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Sarah M. Fleming and Susan Loepp



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Let  $(T, \mathfrak{m})$  be a complete local (Noetherian) domain such that  $\text{depth } T > 1$ . In addition, suppose  $T$  contains the rationals,  $|T| = |T/\mathfrak{m}|$ , and the set of all principal height-1 prime ideals of  $T$  has the same cardinality as  $T$ . We construct a universally catenary local unique factorization domain  $A$  such that the completion of  $A$  is  $T$  and such that there exist uncountably many height-1 prime ideals  $\mathfrak{q}$  of  $A$  such that  $(T/(\mathfrak{q} \cap A)T)_{\mathfrak{q}}$  is a field. Furthermore, in the case where  $T$  is a normal domain, we can make  $A$  “close” to excellent in the following sense: the formal fiber at every prime ideal of  $A$  of height not equal to 1 is geometrically regular, and uncountably many height-1 prime ideals of  $A$  have geometrically regular formal fibers.

## 1. Introduction

One important area of study in commutative algebra is completions of local rings. Many mathematicians have worked to characterize the completions of certain classes of rings. For example, Heitmann [1993] found surprisingly weak necessary and sufficient conditions for a ring to be the completion of a local unique factorization domain (UFD), and Loepp [2003] characterized the completions of local excellent integral domains of characteristic zero. The question of when a ring is the completion of an excellent UFD, the natural extension of these two results, is still open. Bryk et al. [2005] obtained a partial result by constructing what they term “almost excellent UFDs”. Boocher et al. [2010] also attempted to characterize the completions of excellent UFDs and arrived at several necessary conditions for the construction.

One technique for studying the relationship between a ring and its completion is to use the formal fibers, which encode information about the prime ideals of the ring.

**Definition 1.1.** Let  $A$  be a local (Noetherian) ring and let  $\hat{A}$  denote the completion of  $A$  with respect to its maximal ideal. For a prime ideal  $\mathfrak{p}$  of  $A$ , the *formal fiber* of  $A$  at  $\mathfrak{p}$  is given by the set  $\text{Spec}(\hat{A} \otimes_A \kappa(\mathfrak{p}))$ , where  $\kappa(\mathfrak{p})$  is the residue field  $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ . For a local domain  $A$ , the *generic formal fiber* of  $A$  is the formal fiber of  $A$  at the zero ideal, which is the set  $\text{Spec}(\hat{A} \otimes_A K)$ , where  $K$  is the quotient field of  $A$ .

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It is also possible to characterize the formal fibers in terms of the prime ideals of  $A$  and  $\hat{A}$ . Let  $\varphi : \text{Spec } \hat{A} \rightarrow \text{Spec } A$  be the map that sends a prime ideal  $\mathfrak{q} \in \text{Spec } \hat{A}$  to  $\mathfrak{q} \cap A$ . There is a one-to-one correspondence between the inverse image of an ideal  $\mathfrak{p}$  under this morphism and the elements of  $\text{Spec}(\hat{A} \otimes_A \kappa(\mathfrak{p}))$ : the prime ideals of  $\text{Spec}(\hat{A} \otimes_A \kappa(\mathfrak{p}))$  are of the form  $\mathfrak{q} \otimes_A \kappa(\mathfrak{p})$  where  $\mathfrak{q} \cap A = \mathfrak{p}$ .

Our ultimate goal is to characterize the completions of excellent UFDs, but in order to break this problem down into a more manageable one, we consider what it means for a ring to be “close” to excellent. In order for a ring  $A$  to be excellent, all of its formal fibers must be geometrically regular. What if there were a way to get “most” of the formal fibers of  $A$  to be geometrically regular?

We first work towards getting a large number of the formal fibers of  $A$  at height-1 prime ideals to be geometrically regular—in fact, uncountably many. More specifically, we pose the following question, letting  $\mathcal{S}$  denote the set of principal height-1 prime ideals of  $T$ .

**Question 1.2.** Let  $T$  be a complete local domain that contains the rationals. When does there exist a local universally catenary UFD  $A$  such that  $\hat{A} \cong T$  and there is an uncountable subset  $\mathcal{S}'$  of  $\mathcal{S}$  such that

- (1) if  $\mathfrak{q} \in \mathcal{S}'$ , then  $\mathfrak{q} \cap A \neq (0)$ ,
- (2) if  $\mathfrak{q}, \mathfrak{q}' \in \mathcal{S}'$  with  $\mathfrak{q} \neq \mathfrak{q}'$ , then  $\mathfrak{q} \cap A \neq \mathfrak{q}' \cap A$ , and
- (3) if  $\mathfrak{q} \in \mathcal{S}'$ , then the formal fiber of  $A$  at  $\mathfrak{q} \cap A$  is geometrically regular?

Letting  $\mathfrak{m}$  denote the maximal ideal of  $T$ , we prove that, if  $\dim T = 2$ ,  $\text{depth } T > 1$ , and  $|T| = |T/\mathfrak{m}| = |\mathcal{S}|$ , then there does exist a universally catenary UFD  $A$  whose completion is  $T$  that satisfies the above conditions. If  $\dim T > 2$ , then we prove that there exists a universally catenary UFD  $A$  with  $\hat{A} = T$  that satisfies conditions (1) and (2) and a weaker version of (3): if  $\mathfrak{q}$  is in  $\mathcal{S}'$ , then  $(T/(\mathfrak{q} \cap A)T)_{\mathfrak{q}}$  is a field. Our construction is a first step towards constructing an excellent UFD; however, we have no control over (i) the generic formal fiber or (ii) the formal fibers at prime ideals of  $A$  of height greater than 1.

If we assume  $T$  is a complete local normal domain containing the rationals with  $\text{depth } T > 1$  and  $|T| = |T/\mathfrak{m}| = |\mathcal{S}|$ , then we show there exists a local universally catenary UFD  $A$  where  $\hat{A} = T$  and such that  $A$  satisfies conditions (1), (2), and (3), along with the additional condition that the formal fibers of  $A$  at every prime ideal of height not equal to 1 is geometrically regular.

Thus, given a complete local normal domain  $T$  containing the rationals where  $\text{depth } T > 1$  and  $|T| = |T/\mathfrak{m}| = |\mathcal{S}|$ , we construct a local universally catenary UFD  $A$  with completion  $T$  that is “close” to excellent; “close” in the sense that  $A$  is universally catenary and the formal fiber at every prime ideal—except perhaps uncountably many height-1 prime ideals—is geometrically regular. Further, we

construct  $A$  so that the formal fibers at uncountably many of the height-1 prime ideals of  $A$  are geometrically regular. This is a new result, as previously, mathematicians have only been able to construct UFDs with countably many of the height-1 prime ideals having geometrically regular formal fibers; see [Boocher et al. 2010].

## 2. Preliminaries

In this paper, all rings are assumed to be commutative with unity and local rings are assumed to be Noetherian. If a ring has exactly one maximal ideal, but is not necessarily Noetherian, we say it is quasilocal. Often, we denote a quasilocal ring  $R$  with maximal ideal  $\mathfrak{m}$  by  $(R, \mathfrak{m})$ . We recall the definition of geometrically regular and what it means for a local ring to be excellent.

**Definition 2.1.** Let  $k$  be a field, and let  $B$  be a Noetherian  $k$ -algebra. Then  $B$  is called *geometrically regular over  $k$*  if, for every finite field extension  $k \subseteq L$ , the ring  $B \otimes_k L$  is regular.

**Definition 2.2.** A local ring  $A$  is *excellent* if it satisfies the following conditions:

- (1)  $A$  is universally catenary.
- (2) The formal fibers of  $A$  are geometrically regular; that is, for every  $\mathfrak{p} \in \text{Spec } A$ , the ring  $\hat{A} \otimes_A L$  is regular for all finite field extensions  $L$  of  $\kappa(\mathfrak{p})$ .

Theorem 31.7 in [Matsumura 1986] states that, if  $A$  is a local ring, then it is formally catenary if and only if it is universally catenary. In addition, if  $A$  is a domain, formally equidimensional and universally catenary are equivalent, so a domain  $A$  is universally catenary if and only if its completion  $T$  is equidimensional by Theorem 31.6 in [Matsumura 1986]. Therefore, in order for a ring  $T$  to be the completion of a universally catenary domain,  $T$  must be equidimensional.

To show that the formal fibers of  $A$  are geometrically regular, it is enough to consider purely inseparable finite field extensions  $L$  of  $\kappa(\mathfrak{p})$  for every  $\mathfrak{p} \in \text{Spec } A$ . If  $A$  contains the rationals, then  $\kappa(\mathfrak{p})$  is a field of characteristic zero for every  $\mathfrak{p} \in \text{Spec } A$ , so the only purely inseparable finite field extension of  $\kappa(\mathfrak{p})$  is  $\kappa(\mathfrak{p})$  itself. Thus, if  $A$  contains the rationals, we need show only that the ring  $\hat{A} \otimes_A \kappa(\mathfrak{p})$  is geometrically regular for every  $\mathfrak{p} \in \text{Spec } A$ .

Furthermore, if  $\hat{A}$  contains the rationals, it is sufficient to show that, for every  $\mathfrak{p} \in \text{Spec } A$ , if  $\mathfrak{q} \in \text{Spec } \hat{A}$  and  $\mathfrak{q} \cap A = \mathfrak{p}$ , then  $(\hat{A}/\mathfrak{p}\hat{A})_{\mathfrak{q}}$  is a regular local ring. This is a much easier definition to work with than the tensor product definition, and it is the definition we use since our  $\hat{A}$  contains the rationals.

We now establish several necessary conditions for a ring  $T$  to be the completion of an excellent local UFD  $A$ . For the rest of this section, let  $A$  denote an excellent local UFD. We begin with:

**Theorem 2.3** [Heitmann 1993, Theorem 1]. *Let  $R$  be an integrally closed local domain. Then no element of the prime subring of  $R$  is a zero divisor in  $\widehat{R}$ . Moreover,  $\widehat{R}$  is either a field, a discrete valuation ring, or a ring with depth at least 2.*

In order for a complete local ring  $T$  to be the completion of an excellent local UFD,  $T$ , of course, must be the completion of a UFD. Note that unique factorization domains are integrally closed. Thus, if  $T$  has Krull dimension at least 2, then, by Theorem 2.3, it must have depth at least 2. In addition,  $T$  must satisfy the condition that no element of the prime subring of  $T$  is a zero divisor.

Since we know that the completion map is faithfully flat and regular if  $A$  is excellent,  $A$  is normal if and only if  $\widehat{A}$  is normal. Since  $A$  is a UFD, it is normal. Thus, if  $T$  is the completion of an excellent local UFD, it must be normal.

Furthermore, we have already established that a domain is universally catenary if and only if it is formally equidimensional; thus if  $A$  is an excellent domain,  $\widehat{A}$  must be equidimensional. So for  $T$  to be the completion of an excellent local UFD,  $T$  must be equidimensional.

To summarize, suppose  $T$  has Krull dimension at least 2 and is the completion of an excellent local UFD. Then the following statements must be true:

- (1) No element of the prime subring of  $T$  is a zero divisor in  $T$  and  $\text{depth } T > 1$ .
- (2)  $T$  is normal.
- (3)  $T$  is equidimensional.

Because of these necessary conditions, we often assume that  $T$  satisfies the above three conditions.

Given such a complete local ring  $T$ , we now discuss how we construct an “almost excellent” local UFD  $A$  whose completion is  $T$ . Our general construction is as follows: we start with a localization of the prime subring of  $T$  and then build up intermediate subrings, adjoining generators of prime ideals in  $\mathcal{S}$ , the set of principal height-1 prime ideals of  $T$ , along the way. Here we present, without proof, several lemmas we use throughout our construction.

First, we must show that the completion of the subring we construct is in fact  $T$ . In order to show that the constructed subring is Noetherian and has completion  $T$ , we use the following lemma.

**Lemma 2.4** [Heitmann 1994, Proposition 1]. *Let  $(R, \mathfrak{m} \cap R)$  be a quasilocal subring of a complete local ring  $(T, \mathfrak{m})$ . If  $R \rightarrow T/\mathfrak{m}^2$  is surjective, and if  $\mathfrak{a}T \cap R = \mathfrak{a}$  for every finitely generated ideal  $\mathfrak{a}$  of  $R$ , then  $R$  is Noetherian and the natural homomorphism  $\widehat{R} \rightarrow T$  is an isomorphism.*

Lemma 2.5 yields restrictions on the cardinality of the constructed subrings.

**Lemma 2.5** [Charters and Loepf 2004, Lemma 2.2]. *Let  $(T, \mathfrak{m})$  be a complete local ring of dimension at least 1. Let  $\mathfrak{p}$  be a nonmaximal prime ideal of  $T$ . Then  $|T/\mathfrak{p}| = |T| \geq 2^{\aleph_0}$ .*

Throughout the construction, we adjoin infinitely many elements of  $T$  to intermediate subrings while preserving cardinality properties and properties of certain prime ideals. (We elaborate on these properties in the following section.) Lemmas 2.6 and 2.7 from [Heitmann 1993] allow us to do so by avoiding certain cosets of certain prime ideals. Lemma 2.6 deals with the situation where the number of prime ideals to avoid is countable, whereas Lemma 2.7 is for avoiding uncountably many prime ideals.

**Lemma 2.6** [Heitmann 1993, Lemma 2]. *Let  $(T, \mathfrak{m})$  be a complete local ring and let  $D$  be a subset of  $T$ . Suppose  $C$  is a subset of  $T$  such that  $\mathfrak{m} \notin C$ , and suppose  $\mathfrak{a}$  is an ideal of  $T$  such that  $\mathfrak{a} \not\subseteq \mathfrak{p}$  for all  $\mathfrak{p} \in C$ . If  $C$  and  $D$  are countable, then*

$$\mathfrak{a} \not\subseteq \bigcup \{t + \mathfrak{p} \mid t \in D, \mathfrak{p} \in C\},$$

where  $t + \mathfrak{p}$  denotes the coset of  $\mathfrak{p}$  in  $T$ ; that is,  $t + \mathfrak{p} = \{t + x \mid x \in \mathfrak{p}\}$ .

**Lemma 2.7** [Heitmann 1993, Lemma 3]. *Let  $(T, \mathfrak{m})$  be a local ring and let  $D$  be a subset of  $T$ . Suppose  $C$  is a subset of  $T$  and  $\mathfrak{a}$  is an ideal of  $T$  such that  $\mathfrak{a} \not\subseteq \mathfrak{p}$  for all  $\mathfrak{p} \in C$ . If  $|C \times D| < |T/\mathfrak{m}|$ , then*

$$\mathfrak{a} \not\subseteq \bigcup \{t + \mathfrak{p} \mid t \in D, \mathfrak{p} \in C\}.$$

The following lemma from [Loepp 1997] allows us to adjoin generators of principal prime ideals of  $T$  to intermediate subrings. (We require only a weaker version of this statement.)

**Lemma 2.8** [Loepp 1997, Lemma 4]. *Suppose  $(T, \mathfrak{m})$  is a local ring with infinite residue field. Let  $C_1$  and  $C_2$  be subsets of  $T$  and let  $u, w \in T$  such that  $u \notin \mathfrak{p}$  for every  $\mathfrak{p} \in C_1$  and  $w \notin \mathfrak{q}$  for every  $\mathfrak{q} \in C_2$ . Also, suppose  $D_1$  and  $D_2$  are subsets of  $T$ . If  $|C_1 \times D_1| < |T/\mathfrak{m}|$  and  $|C_2 \times D_2| < |T/\mathfrak{m}|$ , then there exists a unit  $x \in T$  such that*

$$ux \notin \bigcup \{t + \mathfrak{p} \mid t \in D_1, \mathfrak{p} \in C_1\} \quad \text{and} \quad wx^{-1} \notin \bigcup \{t + \mathfrak{p} \mid t \in D_2, \mathfrak{p} \in C_2\}.$$

### 3. The construction

We follow closely the construction of a UFD in [Heitmann 1993], so several lemmas and definitions are taken directly from that work.

**Definition 3.1.** Let  $(T, \mathfrak{m})$  be a complete local ring, and let  $(R, R \cap \mathfrak{m})$  be a quasilo-cal unique factorization domain contained in  $T$  satisfying the following:

- (i)  $|R| \leq \sup(\aleph_0, |T/\mathfrak{m}|)$  with equality only if  $T/\mathfrak{m}$  is countable.
- (ii)  $\mathfrak{q} \cap R = (0)$  for all  $\mathfrak{q} \in \text{Ass } T$ .
- (iii) If  $t \in T$  is regular and  $\mathfrak{q} \in \text{Ass}(T/tT)$ , then  $\text{ht}(\mathfrak{q} \cap R) \leq 1$ .

Then  $R$  is called an  $N$ -subring of  $T$ .

We construct “good” ring extensions of  $N$ -subrings inside  $T$  by adjoining elements, using the techniques and terminology from [Heitmann 1993], so that the eventual result is a UFD.

**Definition 3.2.** If  $R$  and  $S$  are  $N$ -subrings of  $T$  with  $R \subseteq S$ , we say that  $S$  is an  $A$ -extension of  $R$  if

- (i) prime elements of  $R$  are prime in  $S$ , and
- (ii)  $|S| \leq \sup(\aleph_0, |R|)$ .

We now mimic Heitmann’s construction of a UFD that has completion  $T$ , adding on additional steps that guarantee that our UFD satisfies the additional conditions (1)–(3) of Question 1.2. We start with a lemma from [Heitmann 1993]. Since several of our later proofs will be based on the proof of Lemma 3.3, we include a detailed proof of the lemma here.

**Lemma 3.3** [Heitmann 1993, Lemma 5]. *Let  $(T, \mathfrak{m})$  be a complete local ring with depth  $T > 1$ , let  $R$  be an  $N$ -subring of  $T$ , and let  $t \in T$ . Then there exists an infinite  $A$ -extension  $S$  of  $R$  such that  $t + \mathfrak{m}^2 \in \text{Image}(S \rightarrow T/\mathfrak{m}^2)$ .*

*Proof.* Note that this proof is taken almost directly from the proof of Lemma 3.5 in [Fleming et al. 2019]. Let

$$C = \{\mathfrak{p} \in \text{Spec } T \mid \mathfrak{p} \in \text{Ass}(T/rT) \text{ with } 0 \neq r \in R\} \cup \text{Ass } T.$$

Since depth  $T > 1$ , we have  $\mathfrak{m} \notin C$ , so  $\mathfrak{m}^2 \not\subseteq \mathfrak{p}$  for every  $\mathfrak{p} \in C$ . For every  $\mathfrak{p} \in C$ , define a subset  $D_{(\mathfrak{p})}$  of  $T$ :  $D_{(\mathfrak{p})}$  is to be a full set of distinct coset representatives  $u$  in  $T$  for those cosets  $u + \mathfrak{p}$  in  $T/\mathfrak{p}$  such that  $(u + t) + \mathfrak{p} \in T/\mathfrak{p}$  is algebraic over  $R/(\mathfrak{p} \cap R)$ . (Note that the map  $R/(\mathfrak{p} \cap R) \rightarrow T/\mathfrak{p}$  is an injection, so this language makes sense.) Set  $D = \bigcup_{\mathfrak{p} \in C} D_{(\mathfrak{p})}$ . If  $R$  is countable, then  $C$  is countable, and therefore  $D$  is also countable. Otherwise,  $|R| < |T/\mathfrak{m}|$  and so  $|C \times D| < |T/\mathfrak{m}|$ . By Lemma 2.6 if  $R$  is countable and by Lemma 2.7 otherwise, choose  $x \in \mathfrak{m}^2$  such that  $(x + t) + \mathfrak{p}$  is transcendental over  $R/(\mathfrak{p} \cap R)$  for all  $\mathfrak{p} \in C$ . Define  $R' = R[x + t]_{\mathfrak{m} \cap R[x + t]}$ . We claim that  $R'$  is an  $N$ -subring. In fact,  $R'$  is an  $A$ -extension of  $R$ .

Suppose  $\mathfrak{p} \in C$ . We show that  $\mathfrak{p} \cap R' = (\mathfrak{p} \cap R)R'$ . Since  $R' = R[x + t]_{\mathfrak{m} \cap R[x + t]}$ , elements of  $\mathfrak{p} \cap R'$  are of the form  $uf$ , where  $u \in R'^{\times}$  is a unit and  $f \in R[x + t]$ . Since  $u$  is a unit, this gives us that  $f \in \mathfrak{p}$ . We can treat  $f$  as a polynomial in  $x + t$  over  $R$ . Because  $x + t + \mathfrak{p}$  is transcendental over  $R/(\mathfrak{p} \cap R)$ , each of the coefficients of  $f$  must be in  $\mathfrak{p} \cap R$ . Then  $f \in (\mathfrak{p} \cap R)R[x + t]$  and  $uf \in (\mathfrak{p} \cap R)R'$ . Therefore,  $\mathfrak{p} \cap R' \subseteq (\mathfrak{p} \cap R)R'$ . The opposite containment is clear; thus, we have equality.

Since  $R$  is a UFD and  $x + t$  is transcendental over  $R$ , it follows that  $R[x + t]$  is a UFD. Any localization of a UFD is a UFD, so  $R'$  is also a UFD. In creating  $R'$ , we are simply adjoining a transcendental element and localizing; thus,  $|R'| = \sup(\aleph_0, |R|)$  and  $R'$  satisfies condition (i) of Definition 3.1. Suppose  $\mathfrak{q} \in \text{Ass } T$ . Then, since  $R$

is an  $N$ -subring, we have  $\mathfrak{q} \cap R' = (\mathfrak{q} \cap R)R' = (0)$ , so  $R'$  satisfies condition (ii) of Definition 3.1.

Let  $\mathfrak{p} \in \text{Ass}(T/rT)$  for some regular  $r \in R$ . First suppose  $\mathfrak{p} \cap R = (0)$ . Then, in  $R'_{\mathfrak{p} \cap R'} = R[x+t]_{\mathfrak{p} \cap R[x+t]}$ , all nonzero elements of  $R$  are units, so  $R'_{\mathfrak{p} \cap R'}$  is isomorphic to  $k[X]$  with additional elements inverted where  $k$  is a field and  $X$  is an indeterminate. Therefore,  $\dim R'_{\mathfrak{p} \cap R'} \leq 1$  and we have  $\text{ht}(\mathfrak{p} \cap R') \leq 1$ .

Now suppose  $\mathfrak{p} \cap R = aR$  for some nonzero  $a \in R$ . By Theorem 6.2 of [Matsumura 1986], we know that  $\mathfrak{p} \in \text{Ass}(T/rT)$  if and only if  $\mathfrak{p}T_{\mathfrak{p}} \in \text{Ass}(T_{\mathfrak{p}}/rT_{\mathfrak{p}})$ . Thus  $\mathfrak{p}T_{\mathfrak{p}}$  consists of zero divisors of  $T_{\mathfrak{p}}/rT_{\mathfrak{p}}$ , and  $T_{\mathfrak{p}}/rT_{\mathfrak{p}}$  consists only of zero divisors and units. Therefore,  $T_{\mathfrak{p}}/rT_{\mathfrak{p}}$  has depth zero, and since  $a \in R$  is regular and in  $\mathfrak{p}$ , we know  $T_{\mathfrak{p}}/aT_{\mathfrak{p}}$  must also have depth zero. Then  $\mathfrak{p}T_{\mathfrak{p}}$  consists only of zero divisors of  $T_{\mathfrak{p}}/aT_{\mathfrak{p}}$ , so  $\mathfrak{p}T_{\mathfrak{p}} \in \text{Ass}(T_{\mathfrak{p}}/aT_{\mathfrak{p}})$ . Thus  $\mathfrak{p} \in \text{Ass}(T/aT)$  and  $\mathfrak{p} \in C$ , so  $\mathfrak{p} \cap R' = (\mathfrak{p} \cap R)R' = aR'$ .

We now show that any principal prime ideal of  $R'$  must have height at most 1. Suppose that there exists some prime ideal  $\mathfrak{q}$  of  $R'$  such that  $(0) \subseteq \mathfrak{q} \subsetneq aR'$ . We show that  $\mathfrak{q} = (0)$ . Let  $y \in \mathfrak{q}$ . Then  $y = ar_1$  for some  $r_1 \in R'$ . Since  $\mathfrak{q} \subsetneq aR'$ , we must have  $a \notin \mathfrak{q}$ . Thus  $r_1 \in \mathfrak{q} \subsetneq aR'$ , so we can write  $r_1$  as  $r_1 = ar_2$  for some  $r_2 \in R'$ . Again, since  $a \notin \mathfrak{q}$  but  $r_1 \in \mathfrak{q}$ , we must have  $r_2 \in \mathfrak{q} \subsetneq aR'$ , so we can write  $r_2$  as  $r_2 = ar_3$ . This gives us that  $y = ar_1 = a^2r_2 = a^3r_3$ . Continue this process. Looking in  $T$ , we have  $y \in \bigcap_{i=1}^{\infty} a^i T$ ; however, since  $T$  is Noetherian, we know that  $\bigcap_{i=1}^{\infty} a^i T = (0)$ . Therefore,  $y = 0$  and  $\mathfrak{q} = (0)$ , so the only prime ideal that  $aR'$  contains is the zero ideal. Thus  $\text{ht}(aR') \leq 1$ .

We have thus shown that  $R'$  is an  $N$ -subring; to show it is an  $A$ -extension of  $R$ , we must simply demonstrate that the cardinality condition is satisfied and that prime elements of  $R$  remain prime in  $R'$ . The cardinality condition is satisfied since  $|R'| = \sup(\aleph_0, |R|)$ . Furthermore, since  $x+t$  is transcendental over  $R$ , prime elements of  $R$  remain prime in  $R'$ , and  $S = R'$  is our desired  $A$ -extension.  $\square$

We have shown that, for an element  $t \in T$ , there exists an  $A$ -extension  $S$  of an  $N$ -subring  $R$  such that  $t + \mathfrak{m}^2$  is in the image of the map from  $S$  to  $T/\mathfrak{m}^2$ . In our construction, we apply this lemma infinitely often in order to make the map from our final subring  $A$  to  $T/\mathfrak{m}^2$  a surjection.

For the stronger case of our theorem, we make the formal fibers geometrically regular at prime ideals  $\mathfrak{q}$  of  $T$  of height greater than or equal to 2. To do so, we make the map from  $A$  to  $T/\mathfrak{q}$  a surjection, and we use a similar lemma with the change that we consider  $\mathfrak{q}$  rather than  $\mathfrak{m}^2$ .

**Lemma 3.4.** *Let  $(T, \mathfrak{m})$  be a complete local normal domain, let  $R$  be an  $N$ -subring of  $T$ , and let  $t \in T$ . Suppose  $\mathfrak{q}$  is an ideal of  $T$  such that  $\text{ht } \mathfrak{q} \geq 2$ . Then there exists an infinite  $A$ -extension  $S$  of  $R$  such that  $t + \mathfrak{q} \in \text{Image}(S \rightarrow T/\mathfrak{q})$ . Moreover, if  $t \in \mathfrak{q}$ , we have  $\mathfrak{q} \cap S \neq (0)$ .*

*Proof.* The proof of this lemma is similar to the proof of the previous lemma. Let

$$C = \{\mathfrak{p} \in \text{Spec } T \mid \mathfrak{p} \in \text{Ass}(T/rT) \text{ with } 0 \neq r \in R\} \cup \text{Ass } T,$$

and for all  $\mathfrak{p} \in C$ , let  $D_{(\mathfrak{p})}$  be a set of coset representatives of the cosets  $u + \mathfrak{p} \in T/\mathfrak{p}$  that make  $u + t + \mathfrak{p}$  algebraic over  $R/(\mathfrak{p} \cap R)$ . Let  $D = \bigcup_{\mathfrak{p} \in C} D_{(\mathfrak{p})}$ .

We note that, since  $T$  is a domain, the only associated prime ideal of  $T$  is the zero ideal. Thus  $\mathfrak{q} \notin \text{Ass } T$ . Since  $T$  is normal, it satisfies Serre's  $(S_2)$  condition, and furthermore,  $T$  is a normal domain, so  $(S_2)$  is equivalent to every prime divisor of a nonzero principal ideal having height 1. Therefore, all elements of  $\text{Ass}(T/rT)$  for all  $0 \neq r \in R$  have height 1. Since  $\mathfrak{q}$  has height strictly greater than 1,  $\mathfrak{q}$  cannot be contained in any element of  $C$ . We thus apply [Lemma 2.6](#) if  $R$  is countable and [Lemma 2.7](#) otherwise to choose  $x \in \mathfrak{q}$  such that  $x + t + \mathfrak{p}$  is transcendental over  $R/(\mathfrak{p} \cap R)$  for all  $\mathfrak{p} \in C$ . Define  $S = R[x + t]_{\mathfrak{m} \cap R[x+t]}$ . Using the same proof as in the previous lemma,  $S$  is an  $A$ -extension of  $R$ .

We now must show that, if  $t \in \mathfrak{q}$ , we have  $\mathfrak{q} \cap S \neq (0)$ . Suppose  $t \in \mathfrak{q}$ . Then  $x + t \in S$  for some  $x \in \mathfrak{q}$  that makes  $x + t + \mathfrak{p}$  transcendental over  $R/(\mathfrak{p} \cap R)$  for all  $\mathfrak{p} \in C$ . Since  $x + t + \mathfrak{p}$  is transcendental,  $x + t \neq 0$ . Therefore,  $x + t$  is a nonzero element of  $\mathfrak{q}$ , so  $0 \neq x + t \in \mathfrak{q} \cap S$  and  $\mathfrak{q} \cap S \neq (0)$ .  $\square$

Given a complete local domain  $T$ , we want to adjoin the generators of infinitely many principal height-1 prime ideals of  $T$  to an  $N$ -subring  $R$  of  $T$ . To do so, we suppose that a prime ideal  $pT$  of  $T$  has zero intersection with an  $N$ -subring  $R$  and show that there exists an  $A$ -extension of  $R$  that contains a generator of  $pT$ .

**Lemma 3.5.** *Let  $(T, \mathfrak{m})$  be a complete local domain of dimension at least 1 such that  $|T/\mathfrak{m}| = |T|$ . Let  $R$  be an  $N$ -subring of  $T$ , and let  $p$  be a nonzero prime element of  $T$  such that  $pT \cap R = (0)$ . Then there exists an  $A$ -extension  $S$  of  $R$  such that  $pT \cap S = puS$  for some unit  $u \in T$ .*

*Proof.* Let

$$C = \{\mathfrak{p} \in \text{Spec } T \mid \mathfrak{p} \in \text{Ass}(T/rT) \text{ with } 0 \neq r \in R\} \cup \text{Ass } T.$$

For this proof, as in the proof of [Lemma 3.3](#), each  $\mathfrak{p} \in C$  corresponds to a subset  $D_{(\mathfrak{p})}$  of  $T$ . Here  $D_{(\mathfrak{p})}$  is a full set of distinct coset representatives  $u$  in  $T$  for those cosets  $u + \mathfrak{p}$  such that  $(pu) + \mathfrak{p} \in T/\mathfrak{p}$  is algebraic over  $R/(\mathfrak{p} \cap R)$ . Define  $D = \bigcup_{\mathfrak{p} \in C} D_{(\mathfrak{p})}$ . By [Lemma 2.5](#),  $T$  is uncountable, and so  $T/\mathfrak{m}$  is also uncountable. It follows that  $|C \times D| < |T/\mathfrak{m}|$ . Furthermore, we know that  $pT \notin C$ :  $pT$  cannot be an associated prime ideal of  $T$  because it has height greater than zero, and  $pT$  cannot be an associated prime ideal of  $T/rT$  for some nonzero  $r \in R$  because  $pT \cap R = (0)$ . Therefore, we can use [Lemma 2.8](#) to find a unit  $u \in T$  such that

$$pu \notin \bigcup \{t + \mathfrak{p} \mid t \in D, \mathfrak{p} \in C\}.$$

We claim that  $S = R[pu]_{\mathfrak{m} \cap R[pu]}$  is an  $N$ -subring. Parts of this proof mimic the proof of [Lemma 3.3](#).

Suppose  $\mathfrak{p} \in C$ . As in the proof of [Lemma 3.3](#), elements of  $\mathfrak{p} \cap S$  are of the form  $vf$  where  $v \in S^\times$  is a unit and  $f \in R[pu]$ . Since  $v$  is a unit, this gives us that  $f \in \mathfrak{p}$ . Note that  $f$  is a polynomial in  $pu$  over  $R$ . Because  $pu + \mathfrak{p}$  is transcendental over  $R/(\mathfrak{p} \cap R)$ , each of the coefficients of  $f$  must be in  $\mathfrak{p} \cap R$ . Then  $f \in (\mathfrak{p} \cap R)R[pu]$  and  $vf \in (\mathfrak{p} \cap R)S$ . Therefore,  $\mathfrak{p} \cap S \subseteq (\mathfrak{p} \cap R)S$ . The opposite containment is clear; thus, we have equality.

We now show that  $S$  is an  $N$ -subring. Since  $R$  is a UFD and  $pu$  is transcendental over  $R$ , we have that  $R[pu]$  is a UFD; furthermore, any localization of a UFD is a UFD, so  $S$  is also a UFD. The ring  $S$  satisfies the cardinality condition by the same argument as in [Lemma 3.3](#): since we are adjoining a transcendental element and localizing, we have  $|S| = \sup(\aleph_0, |R|)$ . Furthermore, if  $\mathfrak{q} \in \text{Ass } T$ , then  $\mathfrak{q} \in C$  so we have  $\mathfrak{q} \cap S = (\mathfrak{q} \cap R)S = (0)$ .

It remains to show that, for all regular  $r \in R$  we have that  $\mathfrak{p} \in \text{Ass}(T/rT)$  satisfies  $\text{ht}(\mathfrak{p} \cap S) \leq 1$ . This part of the proof follows exactly the proof of [Lemma 3.3](#). Thus  $S$  is an  $N$ -subring. Furthermore, since we are adjoining a single transcendental element, prime elements of  $R$  remain prime in  $S$ . Thus  $S$  is in fact an  $A$ -extension of  $R$ .

Clearly,  $puS \subseteq pT \cap S$ . Suppose  $f \in pT \cap R[pu]$ . Then

$$f = r_n(pu)^n + \cdots + r_1(pu) + r_0$$

for some  $r_i \in R$ . Since  $f \in pT$ , we have  $r_0 \in pT$  and so  $r_0 \in pT \cap R = (0)$ . It follows that  $f \in (pu)R[pu]$  and we have  $pT \cap R[pu] \subseteq (pu)R[pu]$ . Therefore,  $pT \cap S \subseteq (pu)S$  and we have  $pT \cap S = puS$  as desired.  $\square$

In order to ensure that our final subring  $A$  is Noetherian and has completion  $T$ , we must close up all finitely generated ideals. In other words, if  $\mathfrak{a}$  is a finitely generated ideal of  $A$ , we must show that  $\mathfrak{a}T \cap A = \mathfrak{a}$ . This will allow us to use [Lemma 2.4](#). The next two lemmas will help us close up finitely generated ideals.

**Lemma 3.6** [[Heitmann 1993](#), Lemma 4]. *Let  $(T, \mathfrak{m})$  be a complete local ring, and let  $R$  be an  $N$ -subring of  $T$ . Suppose  $c \in R$ , and let  $\mathfrak{a}$  be a finitely generated ideal of  $R$  with  $c \in \mathfrak{a}T$ . Then there exists an  $A$ -extension  $S$  of  $R$  such that  $c \in \mathfrak{a}S$ .*

Once we have our chain of intermediate subrings, we take their union to produce our final subring  $A$ . The following lemma helps us control this union.

**Lemma 3.7** [[Heitmann 1993](#), Lemma 6]. *Let  $(T, \mathfrak{m})$  be a complete local ring and let  $R_0$  be an  $N$ -subring of  $T$ . Let  $\Omega$  be a well-ordered set with least element 0 and assume either  $\Omega$  is countable or, for all  $\alpha \in \Omega$ , we have  $|\{\beta \in \Omega \mid \beta < \alpha\}| < |T/\mathfrak{m}|$ . Suppose  $\{R_\alpha \mid \alpha \in \Omega\}$  is an ascending collection of rings such that, if  $\alpha$  is a limit*

ordinal, then  $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$ , while if  $\alpha = \beta + 1$  is a successor ordinal, then  $R_\alpha$  is an  $A$ -extension of  $R_\beta$ .

Then  $S = \bigcup_{\beta \in \Omega} R_\beta$  satisfies all the conditions to be an  $N$ -subring of  $T$  except perhaps the cardinality condition. Instead,  $|S| \leq \sup(\aleph_0, |R_0|, |\Omega|)$ . Furthermore, elements which are prime in some  $R_\beta$  remain prime in  $S$ .

Now we prove that, given an  $N$ -subring  $R$  of  $T$ , there exists an  $A$ -extension  $S$  of  $R$  that satisfies all of the conditions we need our final subring  $A$  to satisfy.

**Lemma 3.8.** *Let  $(T, \mathfrak{m})$  be a complete local domain with  $\text{depth } T > 1$  such that  $|T/\mathfrak{m}| = |T|$ , and let  $R$  be an  $N$ -subring of  $T$ . Let  $p$  be a nonzero prime element of  $T$  such that  $pT \cap R = (0)$ , and let  $t + \mathfrak{m}^2 \in T/\mathfrak{m}^2$ . Then there exists an infinite  $A$ -extension  $S$  of  $R$  such that  $t + \mathfrak{m}^2 \in \text{Image}(S \rightarrow T/\mathfrak{m}^2)$ ,  $pT \cap S = puS$  for some unit  $u \in T$ , and, for every finitely generated ideal  $\mathfrak{a}$  of  $S$ , we have  $\mathfrak{a}T \cap S = \mathfrak{a}$ .*

*Proof.* First, use [Lemma 3.5](#) to obtain an  $A$ -extension  $R'$  of  $R$  such that  $pT \cap R' = puR'$  for some unit  $u \in T$ . Then use [Lemma 3.3](#) to obtain an infinite  $A$ -extension  $R_0$  of  $R'$  such that  $t + \mathfrak{m}^2 \in \text{Image}(R_0 \rightarrow T/\mathfrak{m}^2)$ . We must now prove the statement about finitely generated ideals. Let

$$\Omega = \{(\mathfrak{a}, c) \mid \mathfrak{a} \text{ is a finitely generated ideal of } R_0 \text{ and } c \in \mathfrak{a}T \cap R_0\}.$$

Then  $|\Omega| = |R_0|$ , so since  $|T/\mathfrak{m}| = |T|$  and  $T$  is uncountable, we have  $|\Omega| < |T/\mathfrak{m}|$ . Let  $0$  designate the initial element of  $\Omega$ , and well-order  $\Omega$  in a such a way that it does not have a maximal element. Then  $\Omega$  satisfies the hypotheses of [Lemma 3.7](#). We define an increasing chain of intermediate subrings recursively with one subring for each element of  $\Omega$ . We begin with  $R_0$ . If  $\alpha = \beta + 1$  is a successor ordinal and  $\beta = (\mathfrak{a}, c)$  then we choose  $R_\alpha$  to be an  $A$ -extension of  $R_\beta$  given by [Lemma 3.6](#) such that  $c \in \mathfrak{a}R_\alpha$ . If  $\alpha$  is a limit ordinal, define  $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$  and set  $S_0 = \bigcup_{\alpha \in \Omega} R_\alpha$ . Then by [Lemma 3.7](#),  $S_0$  is an  $A$ -extension of  $R_0$  and if  $\mathfrak{a}$  is a finitely generated ideal of  $R_0$  with  $c \in \mathfrak{a}T \cap R_0$ , then  $(\mathfrak{a}, c) = \beta$  for some  $\beta \in \Omega$ . Then for some  $\alpha > \beta$ , we have  $c \in \mathfrak{a}R_\alpha \subseteq \mathfrak{a}S_0$ . Thus  $\mathfrak{a}T \cap R_0 \subseteq \mathfrak{a}S_0$ .

We repeat this process to obtain an  $A$ -extension  $S_1$  of  $S_0$  such that  $\mathfrak{a}T \cap S_0 \subseteq \mathfrak{a}S_1$  for every finitely generated ideal  $\mathfrak{a}$  of  $S_0$ . We continue recursively to obtain an ascending chain  $S_0 \subseteq S_1 \subseteq \dots$  such that  $\mathfrak{a}T \cap S_n \subseteq \mathfrak{a}S_{n+1}$  for every finitely generated ideal  $\mathfrak{a}$  of  $S_n$ . Let  $S = \bigcup S_i$ . By [Lemma 3.7](#),  $S$  is an  $A$ -extension of  $S_0$  (hence also of  $R$ ). Since we have  $t + \mathfrak{m}^2 \in \text{Image}(R_0 \rightarrow T/\mathfrak{m}^2)$  and  $R_0 \subset S$ , we must have  $t + \mathfrak{m}^2 \in \text{Image}(S \rightarrow T/\mathfrak{m}^2)$ . If  $\mathfrak{a}$  is a finitely generated ideal of  $S$ , then there must exist some  $S_n$  that contains a generating set  $\{a_1, \dots, a_k\}$  for  $\mathfrak{a}$ . If  $c \in \mathfrak{a}T \cap S$ , then there exists some  $m \geq n$  such that  $c \in S_m$ . Then

$$c \in (a_1, \dots, a_k)T \cap S_m \subseteq (a_1, \dots, a_k)S_{m+1} \subseteq \mathfrak{a}.$$

Thus  $\mathfrak{a}T \cap S \subseteq \mathfrak{a}$ . Since the reverse containment is clear, we have equality, and  $\mathfrak{a}T \cap S = \mathfrak{a}$ . Furthermore,  $puS$  is a finitely generated ideal of  $S$ , so by what we have just shown, we must have  $(puS)T \cap S = puS$ . But we know that  $(puS)T = puT = pT$  since  $u$  is a unit in  $T$ . Thus  $pT \cap S = puS$ , and  $S$  satisfies all the properties in the statement of the theorem.  $\square$

We now present the analogous lemma for the stronger case of our theorem.

**Lemma 3.9.** *Let  $(T, \mathfrak{m})$  be a complete normal local domain with  $\text{depth } T > 1$  such that  $|T/\mathfrak{m}| = |T|$ , and let  $R$  be an  $N$ -subring of  $T$ . Let  $p$  be a nonzero prime element of  $T$  such that  $pT \cap R = (0)$ , and let  $t + \mathfrak{q} \in T/\mathfrak{q}$  for some ideal  $\mathfrak{q}$  of  $T$  of height at least 2. Then there exists an infinite  $A$ -extension  $S$  of  $R$  such that*

- (1)  $t + \mathfrak{q} \in \text{Image}(S \rightarrow T/\mathfrak{q})$  with  $\mathfrak{q} \cap S \neq (0)$  if  $t \in \mathfrak{q}$ ,
- (2)  $pT \cap S = puS$  for some unit  $u \in T$ , and,
- (3) for every finitely generated ideal  $\mathfrak{a}$  of  $S$ , we have  $\mathfrak{a}T \cap S = \mathfrak{a}$ .

*Proof.* Follow the proof of [Lemma 3.8](#) replacing the use of [Lemma 3.3](#) with [Lemma 3.4](#).  $\square$

#### 4. The main theorems

We are now ready to prove our main theorems.

**Theorem 4.1.** *Let  $(T, \mathfrak{m})$  be a complete local domain such that  $\text{depth } T > 1$ . Let  $\mathcal{S}$  be the set of principal height-1 prime ideals of  $T$ . Suppose that  $|T| = |T/\mathfrak{m}| = |\mathcal{S}|$  and  $\mathbb{Q} \subseteq T$ . Then there exists a local unique factorization domain  $A$  such that  $\hat{A} \cong T$  and  $A$  satisfies the following conditions:*

- (1)  $A$  is universally catenary.
- (2) There is an uncountable subset  $\mathcal{S}'$  of  $\mathcal{S}$  such that
  - (a) if  $\mathfrak{q} \in \mathcal{S}'$ , then  $\mathfrak{q} \cap A \neq (0)$ ,
  - (b) if  $\mathfrak{q}, \mathfrak{q}' \in \mathcal{S}'$  with  $\mathfrak{q} \neq \mathfrak{q}'$ , then  $\mathfrak{q} \cap A \neq \mathfrak{q}' \cap A$ , and
  - (c) if  $\mathfrak{q} \in \mathcal{S}'$ , then  $(T/(\mathfrak{q} \cap A)T)_{\mathfrak{q}}$  is a field.

*Proof.* Let  $\Omega = T/\mathfrak{m}^2$  be well-ordered so that 0 is its initial element and each element of  $\Omega$  has fewer than  $|\Omega|$  predecessors. Then  $\Omega$  satisfies the hypotheses of [Lemma 3.7](#) since  $|T| \geq |T/\mathfrak{m}^2| \geq |T/\mathfrak{m}|$  implies  $|T/\mathfrak{m}^2| = |T/\mathfrak{m}|$ . We now recursively define a family of rings  $\{R_\alpha \mid \alpha \in \Omega\}$  satisfying the hypotheses of [Lemma 3.7](#).

Let  $R_0$  be  $\mathbb{Q}$ , and note that  $R_0$  is an  $N$ -subring of  $T$ .

Whenever  $\alpha = \beta + 1$  is a successor ordinal, use [Lemma 3.8](#) to construct an  $A$ -extension  $R_\alpha$  of  $R_\beta$  such that  $\beta = t + \mathfrak{m}^2 \in \text{Image}(R_\alpha \rightarrow T/\mathfrak{m}^2)$ ,  $pT \cap R_\alpha = puR_\alpha$  for some unit  $u \in T$  and some nonzero prime element  $p$  of  $T$ , and, for every

finitely generated ideal  $\mathfrak{a}$  of  $R_\alpha$ , we have  $\mathfrak{a}T \cap R_\alpha = \mathfrak{a}$ . In order to do so, we must first justify why, at each step, there exists some nonzero prime element  $p$  of  $T$  such that  $pT \cap R_\beta = (0)$ . Since  $|T| = |T/\mathfrak{m}|$ , we have  $|T/\mathfrak{m}| = |T/\mathfrak{m}^2|$ . Thus  $|R_\beta| < |T/\mathfrak{m}^2| = |T/\mathfrak{m}| = |\mathcal{S}|$ . Since each element of  $R_\beta$  can be contained in only finitely many height-1 prime ideals of  $T$ , there are at most  $|R_\beta|$  height-1 prime ideals of  $T$  having nonzero intersection with  $R_\beta$ . Furthermore, we know that  $|R_\beta| \geq \aleph_0$  because  $\mathbb{Q} \subset R_\beta$ . Therefore, there are at most  $\max(\aleph_0, |R_\beta|) = |R_\beta|$  elements of  $\mathcal{S}$  having nonzero intersection with  $R_\beta$ . Since  $|\mathcal{S}| > |R_\beta|$ , there must exist at least one nonzero element of  $\mathcal{S}$  whose intersection with  $R_\beta$  is  $(0)$ . Thus, at every step, there is a nonzero prime element  $p$  of  $T$  with the property that  $pT \cap R_\beta = (0)$ , and so the hypotheses of [Lemma 3.8](#) are satisfied. Therefore, we can in fact use [Lemma 3.8](#) to construct an  $A$ -extension  $R_\alpha$  of  $R_\beta$  with our desired properties.

If  $\alpha$  is a limit ordinal, let  $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$ . We claim  $A = \bigcup_{\alpha \in \Omega} R_\alpha$  is our desired subring.

First, we claim that  $A$  is Noetherian and  $\hat{A} \cong T$ . By construction, the map  $A \rightarrow T/\mathfrak{m}^2$  is surjective. Let  $\mathfrak{a} = (a_1, \dots, a_n)$  be a finitely generated ideal of  $A$ , and let  $x \in \mathfrak{a}T \cap A$ . Then there exists an  $\alpha \in \Omega$  such that  $\alpha$  is a successor ordinal and  $x, a_1, \dots, a_n \in R_\alpha$ . Then

$$x \in (a_1, \dots, a_n)T \cap R_\alpha = (a_1, \dots, a_n)R_\alpha \subseteq \mathfrak{a}.$$

It follows that  $\mathfrak{a}T \cap A = \mathfrak{a}$  for all finitely generated ideals  $\mathfrak{a}$  of  $A$ . By [Lemma 2.4](#), we have that  $A$  is Noetherian and  $\hat{A} \cong T$ .

By [Lemma 3.7](#),  $A$  is a UFD, and since  $T$  is a domain, it is equidimensional, and so  $A$  is universally catenary. Define

$$\mathcal{S}' = \{pT \in \mathcal{S} \mid pT \cap A = puA \text{ for some unit } u \in T\}.$$

Then, by construction,  $\mathcal{S}'$  is uncountable, if  $\mathfrak{q} \in \mathcal{S}'$ , then  $\mathfrak{q} \cap A \neq (0)$ , and if  $\mathfrak{q}, \mathfrak{q}' \in \mathcal{S}'$  with  $\mathfrak{q} \neq \mathfrak{q}'$ , then  $\mathfrak{q} \cap A \neq \mathfrak{q}' \cap A$ . Now we show that, if  $\mathfrak{q} \in \mathcal{S}'$ , then  $(T/(\mathfrak{q} \cap A)T)_{\mathfrak{q}}$  is a field. Suppose  $\mathfrak{q} = pT$  and  $\mathfrak{q} \cap A = puA$  for some unit  $u \in T$ . Then

$$(T/(\mathfrak{q} \cap A)T)_{\mathfrak{q}} \cong (T/puT)_{\mathfrak{q}} = (T/pT)_{\mathfrak{q}} = (T/\mathfrak{q})_{\mathfrak{q}} = T_{\mathfrak{q}}/\mathfrak{q}T_{\mathfrak{q}}$$

is a field. Thus the  $A$  we construct does indeed satisfy the desired properties stated in the theorem. □

**Corollary 4.2.** *Let  $(T, \mathfrak{m})$  be a complete local domain such that  $\dim T = 2$  and  $\text{depth } T = 2$ . Let  $\mathcal{S}$  be the set of principal height-1 prime ideals of  $T$ . Suppose that  $|T| = |T/\mathfrak{m}| = |\mathcal{S}|$  and  $\mathbb{Q} \subseteq T$ . Then there exists a local universally catenary unique factorization domain  $A$  such that  $\hat{A} \cong T$  and  $A$  satisfies conditions (1)–(3) of [Question 1.2](#).*

*Proof.* By [Theorem 4.1](#), there exists a local UFD  $A$  that satisfies conditions (1) and (2) of [Question 1.2](#) and a modified version of condition (3). Thus we need only show that condition (3) holds. Suppose  $\mathfrak{q} \in S'$  with  $\mathfrak{q} = pT$ . Then,  $\mathfrak{q} \cap A = puA$  for some unit  $u$  of  $T$ . Since  $T$  has dimension 2, if  $Q$  is a prime ideal of  $T$  with  $Q \cap A = puA$ , then  $Q$  has height 1 and  $Q = pT = \mathfrak{q}$ . Therefore, in order to show that the formal fiber of  $A$  at  $\mathfrak{q} \cap A$  is geometrically regular, it suffices to show that the ring  $(T/(\mathfrak{q} \cap A)T)_{\mathfrak{q}}$  is a regular local ring. By [Theorem 4.1](#), we know that  $(T/(\mathfrak{q} \cap A)T)_{\mathfrak{q}}$  is a field, and hence a regular local ring. Thus  $A$  satisfies conditions (1)–(3) as desired.  $\square$

We now impose the condition that  $T$  is a normal domain so we can make the generic formal fiber of  $A$  and all formal fibers at prime ideals of height greater than 1 be geometrically regular.

**Theorem 4.3.** *Let  $(T, \mathfrak{m})$  be a normal complete local domain and  $\text{depth } T > 1$ . Let  $S$  be the set of principal height-1 prime ideals of  $T$ . Suppose that  $|\mathcal{S}| = |T| = |T/\mathfrak{m}|$  and  $\mathbb{Q} \subseteq T$ . Then there exists a local unique factorization domain  $A$  such that  $\hat{A} \cong T$  and  $A$  satisfies the following conditions:*

- (1)  $A$  is universally catenary.
- (2) If  $\mathfrak{p} \in \text{Spec } A$  with the height of  $\mathfrak{p}$  not equal to 1, the formal fiber of  $A$  at  $\mathfrak{p}$  is geometrically regular.
- (3) There is an uncountable subset  $S'$  of  $S$  such that
  - (a) if  $\mathfrak{q} \in S'$ , then  $\mathfrak{q} \cap A \neq (0)$ ,
  - (b) if  $\mathfrak{q}, \mathfrak{q}' \in S'$  with  $\mathfrak{q} \neq \mathfrak{q}'$ , then  $\mathfrak{q} \cap A \neq \mathfrak{q}' \cap A$ , and
  - (c) if  $\mathfrak{q} \in S'$ , then the formal fiber of  $A$  at  $\mathfrak{q} \cap A$  is geometrically regular.

*Proof.* Let

$$\Omega = \{t + \mathfrak{q} \in T/\mathfrak{q} \mid \mathfrak{q} \text{ is an ideal of } T \text{ with } \text{ht } \mathfrak{q} \geq 2\}$$

be well-ordered so that 0 is its initial element and each element of  $\Omega$  has fewer than  $|\Omega|$  predecessors. Then  $\Omega$  satisfies the hypotheses of [Lemma 3.7](#). Also, note that the height of  $\mathfrak{m}^2$  is at least 2. We now recursively define a family of rings  $\{R_\alpha \mid \alpha \in \Omega\}$  satisfying the hypotheses of [Lemma 3.7](#).

Let  $R_0$  be  $\mathbb{Q}$ , and note that  $R_0$  is an  $N$ -subring of  $T$ .

Whenever  $\alpha = \beta + 1$  is a successor ordinal, we use [Lemma 3.9](#) to construct an  $A$ -extension  $R_\alpha$  of  $R_\beta$  such that  $\beta = t + \mathfrak{q} \in \text{Image}(R_\alpha \rightarrow T/\mathfrak{q})$  with  $\mathfrak{q} \cap R_\alpha \neq (0)$  if  $t \in \mathfrak{q}$ ,  $pT \cap R_\alpha = puR_\alpha$  for some unit  $u \in T$  and some nonzero prime element  $p$  in  $T$  with  $pT \cap R_\beta = (0)$ , and, for every finitely generated ideal  $\mathfrak{a}$  of  $R_\alpha$ , we have  $\mathfrak{a}T \cap R_\alpha = \mathfrak{a}$ . In order to do so, we must first justify why at each step, there exists a nonzero prime element  $p$  of  $T$  such that  $pT \cap R_\beta = (0)$ . The justification for this is the same argument that is used in the proof of [Theorem 4.1](#).

If  $\alpha$  is a limit ordinal, let  $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$ . We claim  $A = \bigcup_{\alpha \in \Omega} R_\alpha$  is our desired subring.

The proof that  $A$  is Noetherian and  $\hat{A} \cong T$  is the same argument that is used in the proof of [Theorem 4.1](#).

By [Lemma 3.7](#),  $A$  is a UFD. Since  $T$  is a domain, it is equidimensional; hence,  $A$  is formally equidimensional and therefore universally catenary. In addition, because the map  $A \rightarrow T/\mathfrak{q}$  is surjective for all ideals  $\mathfrak{q}$  of  $T$  of height at least 2, we claim that the formal fibers at all prime ideals of  $A$  of height at least 2 are geometrically regular. Let  $\mathfrak{p}$  be a prime ideal of  $A$  with height at least 2. Then, by the going down theorem,  $\mathfrak{p}T$  is an ideal of  $T$  of height at least 2. Because the map  $A \rightarrow T/\mathfrak{q}$  is surjective for all ideals  $\mathfrak{q}$  of  $T$  of height at least 2, we have that the map  $A \rightarrow T/\mathfrak{p}T$  is a surjection; hence by the first isomorphism theorem, we obtain  $A/\mathfrak{p} \cong T/\mathfrak{p}T$ . Since  $\mathfrak{p}$  is a prime ideal of  $A$ , we know  $A/\mathfrak{p}$  is a domain, so  $T/\mathfrak{p}T$  is a domain. Therefore,  $\mathfrak{p}T$  is a prime ideal of  $T$ . In fact, as  $A/\mathfrak{p}$  is complete,  $\mathfrak{p}T$  is the only prime ideal of  $T$  whose intersection with  $A$  is  $\mathfrak{p}$ , so, in order to prove that the formal fiber of  $A$  at  $\mathfrak{p}$  is geometrically regular, we need only prove that the ring  $(T/\mathfrak{p}T)_{\mathfrak{p}T}$  is a regular local ring. Now,  $(T/\mathfrak{p}T)_{\mathfrak{p}T} \cong T_{\mathfrak{p}}/\mathfrak{p}T_{\mathfrak{p}}$  is a field, and hence a regular local ring. This implies that the formal fiber of  $A$  at  $\mathfrak{p}$  is geometrically regular.

We now show that the generic formal fiber of  $A$  is geometrically regular. Since  $\mathbb{Q} \subseteq T$ , it is enough to show that  $(T/(0)T)_{\mathfrak{q}}$  is a regular local ring for all  $\mathfrak{q} \in \text{Spec } T$  such that  $\mathfrak{q} \cap A = (0)$ . This is equivalent to showing that  $T_{\mathfrak{q}}$  is a regular local ring for all  $\mathfrak{q} \in \text{Spec } T$  such that  $\mathfrak{q} \cap A = (0)$ . Recall that a ring  $T$  satisfies Serre's  $(R_1)$  condition if  $T_{\mathfrak{p}}$  is regular for all prime ideals  $\mathfrak{p}$  of  $T$  with height at most 1. By [Theorem 23.8](#) in [\[Matsumura 1986\]](#), since  $T$  is normal,  $T$  satisfies Serre's  $(R_1)$  condition. Thus, for all  $\mathfrak{q} \in \text{Spec } T$  with  $\text{ht } \mathfrak{q} \leq 1$ ,  $T_{\mathfrak{q}}$  is a regular local ring. We claim that this is enough; in other words, we claim that all elements of the generic formal fiber of  $A$  have height at most 1. By our use of [Lemma 3.9](#) in the construction of  $A$ , for all prime ideals  $\mathfrak{q}$  of  $T$  with height at least 2,  $\mathfrak{q} \cap A \neq (0)$ ; thus  $\mathfrak{q}$  cannot be in the generic formal fiber of  $A$ . Therefore, all elements of the generic formal fiber of  $A$  have height at most 1 and the generic formal fiber of  $A$  is geometrically regular.

Define

$$\mathcal{S}' = \{pT \in \mathcal{S} \mid pT \cap A = puA \text{ for some unit } u \in T\}.$$

Then, by construction,  $\mathcal{S}'$  is uncountable, if  $\mathfrak{q} \in \mathcal{S}'$ , then  $\mathfrak{q} \cap A \neq (0)$  and if  $\mathfrak{q}, \mathfrak{q}' \in \mathcal{S}'$  with  $\mathfrak{q} \neq \mathfrak{q}'$ , then  $\mathfrak{q} \cap A \neq \mathfrak{q}' \cap A$ . Let  $\mathfrak{q} = pT \in \mathcal{S}'$ . Then  $\mathfrak{q} \cap A = puA$  for some unit  $u$  in  $T$ . Suppose  $Q$  is a prime ideal of  $T$  such that  $Q \cap A = puA$ . We claim  $Q = pT = \mathfrak{q}$ . If the height of  $Q$  is at least 2, then, by construction,  $A \rightarrow T/Q$  is surjective, and so  $A/puA \cong T/Q$ . Since  $A$  is catenary, the dimension of  $A/puA$  is  $\dim A - 1$ . But the dimension of  $T/Q$  is less than  $\dim T - 1$ , a contradiction. It follows that  $Q$  is a height-1 prime ideal of  $T$ . Hence, as  $pT \subseteq Q$ , we have

$Q = pT = q$ . Therefore, to show that the formal fiber of  $A$  at  $q \cap A$  is geometrically regular, it suffices to show that  $(T/(q \cap A)T)_{\mathfrak{q}}$  is a field. This argument is given in the proof of [Theorem 4.1](#).  $\square$

We present an example of a class of rings that satisfies the hypotheses of [Theorem 4.3](#).

**Example 4.4.** Let  $T = \mathbb{C}[[x_1, x_2, \dots, x_n]]/(x_1x_2 - x_3^2)$ , where  $n \geq 3$ . Then  $T$  is the completion of a local UFD  $A$  satisfying the properties in [Theorem 4.3](#).

*Proof.* We must show that  $T$  satisfies the hypotheses of [Theorem 4.3](#). Note that  $T$  is a complete local ring. Since  $(x_1x_2 - x_3^2)$  is a prime ideal of  $\mathbb{C}[[x_1, x_2, \dots, x_n]]$ ,  $T$  is a domain. One can check that  $T$  is normal and  $\text{depth } T > 1$ . Now,  $|T| = |T/\mathfrak{m}| = |\mathbb{C}|$ , and we must show that  $|T| = |\mathcal{S}|$ . Let  $\mathfrak{q}_\alpha = (x_1 + \alpha x_2)$ . We claim that for all nonzero  $\alpha \in \mathbb{C}$ , the ideal  $\mathfrak{q}_\alpha$  is prime. To prove this, consider the ring

$$T/\mathfrak{q}_\alpha = \mathbb{C}[[x_1, x_2, \dots, x_n]]/(x_1x_2 - x_3^2, \mathfrak{q}_\alpha) = \mathbb{C}[[x_1, x_2, \dots, x_n]]/(-\alpha x_2^3 - x_3^2).$$

Since  $-\alpha x_2^3 - x_3^2$  is an irreducible polynomial in  $\mathbb{C}[[x_1, x_2, \dots, x_n]]$ , the ideal  $(-\alpha x_2^3 - x_3^2)$  is prime in  $\mathbb{C}[[x_1, x_2, \dots, x_n]]$ ; hence  $T/\mathfrak{q}_\alpha$  is a domain. Therefore,  $\mathfrak{q}_\alpha$  is a prime ideal of  $T$ , and one can verify that  $\mathfrak{q}_\alpha = \mathfrak{q}_\beta$  if and only if  $\alpha = \beta$ . We then consider the set of prime ideals  $\mathcal{Q} = \{\mathfrak{q}_\alpha \mid \alpha \in \mathbb{C} \text{ and } \alpha \neq 0\} \subseteq \mathcal{S}$ . We thus have the following string of inequalities:

$$|\mathbb{C}| = |\mathcal{Q}| \leq |\mathcal{S}| \leq |T| = |\mathbb{C}|.$$

Therefore, we obtain that  $|\mathcal{S}| = |T|$ , and so  $|\mathcal{S}| = |T| = |T/\mathfrak{m}|$ . We can thus apply [Theorem 4.3](#) to  $T$ .  $\square$

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[fleming.sarah.m@gmail.com](mailto:fleming.sarah.m@gmail.com)

*Williams College, Williamstown, MA, United States*

[sloepp@williams.edu](mailto:sloepp@williams.edu)

*Department of Mathematics and Statistics, Williams College,  
Williamstown, MA, United States*

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