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Structured sequences and matrix ranks

Charles Johnson, Yaoxian Qu, Duo Wang and John Wilkes

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We consider infinite sequences from a field and all matrices whose rows consist of distinct consecutive subsequences. We show that these matrices have bounded rank if and only if the sequence is a homogeneous linear recurrence; moreover, it is a k-term linear recurrence if and only if the maximum rank is k. This means, in particular, that the ranks of matrices from the sequence of primes are unbounded. Though not all matrices from the primes have full rank, because of the Green–Tao theorem, we conjecture that square matrices whose entries are a consecutive sequence of primes do have full rank.

1. Introduction

A familiar way to obtain a simple example of a rank-deficient matrix is to array the positive integers as a matrix, wrapping at the end of each row.

For example,

$$\text{rank} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 2.$$

For $m, n \geq 2$, if we start at any integer and array mn consecutive integers as an m-by-n matrix in this way, the rank is 2. In fact, every submatrix of size at least 2-by-2 will have rank 2.

For convenience, we consider *real sequences* $a_1, a_2, a_3, \ldots$, though much of what we say holds over any field. We refer to this sequence as a. The sequence a is a *k-term linear recurrence* if it is of the form

$$a_{i+k} = b_1 a_i + b_2 a_{i+1} + b_3 a_{i+2} + \cdots + b_k a_{i+k-1},$$

in which b_i is a constant, $i = 1, \ldots, k$ [Brousseau 1971]. If we suppress the value of k, we just say "linear recurrence". Some authors refer to the above as a homogeneous linear recurrence, whereas a nonhomogeneous linear recurrence

allows a constant term. However, a k-term nonhomogeneous linear recurrence is also a $2k$-term homogeneous one (which also follows from our Theorem 2.3).

A geometric progression is just the case when $k = 1$, and arithmetic progressions are a special case of 2-term linear recurrences. The positive integers are a very simple arithmetic progression.

In general, if a sequence is a k-term (linear) recurrence, it is entirely determined by its first k terms, and the collection of such sequences (with fixed coefficients) is k-dimensional, including degenerate cases.

Definition 1.1. $M = (m_{ij})$ is a *traditional matrix* of the sequence a if its m_{11} entry is a_j and the remaining entries are consecutive:

$$M_{m,n}(j) = \begin{pmatrix} a_j & a_{j+1} & \cdots & a_{j+n-1} \\ a_{j+n} & a_{j+n+1} & \cdots & a_{j+2n-1} \\ \vdots & & & \vdots \\ a_{j+(m-1)+n} & a_{j+m+n} & \cdots & a_{j+mn-1} \end{pmatrix}.$$

Definition 1.2. M is a *nontraditional matrix* of the sequence a if each row is simply a consecutive subsequence and each begins with a distinct element of the sequence.

For example,

$$M = \begin{pmatrix} a_3 & a_4 & a_5 & a_6 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_5 & a_6 & a_7 & a_8 \\ a_{20} & a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

In either event, we say that M is a matrix of the sequence. For any infinite sequence a, let $M(a)$ be the set of all matrices of the sequence a.

2. Linear recurrences

We now consider sequences that are linear k-term recurrences and we consider the matrices of such sequences. Sometimes, we suppress the "k" and just refer to a "linear recurrence". All facts about ranks of matrices that we use may be found, for example, in [Lay 2006]. We first note:

Lemma 2.1. *If the sequence a is any k-term linear recurrence, then for any matrix $A \in M(a)$, we have* $\mathrm{rank}(A) \leq k$.

Proof. Let

$$A = \begin{pmatrix} a_j & a_{j+1} & \cdots & a_{j+k-1} & a_{j+k} & \cdots & a_{j+n-1} \\ a_i & a_{i+1} & \cdots & a_{i+k-1} & a_{i+k} & \cdots & a_{i+n-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+k-1} & a_{m+k} & \cdots & a_{m+n-1} \end{pmatrix},$$

where
$$a_{j+k} = b_1 a_j + b_2 a_{j+1} + \cdots + b_k a_{j+k-1},$$
$$a_{i+k} = b_1 a_i + b_2 a_{i+1} + \cdots + b_k a_{i+k-1},$$
$$\vdots$$
$$a_{m+k} = b_1 a_m + b_2 a_{m+1} + \cdots + b_k a_{m+k-1}.$$

So the $(k+1)$-th column is a linear combination of the first k columns. Similarly, columns after the $(k+1)$-th column are linear combinations of the k columns before them. Thus at most the first k columns are linearly independent, so $\operatorname{rank}(A) \leq k$. $\square$

Example 2.2. The Fibonacci sequence is a 2-term linear recurrence. Here is a traditional matrix of the Fibonacci sequence:

$$\operatorname{rank} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 5 & 8 & 13 & 21 \\ 34 & 55 & 89 & 144 \\ 233 & 377 & 610 & 987 \end{pmatrix} = 2.$$

Theorem 2.3. *A sequence a is a linear recurrence if and only if matrices in $M(a)$ are of bounded rank. Moreover, if the largest rank of a matrix in $M(a)$ is k, then the linear recurrence is a k-term one.*

Proof. The forward implication is implied by Lemma 2.1.

For the converse, the bounded-rank hypothesis means that there is a largest rank among matrices in $M(a)$; call it k. Choose a matrix $A \in M(a)$ such that

- $\operatorname{rank}(A) = k$,
- no other matrix in $M(a)$ with rank k has fewer columns,
- given this number of columns, no matrix with rank k has fewer rows.

Suppose that A is m-by-n. Notice that any matrix in $M(a)$ of which A is a submatrix must have rank k. Suppose that $A' \in M(a)$ is m-by-$(n+1)$ and that the first n columns of A' are A, and the last column of A' is a linear combination of the first n columns:

$$a_{i+n} = b_1 a_i + b_2 a_{i+1} + \cdots + b_n a_{i+n-1} \tag{$*$}$$

(for any row of A' of which a_i is the first entry) and $\dim \operatorname{Nul}(A') = n+1-k$. Now, if we consider any $\bar{A} \in M(a)$ whose first m rows are those of A', $\operatorname{rank}(\bar{A}) = k$, we have $\dim \operatorname{Nul}(\bar{A}) = n+1-k$, and, since $\operatorname{Nul}(\bar{A}) \subseteq \operatorname{Nul}(A')$, we have $\operatorname{Nul}(\bar{A}) = \operatorname{Nul}(A')$. This means that any allowed row added to A' must satisfy the same linear relations as satisfied collectively by the rows of A'. But, we were allowed to add any row, as long as that row was a consecutive subsequence of a (of length $n+1$) that was not already a row of A'. This means that a satisfies $(*)$ for any $i \geq 1$, and thus a is a linear recurrence, which completes the proof. $\square$

3. Ranks less than k in k-term sequences

So far we know that if a is a linear recurrence, then it is a k-term (and not less) linear recurrence if and only if the maximum rank among matrices in $M(a)$ is k. We might refer to a "rank-k linear recurrence" to be absolutely unambiguous, as a recurrence presented as a k-term linear recurrence might actually have lower rank and be an h-term recurrence for some $h < k$.

This leaves an interesting question of detail. What about the ranks of m-by-n matrices in $M(a)$, when $k \leq m, n$; can they have rank$< k$, and, if so, under what circumstances? Notice that $M(a)$ is closed under the extraction of submatrices in which the column indices are consecutive and that row indices may as well be taken to be consecutive. So, we are really asking about ranks of contiguous submatrices of matrices in $M(a)$. Are they always full, subject to being no more than k?

Interestingly, they are, with some modest exceptions; we consider in detail here the case $k = 2$.

Example 3.1. Let a be the sequence which is generated by the 2-term recurrence

$$a_{i+2} = -a_i - \sqrt{2}a_{i+1}.$$

But

$$\begin{pmatrix} a_1 & a_2 \\ a_5 & a_6 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

lies in $M(a)$ and has rank 1.

Suppose we have a 2-term recurrence $C_1 a_n + C_2 a_{n+1} = a_{n+2}$, and suppose that there exists a contiguous submatrix in $M(a)$ with rank 1:

$$\operatorname{rank} \begin{pmatrix} a_i & a_{i+1} \\ a_j & a_{j+1} \end{pmatrix} = 1, \quad j > i.$$

Lemma 3.2. *If we right-extend the submatrix above by adding columns with subsequent elements or left-extend the submatrix by adding columns with preceding elements in the recurrence, then the rank of the submatrix does not change, and*

$$\begin{pmatrix} a_{i+k} \\ a_{j+k} \end{pmatrix} = r_k \begin{pmatrix} a_{i+k-1} \\ a_{j+k-1} \end{pmatrix}, \quad k \in \mathbb{Z}, \ r_k \neq 0,$$

where

$$r_{n+1} = \frac{C_1}{r_n} + C_2, \quad r_1 = b,$$

in which r_k is the ratio between entries in one column of the submatrix and the next.

Proof. Since

$$\operatorname{rank} \begin{pmatrix} a_i & a_{i+1} \\ a_j & a_{j+1} \end{pmatrix} = 1,$$

we have

$$\begin{pmatrix} a_{i+1} \\ a_{j+1} \end{pmatrix} = b \begin{pmatrix} a_i \\ a_j \end{pmatrix},$$

and since

$$C_1 \begin{pmatrix} a_i \\ a_j \end{pmatrix} + C_2 \begin{pmatrix} a_{i+1} \\ a_{j+1} \end{pmatrix} = \begin{pmatrix} a_{i+2} \\ a_{j+2} \end{pmatrix},$$

we have

$$\frac{C_1}{b} \begin{pmatrix} a_{i+1} \\ a_{j+1} \end{pmatrix} + C_2 \begin{pmatrix} a_{i+1} \\ a_{j+1} \end{pmatrix} = \begin{pmatrix} a_{i+2} \\ a_{j+2} \end{pmatrix}.$$

Thus

$$\left(\frac{C_1}{b} + C_2 \right) \begin{pmatrix} a_{i+1} \\ a_{j+1} \end{pmatrix} = \begin{pmatrix} a_{i+2} \\ a_{j+2} \end{pmatrix}.$$

Suppose $k = 1$; then

$$\left(\frac{C_1}{b} + C_2 \right) \begin{pmatrix} a_{i+1} \\ a_{j+1} \end{pmatrix} = \begin{pmatrix} a_{i+2} \\ a_{j+2} \end{pmatrix}.$$

Suppose $k = x$; then

$$r_x \begin{pmatrix} a_{i+x-1} \\ a_{j+x-1} \end{pmatrix} = \begin{pmatrix} a_{i+x} \\ a_{j+x} \end{pmatrix}.$$

Since

$$C_1 \begin{pmatrix} a_{i+x-1} \\ a_{j+x-1} \end{pmatrix} + C_2 \begin{pmatrix} a_{i+x} \\ a_{j+x} \end{pmatrix} = \begin{pmatrix} a_{i+x+1} \\ a_{j+x+1} \end{pmatrix},$$

we have

$$\frac{C_1}{r_x} \begin{pmatrix} a_{i+x} \\ a_{j+x} \end{pmatrix} + C_2 \begin{pmatrix} a_{i+x} \\ a_{j+x} \end{pmatrix} = \begin{pmatrix} a_{i+x+1} \\ a_{j+x+1} \end{pmatrix}.$$

Thus

$$r_{x+1} \begin{pmatrix} a_{i+x} \\ a_{j+x} \end{pmatrix} = \begin{pmatrix} a_{i+x+1} \\ a_{j+x+1} \end{pmatrix}$$

when $k = x + 1$. Thus for $k \in \mathbb{N}$, we have

$$\begin{pmatrix} a_{i+k} \\ a_{j+k} \end{pmatrix} = r_k \begin{pmatrix} a_{i+k-1} \\ a_{j+k-1} \end{pmatrix}$$

by mathematical induction. Similarly, we can easily see that for $k \in \mathbb{Z}$,

$$\begin{pmatrix} a_{i+k} \\ a_{j+k} \end{pmatrix} = r_k \begin{pmatrix} a_{i+k-1} \\ a_{j+k-1} \end{pmatrix}.$$

Thus, every added column is linearly dependent to the previous one, which means that the submatrix is still rank-1. $\square$

Lemma 3.3. *There must exist k such that $r_1 = C_1/r_k + C_2$.*

Proof. Since extending the submatrix does not change its rank, the following matrix is still rank-1:

$$\begin{pmatrix} a_i & a_{i+1} & \cdots & a_j & a_{j+1} \\ a_j & a_{j+1} & \cdots & a_{2j-i} & a_{2j-i+1} \end{pmatrix}. \tag{1}$$

Thus,

$$r_{j-i+1} \begin{pmatrix} a_j \\ a_{2j-1} \end{pmatrix} = \begin{pmatrix} a_{j+1} \\ a_{2j-i+1} \end{pmatrix}.$$

Since $a_{j+1}/a_j = b$, we have $r_{j-i+1} = b$. Thus, when $k = j - i + 1$, we have $r_1 = C_1/r_k + C_2$. □

Since $r_{n+1} = C_1/r_n + C_2$, we know that $\binom{r_{n+1}}{1}$ must be parallel to

$$\begin{pmatrix} C_2 & C_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_n \\ 1 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} r_{n+1} \\ 1 \end{pmatrix} = P_n \begin{pmatrix} C_2 & C_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_n \\ 1 \end{pmatrix}, \quad P_n = \frac{1}{r_n}.$$

Hence

$$\begin{pmatrix} r_n \\ 1 \end{pmatrix} = P_1 P_2 \cdots P_n \begin{pmatrix} C_2 & C_1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} r_1 \\ 1 \end{pmatrix}.$$

According to Lemma 3.3, there must exist k such that

$$\begin{pmatrix} r_1 \\ 1 \end{pmatrix} = \begin{pmatrix} r_k \\ 1 \end{pmatrix} = P_1 P_2 \cdots P_k \begin{pmatrix} C_2 & C_1 \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} r_1 \\ 1 \end{pmatrix},$$

which means that $\binom{r_1}{1}$ is an eigenvector of

$$\begin{pmatrix} C_2 & C_1 \\ 1 & 0 \end{pmatrix}^k.$$

Theorem 3.4. *If there exists a rank-1 contiguous submatrix in $M(a)$, then a does not degenerate into a geometric recurrence only when*

$$\begin{pmatrix} C_2 & C_1 \\ 1 & 0 \end{pmatrix}^k =: C^k$$

is diagonalizable and has distinct eigenvalues.

Proof. Suppose that C^k is diagonalizable. If C^k does not have distinct eigenvalues, then C must have exactly the same eigenvectors as C^k. Thus $\binom{r_1}{1}$ is an eigenvector of C. Thus

$$\begin{pmatrix} r_1 \\ 1 \end{pmatrix} C = \lambda \begin{pmatrix} r_1 \\ 1 \end{pmatrix},$$

and hence

$$r_1^2 = C_2 r_1 + C_1, \quad r_1 = \frac{C_1}{r_1} + C_2, \quad r_1 = r_2.$$

When $n = 1$, we have $r_n = r_1$ and when $n = 2$, we have $\text{r}_n = r_1$; therefore suppose when $n = x$ we have $r_n = r_1$.

Thus $C_1/r_x + C_2 = r_{x+1} = r_1$ when $n = x + 1$, $r_n = r_1$, so $r_n = r_1$ for $n \in \mathbb{N}$. Similarly, we can see that $r_n = r_1$ for $n \in \mathbb{Z}$. Thus the ratio between every pair of consecutive elements in a is b, which means that a is a geometric progression.

When C^k is not diagonalizable, C^k does not have the same eigenvalues since it has only one eigenvalue. Thus C has exactly the same eigenvalue as C^k. Thus $\binom{r_1}{1}$ is an eigenvector of C. Thus $r_1 = r_2$, and as described above, we can see that a is a geometric progression.

Thus, the case that a does not degenerate into a geometric recurrence, illustrated in Example 3.1, can only happen when C^k is diagonalizable and has distinct eigenvalues. □

4. Prime sequences

We now turn to the sequence of primes and subsequences thereof. Let p be the sequence: 2, 3, 5, 7, 11, 13, 17, 19, 21, ... The primes are not a linear recurrence and seem arithmetically very unstructured. We generated many examples of traditional matrices of primes (up to 100-by-100) and they were always full rank. This suggests:

Conjecture 4.1. If A is an m-by-n *traditional matrix* of primes, then rank $A = \min\{m, n\}$.

This seems difficult to prove, perhaps for the following reason. Suppose that q is a subsequence of p. Again, experiments always turned up full rank for traditional matrices of q.

Observation 4.2. Because of unique factorization, every m-by-n submatrix, with $m, n \geq 2$, of a traditional matrix of q has at least rank 2. Now 2-by-2 minors are of the form $q_i q_j - q_t q_s$ with i, j, t, s distinct and, so, cannot be 0, else $q_i q_j$ and $q_t q_s$ would be distinct prime factorizations of the same integer. However, there exist subsequences and an m-by-n traditional matrix A with $m, n > 2$ such that rank$(A) = 2$. For instance,

$$\text{rank} \begin{pmatrix} 3 & 5 & 13 \\ 7 & 11 & 29 \\ 17 & 31 & 79 \end{pmatrix} = 2.$$

The above example is not isolated.

Theorem 4.3 [Green and Tao 2008]. *The prime numbers contain infinitely many arithmetic progressions of length k for each positive integer $k > 1$.*

This means that $M(p)$ contains many "large" matrices of "low rank".

Though we cannot prove Conjecture 4.1, a weaker version does follow from our results about linear recurrences,

Corollary 4.4. *The set of ranks of all matrices of the sequence p of primes is unbounded.*

Proof. Since p is an infinite sequence, according to Theorem 4.3, if the matrices in $M(p)$ had bounded rank, then $M(p)$ would be a linear recurrence. This is not true. Thus, p must include matrices of unbounded rank. $\qquad\square$

References

[Brousseau 1971] A. Brousseau, *Linear recursion and Fibonacci sequences*, Fibonacci Association, San Jose, CA, 1971.

[Green and Tao 2008] B. Green and T. Tao, "The primes contain arbitrarily long arithmetic progressions", *Ann. of Math.* (2) **167**:2 (2008), 481–547. MR Zbl

[Lay 2006] D. C. Lay, *Linear algebra and its applications*, 3rd ed., Pearson/Addison-Wesley, Boston, MA, 2006.

crjohn@wm.edu   *Department of Mathematics, The College of William & Mary, Williamsburg, VA, United States*

quyaoxian@gmail.com   *Department of Mathematics, University of Notre Dame, South Bend, IN, United States*

dwang02@email.wm.edu   *Department of Mathematics, The College of William & Mary, Williamsburg, VA, United States*

jcwilkes@email.wm.edu   *Department of Mathematics, The College of William & Mary, Williamsburg, VA, United States*

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