involve
a journal of mathematics

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2020 vol. 13, no. 2

INVOLVE 13:2 (2020)
dx.doi.org/10.2140/involve.2020.13.301

Numerical semigroup tree of multiplicities 4 and 5

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(Communicated by Vadim Ponomarenko)

A numerical semigroup S is a cofinite submonoid of the nonnegative integers under addition. The cardinality of the complement of S in the nonnegative integers is called the genus. The smallest nonzero element of S is the multiplicity of S. There is an extensive literature about the tree of numerical semigroups, which has been used to count numerical semigroups by genus, yet the structure of the tree itself has not been described in the literature. In this paper, we completely describe the structure of the subtrees of the numerical semigroup tree of multiplicities 4 and 5. We conclude with an application of these numerical semigroup trees' structure.

1. Introduction

Let $\mathbb{N}$ denote the nonnegative integers. In the study of numerical semigroups, the question of how many ways we can form a numerical semigroup by removing g elements from $\mathbb{N}$ is particularly important. This question is equivalent to finding a formula for the number of numerical semigroups with genus g, denoted by $N(g)$. The tree of numerical semigroups has been a powerful tool for making conjectures and answering questions in this field, and it has been especially useful in analyzing the behavior of the function $N(g)$. Bras-Amorós [2008] conjectured and Zhai [2013] proved that asymptotically $N(g)$ exhibits Fibonacci-like behavior. That is,

$$\lim_{g\to\infty}\frac{N(g)}{\phi^g}=\beta,$$

where ϕ is the golden ratio $\frac{1}{2}(1+\sqrt{5})$ and β is a constant greater than or equal to 3.78. In particular, this means that

$$\lim_{g\to\infty}\frac{N(g-1)+N(g-2)}{N(g)}=1 \quad\text{and}\quad \lim_{g\to\infty}\frac{N(g)}{N(g-1)}=\phi.$$

MSC2010: primary 05A15, 20M14; secondary 11D07.
Keywords: numerical semigroup, tree of numerical semigroups.

Fromentin and Hivert [2016] have computed $N(g)$ for $g \leq 67$ and have extended this to $g \leq 70$ in unpublished computations.[1] This was further extended to $g \leq 71$ by Bras-Amorós and Fernández-González [2019]. Nevertheless, fundamental questions about $N(g)$ remain open. Two related open questions are the validity of the *strong genus conjecture* [Bras-Amorós 2008] and the *weak genus conjecture*.

Conjecture 1. *For* $g \geq 2,\ N(g) \geq N(g-1) + N(g-2)$.

Conjecture 2. *For* $g \geq 1,\ N(g) \geq N(g-1)$.

Several authors have approached the weak genus conjecture by considering the functions $N(m, g)$ which count the numerical semigroups of multiplicity m and genus g. It is immediate to observe that $N(g) = \sum_{m=1}^{g+1} N(m, g)$, which led Kaplan [2012] to conjecture the following, which clearly implies Conjecture 2.

Conjecture 3. *For* $m \geq 2,\ N(m, g) \geq N(m, g-1)$.

In this paper, we give a complete description of the structure of the numerical semigroup tree in multiplicities 4 and 5. The paper is organized as follows. In Section 2 we review some preliminaries about numerical semigroups and the numerical semigroup tree. Section 3 contains the main results of the paper about the structure of the tree. In Section 4 we use the main result in multiplicity 4 to give an alternate proof of the quasipolynomial expression for $N(4, g)$.

2. Preliminaries

In this section we define terms and state prior results that will be used throughout the rest of this paper. Readers familiar with the fundamentals of numerical semigroups can continue to Section 3 and refer to this section as needed. A good general reference for numerical semigroups is [Rosales and García-Sánchez 2009].

A subset S of $\mathbb{N}$ is called a *numerical semigroup* if it is cofinite in $\mathbb{N}$, contains 0, and is closed under addition. We sometimes refer to numerical semigroups as semigroups for brevity. The cardinality of $\mathbb{N} \setminus S$ is called the *genus*, $g(S)$, and the smallest nonzero element of S is called the *multiplicity*, $m(S)$. The largest element of $\mathbb{N} \setminus S$ is called the *Frobenius number*, $F(S)$. It is well known that the number of numerical semigroups of any fixed genus is finite. We can thus define a function $N : \mathbb{N} \to \mathbb{N}$ where $N(g)$ is the number of numerical semigroups of genus g. The *minimal generating set* of S is the unique smallest subset of S such that every element of S is an $\mathbb{N}$-linear combination of the numbers in the set. We write

$$S = \langle n_1, \ldots, n_t \rangle = \{a_1 n_1 + \cdots + a_t n_t \mid a_1, \ldots, a_t \in \mathbb{N}\}. \tag{1}$$

The multiplicity of S is always the smallest element of the minimal generating

[1] See https://github.com/hivert/NumericMonoid.

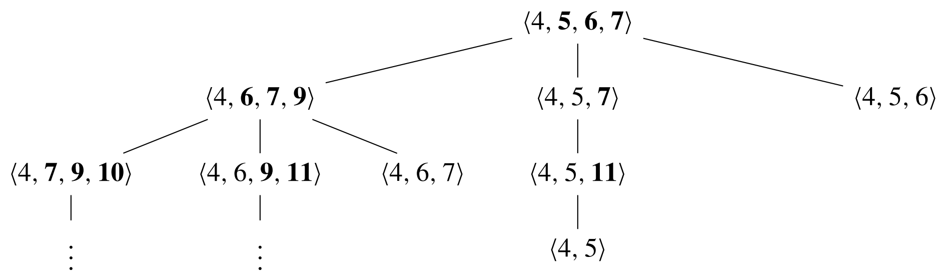

Figure 1. Portion of numerical semigroup tree under $\langle 4, 5, 6, 7\rangle$.

set. The generators in the minimal generating set that are larger than the Frobenius number are called *effective generators*. The *Apéry set* of a numerical semigroup with respect to its multiplicity is defined as

$$\mathrm{Ap}(m, S) = \{0,\, k_1 m + 1,\, k_2 m + 2, \ldots, k_{m-1} m + (m-1)\}, \tag{2}$$

where $\{k_1, \ldots, k_{m-1}\}$ are positive integers such that $k_i m + i$ is the smallest positive integer in S congruent to i modulo m. This uniquely associates to each numerical semigroup of multiplicity m an $(m-1)$-tuple, $(k_1, \ldots, k_{m-1})$, called the *Apéry coordinates* of S.

Numerical semigroups can be organized into a tree by letting the vertices be the set of numerical semigroups, with root $\mathbb{N}$. The parent of a vertex S is given by $S \cup \{F(S)\}$. This gives a unique path from any vertex to the root. A portion of the tree can be seen in Figure 1. The bold numbers in the figure indicate which generators are effective. Notice that by this construction, any child of S has genus one more than $g(S)$. This form of construction has been explored extensively, and an example of it can be seen in [Bras-Amorós and Bulygin 2009].

Every numerical semigroup with genus $g(S)$ can be obtained from a numerical semigroup of genus $g(S)-1$ by removing an effective generator from the semigroup. The numerical semigroup tree is naturally sorted by multiplicity, with the numerical semigroups of multiplicity m forming a subtree rooted at $(\{0\} \cup [m, \infty)) \cap (\mathbb{Z})$. All of the numerical semigroups of this form fall on the same branch of the tree, and this special branch is called the *ordinary* branch [Bras-Amorós and Bulygin 2009]. Some numerical semigroups contain no effective generators and hence no children in the tree. These numerical semigroups are called *leaves*.

3. Numerical semigroup tree structure

The tree structure of multiplicity $m = 3$ is known; see for instance [García-Sánchez et al. 2018]. In this section we give a complete description of the structure of the subtrees of multiplicities 4 and 5. Throughout our discussion, we use the numerical semigroup and its corresponding vertex in the tree interchangeably. Throughout the section when we refer to a branch, we mean a connected subtree. When we refer to the length of a linear graph, we mean the number of edges in that graph. Note that whenever k appears without a subscript in this section it is the smallest Apéry coordinate of the current numerical semigroup under discussion.

3.1. ***The multiplicity-4 subtree.*** The root of this subtree is the semigroup $\langle 4, 5, 6, 7\rangle$. By successively removing the smallest effective generator greater than 4, we create an infinite branch of this subtree, which we call the principal branch. It is easy to see that a multiplicity-4 semigroup lies on the principal branch if and only if it has three effective generators.

To organize the analysis of this subtree, we divide the branches rooted on the principal branch into three classes based on the congruence class of the first effective generator of the root vertex. We show in Lemma 13 that when the first effective generator is even, the branch is infinite. The other branches of the numerical semigroup tree are finite. In a sequence of lemmas, we describe the lengths of all finite branches of the subtree and the structure of the infinite branches. These results are compiled in Theorem 15.

We begin with an example to demonstrate the patterns that we state generally in the lemmas below. In Figure 2 we see the branch of the multiplicity-4 subtree rooted at $\langle 4, 15, 17, 18\rangle$. The bold numbers in the figure are effective generators. Observe that the child formed by removing the effective generator e either has $e+4$ as a new generator or has one fewer generator than its parent.

There are a few observations to make about this branch. First notice that by successively removing the second-largest effective generator, we create the linear branch from the root to $\langle 4, 15, 29\rangle$. To describe the structure of the branch rooted at $\langle 4, 15, 17, 18\rangle$, it is sufficient to know the length of this main subbranch and the lengths of each linear branch descending from it. Each of these linear branches follows one of three patterns.

The pattern of lengths we observe in Figure 2 is $X,\ 3,\ X-1,\ 3,\ X-2, \ldots$. Starting at the first branch, rooted at $\langle 4, 15, 17, 18\rangle$, every second branch's length decreases by 1, and starting at the second branch, rooted at $\langle 4, 15, 18, 21\rangle$, every second branch has length 3. Finally the last linear branch, when the semigroup on the main branch has lost an effective generator, has a distinct length. In the case of Figure 2, this subbranch begins at $\langle 4, 15, 29\rangle$ and has length 4. Describing similar patterns for general branches of the multiplicity-4 subtree allows us to describe its full structure.

We consider three cases for the roots of the subtrees. First consider the case where the root of the subtree has the form $\langle 4, 4k+1, 4k+2, 4k+3\rangle$.

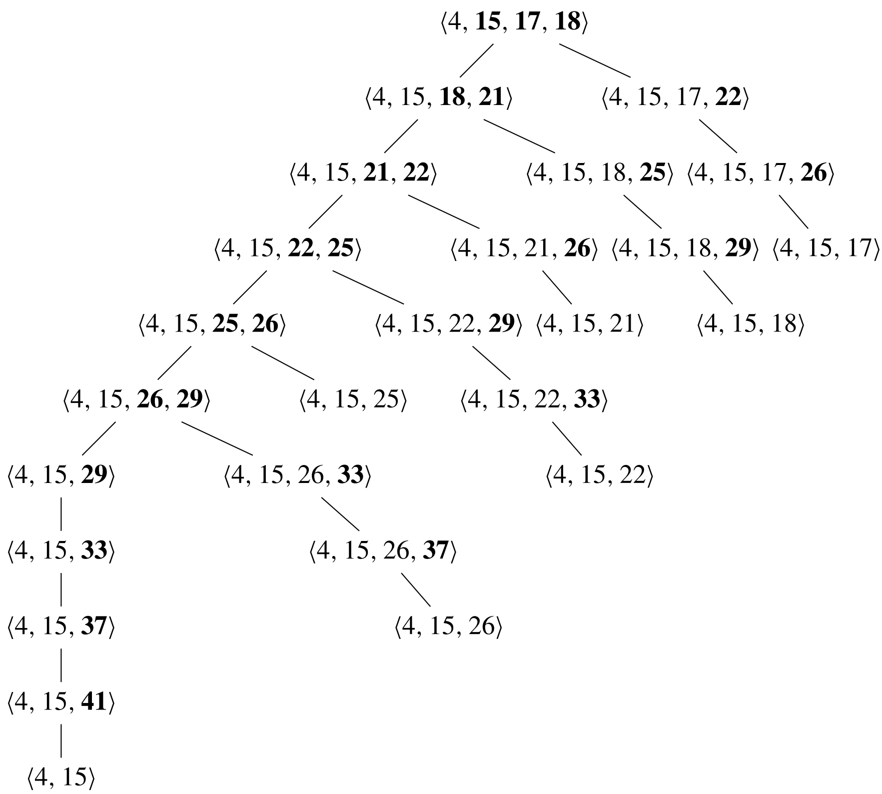

Figure 2. Numerical semigroup tree rooted at $\langle 4, 15, 17, 18\rangle$.

Definition 4. We define the main branch of any subtree rooted on the principal branch to be the set of semigroups formed by consecutively removing the second-largest effective generator.

Since we primarily care about the lengths of the branches, we can make a more abstract diagram that shows this information directly. We call such pictures *comb diagrams* for their shape, as seen in Figure 3. More precisely, a comb diagram is a depiction of some subtree of the numerical semigroup tree that shows the lengths of the subbranches. Typically, we take the root of the diagram to be a numerical semigroup from the principal branch. We use comb diagrams to illustrate patterns in the linear branch lengths. For instance, in Figure 3, we notice that the decreasing pattern begins with length $k-1$ and continues until the length is equal to 1. Note that this is a different case from the example in Figure 2, whose root was of the form $\langle 4, 4k+3, 4(k+1)+1, 4(k+1)+2\rangle$. We can also note that the constant-length branches are length k and the final linear branch that occurs when the main branch has lost a generator is length $k+1$. Finally, observe that by asserting that the last

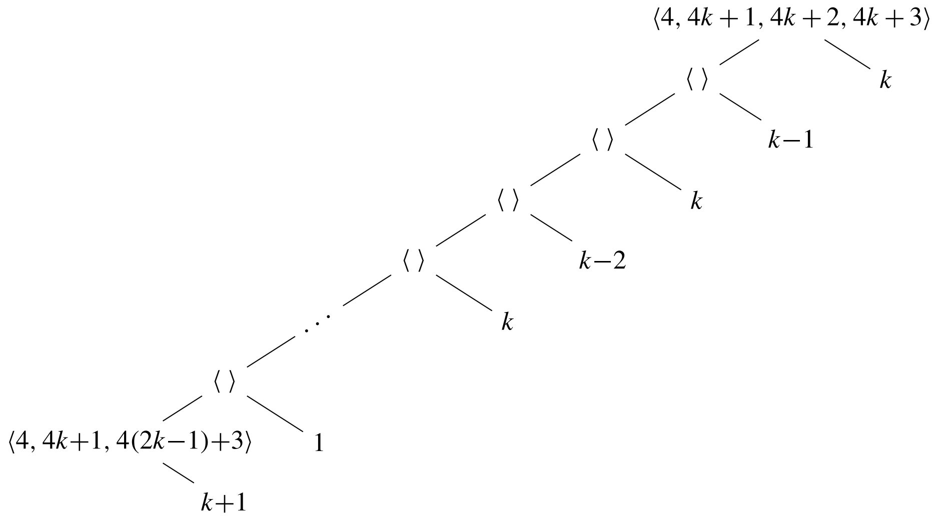

Figure 3. Comb diagram of the subtree rooted at $\langle 4, 4k+1, 4k+2, 4k+3\rangle$.

element of the main branch is $\langle 4, 4k+1, 4(2k-1)+3\rangle$, we have tacitly asserted that the length of the main branch is $2k-1$.

In the following lemmas we codify the information shown in Figure 3. In particular, we describe the structure of the subtree whose root lies on the principal branch of the multiplicity-4 subtree and has the form $\langle 4, 4k+1, 4k+2, 4k+3\rangle$. Every semigroup in this branch has 4 and $4k+1$ as generators. We consider several cases for the other minimal generators. First we describe when the linear branches have fixed length.

Lemma 5. *Let $S = \langle 4, \alpha, \beta, \gamma\rangle$ be on the main branch under $\langle 4, 4k+1, 4k+2, 4k+3\rangle$, where $\alpha = 4k+1$ and $\gamma = \beta + 1$. Then the linear branch rooted at S has length k.*

Proof. First note that the hypotheses imply that $\beta \equiv 2 \pmod 4$ and $\gamma \equiv 3 \pmod 4$. Let the corresponding Apéry coordinates be (k, k_2, k_3). Since β and γ are always effective, we know $k_2 = k_3$, and by hypothesis $k \le k_2$. The branch terminates at the semigroup $\langle 4, \alpha, \beta\rangle$. Each semigroup in this branch has $\gamma + 4l$ as the element of its Apéry set congruent to 3 modulo 4, where l is some nonnegative integer, and the other elements of its Apéry set are unchanged. Since $\alpha+\beta = 4k+4k_2+3 = \alpha+\gamma$, we have $\gamma+4l = \alpha+\beta$ when $l = k$. It is straightforward to show that $\alpha+\beta$ is the smallest element of $\langle 4, \alpha, \beta\rangle$ congruent to 3 modulo 4. Hence the length of the branch is k. $\square$

In order to prove Lemma 5 we used relations between Apéry coordinates to show which semigroup in the branch was a leaf. We found a linear combination of other generators in the same congruence class of the effective generator we removed and observed that this example was minimal. This technique is used throughout the rest of the paper, with deviations noted when applicable.

Lemma 6. *Let* $S = \langle 4, \alpha, \beta, \gamma \rangle$ *be on the main branch under* $\langle 4, 4k+1, 4k+2, 4k+3 \rangle$, *where* $\alpha = 4k + 1$ *and* $\gamma = \beta + 3 = 4k_2 + 2$. *Then the linear branch rooted at S has length* $l = 2k - k_2$, *where* k_2 *is the largest Apéry coordinate.*

Proof. Note that $\beta \equiv 3 \pmod 4$ and $\gamma \equiv 2 \pmod 4$; thus $k_2 = k_3 + 1$. The linear branch terminates at $\langle 4, \alpha, \beta \rangle$. As in Lemma 5, observe that when $l = 2k - k_2$, we have $\gamma + 4l = 4(k_2) + 2 + 4(2k - k_2) = 2\alpha \in \langle 4, \alpha, \beta \rangle$. This is the smallest linear combination of $\langle 4, \alpha, \beta \rangle$ that is congruent to 2 modulo 4. □

Lemma 7. $S = \langle 4, 4k + 1, 4(2k - 1) + 3 \rangle$ *is the last element of the main branch. Hence, the length of the main branch is* $2k - 1$.

Proof. For any semigroup S on the main branch, the Apéry set is

$$\mathrm{Ap}(4, S) = \{0, 4k + 1, 4(c) + 2, 4(c) + 3\}$$

or

$$\mathrm{Ap}(4, S) = \{0, 4k + 1, 4(c + 1) + 2, 4(c) + 3\}$$

for some $c \geq k$. Since the Apéry set elements congruent to 2 and 3 modulo 4 are within 4 of each other, we need only consider the span of 4 and $4k + 1$ to see if a potential generator is redundant. Clearly the smallest element of $\langle 4, 4k + 1 \rangle$ congruent to 2 or 3 is $2(4k + 1)$. This first occurs when

$$\mathrm{Ap}(4, S) = \{0, 4k + 1, 4(c + 1) + 2, 4(c) + 3\}$$

and $c = (2k - 1)$. When $c = (2k - 1)$, the generator equivalent to $4(c + 1) + 2$ is redundant, and we have $S = \langle 4, 4k + 1, 4(2k - 1) + 3 \rangle$. Since c is initially k and increases by 1 every other semigroup, the total length of the main branch is $2(c - k) + 1$, where $c = 2k - 1$. Simplifying, the total length of the main branch is $2(2k - 1 - k) + 1 = 2k - 1$. □

Lemma 8. *Let* $S = \langle 4, \alpha, \beta \rangle$, *where* $\alpha = 4k + 1$ *and* $\beta = 4(2k - 1) + 3$. *Then the branch rooted at S has length* $k + 1$, *where k is the smallest Apéry coordinate.*

Proof. The Apéry coordinates of S are $(k, 2k, 2k - 1)$. The leaf in this branch is $\langle 4, 4k + 1 \rangle$, and the smallest element of this set congruent to 3 modulo 4 is 3α. So $\beta + 4(k + 1) = 4(2k - 1) + 3 + 4(k + 1) = 3\alpha \in \langle 4, \alpha \rangle$, and the length of the branch is $k + 1$. □

Lemmas 5–8 fully describe the subtrees rooted at numerical semigroups where the first effective generator is congruent to $1 \pmod 4$. The subtrees rooted at numerical semigroups with first effective generator congruent to $3 \pmod 4$ are similar. We omit the proofs, which are analogous to the proofs of Lemmas 5–8.

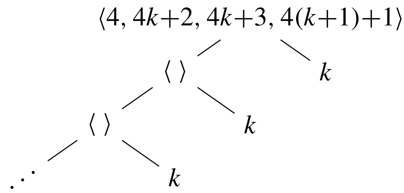

Figure 4. Comb diagram of the subtree rooted at $\langle 4, 4k+2, 4k+3, 4(k+1)+1\rangle$.

Lemma 9. *Let* $S = \langle 4, \alpha, \beta, \gamma\rangle$ *be on the main branch under* $\langle 4, 4k+3, 4(k+1)+1, 4(k+1)+2\rangle$, *where* $\alpha = 4k+3$ *and* $\gamma = \beta + 1 = 4k_2 + 2$. *Then the linear branch rooted at S has length* $1 + 2k - k_2$.

Lemma 10. *Let* $S = \langle 4, \alpha, \beta, \gamma\rangle$ *be on the main branch under* $\langle 4, 4k+3, 4(k+1)+1, 4(k+1)+2\rangle$, *where* $\alpha = 4k+3$ *and* $\gamma = \beta + 3 = 4k_1 + 1$. *Then the linear branch rooted at S has length k*, *where k is the smallest Apéry coordinate.*

Lemma 11. $S = \langle 4, 4k+3, 4(2k+1)+1\rangle$ *is the last element of the main branch. Hence*, *the length of the main branch is* $2k$.

Lemma 12. *Let* $S = \langle 4, \alpha, \beta\rangle$, *where* $\alpha = 4k+3$ *and* $\beta = 4(2k+1)+1$. *Then the branch rooted at S has length* $k+1$, *where k is the smallest Apéry coordinate.*

The subtree rooted at numerical semigroups where the first effective generator is congruent to 2 (mod 4) are infinite, but they still exhibit regular behavior that we can describe. Figure 4 shows a comb diagram for this case.

Lemma 13. *Let* $S = \langle 4, \alpha, \beta, \gamma\rangle$, *where* $\alpha = 4k+2$ *and* β *and* γ *are effective generators. Then the main branch of the subtree rooted at S is infinite.*

Proof. Each element on the main branch has the form $\langle 4, 4k+2, 4c+3, 4(c+1)+1\rangle$ or the form $\langle 4, 4k+2, 4(c+1)+1, 4(c+1)+3\rangle$, where $c \geq k$. In either case, every element of S less than the conductor is even. Hence by [Bras-Amorós and Bulygin 2009, Theorem 12], S lies on an infinite chain. Since this is true for every semigroup of these forms, the chain comprising them is infinite. □

Lemma 14. *Let* $S = \langle 4, \alpha, \beta, \gamma\rangle$ *be on the main branch under* $\langle 4, 4k+2, 4k+3, 4(k+1)+1\rangle$. *Then the linear branch rooted at S has length k.*

Proof. This is analogous to Lemma 5. □

We compile these results in one theorem for ease of reference.

Theorem 15. *The numerical semigroup tree rooted at* $S = \langle 4, 5, 6, 7\rangle$ *has the following properties*:

(1) *For* $S = \langle 4, \alpha, \beta, \gamma\rangle$, *where* $\alpha < \beta < \gamma$ *are all effective minimal generators*, $0 < \gamma - \alpha \leq 3$.

(2) *Let S be a numerical semigroup with multiplicity $m = 4$, and three effective generators $\alpha < \beta < \gamma$. Then, the descendant $S \setminus \{\alpha\}$ has three effective generators, the descendant $S \setminus \{\beta\}$ has at most two effective generators, and $S \setminus \{\gamma\}$ has at most one effective generator.*

(3) *The subbranches of the numerical semigroup tree rooted at numerical semigroups in the form of $S = \langle 4, 4k+1, 4k+2, 4k+3\rangle$ have the following properties*:

- (a) *Let $S = \langle 4, \alpha, \beta, \gamma\rangle$ be on the main branch under $\langle 4, 4k+1, 4k+2, 4k+3\rangle$, where $\alpha = 4k+1$ and $\gamma = \beta + 1$. Then the linear branch rooted at S has length k.*
- (b) *Let $S = \langle 4, \alpha, \beta, \gamma\rangle$ be on the main branch under $\langle 4, 4k+1, 4k+2, 4k+3\rangle$, where $\alpha = 4k+1$ and $\gamma = \beta + 3 = 4k_2 + 2$. Then the linear branch rooted at S has length $l = 2k - k_2$, where k_2 is the largest Apéry coordinate.*
- (c) *$S = \langle 4, 4k+1, 4(2k-1)+3\rangle$ is the last element of the main branch. Hence, the length of the main branch is $2k-1$.*
- (d) *Let $S = \langle 4, \alpha, \beta\rangle$, where $\alpha = 4k+1$ and $\beta = 4(2k-1)+3$. Then the branch rooted at S has length $k+1$, where k is the smallest Apéry coordinate.*

(4) *The subbranches of the numerical semigroup tree rooted at numerical semigroups in the form of $S = \langle 4, 4k+2, 4k+3, 4(k+1)+1\rangle$ have the following properties*:

- (a) *Let $S = \langle 4, \alpha, \beta, \gamma\rangle$, where $\alpha = 4k+2$ and β and γ are effective generators. Then the main branch of the subtree rooted at S is infinite.*
- (b) *Let $S = \langle 4, \alpha, \beta, \gamma\rangle$ be on the main branch under $\langle 4, 4k+2, 4k+3, 4(k+1)+1\rangle$. Then the linear branch rooted at S has length k.*

(5) *The subbranches of the numerical semigroup tree rooted at numerical semigroups in the form of $S = \langle 4, 4k+3, 4(k+1)+1, 4(k+1)+2\rangle$ have the following properties*:

- (a) *Let $S = \langle 4, \alpha, \beta, \gamma\rangle$ be on the main branch under $\langle 4, 4k+3, 4(k+1)+1, 4(k+1)+2\rangle$, where $\alpha = 4k+3$ and $\gamma = \beta + 1 = 4k_2 + 2$. Then the linear branch rooted at S has length $1 + 2k - k_2$.*
- (b) *Let $S = \langle 4, \alpha, \beta, \gamma\rangle$ be on the main branch under $\langle 4, 4k+3, 4(k+1)+1, 4(k+1)+2\rangle$, where $\alpha = 4k+3$ and $\gamma = \beta + 3 = 4k_1 + 1$. Then the linear branch rooted at S has length k, where k is the smallest Apéry coordinate.*
- (c) *$S = \langle 4, 4k+3, 4(2k+1)+1\rangle$ is the last element of the main branch. Hence, the length of the main branch is $2k$.*
- (d) *Let $S = \langle 4, \alpha, \beta\rangle$, where $\alpha = 4k+3$ and $\beta = 4(2k+1)+1$. Then the branch rooted at S has length $k+1$, where k is the smallest Apéry coordinate.*

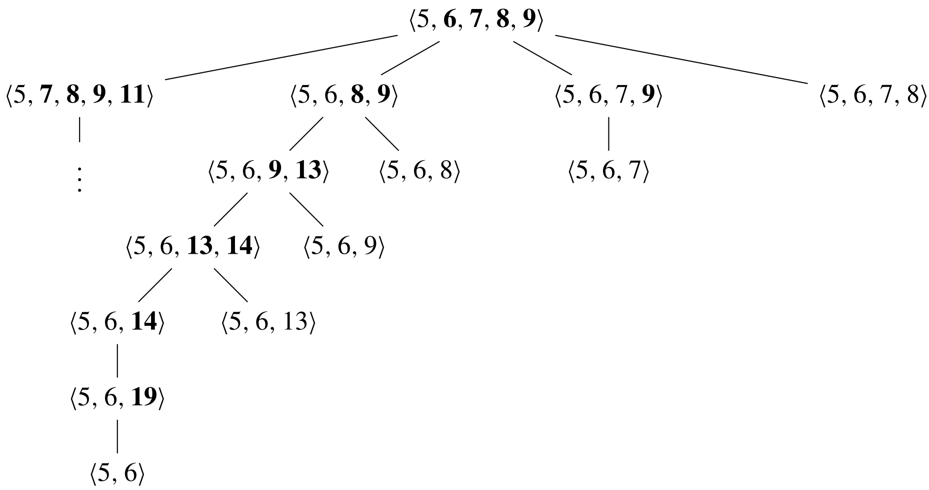

Figure 5. Portion of the numerical semigroup tree under $S = \langle 5, 6, 7, 8, 9\rangle$.

Proof. Parts (1) and (2) are straightforward. The rest of the theorem follows immediately from Lemmas 5–14. □

3.2. ***The subtree rooted at*** $\boldsymbol{S = \langle 5, 6, 7, 8, 9\rangle}$**.** The numerical semigroup tree rooted at $\langle 5, 6, 7, 8, 9\rangle$ is similar to the subtrees rooted at $\langle 3, 4, 5\rangle$ and $\langle 4, 5, 6, 7\rangle$; see Figure 5. We again define an infinite principal branch by successively removing the smallest generator greater than 5. Besides the principal branch, there are no infinite branches. This is because 5 is prime.

The key insight that we use throughout this section is that when we ignore removing the largest effective generator, we get a pattern very similar to what we had in multiplicity 4. We break the branches of the tree into four classes based on the congruence class of the first effective generator. The patterns of lengths act similarly when the first effective generator is congruent to 1 (mod 5) and 2 (mod 5), and again when the first effective generator is congruent to 3 (mod 5) and 4 (mod 5). We omit redundant proofs in our analysis below. Once we describe these patterns we address the largest effective generator in each case.

Each class in multiplicity 5 behaves like the branches rooted at $\langle 4, 4k+1, 4k+2, 4k+3\rangle$ from the multiplicity-4 tree. There is a main branch, and from this branch descend subbranches whose lengths follow a simple pattern. We demonstrate each pattern separately, again using Apéry coordinates to find when a new generator would be redundant.

Here, in Figure 6, we see the comb diagram of the subtree rooted at the numerical semigroup $S = \langle 5, 12, 13, 14, 16\rangle$. Note that in multiplicity 5 the leaves in our comb diagrams are semigroups with exactly one effective generator. There is a

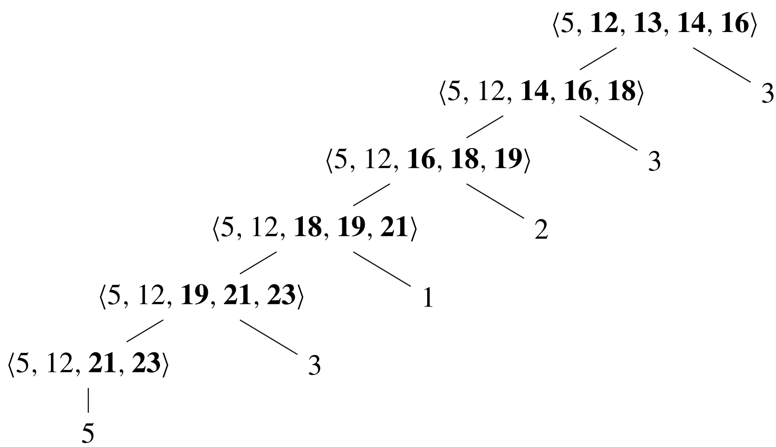

Figure 6. Portion of the comb diagram under $S = \langle 5, 12, 13, 14, 16\rangle$.

linear branch descending from each semigroup in the diagram, which the diagram does not address. We describe the patterns of lengths in these diagrams, just as we did in multiplicity 4. However, with multiplicity 5, the constant-length subbranches occur every third branch instead of every other branch as they did in multiplicity 4. The lengths of the other subbranches decrease by 1 until there is a subbranch of length 1. Finally, the subbranch that occurs when the main branch has just lost a generator has length $2k + 1$, where k is the smallest Apéry coordinate of S.

Definition 16. We define the main branch of any subtree rooted on the principal branch of the multiplicity 5 tree to be the set of semigroups formed by consecutively removing the third-largest effective generator.

Lemma 17. *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+2, 5k+4, 5(k+1)+1, 5(k+1)+3\rangle$, *where* $\alpha = 5k+2,\ \beta = 5k_4+4,\ \gamma = 5(k_4+1)+1$, *and* $\delta = 5(k_4+1)+3$. *Then the linear branch rooted at S generated by consecutively removing the second-largest generator loses an effective generator at length* $2k-1$.

Proof. The second-largest effective generator is always congruent to 1 or 3 modulo 5, alternating with each removal. The branch of the comb diagram terminates when the new semigroup has only one effective generator, which occurs the first time $\gamma + 5(h+1)$ or $\delta + 5(h+1)$ is in $\langle 5, \alpha, \beta\rangle$. It is easy to verify that if γ becomes redundant first, the branch has length $2h+1$, and if δ becomes redundant first, it has length $2h+2$. We know that $\gamma = 5k_1 + 1$ and $\delta = 5k_3 + 3$. Thus, in order for γ to become redundant, it must be equal to 3 copies of α, or $\alpha + \beta$. In order for δ to become redundant, it must be equal to 4 copies of α, or $\alpha + \gamma$. Since $\alpha < \beta < \gamma$ and $\beta < 2\alpha$, we know that $\alpha + \beta < 3\alpha$, also note $\alpha + \beta < \alpha + \gamma$, and so γ becomes redundant first when the new generator would be $\alpha+\beta$. We now set $\gamma+5(h+1)=\alpha+\beta=5k_2+5k_4+6$ and solve for h. Since $k_4=k_1-1$, we get $h=k-1$. Thus, the total length of the branch until we lose an effective generator is $2h+1=2(k-1)+1=2k-1$. □

The other proofs in this section are similar to this one. In each, we determine which generator becomes redundant first. We then express the length of the branch in terms of the Apéry coordinates of the numerical semigroup, as we did in multiplicity 4. In the following proofs, we omit the restatement of this framework.

Lemma 18. *Let $S=\langle 5,\alpha,\beta,\gamma,\delta\rangle$ be on the main branch under $\langle 5, 5k+2, 5k+4, 5(k+1)+1, 5(k+1)+3\rangle$, where $\alpha=5k_2+2$ and $k_3=k_4$. Then the branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length $4k_2-k_4-k_1$, where k_1,k_2,k_3,k_4 are the Apéry coordinates.*

Proof. Let $S=\langle 5,\alpha,\beta,\gamma,\delta\rangle$ with at least two effective generators, $\gamma<\delta$, where $\alpha=5k_2+2$ and $k_3=k_4$. We consider two cases. The first is when $k_1=k_4+1$, and the second is when $k_1=k_4$.

Case 1: Let $k_1=k_4+1$. Then we know that $\gamma=5k_4+4$ and $\delta=5k_1+1$. Now γ becomes redundant when 2α would be the new generator, and δ becomes redundant when the smaller of 3α and 2β would be the new generator. Thus, γ will become redundant before δ. We set $\gamma+5(h+1)=2\alpha$ and solve for h. We get $h=2k_2-k_4-1$. The total length of the branch until we lose an effective generator is $2h+1=4k_2-2k_4-1=4k_2-k_4-k_1$ since by hypothesis we know that $k_1=k_4+1$.

Case 2: Let $k_1=k_4$. Then we know that $\gamma=5k_3+3$, and $\delta=5k_4+4$. γ becomes redundant when $\alpha+\beta$ would be the new generator. Now δ becomes redundant when 2α would be the new generator. Thus, δ will become redundant before γ. We set $\delta+5(h+1)=2\alpha$ and solve for h. We get $h=2k_2-k_4-1$. The total length of the branch until we lose an effective generator is $2h+2=4k_2-2k_4=4k_2-k_4-k_1$ since by hypothesis we know that $k_1=k_4$. □

Lemma 19. *$S=\langle 5, 5k+2, 5(2k)+1, 5(2k)+3\rangle$ is the last element of the main branch. Hence, the length of the main branch is $3k-1$.*

Proof. Using Definition 16, we define the Apéry set on the main branch as

$$\mathrm{Ap}(5,S)=\{0, 5(c+1)+1, 5k+2, 5(c+1)+3, 5c+4\},$$
$$\mathrm{Ap}(5,S)=\{0, 5c+1, 5k+2, 5c+3, 5c+4\},$$

or

$$\mathrm{Ap}(5,S)=\{0, 5(c+1)+1, 5k+2, 5c+3, 5c+4\}$$

for some $c\geq k$. Since $\alpha=5k+2$, we know the generator equivalent to $5c+4$ will become redundant first, because $2\alpha\equiv 4 \pmod 5$ is the smallest element of $\langle 5,\alpha\rangle$

congruent to 1, 3, or 4 (mod 5). The first value of c for which $5c+4=2\alpha=10k+4$ is when $c=2k$. Since we increase c by 1 every third semigroup, we know the total length of the main branch is $3(c-k)-1$ when the generator equivalent to $5c+4$ becomes redundant first. Thus, the total length of the main branch is $3(c-k)-1=3(2k-k)-1=3k-1$. □

Lemma 20. *Let $S=\langle 5, 5k+2, 5(2k)+1, 5(2k)+3\rangle$. Then the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length $2k+1$.*

Proof. Each semigroup on this branch has the form $\langle 5, \alpha, \beta, \gamma\rangle$, where $\alpha=5k+2$, $k_3 \geq 2k$, and $k_1=k_3$ or k_3+1. The generator congruent to 3 (mod 5) becomes redundant when 4α would be the new generator. The generator congruent to 1 (mod 5) becomes redundant when 3α would be the new generator, and hence will become redundant first. We set $5(2k)+1+5(h+1)=3\alpha$ and solve for h. We get $h=k$. Since h increases by 1 every second time we remove the second-largest generator, the total length of the branch until we lose an effective generator is $2k+1$. □

Lemmas 17–20 fully describe the patterns of length for the subbranches rooted at numerical semigroups with first effective generator congruent to 2 (mod 5). The remaining subsets of subbranches are shown in a similar manner.

Lemma 21. *Let $S=\langle 5, \alpha, \beta, \gamma, \delta\rangle$ be on the main branch under $\langle 5, 5k+1, 5k+2, 5k+3, 5k+4\rangle$, where $\alpha=5k+1$:*

(1) *In the case, $k_2=k_4$, the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length $2k-1$.*

(2) *In the case $k_4=k_2-1$, the branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length $4k_1-k_2-k_3-1$, where k_1, k_2, k_3, k_4 are the Apéry coordinates of S.*

(3) *$\langle 5, 5k+1, 5(2k-1)+3, 5(2k-1)+4\rangle$ is the last element of the main branch under $\langle 5, 5k+1, 5k+2, 5k+3, 5k+4\rangle$. Hence, the length of the main branch is $3k-2$.*

(4) *The linear branch under $\langle 5, 5k+1, 5(2k-1)+3, 5(2k-1)+4\rangle$ which is generated by consecutively removing the second-largest generator loses an effective generator at length $2k+1$.*

Lemma 22. *Let $S=\langle 5, \alpha, \beta, \gamma, \delta\rangle$ be on the main branch under $\langle 5, 5k+3, 5k+4, 5(k+1)+1, 5(k+1)+2\rangle$, where $\alpha=5k+3$:*

(1) *In the case $k_1=k_4$, the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length $2k$.*

(2) *In the case* $k_4 = k_1 - 1$, *the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $4k_3 - k_4 - k_2$, *where* k_1, k_2, k_3, k_4 *are the Apéry coordinates of S.*

(3) $\langle 5, 5k+3, 5(2k)+2, 5(2k)+4\rangle$ *is the last element of the main branch under* $\langle 5, 5k+3, 5k+4, 5(k+1)+1, 5(k+1)+2\rangle$. *Hence, the length of the main branch is* $3k-1$.

(4) *The linear branch under* $\langle 5, 5k+3, 5(2k)+2, 5(2k)+4\rangle$ *which is generated by consecutively removing the second-largest generator loses an effective generator at length* $2k+2$.

Lemma 23. *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+4, 5(k+1)+1, 5(k+1)+2, 5(k+1)+3\rangle$, *where* $\alpha = 5k+4$:

(1) *In the case* $k_3 = k_2 - 1$, *the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $2k$.

(2) *In the case* $k_2 = k_3$, *the branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $4k_4 - k_2 - k_1 + 2$, *where* k_1, k_2, k_3, k_4 *are the Apéry coordinates.*

(3) $\langle 5, 5k+4, 5(2k+1)+1, 5(2k+1)+2\rangle$ *is the last element of the main branch under* $\langle 5, 5k+4, 5(k+1)+1, 5(k+1)+2, 5(k+1)+3\rangle$. *Hence, the length of the main branch is* $3k$.

(4) *The linear branch under* $\langle 5, 5k+4, 5(2k+1)+1, 5(2k+1)+2\rangle$ *generated by consecutively removing the second-largest generator loses an effective generator at length* $2k+2$.

Until now we have disregarded the linear branches descending from each semigroup in the comb diagram. To fully describe the structure of the multiplicity-5 numerical semigroup tree, we must describe the lengths of these branches as well. We do this in a sequence of lemmas based on the congruence class modulo 5 of the largest generator.

Lemma 24. *For* $S = \langle 5, 5a+1, 5b+2, 5c+3, 5d+4\rangle$, *with Apéry coordinates* (a, b, c, d) *and largest generator* $\delta \equiv 1 \pmod 5$, *the length of the branch generated by consecutively removing the largest generator is* $\min(b+d-a+1, 2c-a+1)$.

Proof. Let $S = \langle 5, 5a+1, 5b+2, 5c+3, 5d+4\rangle$, where (a, b, c, d) are the Apéry coordinates of S, and let the largest generator be $\delta \equiv 1 \pmod 5$. The Apéry coordinates of each numerical semigroup on this branch are in the form $(a+k, b, c, d)$, where ℓ is the distance down the branch from S. A leaf occurs on this branch when

$$5(a+\ell)+1 \in \langle 5, 5b+2, 5c+3, 5d+4\rangle$$

or in other words when

$$5(a+\ell)+1 = 5t_1 + (5b+2)t_2 + (5c+3)t_3 + (5d+4)t_4 \quad \text{for } t_i \in \mathbb{N}.$$

If $t_2 = t_4 = 1$ and all $t_i = 0$, then we have $5b + 5d + 6 \equiv 1 \pmod 5$ with length $b + d - a + 1$. We can also have $t_3 = 2$ and all other $t_i = 0$; then we have $10c + 6 \equiv 1 \pmod 5$ with length $(2c - a + 1)$. A similar argument to those used in previous lemmas shows that one of these is the smallest element of $\langle 5, b, c, d\rangle$ that is congruent to $1 \pmod 5$. Thus, the length of the subbranch is $\min(b+d-a+1, 2c-a+1)$. □

Lemma 25. *Let* $S = \langle 5, 5a+1, 5b+2, 5c+3, 5d+4\rangle$, *with Apéry coordinates* (a, b, c, d). *Denote the largest generator of* S *by* δ:

(1) *When* $\delta \equiv 2 \pmod 5$, *the length of the branch generated by consecutively removing the largest generator is* $\min(2a-b, c+d-b+1)$.

(2) *When* $\delta \equiv 3 \pmod 5$, *the length of the branch generated by consecutively removing the largest generator is* $\min(a+b-c, 2d-c+1)$.

(3) *When* $\delta \equiv 4 \pmod 5$, *the length of the branch generated by consecutively removing the largest generator is* $\min(a+c-d, 2b-d)$.

We compile these results in one theorem for ease of reference.

Theorem 26. *The numerical semigroup tree rooted at* $S = \langle 5, 6, 7, 8, 9\rangle$ *has the following properties*:

(1) *For* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$, *where* $\alpha < \beta < \gamma < \delta$ *are effective*, $0 < \delta - \alpha \le 4$.

(2) *For* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$, *with effective generators*, $\alpha < \beta < \gamma < \delta$, $S \setminus \{\beta\}$ *has at most three effective generators*, $S \setminus \{\gamma\}$ *has at most two effective generators, and* $S \setminus \{\delta\}$ *has at most one effective generator.*

(3) *The subbranches of the numerical semigroup tree rooted at numerical semigroups in the form of* $S = \langle 5, 5k+1, 5k+2, 5k+3, 5k+4\rangle$ *have the following properties*:

(a) *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+1, 5k+2, 5k+3, 5k+4\rangle$, *where* $\alpha = 5k+1$ *and* $\beta = 5k+2$. *Then the linear branch rooted at* S *generated by consecutively removing the second-largest generator will lose an effective generator at length* $2k-1$.

(b) *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+1, 5k+2, 5k+3, 5k+4\rangle$, *where* $\alpha = 5k+1$, *and* $k_4 = k_2 - 1$. *Then the branch rooted at* S *generated by consecutively removing the second-largest generator will lose an effective generator at length* $4k_1 - k_2 - k_3 - 1$, *where* k_1, k_2, k_3, k_4 *are the Apéry coordinates.*

(c) $S = \langle 5, 5k+1, 5(2k-1)+3, 5(2k-1)+4\rangle$ *is the last element of the main branch. Hence, the length of the main branch is* $3k-2$.

(d) *Let* $S = \langle 5, \alpha, \beta, \gamma\rangle$ *be on the main branch under* $\langle 5, 5k+1, 5(2k-1)+3, 5(2k-1)+4\rangle$, *where* $\alpha = 5k+1$. *Then the linear branch rooted at S generated by consecutively removing the second-largest generator loses an effective generator at length* $2k+1$.

(4) *The subbranches of the numerical semigroup tree rooted at numerical semigroups in the form of* $S = \langle 5, 5k+2, 5k+3, 5k+4, 5(k+1)+1\rangle$ *have the following properties*:

(a) *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+2, 5k+4, 5(k+1)+1, 5(k+1)+3\rangle$, *where* $\alpha = 5k+2$, $\beta = 5k_4+4$, $\gamma = 5(k_4+1)+1$, *and* $\delta = 5(k_4+1)+3$. *Then the linear branch rooted at S generated by consecutively removing the second-largest generator loses an effective generator at length* $2k-1$.

(b) *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+2, 5k+4, 5(k+1)+1, 5(k+1)+3\rangle$, *where* $\alpha = 5k_2+2$ *and* $k_3 = k_4$. *Then the branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $4k_2 - k_4 - k_1$, *where* k_1, k_2, k_3, k_4 *are the Apéry coordinates.*

(c) $S = \langle 5, 5k+2, 5(2k)+1, 5(2k)+3\rangle$ *is the last element of the main branch. Hence, the length of the main branch is* $3k-1$.

(d) *Let* $S = \langle 5, \alpha, \beta, \gamma\rangle$ *be on the main branch under* $\langle 5, 5k+2, 5(2k)+1, 5(2k)+3\rangle$, *where* $\alpha = 5k+2$. *Then the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $2k+1$.

(5) *The subbranches of the numerical semigroup tree rooted at numerical semigroups in the form of* $S = \langle 5, 5k+3, 5k+4, 5(k+1)+1, 5(k+1)+2\rangle$ *have the following properties*:

(a) *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+3, 5k+4, 5(k+1)+1, 5(k+1)+2\rangle$, *where* $\alpha = 5k+3$, $\beta = 5(k+1)+1$, $\gamma = 5(k+1)+2$, *and* $\delta = 5(k+1)+4$. *Then the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $2k$.

(b) *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+3, 5k+4, 5(k+1)+1, 5(k+1)+2\rangle$, *where* $\alpha = 5k+3$. *Then the branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $4k_3 - k_4 - k_2$, *where* k_1, k_2, k_3, k_4 *are the Apéry coordinates.*

(c) $S = \langle 5, 5k+3, 5(2k)+2, 5(2k)+4\rangle$ *is the last element of the main branch. Hence, the length of the main branch is* $3k-1$.

(d) *Let* $S = \langle 5, \alpha, \beta, \gamma\rangle$ *be on the main branch under* $\langle 5, 5k+3, 5(2k)+2, 5(2k)+4\rangle$, *where* $\alpha = 5k+3$. *Then the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $2k+2$.

(6) *The subbranches of the numerical semigroup tree rooted at numerical semigroups in the form of* $S = \langle 5, 5k+4, 5(k+1)+1, 5(k+1)+2, 5(k+1)+3\rangle$ *have the following properties*:

(a) *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+4, 5(k+1)+1, 5(k+1)+2, 5(k+1)+3\rangle$, *where* $\alpha = 5k+4$ *and* $k_3 = k_2 - 1$. *Then the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $2k$.

(b) *Let* $S = \langle 5, \alpha, \beta, \gamma, \delta\rangle$ *be on the main branch under* $\langle 5, 5k+4, 5(k+1)+1, 5(k+1)+2, 5(k+1)+3\rangle$, *where* $\alpha = 5k+4$, *and* $k_2 = k_3$. *Then the branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $4k_4 - k_2 - k_1 + 2$, *where* k_1, k_2, k_3, k_4 *are the Apéry coordinates.*

(c) $S = \langle 5, 5k+4, 5(2k+1)+1, 5(2k+1)+2\rangle$ *is the last element of the main branch. Hence, the length of the main branch is* $3k$.

(d) *Let* $S = \langle 5, \alpha, \beta, \gamma\rangle$ *be on the main branch under* $\langle 5, 5k+4, 5(2k+1)+1, 5(2k+1)+2\rangle$, *where* $\alpha = 5k+4$. *Then the linear branch rooted at S generated by consecutively removing the second-largest generator will lose an effective generator at length* $2k+2$.

(7) *The linear branches formed by successively removing the largest generator from a numerical semigroup anywhere in the multiplicity* 5 *tree have the following properties*:

(a) *For* $S = \langle 5, 5a+1, 5b+2, 5c+3, 5d+4\rangle$, *with Apéry coordinates* (a, b, c, d) *and largest generator* $\delta \equiv 1 \pmod 5$, *the length of the branch generated by consecutively removing the largest generator is* $\min(b+d-a+1, 2c-a+1)$.

(b) *For* $S = \langle 5, 5a+1, 5b+2, 5c+3, 5d+4\rangle$, *with Apéry coordinates* (a, b, c, d) *and largest generator* $\delta \equiv 2 \pmod 5$, *the length of the branch generated by consecutively removing the largest generator is* $\min(2a-b, c+d-b+1)$.

(c) *For* $S = \langle 5, 5a+1, 5b+2, 5c+3, 5d+4\rangle$, *with Apéry coordinates* (a, b, c, d) *and largest generator* $\delta \equiv 3 \pmod 5$, *the length of the branch generated by consecutively removing the largest generator is* $\min(a+b-c, 2d-c+1)$.

(d) *For* $S = \langle 5, 5a+1, 5b+2, 5c+3, 5d+4\rangle$, *with Apéry coordinates* (a, b, c, d) *and largest generator* $\delta \equiv 4 \pmod 5$, *the length of the branch generated by consecutively removing the largest generator is* $\min(a+c-d, 2b-d)$.

Proof. Parts (1) and (2) are straightforward. The rest of the theorem is a direct consequence of Lemma 17–Lemma 25. □

4. Counting $N(4, g)$

Understanding the structure of the numerical semigroup tree allows us to count the number of semigroups by genus. In this section we provide an alternate proof for a known quasipolynomial expression for $N(4, g)$ when $g \geq 9$. Formulas for $N(4, g)$ and $N(5, g)$ are already known, but they were arrived at by computational methods. In Theorem 16 of [García-Sánchez et al. 2018], there is an equation for counting numerical semigroups of multiplicity $m = 4$ for genus greater than $g = 9$. A similar, much longer, formula for $N(5, g)$ is also shown in that work. Another noncomputational proof of the fact that $N(4, g)$ is nondecreasing was recently found by Eliahou and Fromentin [2019] using the related tool of gapset filtrations. Their approach is in some sense dual to considering the tree of numerical semigroups, but they establish their result without finding the quasipolynomial expression for $N(4, g)$.

Theorem 27 [García-Sánchez et al. 2018, Theorem 16]. *Let g be an integer greater than* 9. *Then*

$$\begin{aligned}
N(4, g) = {} & \left\lfloor \tfrac{g}{4} \right\rfloor^2 - \tfrac{3}{2}\left\lfloor \tfrac{g}{3} \right\rfloor^2 + \left\lfloor \tfrac{g+1}{5} \right\rfloor^2 - \tfrac{3}{2}\left\lfloor \tfrac{g+1}{3} \right\rfloor^2 - \tfrac{3}{2}\left\lfloor \tfrac{g+2}{6} \right\rfloor^2 - \tfrac{3}{2}\left\lfloor \tfrac{g+2}{5} \right\rfloor^2 \\
& + \tfrac{3}{2}\left\lfloor \tfrac{g+2}{4} \right\rfloor^2 - \tfrac{3}{2}\left\lfloor \tfrac{g+2}{3} \right\rfloor^2 - \tfrac{3}{2}\left\lfloor \tfrac{g+5}{6} \right\rfloor^2 + \tfrac{3}{2}\left\lfloor \tfrac{2g+1}{5} \right\rfloor^2 - \tfrac{3}{2}\left\lfloor \tfrac{2g+4}{5} \right\rfloor^2 \\
& + \left(g - \tfrac{3}{2}\right)\left\lfloor \tfrac{g}{3} \right\rfloor - \left\lfloor \tfrac{g}{5} \right\rfloor\left\lfloor \tfrac{2g}{5} \right\rfloor + \left\lfloor \tfrac{g}{4} \right\rfloor\left(1 - \left\lfloor \tfrac{g}{2} \right\rfloor\right) + \left(\tfrac{1}{4} + \left\lfloor \tfrac{g}{2} \right\rfloor\right)\left\lfloor \tfrac{g}{2} \right\rfloor \\
& + \left\lfloor \tfrac{g+1}{5} \right\rfloor + \left(g - \tfrac{1}{2}\right)\left\lfloor \tfrac{g+1}{3} \right\rfloor + \left(\left\lfloor \tfrac{g}{2} \right\rfloor + \tfrac{1}{2}\right)\left\lfloor \tfrac{g+2}{6} \right\rfloor + \left\lfloor \tfrac{2g}{5} \right\rfloor\left\lfloor \tfrac{g+2}{5} \right\rfloor \\
& - \left\lfloor \tfrac{g}{2} \right\rfloor\left\lfloor \tfrac{g+2}{4} \right\rfloor + \left(g + \tfrac{1}{2}\right)\left\lfloor \tfrac{g+2}{3} \right\rfloor + \left(-\left\lfloor \tfrac{g}{2} \right\rfloor + g + \tfrac{1}{2}\right)\left\lfloor \tfrac{g+5}{6} \right\rfloor \\
& + \left(\left\lfloor \tfrac{g}{5} \right\rfloor - g + 1\right)\left\lfloor \tfrac{2g+1}{5} \right\rfloor + \left(-\left\lfloor \tfrac{g+1}{5} \right\rfloor + g + 1\right)\left\lfloor \tfrac{2g+4}{5} \right\rfloor - \tfrac{5g^2}{8} - \tfrac{3g}{8}.
\end{aligned}$$

In [Alhajjar et al. 2019], the authors found the following quasipolynomial expression for $N(4, g)$, when $g \geq 9$, using observations about Kunz polytopes:

$$N(4, g) = \begin{cases} \frac{g^2}{12} + \frac{g}{2}, & g \equiv 0 \pmod 6, \\ \frac{g^2}{12} + \frac{g}{2} - \frac{7}{12}, & g \equiv 1, 5 \pmod 6, \\ \frac{g^2}{12} + \frac{g}{2} - \frac{1}{3}, & g \equiv 2, 4 \pmod 6, \\ \frac{g^2}{12} + \frac{g}{2} - \frac{1}{4}, & g \equiv 3 \pmod 6. \end{cases} \tag{3}$$

We demonstrate an alternate proof of this formula using the structure of the semigroup tree directly.

Theorem 28. $N(4,3) = 1$, *and for any* $g \geq 4$,

$$N(4,g) = \begin{cases} \frac{g^2}{12} + \frac{g}{2}, & g \equiv 0 \pmod 6, \\ \frac{g^2}{12} + \frac{g}{2} - \frac{7}{12}, & g \equiv 1, 5 \pmod 6, \\ \frac{g^2}{12} + \frac{g}{2} - \frac{1}{3}, & g \equiv 2, 4 \pmod 6, \\ \frac{g^2}{12} + \frac{g}{2} - \frac{1}{4}, & g \equiv 3 \pmod 6. \end{cases}$$

Proof. The only semigroup of multiplicity 4 and genus 3 is $\langle 4, 5, 6, 7\rangle$. In order to count the semigroups of multiplicity 4 by genus, we consider the structure of the tree of semigroups of multiplicity 4. The structure of our argument is to first count the number of semigroups on each subtree rooted on the principal branch then add these counting functions together. As we observed in Theorem 15, the behavior of the branch is determined by the congruence class modulo 4 of the second generator of the root. Throughout the proof we use many facts about the structure of the tree. All of these are immediate consequences of Theorem 15.

First consider the branch rooted at $\langle 4, 4k+2, 4k+3, 4(k+1)+1\rangle$. This branch is composed of length k linear subbranches descending from a main branch, with one new linear subbranch in each genus. Observe that the genus of the root is $3k+1$. The first linear subbranch terminates at genus $4k+1$. Until that point there is one new subbranch in each genus. From genus $4k+1$ onward, one new subbranch begins, and one subbranch ends. Thus the total number of semigroups is constant. The number of semigroups on this branch is

$$\#_{2,k}(g) = \begin{cases} \max\{0, g-3k\}, & g \leq 4k, \\ k+1, & g > 4k. \end{cases} \tag{4}$$

The total number of semigroups on any branch of this form is $\#_2(g) = \sum_{k=1}^{\infty} \#_{2,k}(g)$. To count this, we need only consider how many branches have stabilized at each genus and how many are still growing.

At genus g, $\lfloor \frac{g-1}{4} \rfloor$ branches have stabilized, yielding $2+3+4+\cdots+\left(\lfloor \frac{g-1}{4} \rfloor + 1\right)$, which sums to

$$\tfrac{1}{2}\left(\lfloor \tfrac{g-1}{4} \rfloor + 1\right)\left(\lfloor \tfrac{g-1}{4} \rfloor + 2\right) - 1.$$

The branches that have not yet stabilized begin at each genus $3k+1$, $k \geq 1$. So at some genus g, all of the growing branches have the same number of vertices modulo 3. Hence the vertices on the growing branches are either

$$1+4+7+\cdots+(3n+1), \quad 2+5+8+\cdots+(3n+2), \quad \text{or} \quad 3+6+9+\cdots+3n$$

in number. Branches stabilize in genus $4k+1$, with $k+1$ vertices. So the first time there are k vertices in a growing branch is at genus $4k$. Thus each sum becomes one term longer when the genus is increased by 12. Consider the case where the genus is congruent to 0 modulo 3. In this case, the sum is

$$3+6+\cdots+3\left\lfloor \tfrac{g}{12}\right\rfloor = \tfrac{1}{2}\left(3\left\lfloor \tfrac{g}{12}\right\rfloor+3\right)\left(\left\lfloor \tfrac{g}{12}\right\rfloor\right).$$

The other cases work similarly. Since every branch is either stabilized or growing, this accounts for every vertex at each genus. So we have the sum

$$\#_2(g) = \begin{cases} \frac{1}{2}\left(\left\lfloor \frac{g-1}{4}\right\rfloor+1\right)\left(\left\lfloor \frac{g-1}{4}\right\rfloor+2\right)-1+\left(3\left\lfloor \frac{g+8}{12}\right\rfloor-1\right)\frac{1}{2}\left\lfloor \frac{g+8}{12}\right\rfloor, & g\equiv 1 \pmod 3,\\ \frac{1}{2}\left(\left\lfloor \frac{g-1}{4}\right\rfloor+1\right)\left(\left\lfloor \frac{g-1}{4}\right\rfloor+2\right)-1+\left(3\left\lfloor \frac{g+4}{12}\right\rfloor+1\right)\frac{1}{2}\left\lfloor \frac{g+4}{12}\right\rfloor, & g\equiv 2 \pmod 3,\\ \frac{1}{2}\left(\left\lfloor \frac{g-1}{4}\right\rfloor+1\right)\left(\left\lfloor \frac{g-1}{4}\right\rfloor+2\right)-1+\left(3\left\lfloor \frac{g}{12}\right\rfloor+3\right)\frac{1}{2}\left\lfloor \frac{g}{12}\right\rfloor, & g\equiv 0 \pmod 3.\end{cases}$$

Next consider the branch rooted at $\langle 4, 4k+1, 4k+2, 4k+3\rangle$, which has genus $3k$. We can observe from the structure of this branch that the first leaves occur in genus $4k$, and the function counting the number of leaves in each genus is

$$\ell_{1,k}(g) = \begin{cases} 2, & 4k \le g \le 5k-2,\ g \text{ even},\\ 1, & 4k \le g \le 5k-2,\ g \text{ odd},\\ 1, & 5k-1 \le g \le 6k,\ g \text{ even},\\ 0, & \text{otherwise}.\end{cases}$$

From genus $3k$ to genus $4k$, there is one additional semigroup in each genus. From genus $4k$ to genus $5k-2$, there are two leaves in each even genus and one in each odd genus. So the number of semigroups in the branch of genus g decreases by 1 whenever g is even. From genus $5k-1$ to genus $6k$, there are no new subbranches, and there is a leaf in each even genus, so the number of semigroups decreases by 1 in each even genus. The largest genus of any semigroup on this branch is $6k$, the genus of $\langle 4, 4k+1\rangle$. Hence we can write the counting function for this branch:

$$\#_{1,k}(g) = \begin{cases} \max\{g+1-3k, 0\}, & g \le 4k,\\ \left\lfloor \frac{6k+2-g}{2}\right\rfloor, & 4k < g \le 6k,\\ 0, & \text{otherwise}.\end{cases}$$

We compute the sum of these functions to find the total number of semigroups on any branch of this form, $\#_1(g) = \sum_{k=1}^{\infty} \#_{1,k}(g)$. At any genus each branch is either growing, declining, or empty. In particular $\langle 4, 4k+1, 4k+2, 4k+3\rangle$ is growing for $3k \le g \le 4k$ and declining for $4k+1 \le g \le 6k$. Consider $g \equiv 0 \pmod 6$. At genus g, $\left\lfloor \frac{g}{12}\right\rfloor + 1$ branches are growing, yielding

$$1+4+7+\cdots+\left(3\left\lfloor \tfrac{g}{12}\right\rfloor+1\right),$$

which sums to

$$\tfrac{1}{2}\left(3\left\lfloor \tfrac{g}{12}\right\rfloor+2\right)\left(\left\lfloor \tfrac{g}{12}\right\rfloor+1\right).$$

There are also $\left\lfloor \frac{g-5}{12}\right\rfloor + 1$ declining branches at genus g, yielding

$$1+4+7+\cdots+\left(3\left\lfloor \tfrac{g-5}{12}\right\rfloor+1\right),$$

which sums to

$$\tfrac{1}{2}\big(3\lfloor\tfrac{g-5}{12}\rfloor+2\big)\big(\lfloor\tfrac{g-5}{12}\rfloor+1\big).$$

Since every branch is either empty, growing, or declining, the sum of these two values accounts for every vertex at this genus. The other cases are similar, yielding the counting function:

$$\#_1(g)=\begin{cases}\frac{1}{2}\big((3\lfloor\frac{g}{12}\rfloor+2)(\lfloor\frac{g}{12}\rfloor+1)+(3\lfloor\frac{g-5}{12}\rfloor+2)(\lfloor\frac{g-5}{12}\rfloor+1)\big), & g\equiv 0 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g-4}{12}\rfloor+4)(\lfloor\frac{g-4}{12}\rfloor+1)+(3\lfloor\frac{g}{12}\rfloor+3)(\lfloor\frac{g}{12}\rfloor)\big), & g\equiv 1 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g+4}{12}\rfloor+3)(\lfloor\frac{g+4}{12}\rfloor)+(3\lfloor\frac{g}{12}\rfloor+3)(\lfloor\frac{g}{12}\rfloor)\big), & g\equiv 2 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g}{12}\rfloor+2)(\lfloor\frac{g}{12}\rfloor+1)+(3\lfloor\frac{g-9}{12}\rfloor+4)(\lfloor\frac{g-9}{12}\rfloor+1)\big), & g\equiv 3 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g-4}{12}\rfloor+4)(\lfloor\frac{g-4}{12}\rfloor+1)+(3\lfloor\frac{g-9}{12}\rfloor+4)(\lfloor\frac{g-9}{12}\rfloor+1)\big), & g\equiv 4 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g+4}{12}\rfloor+3)(\lfloor\frac{g+4}{12}\rfloor)+(3\lfloor\frac{g-5}{12}\rfloor+2)(\lfloor\frac{g-5}{12}\rfloor+1)\big), & g\equiv 5 \pmod 6.\end{cases}$$

The branches with root $\langle 4, 4k+3, 4(k+1)+1, 4(k+1)+2\rangle$ behave similarly to the previous case. Repeating the argument, we obtain the following counting function, valid for $g \geq 4$:

$$\#_3(g)=\begin{cases}\frac{1}{2}\big((3\lfloor\frac{g-4}{12}\rfloor+4)(\lfloor\frac{g-4}{12}\rfloor+1)+(3\lfloor\frac{g-7}{12}\rfloor+4)(\lfloor\frac{g-7}{12}\rfloor+1)\big), & g\equiv 0 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g+4}{12}\rfloor+3)(\lfloor\frac{g+4}{12}\rfloor)+(3\lfloor\frac{g-7}{12}\rfloor+4)(\lfloor\frac{g-7}{12}\rfloor+1)\big), & g\equiv 1 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g}{12}\rfloor+2)(\lfloor\frac{g}{12}\rfloor+1)+(3\lfloor\frac{g-3}{12}\rfloor+2)(\lfloor\frac{g-3}{12}\rfloor+1)\big), & g\equiv 2 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g-4}{12}\rfloor+4)(\lfloor\frac{g-4}{12}\rfloor+1)+(3\lfloor\frac{g-3}{12}\rfloor+2)(\lfloor\frac{g-3}{12}\rfloor+1)\big), & g\equiv 3 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g+4}{12}\rfloor+3)(\lfloor\frac{g+4}{12}\rfloor)+(3\lfloor\frac{g+1}{12}\rfloor+3)(\lfloor\frac{g+1}{12}\rfloor)\big), & g\equiv 4 \pmod 6,\\ \frac{1}{2}\big((3\lfloor\frac{g}{12}\rfloor+2)(\lfloor\frac{g}{12}\rfloor+1)+(3\lfloor\frac{g+1}{12}\rfloor+3)(\lfloor\frac{g+1}{12}\rfloor)\big), & g\equiv 5 \pmod 6\end{cases}$$

Finally we sum these functions. $N(4, g) = \#_1(g) + \#_2(g) + \#_3(g)$. By working modulo 12, the least common multiple of the denominators, we can eliminate the floor functions. Then a simple computation reduces the function to

$$N(4,g)=\begin{cases}\frac{g^2}{12}+\frac{g}{2}, & g\equiv 0 \pmod 6,\\ \frac{g^2}{12}+\frac{g}{2}-\frac{7}{12}, & g\equiv 1,5 \pmod 6,\\ \frac{g^2}{12}+\frac{g}{2}-\frac{1}{3}, & g\equiv 2,4 \pmod 6,\\ \frac{g^2}{12}+\frac{g}{2}-\frac{1}{4}, & g\equiv 3 \pmod 6.\end{cases}$$ □

Theorem 28 uses the structure of the numerical semigroup tree in multiplicity 4 to find a concise quasipolynomial expression for $N(4, g)$. We suspect that a similar approach could yield a relatively concise quasipolynomial expression for $N(5, g)$, but the computations are much more complicated.

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Received: 2019-08-06 Revised: 2019-12-16 Accepted: 2020-01-15

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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2020 vol. 13 no. 2

1944-4176(2020)13:2;1-S