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Discrete Morse functions, vector fields, and homological sequences on trees

Ian Rand and Nicholas A. Scoville

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We construct a discrete Morse function which induces both a specified gradient vector field and homological sequence on a given tree. After reviewing the basics of discrete Morse theory, we provide an algorithm to construct a discrete Morse function on a tree inducing a desired gradient vector field and homological sequence. We prove that our algorithm is correct, and conclude with an example to illustrate its use.

1. Introduction

Let f and g be two discrete Morse functions defined on a 1-dimensional simplicial complex, i.e., a graph. Inspired by Nicolaescu [2008], R. Ayala et al. [2009a] studied the homological sequence of a discrete Morse function by introducing the notion of f and g being homologically equivalent, and they counted the number of excellent discrete Morse functions on all graphs [Ayala et al. 2011]. This result was used to compute the number of excellent discrete Morse functions on all collapsible 2-dimensional complexes in [Agiorgousis et al. 2019]. Furthermore, R. Ayala et al. [2008] studied gradient vector fields on trees by considering discrete Morse functions up to Forman equivalence. In this paper, we desire to combine these two ideas by investigating the combination of the two notions of equivalence. To this end, let $\mathcal{V}$ be a fixed gradient vector field on a graph with $m > 1$ critical values. Which homological sequences can arise from a discrete Morse function that induces $\mathcal{V}$ as its gradient vector field? Conversely, fix a homological sequence B with $m > 1$ critical values. Which discrete vector fields can arise from a discrete Morse function with B as its homological sequence? In the case that G is a tree, we show in Proposition 3.5 that any feasible gradient vector field and homological sequence with the same number of critical values can be realized by a discrete Morse function. This is via a simple algorithmic construction, the details of which are given in Algorithm 2. In addition, this algorithm provides a new proof of Theorem 6.1 in

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[Ayala et al. 2009a] in the case where G is a tree. The layout of the paper is as follows. Section 2 is devoted to introducing the notation and terminology that will be used, as well as background results. Section 3 is the main section where we give an algorithm that yields a desired homological sequence and gradient vector field on a tree. Finally, we give an example illustrating the utility of our algorithm.

2. Background

We now establish notation and terminology that will be used in the body of this paper. Our main reference for graph theory is [Chartrand 1985], while we refer the reader to [Forman 2002; Scoville 2019] for the basics of discrete Morse theory.

Graphs.

Definition 2.1. Let $V \neq \varnothing$ be a finite set called the *vertex set*. A *graph* $G = (V, E)$ is a collection of distinct subsets of V of size 2, denoted by $E(G)$, called the *edge set*. Elements of $V = V(G)$ are *vertices*, while elements of E are called *edges*. If $e = \{a, b\}$ is any edge, a and b are *endpoints* of e. We also say that a and b are *incident* with edge e. We write $ab = \{a, b\}$ for an edge when there is no possibility of confusion.

Definition 2.2. Let u, v be two distinct vertices of G. A (u, v)-*path* is a sequence

$$u = u_0,\ e_0,\ u_1,\ e_1,\ \ldots,\ e_n,\ u_{n+1} = v$$

of distinct vertices and edges such that u_i, u_{i+1} are the endpoints of e_i. Suppose G is a graph such that for every pair of distinct vertices u, v there exists a unique (u, v)-path. Then G is a *tree*. A disconnected graph in which each component is a tree is called a *forest*.

There are several other characterizations of trees. They will be utilized below without further reference.

Theorem 2.3 (characterization of trees). *Let G be a connected graph with v vertices and e edges. The following are equivalent*:

(a) *G is a tree.*

(b) $v = e + 1$.

(c) *G contains no cycles.*

(d) $b_1(G) = 0$, *where b_1 is the first Betti number of G* (see [Ferrario and Piccinini 2011, Chapter II.4]).

(e) *The removal of any edge from G results in a disconnected graph.*

Proofs of the equivalence of the statements may be found in any graph theory textbook; e.g., [Chartrand et al. 2016, Chapter 2.2].

Discrete Morse theory. We now introduce discrete Morse theory on graphs.

Definition 2.4. Let G be a graph. A *discrete Morse function* G is a function $f : G \to \mathbb{R}$ such that for every $v \in G$ we have

$$|\{e \in E : f(e) \geq f(v),\ v \text{ an endpoint of } e\}| \leq 1$$

and for every $e \in G$

$$|\{v \in V : f(v) \geq f(e),\ v \text{ an endpoint of } e\}| \leq 1.$$

If vertex v satisfies

$$|\{e \in E : f(e) \geq f(v),\ v \text{ an endpoint of } e\}| = 0$$

then v is a *critical vertex* and the value $f(v)$ is a *critical value*. If an edge e satisfies

$$|\{v \in V : f(v) \geq f(e),\ v \text{ an endpoint of } e\}| = 0$$

then e is a *critical edge* and $f(e)$ is a *critical value*. A vertex or edge that is not critical is called a *regular vertex* or *regular edge*, respectively. If all the critical values of f are distinct, f is said to be *excellent*.

We will assume that all discrete Morse functions are excellent. The critical values of a discrete Morse function tell us how to "build" the graph G in stages. This is formally accomplished through the level subcomplexes.

Definition 2.5. Let $f : G \to \mathbb{R}$ be a discrete Morse function, $c \in \mathbb{R}$. The smallest subgraph of G containing all v, e such that $f(e), f(v) \leq c$ is called a *level subcomplex*, denoted by $G(c)$.

Although it can be defined more generally, for the purposes of this paper we restrict the following definition to the case when $G = T$ is a tree. Recall that $b_0(T)$ is the 0*-th Betti number of* T; i.e., $b_0(T)$ is the number of connected components of T.

Definition 2.6. Let $f : T \to \mathbb{R}$ be an excellent discrete Morse function with critical values $c_0 < c_1 < c_2 < \cdots < c_{m-1}$. The *homological sequence* of f, denoted by B_f, is given by $b_0(T(c_0)), b_0(T(c_1)), \ldots, b_0(T(c_{m-1}))$. We write $B(i) = B_f(i) := b_0(T(c_i))$ when the discrete Morse function and tree are clear from the context. Two discrete Morse functions $f, g : T \to \mathbb{R}$ are said to be *homologically equivalent* if f and g induce the same homological sequence.

In other words, the homological sequence records the number of connected components at each level subcomplex, associating a sequence to a discrete Morse function. For an excellent discrete Morse function, exactly one single component will be added or removed at each level subcomplex. Formally:

Proposition 2.7 [Ayala et al. 2009a]. *If f is an excellent discrete Morse function with m critical values on a connected tree T, then the homological sequence of f satisfies* $|B(i+1) - B(i)| = 1$.

In addition to homological equivalence, there is the original notion of equivalence of two discrete Morse functions due to Robin Forman [1998].

Definition 2.8. Let $f, g : G \to \mathbb{R}$ be discrete Morse functions. Then f and g are *Forman equivalent* if for every pair (v, e) consisting of a vertex v and an incident edge e, we have $f(v) < f(e)$ if and only if $g(v) < g(e)$.

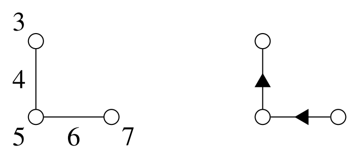
It turns out that this definition has a nice geometric characterization by passing to the gradient vector field.

Definition 2.9. Let $f : G \to \mathbb{R}$ be a discrete Morse function. The *induced gradient vector field* is defined by

$$\mathcal{V}_f := \{(v, e) : f(v) \geq f(e), v \text{ is incident with } e\}.$$

Remark 2.10. In light of the above, we may view a discrete Morse function as a pairing of a vertex with an incident edge (a regular pair) with each vertex and edge appearing in at most one pair. It then follows that any vertex or edge not in a regular pair is critical.

Example 2.11. Let f be the discrete Morse function defined on the graph to the left. The induced gradient vector field is shown on the right:



As mentioned above, the following theorem, due to Ayala et al., shows that Forman equivalence can be characterized in terms of the gradient vector field.

Theorem 2.12 [Ayala et al. 2009b, Theorem 3.1]. *Two discrete Morse functions f and g defined on G are equivalent if and only if* $\mathcal{V}_f = \mathcal{V}_g$.

An immediate corollary of this result is that a gradient vector field is completely determined by the critical simplices. We will use this fact below.

Corollary 2.13. *Two discrete Morse functions f and g are Forman equivalent if and only if they have the same critical simplices.*

3. Main result

In this section, we combine the notions of Forman equivalence and homological equivalence to produce a discrete Morse function on a tree with a desired gradient vector field and homological equivalence.

Definition 3.1. Let T be a tree, $v \in V(T)$, and $f : T \to \mathbb{R}$ a discrete Morse function. We say that f *roots* T *in* v or T *is rooted in* v if v is the unique critical simplex of f. Such a vertex is called the *root* of f.

Lemma 3.2. *Let* $f, g : T \to \mathbb{R}$ *both root* T *in* v. *Then* f *and* g *are Forman equivalent.*

Proof. By Corollary 2.13, a gradient vector field is uniquely determined by its critical simplices. Since by definition both f and g have the exact same set of critical simplices (namely, a single vertex), the result follows. □

We first give an algorithm to explicitly construct a discrete Morse function on a tree T rooted in a particular vertex v. If G is a graph and u, v are vertices of G, the *distance* between u and v is the minimum number of edges traversed over all paths from u to v.

Algorithm 1 (rooted-tree algorithm).
Input: A tree T with a given root vertex v_0 and a nonnegative integer k.
Output: A discrete Morse function $f : T \to \mathbb{R}$ rooted in v_0 with $f(v_0) = k$.

(1) Label $f(v_0) = k$.

(2) For any vertex $u \neq v_0$, label $f(u) = d(v_0, u) + k$.

(3) For any edge $e = uw$, label $f(e) = \max\{f(u), f(w)\}$.

Note that hidden in (2) of Algorithm 1 is the breadth-first search algorithm, which computes the distances from a fixed vertex to all other vertices. The time complexity for this is $O(|E|)$ [Agnarsson and Greenlaw 2007, p. 400]. Since the rest of the rooted-tree algorithm is constant time, the total cost of Algorithm 1 is the same.

Proposition 3.3. *Algorithm* 1 *is correct*; *that is*, *it produces a discrete Morse function rooted in* v_0.

Proof. Clearly $f(v_0) = k$ is critical since, for any edge uv_0, we have $f(uv_0) = \max\{f(u), f(v_0)\} = k+1 > k = f(v_0)$. Let $u \neq v_0$ be any vertex in T. We claim that there exists a unique edge $e = v_0 w$ such that $f(e) \leq f(v_0)$. If so, then f satisfies the definition of a discrete Morse function on each vertex and each vertex (other than v_0) is regular. By induction on n, the distance from a vertex to v_0, we see that there exists an edge e incident with u such that $f(e) = f(u)$. If e' is another edge such that $f(e') = f(u)$, then we obtain two paths from u to v_0, namely, one through e and another through e', contradicting the fact that paths in a tree are unique.

Now let $e = uw$ be any edge. Since the shortest path from v_0 to any other vertex u is unique, either $f(u) > f(w)$ or $f(u) < f(w)$. Hence each edge satisfies the definition of a discrete Morse function and is regular. We conclude that f is a discrete Morse function rooted in v_0. □

Lemma 3.4. *Let T be a tree and $f : T \to \mathbb{R}$ a discrete Morse function on T with m critical vertices. Let C_e be the set of critical edges of f. Then $T - C_e$ is a forest with exactly $T_1, T_2, \dots, T_m$ trees, each of which contains exactly one critical vertex.*

Proof. Clearly $T - C_e$ is a forest. It remains to show that each tree in the forest has exactly one critical vertex. If so, then the number of trees is in bijective correspondence with the number of critical vertices and hence there would be m trees. Now the discrete Morse function on T restricts to a discrete Morse function on any T_i. By Remark 2.10, regular simplices come in vertex/edge pairs. Since all critical edges have been removed, T_i is partitioned into vertex/edge pairs along with critical vertices. But since $|V(T_i)| = |E(T_i)| + 1$, such a partition can only happen if there is exactly one critical vertex. □

It then follows from Lemma 3.2 that the gradient vector field on each tree in the forest is uniquely determined.

Before giving the main algorithm, observe that a homological sequence from an excellent discrete Morse function on a tree T must start with 1 and end with 1. Furthermore, by Proposition 2.7, each value increases or decreases by exactly 1, never going below 1. Such a sequence is known as a *Dyck path* [Brualdi 2010]. This was first observed in [Ayala et al. 2009a].

We are now ready to give an algorithm which constructs an excellent discrete Morse function on a tree T whose induced homological sequence B is a given Dyck path beginning and ending at 1 and which induces a given gradient vector field $\mathcal{V}$. Subdivide T if more critical values are needed and by abuse of notation, call the resulting graph T. We call a Dyck path and a gradient vector field $\mathcal{V}$ *consistent* if the number of points in the Dyck path is the same as the number of critical simplices of $\mathcal{V}$.

Algorithm 2 (homological and Forman algorithm).
Input: A tree T with a given Dyck path beginning and ending at 1 and gradient vector field $\mathcal{V}$ consistent with the Dyck path.
Output: An excellent discrete Morse function $f : T \to \mathbb{R}$ whose homological sequence B is the given Dyck path and gradient vector field is $\mathcal{V}$.

(a) Remove all critical edges from T. Initialize $n = -1$.

(b) Choose an unlabeled tree T' and apply Algorithm 1 with root vertex the unique critical vertex of T' with $k = n + 1 = 0$.

(c) If b_0 increases, choose an unlabeled tree T' such that there exists a critical edge between T' and a labeled tree. Apply Algorithm 1 with root vertex the unique critical vertex of T' with $k = n + 1$ and go to step (e). Otherwise, go to step (d).

(d) If b_0 decreases, choose a critical edge connecting two labeled trees, and label this edge $n+1$. Go to step (e).

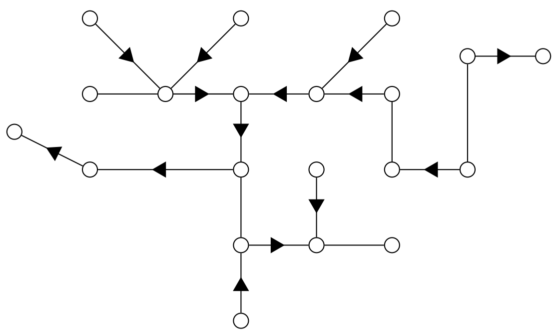
(e) If all vertices and edges are labeled, terminate. Otherwise, go to step (c).

Proposition 3.5. *Algorithm 2 is correct.*

Proof. By Algorithm 1, $T - C_e$ is labeled with a discrete Morse function with precisely the desired critical vertices. By step (d), each of the edges in C_e is labeled strictly greater than its two incident vertices, and hence the edges in C_e are the critical edges of f. Since f is a discrete Morse function with precisely the desired critical simplices, Theorem 2.12 implies that $\mathcal{V}_f = \mathcal{V}$.

To see that $B = B_f$, we note that when b_0 increases, a new tree is added in step (c) which increases b_0, and when b_0 decreases, two trees become a single tree in step (d). Thus the result. □

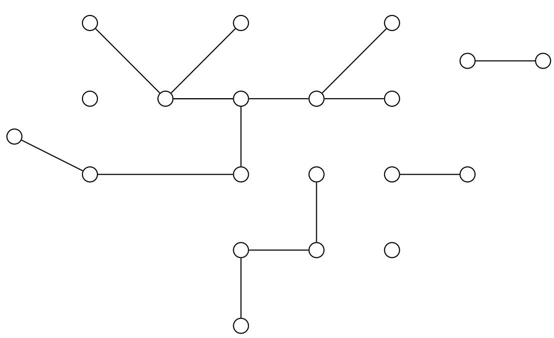
Example 3.6. We give an example to illustrate the construction of a discrete Morse function with a desired homological sequence and gradient vector field. Let T be the tree with desired gradient vector field given below:



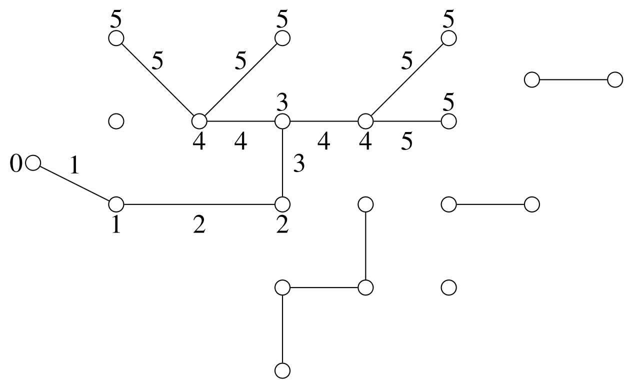
We use Algorithm 2 to construct a discrete Morse function on T inducing the above gradient vector field with homological sequence (Dyck path)

$$B_0 : 1\ 2\ 3\ 2\ 1\ 2\ 3\ 4\ 3\ 2\ 1.$$

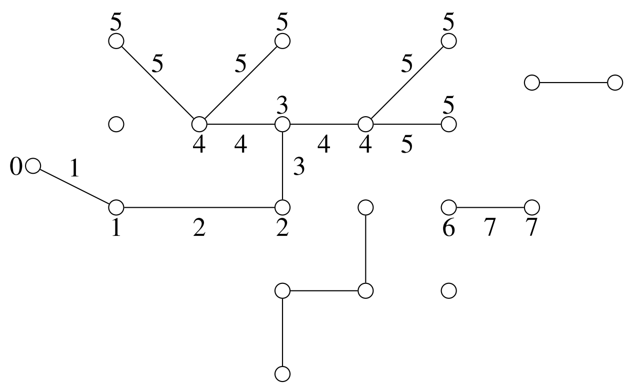
Step (a) of 2 is to remove all critical edges and initialize $n = -1$:



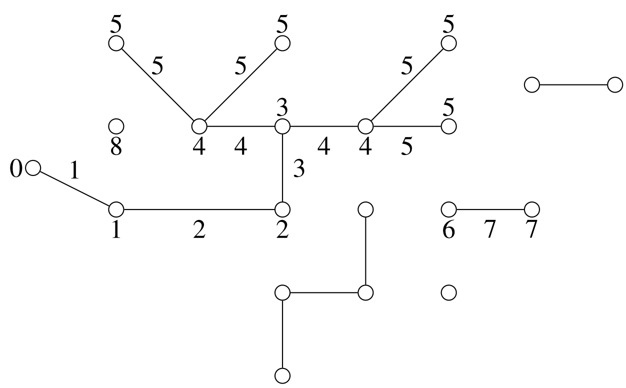
For step (b) of Algorithm 2, we pick a tree and apply Algorithm 1 with $k = n+1 = 0$. This yields the following labeling:



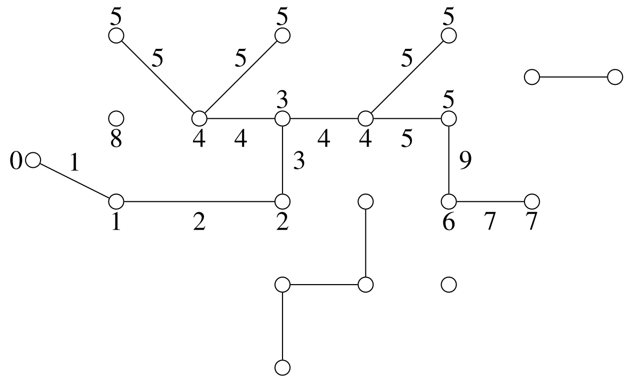
Because B_0 increases in the next stage of the specified homological sequence, we apply step (c) of Algorithm 2 with $k = n+1 = 6$:



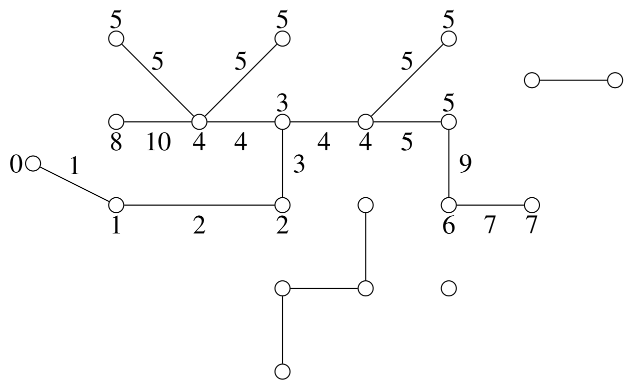
The homological sequence increases again, so we repeat with $k = n+1 = 8$ on a tree with a critical edge connecting to a labeled tree:



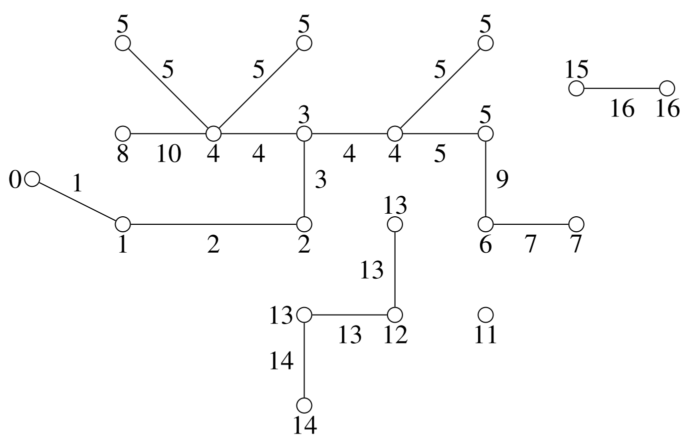
The desired homological sequence decreases at this point, so by step (d), we choose a critical edge connecting two labeled trees and label this $n+1=9$:



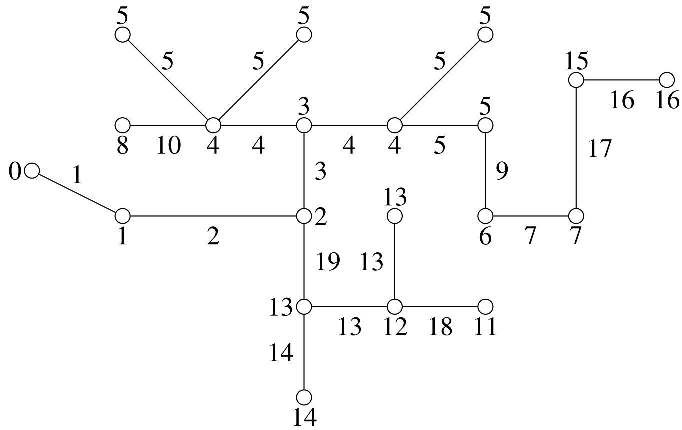
The sequence decreases again, so we repeat:



The homological sequence increase three times in a row, which we show all at once:



Finally, B_0 decreases three times in a row, which we show all at once:



It is now easy to check that the discrete Morse function has both the desired induced gradient vector field and homological sequence.

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References

[Agiorgousis et al. 2019] M. Agiorgousis, B. Green, A. Onderdonk, K. Rich, and N. A. Scoville, "Homologically equivalent discrete Morse functions", *Topology Proc.* **54** (2019), 283–294. MR Zbl

[Agnarsson and Greenlaw 2007] G. Agnarsson and R. Greenlaw, *Graph theory: modeling, applications, and algorithms*, Pearson Prentice Hall, Upper Saddle River, NJ, 2007. MR Zbl

[Ayala et al. 2008] R. Ayala, L. M. Fernández, A. Quintero, and J. A. Vilches, "A note on the pure Morse complex of a graph", *Topology Appl.* **155**:17-18 (2008), 2084–2089. MR Zbl

[Ayala et al. 2009a] R. Ayala, L. M. Fernández, D. Fernández-Ternero, and J. A. Vilches, "Discrete Morse theory on graphs", *Topology Appl.* **156**:18 (2009), 3091–3100. MR Zbl

[Ayala et al. 2009b] R. Ayala, L. M. Fernández, and J. A. Vilches, "Characterizing equivalent discrete Morse functions", *Bull. Braz. Math. Soc.* (*N.S.*) **40**:2 (2009), 225–235. MR Zbl

[Ayala et al. 2011] R. Ayala, D. Fernández-Ternero, and J. A. Vilches, "The number of excellent discrete Morse functions on graphs", *Discrete Appl. Math.* **159**:16 (2011), 1676–1688. MR Zbl

[Brualdi 2010] R. A. Brualdi, *Introductory combinatorics*, 5th ed., Pearson Prentice Hall, Upper Saddle River, NJ, 2010. MR Zbl

[Chartrand 1985] G. Chartrand, *Introductory graph theory*, Dover, New York, 1985. MR

[Chartrand et al. 2016] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, 6th ed., CRC Press, Boca Raton, FL, 2016. MR

[Ferrario and Piccinini 2011] D. L. Ferrario and R. A. Piccinini, *Simplicial structures in topology*, Springer, 2011. MR Zbl

[Forman 1998] R. Forman, “Morse theory for cell complexes”, *Adv. Math.* **134**:1 (1998), 90–145. MR Zbl

[Forman 2002] R. Forman, “A user’s guide to discrete Morse theory”, *Sém. Lothar. Combin.* **48** (2002), art. id. B48c. MR Zbl

[Nicolaescu 2008] L. I. Nicolaescu, “Counting Morse functions on the 2-sphere”, *Compos. Math.* **144**:5 (2008), 1081–1106. MR Zbl

[Scoville 2019] N. A. Scoville, *Discrete Morse theory*, Student Mathematical Library **90**, American Mathematical Society, Providence, RI, 2019. MR Zbl

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i.bradford.r@gmail.com *University of Deleware, Newark, DE, United States*

nscoville@ursinus.edu *Mathematics and Computer Science Department, Ursinus College, Collegeville, PA, United States*

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