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Connectedness of digraphs from  
quadratic polynomials

Siji Chen and Sheng Chen





# Connectedness of digraphs from quadratic polynomials

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(Communicated by Vadim Ponomarenko)

Suppose that  $f(x) = x(x - k)$ , where  $k$  is an odd positive integer. First, an infinite digraph  $G_k = (V, E)$  is defined, where the vertex set is  $V = \mathbb{Z}$  and the edge set is  $E = \{(x, y) \mid x, y \in \mathbb{Z}, f(x) = f(2y)\}$ . Then the following results are proved: if  $k = 1$ , then the digraph  $G_k$  is weakly connected; if  $p$  is a safe prime, i.e., both  $p$  and  $q = (p - 1)/2$  are primes, then the number  $w_p$  of weakly connected components of the digraph  $G_p$  is 2. Finally, a conjecture that there are infinitely many primes  $p$  such that  $w_p = 2$  is presented.

## 1. Introduction

Denote by  $\text{Int}(\mathbb{Z}) = \{f(x) \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$  the ring of integer-valued polynomials on  $\mathbb{Z}$ . Chapman and Ponomarenko [2011] completely characterized the pairs of polynomials  $(f, g)$  in  $\text{Int}(\mathbb{Z})$  such that  $f(\mathbb{Z}) = g(\mathbb{Z})$ . Motivated by this work, in this paper we study the digraph  $G_k = (V, E)$  defined by

$$V = \mathbb{Z}, \quad E = \{(x, y) \mid x, y \in \mathbb{Z}, f(x) = g(y)\},$$

where  $f(x) = x(x - k)$  and  $g(x) = f(2x)$ . If  $k$  is odd, then by Lemma 1 in [Chapman and Ponomarenko 2011], it is easy to see that the digraph  $G_k$  is an infinite digraph in which each vertex has out-degree 1 and in-degree 2. In fact, on one hand, since  $f(x) = f(k - x)$ , for every vertex  $x \in \mathbb{Z}$  we have  $(x, y)$  is the unique edge from  $x$  where

$$y = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (k - x)/2 & \text{if } x \text{ is odd,} \end{cases}$$

and on the other hand, for every vertex  $y$ , there are two edges  $(2y, y)$  and  $(k - 2y, y)$  into  $y$ .

For general definitions about digraphs, we can refer to [Bang-Jensen and Gutin 2009]. Recall that a digraph is weakly connected if the underlying graph is connected.

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## 2. Results

**Theorem 1.** *If  $k = 1$ , then the digraph  $G_k$  is weakly connected.*

*Proof.* To prove the result, it suffices to prove the following statement by induction on positive integer  $n$ : every vertex  $x \in \mathbb{Z}$  such that  $|x/2| \leq n$  is weakly connected to the vertex 0.

If  $n = 1$ , then  $|x/2| \leq n$  implies  $x = -2, x = -1, x = 0, x = 1, \text{ or } x = 2$ . Since  $(-2, -1), (-1, 1), (2, 1), (1, 0), \text{ and } (0, 0)$  are edges in  $G_1$ , the statement is true.

Suppose the statement is true for some  $n \geq 1$ , that is, every vertex  $x \in \mathbb{Z}$  such that  $|x/2| \leq n$  is weakly connected to the vertex 0. Then for a vertex  $x \in \mathbb{Z}$  such that  $n < |x/2| \leq n + 1$ , we have  $x = 2n+2, 2n+1, -2n-2, \text{ or } -2n-1$ . Note that the following are edges in  $G_1$ :

$$(2n + 2, n + 1), \quad (2n + 1, -n), \quad (-2n - 2, -n - 1), \quad (-2n - 1, n + 1).$$

Since  $|n/2| \leq n$  and  $|(n+1)/2| \leq n$ , we know that vertex  $x$  such that  $n < |x/2| \leq n + 1$  is connected to some vertex  $y$  such that  $|y/2| \leq n$ . Therefore it is weakly connected to the vertex 0. □

**Theorem 2.** *If  $p$  is a safe prime, that is, both  $p$  and  $q = (p - 1)/2$  are primes, then the digraph  $G_p$  has precisely two weakly connected components.*

*Proof.* Suppose that  $p = 2q + 1$  is a safe prime. We prove the result in four steps.

Step 1: Take

$$U_1 = \{x \in \mathbb{Z} \mid p \text{ divides } x\},$$

$$U_2 = \{x \in \mathbb{Z} \mid p \text{ does not divide } x\}.$$

We use  $G_{p,U_1}$  and  $G_{p,U_2}$  to denote the two subdigraphs induced by the vertex subsets  $U_1$  and  $U_2$  respectively. Since there is no edge  $(x, y)$  such that  $x \in U_1$  and  $y \in U_2$ , or  $x \in U_2$  and  $y \in U_1$ , we know  $G_p$  is a disjoint union of the subdigraphs  $G_{p,U_1}$  and  $G_{p,U_2}$ .

Step 2: There is an isomorphism of digraphs from  $G_{p,U_1}$  to  $G_1$  defined by the mapping of vertex sets:  $U_1 \rightarrow \mathbb{Z} : x \mapsto x/p$ . By Theorem 1, we know that  $G_{p,U_1}$  is weakly connected.

Step 3: Take  $W = \{1, 2, \dots, p - 1\}$ . In this step, we aim to prove the following statement by induction on positive integers  $n$ :

Every vertex  $x \in U_2$  with  $|x/p| < n$  can be weakly connected to some vertex  $y \in W$ .

First, if  $n = 1$ , then the condition  $x \in U_2$  with  $|x/p| < 1$  is equivalent to  $x \in \mathbb{Z}$  with  $-p < x < 0$  or  $0 < x < p$ . If  $x \in U_2$  is odd, then  $x$  is connected to  $y = (p - x)/2 \in W$ . Otherwise,  $x \in U_2$  is even, say  $x = 2^r s$ , where  $r$  is a nonnegative integer and  $s$  is

an odd integer. Then  $x$  is connected to  $s$  and thus connected to some vertex in  $W$ , and thus the statement is true for  $n = 1$ .

Next, suppose the statement is true for some positive integer  $n$ , that is, every  $x \in U_2$  with  $|x/p| < n$  is weakly connected to some vertex  $y \in W$ . For even  $x \in U_2$  with  $n < |x/p| < n+1$ , the terminal vertex  $y = x/2$  of the unique edge from  $x$  satisfies  $|y/p| < (n+1)/2 \leq n$ . For odd  $x \in U_2$  with  $n < |x/p| < n+1$ , both  $(x, z)$  and  $(z, y)$  are edges, where  $z = (p - x)/2$ , and one of the following holds:  $y = (p - z)/2$  or  $y = z/2$ . So  $y = (p \pm x)/4$ , which satisfies  $|y/p| < (1 + (n + 1))/4 < n$ . Therefore  $x \in U_2$  is always weakly connected to some vertex in  $W$ . Thus the statement is true for  $n + 1$ .

Step 4: Denote by  $G_{p,W}$  the subdigraph induced by the vertex set

$$W = \{1, 2, \dots, p - 1\} = \{1, 2, \dots, 2q\}.$$

The edge set of  $G_{p,W}$  is

$$\{(2y, y) \mid y = 1, 2, \dots, q\} \cup \{(p - 2y, y) \mid y = 1, 2, \dots, q\}.$$

For any  $x \in W$ , we have some  $y \in \{1, 2, \dots, q\}$  such that both  $(x, y)$  and  $(p - x, y)$  are edges in  $G_{p,W}$ . So  $x \in W$  is weakly connected to  $p - x \in W$ . For any  $y \in \{1, 2, \dots, q\}$ , we have  $(2y, y)$  is an edge in  $G_{p,W}$  and thus  $y$  is weakly connected to  $2y$ . For any  $y \in \{q + 1, q + 2, \dots, q + q\}$ , we know  $(2p - 2y, p - y)$  is an edge in  $G_{p,W}$  and thus  $y, p - y, 2p - 2y, p - (2p - 2y) = 2y - p$  are weakly connected. That is to say, any  $y \in W$  is weakly connected to  $2y \in W$  or  $2y - p \in W$ .

Denote by  $U(\mathbb{F}_p)$  the multiplicative group of the invertible elements of the finite field  $\mathbb{F}_p = (\mathbb{Z}, +, \cdot)/p\mathbb{Z}$ . Since  $p$  is a safe prime, the unit group  $U(\mathbb{F}_p)$  is a cyclic group of order  $p - 1 = 2q$ , which can be generated by the set  $S = \{-1, 2\}$ . Therefore the digraph  $C(U(\mathbb{F}_p), S)$  with vertex set  $U(\mathbb{F}_p)$  and edge set

$$\{(a, b) \mid a, b \in U(\mathbb{F}_p), a = -b \text{ or } a = 2b\}$$

is connected.

Using elementary number theory, it follows that  $G_{p,W}$  is weakly connected.  $\square$

### 3. A conjecture

Safe primes appear in many places of mathematical theory and applications; see, e.g., [OEIS; von zur Gathen and Shparlinski 2013]. Although a folklore conjecture states that infinitely many Sophie Germain primes  $q$  exist, i.e.,  $q$  and  $p = 2q + 1$  are prime, a proof of this conjecture is not yet in sight; see [von zur Gathen and Shparlinski 2013]. Since we do not know whether the conjecture that there are infinitely many safe primes is true or not, we present the following slightly weaker *conjecture*:

**Conjecture.** There are infinitely many primes  $p$  such that the digraph  $G_p$  has exactly two weakly connected components.

**Remark.** For primes  $p = 13$ ,  $p = 17$ ,  $p = 31$ , which are not safe primes, the number of weakly connected components of  $G_p$  are 2, 3, 4 respectively. We may use the result in [Chen and Chen 2015] to study the number of weakly connected components of  $G_k$  for general nonprime odd integers  $k$ . However, an explicit formula for  $w_k$  seems out of reach.

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