

involve

a journal of mathematics

On arithmetical structures on complete graphs

Zachary Harris and Joel Louwsma



On arithmetical structures on complete graphs

Zachary Harris and Joel Louwsma

(Communicated by Joshua Cooper)

An arithmetical structure on the complete graph K_n with n vertices is given by a collection of n positive integers with no common factor, each of which divides their sum. We show that, for all positive integers c less than a certain bound depending on n , there is an arithmetical structure on K_n with largest value c . We also show that, if each prime factor of c is greater than $(n+1)^2/4$, there is no arithmetical structure on K_n with largest value c . We apply these results to study which prime numbers can occur as the largest value of an arithmetical structure on K_n .

1. Introduction

How can one have a collection of positive integers, with no common factor, each of which divides their sum? For example, 105, 70, 15, 14, and 6 sum to 210, which is divisible by each of these numbers. Introducing notation, we seek positive integers r_1, r_2, \dots, r_n with no common factor such that

$$r_j \mid \sum_{i=1}^n r_i \quad \text{for all } j. \quad (1)$$

It is well known that finding such r_i is equivalent to finding positive integer solutions of the Diophantine equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = 1. \quad (2)$$

Indeed, given r_1, r_2, \dots, r_n satisfying (1), dividing both sides of the equation $r_1 + r_2 + \dots + r_n = \sum_{i=1}^n r_i$ by $\sum_{i=1}^n r_i$ gives a solution to (2), and, given a solution of (2), the numbers $\text{lcm}(x_1, x_2, \dots, x_n)/x_i$ satisfy (1) and have no common factor.

Our interest in this question stems from an interest in arithmetical structures. An *arithmetical structure* on a finite, connected graph is an assignment of positive integers to the vertices such that:

MSC2010: primary 11D68; secondary 05C50, 11A41.

Keywords: arithmetical structure, complete graph, Diophantine equation, Laplacian matrix, prime number.

- (a) At each vertex, the integer there is a divisor of the sum of the integers at adjacent vertices (counted with multiplicity if the graph is not simple).
- (b) The integers used have no nontrivial common factor.

Arithmetical structures were introduced in [Lorenzini 1989] to study intersections of degenerating curves in algebraic geometry. The usual definition, easily seen to be equivalent to the one given here, is formulated in terms of matrices. From that perspective, an arithmetical structure may be regarded as a generalization of the *Laplacian matrix*, which encodes many important properties of a graph. Notions in this direction that have received a significant amount of attention include the sandpile group and the chip-firing game; for details, see [Corry and Perkinson 2018; Klivans 2019; Glass and Kaplan 2019].

On the *complete graph* K_n with n vertices, positive integers r_1, r_2, \dots, r_n with no common factor give an arithmetical structure if and only if

$$r_j \mid \sum_{\substack{i=1 \\ i \neq j}}^n r_i \quad \text{for all } j;$$

it is immediate that this condition is equivalent to (1). Therefore, in this language, the opening question of this paper seeks arithmetical structures on complete graphs.

Lemma 1.6 of [Lorenzini 1989] shows that there are finitely many arithmetical structures on any finite, connected graph, but this result does not give a bound on the number of structures. Several recent papers [Braun et al. 2018; Archer et al. 2020; Glass and Wagner 2019] count arithmetical structures on various families of graphs, including path graphs, cycle graphs, bidents, and certain path graphs with doubled edges. However, counting arithmetical structures on complete graphs is a difficult problem. The number of arithmetical structures on K_n for $n \leq 8$ is given in [Sloane 1991]. For general n , bounds have been obtained in [Erdős and Graham 1980; Sándor 2003; Browning and Elsholtz 2011; Konyagin 2014], which work from the perspective of the Diophantine equation (2). Other papers such as [Burshtein 2007; 2008; Arce-Nazario et al. 2013] determine, for specific n , the number of solutions of (2) satisfying certain additional conditions on the x_i .

It is conjectured in [Corrales and Valencia 2018, Conjecture 6.10] that, for any connected, simple graph G with n vertices, the number of arithmetical structures on G is at most the number of arithmetical structures on K_n . To approach this conjecture, one would like a better understanding of the types of arithmetical structures that occur on complete graphs. In this direction, this paper studies which positive integers can occur as the largest value of an arithmetical structure on K_n . Clearly the r_i of an arithmetical structure can be permuted; in the following we make the assumption $r_1 \geq r_2 \geq \dots \geq r_n$. We construct arithmetical structures to show that r_1 can take certain values and give obstructions to show that it cannot take other values.

Our primary construction theorem ([Theorem 1](#)) shows that r_1 can take any value up to a certain bound depending on n . More specifically, r_1 can be any positive integer less than or equal to $\max_{k \in \mathbb{Z}_{>0}} (2^k n - (k + 2^k - 2)2^k - 1)$. This bound improves somewhat if we restrict attention to prime numbers; r_1 can be any prime number less than or equal to $\max_{k \in \mathbb{Z}_{>0}} (2^k n - (k + 2^k - 3)2^k - 3)$.

We also prove an obstruction theorem ([Theorem 7](#)) that shows r_1 cannot take any value all of whose prime factors are greater than $(n+1)^2/4$. Restricting attention to prime numbers, this bound improves to show that r_1 cannot be any prime number greater than $n^2/4 + 1$ ([Theorem 8](#)).

The final section focuses on the possible prime values r_1 can take. We explicitly check prime numbers in the gap between the bound of [Theorem 1](#) and the bound of [Theorem 8](#), showing that r_1 can take some of these values but not others. In particular, we observe that there can be prime numbers p_1 and p_2 with $p_1 < p_2$ such that there is an arithmetical structure on K_n with $r_1 = p_2$ but no arithmetical structure on K_n with $r_1 = p_1$.

2. Construction

In this section, we show how to construct arithmetical structures on complete graphs with certain values of r_1 . Our main construction theorem is the following.

Theorem 1. (a) *For any positive integer $c \leq \max_{k \in \mathbb{Z}_{>0}} (2^k n - (k + 2^k - 2)2^k - 1)$, there is an arithmetical structure on K_n with $r_1 = c$.*
 (b) *For any prime number $p \leq \max_{k \in \mathbb{Z}_{>0}} (2^k n - (k + 2^k - 3)2^k - 3)$, there is an arithmetical structure on K_n with $r_1 = p$.*

We establish [Propositions 2, 4, and 5](#) on the way to proving [Theorem 1](#).

Proposition 2. *For any positive integer $c \leq n - 1$, there is an arithmetical structure on K_n with $r_1 = c$.*

Proof. Let

$$r_i = \begin{cases} c & \text{for } i \in \{1, 2, \dots, n - c\}, \\ 1 & \text{for } i \in \{n - c + 1, n - c + 2, \dots, n\}. \end{cases}$$

Then

$$\sum_{i=1}^n r_i = c(n - c) + c = c(n - c + 1).$$

Since this is divisible by both c and 1, we have thus produced an arithmetical structure on K_n . \square

Before turning to [Propositions 4 and 5](#), we establish the following lemma.

Lemma 3. (a) *Let $k \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$ with $k \leq \ell$. Every integer c satisfying $\ell \leq c \leq (\ell - k + 1)2^k - 1$ can be expressed as $\sum_{j=1}^{\ell} 2^{k_j}$ for some $k_j \in \{0, 1, \dots, k\}$, where $k_j = 0$ for some $j \in \{1, 2, \dots, \ell\}$.*

(b) Let $k \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$ with $k \leq \ell$. Every odd integer c satisfying $\ell \leq c \leq (\ell - k + 2)2^k - 3$ can be expressed as $\sum_{j=1}^{\ell} 2^{k_j}$ for some $k_j \in \{0, 1, \dots, k\}$, where $k_j = 0$ for some $j \in \{1, 2, \dots, \ell\}$.

Proof. To show (a), we proceed by induction on c . In the base case, $c = \ell$, we can let $k_j = 0$ for all j and have $c = \sum_{j=1}^{\ell} 2^{k_j}$. Now suppose $c = \sum_{j=1}^{\ell} 2^{k_j}$ for $c \leq (\ell - k + 1)2^k - 2$. Then $k_j = k$ for at most $\ell - k$ values of j , meaning $k_j < k$ for at least k values of j . If among these values we had each of $0, 1, \dots, k - 1$ only once, we would then have $\sum_{j=1}^{\ell} 2^{k_j} = (\ell - k + 1)2^k - 1 > (\ell - k + 1)2^k - 2$. Therefore there is some $b < k$ for which $k_{j_1} = b = k_{j_2}$ for some $j_1 \neq j_2$. Define

$$k'_j = \begin{cases} k_j + 1 & \text{for } j = j_1, \\ 0 & \text{for } j = j_2, \\ k_j & \text{otherwise.} \end{cases}$$

Then $\sum_{j=1}^{\ell} 2^{k'_j} = \sum_{j=1}^{\ell} 2^{k_j} + 1 = c + 1$. The result follows.

For (b), first note that if $c \leq (\ell - k + 1)2^k - 1$ then the result follows from (a). Therefore, assume $c > (\ell - k + 1)2^k - 1$ and let $c' = c - ((\ell - k + 1)2^k - 1)$, noting that $c' \leq 2^k - 2$. Since c is odd, c' must be even. Therefore c' can be written in the form $\sum_{j=1}^{k-1} s_j 2^j$, where each s_j is either 0 or 1; the s_j are iteratively determined in reverse by letting $s_j = 1$ if $c' - \sum_{i=j+1}^{k-1} s_i 2^i \geq 2^j$ and letting $s_j = 0$ otherwise. Define

$$k_j = \begin{cases} 0 & \text{for } j = 1, \\ j - 1 + s_{j-1} & \text{for } j \in \{2, 3, \dots, k\}, \\ k & \text{for } j \in \{k+1, k+2, \dots, \ell\}. \end{cases}$$

Then

$$\sum_{j=1}^{\ell} 2^{k_j} = \sum_{j=1}^k 2^{j-1} + \sum_{j=2}^k s_{j-1} 2^{j-1} + \sum_{j=k+1}^{\ell} 2^k = 2^k - 1 + c' + (\ell - k)2^k = c. \quad \square$$

We use [Lemma 3](#) to prove [Proposition 4](#) and [5](#).

Proposition 4. Fix $n \geq 2$. For any positive integer k satisfying $k + 2^k - 1 \leq n$ and any positive integer c satisfying $n - 2^k + 1 \leq c \leq (n - k - 2^k + 2)2^k - 1$, there is an arithmetical structure on K_n with $r_1 = c$.

Proof. Let $r_i = c$ for all $i \in \{1, 2, \dots, 2^k - 1\}$. Let $\ell = n - 2^k + 1$, noting that our assumptions guarantee that $k \leq \ell$ and $\ell \leq c \leq (\ell - k + 1)2^k - 1$. [Lemma 3\(a\)](#) then shows how to write $c = \sum_{j=1}^{\ell} 2^{k_j}$. We use the values 2^{k_j} , in decreasing order, to define r_i for $i \in \{2^k, 2^k + 1, \dots, n\}$, noting that $r_n = 1$. Then

$$\sum_{i=1}^n r_i = (2^k - 1)c + c = 2^k c.$$

Since $2^k c$ is divisible by c and by $2^{k'}$ for all $k' \in \{0, 1, \dots, k\}$, we have thus produced an arithmetical structure on K_n . \square

Although we imposed the condition $k + 2^k - 1 \leq n$ here to ensure that $k \leq \ell$ in the proof, this does not restrict possible values of r_1 , as we show in the proof of [Theorem 1](#). Together with [Proposition 2](#), [Proposition 4](#) with $k = 1$ allows us to construct arithmetical structures on K_n with r_1 taking any value up to $2n - 3$. When $k = 2$, the bound is $4n - 17$; when $k = 3$, the bound is $8n - 73$; and when $k = 4$, the bound is $16n - 289$. If we restrict attention to prime r_1 , these bounds can be improved slightly, as the following proposition shows.

Proposition 5. *Fix $n \geq 2$. For any positive integer k satisfying $k + 2^k - 1 \leq n$ and any prime number p satisfying $n - 2^k + 1 \leq p \leq (n - k - 2^k + 3)2^k - 3$, there is an arithmetical structure on K_n with $r_1 = p$.*

Proof. If $p = 2$ (and $n \geq 3$), [Proposition 2](#) gives an arithmetical structure on K_n with $r_1 = p$. Therefore suppose p is odd. Let $r_i = p$ for all $i \in \{1, 2, \dots, 2^k - 1\}$. Let $\ell = n - 2^k + 1$, noting that our assumptions guarantee $k \leq \ell$ and $\ell \leq p \leq (\ell - k + 2)2^k - 3$. [Lemma 3\(b\)](#) then shows how to write $p = \sum_{j=1}^{\ell} 2^{k_j}$. As in the proof of [Proposition 4](#), we use the values 2^{k_j} , in decreasing order, to define r_i for $i \in \{2^k, 2^k + 1, \dots, n\}$, noting that $r_n = 1$. Then

$$\sum_{i=1}^n r_i = (2^k - 1)p + p = 2^k p,$$

which is divisible by p and by $2^{k'}$ for all $k' \in \{0, 1, \dots, k\}$. Therefore we have produced an arithmetical structure on K_n . \square

For example, when $k = 1$, [Proposition 5](#) allows us to construct arithmetical structures on K_n with r_1 taking prime values as large as $2n - 3$. When $k = 2$, the bound is $4n - 15$; when $k = 3$, the bound is $8n - 67$; and when $k = 4$, the bound is $16n - 275$.

We are now prepared to complete the proof of [Theorem 1](#).

Proof of Theorem 1. The necessary constructions are given in [Propositions 2, 4](#), and [5](#). It remains only to show that, for each n , values of k that maximize the upper bounds in [Propositions 4](#) and [5](#) satisfy $k + 2^k - 1 \leq n$.

The upper bound $(n - k - 2^k + 2)2^k - 1$ in [Proposition 4](#) is linear in n with slope 2^k . A straightforward calculation shows that the bound with slope 2^{k-1} coincides with the bound with slope 2^k when $n = k + 3 \cdot 2^{k-1} - 1$ and that the bound with slope 2^k coincides with the bound with slope 2^{k+1} when $n = k + 3 \cdot 2^k$. Therefore the bound with slope 2^k is maximal exactly when n is between $k + 3 \cdot 2^{k-1} - 1$ and $k + 3 \cdot 2^k$. When the bound is maximized, we therefore have that $n \geq k + 3 \cdot 2^{k-1} - 1 \geq k + 2^k - 1$, meaning the condition of [Proposition 4](#) is satisfied. This proves (a).

The argument for (b) is very similar. The upper bound $(n - k - 2^k + 3)2^k - 3$ in [Proposition 5](#) is maximal for n between $k + 3 \cdot 2^{k-1} - 2$ and $k + 3 \cdot 2^k - 1$. When the bound is maximized, we then have that $n \geq k + 3 \cdot 2^{k-1} - 2 \geq k + 2^k - 1$, meaning the condition of [Proposition 5](#) is satisfied. \square

We conclude this section by giving another construction that allows us to produce some arithmetical structures with values of r_1 other than those guaranteed by [Theorem 1](#).

Proposition 6. *For any positive integer $k \leq n - 1$, there is an arithmetical structure on K_n with $r_1 = k(n - k) + 1$.*

Proof. Let

$$r_i = \begin{cases} k(n - k) + 1 & \text{for } i \in \{1, 2, \dots, k - 1\}, \\ k & \text{for } i \in \{k, k + 1, \dots, n - 1\}, \\ 1 & \text{for } i = n. \end{cases}$$

Then

$$\sum_{i=1}^n r_i = (k - 1)(k(n - k) + 1) + k(n - k) + 1 = k(k(n - k) + 1).$$

Since this is divisible by $k(n - k) + 1$, k , and 1, we have thus produced an arithmetical structure on K_n . \square

For example, when $n = 13$, [Theorem 1](#) guarantees that we can find an arithmetical structure on K_n with $r_1 = p$ for all prime $p \leq 37$. By taking $k = 5$ in [Proposition 6](#), we can also produce an arithmetical structure with $r_1 = 41$. By taking $k = 6$, we can produce an arithmetical structure with $r_1 = 43$. The results of this section cannot be extended too much further, as we show in the following section.

3. Obstruction

We next prove obstruction results that complement our constructions in the previous section. Our first result shows that r_1 cannot be a product of primes all of which are too large.

Theorem 7. *Suppose $c \geq 2$ is an integer with prime factorization $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 < p_2 < \cdots < p_k$ and $a_i \geq 1$ for all i . If $p_1 > (n + 1)^2/4$, then there is no arithmetical structure on K_n with $r_1 = c$.*

Proof. Suppose we have an arithmetical structure on K_n with $r_1 = c$. Knowing that $r_1 \mid \sum_{i=1}^n r_i$, we define $b = \sum_{i=1}^n r_i / r_1$. Then $\sum_{i=1}^n r_i = bc$, meaning that $r_i \mid bp_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for all i . Let m be the largest value of i for which $r_i = c$. For all $i \in \{m + 1, m + 2, \dots, n\}$, we have $r_i < c$, which implies that $r_i \leq bp_1^{a_1-1} p_2^{a_2} \cdots p_k^{a_k}$.

This means $\sum_{i=m+1}^n r_i \leq (n-m)bp_1^{a_1-1}p_2^{a_2} \cdots p_k^{a_k}$. We also have that $\sum_{i=m+1}^n r_i = (b-m)p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$. Therefore

$$(b-m)p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k} \leq (n-m)bp_1^{a_1-1}p_2^{a_2} \cdots p_k^{a_k},$$

and hence $(b-m)p_1 \leq (n-m)b$. When $b = m$, there is only one arithmetical structure on K_n , namely that with $r_i = 1$ for all i , so the desired structure cannot arise in this case. Therefore we assume $b > m$, in which case we have

$$p_1 \leq \frac{(n-m)b}{b-m}.$$

When $b = m + 1$, this gives $p_1 \leq (n-b+1)b$. It is a simple calculus exercise to show this bound is maximized when $b = (n+1)/2$. It follows that $p_1 \leq (n+1)^2/4$.

When $b \geq m + 2$, we have

$$p_1 \leq \frac{(n-m)b}{b-m} = \frac{nb - mb + nm - nm}{b-m} = n + \frac{m(n-b)}{b-m} \leq n + \frac{m(n-m-2)}{2}.$$

It is a simple calculus exercise to show this bound is maximized when $m = n/2 - 1$, so therefore

$$p_1 \leq n + \frac{(n/2-1)(n/2-1)}{2} = \frac{n^2}{8} + \frac{n}{2} + \frac{1}{2} = \frac{(n+1)^2}{4} - \frac{n^2-1}{4} \leq \frac{(n+1)^2}{4}.$$

The result follows. \square

If we restrict attention to arithmetical structures where r_1 is a prime number, then [Theorem 7](#) can be improved to [Theorem 8](#). The general outline of the proof is similar, with some of the bounds improved.

Theorem 8. *If p is a prime number with $p > n^2/4 + 1$, then there is no arithmetical structure on K_n with $r_1 = p$.*

Proof. If $p = 2$, the hypothesis of the theorem is only satisfied for $n = 1$, and there is no arithmetical structure on K_1 with $r_1 = 2$. Suppose we have an arithmetical structure on K_n with $r_1 = p$, where $p \geq 3$. Knowing that $r_1 \mid \sum_{i=1}^n r_i$, we define $b = \sum_{i=1}^n r_i / r_1$. Then $\sum_{i=1}^n r_i = bp$, meaning that $r_i \mid bp$ for all i . Let m be the largest value of i for which $r_i = p$. We can only have $b = m$ if $r_i = 1$ for all i , but then r_1 is not prime. We consider two cases: when $b = m + 1$ and when $b \geq m + 2$.

Case I: $b = m + 1$. For all $i \in \{m+1, m+2, \dots, n\}$, we have that $r_i \mid bp$ and $r_i < p$, so therefore $r_i \mid b$. Whenever $r_i < b$, this means $r_i \leq b/2$. If $r_{n-1}, r_n < b$, we would then have that $\sum_{i=m+1}^n r_i \leq (n-m-1)b$. If instead $r_i = b$ for all $i \in \{m+1, m+2, \dots, n-1\}$, we would have that $r_n \mid r_i$ for all $i \in \{m+1, m+2, \dots, n\}$. Since $\sum_{i=m+1}^n r_i = (b-m)p = p$, this would also mean $r_n \mid p$, and hence that $r_n \mid r_i$ for all i . Therefore we would need to have $r_n = 1$, meaning that $\sum_{i=m+1}^n r_i \leq$

$(n - m - 1)b + 1$. Regardless of the value of r_{n-1} , we thus have

$$p = \sum_{i=m+1}^n r_i \leq (n - m - 1)b + 1 = (n - b)b + 1 = nb - b^2 + 1.$$

It is a simple calculus exercise to show that this bound is maximized when $b = n/2$. Hence we have that $p \leq n^2/4 + 1$.

Case II: $b \geq m + 2$. We have that $r_i \leq b$ for all $i \in \{m + 1, m + 2, \dots, n\}$ and $\sum_{i=m+1}^n r_i = (b - m)p$, so therefore $(b - m)p \leq (n - m)b$. As in the proof of [Theorem 7](#), this yields

$$p \leq \frac{(n - m)b}{b - m} \leq n + \frac{m(n - m - 2)}{2},$$

and this bound is maximized when $m = n/2 - 1$. Therefore

$$p \leq n + \frac{(n/2 - 1)(n/2 - 1)}{2} = \frac{n^2}{8} + \frac{n}{2} + \frac{1}{2} = \frac{n^2}{4} + 1 - \frac{(n - 2)^2}{8} < \frac{n^2}{4} + 1.$$

We have thus shown that in all cases we must have $p \leq n^2/4 + 1$. \square

For even n , we can choose $k = n/2$ in [Proposition 6](#) and get an arithmetical structure on K_n with $r_1 = n^2/4 + 1$. For odd n , we can choose $k = (n - 1)/2$ and get an arithmetical structure on K_n with $r_1 = (n^2 - 1)/4 + 1$. As some of these values of r_1 are prime, the bound in [Theorem 8](#) therefore cannot be improved.

There are arithmetical structures for which r_1 takes composite values larger than the bound given in [Theorem 8](#). For instance, the example in the opening paragraph of this paper gives an arithmetical structure on K_5 with $r_1 = 105$.

4. Prime r_1

This section considers the possible prime values r_1 can take in an arithmetical structure on K_n . [Theorem 1\(b\)](#) guarantees that r_1 can take any prime value up to $2^k n - (k + 2^k - 3)2^k - 3$ for any k . [Theorem 8](#) says that r_1 cannot take any prime value larger than $n^2/4 + 1$. These bounds are not too far from each other. The function $n^2/4 + 1$ has linear approximations of the form $2^k n - 2^{2k} + 1$. When k is 1 or 2, this linear approximation coincides with the bound from [Theorem 1\(b\)](#). In general, it differs from this bound by $(k - 3)2^k + 4$.

[Proposition 6](#) shows that r_1 can take some of the prime values in the gap between the bound of [Theorem 1\(b\)](#) and the bound of [Theorem 8](#). We can check by hand whether it can take other prime values; to illustrate how to do this, we explain why there is no arithmetical structure on K_{18} with $r_1 = 79$. Suppose there were such a structure, and let $b = \sum_{i=1}^{18} r_i / r_1$, so that $\sum_{i=1}^{18} r_i = 79b$. Let m be the largest value of i for which $r_i = 79$. Then $\sum_{i=m+1}^{18} r_i = 79(b - m)$. For all $i \in \{m + 1, m + 2, \dots, 18\}$, we have that $r_i \mid 79b$ and $r_i < 79$. Therefore $r_i \mid b$,

n	yes, Theorem 1(b)	no, Theorem 8	yes, Proposition 6	yes, other	no, other
3	$p \leq 3$	$p > 3.25$			
4	$p \leq 5$	$p > 5$			
5	$p \leq 7$	$p > 7.25$			
6	$p \leq 9$	$p > 10$			
7	$p \leq 13$	$p > 13.25$			
8	$p \leq 17$	$p > 17$			
9	$p \leq 21$	$p > 21.25$			
10	$p \leq 25$	$p > 26$			
11	$p \leq 29$	$p > 31.25$	31		
12	$p \leq 33$	$p > 37$	37		
13	$p \leq 37$	$p > 43.25$	41, 43		
14	$p \leq 45$	$p > 50$		47	
15	$p \leq 53$	$p > 57.25$	57		
16	$p \leq 61$	$p > 65$			
17	$p \leq 69$	$p > 73.25$	71, 73		
18	$p \leq 77$	$p > 82$			79
19	$p \leq 85$	$p > 91.25$	89		
20	$p \leq 93$	$p > 101$	97, 101		
21	$p \leq 101$	$p > 111.25$	109		103, 107
22	$p \leq 109$	$p > 122$	113		
23	$p \leq 117$	$p > 133.25$	127, 131		
24	$p \leq 125$	$p > 145$		127, 131, 137, 139	
25	$p \leq 133$	$p > 157.25$	137, 151, 157	139, 149	
26	$p \leq 141$	$p > 170$		149, 151, 157, 163	167
27	$p \leq 149$	$p > 183.25$	163, 181	151, 157, 167, 173	179

Table 1. Possible prime r_1 in arithmetical structures on K_n for $n \leq 27$.

and hence $r_i \leq b$. This means that $\sum_{i=m+1}^{18} r_i \leq (18-m)b$, so we must have $79(b-m) \leq (18-m)b$. If $b \geq m+2$, we would have

$$61b - 79m + mb \geq 61(m+2) - 79m + m(m+2) = (m-8)^2 + 58 > 0,$$

which would imply that $79(b-m) > (18-m)b$. Therefore we cannot have $b \geq m+2$. Since $b = m$ is only possible if $r_1 = 1$, it therefore remains to consider whether we can have $b = m+1$. In this case, we would have $\sum_{i=m+1}^{18} r_i \leq (18-m)(m+1)$. This bound is less than 79 except when m satisfies $6 \leq m \leq 11$. If $m = 6$, we would need to have 12 divisors of 7 that sum to 79, but this is not possible. If $m = 7$, we would need to have 11 divisors of 8 that sum to 79, but this is not possible. If $m = 8$, we would need to have 10 divisors of 9 that sum to 79, but this is not

possible. If $m = 9$, we would need to have 9 divisors of 10 that sum to 79, but this is not possible. If $m = 10$, we would need to have 8 divisors of 11 that sum to 79, but this is not possible. If $m = 11$, we would need to have 7 divisors of 12 that sum to 79, but this is not possible. Therefore there is no arithmetical structure on K_{18} with $r_1 = 79$. A similar approach can be used to either find arithmetical structures with other prime values of r_1 or to show that they do not exist. We have done this for all $n \leq 27$; the results are shown in Table 1.

We conclude by noting that, on K_{27} , there is no arithmetical structure with $r_1 = 179$, whereas there is an arithmetical structure with $r_1 = 181$. This shows that there is not a cutoff function $f(n)$ such that, for each n , there is an arithmetical structure on K_n with $r_1 = p$ for all primes $p \leq f(n)$ and no such structure for any $p > f(n)$. Therefore, while one could attempt to improve the bound of Theorem 1(b), the possible prime values of r_1 cannot be fully explained by a result of this form.

Acknowledgments

We would like to thank Nathan Kaplan for a helpful conversation. Harris was supported by a Niagara University Undergraduate Student Summer Support Grant.

References

- [Arce-Nazario et al. 2013] R. Arce-Nazario, F. Castro, and R. Figueroa, “On the number of solutions of $\sum_{i=1}^{11} 1/x_i = 1$ in distinct odd natural numbers”, *J. Number Theory* **133**:6 (2013), 2036–2046. [MR](#) [Zbl](#)
- [Archer et al. 2020] K. Archer, A. C. Bishop, A. Diaz-Lopez, L. D. García Puente, D. Glass, and J. Louwsma, “Arithmetical structures on bidents”, *Discrete Math.* **343**:7 (2020), 111850. [MR](#)
- [Braun et al. 2018] B. Braun, H. Corrales, S. Corry, L. D. García Puente, D. Glass, N. Kaplan, J. L. Martin, G. Musiker, and C. E. Valencia, “Counting arithmetical structures on paths and cycles”, *Discrete Math.* **341**:10 (2018), 2949–2963. [MR](#) [Zbl](#)
- [Browning and Elsholtz 2011] T. D. Browning and C. Elsholtz, “The number of representations of rationals as a sum of unit fractions”, *Illinois J. Math.* **55**:2 (2011), 685–696. [MR](#) [Zbl](#)
- [Burshtein 2007] N. Burshtein, “The equation $\sum_{i=1}^9 1/x_i = 1$ in distinct odd integers has only the five known solutions”, *J. Number Theory* **127**:1 (2007), 136–144. [MR](#) [Zbl](#)
- [Burshtein 2008] N. Burshtein, “All the solutions of the equation $\sum_{i=1}^{11} 1/x_i = 1$ in distinct integers of the form $x_i \in 3^\alpha 5^\beta 7^\gamma$ ”, *Discrete Math.* **308**:18 (2008), 4286–4292. [MR](#) [Zbl](#)
- [Corrales and Valencia 2018] H. Corrales and C. E. Valencia, “Arithmetical structures on graphs”, *Linear Algebra Appl.* **536** (2018), 120–151. [MR](#) [Zbl](#)
- [Corry and Perkinson 2018] S. Corry and D. Perkinson, *Divisors and sandpiles: an introduction to chip-firing*, American Mathematical Society, Providence, RI, 2018. [MR](#) [Zbl](#)
- [Erdős and Graham 1980] P. Erdős and R. L. Graham, *Old and new problems and results in combinatorial number theory*, Monographies de L’Enseignement Mathématique **28**, Université de Genève, L’Enseignement Mathématique, Geneva, 1980. [MR](#) [Zbl](#)
- [Glass and Kaplan 2019] D. Glass and N. Kaplan, “Chip-firing games and critical groups”, preprint, 2019. [arXiv](#)

[Glass and Wagner 2019] D. Glass and J. Wagner, “Arithmetical structures on paths with a doubled edge”, preprint, 2019. [arXiv](https://arxiv.org/abs/1908.03001)

[Klivans 2019] C. J. Klivans, *The mathematics of chip-firing*, CRC Press, Boca Raton, FL, 2019. [MR](#) [Zbl](#)

[Konyagin 2014] S. V. Konyagin, “Double exponential lower bound for the number of representations of unity by Egyptian fractions”, *Mat. Zametki* **95**:2 (2014), 312–316. In Russian; translated in *Math. Notes* **95**:1-2 (2014), 277–281. [MR](#) [Zbl](#)

[Lorenzini 1989] D. J. Lorenzini, “Arithmetical graphs”, *Math. Ann.* **285**:3 (1989), 481–501. [MR](#) [Zbl](#)

[Sándor 2003] C. Sándor, “On the number of solutions of the Diophantine equation $\sum_{i=1}^n 1/x_i = 1$ ”, *Period. Math. Hungar.* **47**:1-2 (2003), 215–219. [MR](#) [Zbl](#)

[Sloane 1991] N. J. A. Sloane, “Egyptian fractions: number of solutions of $1 = 1/x_1 + \dots + 1/x_n$ in positive integers”, entry A002967 in *The On-Line Encyclopedia of Integer Sequences* (<http://oeis.org>), 1991.

Received: 2019-09-15 Revised: 2020-01-01 Accepted: 2020-01-06

zharris@mail.niagara.edu *Department of Mathematics, Niagara University,
Niagara University, NY, United States*

jlouwsma@niagara.edu *Department of Mathematics, Niagara University,
Niagara University, NY, United States*

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

MANAGING EDITOR

Kenneth S. Berenhaut Wake Forest University, USA

BOARD OF EDITORS

Colin Adams	Williams College, USA	Robert B. Lund	Clemson University, USA
Arthur T. Benjamin	Harvey Mudd College, USA	Gaven J. Martin	Massey University, New Zealand
Martin Bohner	Missouri U of Science and Technology, USA	Mary Meyer	Colorado State University, USA
Amarjit S. Budhiraja	U of N Carolina, Chapel Hill, USA	Frank Morgan	Williams College, USA
Pietro Cerone	La Trobe University, Australia	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran
Scott Chapman	Sam Houston State University, USA	Zuhair Nashed	University of Central Florida, USA
Joshua N. Cooper	University of South Carolina, USA	Ken Ono	Univ. of Virginia, Charlottesville
Jem N. Corcoran	University of Colorado, USA	Yuval Peres	Microsoft Research, USA
Toka Diagana	University of Alabama in Huntsville, USA	Y.-F. S. Pétermann	Université de Genève, Switzerland
Michael Dorff	Brigham Young University, USA	Jonathon Peterson	Purdue University, USA
Sever S. Dragomir	Victoria University, Australia	Robert J. Plemmons	Wake Forest University, USA
Joel Foisy	SUNY Potsdam, USA	Carl B. Pomerance	Dartmouth College, USA
Errin W. Fulp	Wake Forest University, USA	Vadim Ponomarenko	San Diego State University, USA
Joseph Gallian	University of Minnesota Duluth, USA	Bjorn Poonen	UC Berkeley, USA
Stephan R. Garcia	Pomona College, USA	Józeph H. Przytycki	George Washington University, USA
Anant Godbole	East Tennessee State University, USA	Richard Rebarber	University of Nebraska, USA
Ron Gould	Emory University, USA	Robert W. Robinson	University of Georgia, USA
Sat Gupta	U of North Carolina, Greensboro, USA	Javier Rojo	Oregon State University, USA
Jim Haglund	University of Pennsylvania, USA	Filip Saidak	U of North Carolina, Greensboro, USA
Johnny Henderson	Baylor University, USA	Hari Mohan Srivastava	University of Victoria, Canada
Glenn H. Hurlbert	Virginia Commonwealth University, USA	Andrew J. Sterge	Honorary Editor
Charles R. Johnson	College of William and Mary, USA	Ann Trenk	Wellesley College, USA
K. B. Kulasekera	Clemson University, USA	Ravi Vakil	Stanford University, USA
Gerry Ladas	University of Rhode Island, USA	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy
David Larson	Texas A&M University, USA	John C. Wierman	Johns Hopkins University, USA
Suzanne Lenhart	University of Tennessee, USA	Michael E. Zieve	University of Michigan, USA
Chi-Kwong Li	College of William and Mary, USA		

PRODUCTION

Silvio Levy, Scientific Editor

Cover: Alex Scorpan

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2020 is US \$205/year for the electronic version, and \$275/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**

nonprofit scientific publishing

<http://msp.org/>

© 2020 Mathematical Sciences Publishers

involve

2020

vol. 13

no. 2

Arithmetic functions of higher-order primes	181
KYLE CZARNECKI AND ANDREW GIDDINGS	
Spherical half-designs of high order	193
DANIEL HUGHES AND SHAYNE WALDRON	
A series of series topologies on \mathbb{N}	205
JASON DEVITO AND ZACHARY PARKER	
Discrete Morse functions, vector fields, and homological sequences on trees	219
IAN RAND AND NICHOLAS A. SCOVILLE	
An explicit third-order one-step method for autonomous scalar initial value problems of first order based on quadratic Taylor approximation	231
THOMAS KRAINER AND CHENZHANG ZHOU	
New generalized secret-sharing schemes with points on a hyperplane using a Wronskian matrix	257
WESTON LOUCKS AND BAHATTIN YILDIZ	
Generalized Cantor functions: random function iteration	281
JORDAN ARMSTRONG AND LISBETH SCHAUBROECK	
Numerical semigroup tree of multiplicities 4 and 5	301
ABBY GRECO, JESSE LANSFORD AND MICHAEL STEWARD	
Enumerating diagonalizable matrices over \mathbb{Z}_{p^k}	323
CATHERINE FALVEY, HEEWON HAH, WILLIAM SHEPPARD, BRIAN SITTINGER AND RICO VICENTE	
On arithmetical structures on complete graphs	345
ZACHARY HARRIS AND JOEL LOUWSMA	
Connectedness of digraphs from quadratic polynomials	357
SIJI CHEN AND SHENG CHEN	