes the case $\gcd(6, f) = 2$.

The remaining cases are similar, using $\mathfrak{p}_3^{-1} = \mathcal{O}_{D/9}$ if $\gcd(6, f) = 3$ and $\mathfrak{b}^{-1} = \mathcal{O}_{D/36}$ if $\gcd(6, f) = 6$. □

4.4 Siegel modular embeddings

Let $X_D(\mathfrak{b})$ parametrize a component of the moduli space of $(1, n)$–polarized abelian varieties with a choice of real multiplication by $\mathcal{O}_D$ as above. The forgetful map $X_D(\mathfrak{b}) \to \mathcal{A}_{2,(d_1,d_2)}$ to the moduli space of (d_1, d_2)–polarized abelian varieties can be lifted to a holomorphic map $\psi\colon \mathbb{H}^2 \to \mathbb{H}_2$ which is equivariant with respect to a homomorphism $\Psi\colon \mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee) \to G_P$, where $P := P_{d_1,d_2} = \left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)$ and G_P is the symplectic group for the polarization type (d_1, d_2) (see [3, Section 8.2]),

$$G_P = \left\{ M \in \mathrm{Sp}_4(\mathbb{Q}) : M^T \left(\begin{smallmatrix} I_2 & 0 \\ 0 & P \end{smallmatrix}\right) \mathbb{Z}^4 \subseteq \left(\begin{smallmatrix} I_2 & 0 \\ 0 & P \end{smallmatrix}\right) \mathbb{Z}^4 \right\}.$$

Such a lift (ψ, Ψ) is called a *Siegel modular embedding*, and will be used to pull back classical theta functions, given in standard coordinates on the universal family over $\mathbb{H}_2/G_P$, to $X_D(\mathfrak{b})$. We note in passing that there are two useful conventions for symplectic groups in the case of nonprincipal polarizations. The other symplectic group

$$\mathrm{Sp}_{2g}^P(\mathbb{Z}) = \left\{ M \in \mathbb{Z}^{2g \times 2g} : M \cdot \left(\begin{smallmatrix} 0 & P \\ -P & 0 \end{smallmatrix}\right) \cdot M^T = \left(\begin{smallmatrix} 0 & P \\ -P & 0 \end{smallmatrix}\right) \right\}$$

is convenient, since it has integral entries. Conjugation by $\left(\begin{smallmatrix} I_2 & 0 \\ 0 & P \end{smallmatrix}\right)$ takes $\mathrm{Sp}_{2g}^P(\mathbb{Z})$ into G_p. Whereas the action of G_P is the standard action, the group $\mathrm{Sp}_{2g}^P(\mathbb{Z})$ acts on $\mathbb{H}_2$ by

$$Z \mapsto (AZ + BP)(P^{-1}CZ + P^{-1}DP)^{-1} \quad \text{for } M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}^P(\mathbb{Z}).$$

In order to construct Siegel modular embeddings, one needs to find an appropriate $\mathbb{Z}$–basis of $\mathfrak{b} \oplus \mathcal{O}_D^\vee$. Let $\mathfrak{c}$ be any fractional $\mathcal{O}_D$–ideal and $\boldsymbol{\eta} = (\eta_1, \eta_2)$ an ordered basis of $\mathfrak{c}^\vee$, and define the matrices

$$B = B_{\boldsymbol{\eta}} = \begin{pmatrix} \eta_1 & \eta_1^\sigma \\ \eta_2 & \eta_2^\sigma \end{pmatrix} \quad \text{and} \quad C = (B_{\boldsymbol{\eta}}^{-1} P)^T = \begin{pmatrix} \nu_1 & \nu_1^\sigma \\ \nu_2 & \nu_2^\sigma \end{pmatrix}. \tag{6}$$

We say that (η_1,η_2) is a basis *symplectically adapted to* P (or a (d_1,d_2)*–symplectically adapted basis*) if (ν_1,ν_2) is the basis of an $\mathcal{O}_D$–ideal. In this case we may factor the ideal as $\mathfrak{c}\mathfrak{b}$, where $\mathfrak{b}$ is necessarily an ideal of norm $n=d_1\cdot d_2$. Accordingly, the basis $\boldsymbol{\eta}$ determines the rank-2 $\mathcal{O}_D$–module $\mathfrak{c}\mathfrak{b}\oplus\mathfrak{c}^\vee$ that, provided with the trace pairing, becomes a (d_1,d_2)–polarized module with symplectic basis

$$(\nu_1,0),\quad (\nu_2,0),\quad (0,\eta_1),\quad (0,\eta_2).$$

We do not necessarily assume $d_1\mid d_2$ here.

To give an example in the particular case of $\mathfrak{c}=\mathcal{O}_D$ and $\mathfrak{b}=\langle\frac{1}{2}(r+\sqrt{D}),n\rangle$ an ideal of norm n, we can always use the basis $\boldsymbol{\eta}=\frac{1}{\sqrt{D}}\langle 1,\frac{1}{2}(-r+\sqrt{D})\rangle$ of $\mathcal{O}_D^\vee$, which is $(1,n)$–symplectically adapted to $P_{1,n}$ and such that the first column of $(B_{\boldsymbol{\eta}}^{-1}P_{1,n})^T$ agrees with the given basis of $\mathfrak{b}$.

The period matrix for $T_{\boldsymbol{\tau}}=\mathbb{C}^2/\Lambda_{\mathfrak{b},\boldsymbol{\tau}}$ with respect to eigenforms for the $\mathcal{O}_D$–action becomes

$$\Pi_{\boldsymbol{u}}=\left(\begin{pmatrix}\tau_1&0\\0&\tau_2\end{pmatrix}\cdot B_{\boldsymbol{\eta}}^T\;\middle|\;C^T\right).$$

We refer to the corresponding coordinates of $\mathbb{C}^2$ as *eigenform coordinates* $\boldsymbol{u}=(u_1,u_2)$. By multiplying on the left by $B_{\boldsymbol{\eta}}$, one gets the period matrix in *standard coordinates* $\boldsymbol{v}=B_{\boldsymbol{\eta}}\cdot\boldsymbol{u}$,

$$\Pi_{\boldsymbol{v}}=\left(\Omega_{\boldsymbol{\tau}}\;\middle|\;\begin{matrix}1&0\\0&n\end{matrix}\right),\quad\text{where } \Omega_{\boldsymbol{\tau}}=B_{\boldsymbol{\eta}}\cdot\begin{pmatrix}\tau_1&0\\0&\tau_2\end{pmatrix}\cdot B_{\boldsymbol{\eta}}^T\in\mathbb{H}_2.$$

Let us remark that, with the notation of Section 5.6, one can assume that the columns of $\Pi_{\boldsymbol{v}}$ correspond to the lattice vectors λ_1, λ_2, μ_1 and μ_2, respectively.

We claim that the following is a well-defined homomorphism:

$$\text{(7)}\quad \Psi\colon \mathrm{SL}(\mathfrak{b}\oplus\mathcal{O}_D^\vee)\to G_P,\quad \delta=\begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto\begin{pmatrix}B_{\boldsymbol{\eta}}&0\\0&B_{\boldsymbol{\eta}}^{-T}\end{pmatrix}\begin{pmatrix}\hat a&\hat b\\\hat c&\hat d\end{pmatrix}\begin{pmatrix}B_{\boldsymbol{\eta}}^{-1}&0\\0&B_{\boldsymbol{\eta}}^T\end{pmatrix}.$$

Here we denote by $\hat k$ the matrix $\left(\begin{smallmatrix}k&0\\0&k^\sigma\end{smallmatrix}\right)$ for $k\in K$. The claim can be easily checked by studying the action on integral column vectors of the four blocks forming $\Psi(\delta)$.

It is clear that (ψ,Ψ) defined by $\psi\colon\mathbb{H}^2\to\mathbb{H}_2$, $\boldsymbol{\tau}=(\tau_1,\tau_2)\mapsto\Omega_{\boldsymbol{\tau}}$, and Ψ as above induces the forgetful map $X_D(\mathfrak{b})\to\mathcal{A}_{2,(1,n)}$ and is therefore a Siegel modular embedding.

We finish this section with a criterion for some specific bases to be symplectically adapted. Recall that to a given triple of integers $Q=(a,b,c)$ such that $D=b^2-4ac$, one can associate the fractional ideal $\mathfrak{a}^\vee = \langle 1, \lambda_Q \rangle$ of $\mathcal{O}_D$, where $\lambda_Q = \frac{1}{2a}(-b+\sqrt{D})$ is the quadratic irrationality of Q.

Lemma 4.4 *Let (d_1, d_2) be the type of a polarization such that* $\gcd(d_1, d_2) = 1$.

The basis $(1, \lambda_Q)$ is a (d_1, d_2)–symplectically adapted basis of $\mathfrak{a}^\vee$ if and only if $a \equiv 0 \bmod d_1$ and $c \equiv 0 \bmod d_2$. Moreover, $\mathfrak{a}\mathfrak{b} = \frac{a}{\sqrt{D}}\langle d_2, -d_1\lambda^\sigma \rangle$.

Note that the choice of the type (d_1, d_2) of the polarization does not follow the usual convention $d_1 \mid d_2$ except in the case $d_1 = 1$.

Proof Let

$$B = \begin{pmatrix} 1 & 1 \\ \frac{1}{2a}(-b+\sqrt{D}) & \frac{1}{2a}(-b-\sqrt{D}) \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

The basis $(1, \lambda_Q)$ is a (d_1, d_2)–symplectically adapted basis if and only if the columns of $(B^{-1}P)^T$ generate an ideal. This is equivalent to the existence of an integral matrix R satisfying

$$\begin{pmatrix} \frac{1}{2}(D+\sqrt{D}) & 0 \\ 0 & \frac{1}{2}(D-\sqrt{D}) \end{pmatrix} B^{-1}P = B^{-1}PR.$$

Now a simple calculation shows that

$$P^{-1}B \begin{pmatrix} \frac{1}{2}(D+\sqrt{D}) & 0 \\ 0 & \frac{1}{2}(D-\sqrt{D}) \end{pmatrix} B^{-1}P = \begin{pmatrix} \frac{1}{2}(D+b) & ad_2/d_1 \\ -cd_1/d_2 & \frac{1}{2}(D-b) \end{pmatrix}.$$

Since $b \equiv D \bmod 2$, the claim follows. The generators of $\mathfrak{a}\mathfrak{b}$ correspond to the first column of the matrix $(B^{-1}P)^T$. □

4.5 Cusps of $X_D(\mathfrak{b})$

Cusps of $X_D(\mathfrak{b})$ are orbits of $\mathbb{P}^1(K)$ under the action of $\mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee)$. Via the map $(\alpha : \beta) \mapsto \mathfrak{a} = \alpha\mathcal{O}_D + \beta\sqrt{D}\mathfrak{b}^{-1}$, they correspond to ideal classes of invertible $\mathcal{O}_D$–ideals $\mathfrak{a}$ (see [10, Section I.4]). In order to study the behavior of modular forms around the different cusps and to avoid the problem of changing coordinates in $\mathrm{SL}(\mathfrak{b}\oplus\mathcal{O}_D^\vee)\backslash\mathbb{H}^2$, one can instead change the Hilbert modular surface in the following way.

Let $\mathfrak{a}$ be an invertible $\mathcal{O}_D$–ideal. The trace pairing defined in the previous subsection induces again a symplectic pairing of type $(1, n)$ on the "shifted" $\mathcal{O}_D$–module $\mathfrak{a}\mathfrak{b} \oplus \mathfrak{a}^\vee$.

In particular, one can define a lattice $\Lambda^{\mathfrak{a}}_{\mathfrak{b},\boldsymbol{\tau}}$ for each $\boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathbb{H}^2$ as above and the Hilbert modular surface

$$X^{\mathfrak{a}}_D := X^{\mathfrak{a}}_D(\mathfrak{b}) = \mathrm{SL}(\mathfrak{a}\mathfrak{b} \oplus \mathfrak{a}^\vee)\backslash\mathbb{H}^2,$$

where

$$\mathrm{SL}(\mathfrak{a}\mathfrak{b} \oplus \mathfrak{a}^\vee) = \begin{pmatrix} \mathcal{O}_D & \sqrt{D}\,\mathfrak{a}^2\mathfrak{b} \\ \frac{1}{\sqrt{D}}\,\mathfrak{a}^{-2}\mathfrak{b}^{-1} & \mathcal{O}_D \end{pmatrix} \cap \mathrm{SL}_2(K),$$

parametrizes $(1,n)$–polarized abelian surfaces with a choice of real multiplication by $\mathcal{O}_D$ too. In fact, for any element

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \begin{pmatrix} \mathfrak{a} & \sqrt{D}\,\mathfrak{a}\mathfrak{b} \\ \frac{1}{\sqrt{D}}\,(\mathfrak{a}\mathfrak{b})^{-1} & \mathfrak{a}^{-1} \end{pmatrix} \cap \mathrm{SL}_2(K), \tag{8}$$

the map

$$\phi\colon \mathbb{H}^2 \to \mathbb{H}^2, \quad (\tau_1, \tau_2) \mapsto (M\tau_1, M^\sigma \tau_2),$$

is equivariant with respect to the action of $U \in \mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee)$ on its domain and $MUM^{-1} \in \mathrm{SL}(\mathfrak{a}\mathfrak{b} \oplus \mathfrak{a}^\vee)$ on its range. Via the map ϕ, the cusp of $X_D(\mathfrak{b})$ corresponding to $\mathfrak{a}$ is sent to the cusp at infinity of $X^{\mathfrak{a}}_D(\mathfrak{b})$.

The matrices defined in the last section for the usual Hilbert modular group can be changed accordingly. Let $\boldsymbol{\xi} = (\xi_1, \xi_2)$ now be an ordered basis of $\mathfrak{a}^\vee_D$ that is symplectically adapted to P and such that the first column of $(B_{\boldsymbol{\xi}}^{-1} P)^T$ forms a basis of the ideal $\mathfrak{a}\mathfrak{b}$. Then the matrix $B_{\boldsymbol{\xi}}$ determines a Siegel modular embedding $(\psi_{\mathfrak{a}}, \Psi_{\mathfrak{a}})$ by setting $\psi_{\mathfrak{a}}(\tau_1, \tau_2) = B_{\boldsymbol{\xi}} \left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right) B_{\boldsymbol{\xi}}^T$ and by defining $\Psi_{\mathfrak{a}}$ as in (7).

As expected, by changing the cusp at infinity we are changing the Hilbert modular surface, but the Siegel modular embedding $(\psi_{\mathfrak{a}}, \Psi_{\mathfrak{a}})$ and the general one (ψ, Ψ) constructed in the last section are compatible.

Proposition 4.5 *Let $\boldsymbol{\eta} = (\eta_1, \eta_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$ be symplectically adapted bases of $\mathcal{O}_D^\vee$ and $\mathfrak{a}^\vee$ determining the $\mathcal{O}_D$–modules $\mathfrak{b} \oplus \mathcal{O}_D^\vee$ and $\mathfrak{a}\mathfrak{b} \oplus \mathfrak{a}^\vee$, respectively. Moreover, let M be the matrix in* (8) *and define the matrix $\widetilde{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by*

$$a = B_{\boldsymbol{\xi}} \hat{\alpha} B_{\boldsymbol{\eta}}^{-1}, \quad b = B_{\boldsymbol{\xi}} \hat{\beta} B_{\boldsymbol{\eta}}^T, \quad c = B_{\boldsymbol{\xi}}^{-T} \hat{\gamma} B_{\boldsymbol{\eta}}^{-1}, \quad d = B_{\boldsymbol{\xi}}^{-T} \hat{\delta} B_{\boldsymbol{\eta}}^T.$$

Then $\widetilde{M}$ belongs to the symplectic group G_P and the left action map

$$\widetilde{\psi}(\Omega) = \widetilde{M} \cdot \Omega := (a\Omega + b)(c\Omega + d)^{-1}$$

lifts the map ϕ to the Siegel upper halfspace, ie $\widetilde{\psi} \circ \psi = \psi_{\mathfrak{a}} \circ \phi$.

Proof Proceeding as in the last section, one can easily check that $\widetilde{M} \in G_P$. Now, by definition and using the abbreviation $\boldsymbol{\tau} = (\tau_1, \tau_2)$ we have

$$\begin{aligned}\psi_{\mathfrak{a}} \circ \phi(\tau_1, \tau_2) &= B_{\boldsymbol{\xi}} \left(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^\sigma \end{smallmatrix}\right)\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^\sigma \end{smallmatrix}\right)\right) \cdot \left(\left(\begin{smallmatrix} \gamma & 0 \\ 0 & \gamma^\sigma \end{smallmatrix}\right)\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \delta & 0 \\ 0 & \delta^\sigma \end{smallmatrix}\right)\right)^{-1} B_{\boldsymbol{\xi}}^T \\ &= (B_{\boldsymbol{\xi}} \hat{\alpha} B_{\boldsymbol{\eta}}^{-1} \psi(\boldsymbol{\tau}) + B_{\boldsymbol{\xi}} \hat{\beta} B_{\boldsymbol{\eta}}^T) \cdot (B_{\boldsymbol{\xi}}^{-T} \hat{\gamma} B_{\boldsymbol{\eta}}^{-1} \psi(\boldsymbol{\tau}) + B_{\boldsymbol{\xi}}^{-T} \hat{\delta} B_{\boldsymbol{\eta}}^T)^{-1}\end{aligned}$$

and thus the map $\widetilde{\psi}$ has the required commutation property. □

5 Line bundles on $(1, n)$–polarized abelian surfaces

Classical theta functions are sections of line bundles on the abelian surface $T = \mathbb{C}^2/\Lambda$, where $\Lambda = \Pi\mathbb{Z}^4$ is the period lattice generated by the period matrix $\Pi = (\Omega, P_{1n})$. They are given by the Fourier expansion

$$\vartheta\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \colon \mathbb{H}_2 \times \mathbb{C}^2 \to \mathbb{C}, \quad \vartheta\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}(\Omega, \boldsymbol{v}) = \sum_{\boldsymbol{x} \in \mathbb{Z}^2 + c_1} \boldsymbol{e}(\boldsymbol{x}^T \Omega \boldsymbol{x}) \boldsymbol{e}(2\boldsymbol{x}^T(\boldsymbol{v} + c_2)),$$

where $\boldsymbol{e}(t) = e^{\pi i t}$. (We consider all vectors inside the formula as column vectors). The argument c is called the *characteristic* of the theta functions. Theta functions that differ only in their characteristics correspond to sections of line bundles that are translates of each other. For the moment we think of Ω fixed and consider the dependence on Ω in the image of a Siegel modular embedding starting from Section 5.3

The purpose of this section is to give a basis of sections of a line bundle on a $(1, n)$–polarized abelian surface for a characteristic chosen with the application in Lemma 6.5 in mind. Moreover we compute the Fourier expansions of these line bundles with respect to a symplectically adapted basis. The main goal are consequently the Fourier expansions in Proposition 5.5 and the relation among the values of these theta functions at 2–torsion points in (16). The miraculous reduction of the number of constraints appearing in the next section relies on this.

Most statements in this section are essentially in Sections 3.1, 4.6 and 4.7 of [3] and which we rewrite for our purposes. Since this reference uses the (equivalent) language of canonical (as opposed to classical) theta functions, we provide a short introduction and conversion between the languages.

5.1 Canonical theta functions

Let V be a complex vector space and let Λ be a lattice in V. To a line bundle $\mathcal{L}$ on the complex torus $T = V/\Lambda$ one associates its first Chern class $H = c_1(\mathcal{L})$, which we view as a Hermitian form on V whose imaginary part takes integral values on Λ. To a line bundle $\mathcal{L}$ one can associate a semicharacter $\chi\colon \Lambda \to S^1$ such that conversely $\mathcal{L}$ is the line bundle associated with (see [3, Appendix B]) the *canonical factor of automorphy*

$$a_{\mathcal{L}}(\lambda, \boldsymbol{u}) = \chi(\lambda)\exp(\pi H(\boldsymbol{u}, \lambda)) \in Z^1(\Lambda, H^0(\mathcal{O}_T^*)), \quad \boldsymbol{u} \in V,\ \lambda \in \Lambda. \tag{9}$$

This correspondence can be made more concrete in the case that H is positive-definite, ie the line bundle $\mathcal{L}$ is ample on T and hence T an abelian variety. A *decomposition* of V for H is a direct sum $V = V_1 \oplus V_2$ such that $\Lambda_i := V_i \cap \Lambda$ are isotropic with respect to $E = \operatorname{Im} H$. For such a decomposition there is a standard semicharacter

$$\chi_0(\boldsymbol{u}) = \exp(\pi i E(u_1, u_2)), \quad \text{where } \boldsymbol{u} = u_1 + u_2,\ u_i \in V_i, \tag{10}$$

with associated line bundle $\mathcal{L}_0 = \mathcal{L}(H, \chi_0)$. For every other line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = H$ there is a point $c \in V$ such that $\mathcal{L} = t_c^*\mathcal{L}_0$. The point is called the *characteristic* of $\mathcal{L}$ for the chosen decomposition. It is uniquely determined up to translation by an element in

$$\Lambda(H) = \{\boldsymbol{u} \in V \mid E(\boldsymbol{u}, \lambda) \in \mathbb{Z}\}.$$

(Here and in the sequel we often write eg $\Lambda(\mathcal{L})$ and $\Lambda(H)$ interchangeably for notions depending only on the first Chern class of the line bundle.) Consequently, characteristics for a given decomposition are in bijection with $V/\Lambda(H)$.

For a given line bundle $\mathcal{L}$ the global sections $H^0(T, \mathcal{L})$ can be identified with functions $\vartheta\colon V \to \mathbb{C}$, $\vartheta(\boldsymbol{u}+\lambda) = f(\lambda, u)\vartheta(u)$, where f is a factor of automorphy for $\mathcal{L}$. More concretely, in the case $f = a_{\mathcal{L}}$ as in (9) the functions

$$\vartheta\colon V \to \mathbb{C}, \quad \vartheta(\boldsymbol{u}+\lambda) = a_{\mathcal{L}}(\lambda, \boldsymbol{u})\vartheta(\boldsymbol{u}),$$

are called *canonical theta functions* for $\mathcal{L}$, which we now construct. We define, for every $c \in V$,

$$\begin{aligned}\vartheta^c(\boldsymbol{u}) = \exp\bigl(-\pi H(\boldsymbol{u}, c) - \tfrac{\pi}{2}H(c,c) + \tfrac{\pi}{2}\mathcal{B}(\boldsymbol{u}+c, \boldsymbol{u}+c)\bigr)&\\ \cdot \sum_{\lambda\in\Lambda_1} \exp\bigl(\pi(H-\mathcal{B})(\boldsymbol{u}+c, \lambda) - \tfrac{\pi}{2}(H-\mathcal{B})(\lambda,\lambda)\bigr)&,\end{aligned} \tag{11}$$

where $\mathcal{B}$ is the symmetric bilinear extension of $H|_{V_2}$. For every $w \in K(\mathcal{L})$ we use the bilinear extension

$$a_{\mathcal{L}}(\boldsymbol{u}, \boldsymbol{v}) = \chi_0(\boldsymbol{u})\exp\bigl(2\pi i E(c, \boldsymbol{u}) + \pi H(\boldsymbol{v}, \boldsymbol{u}) + \tfrac{\pi}{2}H(\boldsymbol{u}, \boldsymbol{u})\bigr) \tag{12}$$

of $a_{\mathcal{L}}$ to a function $V \times V \to \mathbb{C}$ and we set

$$\vartheta_w^c(\boldsymbol{u}) = a_{\mathcal{L}}(w, \boldsymbol{u})^{-1} \vartheta^c(\boldsymbol{u} + w). \tag{13}$$

Let us denote by $K(H)$ the kernel $\ker(\phi_{\mathcal{L}}) = \Lambda(H)/\Lambda$ of the canonical isogeny $\phi_{\mathcal{L}}\colon T \to T^\vee$ defined by $\mathcal{L}$. For the following theorem we note that the choice of a decomposition $V = V_1 \oplus V_2$ induces direct-sum decompositions of the lattice of integral points $\Lambda(H) = \Lambda(H)_1 \oplus \Lambda(H)_2$ and of $K(H) = K(H)_1 \oplus K(H)_2$, where $K(H)_i = \Lambda(H)_i/(\Lambda \cap \Lambda(H)_i)$. In this notation, Theorem 3.2.7 of [3] gives:

Theorem 5.1 *The function ϑ_w^c is a canonical theta function for $\mathcal{L} = t_c^*\mathcal{L}_0$. More precisely, if c is a characteristic with respect to a decomposition of V then the set $\{\vartheta_w^c : w \in K(\mathcal{L})_1\}$ is a basis of $H^0(\mathcal{L})$.*

Next we prove that actually the theta function ϑ_w^c only depends on the $K(\mathcal{L})_1$ component of w. This fact will be crucial to get extra relations between the values of theta functions at torsion points.

Lemma 5.2 *Let $w = w_1 + w_2 \in \Lambda(H)/\Lambda$. Then $\vartheta_w^c = \vartheta_{w_1}^c$.*

Proof The definition of the canonical theta function implies

$$\vartheta_w^c(\boldsymbol{u}) = \exp\bigl(-\pi H(\boldsymbol{u}, c) - \tfrac{\pi}{2} H(c,c)\bigr)\vartheta_w^0(\boldsymbol{u})$$

and hence it is enough to prove the claim for the characteristic 0. By the definition (13) of ϑ_w^0 and the properties of the factor $a_{\mathcal{L}}$ (see [3, Lemma 3.1.3]),

$$\begin{aligned}\vartheta_w^0(\boldsymbol{u}) &= a_{\mathcal{L}_0}(w_1 + w_2, \boldsymbol{u})^{-1}\vartheta_0^0(\boldsymbol{u} + w_1 + w_2)\\ &= a_{\mathcal{L}_0}(w_1, \boldsymbol{u})^{-1} a_{\mathcal{L}_0}(w_2, w_1 + \boldsymbol{u})^{-1}\vartheta_0^0(\boldsymbol{u} + w_1 + w_2).\end{aligned}$$

Applying the Fourier expansion (11) of ϑ_0^0, using (12) and $\chi_0(w_2) = 1$, we obtain

$$\begin{aligned}\vartheta_w^0(\boldsymbol{u}) = a_{\mathcal{L}_0}(w_1, \boldsymbol{u})^{-1} &\exp\bigl(-\pi(H-\mathcal{B})\bigl(\boldsymbol{u}+w_1+\tfrac{1}{2}w_2, w_2\bigr)+\tfrac{\pi}{2}\mathcal{B}(\boldsymbol{u}+w_1, \boldsymbol{u}+w_1)\bigr)\\ &\cdot \sum_{\lambda\in\Lambda_1} \exp\bigl(\pi(H-\mathcal{B})(\boldsymbol{u}+w_1, \lambda) + \pi(H-\mathcal{B})(w_2,\lambda) - \tfrac{\pi}{2}(H-\mathcal{B})(\lambda,\lambda)\bigr).\end{aligned}$$

Now [3, Lemma 3.2.2] implies $\pi(H-\mathcal{B})\bigl(\boldsymbol{u}+w_1+\frac{1}{2}w_2, w_2\bigr) = 0$, since $w_2 \in V_2$, and $\pi(H-\mathcal{B})(w_2,\lambda) = 2\pi i E(w_2,\lambda) \in 2\pi i\mathbb{Z}$, since $w_2 \in \Lambda(H)$. Applying (11) and (13) again we obtain

$$\begin{aligned}\vartheta_w^0(\boldsymbol{u}) &= a_{\mathcal{L}_0}(w_1,\boldsymbol{u})^{-1}\exp\bigl(\tfrac{\pi}{2}\mathcal{B}(\boldsymbol{u}+w_1,\boldsymbol{u}+w_1)\bigr)\\ &\qquad\cdot\sum_{\lambda\in\Lambda_1}\exp\bigl(\pi(H-\mathcal{B})(\boldsymbol{u}+w_1,\lambda)-\tfrac{\pi}{2}(H-\mathcal{B})(\lambda,\lambda)\bigr)\\ &= a_{\mathcal{L}_0}(w_1,\boldsymbol{u})^{-1}\vartheta_0^0(\boldsymbol{u}+w_1)=\vartheta_{w_1}^0(\boldsymbol{u}),\end{aligned}$$

as claimed. □

5.2 Specialization to $(1,6)$–polarization

From now on we suppose $\dim(T)=2$ and that $\mathcal{L}$ is a line bundle of *type* $(1,6)$, ie there exists a decomposition $V=V_1\oplus V_2$ for $H=c_1(\mathcal{L})$ and bases $\Lambda_1=\langle\lambda_1,\lambda_2\rangle$ and $\Lambda_2=\langle\mu_1,\mu_2\rangle$ in which $\operatorname{Im} H$ has a representation

$$\operatorname{Im} H=\begin{pmatrix}0 & P\\ -P & 0\end{pmatrix},\quad \text{where } P=\operatorname{diag}(1,6). \tag{14}$$

Under these assumptions, $\Lambda(H)=\bigl\langle\lambda_1,\tfrac{1}{6}\lambda_2,\mu_1,\tfrac{1}{6}\mu_2\bigr\rangle$ and $K(H)=\Lambda(H)/\Lambda\cong(\mathbb{Z}/6\mathbb{Z})^2$.

Recall that a divisor D on T is *symmetric* if $(-1)^*D=D$. A line bundle $\mathcal{L}$ is defined to be *symmetric* if the corresponding semicharacter χ takes values in ±1. This notion is designed so that the line bundle $\mathcal{L}=\mathcal{O}(D)$ of a symmetric divisor is symmetric (see [3, Section 4.7]). For such a line bundle, $(-1)^*$ induces an involution on $H^0(\mathcal{L})$, and hence on the vector space generated by canonical theta functions.

With the application to Prym varieties in mind, we focus on the line bundle $\mathcal{L}=t_c^*\mathcal{L}_0$ of characteristic $c=\frac{1}{2}\lambda_1+\frac{1}{2}\mu_1$. The space $H^0(\mathcal{L})$ is generated by $\{\vartheta^c_{j\lambda_2/6}:j=0,\dots,5\}$ and, in this situation, the inverse formula [3, Formula 4.6.4] gives $(-1)^*\vartheta_w^c=(-1)\cdot\vartheta^c_{-w}$ for all $w\in K(H)_1$. Consequently, the spaces of even and odd theta functions are given respectively by

$$\begin{aligned}H^0(\mathcal{L})_+&=\langle\vartheta^c_{\lambda_2/6}-\vartheta^c_{5\lambda_2/6},\ \vartheta^c_{2\lambda_2/6}-\vartheta^c_{4\lambda_2/6}\rangle,\\ H^0(\mathcal{L})_-&=\langle\theta_0=\vartheta^c_0,\ \theta_1=\vartheta^c_{\lambda_2/6}+\vartheta^c_{5\lambda_2/6},\ \theta_2=\vartheta^c_{2\lambda_2/6}+\vartheta^c_{4\lambda_2/6},\ \theta_3=\vartheta^c_{\lambda_2/2}\rangle.\end{aligned} \tag{15}$$

We will need the following result relating the values of odd theta functions at certain 2–torsion points, more precisely the set of 2–torsion points in the kernel $K(H)$ of the map $\phi_{\mathcal{L}}$ to the dual torus.

Lemma 5.3 *Let $\theta_0(\boldsymbol{u}),\dots,\theta_3(\boldsymbol{u})$ be the generators of $H^0(\mathcal{L})_-$. Then*

$$\begin{aligned}
\theta_0(\boldsymbol{u}) &= a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2,\boldsymbol{u}\big)^{-1}\theta_3\big(\boldsymbol{u}+\tfrac{1}{2}\lambda_2\big) = a_{\mathcal{L}}\big(\tfrac{1}{2}\mu_2,\boldsymbol{u}\big)^{-1}\theta_0\big(\boldsymbol{u}+\tfrac{1}{2}\mu_2\big),\\
\theta_1(\boldsymbol{u}) &= a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2,\boldsymbol{u}\big)^{-1}\theta_2\big(\boldsymbol{u}+\tfrac{1}{2}\lambda_2\big) = -a_{\mathcal{L}}\big(\tfrac{1}{2}\mu_2,\boldsymbol{u}\big)^{-1}\theta_1\big(\boldsymbol{u}+\tfrac{1}{2}\mu_2\big),\\
\theta_2(\boldsymbol{u}) &= a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2,\boldsymbol{u}\big)^{-1}\theta_1\big(\boldsymbol{u}+\tfrac{1}{2}\lambda_2\big) = a_{\mathcal{L}}\big(\tfrac{1}{2}\mu_2,\boldsymbol{u}\big)^{-1}\theta_2\big(\boldsymbol{u}+\tfrac{1}{2}\mu_2\big),\\
\theta_3(\boldsymbol{u}) &= a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2,\boldsymbol{u}\big)^{-1}\theta_0\big(\boldsymbol{u}+\tfrac{1}{2}\lambda_2\big) = -a_{\mathcal{L}}\big(\tfrac{1}{2}\mu_2,\boldsymbol{u}\big)^{-1}\theta_3\big(\boldsymbol{u}+\tfrac{1}{2}\mu_2\big).
\end{aligned}$$

Proof For any $w = w_1 + w_2$ and $\widetilde{w} = \widetilde{w}_1 + \widetilde{w}_2 \in \Lambda(H)/\Lambda$ we find, using (13) and the transformation law of the canonical factor of automorphy (cf Exercise 3.7(2) in [3]), that

$$\vartheta^c_w(\boldsymbol{u}) = \exp(2\pi i\ \mathrm{Im}\, H(\widetilde{w}_1,\widetilde{w}_2 - w_2))a_{\mathcal{L}_X}(w-\widetilde{w},\boldsymbol{u})^{-1}\vartheta^c_{\widetilde{w}}(\boldsymbol{u}+w-\widetilde{w}).$$

The first equalities claimed in the lemma are a direct application of this formula to $\widetilde{w} = \frac{1}{6}j\lambda_2$ and $w = \frac{1}{6}(j+3)\lambda_2$, where indices should be taken $\bmod 6$.

The second ones follow from the same formula applied to $\widetilde{w} = \frac{1}{6}j\lambda_2$ and $w = \frac{1}{6}j\lambda_2 + \frac{1}{2}\mu_2$ together with the fact that, by Lemma 5.2, $\theta^c_{\widetilde{w}} = \theta^c_w$. □

5.3 Partial derivatives at 2–torsion points

So far the computations were for a general abelian surface and we now restrict to real multiplication loci, ie to a period matrix $\Omega_{\boldsymbol{\tau}} = \psi(\boldsymbol{\tau})$ in the image of a Siegel modular embedding determined by a (d_1,d_2)–symplectically adapted basis (ω_1,ω_2) as in Section 4.4. Since on a surface with real multiplication there are two eigendirections, which we have given the coordinates u_i, for a general theta function ϑ the partial derivatives

$$D_i\vartheta(\boldsymbol{\tau},\boldsymbol{u}_0) := \frac{\partial}{\partial u_i}\vartheta(\boldsymbol{\tau},\boldsymbol{u})|_{\boldsymbol{u}=\boldsymbol{u}_0}$$

will be of particular interest in the sequel. As a direct consequence of Lemma 5.3 together with the vanishing of the θ_j at the given 2–torsion points we obtain the analogous results for the derivatives $D_i\theta_j$ for $i=1,2$,

$$\begin{aligned}
D_i\theta_0(0) &= a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2,0\big)^{-1}D_i\theta_3\big(\tfrac{1}{2}\lambda_2\big) = a_{\mathcal{L}}\big(\tfrac{1}{2}\mu_2,0\big)^{-1}D_i\theta_0\big(\tfrac{1}{2}\mu_2\big),\\
D_i\theta_1(0) &= a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2,0\big)^{-1}D_i\theta_2\big(\tfrac{1}{2}\lambda_2\big) = -a_{\mathcal{L}}\big(\tfrac{1}{2}\mu_2,0\big)^{-1}D_i\theta_1\big(\tfrac{1}{2}\mu_2\big),\\
D_i\theta_2(0) &= a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2,0\big)^{-1}D_i\theta_1\big(\tfrac{1}{2}\lambda_2\big) = a_{\mathcal{L}}\big(\tfrac{1}{2}\mu_2,0\big)^{-1}D_i\theta_2\big(\tfrac{1}{2}\mu_2\big),\\
D_i\theta_3(0) &= a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2,0\big)^{-1}D_i\theta_0\big(\tfrac{1}{2}\lambda_2\big) = -a_{\mathcal{L}}\big(\tfrac{1}{2}\mu_2,0\big)^{-1}D_i\theta_3\big(\tfrac{1}{2}\mu_2\big).
\end{aligned} \tag{16}$$

Another consequence is that an odd theta function behaves near those nontrivial 2–torsion points like an odd function in the following sense:

Corollary 5.4 *Let $f \in H^0(\mathcal{L})_-$ be an odd theta function, let Q be one of the 2–torsion points $\{0, \frac{1}{2}\lambda_2, \frac{1}{2}\mu_2, \frac{1}{2}(\lambda_2+\mu_2)\}$ and fix $i=1$ or $i=2$. If $D_i^{2k-1} f(Q) = 0$ for all $k = 1, \dots, n$, then $D_i^{2n} f(Q) = 0$.*

Proof The proof is trivial for $Q = 0$ since f is an odd function of $\mathbb{C}^2$. To discuss the other 2–torsion points, write $f = f_1 + f_2$, where $f_1 \in \langle \theta_0, \theta_2 \rangle$ and $f_2 \in \langle \theta_1, \theta_3 \rangle$. For $Q = \frac{1}{2}\mu_2$ we can write, for each $N > 0$,

$$D_i^N f\big(\tfrac{1}{2}\mu_2\big) = \sum_{j=0}^{N} \binom{N}{j} D_i^{N-j} a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2, 0\big)\big(D_i^j f_1(0) - D_i^j f_2(0)\big)$$

by Lemma 5.3. Since we work with a space of odd theta functions, $D_i^{2k}\theta_j(0) = 0$ for every k and $j \in \{0,1,2,3\}$. Consequently, we can use this formula inductively to show that the hypothesis $D_i^{2k-1} f\big(\frac{1}{2}\mu_2\big) = 0$ for all $k = 1, \dots, n$ holds if and only if $D_i^{2k-1} f_1(0) - D_i^{2k-1} f_2(0) = 0$ for all $k = 1, \dots, n$. As a consequence,

$$D_i^{2n} f\big(\tfrac{1}{2}\mu_2\big) = \sum_{k=1}^{n} \binom{2n}{2k-1} D_i^{2n-2k+1} a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2, 0\big)\big(D_i^{2k-1} f_1(0) - D_i^{2k-1} f_2(0)\big) = 0.$$

For $Q = \frac{1}{2}\lambda_2$, we write $\tilde{f}_j$ for f_j with θ_0 and θ_1 exchanged with θ_3 and θ_2, respectively. With this notation,

$$D_i^N f\big(\tfrac{1}{2}\lambda_2\big) = \sum_{j=0}^{N} \binom{N}{j} D_i^{N-j} a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2, 0\big)\big(D_i^j \tilde{f}_1(0) - D_i^j \tilde{f}_2(0)\big).$$

Again, the hypothesis $D_i^{2k-1} f\big(\frac{1}{2}\lambda_2\big) = 0$ for all $k = 1, \dots, n$ holds if and only if $D_i^{2k-1} \tilde{f}_1(0) - D_i^{2k-1} \tilde{f}_2(0) = 0$ for all $k = 1, \dots, n$. Consequently,

$$D_i^{2n} f\big(\tfrac{1}{2}\lambda_2\big) = \sum_{k=1}^{n} \binom{2n}{2k-1} D_i^{2n-2k+1} a_{\mathcal{L}}\big(\tfrac{1}{2}\lambda_2, 0\big)\big(D_i^{2k-1} \tilde{f}_1(0) - D_i^{2k-1} \tilde{f}_2(0)\big) = 0.$$

The proof for $Q = \frac{1}{2}(\lambda_2 + \mu_2)$ follows the same lines. □

5.4 Fourier expansions

For a $(1,6)$–symplectically adapted basis $\boldsymbol{\eta} = (\eta_1, \eta_2)$ we define $\rho_{\boldsymbol{\eta}}(x_1, x_2) = x_1\eta_1 + x_2\eta_2$, hence $\boldsymbol{x}^T B_{\boldsymbol{\eta}} = (\rho_{\boldsymbol{\eta}}(\boldsymbol{x}), \rho_{\boldsymbol{\eta}}^{\sigma}(\boldsymbol{x}))$ for the matrix $B_{\boldsymbol{\eta}}$ used in (6) to define a Siegel modular embedding of the Hilbert modular surface $X_D(\mathfrak{b})$. Recall that the choice of such a basis also determines a decomposition of V using

$$V_1 = \langle (\nu_1, 0), (\nu_2, 0) \rangle_{\mathbb{R}}, \quad V_2 = \langle (0, \eta_1), (0, \eta_2) \rangle_{\mathbb{R}}. \tag{17}$$

We moreover define the shifted lattice $\Lambda_{\epsilon,\delta} = \mathbb{Z}^2 + (\epsilon, \delta)^T$ and abbreviate $\rho = \rho_{\boldsymbol{\eta}}$ if $\boldsymbol{\eta}$ has been fixed.

Proposition 5.5 *The Nullwerte of the derivatives of the theta functions θ_j for $j \in \{0, 1, 2, 3\}$, as defined in (15), have the Fourier expansion*

$$\begin{aligned} \frac{\partial}{\partial u_1}\theta_j(\boldsymbol{\tau}, 0) &= 2\pi i \sum_{\boldsymbol{x} \in \Lambda_{1/2, j/6}} \boldsymbol{e}(x_1)\rho(\boldsymbol{x}) q_1^{\rho(\boldsymbol{x})^2} q_2^{\rho^\sigma(\boldsymbol{x})^2}, \\ \frac{\partial}{\partial u_2}\theta_j(\boldsymbol{\tau}, 0) &= 2\pi i \sum_{\boldsymbol{x} \in \Lambda_{1/2, j/6}} \boldsymbol{e}(x_1)\rho^\sigma(\boldsymbol{x}) q_1^{\rho(\boldsymbol{x})^2} q_2^{\rho^\sigma(\boldsymbol{x})^2}, \end{aligned} \tag{18}$$

where $q_i = \boldsymbol{e}(\tau_i)$ and $\boldsymbol{e}(\cdot) = \exp(\pi i \cdot)$.

Proof By [3, Lemma 8.5.2], the canonical theta function with characteristic c is given by

$$\vartheta^c(\boldsymbol{\tau}, \boldsymbol{v}) = e^{\pi/2 B(\boldsymbol{v}, \boldsymbol{v}) - \pi i c_1^T c_2} \vartheta \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} (\boldsymbol{\tau}, \boldsymbol{v})$$

in terms of classical theta functions. We differentiate this, use that the θ_j are odd, hence vanish at zero, and use the Fourier expansions

$$\begin{aligned} \frac{\partial}{\partial u_1}\vartheta \begin{bmatrix} (\frac{1}{2}, \frac{1}{6}j) \\ (\frac{1}{2}, 0) \end{bmatrix} (\boldsymbol{\tau}, 0) &= 2\pi i \sum_{\boldsymbol{x} \in \Lambda_{3j}} \boldsymbol{e}(x_1)\rho(\boldsymbol{x}) q_1^{\rho(\boldsymbol{x})^2} q_2^{\rho^\sigma(\boldsymbol{x})^2}, \\ \frac{\partial}{\partial u_2}\vartheta \begin{bmatrix} (\frac{1}{2}, \frac{1}{6}j) \\ (\frac{1}{2}, 0) \end{bmatrix} (\boldsymbol{\tau}, 0) &= 2\pi i \sum_{\boldsymbol{x} \in \Lambda_{3j}} \boldsymbol{e}(x_1)\rho^\sigma(\boldsymbol{x}) q_1^{\rho(\boldsymbol{x})^2} q_2^{\rho^\sigma(\boldsymbol{x})^2}. \end{aligned}$$

This immediately gives the expansions for θ_0 and θ_3. For the two remaining generators we moreover use that

$$\frac{\partial}{\partial u_i}\vartheta \begin{bmatrix} (\frac{1}{2}, -\frac{1}{6}j) \\ (\frac{1}{2}, 0) \end{bmatrix} (\boldsymbol{\tau}, 0) = \frac{\partial}{\partial u_i}\vartheta \begin{bmatrix} (\frac{1}{2}, \frac{1}{6}j) \\ (\frac{1}{2}, 0) \end{bmatrix} (\boldsymbol{\tau}, 0) \quad \text{for } j = 1, 2,$$

as we see by changing the order of summation in (18) using the observation that ρ is odd. □

5.5 Derivatives of theta functions as Hilbert modular forms

The set of all Siegel theta functions for characteristics in $\frac{1}{N}\mathbb{Z}$ (with N fixed) satisfies a modular transformation law (see [3, Section 8.4] for the complete formula). This implies that the restriction via a Siegel modular embedding satisfies a modular transformation law for the Hilbert modular group. In general, this action still permutes characteristics, but here we make use of the following fact:

Lemma 5.6 *The space $H^0(\mathcal{L})$ of theta functions of characteristic $c = \frac{1}{2}\lambda_1 + \frac{1}{2}\mu_1$ is preserved by the whole modular group $\mathrm{SL}(\mathfrak{b} \oplus \mathcal{O}_D^\vee)$.*

Proof The action of the modular group on characteristics preserves the set of characteristics corresponding to symmetric line bundles, and the action on theta functions preserves the even and odd subspaces. Let $\mathcal{L}$ be a symmetric line bundle of characteristic c that provides the $(1,6)$–polarization. Since $h^0(L) = 6$, the space of odd theta functions of L has dimension

$$h^0_- = \tfrac{1}{2}(6 - \#S) + \#S^-,$$

by [3, Proposition 4.6.5], where

$$S = \{\overline{w} \in K(L)_1 : 2\overline{w} = 2\overline{c}_1\} \quad \text{and} \quad S^- = \{\overline{w} \in S : e(4\pi i \operatorname{Im} H(w + c_1, c_2)) = -1\}.$$

One now computes that the line bundle of characteristic $\frac{1}{2}\lambda_1 + \frac{1}{2}\mu_1$ is the only one with a 4–dimensional space of odd theta functions. Thus every element of the modular group fixes this characteristic. □

Recall that a Hilbert modular form f of biweight (k, ℓ) with character χ for the subgroup Γ of a Hilbert modular group is a holomorphic function $f\colon \mathbb{H}^2 \to \mathbb{C}$ with the transformation law

$$f(\gamma\tau_1, \gamma^\sigma\tau_2) = \chi(\gamma)(c\tau_1 + d)^k (c^\sigma\tau_2 + d^\sigma)^\ell f(\tau_1, \tau_2)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. The specialization of the theta transformation law implies that for an even theta function ϑ of characteristic c the Nullwert $\vartheta(\boldsymbol{\tau}, 0)$ is a Hilbert modular form of biweight $\left(\frac{1}{2}, \frac{1}{2}\right)$ with some finite character for some finite-index subgroup of the Hilbert modular group. The partial derivatives $D_1\vartheta(\boldsymbol{\tau}, 0)$ and $D_2\vartheta(\boldsymbol{\tau}, 0)$ of odd theta functions are modular forms of biweight $\left(\frac{3}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{3}{2}\right)$, respectively; see [21, Section 9] for the details.

5.6 Line bundles of type $(2,3)$

The usual convention for the type (d_1, d_2) of a polarization is that $d_1 \mid d_2$. However, it will be convenient in our particular case to consider also the polarization type $(2,3)$ rather than just of type $(1,6)$. In this subsection we translate the results between the two different conventions.

Let $\mathcal{L}$ be a line bundle of type $(1,6)$ and let $V = V_1 \oplus V_2$ be a decomposition for $\mathcal{L}$, so that $\Lambda = \Lambda_1 \oplus \Lambda_2 = \langle \lambda_1, \lambda_2 \rangle \oplus \langle \mu_1, \mu_2 \rangle$ gives a symplectic basis of the lattice with canonical $(1,6)$–symplectic matrix, ie the nontrivial intersection are $E(\lambda_1, \mu_1) = 1$ and $E(\lambda_2, \mu_2) = 6$.

The matrices $R_\lambda = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 1 \end{smallmatrix}\right)$ and $R_\mu = \left(\begin{smallmatrix} -2 & 1 \\ 3 & -1 \end{smallmatrix}\right)$ give a change of basis to a symplectic basis $\langle \widetilde{\lambda}_1, \widetilde{\lambda}_2 \rangle \oplus \langle \widetilde{\mu}_1, \widetilde{\mu}_2 \rangle$ with canonical $(2,3)$–symplectic matrix while preserving the chosen decomposition of V. In particular we may identify the characteristics in the two situations and we may identify the basis elements of $H^0(\mathcal{L})$ named in (15) in the two conventions. The distinguished characteristic $c = \frac{1}{2}\lambda_1 + \frac{1}{2}\mu_1 \in \frac{1}{2}\Lambda(H)/\Lambda(H)$ is expressed in the new basis as $c = \frac{1}{6}\widetilde{\lambda}_2 + \frac{1}{6}\widetilde{\mu}_2$ since $\Lambda(H) = \left\langle \frac{1}{2}\widetilde{\lambda}_1, \frac{1}{3}\widetilde{\lambda}_2, \frac{1}{2}\widetilde{\mu}_1, \frac{1}{3}\widetilde{\mu}_2 \right\rangle$.

Now let $\boldsymbol{\eta} = (\eta_1, \eta_2)$ be a $(2,3)$–symplectically adapted basis of $\mathfrak{a}^\vee$, determining the $\mathcal{O}_D$–module $\mathfrak{ab} \oplus \mathfrak{a}^\vee$, and consider the Siegel modular embedding given by the matrix $B := B_{\boldsymbol{\eta}} = \left(\begin{smallmatrix} \eta_1 & \eta_1^\sigma \\ \eta_2 & \eta_2^\sigma \end{smallmatrix}\right)$ as in Section 4.5, so that the cusp $\mathfrak{a}$ of $X_D(\mathfrak{b})$ corresponds to the cusp at infinity of $X_D^{\mathfrak{a}}(\mathfrak{b})$. Then $\boldsymbol{\eta} R_\mu^{-1}$ is $(1,6)$–symplectically adapted, and this base change together with the action of R_λ^{-1} on $\boldsymbol{v}$ preserves the decomposition (17), so that we are in indeed in the situation considered above.

Lemma 5.7 *With $\rho(\boldsymbol{x}) = \rho_{\boldsymbol{\eta}}(\boldsymbol{x}) := x_1\eta_1 + x_2\eta_2$ stemming from a $(2,3)$–symplectically adapted basis, the global sections $\theta_j \in H^0(\mathcal{L})$ of the line bundle with characteristic $c = \frac{1}{6}\widetilde{\lambda}_2 + \frac{1}{6}\widetilde{\mu}_2$ have Fourier expansions as in* (18) *with the lattice coset $\Lambda_{1/2,j/6}$ for the series θ_j replaced by $\Lambda_{j/2,(2j+3)/6}$ and the character $\boldsymbol{e}(x_1)$ replaced by $\boldsymbol{e}(x_2)$.*

Proof By definition, the lattice coset is $\left(\frac{1}{2}, \frac{1}{6}j\right)$ in the basis λ_1, λ_2, which is equal to $\left(\frac{1}{2}j, \frac{1}{6}(2j+3)\right)$ in the basis $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$ and the character is determined by the μ–component of the characteristic. □

6 The Gothic modular form and the Gothic theta function

We now specialize again to curves (X, π_A, π_B) in the Gothic locus. Abel–Prym maps denote, in analogy to the classical Abel–Jacobi map from a curve to its Jacobian, the map from X to its Prym variety. Since the Prym variety is not principally polarized, there are two natural choices that we analyze here: to the Prym variety and to its dual. The main player is the pre-Abel–Prym map $\varphi\colon X \to \mathrm{Prym}^\vee(X, \pi_A, \pi_B)$ to the dual Prym variety defined in Section 3. Since the Prym variety $\mathrm{Prym}^\vee(X)$ of a point in G_D admits real multiplication by $\mathcal{O}_D$, we can see the Teichmüller curve G_D inside some Hilbert modular surface $X_D(\mathfrak{b})$. Let us denote by $G_D(\mathfrak{b})$ the union of those components of the Torelli-image of G_D in $X_D(\mathfrak{b})$ for which du_1 induces the eigenform ω at each point (X, ω).

Our goal is to describe $\varphi(X)$ in terms of theta functions and nearly determine the Torelli-image of G_D.

Theorem 6.1 *The Torelli-image $G_D(\mathfrak{b})$ is contained in the vanishing locus of the Hilbert modular form*

$$\mathcal{G}_D(\boldsymbol{\tau}) := D_2\theta_0(\boldsymbol{\tau},0)\cdot D_2\theta_1(\boldsymbol{\tau},0) - D_2\theta_2(\boldsymbol{\tau},0)\cdot D_2\theta_3(\boldsymbol{\tau},0)$$

of biweight $(1,3)$. Consider the locus

$$\widetilde{\mathrm{Red}}_{23}(\mathfrak{b}) = \{\mathcal{G}_D(\boldsymbol{\tau}) = 0\} \cap \{\mathcal{F}_a(\boldsymbol{\tau}) = 0\} \cap \{\mathcal{F}_b(\boldsymbol{\tau}) = 0\},$$

where we define the modular forms

$$\begin{aligned}\mathcal{F}_a(\boldsymbol{\tau}) &= D_1\theta_0(\boldsymbol{\tau},0)\cdot D_2\theta_2(\boldsymbol{\tau},0) - D_1\theta_2(\boldsymbol{\tau},0)\cdot D_2\theta_0(\boldsymbol{\tau},0),\\ \mathcal{F}_b(\boldsymbol{\tau}) &= D_1\theta_1(\boldsymbol{\tau},0)\cdot D_2\theta_3(\boldsymbol{\tau},0) - D_1\theta_3(\boldsymbol{\tau},0)\cdot D_2\theta_1(\boldsymbol{\tau},0).\end{aligned}$$

Then, for all points in $\{\mathcal{G}_D(\boldsymbol{\tau}) = 0\} \setminus \widetilde{\mathrm{Red}}_{23}(\mathfrak{b})$, the theta function

$$\theta_X(\boldsymbol{u}) = \begin{vmatrix} \theta_0(\boldsymbol{u}) & \theta_1(\boldsymbol{u}) & \theta_2(\boldsymbol{u}) & \theta_3(\boldsymbol{u}) \\ D_1\theta_0(0) & D_1\theta_1(0) & D_1\theta_2(0) & D_1\theta_3(0) \\ D_2\theta_0(0) & 0 & D_2\theta_2(0) & 0 \\ 0 & D_2\theta_1(0) & 0 & D_2\theta_3(0) \end{vmatrix}$$

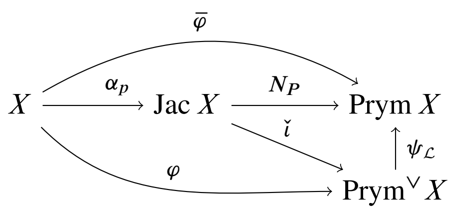
is nonzero and the vanishing locus of this theta function is equal to the pre-Abel–Prym image $\varphi(X)$ of a Gothic Veech surface X.

We will discuss the exceptional set where the modular forms $\mathcal{G}_D$, $\mathcal{F}_a$ and $\mathcal{F}_b$ jointly vanish in Section 7. It is part of the reducible locus, as suggested by the notation and as we will see in Proposition 8.6.

6.1 The Abel–Prym map and the pre-Abel–Prym map

Let (X,ω,π_A,π_B) be a flat surface in the Gothic locus ΩG. For each choice of a "basepoint" $p \in X$ there is the usual Abel–Jacobi map $\alpha_p\colon X \to \operatorname{Jac} X$ centered at p. We will fix once and for all the center of the Abel–Jacobi map to be $p = p_1$, one of the fixed points of J where ω does not vanish.

We have defined $\operatorname{Prym}(X) = \operatorname{Prym}(X,\pi_A,\pi_B)$ as the subvariety complementary to $A^\vee \times B^\vee$, hence there is a natural inclusion $\iota\colon \operatorname{Prym}(X) \to \operatorname{Jac}(X)$. Its dual is thus a quotient map $\iota^\vee\colon \operatorname{Jac}(X) \to \operatorname{Prym}^\vee(X)$ and the norm endomorphism N_P defined in Proposition 3.3 is also such a quotient map. Using (5) we conclude that they fit into the commutative diagram



The composition $\overline{\varphi} := N_P \circ \alpha_{p_1}$ is called the *Abel–Prym map* and the composition $\varphi := \check{\iota}\alpha_{p_1}$ is called the *pre-Abel–Prym map* centered at p_1. By Proposition 3.3 we can write the Abel–Prym map in terms of divisors as

$$\overline{\varphi}(x) = [x^{(1)} - 3J(x^{(1)}) - 2x^{(2)} - 2x^{(3)} + 2p_1 + 2q_1 + 2J(q_1)].$$

Moreover, $\overline{\varphi}(x) = \overline{\varphi}(y)$ if and only if

$$x^{(1)} - 3J(x^{(1)}) - 2x^{(2)} - 2x^{(3)} - y^{(1)} + 3J(y^{(1)}) + 2y^{(2)} + 2y^{(3)} \sim 0. \tag{19}$$

As a consequence of this formula we obtain:

Lemma 6.2 *The Abel–Prym map $\overline{\varphi}$ maps $Z \cup P$ to a single point, ie*

$$\overline{\varphi}(z_i) = \overline{\varphi}(p_i) = 0 \quad \textit{for } i = 1,2,3.$$

Proof Using (19) and the fact that points in $Z \cup P$ are fixed under J, the claim is equivalent to

$$2p_i^{(1)} + 2p_i^{(2)} + 2p_i^{(3)} \sim 2p_j^{(1)} + 2p_j^{(2)} + 2p_j^{(3)}$$

and

$$2p_i^{(1)} + 2p_i^{(2)} + 2p_i^{(3)} \sim 2z_j^{(1)} + 2z_j^{(2)} + 2z_j^{(3)}$$

for any $p_i, p_j \in P$ and $z_j \in Z$. This follows from the preimage diagram (3) and the fact that each of the points involved appears with coefficient 2 in $h^*(e_i)$. □

6.2 The natural line bundles on $\mathrm{Prym}^\vee(X)$

There are several natural line bundles on the Prym varieties. The restriction of the principal polarization on $\mathrm{Jac}\, X$ to $\mathrm{Prym}\, X$ via ι yields a polarization of type $(1,6)$ given by a line bundle, which we denote by $\mathcal{L}$. But we are rather interested in $\mathrm{Prym}^\vee(X)$. There, we first have the bundle $\mathcal{L}_X := \mathcal{O}_{\mathrm{Prym}^\vee X}(\varphi(X))$ generated by the image of the Gothic Veech surface that we are mainly interested in. Second, there is the following general construction.

Let $H = c_1(\mathcal{L})$ and let $\phi_{\mathcal{L}}\colon \mathrm{Prym}\, X \to \mathrm{Prym}^\vee X$ be the isogeny associated with $\mathcal{L}$. Since $\mathcal{L}$ is of type $(1,6)$ there is an isogeny $\psi\colon \mathrm{Prym}^\vee X \to \mathrm{Prym}\, X$ such that $\psi \circ \phi_{\mathcal{L}} = [6]$ (see [3, Section 14.4]). More precisely, $\psi = \psi_{\check{\mathcal{L}}}$ for a line bundle $\check{\mathcal{L}}$ on $\mathrm{Prym}^\vee X$, well defined only up to translations, with the same polarization $H = c_1(\check{\mathcal{L}})$. To fix a precise point of reference, we fix a decomposition for the universal covering V of $\mathrm{Prym}^\vee X$ in which $\mathrm{Im}\, H$ has the form (14). Such a decomposition distinguishes a line bundle in the algebraic class of $\check{\mathcal{L}}$, namely the symmetric line bundle $\check{\mathcal{L}}_0 = L(H, \chi_0)$ of characteristic 0 (see Section 5.1) associated to the semicharacter $\chi_0(v_1 + v_2) = e(\pi i\, \mathrm{Im}\, H(v_1, v_2))$.

Lemma 6.3 *The line bundles $\mathcal{L}_X$ and $\check{\mathcal{L}}_0$ are algebraically equivalent.*

Proof We use the endomorphism $\delta(C, D)$ associated with a curve C and a divisor D of an abelian variety T. It is defined by mapping $a \in T$ to the sum of the intersection points of the curve C translated by a and the divisor D; see [3, Sections 5.4 and 11.6]. By [3, Theorem 11.6.4] we need to show that $\delta(\varphi(X), \check{\mathcal{L}}) = \delta(\check{\mathcal{L}}, \check{\mathcal{L}})$. By [3, Proposition 5.4.7] and Riemann–Roch, $\delta(\check{\mathcal{L}}, \check{\mathcal{L}}) = -6\, \mathrm{id}_{\mathrm{Prym}^\vee X}$. On the other hand,

$$\delta(\varphi(X), \check{\mathcal{L}}) = -\check{\iota} \circ \iota \circ \psi_{\check{\mathcal{L}}} = -\phi_{\mathcal{L}} \circ \psi_{\check{\mathcal{L}}} = -6\, \mathrm{id}_{\mathrm{Prym}^\vee X}$$

by [3, Proposition 11.6.1]. □

6.3 The pre-Abel–Prym map

Next, we study the pre-Abel–Prym map. We write $\mathrm{Prym}^\vee X = V/\Lambda$.

Lemma 6.4 *The pre-Abel–Prym map φ with basepoint p_1 sends the p_i to zero, ie*

$$\varphi(p_1) = \varphi(p_2) = \varphi(p_3) = 0 \quad \textit{for } i = 1, 2, 3.$$

The points in Z are sent to three different nontrivial 2–torsion points in a Lagrangian subspace of Λ, ie $\varphi(Z) = \{\frac{1}{2}\lambda_2, \frac{1}{2}\mu_2, \frac{1}{2}(\lambda_2 + \mu_2)\}$ for some decomposition of V. Moreover, the endomorphism (-1) of $\mathrm{Prym}^\vee X$ induces the involution J on $\varphi(X)$ and φ is injective on $X \setminus P$.

Proof The inclusion $A^\vee \times B^\vee \subset \mathrm{Jac}\, X$ is given in terms of degree-zero divisors D and E by $(D, E) \mapsto D + J(D) + E^{(1)} + E^{(2)} + E^{(3)}$. In particular, on the images of q_i in A and B (as in (3)) this inclusion map is given by $(\overline{q}_1 - \overline{q}_i, e'_i - e'_1) \mapsto [p_i - p_1] = \varphi(p_i)$. This proves that the points p_i are sent to zero.

Next, for each $x \in X$ the divisor $x + J(x) - 2p_1$ belongs to $\pi_A^* \mathrm{Div}^0(A)$, hence maps to zero in $\mathrm{Prym}^\vee X$ and therefore

$$\varphi(x) = [x - p_1] = [-J(x) + p_1] = -\varphi(J(x)).$$

In particular, the points $\varphi(z_i) = [z_i - p_1]$ have order 2 and

$$\sum_{i=1}^{3} \varphi(z_i) = [z_1 + z_2 + z_3 - 3p_1] = 0.$$

As a consequence, all three of the z_i are 2–torsion points and by Lemma 6.2 they moreover lie in $\Lambda(H)$. It remains to exclude that $\varphi(Z) = 0$.

By the preceding Lemma 6.3 and Riemann–Roch, the curve $\varphi(X)$ is of arithmetic genus 7. If $\varphi(X)$ is generically injective then $\varphi(Z) = 0$ would imply that there are 6 branches passing through zero and the arithmetic genus had to be larger than 7, a contradiction. On the other hand, the geometric genus of $\varphi(X)$ is at least 2, since this curve generates $\mathrm{Prym}^\vee X$, hence the degree of φ is at most 3. In this case, the differential ω has to be a pullback of a differential on (the normalization of) the genus 2 curve $\mathrm{Prym}^\vee X$. This is impossible, as discussed in [16, Lemma 6.2]. □

We can now complete the identification of the line bundles begun in Lemma 6.3.

Lemma 6.5 *Let $(X, \omega) \in \Omega G$. With the above choice of a decomposition of V, the line bundles $\mathcal{L}_X$ and $\check{\mathcal{L}}_0$ differ by the characteristic $c = \frac{1}{2}\lambda_1 + \frac{1}{2}\mu_1$, ie $\mathcal{L}_X = t_c^* \check{\mathcal{L}}_0$.*

Proof To compute the characteristic, note that by Lemma 6.4 the image $\varphi(X)$ is a symmetric divisor, that is, $(-1)^*\varphi(X) = \varphi(X)$. The requirement on a line bundle in the algebraic class of $\check{\mathcal{L}}_0$ to be symmetric, narrows the number of choices down to 2^4 possibilities, which agree with the translates of $\check{\mathcal{L}}_0$ by half-integral points. As a consequence, $\mathcal{L}_X = t_c^* \check{\mathcal{L}}_0$ for some half-integral character $c \in \frac{1}{2}\Lambda(H)/\Lambda(H)$.

In order to compute explicitly the characteristic of $\mathcal{L}_X$, let us first note that by Lemma 6.4 the only 2–torsion points in $\varphi(X)$ are $\varphi(z_i)$ for $i=1,2,3$, all of them with multiplicity 1. By [3, Proposition 4.7.2] the semicharacter χ associated to the line bundle $\mathcal{L}_X$ takes the value

$$\chi(\lambda)=(-1)^{\mathrm{mult}_{\lambda/2}(\varphi(X))-\mathrm{mult}_0(\varphi(X))}$$

for each lattice element $\lambda\in\Lambda$. Since $\mathrm{mult}_0(\varphi(X))=3$, we deduce that χ takes values $+1$ at λ_2, μ_2 and $\lambda_2+\mu_2$ and -1 at λ_1, μ_1 and $\lambda_1+\mu_1$.

Recall that $\Lambda(H)=\langle\lambda_1,\frac{1}{6}\lambda_2,\mu_1,\frac{1}{6}\mu_2\rangle$, and let $c=a_1\lambda_1+\frac{1}{6}a_2\lambda_2+b_1\mu_1+\frac{1}{6}b_2\mu_2$, where $a_1,a_2,b_1,b_2\in\{0,\frac{1}{2}\}$. Using the fact that $\chi=\chi_0\cdot\exp(2\pi i\,\mathrm{Im}\,H(c,\cdot))$ and the expression (10) for χ_0, one gets $a_1=b_1=\frac{1}{2}$ and $a_2,b_2=0$. □

6.4 Identifying the theta function

Our main objective now is to describe $\varphi(X)$ as the vanishing locus of some theta function θ_X in $H^0(\mathcal{L}_X)$. For this purpose, we restrict furthermore to the case that (X,ω) is a Gothic eigenform for real multiplication by $\mathcal{O}_D$. This implies that on the Prym variety we have the distinguished eigenform coordinates introduced in Section 4.4.

Lemma 6.6 *Let $(X,\omega)\in\Omega G_D$ for some D. Then $\varphi(X)$ is the vanishing locus of a global section $\theta_X\in H^0(\mathcal{L}_X)_-$ satisfying*

(C1) $D_1\theta_X(0)=0$,

(C2) $D_2\theta_X(0)=0$,

(C3) $D_2\theta_X(\frac{1}{2}\mu_2)=0$,

(C4) $D_2\theta_X(\frac{1}{2}\lambda_2)=0$,

(C5) $D_2\theta_X(\frac{1}{2}(\lambda_2+\mu_2))=0$.

Proof By definition and Lemma 6.5, $\varphi(X)$ is the vanishing locus of some theta function $\theta_X\in H^0(\mathcal{L}_X)$. Since $\mathrm{mult}_0(\varphi(X))=3$, this theta function is necessarily odd by [3, Lemma 4.7.1] and the comments after that lemma.

Since θ_X is an odd function, both θ_X and its second derivatives vanish at 0. Since $\mathrm{mult}_0(\varphi(X))=3$, also its first derivatives must vanish, that is, $D_1\theta_X(0)=D_2\theta_X(0)=0$.

Let us assume that du_1 is the eigenform in $\Omega\mathcal{M}_4(2^3,0^3)$. Note that the condition of this eigenform having a zero of order k at a point p translates into $\partial^j\theta_X/\partial u_2^j$ vanishing at $\varphi(p)$ for $j=0,\dots,k$. □

Recall the definition of the generators θ_0, θ_1, θ_2 and θ_3 of $H^0(\mathcal{L}_X)_-$ from (15), and let $\theta_X(\boldsymbol{u}) = \sum_i a_i\theta_i(\boldsymbol{u})$ be a theta function cutting out $\varphi(X)$. By (16), the conditions in Lemma 6.6 correspond to the system of equations

$$\text{(C1)} \qquad a_0D_1\theta_0(0) + a_1D_1\theta_1(0) + a_2D_1\theta_2(0) + a_3D_1\theta_3(0) = 0,$$

$$\text{(C2)} \qquad a_0D_2\theta_0(0) + a_1D_2\theta_1(0) + a_2D_2\theta_2(0) + a_3D_2\theta_3(0) = 0,$$

$$\text{(C3)} \qquad a_0D_2\theta_0(0) - a_1D_2\theta_1(0) + a_2D_2\theta_2(0) - a_3D_2\theta_3(0) = 0,$$

$$\text{(C4)} \qquad a_0D_2\theta_3(0) + a_1D_2\theta_2(0) + a_2D_2\theta_1(0) + a_3D_2\theta_0(0) = 0,$$

$$\text{(C5)} \qquad a_0D_2\theta_3(0) - a_1D_2\theta_2(0) + a_2D_2\theta_1(0) - a_3D_2\theta_0(0) = 0.$$

Note that conditions (C2)–(C3) and conditions (C4)–(C5) can be rephrased as

$$\text{(20)} \qquad \begin{cases} a_0D_2\theta_0(0) + a_2D_2\theta_2(0) = 0, \\ a_1D_2\theta_1(0) + a_3D_2\theta_3(0) = 0, \end{cases} \quad \text{and} \quad \begin{cases} a_0D_2\theta_3(0) + a_2D_2\theta_1(0) = 0, \\ a_1D_2\theta_2(0) + a_3D_2\theta_0(0) = 0, \end{cases}$$

respectively. This already allows us to get some necessary conditions on the derivatives of theta functions for a point to belong to the Gothic locus.

Proposition 6.7 *If the point $\boldsymbol{\tau} \in \mathbb{H}^2$ has the property that there is a nonzero odd theta function $\theta_X(\boldsymbol{u}) = \sum_i a_i(\boldsymbol{\tau})\theta_i(\boldsymbol{u})$ on $T_{\boldsymbol{\tau}}$ satisfying* (C2)–(C5)*, then $\mathcal{G}_D(\boldsymbol{\tau}) = 0$. In particular, for any $(X,\omega) \in \Omega G_D$, the Prym variety* $\mathrm{Prym}^\vee X$ *belongs to the vanishing locus of the Gothic modular form $\mathcal{G}_D(\boldsymbol{\tau})$.*

Proof By (20), the coefficients must satisfy

$$M\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} := \begin{pmatrix} D_2\theta_0(0) & 0 & D_2\theta_2(0) & 0 \\ 0 & D_2\theta_1(0) & 0 & D_2\theta_3(0) \\ D_2\theta_3(0) & 0 & D_2\theta_1(0) & 0 \\ 0 & D_2\theta_2(0) & 0 & D_2\theta_0(0) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system of equations must have a nontrivial solution, and therefore

$$\det(M) = (D_2\theta_0(0)\cdot D_2\theta_1(0) - D_2\theta_2(0)\cdot D_2\theta_3(0))^2 = \mathcal{G}_D(\boldsymbol{\tau})^2 = 0.$$

The second claim follows from Lemma 6.6. □

6.5 The vanishing locus of the Gothic modular form

We now start in the converse direction and analyze the vanishing locus of the Gothic modular form $\mathcal{G}_D$. For this purpose we note that the theta function θ_X defined in Theorem 6.1 equals

$$\theta_X(\boldsymbol{u}) = \Theta_a\mathcal{F}_b - \Theta_b\mathcal{F}_a,$$

where

$$\Theta_a(\boldsymbol{\tau},\boldsymbol{u}) = \theta_0(\boldsymbol{\tau},\boldsymbol{u})\cdot D_2\theta_2(\boldsymbol{\tau},0) - \theta_2(\boldsymbol{\tau},\boldsymbol{u})\cdot D_2\theta_0(\boldsymbol{\tau},0),$$
$$\Theta_b(\boldsymbol{\tau},\boldsymbol{u}) = \theta_1(\boldsymbol{\tau},\boldsymbol{u})\cdot D_2\theta_3(\boldsymbol{\tau},0) - \theta_3(\boldsymbol{\tau},\boldsymbol{u})\cdot D_2\theta_1(\boldsymbol{\tau},0)$$

and where $\mathcal{F}_a(\boldsymbol{\tau}) = D_1\Theta_a|_{\boldsymbol{u}=0}$ and $\mathcal{F}_b(\boldsymbol{\tau}) = D_1\Theta_b|_{\boldsymbol{u}=0}$ as defined in Theorem 6.1, too.

Proof of Theorem 6.1 The nonvanishing of θ_X on the complement of $\widetilde{\mathrm{Red}}_{23}$ follows from the factorization given above and the linear independence of the θ_i. Given Proposition 6.7 it remains to show that on the complement of $\widetilde{\mathrm{Red}}_{23}$ the divisor $Y = Y_{\boldsymbol{\tau}} := \{\theta_X = 0\}$ is indeed the φ–image of a Gothic Veech surface.

We first check the conditions (C1)–(C5) for Y. Differentiating θ_X implies that Y satisfies (C1) using the second row of the defining matrix, and Y satisfies (C2) and (C3) in the reformulation (20), as can be seen from the last two rows. From (16) we deduce

$$D_2\theta_X\big(\tfrac{1}{2}\lambda_2\big) = \begin{vmatrix} D_2\theta_3(0) & D_2\theta_2(0) & D_2\theta_1(0) & D_2\theta_0(0)\\ D_1\theta_0(0) & D_1\theta_1(0) & D_1\theta_2(0) & D_1\theta_3(0)\\ D_2\theta_0(0) & 0 & D_2\theta_2(0) & 0\\ 0 & D_2\theta_1(0) & 0 & D_2\theta_3(0)\end{vmatrix} = (\mathcal{F}_b+\mathcal{F}_a)\mathcal{G}_D,$$

$$D_2\theta_X\big(\tfrac{1}{2}(\lambda_2+\mu_2)\big) = \begin{vmatrix} -D_2\theta_3(0) & D_2\theta_2(0) & -D_1\theta_3(0) & D_2\theta_0(0)\\ D_1\theta_0(0) & D_1\theta_1(0) & D_1\theta_2(0) & D_1\theta_3(0)\\ D_2\theta_0(0) & 0 & D_2\theta_2(0) & 0\\ 0 & D_2\theta_1(0) & 0 & D_2\theta_3(0)\end{vmatrix} = (\mathcal{F}_b-\mathcal{F}_a)\mathcal{G}_D.$$

We deduce that for Y the conditions (C4)–(C5) hold as well.

Since θ_X is a section of the line bundle $\mathcal{L}_X$ of characteristic $c = \frac{1}{2}\lambda_1 + \frac{1}{2}\mu_1$, the argument in Lemma 6.5 implies that the multiplicity of $Y_{\boldsymbol{\tau}}$ at each point in the set $S^+ = \big\{\frac{1}{2}\lambda_2, \frac{1}{2}\mu_2, \frac{1}{2}(\lambda_2+\mu_2)\big\}$ is odd, in particular Y contains these points. By [3, Proposition 4.7.5(a)] each of the other 2–torsion points is not contained in Y or Y has even multiplicity there.

The case Y reduced with zero as its only singular point Then $X_{\boldsymbol{\tau}} = Y_{\boldsymbol{\tau}}^{\mathrm{norm}}$ is the desingularization at zero. We check the properties of a Gothic eigenform. Since $Y_{\boldsymbol{\tau}}$ is nonsingular at S^+ in the case under consideration, the one-form du_1 is an eigenform for real multiplication and has double zero at each of the three the points in S^+.

The involution (-1) on $T_{\boldsymbol{\tau}}$ induces an involution J on $X_{\boldsymbol{\tau}}$ that has six fixed points. The quotient $A = A_{\boldsymbol{\tau}} = X_{\boldsymbol{\tau}}/J$ is therefore a smooth curve of genus 1. The complement $T_{\boldsymbol{\tau}}^{\perp}$ of $T_{\boldsymbol{\tau}}$ in $\mathrm{Jac}(X)$ is $(1,6)$–polarized (see [3, Corollary 12.1.5] or the proof of Proposition 3.1). The pullback of the theta divisor on $\mathrm{Jac}(X)$ to $A^\vee$ has degree 2 since $X_{\boldsymbol{\tau}} \to A_{\boldsymbol{\tau}}$ is a double covering. We claim that the restriction of the theta divisor on $\mathrm{Jac}(X)$ to the complement $B^\vee$ of $A^\vee$ in $T_{\boldsymbol{\tau}}^\perp$ has degree 3. In fact, we may view $B^\vee$ as the complement to the image of the addition map $+\colon A^\vee \times T_{\boldsymbol{\tau}}^\perp \to \mathrm{Jac}(X)$. If $+$ factors through an isogeny (necessarily of exponent 2), then the image is $(1,1,3)$–polarized, hence $B^\vee$ has a polarization of type (3), again by [3, Corollary 12.1.5]. The case that $+$ is injective, hence that the image is $(1,2,6)$–polarized, contradicts loc. cit. Consequently, the map $\pi_{\boldsymbol{B}}\colon X \to \mathrm{Jac}(X) \to B$ is a covering of degree 3.

We claim that the map $\pi_{\boldsymbol{B}}\colon X_{\boldsymbol{\tau}} \to B_{\boldsymbol{\tau}}$ is odd. In fact, writing $j = (-1)$ on the elliptic curve $B_{\boldsymbol{\tau}}$ we compute that

$$j \circ \pi_{\boldsymbol{B}}(x) = [p_1 - x] = [p_1 - x] + [J(x) + x - 2p_1] = [J(x) - p_1] = \pi_{\boldsymbol{B}}(J(x)),$$

since $x + J(x) - 2p_1 \in A^\vee$. This argument also shows that the images of the points in P and Z are 2–torsion points in any quotient of $\mathrm{Jac}(X_{\boldsymbol{\tau}})/(A_{\boldsymbol{\tau}}^\vee)$, in particular in $B_{\boldsymbol{\tau}}$. Since $|\pi_{\boldsymbol{B}}(Z)| = 1$ on points in the Gothic locus, we deduce that $|\pi_{\boldsymbol{B}}(Z)| = 1$ over all of $X_{\boldsymbol{D}}$. We have indeed checked that $(X, du_1, \pi_{\boldsymbol{A}}, \pi_{\boldsymbol{B}})$ has all the Gothic properties under our assumptions on Y.

The case Y reduced with other singularities besides zero This does not occur. In fact, if $Y = \sum Y_i$ then $Y^2 = 12$ for a line bundle of type $(1,6)$ by Riemann–Roch. A triple point such as zero contributes 6 to Y^2. Each of the points in S^+ is either a triple point or du_1 has a double zero there, contributing 2 to Y^2 by increasing the genus of the component passing through this point. The total count implies that Y is nonsingular at S^+ and also nonsingular elsewhere besides zero, since the three double zeros at S^+ and the contribution at $0 \in T_{\boldsymbol{\tau}}$ already add up to 12.

The case Y nonreduced The above counting argument has to be refined for Y nonreduced, since eg a triple point might consist of $2Y_1$ and Y_2 intersecting transversally, hence contributing only 4 to Y^2. We first note that there are at most two branches through zero, since if Y contained nonreduced $a_1Y_1 + a_2Y_2 + a_3Y_3$ all meeting at zero, the odd multiplicity at the origin implies that $a_1 + a_2 + a_3$ is at least 5, and therefore $Y^2 > 12$.

We now write $Y = a_1Y_1 + a_2Y_2 + Y_R$ with $a_1 \geq a_2$, with Y_1 and Y_2 irreducible and passing through zero while Y_R is potentially reducible with no component passing through zero. In particular, $a_1 + a_2$ is odd.

Case $(a_1, a_2) = (3, 2)$ In this case 0 is the only intersection point of Y_1 and Y_2 and $Y_i^2 = 0$, so both components are elliptic curves. Consider the product $Y_1 \times Y_2$

with the polarization $2p_1^*\mathcal{O}_{Y_1}(0) \otimes 3p_2^*\mathcal{O}_{Y_2}(0)$. The addition map $Y_1 \times Y_2 \to T_{\boldsymbol{\tau}}$ is an isomorphism at the level of complex tori since $Y_1 \cdot Y_2 = 1$, and the pullback of $\mathcal{L}_X = \mathcal{O}_{T_{\boldsymbol{\tau}}}(Y)$ agrees with the $(2,3)$–product polarization. In particular, the map is an isomorphism of abelian surfaces and hence we are in $\widetilde{\mathrm{Red}}_{23}$ (see Proposition 8.6).

Case $(a_1, a_2) = (6, 1)$ Again 0 is the only intersection point, and $Y_i^2 = 0$, so both components are again elliptic curves. Odd parity of the theta function implies that $S^+ \subset Y_2$, but then du_1 induces an abelian differential on Y_2 with three zeroes of order ≥ 2, which is a contradiction.

Case $(a_1, a_2) = (4, 1)$ Again 0 is the only intersection point, and the case $Y_i^2 = 0$ for $i = 1, 2$ yields the same contradiction as in the case before. Hence we have $Y_2^2 = 4$ and $S^+ \subset Y_2$. This implies that on the one hand Y_2 has genus 3, and on the other du_1 induces an abelian differential on Y_2 with three zeroes of order ≥ 2, which is again a contradiction.

Case $(a_1, a_2) = (2, 1)$ We have the following possibilities:

(1) $\boldsymbol{Y_1 \cdot Y_2 = 1,\ Y_2^2 = 4}$ The curve Y_2 has genus 3 and du_1 induces an abelian differential on it with three zeroes of order ≥ 2.

(2) $\boldsymbol{Y_1 \cdot Y_2 = 2,\ Y_2^2 = 0}$ **or** $\boldsymbol{2}$ The curve Y_2 has genus 1 (or 2) and du_1 induces an abelian differential on it with two zeroes of order ≥ 2.

(3) $\boldsymbol{Y_1 \cdot Y_2 = 3,\ Y_2^2 = 0}$ The curve Y_2 has genus 1 and du_1 induces an abelian differential on it with a zero of order ≥ 2.

All these cases yield contradictions with the genus of the curve Y_2 and this completes the claim. □

7 Modular curves and the reducible locus

The main result in this section is an explicit parametrization of the *reducible locus*, the locus where the $(1,6)$–polarized abelian varieties with real multiplication split as a product of two elliptic curves E_1 and E_2, which are necessarily isogenous. This locus is a union of modular curves (also known as Hirzebruch–Zagier cycles or Shimura curves), in fact exclusively noncompact modular curves.

There are interesting similarities and differences to the reducible locus in the principally polarized case and the well-studied case of genus 2 Teichmüller curves. The main similarity is that the Teichmüller curves are disjoint from the reducible locus in both situations, Gothic and genus 2. The two cases also agree in the fact that the reducible locus has many components, several but not all of which can be distinguished by the precise endomorphism ring.

The main difference starts with the fact that the reducible locus decomposes into two subloci that can already be distinguished by degree of restriction of the polarization line bundle to E_1 and E_2. Since the product of these degrees is 6, the reducible locus decomposes into Red_{23} and Red_{16}, where the indices give the degree of the restricted line bundles. These loci are indeed disjoint, as we show in Section 7.3. The main result of this section is a description of the components of Red_{23} and a computation of their volumes.

7.1 Modular curves on Hilbert modular surfaces

The reducible locus consists of modular curves (also known as Hirzebruch–Zagier cycles or Shimura curves). Modular curves are the images of graphs of Möbius transformations in $\mathbb{H}^2$ that descend to algebraic curves in the Hilbert modular surface. We recall the precise definition, adapted to our Hilbert modular surfaces $X_D^{\mathfrak{a}}(\mathfrak{b}) = \mathrm{SL}(\mathfrak{a}\mathfrak{b} \oplus \mathfrak{a}^\vee)\backslash\mathbb{H}^2$. Let us define the ideal $M = \sqrt{D}\,\mathfrak{a}^2\mathfrak{b}$. We say that $U \in \mathrm{SL}_2(K)$ is a *generator matrix* for the Hilbert modular group $\mathrm{SL}(\mathfrak{a}\mathfrak{b} \oplus \mathfrak{a}^\vee)$ if it is of the form

$$U = \begin{pmatrix} a\sqrt{D} & \mu \\ -\mu^\sigma & Ab\sqrt{D} \end{pmatrix}, \quad \text{where } a, b \in \mathbb{Z},\ \mu \in M \text{ and } A = N(M),$$

and we define the modular curves F_U to be the image in $X_D^{\mathfrak{a}}(\mathfrak{b})$ of the set

$$\left\{(\tau_1, \tau_2) \in \mathbb{H}^2 : (\,\tau_2 \;\; 1\,)U\begin{pmatrix}\tau_1 \\ 1\end{pmatrix} = a\sqrt{D}\tau_1\tau_2 - \mu^\sigma\tau_1 + \mu\tau_2 + Ab\sqrt{D} = 0\right\}.$$

The generator matrix U is *primitive* if it is not divisible by any natural number $m > 1$. For any integer $N > 0$, the *modular curve* F_N is defined as the union

$$F_N = \bigcup_{\substack{U \text{ primitive} \\ \det(U) = AN}} F_U.$$

The components of F_N and their geometry (cusps, fixed points) were intensely studied by Hirzebruch and his students (see the survey in [10, Chapter V]). Most notably the volumes of the union

$$T_N = \bigcup_{\det(U) = AN} F_U = \bigcup_{\ell^2 \mid N} F_{N/\ell^2}$$

are the coefficients of a modular form, in fact an Eisenstein series of weight 2 for some character.

This however does not yet yield formulas for the volume of Red_{23}, since the latter turns out to be a union of modular curves, but not of the entire curves F_N. In fact, F_N can be decomposed as the union of the curves $F_N(\nu)$ for $\nu \in M/\sqrt{D}M$, where

$$F_N(\nu) = \bigcup \{F_U : U \text{ is primitive with } \det(U) = AN \text{ and } \nu(U) = \nu\}.$$

In the case of abelian surfaces with principal polarization the reducible locus was written in terms of $F_N(\nu)$ by [15]. However, the $F_N(\nu)$ are sometimes still reducible and this decomposition does not directly yield a volume formula, so we proceed differently for our $(1,6)$–polarization.

7.2 The $(2,3)$–reducible locus

Let us define the $(2,3)$*–reducible locus* Red_{23} as the locus inside the moduli space $\mathcal{A}_{2,(2,3)}$ of $(2,3)$–polarized abelian surfaces consisting of products $E_1 \times E_2$ of elliptic curves with the natural $(2,3)$–polarization $2p_1^*\mathcal{O}_{E_1}(0) \otimes 3p_2^*\mathcal{O}_{E_2}(0)$. For each $\mathcal{O}_D$–ideal $\mathfrak{b}$ of norm 6, we will write $\mathrm{Red}_{23}(\mathfrak{b})$ for the pullback of Red_{23} to $X_D(\mathfrak{b})$.

Theorem 7.1 *Let $D = f^2 D_0$ be a positive quadratic discriminant with conductor f. There is a bijective correspondence between irreducible components of the $(2,3)$–reducible locus with a chosen proper real multiplication by $\mathcal{O}_D$ and the set of prototypes*

$$\mathcal{P}_D = \{[\ell, e, m] \in \mathbb{Z}^3 : \ell, m > 0,\ D = e^2 + 24\ell^2 m \text{ and } \gcd(e, \ell, f) = 1\}.$$

More precisely, the component parametrized by the prototype $\mathcal{P} = [\ell, e, m]$ is the image of a Shimura curve in the Hilbert modular surface $X_D^{\mathfrak{a}}(\mathfrak{b})$ corresponding to the ideals $\mathfrak{a} = \frac{1}{\sqrt{D}}\big(2\ell, \frac{1}{2}(e + \sqrt{D})\big)$ and $\mathfrak{b} = \big(6, \frac{1}{2}(r + \sqrt{D})\big)$, where

$$\begin{cases} e & \text{if } D \equiv 0 \bmod 2, \\ e + 6 & \text{if } D \equiv 1 \bmod 2. \end{cases}$$

The image in $\mathcal{A}_{2,(2,3)}$ of the Shimura curve given by $\mathcal{P} = [\ell, e, m]$ is isomorphic to $\Gamma_0(m)\backslash\mathbb{H}$.

We split the proof into a series of lemmas.

Lemma 7.2 *The period matrix of an abelian surface parametrizing a point in* Red_{23} *with real multiplication can be assumed to be*

$$\Pi_m(\tau) = \left(\begin{matrix} 2\tau & 0 \\ 0 & 3m\tau \end{matrix}\,\middle|\, \begin{matrix} 2 & 0 \\ 0 & 3 \end{matrix}\right) \quad \text{for some } \tau \in \mathbb{H} \text{ and } 0 < m \in \mathbb{Z}$$

with the polarization given by the standard form $\left(\begin{smallmatrix} 0 & P_{23} \\ -P_{23} & 0 \end{smallmatrix}\right)$.

Proof Since we will be interested in the components of this locus that lie in some Hilbert modular surface, let us assume furthermore that E_1 and E_2 are isogenous elliptic curves, so the left block of the period matrix $\Pi_{m,n}(\tau)$ is a diagonal matrix with entries $(2\tau, 3(m\tau+n))$ with $m,n \in \mathbb{Q}$. Positive-definiteness of the period matrix implies $m>0$. We define the matrices

$$M_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & aU+cV & 0 & (dV+b)/L \\ b & 0 & dU & 0 \\ 0 & cL & 0 & d \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} x & 0 & y & 0 \\ 0 & xq & 0 & yp \\ p & 0 & q & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We first argue that we can take $n=0$. Write $m=U/L$ and $n=V/L$ with $\gcd(U,V,L)=1$. Take d such that $\gcd(dU, L-dV)=1$. (To show the existence, consider d_i with $\gcd(d_i,L)=1$. Among a collection of d_i with $\gcd(d_i-d_j,U)=1$ with more elements than B has divisors, one will work.) Let $b=L-dV$ and take a and c such that $adU-c(L-dV)=1$. Then the matrix M_1 has integral coefficients, belongs to the symplectic group $\mathrm{Sp}^P_{2g}(\mathbb{Z})$ and takes $\Pi_{m,n}(\tau)$ to $\Pi_{m,0}(\tau')$ for some τ'.

To show that we may assume $m \in \mathbb{Z}$, we write $m=p/q$ and take $x,y \in \mathbb{Z}$ such that $xq-yp=1$. Then the matrix M_2 belongs to $\mathrm{Sp}^P_{2g}(\mathbb{Z})$ and takes $\Pi_{p/q,n}(\tau)$ to $\Pi_{pq,0}(\tau')$ for some τ'. □

Lemma 7.3 *An abelian surface in the* $(2,3)$*–reducible locus contains a unique elliptic curve with a polarization of type* (2) *and a unique elliptic curve with a polarization of type* (3).

In particular, a matrix $M \in \mathrm{Sp}^P_{2g}(\mathbb{Z})$ *taking the locus* $\{\Pi_m(\tau) : \tau \in \mathbb{H}\}$ *into some locus* $\{\Pi_{m_2}(\tau), \tau \in \mathbb{H}\}$ *consists of matrices diagonal in each of its four blocks* (like the matrices M_1 and M_2 above).

Proof The type of a polarization is translation invariant. So we may assume that the elliptic curve in question passes through the origin. Such an elliptic curve E in a product of elliptic curves is determined by a rational slope in the universal cover. We may assume this slope is $(2x,3y)$ with $x,y \in \mathbb{Z}$ coprime and both different from zero, since we already know the polarizations of the curves with slope $(1,0)$ and $(0,1)$. If we denote by a_1, a_2, b_1, b_2 the symplectic basis corresponding to the column vectors of $\Pi_m(\tau)$, lattice points in E are given by the multiples of $f_1 = xa_1 + (y/m)a_2$ and $f_2 = xb_1 + yb_2$ that have integral coefficients. This implies that $m \mid y$ and that the type of the polarization on E is $\langle f_1, f_2\rangle = 2x^2 + 3my^2$, therefore proving the claim. □

Lemma 7.4 *The analytic representation of real multiplication by $\gamma = \frac{1}{2}(D+\sqrt{D})$ on an abelian surface with period matrix $\Pi_m(\tau)$ with $m \in \mathbb{Z}$ is given by*

$$A_\gamma = \begin{pmatrix} \frac{1}{2}(D+e) & 2\ell \\ 3\ell m & \frac{1}{2}(D-e) \end{pmatrix},$$

with $e, \ell \in \mathbb{Z}$ and $D = e^2 + 24\ell^2 m$.

The real multiplication defined by $[\ell, e, m]$ and $[-\ell, e, m]$ are equivalent, whereas the real multiplication defined by $[\ell, e, m]$ and $[-\ell, -e, m]$ are Galois conjugate.

Proof The abelian surface $T_{\tau,m}$ given by the period matrix $\Pi_m(\tau)$ admits real multiplication by $\mathcal{O}_D$, if and only if there are matrices $A_\gamma \in \mathrm{GL}_2(\mathbb{Q})$ and $R_\gamma \in \mathrm{Sp}(4,\mathbb{Z})$ that are the analytic and rational representations of $\gamma = \frac{1}{2}(D+\sqrt{D})$, ie such that $A\Pi_m(\tau) = \Pi_m(\tau)R_\gamma$, $\mathrm{tr}(A_\gamma) = D$ and $\det(A_\gamma) = \frac{1}{4}(D^2 - D)$. Together with the self-adjointness of R_γ this implies that

$$A_\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad R_\gamma = \begin{pmatrix} a & \frac{3}{2}bm & 0 & 0 \\ \frac{2c}{3m} & d & 0 & 0 \\ 0 & 0 & a & \frac{3}{2}b \\ 0 & 0 & \frac{2}{3}c & d \end{pmatrix},$$

where $d = D - a$ and $ad - bc = \frac{1}{4}(D^2 - D)$, and moreover that $c = \frac{3}{2}bm \in 3\mathbb{Z}$. Integrality of R_γ implies that $a, d, \ell = \frac{1}{2}b \in \mathbb{Z}$ and we set $e = 2a - D$.

Finally, the real multiplications defined by $[\ell, e, m]$ and $[-\ell, e, m]$ are conjugate under the isomorphism $-\mathrm{Id}|_{E_2}$. The claim about Galois conjugation is obvious. □

Proof of Theorem 7.1 Suppose we are given a tuple $[\ell, e, m]$ as in the theorem. We check that the real multiplication on the locus of matrices $\Pi_m(\tau)$ given by Lemma 7.4 is indeed proper. The action is not proper if γ/k also acts for some $1 < k \in \mathbb{Z}$, ie if all the entries of R_γ are divisible by k. This implies $k \mid \gcd(e, \ell, f)$ and conversely this divisibility is also sufficient for the action to be nonproper.

Next we show that the images in $\mathcal{A}_{2,(2,3)}$ of the loci given by $\Pi_m(\tau)$ for $m \in \mathbb{Z}$ are pairwise disjoint. Otherwise there exists a symplectic matrix taking the locus $\Pi_m(\tau)$ into $\Pi_{m_2}(\tau)$. By Lemma 7.3 this matrix is diagonal in each block. It suffices thus consider only matrices of the form

$$M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & ka/m & 0 & kb \\ b & 0 & d & 0 \\ 0 & c/k & 0 & md/k \end{pmatrix}$$

with integral entries and $ad-bc=1$, where a priori $k \in \mathbb{Q}$ and $m_2 = k^2/m$, which implies $k \in \mathbb{Z}$. Since c and d have no common divisor, this implies $k \mid m$, hence $k = m = m_2$.

This argument also gives the stabilizer of the locus $\{\Pi_m(\tau) : \tau \in \mathbb{H}\}$ in the symplectic group. Such a matrix is of the form of M with $k=m$ and integrality of the entries implies that the quotient curve is isomorphic to $\Gamma_0(m)\backslash\mathbb{H}$.

Finally, we determine for each component in Red_{23} with chosen real multiplication given by prototype $\mathcal{P} = [\ell, e, m]$ a Hilbert modular surface and a Siegel modular embedding that maps to this component. To exhibit a Siegel modular embedding, we need to find eigenform coordinates, ie a matrix that diagonalizes the analytic representation of real multiplication given by A_γ in Lemma 7.4. Such a matrix is given by

$$V_{\mathcal{P}} = \begin{pmatrix} 1 & 1 \\ \lambda & \lambda^\sigma \end{pmatrix}, \quad \text{where } \lambda := \lambda_{\mathcal{P}} = \frac{-e+\sqrt{D}}{4\ell}.$$

Indeed, associated to the prototype $\mathcal{P}$ one can produce the quadratic form $Q_{\mathcal{P}} = [2\ell, e, -3\ell m]$ of discriminant D, so that λ is precisely the quadratic irrationality of $Q_{\mathcal{P}}$, and Lemma 4.4 ensures that the first column $(1,\lambda)$ of the matrix $V_{\mathcal{P}} = B_{\boldsymbol{\eta}}$ is a $(2,3)$–symplectically adapted basis for a fractional ideal $\mathfrak{a}^\vee$ of $\mathcal{O}_D$, and the first column of the matrix $(V_{\mathcal{P}}^{-1} P_{23})^T$ is a basis $\frac{1}{\sqrt{D}}(-4\ell\lambda^\sigma, 6\ell)$ of the ideal $\mathfrak{a}\mathfrak{b}$. A simple calculation shows that $\mathfrak{a} = \frac{1}{\sqrt{D}}\langle 2\ell, -2\ell\lambda^\sigma\rangle$ and therefore, writing $\mathfrak{b} = \big\langle 6, \frac{1}{2}(r+\sqrt{D})\big\rangle$ for $r \in \mathbb{Z}$, the following equality of ideals determines r:

$$\begin{aligned}\sqrt{D}\,\mathfrak{a}\mathfrak{b} &= \langle 4\ell\lambda^\sigma, 6\rangle \\ &= \langle 2\ell, -2\ell\lambda^\sigma\rangle\big\langle 6, \tfrac{1}{2}(r+\sqrt{D})\big\rangle \\ &= \big\langle 12\ell, -12\ell\lambda^\sigma, \ell(r+\sqrt{D}), \tfrac{1}{4}((D+er)+\sqrt{D}(e+r))\big\rangle.\end{aligned}$$

To verify this, it is enough to prove that the second ideal lies in the first one, and one checks that this holds for r as stated in the theorem. □

In order to translate the theorem into Euler characteristics, we define another set of prototypes, closely related to standard quadratic irrationalities. For a quadratic discriminant $D = f^2 D_0$ with conductor f, we let

$$\mathcal{P}_k(D) = \big\{[a,b,c] \in \mathbb{Z}^3 : a > 0 > c,\ D = b^2 - 4\cdot k\cdot ac \text{ and } \gcd(f, b, c/c_0) = 1\big\}, \tag{21}$$

where c_0 is the square-free part of c.

The following result gives an explicit formula for the Euler characteristics of the reducible loci $\mathrm{Red}_{23}(\mathfrak{b})$ in terms of prototypes.

Lemma 7.5 *The Euler characteristic of the reducible locus* $\mathrm{Red}_{23}(\mathfrak{b})$ *in the Hilbert modular surface* $X_D(\mathfrak{b})$ *is given by*

$$\chi(\mathrm{Red}_{23}(\mathfrak{b})) = -\frac{1}{6k} \sum_{[a,b,c]\in\mathcal{P}_6(D)} a$$

for each of the ideals $\mathfrak{b}$ *of norm* 6 *in* $\mathcal{O}_D$, *where* k *is the number of* $\mathcal{O}_D$*–ideals of norm* 6.

Proof By Theorem 7.1, the different components of Red_{23} in $\mathcal{A}_{2,(2,3)}$ are isomorphic to certain $\Gamma_0(m)\backslash\mathbb{H}$. Note that

$$\chi(\Gamma_0(m)) = -\frac{m}{6} \prod_{\substack{p\,|\,m \\ p \text{ prime}}} \Bigl(1+\frac{1}{p}\Bigr).$$

Moreover, it is easy to show that, for each D,

$$\sum_{[\ell,e,m]\in\mathcal{P}_D} \chi(\Gamma_0(m)) = -\frac{1}{6} \sum_{[a,b,c]\in\mathcal{P}_6(D)} a.$$

Let us now suppose that $D \equiv 4, 9, 16 \bmod 24$, so that there exist two ideals $\mathfrak{b} \neq \mathfrak{b}^\sigma$ of norm 6. This implies that precomposition of a chosen real multiplication $\mathcal{O}_D \to \mathrm{End}_{T_\tau}$ with Galois conjugation gives a point on a different Hilbert modular surface, the one with the conjugate $\mathfrak{b}^\sigma$. Each component of $\mathcal{P}_D$ is in the image of some Hilbert modular surface $X_D^{\mathfrak{a}}(\mathfrak{b})$ with $\mathfrak{b}$ determined in Theorem 7.1 and thus, by the change of cusp explained in Section 4.5, also on the standard Hilbert modular surface $X_D(\mathfrak{b})$. Precomposition with Galois conjugation corresponds to $e \mapsto -e$. Consequently, on $X_D^{\mathfrak{a}}(\mathfrak{b})$,

$$\chi(\mathrm{Red}_{23}(\mathfrak{b})) = \frac{1}{2} \sum_{[\ell,e,m]\in\mathcal{P}_D} \chi(\Gamma_0(m)) = -\frac{1}{2}\cdot\frac{1}{6} \sum_{[a,b,c]\in\mathcal{P}_6(D)} a.$$

In the case $D \equiv 0, 12 \bmod 24$ there exists only one ideal $\mathfrak{b} = \mathfrak{b}^\sigma$ of norm 6, and thus the map $X_D(\mathfrak{b}) \to \mathcal{A}_{2,(2,3)}$ is generically two-to-one onto its image. In the particular case of $\mathrm{Red}_{23}(\mathfrak{b})$, components corresponding to prototypes $[\ell, e, m]$ and $[\ell, -e, m]$ are sent to the same component of Red_{23}, whereas components corresponding to prototypes $[\ell, 0, m]$ lie in the ramification locus of $X_D(\mathfrak{b}) \to \mathcal{A}_{2,(2,3)}$. As a consequence,

$$\chi(\mathrm{Red}_{23}(\mathfrak{b})) = \sum_{[\ell,e,m]\in\mathcal{P}_D} \chi(\Gamma_0(m)) = -\frac{1}{6} \sum_{[a,b,c]\in\mathcal{P}_6(D)} a.$$

Finally, if $D \equiv 1 \bmod 24$, there exist four ideals $\mathfrak{b}_1$, $\mathfrak{b}_1^\sigma$, $\mathfrak{b}_2$ and $\mathfrak{b}_2^\sigma$ of norm 6. For the same reason as above, the forgetful map from $\mathrm{Red}_{23}(\mathfrak{b}_i)$ to $\mathcal{A}_{2,(2,3)}$ is an isomorphism onto its image. Precomposition with Galois conjugation corresponds again to $e \mapsto -e$. We conclude

$$\chi(\mathrm{Red}_{23}(\mathfrak{b}_1)) + \chi(\mathrm{Red}_{23}(\mathfrak{b}_2)) = \frac{1}{2} \sum_{[\ell,e,m]\in\mathcal{P}_D} \chi(\Gamma_0(m)) = -\frac{1}{2}\cdot\frac{1}{6} \sum_{[a,b,c]\in\mathcal{P}_6(D)} a.$$

Using Lemma 7.6, we deduce that $\chi(\mathrm{Red}_{23}(\mathfrak{b}_1)) = \chi(\mathrm{Red}_{23}(\mathfrak{b}_2))$ and the result follows. □

Lemma 7.6 *For $D \equiv 1 \bmod 24$ not a square,*

$$\sum_{\substack{b\equiv 1,11 \bmod 12\\ 0<b<\sqrt{D}}} \sigma_1\big(\tfrac{1}{24}(D-b^2)\big) = \sum_{\substack{b\equiv 5,7 \bmod 12\\ 0<b<\sqrt{D}}} \sigma_1\big(\tfrac{1}{24}(D-b^2)\big).$$

Proof Recall the definition

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \sum_{b\geq 1}\Big(\frac{12}{b}\Big)q^{b^2/24}$$

of the Dedekind η–function and recall that

$$E_2(q) = -\frac{1}{24} + \sum_{n\geq 1}\sigma_1(n)q^n = \frac{\eta'(q)}{\eta(q)},$$

where $' = q\frac{\partial}{\partial q}$. The statement of the lemma is now equivalent to

$$0 = [q^{D/24}](E_2(q)\eta(q)) = [q^{D/24}]\eta'(q),$$

which obviously holds for D nonsquare by definition of η. □

Finally, we relate the components given by prototypes at least coarsely to the usual classification of modular curves.

Proposition 7.7 *Let $\mathcal{P} = [\ell, e, m] \in \mathcal{P}_D$ be a prototype for real multiplication by $\mathcal{O}_D$ belonging to $X_D^{\mathfrak{a}}(\mathfrak{b})$. The corresponding component $F_{\mathcal{P}}$ of $\mathrm{Red}_{23}(\mathfrak{b})$ is an irreducible component of the modular curve $F_{g^2m}(\mu)$, where $\mu = g(e+\sqrt{D})/\sqrt{D}$ and $g = \gcd(e,\ell)$.*

Proof By the proof of Theorem 7.1, we know that the Siegel modular embedding determined by this prototype is given by $V_{\mathcal{P}}$, and therefore one has

$$\begin{pmatrix} 2\tau & 0 \\ 0 & 3m\tau \end{pmatrix} = V_{\mathcal{P}} \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} V_{\mathcal{P}}^T = \begin{pmatrix} \tau_1 + \tau_2 & \lambda\tau_1 + \lambda^\sigma\tau_2 \\ \lambda\tau_1 + \lambda^\sigma\tau_2 & \lambda^2\tau_1 + (\lambda^\sigma)^2\tau_2 \end{pmatrix}$$

for a curve $(\tau_1, \tau_2) = (\tau_1(\tau), \tau_2(\tau))$ in $\mathbb{H}^2$, where $\lambda := \lambda_{\mathcal{P}} = (-e + \sqrt{D})/(4\ell)$.

In particular, this curve necessarily lies in the curve $\lambda\tau_1 + \lambda^\sigma\tau_2 = 0$, which obviously agrees with $F_{g^2m}(\mu)$. The only thing left to prove is that μ is primitive in M and $N(\mu) = N(M)g^2m$.

From the calculations in the proof of Theorem 7.1, one gets

$$M = \sqrt{D}\,\mathfrak{a}^2\mathfrak{b} = \frac{4\ell}{\sqrt{D}}\langle 3\ell, \ell\lambda^\sigma, e\lambda^\sigma\rangle,$$

and $N(M) = 24\ell^2/D$. Since $\mu = (4\ell/\sqrt{D})g\lambda^\sigma$, it is clear that μ is primitive in M, and $N(\mu) = 24\ell^2g^2m/D$. □

7.3 The $(1,6)$–reducible locus

To put the results of the previous section in perspective we compare here loci of reducible abelian surfaces according to their polarization. The moduli space $\mathcal{A}_{2,(1,6)}$ of $(1,6)$–polarized abelian surfaces is of course isomorphic to $\mathcal{A}_{2,(2,3)}$ used in the previous section, an isomorphism being induced by multiplication of period matrices by $\operatorname{diag}\left(\frac{1}{2}, 2\right)$ from the left.

In $\mathcal{A}_{2,(1,6)}$ (and by the above isomorphism thus also in $\mathcal{A}_{2,(2,3)}$) one can similarly define the $(1,6)$–*reducible locus* Red_{16} of products $E_1 \times E_2$ of isogenous elliptic curves with the natural $(1,6)$–polarization $p_1^*\mathcal{O}_{E_1}(0) \otimes 6p_2^*\mathcal{O}_{E_2}(0)$. With the arguments of Lemma 7.2 we can put period matrices in Red_{16} in the form

$$\Pi_{\tau,m} = \left(\begin{array}{cc|cc} \tau & 0 & 1 & 0 \\ 0 & m\tau & 0 & 6 \end{array}\right) \quad \text{for some } \tau \in \mathbb{H} \text{ and } 0 < m \in \mathbb{Q},$$

with the polarization given by the standard form $\left(\begin{smallmatrix} 0 & P_{16} \\ -P_{16} & 0 \end{smallmatrix}\right)$. The remaining arguments in the previous section work verbatim in this case as well and yield:

Theorem 7.8 *Let $D = f^2D_0$ be a positive quadratic discriminant with conductor f. There is a bijective correspondence between irreducible components of the $(1,6)$–reducible locus admitting proper real multiplication by $\mathcal{O}_D$ and the set of prototypes $\mathcal{P}_D$ as defined in Theorem 7.1.*

In particular, Red_{23} and Red_{16} have the same Euler characteristics. However:

Proposition 7.9 *The loci* Red_{23} *and* Red_{16} *are disjoint in* $\mathcal{A}_{2,(2,3)}$.

Proof The degrees of elliptic curves on an abelian surface in Red_{23} are the values of the quadratic form $2x^2+3my^2$ for $x, y, \in \mathbb{Z}$, as computed in the proof of Lemma 7.3. This form never takes the value 1. □

8 The divisor of the Gothic modular form

In this section we calculate the vanishing locus of the Gothic modular form.

Theorem 8.1 *Let* $G_D(\mathfrak{b})$ *denote the union of components of the Torelli-image of* G_D *lifted to* $X_D(\mathfrak{b})$ *such that* du_1 *induces the eigenform* ω *at each point* (X,ω). *Then*

$$\mathrm{div}(\mathcal{G}_D) = G_D(\mathfrak{b}) + 2\,\mathrm{Red}_{23}(\mathfrak{b}).$$

The theorem will be a direct consequence of Propositions 8.3 and 8.4 below, together with Theorem 6.1.

8.1 The Fourier expansion of the Gothic modular form

For each cusp $\mathfrak{a} \in X_D(\mathfrak{b})$ let $\boldsymbol{\eta} = (\eta_1, \eta_2)$ be a basis of $\mathfrak{a}^\vee$ that is $(2,3)$–symplectically adapted, determining the $\mathcal{O}_D$–module $\mathfrak{ab} \oplus \mathfrak{a}^\vee$.

We want to write down the Fourier expansion of $\mathcal{G}_D$ around this cusp using the Siegel modular embedding given by the matrix $B := B_{\boldsymbol{\eta}} = \left(\begin{smallmatrix} \eta_1 & \eta_1^\sigma \\ \eta_2 & \eta_2^\sigma \end{smallmatrix}\right)$ as in Section 4.5, so that the cusp $\mathfrak{a}$ of $X_D(\mathfrak{b})$ corresponds to the cusp at infinity of $X_D^{\mathfrak{a}}(\mathfrak{b})$. The stabilizer of ∞ agrees with the subgroup

$$\mathrm{SL}(\mathfrak{ab} \oplus \mathfrak{a}^\vee)_\infty = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & \varepsilon^{-1} \end{pmatrix} : \varepsilon \in \mathcal{O}_D^*,\ \mu \in M := \sqrt{D}\,\mathfrak{a}^2\mathfrak{b} \right\}.$$

For any Hilbert modular form f one has $f(\boldsymbol{\tau}+\mu) = f(\boldsymbol{\tau})$ for $\mu \in M$, and therefore one can write the Fourier expansion

$$f(\boldsymbol{\tau}) = \sum_{\nu \in M^\vee} a_\nu \boldsymbol{e}(\mathrm{tr}(\nu\boldsymbol{\tau})),$$

where $\mathrm{tr}(\nu\boldsymbol{\tau}) = \nu\tau_1 + \nu^\sigma\tau_2$ and $M^\vee = (\sqrt{D}\,\mathfrak{a}^2\mathfrak{b})^\vee = \frac{1}{\sqrt{D}}\mathfrak{a}^\vee(\mathfrak{ab})^{-1}$.

Write $\rho_{\boldsymbol{\eta}}(\boldsymbol{x}) := x_1\eta_1 + x_2\eta_2$ for $\boldsymbol{x} = (x_1, x_2) \in \mathbb{Q}^2$. We will drop the subindex from $\rho_{\boldsymbol{\eta}}$ whenever the choice of basis is clear.

Proposition 8.2 *The Fourier expansion of $\mathcal{G}_D$ around the cusp $\mathfrak{a}$ is given by*

$$\mathcal{G}_D(\tau) = 8\pi^2 i \cdot \Bigg(\sum_{\substack{\boldsymbol{a}\in\Lambda_{0,1/2}\\ \boldsymbol{b}\in\Lambda_{1/2,1/6}}} k_{\boldsymbol{a},\boldsymbol{b}} q_1^{\rho_\eta(\boldsymbol{a})^2+\rho_\eta(\boldsymbol{b})^2} q_2^{\rho_\eta^\sigma(\boldsymbol{a})^2+\rho_\eta^\sigma(\boldsymbol{b})^2} - \sum_{\substack{\boldsymbol{a}\in\Lambda_{1/2,1/2}\\ \boldsymbol{b}\in\Lambda_{0,1/6}}} k_{\boldsymbol{a},\boldsymbol{b}} q_1^{\rho_\eta(\boldsymbol{a})^2+\rho_\eta(\boldsymbol{b})^2} q_2^{\rho_\eta^\sigma(\boldsymbol{a})^2+\rho_\eta^\sigma(\boldsymbol{b})^2} \Bigg),$$

where $k_{\boldsymbol{a},\boldsymbol{b}} = (-1)^{a_2+b_2}\rho_{\boldsymbol{\eta}}^\sigma(\boldsymbol{a})\rho_{\boldsymbol{\eta}}^\sigma(\boldsymbol{b})$ *and* $\Lambda_{\epsilon,\delta} = \mathbb{Z}^2 + (\epsilon,\delta)^T$.

8.2 Vanishing order along Red_{23}

The reducible loci $\mathrm{Red}_{23}(\mathfrak{b})$ turn out to lie in the vanishing locus of the Gothic modular form $\mathcal{G}_D$. We next calculate the corresponding vanishing order.

Recall that, by the results of Section 7.1, the reducible loci $\mathrm{Red}_{23}(\mathfrak{b})$ decompose into different components $F_{\mathcal{P}}$ indexed by prototypes in $\mathcal{P}_D$.

Proposition 8.3 *The Gothic modular form $\mathcal{G}_D$ vanishes to order 2 along the reducible locus* $\mathrm{Red}_{23}(\mathfrak{b})$.

Proof Let $\mathcal{P} = [\ell, e, m] \in \mathcal{P}_D$ be the prototype corresponding to a component $F_{\mathcal{P}} \subset F_{g^2m}(\mu)$ of $\mathrm{Red}_{23}(\mathfrak{b})$ as in Proposition 7.7. Recall from Theorem 7.1 that the curve $F_{\mathcal{P}}$ lives in the Hilbert modular surface $X_D(\mathfrak{b})$ determined by the $(2,3)$–symplectically adapted basis $\mathfrak{a}^\vee = \langle 1, \lambda\rangle$, where $\lambda = (-e+\sqrt{D})/(4\ell)$. Note that λ is precisely the irrationality associated to the quadratic form $Q_{\mathcal{P}} = [2\ell, e, -3\ell m]$ of discriminant D. Moreover, by Proposition 7.7 the curve $F_{\mathcal{P}}$ can be parametrized by $\tau \mapsto (\alpha\tau, \alpha^\sigma\tau)$, where $\alpha = -\frac{1}{4\ell g}\mu = \lambda^\sigma/\sqrt{D}$.

In the chosen basis for $\mathfrak{a}^\vee$, one has $\rho(x_1,x_2) = x_1 + x_2\lambda$ and therefore

$$\mathrm{tr}(\alpha\rho(x_1,x_2)^2) = \frac{1}{2\ell}\big(x_1^2 + \tfrac{3}{2}mx_2^2\big).$$

Now, restricted to $F_{\mathcal{P}}$ the coordinates q_1 and q_2 become q^α and q^{α^σ}, respectively, where $q = \boldsymbol{e}(\tau)$. In particular, up to an $8\pi^2 i$ factor, the expression for $\mathcal{G}_D$ from Proposition 8.2 along $F_{\mathcal{P}}$ reads

$$\mathcal{G}_D(\tau) = \sum_{\substack{\boldsymbol{a}\in\Lambda_{0,1/2}\\ \boldsymbol{b}\in\Lambda_{1/2,1/6}}} (-1)^{a_2+b_2}(a_1+a_2\lambda^\sigma)(b_1+b_2\lambda^\sigma) q^{(1/g)(a(a_1^2+b_1^2)-c(a_2^2+b_2^2))} - \sum_{\substack{\boldsymbol{a}\in\Lambda_{1/2,1/2}\\ \boldsymbol{b}\in\Lambda_{0,1/6}}} (-1)^{a_2+b_2}(a_1+a_2\lambda^\sigma)(b_1+b_2\lambda^\sigma) q^{(1/g)(a(a_1^2+b_1^2)-c(a_2^2+b_2^2))}.$$

Due to the symmetries of the lattices considered, the q–exponents of the terms corresponding to different choices of the signs $\pm a_1$ and $\pm b_1$ are the same. Moreover, the flip $(a_1, a_2; b_1, b_2) \mapsto (b_1, a_2; a_1, b_2)$ gives a bijection between the lattice $\Lambda_{0,1/2} \times \Lambda_{1/2,1/6}$ appearing in the first summand and the lattice $\Lambda_{1/2,1/2} \times \Lambda_{0,1/6}$ appearing in the second one.

As a consequence, the coefficients of the terms corresponding to $(a_1, a_2; b_1, b_2)$ and $(-a_1, a_2; -b_1, b_2)$ in the first lattice and their flipped images $(b_1, a_2; a_1, b_2)$ and $(-b_1, a_2; -a_1, b_2)$ in the second one give (up to a $(-1)^{a_2+b_2}$ factor)

$$\begin{aligned}(a_1b_1 + a_2b_2(\lambda^\sigma)^2) + \lambda^\sigma(a_1b_2 + a_2b_1) + (a_1b_1 + a_2b_2(\lambda^\sigma)^2) - \lambda^\sigma(a_1b_2 + a_2b_1)\\ -(a_1b_1 + a_2b_2(\lambda^\sigma)^2) - \lambda^\sigma(b_1b_2 + a_1a_2) - (a_1b_1 + a_2b_2(\lambda^\sigma)^2) + \lambda^\sigma(b_1b_2 + a_1a_2),\end{aligned}$$

which sums up to zero, and therefore $\mathcal{G}_D$ vanishes along $F_\mathcal{P}$.

In order to determine the vanishing order, we will study the highest order k such that all the k–derivatives of $\mathcal{G}_D$ vanish along $F_\mathcal{P}$. The Fourier expansions of the restriction of the derivatives $\partial^k \mathcal{G}_D/\partial \tau_1^k$ and $\partial^k \mathcal{G}_D/\partial \tau_2^k$ to the Shimura curve $F_\mathcal{P}$ are given by the same series as above, with the coefficients replaced by

$$(-1)^{a_2+b_2}(a_1 + a_2\lambda^\sigma)(b_1 + b_2\lambda^\sigma)(\rho(\boldsymbol{a})^2 + \rho(\boldsymbol{b})^2)^k$$

in the case of $\partial^k \mathcal{G}_D/\partial \tau_1^k$ and the equivalent expression with $(\rho^\sigma(\boldsymbol{a})^2 + \rho^\sigma(\boldsymbol{b})^2)^k$ for $\partial^k \mathcal{G}_D/\partial \tau_2^k$.

The coefficients of $\partial \mathcal{G}_D/\partial \tau_1$ corresponding to $(a_1, a_2; b_1, b_2)$ and $(-a_1, a_2; -b_1, b_2)$ in the first lattice and their flipped images $(b_1, a_2; a_1, b_2)$ and $(-b_1, a_2; -a_1, b_2)$ in the second one are given this time by

$$\begin{aligned}&(a_2\lambda^\sigma + a_1)\cdot(b_2\lambda^\sigma + b_1)\cdot[(a_1^2 + b_1^2 + a_2^2\lambda^2 + b_2^2\lambda^2) + 2\lambda(a_1a_2 + b_1b_2)]\\ &\quad + (a_2\lambda^\sigma - a_1)\cdot(b_2\lambda^\sigma - b_1)\cdot[(a_1^2 + b_1^2 + a_2^2\lambda^2 + b_2^2\lambda^2) - 2\lambda(a_1a_2 + b_1b_2)]\\ &\quad\quad - (a_2\lambda^\sigma + a_1)\cdot(b_2\lambda^\sigma + b_1)\cdot[(a_1^2 + b_1^2 + a_2^2\lambda^2 + b_2^2\lambda^2) + 2\lambda(b_1a_2 + a_1b_2)]\\ &\quad\quad\quad - (a_2\lambda^\sigma - a_1)\cdot(b_2\lambda^\sigma - b_1)\cdot[(a_1^2 + b_1^2 + a_2^2\lambda^2 + b_2^2\lambda^2) - 2\lambda(b_1a_2 + a_1b_2)],\end{aligned}$$

which again sums up to zero. The same calculation for the derivative $\partial \mathcal{G}_D/\partial \tau_2$ shows that it is zero too, and the vanishing order of $\mathcal{G}_D$ along $F_\mathcal{P}$ is therefore at least 2.

Finally, a simple but long calculation shows that the minimum coefficient of $\partial^2 \mathcal{G}_D/\partial \tau_1^2$, given by the terms corresponding to $\boldsymbol{a} = \left(0, \pm\frac{1}{2}\right)$ and $\boldsymbol{b} = \left(\pm\frac{1}{2}, \frac{1}{6}\right)$ in the first lattice and $\boldsymbol{a} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$ and $\boldsymbol{b} = \left(0, \frac{1}{6}\right)$ in the second one, is $-\frac{4}{27}(-1)^{2/3}\lambda^2(\lambda^\sigma)^2$. □

8.3 Vanishing order along G_D

The modular form $\mathcal{G}_D$ vanishes along $G_D(\mathfrak{b})$ by construction. We next prove that it vanishes only with multiplicity 1.

Proposition 8.4 *If the Gothic modular form $\mathcal{G}_D$ vanishes to order >1 at τ_0, then $\tau_0 \in \widetilde{\mathrm{Red}}_{23}(\mathfrak{b})$. In particular, $\mathcal{G}_D$ vanishes to order 1 along the Gothic Teichmüller curve $G_D(\mathfrak{b})$.*

Proof Assume that $\mathcal{G}_D$ vanishes to order strictly larger than 1 at a point $\boldsymbol{\tau}_0 \in G_D(\mathfrak{b})$, so that in particular $\boldsymbol{\tau}_0 \in X_D \setminus \widetilde{\mathrm{Red}}_{23}(\mathfrak{b})$. This is equivalent to both derivatives $\partial\mathcal{G}_D/\partial\tau_1(\boldsymbol{\tau}_0)$ and $\partial\mathcal{G}_D/\partial\tau_2(\boldsymbol{\tau}_0)$ being zero.

Recall the theta functions $\theta_X(\boldsymbol{\tau},\boldsymbol{u})$, $\Theta_a(\boldsymbol{\tau},\boldsymbol{u})$ and $\Theta_b(\boldsymbol{\tau},\boldsymbol{u})$ defined at the beginning of Section 6.5, and the fact that $\operatorname{div}\theta_X(\boldsymbol{\tau}_0) = \varphi(X)$ is the pre-Abel–Prym image of a curve X in the Gothic locus.

By (16), $\mathcal{G}_D(\boldsymbol{\tau})$ is proportional to $D_2\Theta_a\big(\boldsymbol{\tau},\tfrac{1}{2}\lambda_2\big)$ and $D_2\Theta_b\big(\boldsymbol{\tau},\tfrac{1}{2}\lambda_2\big)$ and therefore, by the heat equation (see [3, Proposition 8.5.5]), one has

$$\frac{\partial}{\partial\tau_2}\mathcal{G}_D(\boldsymbol{\tau}_0) = \frac{\partial^3}{\partial u_2^3}\Theta_a(\boldsymbol{\tau}_0,\boldsymbol{u})|_{\boldsymbol{u}=\lambda_2/2} = \frac{\partial^3}{\partial u_2^3}\Theta_b(\boldsymbol{\tau}_0,\boldsymbol{u})|_{\boldsymbol{u}=\lambda_2/2}.$$

In particular, since all the lower-order u_2–derivatives of θ_X vanish, one has

$$\frac{\partial^3}{\partial u_2^3}\theta_X\big(\tfrac{1}{2}\lambda_2\big) = \frac{\partial^3}{\partial u_2^3}\Theta_a\big(\tfrac{1}{2}\lambda_2\big)\cdot\mathcal{F}_b - \frac{\partial^3}{\partial u_2^3}\Theta_b\big(\tfrac{1}{2}\lambda_2\big)\cdot\mathcal{F}_a = 0.$$

Therefore, the differential du_1 induces an abelian differential on X with two double zeroes at $\frac{1}{2}\mu_2$ and $\frac{1}{2}(\lambda_2+\mu_2)$ and a zero of order ≥ 3 at $\frac{1}{2}\lambda_2$, which is a contradiction to X having genus 4. □

Note that we have proved that not even the τ_2–derivative of $\mathcal{G}_D$ vanishes anywhere along $G_D(\mathfrak{b})$. This gives actually a direct proof of the following fact, without knowing that the curves originate as Teichmüller curves.

Corollary 8.5 *The vanishing locus of $\mathcal{G}_D$ is a union of Kobayashi geodesics.*

Proof Being a Kobayashi geodesic is equivalent to always being transversal to one of the two natural foliations of $X_D(\mathfrak{b})$ (see [19, Proposition 1.3]), hence modular curves are obviously Kobayashi geodesics and the nonvanishing of the derivative $\partial/\partial\tau_2\mathcal{G}_D(\boldsymbol{\tau})$ anywhere in $G_D(\mathfrak{b})$ proves the statement. □

Finally, the following result shows that the reducible locus agrees indeed with the locus $\widetilde{\mathrm{Red}}_{23}(\mathfrak{b})$ defined in Theorem 6.1:

Proposition 8.6 *The two definitions of the reducible locus in $X_D(\mathfrak{b})$ agree, that is,*

$$\mathrm{Red}_{23}(\mathfrak{b}) = \{\mathcal{G}_D(\boldsymbol{\tau}) = 0\} \cap \{\mathcal{F}_a(\boldsymbol{\tau}) = 0\} \cap \{\mathcal{F}_b(\boldsymbol{\tau}) = 0\}.$$

Proof By the previous proposition, the only thing left to prove is that the intersection on the right-hand side is included in the reducible locus.

Let $\boldsymbol{\tau}_0 \in \{\mathcal{G}_D(\boldsymbol{\tau}) = 0\} \cap \{\mathcal{F}_a(\boldsymbol{\tau}) = 0\} \cap \{\mathcal{F}_b(\boldsymbol{\tau}) = 0\}$. Assume without loss of generality that Θ_a is nonzero, otherwise take Θ_b. This theta function satisfies

- $D_2\Theta_a(0) = D_1\Theta_a(0) = 0$ by definition and by $\mathcal{F}_a(\boldsymbol{\tau}_0) = 0$, respectively;
- $D_2\Theta_a\big(\frac{1}{2}\mu_2\big) = D_1\Theta_a\big(\frac{1}{2}\mu_2\big) = 0$ by translation to zero via (16);
- $D_2\Theta_a\big(\frac{1}{2}\lambda_2\big) = D_2\Theta_a\big(\frac{1}{2}(\lambda_2 + \mu_2)\big) = 0$, both by translation to zero via (16) and $\mathcal{G}_D(\boldsymbol{\tau}_0) = 0$.

Thus the theta function Θ_a satisfies all the conditions of θ_X in the proof of Theorem 6.1, and additionally $D_1\Theta_a\big(\frac{1}{2}\mu_2\big) = 0$. As a consequence $Y = \operatorname{div}\Theta_a$ is a divisor with self-intersection $Y^2 = 12$ by Riemann–Roch, and multiplicity 3 at the origin and $\frac{1}{2}\mu_2$. Moreover, since at least the first and second (by odd parity) u_2–derivatives vanish at $\frac{1}{2}\lambda_2$ and $\frac{1}{2}(\lambda_2 + \mu_2)$, either du_1 induces an abelian differential with zeroes of order ≥ 2 at those points, or the multiplicity of Y at them is ≥ 3. The same analysis as in the proof of Theorem 6.1 concludes that the only option is $T_{\boldsymbol{\tau}_0} \in \mathrm{Red}_{23}(\mathfrak{b})$ (case $Y = 3Y_1 + 2Y_2$).

Note that the case $T_{\boldsymbol{\tau}_0} \in G_D$ (case $Y = \varphi(X)$ reduced with zero as its only singular point) is not possible due to the extra vanishing of $D_1\Theta_a\big(\frac{1}{2}\mu_2\big)$, which implies multiplicity ≥ 3 at that point. □

9 Modular embedding of G_{12}

This section is independent of the rest of the paper and illustrates the parametrization of the Gothic locus in the language of the modular embeddings. We illustrate this for $D = 12$, the unique case where G_D is a triangle curve and, therefore, the methods of hypergeometric differential equations are available.

A modular embedding for the Fuchsian group Γ with quadratic invariant trace field K is a map $\tau \mapsto (\tau, \varphi(\tau))$ from $\mathbb{H}$ to $\mathbb{H}^2$ such that $\varphi(\gamma\tau) = \gamma^\sigma\varphi(\tau)$. The universal covering of a map $C \to X_D(\mathfrak{b})$ from a Teichmüller curve C with quadratic trace field to the corresponding Hilbert modular surface gives rise to a modular embedding; see eg [21] for more details.

The *hypergeometric differential equation* with parameters $(a,b,c) \in \mathbb{R}$ is given by

$$L(a,b,c)(y) = t(1-t)y'' + (c-(a+b+1)t)y' - aby = 0. \tag{22}$$

Whenever $\frac{1}{l} = |1-c|$, $\frac{1}{m} = |c-a-b|$ and $\frac{1}{n} = |a-b|$ for some $l,m,n \in \mathbb{Z} \cup \{\infty\}$ satisfying $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, the monodromy group of this equation is the Fuchsian triangle group $\Delta(l,m,n)$. If $l = \infty$, ie if $c = 1$, the space of solutions of (22) near $t = 0$ is generated by $y_1(t)$ and $\log(t)y_1(t) + y_2(t)$, where

$$y_1(t) = F(a,b,c;t) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} t^n,$$

$$y_2(t) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r r!} \left(\sum_{k=0}^{r-1} \frac{1}{a+k} + \frac{1}{b+k} - \frac{2}{c+k} \right) t^r.$$

Here $(x)_n$ denotes the Pochhammer symbol and F is the *hypergeometric function* with coefficients (a,b,c) for $c = 1$.

By Proposition 2.4, the Veech group of G_{12} is the triangle group $\Delta = \Delta(\infty,3,6)$, generated by the matrices

$$M_\infty = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad M_6 = \begin{pmatrix} \sqrt{3}+\frac{1}{2} & \frac{5\sqrt{3}}{6}+1 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

where $\alpha = \frac{2\sqrt{3}}{3} + 2$. It is therefore the monodromy group of the hypergeometric differential equation $L := L\bigl(\frac{5}{12}, \frac{1}{4}, 1\bigr) = 0$ corresponding to $\bigl(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}\bigr) = \bigl(0, \frac{1}{3}, \frac{1}{6}\bigr)$ and we let $y_1(t)$ and $y_2(t)$ be the functions defined above. We will also identify the quotient $\Delta \backslash \mathbb{H}$ with $\mathbb{P}^1$ via the function $t \colon \mathbb{H} \to \mathbb{P}^1$ sending the elliptic generators M_∞, M_3 and M_6 of $\Delta(\infty,3,6)$ to 0, 1 and ∞, respectively.

Given that the invariant trace field of Δ is $\mathbb{Q}(\sqrt{3})$, we will also be interested in the "conjugate" differential equation corresponding to the triangle group Δ^σ for the nontrivial element $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{3}))$. One can see that the rotation numbers of the generators of Δ^σ of order 3 and 6 are $e^{4\pi i/3}$ and $e^{2\pi i/6}$, respectively. As a consequence, the differential equation associated to the group Δ^σ is $L_\sigma := L\bigl(\frac{1}{4}, \frac{1}{12}, 1\bigr) = 0$, which corresponds to $\bigl(\frac{1}{l_\sigma}, \frac{1}{m_\sigma}, \frac{1}{n_\sigma}\bigr) = \bigl(0, \frac{2}{3}, \frac{1}{6}\bigr)$. We denote the corresponding functions defining the solutions of L_σ by $\widetilde{y}_1(t)$ and $\widetilde{y}_2(t)$.

By [21, Formula (52)] the modular embedding φ is given in terms of these solutions by

$$\varphi(\tau) = \frac{\alpha^\sigma}{\alpha}\tau + \frac{\alpha^\sigma}{2\pi i}\left(\log\frac{A}{\widetilde{A}} + \frac{\widetilde{f}_2(\tau)}{\widetilde{f}_1(\tau)} + \frac{f_2(\tau)}{f_1(\tau)}\right) \tag{23}$$

for the constants A and $\widetilde{A}$, where $f_i(\tau) = y_i(t(\tau))$ and $\widetilde{f}_i(\tau) = \widetilde{y}_i(t(\tau))$. Since $t(\tau)$ is M_∞–invariant, we can express these functions in terms of the parameter $q = e^{2\pi i\tau/\alpha}$. The constants A and $\widetilde{A}$ are determined by $Q(t) := te^{y_2(t)/y_1(t)} = Aq$ and $\widetilde{Q}(t) := te^{\widetilde{y}_2(t)/\widetilde{y}_1(t)} = \widetilde{A}\widetilde{q}$, where $\widetilde{q} = e^{2\pi i\varphi(\tau)/\alpha^\sigma}$. The main remaining task is thus to determine A and $\widetilde{A}$.

Due to the chosen normalization, the function $t(\tau)$ takes the value 1 at the point i with multiplicity 3 and $t(\tau) \neq 1$ whenever $\mathrm{im}(\tau) > 1$. It follows that the function $1/(t(\tau)-1)$ has a triple pole at $\tau = i$ and that, as a power series in q (resp. Q), the closest singularity to the origin is given by $q_0 = e^{-2\pi/\alpha}$ (resp. $Q_0 = Aq_0$). This implies that, if one writes $1/(t(Q)-1)^{1/3} = \sum b_n Q^n$ as a power series in Q, the quotients b_n/b_{n+1} will tend exponentially fast to Q_0. This yields a high-precision approximation

$$A \approx 33.9797081543461844465412173813877\ldots.$$

The same calculations for $\widetilde{Q} = \widetilde{A}\widetilde{q}$ yield

$$\widetilde{A} \approx 3254.6483182744669365311774168770392\ldots.$$

These constants can be recognized as the "conjugate-in-exponent" pair

$$A = (2+\sqrt{3})^{-6-\sqrt{3}}(1+\sqrt{3})^9(3+\sqrt{3})^3,$$
$$\widetilde{A} = (2+\sqrt{3})^{-6+\sqrt{3}}(1+\sqrt{3})^9(3+\sqrt{3})^3.$$

The resulting modular embedding from formula (23) is approximately

$$\begin{aligned}\varphi(\tau) = -\frac{2(1-\sqrt{3})}{\pi i}\log(2+\sqrt{3}) + (2-\sqrt{3})\tau + \frac{3+\sqrt{3}}{3\pi i}\Big(-\tfrac{1}{6}Aq - \tfrac{5}{1152}A^2q^2 \\ - \tfrac{61}{497\,664}A^3q^3 - \tfrac{713}{382\,205\,952}A^4q^4 - \tfrac{4943}{183\,458\,856\,960}A^5q^5 + \cdots\Big) \\ = 0.963i + 0.268\tau + 9.243\cdot 10^{-7}i\tau^2 - 1.159\cdot 10^{-6}\tau^3 + 8.389\cdot 10^{-7}i\tau^4 \\ - 2.136\cdot 10^{-7}\tau^5 + 4.611\cdot 10^{-7}i\tau^6 + 9.035\cdot 10^{-8}\tau^7 + 1.630\cdot 10^{-7}i\tau^8 \\ + 1.053\cdot 10^{-7}\tau^9 + 2.502\cdot 10^{-8}i\tau^{10} 5.408\cdot 10^{-8}\tau^{11} + \cdots\end{aligned}$$

and the modular transformation can be checked numerically.

Finally, note that the group Δ does not belong to the Hilbert modular group $\mathrm{SL}(\mathfrak{b}\oplus\mathcal{O}_{12}^\vee)$, but the conjugate $\Delta_C = C\Delta C^{-1}$ by the matrix

$$C = \begin{pmatrix} 3\sqrt{3}+9 & -3\sqrt{3}-15 \\ 0 & 1 \end{pmatrix}$$

does. Consequently, the map $\tau \mapsto (\tau, C^\sigma \circ \varphi \circ C^{-1}(\tau))$, where matrices act on $\mathbb{H}$ by Möbius transformations, parametrizes the Teichmüller curve $G_{12} = \Delta_C \backslash \mathbb{H} \to X_{12}(\mathfrak{b})$. Indeed, it can be numerically checked that the image of this map lies in the vanishing locus of the modular form $\mathcal{G}_{12}$ and, since $\mathrm{Red}_{23}(\mathfrak{b})$ is empty in this case, it actually equals $\{\mathcal{G}_{12}(\boldsymbol{\tau}) = 0\}$.

10 Asymptotics of divisor sums

As preparation for computing the asymptotics of volumes and Lyapunov exponents in the next section, we study here for a fundamental discriminant D the asymptotics as $D \to \infty$ of

$$e(D,k) = \sum_{\substack{b^2 \equiv D \bmod 4k \\ |b| \le \sqrt{D}}} \sigma_1\Big(\frac{D-b^2}{4k}\Big) = \sum_{[a,b,c]\in\mathcal{P}_k(D)} a,$$

where $\sigma_1(\cdot)$ is the divisor sum function and where $\mathcal{P}_k(D)$ has been introduced in (21). Our focus is on the cases $k=1$ and $k=6$, but the method works for general k.

Theorem 10.1 *The following asymptotic statements hold*:

$$\begin{aligned} e(D,1) &= \phantom{\frac{1}{50}}\frac{\zeta_{\mathbb{Q}(\sqrt{D})}(-1)}{2\zeta(-3)} + O(D^{5/4}) \\ e(D,6) &= \frac{1}{50}\frac{\zeta_{\mathbb{Q}(\sqrt{D})}(-1)}{2\zeta(-3)} + O(D^{5/4}) \quad \text{for } D \equiv 0, 12 \bmod 24, \\ e(D,6) &= \frac{2}{50}\frac{\zeta_{\mathbb{Q}(\sqrt{D})}(-1)}{2\zeta(-3)} + O(D^{5/4}) \quad \text{for } D \equiv 4, 9, 16 \bmod 24, \\ e(D,6) &= \frac{4}{50}\frac{\zeta_{\mathbb{Q}(\sqrt{D})}(-1)}{2\zeta(-3)} + O(D^{5/4}) \quad \text{for } D \equiv 1 \bmod 24, \end{aligned}$$

as $D \to \infty$ among fundamental discriminants.

Note that $\zeta_{\mathbb{Q}(\sqrt{D})}(-1) > CD^{3/2}$, so the theorem captures indeed the asymptotics for large D. Our proof here follows closely an application of the circle method used by Zagier in [27, Section 4]. To set the stage, we define the one-variable theta series and the Eisenstein series to be the modular forms

$$\theta(\tau) = \sum_{\ell=-\infty}^{\infty} e^{\pi i \ell^2 \tau}, \quad G_2(\tau) = -\frac{1}{24} + \sum_{a=1}^{\infty} \sigma_1(a) e^{2\pi i a\tau}.$$

Then the modular form

$$F(\tau,k) := G_2(2k\tau)\theta(\tau) = \sum_{n=0}^{\infty} e(n,k) e^{\pi i n\tau}$$

has a Fourier expansion with coefficients that generalize the coefficients we are interested in. The basic idea is to compute the Fourier coefficients of $F(\tau,k)$ by integration at small height ϵ. The dominating term of the asymptotics then comes from the expansions near each rational point. Consequently, we use the modular transformation law to obtain the expansions

$$\begin{aligned} \theta\Big(\frac{a}{c} + iy\Big) &= \lambda(a,c)(cy)^{-1/2} + O(y^{-1/2} e^{-\pi/4c^2 y}), \\ G_2\Big(\frac{a}{c} + iy\Big) &= -\zeta(2)(cy)^{-2} + O(y^{-2} e^{-\pi/c^2 y}) \end{aligned} \tag{24}$$

as $y \to \infty$, where $a,c \in \mathbb{Z}$ with $\gcd(a,c) = 1$ and where $\lambda(a,c)$ is a Legendre symbol times a power of i, depending on the parities of a and c. Here we mainly need to know that the Gauss sum

$$\gamma_c(n) = c^{-1/2} \sum_{a=1}^{2c} \lambda(a,c) e^{-\pi i n a/c}$$

is computed in [27, Theorem 2] for D fundamental to be a weakly multiplicative function in c given on prime powers by

$$2^r \mapsto \begin{cases} 1 & \text{if } r \in \{0,1\}, \\ 2\chi(2) & \text{if } r = 2, \\ 2 & \text{if } r = 3 \text{ and } 2 \mid D, \\ 0 & \text{otherwise,} \end{cases} \qquad p^r \mapsto \begin{cases} 1 & \text{if } r = 0, \\ \chi(p) & \text{if } r = 1, \\ -1 & \text{if } r = 2 \text{ and } p \mid D, \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

for odd primes p, where $\chi(m) = \big(\frac{D}{m}\big)$. Define

$$e^*(n,k) = \sum_{c=1}^{\infty} \frac{\gcd(c,2k)^2}{c^2} \gamma_c(n). \tag{26}$$

Lemma 10.2 *For k square-free and D a fundamental discriminant,*

$$\frac{e^*(D,k)}{e^*(D,1)} = \prod_{p \mid k \text{ prime}} \frac{1+\chi(p)}{1+p^{-2}}.$$

Proof Since the summands in (26) are weakly multiplicative in c, the function $e^*(D,1)$ admits an Euler product expansion. For $p \neq 2$, equation (25) directly implies that the local factor is

$$1 + \frac{\chi(p)}{p^2} + \frac{\chi(p)^2 - 1}{p^4} = \frac{1 - p^{-4}}{1 - \chi(p)p^{-2}}.$$

For $p = 2$ the same conclusion holds up to global factor 2 after taking the factor $\gcd(c, 2k)^2$ into account. In total,

$$e^*(D,1) = 2\frac{L(2,\chi)}{\zeta(4)}, \tag{27}$$

where $L(s,\chi) = \zeta_K(s)/\zeta(s)$ is the L–series associated with the character χ. The passage from $\gcd(c,2)$ to $\gcd(c,2k)$ only changes the factors at the primes dividing k. For $p \neq 2$ the local factor now is $1 + \chi(p) + (\chi(p)^2 - 1)p^{-2}$, and the ratio compared to the original factor results in the modification claimed in the lemma. For $p = 2$ the same final conclusion holds. □

Proof of Theorem 10.1 Since F is periodic under $\tau \mapsto \tau + 2$ we can compute the coefficients

$$e(n,k) = \frac{1}{2}\int_{i\epsilon}^{2+i\epsilon} e^{\pi i n\tau} F(\tau,k)\,dy$$

using Cauchy's formula by integration at small height ϵ. We replace the right-hand side in a neighborhood of $\frac{a}{c} \in [0,2)$ by the dominating term in

$$F\Big(\frac{a}{c} + iy, k\Big) = \frac{\zeta(2)}{16\pi^2}\lambda(a,c)\frac{\gcd(c,2k)^2}{k^2c^{5/2}}y^{-5/2} + O(y^{-5/2}e^{-\pi/4c^2y}),$$

obtained as a combination of (24). The sum over all "major arcs" of the circle method is the summation of these neighborhoods. It is computed in [27, Equation (32)] using the integral representation of the Gamma-function to be

$$\overline{e}(n,k) = \frac{\pi^{1/2}\zeta(2)}{16\Gamma\big(\frac{5}{2}\big)}\frac{n^{3/2}}{k^2}\sum_{c=1}^{\infty}\frac{\gcd(c,2k)^2}{c^2}\gamma_c(n). \tag{28}$$

To see that the major arcs indeed give the dominating term, we can argue as in [27, page 81] (referring to Hardy) for any fixed k. The equations (28) and (27) can now be combined as in [27] to the case $k = 1$ of the theorem. The cases for $k = 6$ differ by the factor $1/k^2$ in (28) and the factors in Lemma 10.2. □

11 Volumes and Lyapunov exponents

The results of the previous sections can now be assembled to compute the Euler characteristic of the Gothic Teichmüller curves and their Lyapunov exponents. We first state a more precise version of Theorem 1.1. Recall the definition of κ_D in Proposition 4.3.

Theorem 11.1 *Let D be a nonsquare discriminant. The Gothic Teichmüller curve G_D is nonempty if and only if $D \equiv 0, 1, 4, 9, 12, 16 \bmod 24$.*

For $D \equiv 0, 12 \bmod 24$ the Gothic Teichmüller curve G_D has Euler characteristic

$$-\chi(G_D) = \frac{1}{20}\kappa_D \sum_{[a,b,c]\in\mathcal{P}_1(D)} a - \frac{1}{3} \sum_{[a,b,c]\in\mathcal{P}_6(D)} a.$$

For $D \equiv 4, 9, 16 \bmod 24$ the Gothic Teichmüller curve $G_D = G_D^0 \cup G_D^1$ consists of two subcurves G_D^ϵ of the same volume equal to

$$-\chi(G_D^\epsilon) = \frac{1}{20}\kappa_D \sum_{[a,b,c]\in\mathcal{P}_1(D)} a - \frac{1}{6} \sum_{[a,b,c]\in\mathcal{P}_6(D)} a, \quad \epsilon \in \{0,1\}.$$

For $D \equiv 1 \bmod 24$ there is a decomposition $G_D = G_D^{00} \cup G_D^{01} \cup G_D^{10} \cup G_D^{01}$ of the Gothic Teichmüller curve into four subcurves $G_D^{\epsilon\delta}$ of the same volume equal to

$$-\chi(G_D^{\epsilon\delta}) = \frac{1}{20}\kappa_D \sum_{[a,b,c]\in\mathcal{P}_1(D)} a - \frac{1}{12} \sum_{[a,b,c]\in\mathcal{P}_6(D)} a, \quad \epsilon,\delta \in \{0,1\}.$$

To state the other theorems, we provide a brief introduction to Lyapunov exponents, in particular for flat surfaces (X,ω) in the Gothic locus.

Lyapunov exponents measure the growth rate of cohomology classes in $H^1(X,\mathbb{R})$ under parallel transport along the geodesic flow in $\overline{\mathrm{SL}(2,\mathbb{R})\cdot(X,\omega)}$, the closure of the $\mathrm{SL}(2,\mathbb{R})$–orbit of (X,ω) (see eg [28] or [18] for background). The Lyapunov spectrum of a genus 4 surface consists of Lyapunov exponents $\lambda_1 = 1 \ge \lambda_2 \ge \lambda_3 \ge \lambda_4$ and their negatives.

In the case of flat surfaces (X,ω) in the Gothic locus, the existence of the maps π_A and π_B in (1) decomposes the local system $\mathbb{V}$ with fiber $H^1(X,\mathbb{R})$ over ΩG into local subsystems $\mathbb{V}_A$ and $\mathbb{V}_B$ of rank 2, corresponding to the elliptic curves A and B, and the ("Prym") complement $\mathbb{V}_P$. Since the generating differential of the Gothic form belongs to the Prym part, the exponent $\lambda_1 = 1$ is one of the two positive exponents

$\{1, \lambda_P\}$ of $\mathbb{V}_P$. If we denote by λ_A and λ_B the Lyapunov exponents from the elliptic curves, then the sets

$$\{1, \lambda_2, \lambda_3, \lambda_4\} = \{1, \lambda_A, \lambda_B, \lambda_P\}$$

coincide. Since by definition $\omega^2 = \pi_A^* q$ for some quadratic differential q on A, the double covering formula of Eskin, Kontsevich and Zorich [7] implies

$$\lambda_1 + \lambda_P + \lambda_B - \lambda_A = \tfrac{1}{4} \cdot 3 \cdot \left(\tfrac{1}{1+2} + \tfrac{1}{-1+2}\right) = 1.$$

Theorem 11.2 *The Prym Lyapunov exponent λ_P of a generic surface in the Gothic locus is equal to $\frac{3}{13}$.*

This is a direct consequence of the asymptotics formulas in Theorem 10.1, the following proposition and the convergence of individual Lyapunov exponents [4], since the curves G_D equidistribute towards (the Lebesgue measure on) the Gothic locus by [9].

Proposition 11.3 *The Prym Lyapunov exponent of a Gothic Veech surface on G_D is equal to*

$$\lambda_P(G_D(\mathfrak{b})) = 1 + \frac{\chi(X_D(\mathfrak{b}))}{\chi(G_D(\mathfrak{b}))}.$$

Note that we do not claim that the curves $G_D(\mathfrak{b})$ are connected, although we expect this to be true. Therefore, the statement of the proposition has to be interpreted as the volume-weighted average of the λ_P of the connected components.

Proof of Theorem 11.1 The arguments in the following work for any good compactification $\overline{X_D(\mathfrak{b})}$ of $X_D(\mathfrak{b})$ (see [19]). Since the specific choice of compactification is not relevant, we will denote simply by $[C]$ the class of the closure $\overline{C}$ in $\overline{X_D(\mathfrak{b})}$.

Let $[\omega_i]$ be the classes of the two foliations of the Hilbert modular surface $X_D(\mathfrak{b})$. Then the uniformization of $X_D(\mathfrak{b})$ implies $\chi(X_D(\mathfrak{b})) = [\omega_1] \cdot [\omega_2]$ and the vanishing locus of a modular for of biweight (k, ℓ) has class $\frac{1}{2}(k[\omega_1] + \ell[\omega_2])$.

Theorem 6.1 and Proposition 8.3 together show that the vanishing locus of the Gothic modular form $\mathcal{G}_D$ is a union of Kobayashi curves and the first coordinate can be used as parameter for each of these curves. By Theorem 8.1, $\operatorname{div}(\mathcal{G}_D) = G_D(\mathfrak{b}) + 2\,\mathrm{Red}_{23}(\mathfrak{b})$, where $G_D(\mathfrak{b})$ denotes the union of those components of the Torelli-image of G_D in $X_D(\mathfrak{b})$ for which du_1 induces the eigenform ω at each point (X, ω).

This implies that integration of ω_1 along $\operatorname{div}(\mathcal{G}_D)$ (equivalently, the intersection product $-[\omega_1] \cdot [\operatorname{div}(\mathcal{G}_D)]$) computes the sum of the Euler characteristics of these curves with the multiplicity determined in Propositions 8.3 and 8.4 (see [1, Corollary 10.4] or [19, Proposition 1.3]). We obtain

(29) $$-\tfrac{3}{2}\chi(X_D(\mathfrak{b})) = -[\omega_1]\cdot\left(\tfrac{1}{2}[\omega_1]+\tfrac{3}{2}[\omega_2]\right) = \chi(G_D(\mathfrak{b})) + 2\chi(\mathrm{Red}_{23}(\mathfrak{b})).$$

Proposition 4.3 together with the well-known expression for the Euler characteristic $\chi(X_D)$ of standard Hilbert modular surfaces in terms of prototypes (see [11] for example) give

$$\chi(X_D(\mathfrak{b})) = \frac{1}{30}\kappa_D \sum_{[a,b,c]\in\mathcal{P}_1(D)} a.$$

Formula (29) together with Lemma 7.5 proves the result for $G_D(\mathfrak{b})$. The only thing left to do is to prove the decomposition of G_D into subcurves as claimed.

We claim that, for different ideals $\mathfrak{b}_1$ and $\mathfrak{b}_2$, the images of $G_D(\mathfrak{b}_1)$ and $G_D(\mathfrak{b}_2)$ in $\mathcal{A}_{2,(2,3)}$ are different. In fact, if $\mathfrak{b}_2 \neq \mathfrak{b}_1^\sigma$, the images of the whole Hilbert modular surfaces $X_D(\mathfrak{b}_1)$ and $X_D(\mathfrak{b}_2)$ are disjoint in $\mathcal{A}_{2,(2,3)}$, since the lattices of the corresponding abelian surfaces are not even isomorphic as $\mathcal{O}_D$–modules. On the other hand, if $\mathfrak{b}^\sigma \neq \mathfrak{b}$, the subcurves $G_D(\mathfrak{b})$ and $G_D(\mathfrak{b}^\sigma)$ can both be seen in $X_D(\mathfrak{b})$ as Kobayashi geodesics with ω_1 and ω_2 as parameters, respectively. In particular, if their images under $X_D(\mathfrak{b}) \to \mathcal{A}_{2,(2,3)}$ agreed, their associated eigenforms for real multiplication would map to two different eigenforms on each point $X \in G_D$.

Finally, by construction G_D, is covered by the union of the images of $G_D(\mathfrak{b})$ for the different ideals $\mathfrak{b}$ of norm 6. □

Proof of Proposition 11.3 and Theorem 11.2 The Lyapunov exponent $\lambda_P(C)$ of a Kobayashi geodesic C in $X_D(\mathfrak{b})$ is given by the quotient (see [1] or [19])

$$\lambda_P(C) = \frac{[\omega_2]\cdot[C]}{[\omega_1]\cdot[C]}.$$

The reducible locus $\mathrm{Red}_{23}(\mathfrak{b})$ is a union of Shimura curves, and therefore one has $\lambda_P(\mathrm{Red}_{23}(\mathfrak{b})) = 1$ and $-[\omega_1]\cdot[\mathrm{Red}_{23}(\mathfrak{b})] = -[\omega_2]\cdot[\mathrm{Red}_{23}(\mathfrak{b})] = \chi(\mathrm{Red}_{23}(\mathfrak{b}))$. In the case of the Gothic Teichmüller curves, since $[G_D(\mathfrak{b})] = [\mathrm{div}(\mathcal{G}_D)] - 2[\mathrm{Red}_{23}(\mathfrak{b})]$, one has

$$\lambda_P(G_D(\mathfrak{b})) = \frac{[\omega_2]\cdot\left(\frac{1}{2}[\omega_1]+\frac{3}{2}[\omega_2]-2[\mathrm{Red}_{23}(\mathfrak{b})]\right)}{[\omega_1]\cdot\left(\frac{1}{2}[\omega_1]+\frac{3}{2}[\omega_2]-2[\mathrm{Red}_{23}(\mathfrak{b})]\right)} = \frac{\frac{1}{2}\chi(X_D(\mathfrak{b}))+2\chi(\mathrm{Red}_{23}(\mathfrak{b}))}{\frac{3}{2}\chi(X_D(\mathfrak{b}))+2\chi(\mathrm{Red}_{23}(\mathfrak{b}))}.$$

By Theorem 1.1, this is exactly $1+\chi(X_D(\mathfrak{b}))/\chi(G_D(\mathfrak{b}))$.

Theorem 11.2 follows by taking the limit and using Theorems 11.1 and 10.1 □

D	#	$\chi(X_D(\mathfrak{b}))$	$\chi(\mathrm{Red}_{23}(\mathfrak{b}))$	$\chi(G_D^{\epsilon\delta})$
12	1	$\frac{1}{3}$	0	$-\frac{1}{2}$ *
24	1	1	$-\frac{1}{6}$	$-\frac{7}{6}$
28	2	$\frac{4}{3}$	$-\frac{1}{6}$	$-\frac{5}{3}$
33	2	2	$-\frac{1}{6}$	$-\frac{8}{3}$ *
40	2	$\frac{7}{3}$	$-\frac{1}{6}$	$-\frac{19}{6}$ *
48	1	4	$-\frac{1}{2}$	-5
52	2	5	$-\frac{1}{2}$	$-\frac{13}{2}$
57	2	$\frac{14}{3}$	$-\frac{1}{2}$	-6 †
60	1	4	$-\frac{1}{3}$	$-\frac{16}{3}$
72	1	$\frac{20}{3}$	$-\frac{2}{3}$	$-\frac{26}{3}$ *
73	4	$\frac{22}{3}$	$-\frac{2}{3}$	$-\frac{29}{3}$ †
76	2	$\frac{19}{3}$	$-\frac{2}{3}$	$-\frac{49}{6}$ *
84	1	10	-1	-13 †
88	2	$\frac{23}{3}$	$-\frac{5}{6}$	$-\frac{59}{6}$
96	1	12	-1	-16
97	4	$\frac{34}{3}$	$-\frac{7}{6}$	$-\frac{44}{3}$
105	2	12	$-\frac{4}{3}$	$-\frac{46}{3}$
108	1	12	$-\frac{4}{3}$	$-\frac{46}{3}$
112	2	16	$-\frac{3}{2}$	-21 †
120	1	$\frac{34}{3}$	-1	-15 *
124	2	$\frac{40}{3}$	$-\frac{7}{6}$	$-\frac{53}{3}$ *
129	2	$\frac{50}{3}$	$-\frac{3}{2}$	-22 *
132	1	18	-2	-23
136	2	$\frac{46}{3}$	$-\frac{5}{3}$	$-\frac{59}{3}$
145	4	$\frac{64}{3}$	$-\frac{13}{6}$	$-\frac{83}{3}$
148	2	25	$-\frac{5}{2}$	$-\frac{65}{2}$ †
153	2	$\frac{80}{3}$	$-\frac{8}{3}$	$-\frac{104}{3}$ *
156	1	$\frac{52}{3}$	-2	-22
160	2	28	-3	-36 †
168	1	18	$-\frac{5}{3}$	$-\frac{71}{3}$
172	2	21	-2	$-\frac{55}{2}$ *
177	2	26	$-\frac{5}{2}$	-34
180	1	40	-4	-52 †
184	2	$\frac{74}{3}$	$-\frac{7}{3}$	$-\frac{97}{3}$ *
192	1	32	-3	-42 †
193	4	$\frac{98}{3}$	$-\frac{10}{3}$	$-\frac{127}{3}$
201	2	$\frac{98}{3}$	$-\frac{7}{2}$	-42
204	1	26	$-\frac{8}{3}$	$-\frac{101}{3}$
208	2	40	-4	-52
216	1	36	$-\frac{10}{3}$	$-\frac{142}{3}$
217	4	$\frac{116}{3}$	$-\frac{23}{6}$	$-\frac{151}{3}$
220	2	$\frac{92}{3}$	$-\frac{10}{3}$	$-\frac{118}{3}$
228	1	42	-4	-55
232	2	33	$-\frac{7}{2}$	$-\frac{85}{2}$
240	1	48	-5	-62
241	4	$\frac{142}{3}$	$-\frac{14}{3}$	$-\frac{185}{3}$
244	2	55	$-\frac{11}{2}$	$-\frac{143}{2}$
249	2	46	$-\frac{9}{2}$	-60
252	1	$\frac{128}{3}$	-4	-56 *
264	1	$\frac{112}{3}$	-4	-48
265	4	$\frac{160}{3}$	$-\frac{31}{6}$	$-\frac{209}{3}$
268	2	41	-4	$-\frac{107}{2}$
273	2	$\frac{148}{3}$	-5	-64
276	1	60	-6	-78
280	2	$\frac{134}{3}$	$-\frac{13}{3}$	$-\frac{175}{3}$
288	1	80	-8	-104 †
292	2	66	-7	-85
297	2	72	$-\frac{22}{3}$	$-\frac{280}{3}$
300	1	$\frac{130}{3}$	-4	-57
304	2	76	$-\frac{15}{2}$	-99
312	1	46	-5	-59
313	4	$\frac{200}{3}$	$-\frac{41}{6}$	$-\frac{259}{3}$
316	2	56	$-\frac{11}{2}$	-73
321	2	66	$-\frac{13}{2}$	-86
328	2	54	-5	-71
336	1	80	-8	-104

Table 1: Number of $\mathcal{O}_D$–ideals $\mathfrak{b}$ of norm 6 and volumes of each $X_D(\mathfrak{b})$, $\mathrm{Red}_{23}(\mathfrak{b})$ and $G_D(\mathfrak{b})$, for $D \leq 385$. The cross and the asterisk indicate a Gothic or hexagons model, respectively.

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