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A BERNOULLI–EULER BEAM MODEL BASED ON THE LOCAL GRADIENT THEORY OF ELASTICITY

OLHA HRYTSYNA

Using the total energy balance equation and principle of the frame indifference, a fundamental set of relations of the local gradient continuum model of elastic solids is formulated. The model is based on taking account of non-convective and non-diffusive mass flux related to the changes in the material microstructure. Linear stationary governing equations of the local gradient theory and corresponding boundary conditions are also derived by variational principle. In order to investigate the size-dependent behavior of nano-scale structures, this model is combined with the Bernoulli–Euler beam theory. Deflection of the cantilever beam subjected to the end-point loading under the plane stress conditions is evaluated and compared to the corresponding ones provided by the classical theory and by the strain gradient theory. It is shown that the beam deflection within the local gradient theory is smaller than that predicted by the classical Bernoulli–Euler beam theory. This work may be of special interest for designing the devices utilizing the micro/nano-beam elements.

1. Introduction

It is known that surface and size phenomena play an important role in determining mechanical behaviors of small-scale structures. Laboratory investigations showed that thin films, nanowires, and nanobeams exhibit different physical properties compared with macro-sized structures of the same material [Espinosa et al. 2003; Greer and Nix 2005; Hardwick 1987; Kakunai et al. 1985; Ma and Clarke 1995; Weihs et al. 1988]. Since the classical theory of elasticity cannot explain a specific mechanical behavior of micro/nano-scale structures, various modified mathematical models of elastic continua were developed in the last six decades. Among them there are the couple stress theory [Mindlin and Tiersten 1962; Toupin 1962], strain gradient theories of elasticity [Mindlin 1965], theory of elastic micromorphic materials [Eringen 1999; Eringen and Suhubi 1964; Mindlin 1964], micropolar theory [Eringen 1966; Kafadar and Eringen 1971], microstrech theory [Eringen 1999], nonlocal theory of elasticity with integral-type constitutive relations [Edelen 1969; Eringen 1972], etc. In recent years, modified theories of the elastic continua were effectively adapted in order to investigate the mechanical behavior of elastic micro/nanoscale beams, plates, rods, rings, and shells (see, for example, [Akgöz and Civalek 2011; Lazopoulos and Lazopoulos 2010; Liebold and Müller 2015; Lurie and Solyaev 2019; Repka et al. 2018; Shokrieh and Zibaei 2015; Tahaei Yaghoubi et al. 2018], etc). The papers studied the size-dependent mechanical behaviour of micro-/nano-beams using modified continuum models are reviewed in [Niiranen et al. 2019]. The applications of the surface elasticity theory, pure nonlocal elastic models and nonlocal strain gradient elasticity theories in static and dynamic analysis of nanobeams, nanotubes and nanoplates are presented in [Wang et al.

Keywords: gradient-type theory, continuum mechanics modeling, local gradient elasticity, Bernoulli–Euler beam, micro/nano-beams, size effect.

2016; Farajpour et al. 2018]. Trends in the development of the nonlocal and gradient-type theories of elastic beams are highlighted in reviews [Reddy 2007; Eltaher et al. 2016; Thai et al. 2017; Spagnuolo and Andreaus 2019], etc.

In 1987, Burak proposed a new continuum-thermodynamic approach to the construction of gradient-type continuum theory of elastic solids. The mentioned approach is based on taking account of non-diffusive and non-convective mass flux $\mathbf{J}_{ms}$ associated with the changes in the material microstructure. Such changes in the material microstructure can be observed, for instance, in the vicinity of the newly-created surfaces. Burak related the mentioned non-diffusive and non-convective mass flux to the process referred to as the local mass displacement. [Marchenko et al. 2009] observed this kind of mass flux within the near-surface regions of thin films during their formation. Making use the total energy balance equation and assuming that the mass flux $\mathbf{J}_{ms}$ causes an energy flux $\mu\,\mathbf{J}_{ms}$, where μ is the chemical potential, [Burak 1987] received gradient-type constitutive equations. The additional pair of conjugate variables $(\nabla\mu,\ \mathbf{\Pi}_m)$ related to the local mass displacement was obtained within this theory. Here, $\mathbf{\Pi}_m$ is the vector of local mass displacement related to the vector of mass flux $\mathbf{J}_{ms}$ by the formula [Burak 1987]

$$\mathbf{\Pi}_m(\mathbf{r},\ t) = \int_0^t \mathbf{J}_{ms}(\mathbf{r},\ t')dt' \Rightarrow \mathbf{J}_{ms} = \frac{\partial \mathbf{\Pi}_m}{\partial t}, \tag{1}$$

where $\mathbf{r}$ is the position vector and t denotes the time variable.

This is called the local gradient theory of elasticity. The relations of that theory were applied to describe surface and size effects not covered by the classical theory (see the review article [Hrytsyna et al. 2006] and references therein). In the cited papers, specific studies were carried out in linear approximation with the assumption that convective term in the material time derivative can be neglected.

Two decades later, [Burak et al. 2007] proposed a modified local gradient theory of thermoelasticity in which two additional constitutive parameters related to the local mass displacement were introduced. The mentioned theory contains a new balance-type differential equation that governs the behavior of thermoelastic media in addition to the classical momentum balance equation and entropy balance equation. The local gradient theory was successfully used to describe a near-surface inhomogeneity of mechanical fields [Burak et al. 2008], high-frequency dispersion of longitudinal elastic waves [Kondrat and Hrytsyna 2010], the propagation of antiplane horizontally polarized surface shear waves (SH waves) in homogeneous solids [Hrytsyna 2017], etc.

The objectives of the present paper are (*i*) to derive stationary balance equations and boundary conditions of the local gradient elasticity from the variational principle, (*ii*) to establish a local gradient Bernoulli–Euler linear beam model to incorporate the surface and size effects, and (*iii*) to test the obtained relations on simple problem of a cantilevered beam under the end-point loading.

The paper is organized as follows. In Section 2, the nonlinear local gradient mathematical model of elastic solids is developed based on the total energy balance equation and on the principle of the frame indifference. In Section 3, the stationary balance equations and constitutive relations are formulated for linear approximation. In Section 4, the linear governing equations of local gradient elasticity and corresponding boundary conditions are derived from the variational principle. Based on these relations, the Bernoulli–Euler local gradient beam model is developed in Section 5. The deflection of the cantilever beam loaded by a point force is obtained in this section. The numerical results are discussed in Section 6. Conclusions are drawn in the final Section 7.

2. Basic equations of nonlinear local gradient elasticity

In the classical elasticity, the position of a small body element (body particle) is identified with the position of its center of mass. In such a theory, the change of position of the mass center of the small body element can be caused only by the convective displacement of this element (Figure 1a). The local gradient theory takes into account that a change in the center of mass of the body particle may be induced not only by its convective displacement as a rigid entity (i.e., translational displacement of the particle geometric center) but by the changes of the relative positions of microparticles within this element as well, i. e., the change of its microstructure (Figure 1b). Within the local gradient theory mentioned changes in microstructure are described by non-convective and non-diffusive mass flux $\mathbf{J}_{ms}$. This mass flux was linked with the process of the local mass displacement.

To describe the local mass displacement, new physical quantities associated with this process should be introduced. Following [Burak 1987], alongside with the mass flux $\mathbf{J}_{ms}$, we introduce vector of the local mass displacement $\mathbf{\Pi}_m$. Let us assume that above vectors are related by Eq. (1). Note that the vector $\mathbf{\Pi}_m$ of the local mass displacement has a dimension of the density of a dipole mass moment ($\mathrm{kg}\cdot\mathrm{m}/\mathrm{m}^3$) while its specific quantity $\boldsymbol{\pi}_m = \mathbf{\Pi}_m/\rho$ has the length dimension (m) (here, ρ is the mass density).

For an arbitrary body of finite size (domain (V)), using the relation [Burak et al. 2008]

$$\int_{(V)} \mathbf{\Pi}_m dV = \int_{(V)} \rho_{m\pi}\, \mathbf{r}\, dV, \tag{2}$$

we introduce a new scalar physical quantity $\rho_{m\pi}$ that has the dimension of mass density. Here, $\mathbf{r}$ is the position vector of the material point. From integral relation (2), formula

$$\rho_{m\pi} = -\nabla\cdot\mathbf{\Pi}_m \tag{3}$$

can be obtained [Hrytsyna and Kondrat 2019]. Here, $\nabla = (\partial/\partial\xi_1, \partial/\partial\xi_2, \partial/\partial\xi_3)$ is the nabla operator, where ξ_i, $i=\overline{1,3}$, are the space coordinates.

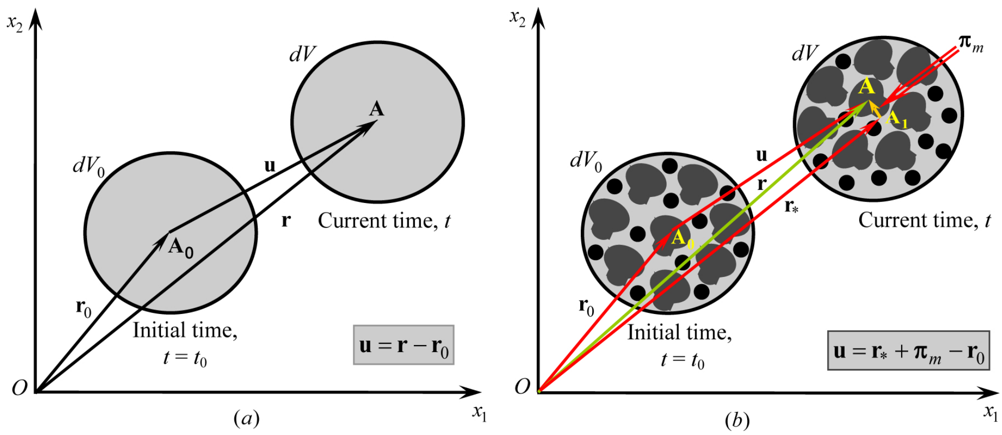

Figure 1. Mass-center displacement of a small volume element due to action of external load: (*a*) the classical theory of elasticity, $\mathbf{u} = \mathbf{r} - \mathbf{r}_0$; (*b*) the local gradient theory of elasticity, $\mathbf{u} = \mathbf{r}_* + \boldsymbol{\pi}_m - \mathbf{r}_0$ where $\boldsymbol{\pi}_m = \mathbf{\Pi}_m\rho$.

Note that in electrodynamics by similar formula $\rho_{e\pi} = -\nabla \cdot \mathbf{\Pi}_e$, the density of an induced charge $\rho_{e\pi}$ was introduced into consideration (here, $\mathbf{\Pi}_e$ is the polarization vector, i.e., the vector of the local displacement of electric charges) [Bredov et al. 1985]. Thus, by analogy with the electrodynamics, we refer to $\rho_{m\pi}$ as the density of an induced mass [Hrytsyna and Kondrat 2019].

By differentiating expression (3) with respect to time and taking formula (1) into account, one can obtain the balance-type differential equation

$$\frac{\partial \rho_{m\pi}}{\partial t} + \nabla \cdot \mathbf{J}_{ms} = 0, \tag{4}$$

for quantities $\rho_{m\pi}$ and $\mathbf{J}_{ms}$ introduced for a description of the local mass displacement.

Within this subsection, we derive the constitutive and balance equations, using the energy conservation law and the principle of frame indifference. We consider an elastic body that occupies the region (V) and is bounded by a smooth surface (Σ). Let us separate from the body a fixed small volume (V') bounded by closed surface (Σ'). We represent the total energy of this volume as the sum of internal ρu and kinetic $\rho \mathbf{v}^2/2$ energies. Here, u denotes the specific internal energy and $\mathbf{v}$ is the velocity of continuum of centre mass. The change in the total energy is caused (*i*) by the convective energy transport $\rho\big(u + \mathbf{v}^2/2\big)\mathbf{v}$ through the body surface, (*ii*) by the energy flux $\hat{\boldsymbol{\sigma}} \cdot \mathbf{v}$ due to the mechanical work of the surface forces, (*iii*) by the energy flux $\mu \mathbf{J}_m$ linked with the mass transport relative to the mass-centre of the small body particle, (*iv*) by the energy flux $\mu_{\boldsymbol{\pi}} \mathbf{J}_{ms}$ related to the material microstructure ordering (i. e., the local mass displacement) as well as (*v*) by the action of mechanical mass forces $\mathbf{F}$. Thus, for the fixed volume (V'), the energy balance equation can be written as follows:

$$\frac{d}{dt}\int_{(V')} \big(\rho u + \tfrac{1}{2}\rho \mathbf{v}^2\big)\, dV = -\oint_{(\Sigma')} \big[\rho\big(u + \tfrac{1}{2}\mathbf{v}^2\big)\mathbf{v} - \hat{\boldsymbol{\sigma}} \cdot \mathbf{v} + \mu \mathbf{J}_m + \mu_\pi\, \mathbf{J}_{ms}\big] \cdot \mathbf{n}\, d\Sigma + \int_{(V')} \rho \mathbf{F} \cdot \mathbf{v}\, dV. \tag{5}$$

Here, $\hat{\boldsymbol{\sigma}}$ represents the classical Cauchy stress tensor; μ_π is the energy measure of the effect of the local mass displacement on the internal energy; $\mathbf{J}_m = \rho(\mathbf{v}_* - \mathbf{v})$; $\mathbf{v}_*$ is the velocity of convective displacement of the small body element; $\mathbf{n}$ is the unit vector normal to the material surface (Σ'), and the dot denotes the scalar product. Note that vectors $\mathbf{v}_*$ and $\mathbf{v}$ are related through the following expression (see Figure 1):

$$\mathbf{v} = \mathbf{v}_* + \frac{1}{\rho}\frac{\partial \mathbf{\Pi}_m}{\partial t}. \tag{6}$$

Note also that by virtue of Eqs. (1) and (6), the mass flux $\mathbf{J}_m$ can be written down as follows:

$$\mathbf{J}_m = -\frac{\partial \mathbf{\Pi}_m}{\partial t}. \tag{7}$$

Making use of the divergence theorem as well as of the expressions (1) and (7), from integral equation (5), in view of the arbitrary of the volume (V') one obtains the local form of the energy conservation law for the elastic continuum

$$\frac{\partial}{\partial t}\big(\rho u + \tfrac{1}{2}\rho \mathbf{v}^2\big) = -\nabla \cdot \big[\rho \mathbf{v}\big(u + \tfrac{1}{2}\mathbf{v}^2\big)\big] + \nabla \cdot (\hat{\boldsymbol{\sigma}} \cdot \mathbf{v}) + \rho \mathbf{F} \cdot \mathbf{v} - \nabla \mu'_\pi \cdot \frac{\partial \mathbf{\Pi}_m}{\partial t} + \mu'_\pi \frac{\partial(-\nabla \cdot \mathbf{\Pi}_m)}{\partial t}. \tag{8}$$

Here, $\mu'_\pi = \mu_\pi - \mu$ is the modified chemical potential.

After some algebra, expression (8) can be written as follows:

$$\rho\frac{du}{dt}+\left(u+\tfrac{1}{2}\mathbf{v}^2\right)\left[\frac{\partial\rho}{\partial t}+(\nabla\cdot\rho\mathbf{v})\right] \\ =\mathbf{v}\cdot\left(-\rho\frac{d\mathbf{v}}{dt}+\nabla\cdot\hat{\boldsymbol{\sigma}}+\rho\mathbf{F}\right)+\hat{\boldsymbol{\sigma}}:(\nabla\otimes\mathbf{v})-\nabla\mu'_\pi\cdot\frac{\partial\boldsymbol{\Pi}_m}{\partial t}+\mu'_\pi\frac{\partial\rho_{m\pi}}{\partial t}. \quad (9)$$

Here, $d/dt=\partial/\partial t+\mathbf{v}\cdot\nabla$ denotes the substantive derivative, and $\otimes$ is the tensor product.

Let us introduce specific quantities $\boldsymbol{\pi}_m=\boldsymbol{\Pi}_m/\rho$ and $\rho_m=\rho_{m\pi}/\rho$. Substituting these two relations into the energy balance equation (9) after some transformations, we obtain

$$\rho\frac{du}{dt}+\rho\nabla\mu'_\pi\cdot\frac{d\boldsymbol{\pi}_m}{dt}-\rho\mu'_\pi\frac{d\rho_m}{dt}-\hat{\boldsymbol{\sigma}}_*:(\nabla\otimes\mathbf{v})=\mathbf{v}\cdot\left(-\rho\frac{d\mathbf{v}}{dt}+\nabla\cdot\hat{\boldsymbol{\sigma}}_*+\rho\mathbf{F}_*\right) \\ -\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})\right)\left(u+\frac{1}{2}\mathbf{v}^2-\rho_m\mu'_\pi+\nabla\mu'_\pi\cdot\boldsymbol{\pi}_m\right), \quad (10)$$

where

$$\hat{\boldsymbol{\sigma}}_*=\hat{\boldsymbol{\sigma}}-\rho(\rho_m\mu'_\pi-\boldsymbol{\pi}_m\cdot\nabla\mu'_\pi)\hat{\mathbf{I}}, \quad (11)$$

$$\mathbf{F}_*=\mathbf{F}+\rho_m\nabla\mu'_\pi-\boldsymbol{\pi}_m\cdot\nabla\otimes\nabla\mu'_\pi, \quad (12)$$

where $\hat{\mathbf{I}}$ represents the unit tensor.

Following [Green and Rivlin 1964], assume that the energy balance equation (10) remains valid under superimposed rigid body translation. As a result from (10), we get the equation of motion and the mass balance equation, namely [Hrytsyna and Kondrat 2019]:

$$\nabla\cdot\hat{\boldsymbol{\sigma}}_*+\rho\mathbf{F}_*=\rho\frac{d\mathbf{v}}{dt}, \quad (13)$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0. \quad (14)$$

From formulae (11)–(13), it follows that the local mass displacement being taken into consideration leads to the modification of the stress tensor $\hat{\boldsymbol{\sigma}}_*$ and to the appearance of an additional nonlinear mass force $\mathbf{F}'_*=\rho_m\nabla\mu'_\pi-\boldsymbol{\pi}_m\cdot\nabla\otimes\nabla\mu'_\pi$ in equation of motion (13).

Since the energy balance equation (10) should be invariant under superimposed rigid body rotation with a constant angular rate, it follows that $\hat{\boldsymbol{\sigma}}_*$ is a symmetric tensor [Hrytsyna and Kondrat 2019].

In view of Eqs. (13) and (14), the balance equation (10) for the internal energy simplifies and can be written as follows:

$$\rho\frac{du}{dt}=\hat{\boldsymbol{\sigma}}_*:\frac{d\hat{\boldsymbol{e}}}{dt}-\rho\nabla\mu'_\pi\cdot\frac{d\boldsymbol{\pi}_m}{dt}+\rho\mu'_\pi\frac{d\rho_m}{dt}. \quad (15)$$

Here, $\hat{\boldsymbol{e}}$ is the strain tensor related to the mechanical displacement vector $\mathbf{u}$ by the formula

$$\hat{\boldsymbol{e}}=\tfrac{1}{2}\left(\nabla\otimes\mathbf{u}+(\nabla\otimes\mathbf{u})^{\mathrm{T}}\right), \quad (16)$$

where superscript T stands for transpose.

Let us introduce a new energy density function $H=H(\hat{\boldsymbol{e}},\ \mu'_\pi,\ \nabla\mu'_\pi)$ through the Legendre transformation $H=u+\rho_m\mu'_\pi+\nabla\mu'_\pi\cdot\boldsymbol{\pi}_m$. Based on (15) we can write the following generalized Gibbs relation:

$$dH = \frac{1}{\rho}\hat{\boldsymbol{\sigma}}_* : d\hat{\boldsymbol{e}} - \rho_m d\mu'_\pi + \boldsymbol{\pi}_m \cdot d\nabla\mu'_\pi. \tag{17}$$

Since H is the function of scalar μ'_π, vector $\nabla\mu'_\pi$, and tensor $\hat{\boldsymbol{e}}$ arguments, its full differential looks as follows:

$$dH = \frac{\partial H}{\partial \hat{\boldsymbol{e}}} d\hat{\boldsymbol{e}} + \frac{\partial H}{\partial \mu'_\pi} d\mu'_\pi + \frac{\partial H}{\partial(\nabla\mu'_\pi)} \cdot d(\nabla\mu'_\pi). \tag{18}$$

Subtracting formulae (17) and (18), we get

$$\left(\frac{\partial H}{\partial \hat{\boldsymbol{e}}} - \frac{1}{\rho}\hat{\boldsymbol{\sigma}}_*\right) d\hat{\boldsymbol{e}} + + \left(\frac{\partial H}{\partial \mu'_\pi} + \rho_m\right) d\mu'_\pi + \left(\frac{\partial H}{\partial(\nabla\mu'_\pi)} - \boldsymbol{\pi}_m\right) \cdot d(\nabla\mu'_\pi) = 0. \tag{19}$$

Equation (19) must hold for arbitrary $d\hat{\boldsymbol{e}}$, $d\mu'_\pi$, and $d(\nabla\mu'_\pi)$. Thus, (19) yields the following gradient-type constitutive equations:

$$\hat{\boldsymbol{\sigma}}_* = \rho \frac{\partial H}{\partial \hat{\boldsymbol{e}}}\bigg|_{\mu'_\pi, \nabla\mu'_\pi}, \quad \rho_m = -\frac{\partial H}{\partial \mu'_\pi}\bigg|_{\hat{\boldsymbol{e}}, \nabla\mu'_\pi}, \quad \boldsymbol{\pi}_m = \frac{\partial H}{\partial(\nabla\mu'_\pi)}\bigg|_{\hat{\boldsymbol{e}},\, \mu'_\pi}, \tag{20}$$

where the subscripts to the right from the vertical lines indicate the variables that are held constant during the differentiation. Note that within the framework of this higher-order theory of elastic media, the set of conjugate variables is complemented by two additional pairs of variables $(\rho_m,\ \mu'_\pi)$ and $(\boldsymbol{\pi}_m,\ \nabla\mu'_\pi)$ related to the local mass displacement.

3. Linear set of equations for stationary approximation

Let us represent the constitutive relations (20) in an explicit form. To this end, we expand the potential H in the vicinity of the natural state ($\hat{\boldsymbol{e}} = 0$, $\hat{\boldsymbol{\sigma}}_* = 0$, $\mu'_\pi = \mu'_{\pi 0}$, $\nabla\mu'_\pi = 0$, $\boldsymbol{\pi}_m = 0$, and $\rho_m = 0$) into a Taylor series. We retain quadratic terms in this decomposition which enables us to get linear constitutive equations. Let $e_{ij} \ll 1$, $\tilde{\mu}'_\pi \ll 1$, and $\nabla\tilde{\mu}'_\pi \ll 1$, where $\tilde{\mu}'_\pi = \mu'_\pi - \mu'_{\pi 0}$. For anisotropic media, we can write:

$$H = H_0 + \frac{1}{2\rho_0}\hat{\mathbf{C}}^{(4)} :: (\hat{\boldsymbol{e}} \otimes \hat{\boldsymbol{e}}) - \frac{1}{2} d_\mu \tilde{\mu}'^2_\pi - \frac{1}{2}\hat{\boldsymbol{\chi}}^m : (\nabla\tilde{\mu}'_\pi \otimes \nabla\tilde{\mu}'_\pi) - \frac{1}{\rho_0}\hat{\boldsymbol{\alpha}}^\mu : \hat{\boldsymbol{e}}\ \tilde{\mu}'_\pi$$
$$- \frac{1}{\rho_0}\hat{\mathbf{g}}^{(3)} \vdots (\hat{\boldsymbol{e}} \otimes \nabla\tilde{\mu}'_\pi) - (\boldsymbol{\gamma}^\mu \cdot \nabla\tilde{\mu}'_\pi)\tilde{\mu}'_\pi. \tag{21}$$

Here, ρ_0 is the reference mass density, $\hat{\mathbf{C}}^{(4)}$ is the fourth-order tensor of the elastic constants, $\hat{\boldsymbol{\alpha}}^\mu$ denotes the tensor of volumetric expansion caused by the local mass displacement, $\hat{\mathbf{g}}^{(3)}$ is the third-order tensor of piezomass constants, d_μ is the isochoric coefficient of the dependency of specific density of the induced mass on perturbation of modified chemical potential $\tilde{\mu}'_\pi$, $\hat{\boldsymbol{\chi}}^m$ denote the tensor characterizing the dependence of vectors of the local mass displacement on $\nabla\tilde{\mu}'_\pi$, $\hat{\boldsymbol{\gamma}}^\mu$ is the vector characterizing the dependence of the induced mass density on $\nabla\tilde{\mu}'_\pi$.

Note that for the linear approximation, tensor $\hat{\boldsymbol{\sigma}}_*$ can be replaced by the ordinary Cauchy stress tensor $\hat{\boldsymbol{\sigma}}$. Therefore, using relations (20) and (21), we obtain the following state equations:

$$\hat{\boldsymbol{\sigma}} = \hat{\mathbf{C}}^{(4)} : \hat{\boldsymbol{e}} - \hat{\boldsymbol{\alpha}}^{\mu}\,\tilde{\mu}'_{\pi} - \nabla\tilde{\mu}'_{\pi}\cdot\hat{\mathbf{g}}^{(3)}, \tag{22a}$$

$$\rho_m = d_{\mu}\tilde{\mu}'_{\pi} + \frac{1}{\rho_0}\hat{\boldsymbol{\alpha}}^{\mu} : \hat{\boldsymbol{e}} + \boldsymbol{\gamma}^{\mu}\cdot\nabla\tilde{\mu}'_{\pi}, \tag{22b}$$

$$\boldsymbol{\pi}_m = -\hat{\boldsymbol{\chi}}^{m}\cdot\nabla\tilde{\mu}'_{\pi} - \boldsymbol{\gamma}^{\mu}\,\tilde{\mu}'_{\pi} - \frac{1}{\rho_0}\hat{\mathbf{g}}^{(3)} : \hat{\boldsymbol{e}}. \tag{22c}$$

Due to the accounting for the local mass displacement, the constitutive equations (22) contain new material constants d_{μ}, $\boldsymbol{\gamma}^{\mu}$, $\hat{\boldsymbol{\alpha}}^{\mu}$, $\hat{\boldsymbol{\chi}}^{m}$, and $\hat{\mathbf{g}}^{(3)}$, characterizing the physical properties of material. The tensors of the second order $\hat{\boldsymbol{\alpha}}^{\mu}$ and $\hat{\boldsymbol{\chi}}^{m}$ have 6 components and vector $\boldsymbol{\gamma}^{\mu}$ has 3 components. The third-order tensor of the piezomass coefficients $\hat{\mathbf{g}}^{(3)}$ contains 18 independent components (they are symmetric about the permutation of the second and third subscripts).

The number of material constants is decreased for isotropic media. For such media, constitutive relations are as follows:

$$\hat{\boldsymbol{\sigma}} = 2G\hat{\boldsymbol{e}} + \big[\big(K - \tfrac{2}{3}G\big)e - K\alpha_{\mu}\tilde{\mu}'_{\pi}\big]\hat{\mathbf{I}}, \tag{23a}$$

$$\rho_m = d_{\mu}\tilde{\mu}'_{\pi} + \frac{K\alpha_{\mu}}{\rho_0}e, \tag{23b}$$

$$\boldsymbol{\pi}_m = -\chi_m\nabla\tilde{\mu}'_{\pi}. \tag{23c}$$

The local gradient theory yields five coefficients for isotropic materials, two of which are classical elastic moduli (i. e., K and G), and α_{μ}, d_{μ}, and χ_m are new material coefficients.

It is worth to note that the stress for isotropic materials is related not only to the strain, as in classical elasticity, but also to the modified chemical potential $\tilde{\mu}'_{\pi}$. Therefore, in general, the local mass displacement can influence the stress-strain state of an isotropic solid body.

For linear stationary approximation, Eqs. (4) and (13) can be written as:

$$\rho_m = -\nabla\cdot\boldsymbol{\pi}_m, \tag{24}$$

$$\nabla\cdot\hat{\boldsymbol{\sigma}} + \rho_0\mathbf{F} = 0. \tag{25}$$

For static fields, the complete linear system of field equations consists of Eqs. (24) and (25), the constitutive relations (22) (or ((23))), and the strain-displacement relation (16).

4. Variational approach

In this section, the variational principle is used to derive the governing equations and corresponding boundary conditions of the linear local gradient elasticity.

Assume that energy density potential H exists for the elastic continuum introduced in Section 2. To take the changes in the material microstructure into account, we assume the potential H to be a C^2-continuous function of the strain tensor $\hat{\boldsymbol{e}}$, modified chemical potential $\tilde{\mu}'_{\pi}$ as well as on its space gradient $\nabla\tilde{\mu}'_{\pi}$, that is: $H(\hat{\boldsymbol{e}}, \tilde{\mu}'_{\pi}, \nabla\tilde{\mu}'_{\pi})$. Let us also assume that constitutive relations are defined by the formulae (20). For elastic solids that occupy the region (V) bounded by a smooth surface (Σ), the variational principle can be expressed by formula

$$\delta\Big(-\int_{(V)} H(\hat{\boldsymbol{e}}, \tilde{\mu}'_\pi, \nabla\tilde{\mu}'_\pi)\,dV + \int_{(V)} W_V\,dV + \int_{(\Sigma)} W_s\,d\Sigma\Big) = 0. \tag{26}$$

The second and third integrals in (26) correspond to the virtual work done by the external body force and surface loading, respectively.

In view of the constitutive equations (20), the first integral in the above relation can be represented as

$$\begin{aligned}\delta\int_{(V)} H(\hat{\boldsymbol{e}}, \tilde{\mu}'_\pi, \nabla\tilde{\mu}'_\pi)\,dV &= \int_{(V)}\Big(\frac{\partial H}{\partial\hat{\boldsymbol{e}}}\delta\hat{\boldsymbol{e}} + \frac{\partial H}{\partial\tilde{\mu}'_\pi}\delta\tilde{\mu}'_\pi + \frac{\partial H}{\partial\nabla\tilde{\mu}'_\pi}\delta\nabla\tilde{\mu}'_\pi\Big)\,dV\\ &= \int_{(V)}\Big(\frac{1}{\rho_0}\hat{\boldsymbol{\sigma}} : \delta\hat{\boldsymbol{e}} - \rho_m\delta\tilde{\mu}'_\pi + \boldsymbol{\pi}_m\cdot\delta\nabla\tilde{\mu}'_\pi\Big)\,dV.\end{aligned}$$

Since $\hat{\boldsymbol{\sigma}}$ is a symmetric tensor, and taking strain-displacement relation (16) and equations

$$\begin{aligned}\boldsymbol{\pi}_m\cdot\delta\nabla\tilde{\mu}'_\pi &= \nabla\cdot(\boldsymbol{\pi}_m\delta\tilde{\mu}'_\pi) - (\nabla\cdot\boldsymbol{\pi}_m)\delta\tilde{\mu}'_\pi,\\ \hat{\boldsymbol{\sigma}} : \delta(\nabla\otimes\mathbf{u}) &= \nabla\cdot(\hat{\boldsymbol{\sigma}}\cdot\delta\mathbf{u}) - (\nabla\cdot\hat{\boldsymbol{\sigma}})\cdot\delta\mathbf{u}\end{aligned}$$

into account, the variation of the potential H can be expressed as follows:

$$\delta\int_{(V)} H\,dV = \int_{(V)}\Big(\frac{1}{\rho_0}\nabla\cdot(\hat{\boldsymbol{\sigma}}\cdot\delta\mathbf{u}) - \frac{1}{\rho_0}(\nabla\cdot\hat{\boldsymbol{\sigma}})\cdot\delta\mathbf{u} - (\rho_m + \nabla\cdot\boldsymbol{\pi}_m)\delta\tilde{\mu}'_\pi + \nabla\cdot(\boldsymbol{\pi}_m\delta\tilde{\mu}'_\pi)\Big)\,dV.$$

The variation of the external mass force is

$$\delta\int_{(V)} W_V\,dV = \int_{(V)} \mathbf{F}\cdot\delta\mathbf{u}\,dV.$$

Thus, by the divergence theorem for smooth surface (Σ), we can rewrite the expression (26) as

$$\int_{(V)}\Big(\frac{1}{\rho_0}(\nabla\cdot\hat{\boldsymbol{\sigma}} + \rho_0\mathbf{F})\cdot\delta\mathbf{u} + (\rho_m + \nabla\cdot\boldsymbol{\pi}_m)\delta\tilde{\mu}'_\pi\Big)\,dV - \int_{(\Sigma)}\Big(\frac{1}{\rho_0}\boldsymbol{\sigma}_n\cdot\delta\mathbf{u} + \pi_{mn}\delta\tilde{\mu}'_\pi - \delta W_s\Big)\,d\Sigma = 0, \tag{27}$$

where $\boldsymbol{\sigma}_n = \mathbf{n}\cdot\hat{\boldsymbol{\sigma}}$, and $\pi_{mn} = \mathbf{n}\cdot\boldsymbol{\pi}_m$. From variational relation (27), the stationary equations (24) and (25) as well as the following boundary conditions ensue:

$$(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{ns})\cdot\delta\mathbf{u} = 0, \quad \forall\mathbf{r}\in(\Sigma), \tag{28}$$

$$(\pi_{mn} - \pi_{ms})\delta\tilde{\mu}'_\pi = 0, \quad \forall\mathbf{r}\in(\Sigma). \tag{29}$$

Here, $\boldsymbol{\sigma}_{ns}$ and π_{ms} are the surface values of the corresponding quantities.

The equations set (16), (22), (24), and (25), along with the boundary conditions (28) and (29) constitute the stationary boundary-value problem for linear local gradient elasticity.

5. Bernoulli–Euler local gradient beam model; cantilever beam bending problem

In this section, based on relations of local gradient elasticity, the bending problem of elastic cantilever nanobeam is studied. The length of the cantilever beam is L, thickness is h and width is b. The beam

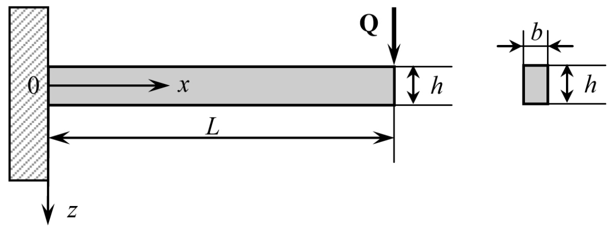

Figure 2. The cantilever beam with rectangular cross section subjected to a tip-point load.

thickness and width are assumed to be much less than the cantilever length, i. e., $L >> h$ and $L >> b$. Assume that beam is fixed at one end and concentrated load $\mathbf{Q}$ is applied at the free end of the cantilever beam (Figure 2). Let the x-axis be along the beam length, the y-axis be along the width of the beam, and the z-axis be along the thickness of the beam. The applied load and the beam geometry are treated in such a way that the plane stress conditions are realized in the body. For this case, the displacement vector and the modified chemical potential are functions of x and z space coordinates only.

For the sake of simplicity, we neglect the effect of the gradient of modified chemical potential on stress distribution $\hat{\boldsymbol{\sigma}}$ as well as on the density of the induced mass ρ_m. In this case, the constitutive equations (22) read as:

$$\hat{\boldsymbol{\sigma}} = \hat{\mathbf{C}}^{(4)} : \hat{\boldsymbol{e}} - \hat{\boldsymbol{\alpha}}^{\mu}\, \tilde{\mu}'_{\pi}, \qquad \rho_m = d_\mu \tilde{\mu}'_{\pi} + \frac{1}{\rho_0}\hat{\boldsymbol{\alpha}}^{\mu} : \hat{\boldsymbol{e}}, \qquad \boldsymbol{\pi}_m = -\hat{\boldsymbol{\chi}}^{m} \cdot \nabla \tilde{\mu}'_{\pi}. \tag{30}$$

In order to derive the relations of local gradient beam theory, we introduce the Bernoulli–Euler hypotheses. Thus, the displacement components are expressed as follows:

$$u_1 = -z w_{,x}\,, \qquad u_2 = 0, \qquad u_3 = w(x). \tag{31}$$

Here, $w(x)$ is the beam deflection, and the comma indicates differentiation with respect to the spatial variables. For this case, there is only one nonzero strain component e_{11}, that is

$$e_{11} = -z w_{,xx}\,. \tag{32}$$

Similarly to formulae (31), we represent the modified chemical potential as the linear function of z-coordinate, namely:

$$\tilde{\mu}'_{\pi}(x, z) = z m(x), \tag{33}$$

where $m(x)$ is unknown function of x-coordinate.

Let us consider the following material property matrices:

$$\begin{bmatrix} C_{11} & C_{13} & 0 \\ C_{13} & C_{33} & 0 \\ 0 & 0 & C_{55} \end{bmatrix}, \begin{bmatrix} \alpha_1^{\mu} \\ \alpha_3^{\mu} \\ 0 \end{bmatrix}, \begin{bmatrix} -\chi_1^m & 0 \\ 0 & -\chi_3^m \end{bmatrix}, \tag{34}$$

where the Voigt notation (i.e., $11 \to 1$, $22 \to 2$, $33 \to 3$, $23 \to 4$, $13 \to 5$, $12 \to 6$) are employed.

In view of relations (32)–(34), constitutive equations (30) may be rewritten as follows:

$$\sigma_{11}(x,z) = -z(C_{11}w_{,xx} + \alpha_1^{\mu} m) = z\sigma_1'(x), \tag{35a}$$

$$\sigma_{33}(x,z) = -z(C_{13}w_{,xx} + \alpha_3^{\mu} m) = z\sigma_3'(x), \tag{35b}$$

$$\sigma_{13} = 0, \tag{35c}$$

$$\rho_m(x,z) = z(d_\mu m - \alpha_1^{\mu}\rho_0^{-1} w_{,xx}) = z\rho_m'(x), \tag{35d}$$

$$\pi_m^1(x,z) = -\chi_1^m z m_{,x} = z\pi_1'(x), \tag{35e}$$

$$\pi_m^3(x) = -\chi_3^m m(x). \tag{35f}$$

To obtain the equations of local gradient beam model, let us consider the variation of the potential H. In view of (21) and (35), we have the following form for this energy:

$$H - H_0 = \frac{1}{\rho_0}\sigma_{11}e_{11} - \rho_m\tilde{\mu}_\pi' + \pi_m^1\,\tilde{\mu}_\pi'{}_{,x} + \pi_m^3\, d\tilde{\mu}_\pi'{}_{,z}\,. \tag{36}$$

By means of the variation of the relation (36), we obtain

$$\begin{aligned}\delta\int_{(V)} H\,dV &= \int_{(V)}\left(\frac{1}{\rho_0}\sigma_{11}\delta e_{11} - \rho_m\,\delta\tilde{\mu}_\pi' + \pi_m^1\,\delta\tilde{\mu}_\pi'{}_{,x} + \pi_m^3\,\delta\tilde{\mu}_\pi'{}_{,z}\right)dV \\ &= \int_0^L\int_{-h/2}^{h/2}\int_0^b\left(-\frac{1}{\rho_0}\sigma_{11}z\,\delta w_{,xx} - (\rho_m + \pi_m^1{}_{,x} + \pi_m^3{}_{,z})\delta\tilde{\mu}_\pi' + (\pi_m^1\,\delta\tilde{\mu}_\pi')_{,x} + (\pi_m^3\,\delta\tilde{\mu}_\pi')_{,z}\right)dx\,dy\,dz.\end{aligned}$$

The above relation may be expressed as

$$\delta\int_{(V)} H\,dV = -b\int_0^L\int_{-h/2}^{h/2}\left[\frac{1}{\rho_0}\sigma_{11}z\,\delta w_{,xx} + (\rho_m + \pi_m^1{}_{,x})\delta\tilde{\mu}_\pi'\right]dx\,dz + b\int_0^L\left(\pi_m^3\,\delta\tilde{\mu}_\pi'\right)\Big|_{z=-\frac{h}{2}}^{z=\frac{h}{2}}dx + b\int_{-h/2}^{h/2}\left(\pi_m^1\,\delta\tilde{\mu}_\pi'\right)\Big|_{x=0}^{x=L}dz.$$

Taking (35) and the relation

$$\sigma_{11}\delta w_{,xx} = (\sigma_{11}\delta w_{,x} - \sigma_{11,x}\,\delta w)_{,x} + \sigma_{11,xx}\,\delta w$$

into account and integrating by parts in the space coordinates, yields

$$\delta\int_{(V)} H\,dV = -\int_0^L\left\{\frac{I}{\rho_0}\sigma_1'{}_{,xx}\,\delta w + \left(I(\rho_m' + \pi_1'{}_{,x}) - S\pi_m^3\right)\delta m\right\}dx + I\left(\pi_1'\delta m\right)\Big|_{x=0}^{x=L} - \frac{I}{\rho_0}\left(\sigma_1'\,\delta w_{,x} - \sigma_1'{}_{,x}\,\delta w\right)\Big|_{x=0}^{x=L}. \tag{37}$$

Here, $S = bh$ is the cross-section area and $I = bh^3/12$ is the second moment of cross-section area.

Due to the arbitrariness of δw and δm, based on the variational relation (37), we obtain the governing equations

$$M_{,xx} = 0, \qquad I(\rho_m' + \pi_1'{}_{,x}) - S\pi_m^3 = 0, \tag{38}$$

with boundary conditions that are defined at the ends $x = 0$ and $x = L$ of the cantilever beam the following quantities:

$$M \text{ or } w,_x\,, \quad M,_x \text{ or } w, \quad I\pi_1' \text{ or } m.$$

Here, $M = -I(C_{11}w,_{xx} + \alpha_1^\mu m)$ is the bending moment.

Substituting (35) into (38) yields the equations to determine the unknown functions $w(x)$ and $m(x)$:

$$C_{11}w,_{xxxx} + \alpha_1^\mu m,_{xx} = 0, \tag{39}$$

$$m,_{xx} - \lambda_0^2(1 + \mathsf{M}_h)m + \frac{\alpha_1^\mu}{\rho_0\chi_1^m} w,_{xx} = 0, \tag{40}$$

where $\mathsf{M}_h = S\chi_3^m/Id_\mu$, and $\lambda_0 = \sqrt{d_\mu/\chi_1^m}$. Note that $l_* = \lambda_0^{-1}$ is the internal material length scale parameter. This kind of parameter appears within the gradient-type theories of elasticity while it is absent in the classical theory.

For cantilever beam, the boundary conditions can be written as:

$$x = 0: \quad w = 0, \quad w,_x = 0, \quad m = 0, \tag{41}$$

$$x = L: \quad M = 0, \quad M,_x = Q, \quad m = 0. \tag{42}$$

Finally, from Eqs. (39) and (40), we get the sixth-order ordinary differential equation for beam deflection

$$(w,_{xx} - \lambda^2 w),_{xxxx} = 0 \tag{43}$$

in contrast to the fourth-order equation of the classical Euler–Bernoulli beam model. Here, $\lambda^2 = \lambda_0^2(1 + \mathsf{M} + \mathsf{M}_h)$, and $\mathsf{M} = (\alpha_1^\mu)^2/\rho_0 d_\mu C_{11}$.

The solution to the formulated boundary-value problems (40)–(43) is as follows:

$$\begin{aligned} w(X) &= \frac{6QL^3}{C_{11}bh^3(1+\Omega)}\Bigg\{X^2 - \frac{X^3}{3} \\ &+ \frac{2\Omega[\exp(\xi)(\exp(-\xi X) - 1 + \xi X) - \exp(-\xi)(\exp(\xi X) - 1 - \xi X)]}{\xi^2[\exp(\xi) - \exp(-\xi)]}\Bigg\}, \\ m(X) &= \frac{12QL\Omega}{\alpha_1^\mu bh^3(1+\Omega)}\left[1 - X - \frac{\exp(\xi(1-X)) - \exp(-\xi(1-X))}{\exp(\xi) - \exp(-\xi)}\right]. \end{aligned} \tag{44}$$

Here, $\xi = L\lambda$, $\Omega = \mathsf{M}/(1 + \mathsf{M}_h)$, and $X = x/L$ is the dimensionless space coordinate.

If the local mass displacement effect is neglected $(1/\lambda \to 0)$, the governing equation (43) can be reduced to a known fourth order differential equation of the classical Bernoulli–Euler beam theory, namely, $w,_{xxxx} = 0$ [Shokrieh and Zibaei 2015]. The analytical solution for classical theory is as follows:

$$w_{cl}(X) = \frac{6QL^3}{C_{11}bh^3}\left(X^2 - \frac{X^3}{3}\right). \tag{45}$$

6. Numerical calculations and discussions

We now discuss the effect of local mass displacement on mechanical response of nanobeam. To illustrate this effect, the deflection of cantilever beam under the end-point load was plotted in Figures 3–6 for material PZT-5H. The concentrated force is Q = 1 nN and the geometry of the nanobeam is $L = 500$ nm, $h = 20$ nm, and $b = 10$ nm. The elastic material constants for PZT-5H are $C_{11} = 12.6 \times 10^{10}$ Pa, $C_{13} = 5.3 \times 10^{10}$ Pa, and $C_{33} = 11.7 \times 10^{10}$ Pa. Unfortunately, the relevant parameters d_μ, α_i^μ, and χ_i^m for PZT-5H are not available in the literature. Hence, the material properties related to the local mass displacement (i. e., internal material length scale parameters) adopted here are: $l_* = \sqrt{\chi_1^m/d_\mu} = 3$ nm, and $l_{*3} = \sqrt{\chi_3^m/d_\mu} = 1$ nm.

Note that based on the Bernoulli–Euler strain-gradient theory [Lurie and Solyaev 2019], the deflection can be calculated as

$$w_{st}(X) = \frac{6QL^5}{c_{11}bh^3\zeta^2 l^2} \times \left(X^2 - \frac{X^3}{3} - \frac{2X}{\zeta^2} + 2\frac{(\exp(\zeta X)-1)(1-\exp(-\zeta)) + (\exp(-\zeta X)-1)(1-\exp(\zeta))}{\zeta^3(\exp(\zeta)-\exp(-\zeta))} \right), \quad (46)$$

where l is the material characteristic length scale parameter, and $\zeta = Ll^{-1}\sqrt{1+12l^2/h^2}$.

Solution (46) satisfies the following boundary conditions:

$$x = 0: \quad w = 0, \quad w_{,x} = 0, \quad M^H = 0,$$
$$x = L: \quad M^H = 0, \quad M + P - M^H_{,x} = 0, \quad (M + P - M^H_{,x})_{,x} = Q,$$

where M is the bending moment,

$$M(x) = \int_0^b \int_{-h/2}^{h/2} z\sigma_{11}(x,z)\,dy\,dz, \qquad \sigma_{11}(x,z) = -C_{11}z\frac{d^2w(x)}{dx^2},$$

M^H is the higher-order bending moment,

$$M^H(x) = \int_0^b \int_{-h/2}^{h/2} z\tau_{111}(x,z)\,dy\,dz, \qquad \tau_{111}(x,z) = -l^2C_{11}z\frac{d^3w(x)}{dx^3},$$

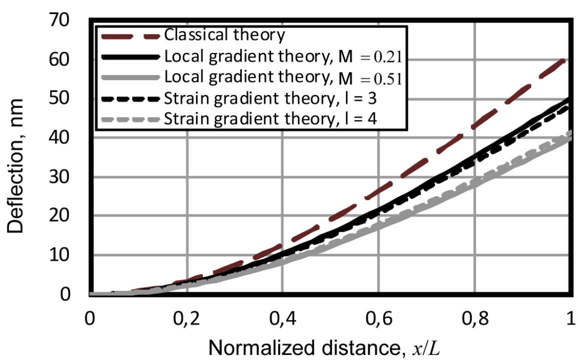

Figure 3. The deflections predicted by the classical theory, local gradient theory and strain gradient theory of the elastic beams.

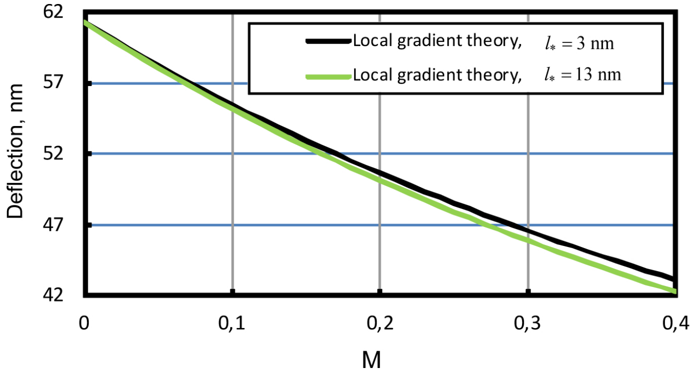

Figure 4. The end-point deflection $w(L)$ versus the coupling factor M for different length scale parameters.

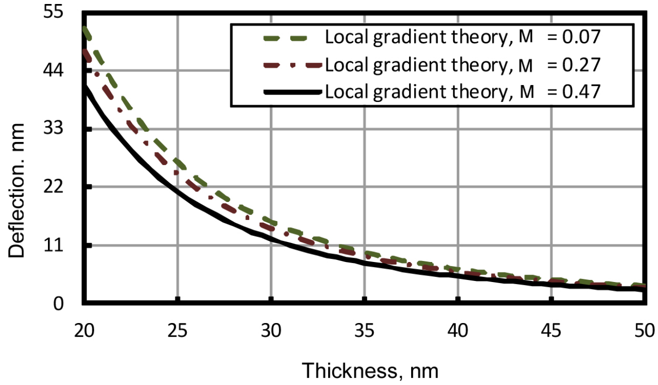

Figure 5. Variation of the end-point deflection $w(L)$ with the beam thickness h.

and P is the higher-order shear forces:

$$P(x) = \int_0^b \int_{-h/2}^{h/2} \tau_{113}(x, z)\, dy\, dz, \qquad \tau_{113} = -l^2 C_{11} \frac{d^2 w(x)}{dx^2}.$$

A comparison of the deflection between the analytical solution (45) for classical theory, solution (46) for strain-gradient theory, and solution (44) for local gradient theory is presented in Figure 3. The deflection predicted by the local gradient beam theory is smaller than that by the classical elastic Bernoulli–Euler beam theory. Such a result describes the experimental data reported in the literature and agrees well with the results obtained within the strain-gradient theory of beam. The graphs in Figure 4 show that the beam deflection essentially depends on the coupling factor M. An increased parameter M the beam deflection decreases.

The deflection of the end-point of cantilever beam versus the different beam thickness and length are shown in Figures 5 and 6, respectively. As it follows from Figure 5, the deflection in the local gradient beam theory increases if the beam thickness decreases. This effect became more significant when the thickness of the nanobeam became comparable to the internal material length scale parameter.

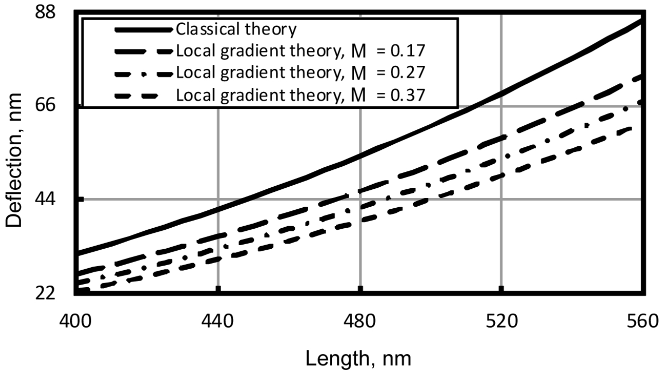

Figure 6. Variation of the end-point deflection $w(L)$ with the beam length L.

Note that the beam stiffness is directly related to the deflection. Therefore, the stiffness based on the relations of local gradient elasticity is higher than that of predicted by classical theory, i. e., the local mass displacement being taken into account stiffens the elastic cantilever nanobeam.

Note that the analytical solution (44) can be used to evaluate material constants pertinent to the local mass displacement. Indeed, based on formula (44), we get

$$\Omega = \frac{QL^3/3 - w(L)C_{11}I}{QL^3 g(\zeta) + w(L)C_{11}I}, \tag{47}$$

where $\zeta = L/l_* >> 1$ and $g(\zeta) = (\mathrm{th}\zeta - \zeta^{-1})/\zeta \approx \zeta^{-1}$. The mechanical deflection $w(L)$ of nanocantilever beam subjected to the end-point loading can be defined from force-displacement measurements with the help of atomic force microscopy or other techniques of mechanical property measurement [Hardwick 1987; Weihs et al. 1988; Liebold and Müller 2015]. The length scale parameter l_* can be evaluated using the methods of molecular dynamics or lattice theory [Askar et al. 1970]. Then, based on formula (47), the material constant Ω can be evaluated using the known values of the concentrated force Q, beam geometric properties (i.e., b, h and L), elastic module C_{11}, beam deflection $w(L)$ and length scale parameter l_*.

7. Conclusion

In the present study, the classical theory of elasticity is extended by taking account of non-diffusive and non-convective mass flux associated with the possible changes in the material microstructure. This flux is related to the process of local mass displacement. New physical quantities associated with this process are introduced and an additional balance equation for these quantities is formulated. Based on the total energy balance equation and on the principle of frame-indifference, a fundamental system of coupled nonlinear equations of local gradient theory of elastic solids is derived. Within the framework of this theory, the set of conjugate variables is complemented by two additional pairs of variables (μ'_π, ρ_m) and $(\boldsymbol{\pi}_m, \nabla\mu'_\pi)$, related to the local mass displacement. The local mass displacement being taken into consideration leads to the appearance of an additional nonlinear mass force in the equation of motion and to the redefinition of the stress tensor $\hat{\boldsymbol{\sigma}}_*$. It is shown that the linear relations of the local gradient elasticity can be also formulated within a framework of the variational principle.

A simple cantilever problem with Bernoulli–Euler kinematics is investigated to illustrate the efficiency of the theory. The governing equations of local gradient Bernoulli–Euler nanobeam model and corresponding boundary conditions are derived from a variational principle. An analytical solution to the resultant differential equation of the sixth order is derived and the effect of the local mass displacement on beam behavior under the end-point load is discussed. It is shown that the deflection within the local gradient theory is smaller than that in the classical Bernoulli–Euler nanobeam model. This indicates that the local mass displacement being taken into account stiffens the nanocantilever beam. The obtained result agrees well with the strain gradient theory as well as with the data of experimental investigations available in the scientific literature. The possibility of the evaluation of new material constants associated with the local gradient elasticity is pointed out. It is shown that the developed local gradient Bernoulli–Euler micro-beam model recovers the classical Bernoulli–Euler beam theory for vanishing length scale parameters.

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OLHA HRYTSYNA: hrytsyna.olha@gmail.com
Institute of Construction and Architecture, Slovak Academy of Sciences, 84503 Bratislava 45, Slovakia

and

Center of Mathematical Modeling, Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, National Academy of Sciences of Ukraine, 79005 Lviv, Ukraine

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