involve

a journal of mathematics

On the sandpile group of Eulerian series-parallel graphs

Kyle Weishaar and James Seibert

msp

2020 vol. 13, no. 3

INVOLVE 13:3 (2020)
https://doi.org/10.2140/involve.2020.13.381

On the sandpile group of Eulerian series-parallel graphs

Kyle Weishaar and James Seibert

(Communicated by Joshua Cooper)

A sandpile configuration is a representation of the current layout of theoretical sand on a graph in which every vertex is assigned a nonnegative integer value. The Abelian sandpile group is a finite group composed of the recurrent sandpile configurations of a graph. We investigate the sandpile group of graphs constructed using the composition rules of series-parallel graphs, and determine the sandpile groups of parallel compositions of path-graphs.

1. Introduction

The Abelian sandpile model was originally introduced as a simple example of self-organized criticality [Bak et al. 1988]. The model was later generalized into a form that can be expressed in terms of a rooted graph [Dhar et al. 1995]. On such a graph, nonnegative integers are assigned to each vertex to represent the current configuration of sand. Toppling happens when a vertex with an amount of sand that is at least its degree transfers one grain of sand to each of its neighboring vertices. The root (also called the sink in this context) is the only vertex that is not allowed to topple. The sink acts as an outlet for sand to escape the system. As a result, the amount of sand on the sink is inconsequential and therefore not included in the configuration of a graph. A configuration with no toppleable vertices is said to be stable. Two configurations can be added by adding the sandpiles on corresponding vertices and then toppling until stable. A configuration is recurrent if it is stable and can be reached from any starting configuration by adding some other configuration. The set of recurrent configurations makes up a finite Abelian group known as the sandpile group. The order of the sandpile group is equal to the number of spanning trees of the graph [Dhar et al. 1995]. The recently published book [Corry and Perkinson 2018, Chapters 6–9] provides a comprehensive introduction to all of these ideas, and much more.

MSC2010: 05C50.
Keywords: sandpile group, Eulerian graph, Laplacian, Smith normal form.

Previous works have classified the sandpile groups of several families of graphs [Cori and Rossin 2000; Hou et al. 2006; Levine 2009; Shen and Hou 2008; Toumpakari 2007]. Furthermore, the sandpile groups of the Cartesian products of many types of graphs have also been identified [Hou et al. 2008; Jacobson et al. 2003; Liang et al. 2008]. A better understanding of how graph compositions affect sandpile groups may allow for the classification of sandpile groups on more complex graphs. The growing catalog of classified sandpile groups can be used to view complicated graphs as the composition of graphs with known sandpile groups. In this spirit, we investigate the sandpile group of series-parallel graphs constructed from graphs with known groups.

This paper is organized as follows. In Section 2 we provide necessary background information. In Section 3 we prove the general form of series compositions involving an Eulerian graph. We present this known result (it appears as an exercise in [Corry and Perkinson 2018]) in order to prepare the reader for the new result on parallel compositions of paths in Section 4. We demonstrate a significant reduction that lends itself to several corollaries. In Section 5 we find the sandpile group of an example series-parallel graph and offer concluding remarks.

2. Background

On a directed graph, the number of edges leaving a vertex is called the outdegree. Similarly, the number of edges entering a vertex is called the indegree. A graph is said to be Eulerian if the outdegree of every vertex is equal to its indegree. A graph with v vertices can be represented by a $v \times v$ matrix known as the Laplacian matrix, L. The Laplacian of a graph is defined by

$$L_{i,j} = \begin{cases} \text{outdegree of vertex } i & \text{if } i = j, \\ -(\text{number of edges directed from } j \text{ to } i) & \text{if } i \neq j. \end{cases}$$

Column j of the Laplacian then reflects the result of toppling vertex j. Specifically, vertex j loses grains of sand equal to its outdegree (represented by the positive number on the diagonal), and each vertex connected to j gains grains of sand equal to the number of edges coming from vertex j (represented by the corresponding off-diagonal entries). The reduced Laplacian is obtained by removing the row and column that corresponds to the sink of the full Laplacian. The choice of sink does not impact the sandpile group of an Eulerian graph [Cori and Rossin 2000]. For such graphs, the reduced Laplacian can be created by removing any corresponding column and row.

The Laplacian of a graph determines its sandpile group in the following manner. We can define an equivalence relation on the additive group $\mathbb{Z}^v$ by declaring two configurations (now allowing negative as well as nonnegative integers) equivalent if one can be reached from the other by a sequence of toppling (or untoppling) moves.

Each equivalence class has a unique representative that is a recurrent configuration. (This nontrivial result is proven in several sources, including [Cori and Rossin 2000; Corry and Perkinson 2018].) The columns of the Laplacian are a complete set of relations using the standard basis vectors as a generating set of the group. In other words, the Laplacian is a presentation matrix for a finitely generated Abelian group. The torsion subgroup of this group is the sandpile group.

In order to identify the sandpile group, we can manipulate the presentation matrix of a group by using the following elementary row and columns operations [Dummit and Foote 2004]:

(1) Add an integer multiple of one row or column to another. This changes the generating set of the group and the corresponding relations.

(2) Multiply a row or column by -1. This replaces the generator with its inverse.

(3) Delete a column of zeros. This eliminates a trivial relation.

(4) Delete a row and a column that form the standard basis vector. This eliminates a redundant generator.

(5) Swapping corresponding rows and columns. This reorders the generators and adjusts the relations accordingly.

These operations do not change the group represented. If a matrix A can be changed into a matrix B by use of these operations, we will say that A is $\mathbb{Z}$-equivalent to B and we will write $A \equiv B$. Note that operations (1), (2), and (5) can be accomplished by left or right multiplication by an invertible matrix.

Every integer matrix has a unique Smith normal form, arrived at using the row and column operations above. The Smith normal form is defined as the diagonal matrix $\mathbb{Z}$-equivalent to the original in which every entry is divisible by the previous entries. For every matrix M with integer entries, there exist invertible matrices P and Q such that PMQ is in Smith normal form. The Smith normal form of the matrix can be written as an $(n+m) \times (n+m)$ matrix

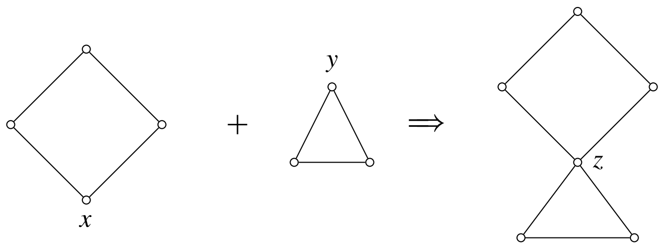
$$\begin{bmatrix} a_{1,1} & & & & & \\ & \ddots & & & & \\ & & a_{n,n} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

in which $a_{i,i}$ divides $a_{i+1,i+1}$. This matrix represents the group

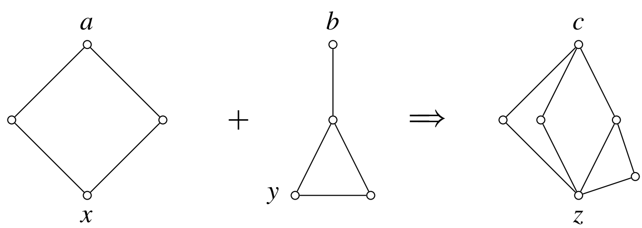
$$\left(\bigoplus_{i=1}^{n} \mathbb{Z}_{a_{i,i}}\right) \oplus \mathbb{Z}^m.$$

We arrive at the sandpile group by discarding the infinite summands and keeping the finite Abelian group that remains (the torsion subgroup). A common technique used to determine the sandpile group of a family of graphs is to show that the Laplacian matrix (or reduced Laplacian) for graphs in the family is equivalent to a predictable Smith normal form.

This paper focuses on the sandpile groups of graphs constructed using the composition rules of series-parallel graphs. A series composition of two graphs, G_1 and G_2, is formed by taking a vertex $x \in G_1$ and a vertex $y \in G_2$ and merging x with y to produce the vertex z, while all other existing vertices and connections remain unchanged:



Similarly, a parallel composition of two graphs, G_1 and G_2, is formed by taking the vertices $a, x \in G_1$ and $b, y \in G_2$, merging a with b to produce the vertex c, and merging x with y to produce the vertex z, while all other existing vertices and connections remain unchanged:



3. Series compositions

Theorem 3.1. *Let G_1 and G_2 be Eulerian graphs with the sandpile groups SG_1 and SG_2, respectively. A series composition of G_1 and G_2 will have a sandpile group of the form $SG_1 \oplus SG_2$.*

Proof. Let G_1 and G_2 be Eulerian graphs with the respective sandpile groups SG_1 and SG_2. The Laplacian of a series composition of G_1 and G_2 will be composed of the values $a_{i,j}$, which come from the Laplacian of G_1, listing the vertex to merge last, and the values $b_{i,j}$, which come from the Laplacian of G_2, listing the vertex to merge first. The merged vertex has outdegree $a_{n,n} + b_{1,1}$ and connections to G_1

and G_2 preserved. Therefore, the Laplacian is of the form

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} & & \\ \vdots & \ddots & \vdots & & \\ a_{n,1} & \cdots & a_{n,n}+b_{1,1} & \cdots & b_{1,m} \\ & & \vdots & \ddots & \vdots \\ & & b_{m,1} & \cdots & b_{m,m} \end{bmatrix}.$$

We will now use the elementary row operations defined in Section 2 to show this Laplacian is $\mathbb{Z}$-equivalent to a useful diagonal matrix. Since G_1 and G_2 are both Eulerian graphs, the series composition of these graphs will also be Eulerian. Therefore, we can let the sink be represented by the n-th row and column of the full Laplacian. As a result, the reduced Laplacian will have the form

$$\left[\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,n-1} & & & \\ \vdots & \ddots & \vdots & & & \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & & & \\ \hline & & & b_{2,2} & \cdots & b_{2,m} \\ & & & \vdots & \ddots & \vdots \\ & & & b_{m,2} & \cdots & b_{m,m} \end{array}\right] = \left[\begin{array}{c|c} A & \\ \hline & B \end{array}\right].$$

Let P_1 and Q_1 be the invertible matrices that put A in Smith normal form. Similarly, let P_2 and Q_2 be the invertible matrices that put B in Smith normal form. Then

$$\left[\begin{array}{c|c} P_1 & \\ \hline & P_2 \end{array}\right] \cdot \left[\begin{array}{c|c} A & \\ \hline & B \end{array}\right] \cdot \left[\begin{array}{c|c} Q_1 & \\ \hline & Q_2 \end{array}\right]$$

would produce the matrix

$$\left[\begin{array}{ccc|ccc} f_1 & & & & & \\ & \ddots & & & & \\ & & f_{n-1} & & & \\ \hline & & & g_1 & & \\ & & & & \ddots & \\ & & & & & g_{m-1} \end{array}\right],$$

where the values $f_1, \ldots, f_{n-1}$ and $g_1, \ldots, g_{m-1}$ are the nonzero entries of the Smith normal form of the Laplacian of G_1 and G_2, respectively. This matrix presents

$$\left(\bigoplus_{i=1}^{n-1} f_i\right) \oplus \left(\bigoplus_{j=1}^{m-1} g_j\right) = SG_1 \oplus SG_2.$$

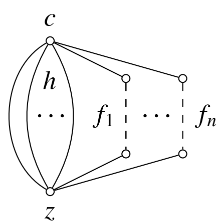
Since the reduced Laplacian of a series composition of the Eulerian graphs is $\mathbb{Z}$-equivalent to a diagonal matrix with the nonzero entries of the Smith normal forms of G_1 and G_2 as values, the sandpile group of a series composition of G_1 and G_2 is $SG_1 \oplus SG_2$. □

Corollary 3.2. *The sandpile group of any series combination of two Eulerian graphs, G_1 and G_2, will be of the form $SG_1 \oplus SG_2$, independent of which vertices are merged.*

Proof. The choice of vertices to merge simply rearranges the rows and columns using the fifth elementary operation in order to list the merged vertices in an overlapping row. The sandpile groups of G_1 and G_2 are unique, independent of the order in which the vertices are listed and independent of the vertices chosen as the sinks [Cori and Rossin 2000]. The sandpile group of the series composition of graphs will be $SG_1 \oplus SG_2$, no matter which vertices are merged. □

4. Parallel composition of paths

We now focus on a family of undirected multigraphs made of the parallel composition of h paths of length 1 and n paths of the varying lengths $f_1, \dots, f_n$, where $f_i \geq 2$. Graphs in this family have the following form:



We define a partitioned matrix that represents the general Laplacian of graphs in this family as follows. The top row of the Laplacian will correspond to the vertex c and it will have the form

$$\begin{bmatrix} n+h & D_1 & \cdots & D_n & -h \end{bmatrix},$$

where each D_i is a $1 \times (f_i - 1)$ matrix of the form $D_i = \begin{bmatrix} -1 & 0 & \cdots & 0 \end{bmatrix}$. The first entry of this row corresponds to the degree of c, which is the sum of the number of paths of length 1 and the number of paths of length 2 or greater. Each D_i matrix corresponds to the path f_i, with its vertices listed from top to bottom. We see that c is only connected to the first vertex of each path. The last entry of the top row of the Laplacian corresponds to the h paths of length 1 connecting the vertices c and z. The Laplacian of an undirected graph is symmetric. Therefore, the first column will be the transpose of the first row.

The bottom row of the Laplacian will correspond to vertex z and have the form

$$\begin{bmatrix} -h & C_1 & \cdots & C_n & n+h \end{bmatrix},$$

where each C_i is a $1 \times (f_i - 1)$ matrix of the form $C_i = \begin{bmatrix} 0 & \cdots & 0 & -1 \end{bmatrix}$. The first entry reflects the h connections from z to c, while the last represents the total degree of $n+h$. Each C_i corresponds to the path f_i and shows that z is only connected to the last vertex in this path. The last column of the Laplacian will be the transpose of the bottom row.

The vertices of the path f_i (excluding c and z, accounted for above) correspond to a diagonal block in the Laplacian, B_i. When $f_i > 2$, B_i is an $(f_i - 1) \times (f_i - 1)$ matrix of the form

$$B_i = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}.$$

Here we see that each vertex in the path has degree 2 and is connected to the vertices immediately before and after it. When $f_i = 2$, B_i is the 1×1 matrix with entry 2, and the corresponding matrices D_i and C_i are 1×1 matrices consisting of entry -1. Putting this all together we see that the full Laplacian of the parallel composition of h paths of length 1 and n paths of the varying lengths $f_1, \ldots, f_n$, where $f_i \geq 2$, is of form

$$\left[\begin{array}{c|c|c|c|c} n+h & D_1 & \cdots & D_n & -h \\ \hline D_1^{\mathsf{T}} & B_1 & & & C_1^{\mathsf{T}} \\ \hline \vdots & & \ddots & & \vdots \\ \hline D_n^{\mathsf{T}} & & & B_n & C_n^{\mathsf{T}} \\ \hline -h & C_1 & \cdots & C_n & n+h \end{array}\right].$$

The empty partitions are zero matrices that reflect the fact that vertices from distinct paths f_i do not connect to each other. Now that we have the Laplacian for this graph of parallel paths, we can use the row operations from Section 2 to simplify it as much as possible. Since the graph is undirected (and therefore Eulerian), we can let the first row and column represent the sink. The resulting reduced Laplacian will be of the form

$$\left[\begin{array}{c|c|c|c} B_1 & & & C_1^{\mathsf{T}} \\ \hline & \ddots & & \vdots \\ \hline & & B_n & C_n^{\mathsf{T}} \\ \hline C_1 & \cdots & C_n & n+h \end{array}\right].$$

The following lemma will allow us to further reduce this matrix.

Lemma 4.1. *An $m \times m$ matrix F_m, where $m \geq 2$, of the form*

$$F_m = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

satisfies

$$F_m \equiv \begin{bmatrix} m & -1 \\ -(m-1) & 2 \end{bmatrix}.$$

Proof. We start by considering cases with small m. When $m = 2$,

$$F_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -(2-1) & 2 \end{bmatrix},$$

which is in the desired form.

When $m = 3$, we have the matrix

$$F_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

We create the standard basis vector in the first row and column by adding the second column to the first, and then adding the first column to the second. The top row will now have 1 as the first entry and zeros everywhere else. Next, we use the first row to eliminate the entries in the rest of the first column:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -2 & 2 \end{bmatrix}.$$

We now use the fourth elementary operation to remove the first row and column, and see the desired form

$$\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -(3-1) & 2 \end{bmatrix}.$$

The $m = 3$ example shows the basic process that we can iterate. If we start with a matrix of the form

$$\begin{bmatrix} k & -1 & & & \\ -(k-1) & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix},$$

we can add $(k-1)$ times the second column to the first column, and then add the first column to the second column to get

$$\begin{bmatrix} 1 & 0 & & \cdots & & 0 \\ k-1 & k+1 & -1 & & & \\ -(k-1) & -k & 2 & -1 & & \\ & & -1 & 2 & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

Now use the first row to eliminate the rest of the entries in the first column, and we can use the fourth elementary operation to remove the first row and column. The result is a matrix of the same form, one dimension smaller, with k increased by 1:

$$\begin{bmatrix} k+1 & -1 & & & \\ -k & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Starting with F_m, we iterate this process $m-2$ times and the result is

$$\begin{bmatrix} m & -1 \\ -(m-1) & 2 \end{bmatrix}. \qquad \square$$

We will apply Lemma 4.1 to reduce the blocks B_i in the Laplacian, which leads to the following significant reduction in size. We initially deal with the case where there are three or more longer paths, and address the cases with fewer long paths separately.

Theorem 4.2. *The reduced Laplacian of an undirected graph made of the parallel composition of h paths of length* 1 *and n paths of the varying lengths $f_1, \ldots, f_n$, where $f_i \geq 2$ and $n \geq 3$, is $\mathbb{Z}$-equivalent to the form*

$$\begin{bmatrix} f_3 & & & -f_1 \\ & \ddots & & \vdots \\ & & f_n & -f_1 \\ f_2 & \cdots & f_2 & f_1+f_2+h(f_1 f_2) \end{bmatrix}.$$

Proof. Recall that the reduced Laplacian of graphs in this family has the form

$$\left[\begin{array}{c|c|c|c} B_1 & & & C_1^\top \\ \hline & \ddots & & \vdots \\ \hline & & B_n & C_n^\top \\ \hline C_1 & \cdots & C_n & n+h \end{array}\right].$$

We begin by treating paths of length 3 or greater separately from paths of length 2. Sorting the paths by length from shortest to longest gives the matrix

$$\left[\begin{array}{c|c|c|c|c|c|c} 2 & & & & & & -1 \\ \hline & \ddots & & & & & \vdots \\ \hline & & 2 & & & & -1 \\ \hline & & & B_m & & & C_m^\intercal \\ \hline & & & & \ddots & & \vdots \\ \hline & & & & & B_n & C_n^\intercal \\ \hline -1 & \cdots & -1 & C_m & \cdots & C_n & n+h \end{array}\right],$$

where f_m through f_n are all greater than 2.

We now utilize Lemma 4.1 to condense all paths with length greater than 2 (paths f_m through f_n). We can do this because none of the row and column operations used in Lemma 4.1 involve the last row or column of the block. The entries in the corresponding rows and columns outside of the block B_i are all zero and unaffected by the operations in Lemma 4.1. We now see that the reduced Laplacian is $\mathbb{Z}$-equivalent to the matrix

$$\left[\begin{array}{c|c|c|cc|c|cc|c} -2 & & & & & & & & -1 \\ \hline & \ddots & & & & & & & \vdots \\ \hline & & -2 & & & & & & -1 \\ \hline & & & -f_m-1 & -1 & & & & 0 \\ & & & -(f_m-2) & 2 & & & & -1 \\ \hline & & & & & \ddots & & & \vdots \\ \hline & & & & & & -f_n-1 & -1 & 0 \\ & & & & & & -(f_n-2) & 2 & -1 \\ \hline -1 & \cdots & -1 & -0 & -1 & \cdots & 0 & -1 & n+h \end{array}\right].$$

We can further condense each 2×2 block along the diagonal using row and column operations similar to those used to prove Lemma 4.1. Corresponding to each 2×2 block we add $(f_i - 2)$ times the second column to the first column. Next, we add the first column to the second column. Every corresponding diagonal block now has a top row with 1 as the first entry and zero everywhere else. We can utilize this to eliminate all other values of each section's first column. The first row and column of each section are now in the form of the standard basis vector and can therefore be eliminated. The resulting matrix is of the form

$$\begin{bmatrix} f_1 & & & -1 \\ & \ddots & & \vdots \\ & & f_n & -1 \\ -(f_1-1) & \cdots & -(f_n-1) & n+h \end{bmatrix}.$$

The paths of length 2 also fit into this form since such paths are represented by a diagonal value of 2 and -1 as the bottom entry. Next, we add $(f_1 - 1)$ times the rightmost column to the leftmost column to put the matrix in the form

$$\begin{bmatrix} 1 & & & & -1 \\ -(f_1-1) & f_2 & & & -1 \\ \vdots & & \ddots & & \vdots \\ -(f_1-1) & & & f_n & -1 \\ (n+h-1)(f_1-1) & -(f_2-1) & \cdots & -(f_n-1) & n+h \end{bmatrix}.$$

From here we can add the leftmost column to the rightmost column. As a result, the top row has a first entry of 1 and zeros everywhere else. Therefore, all the rest of the entries of the leftmost column can be made into zero using row operations. Note the first row and column form the standard basis vector and can therefore be eliminated. The matrix is now of the form

$$\begin{bmatrix} f_2 & & & -f_1 \\ & \ddots & & \vdots \\ & & f_n & -f_1 \\ -(f_2-1) & \cdots & -(f_n-1) & (n+h)f_1-f_1+1 \end{bmatrix}.$$

Adding the bottom row to the top row produces the matrix

$$\begin{bmatrix} 1 & -(f_3-1) & \cdots & -(f_n-1) & (n+h)f_1-2f_1+1 \\ & f_3 & & & -f_1 \\ & & \ddots & & \vdots \\ & & & f_n & -f_1 \\ -(f_2-1) & -(f_3-1) & \cdots & -(f_n-1) & (n+h)f_1-f_1+1 \end{bmatrix}.$$

We can now add $(f_2 - 1)$ times the top row to the bottom row. The first column will have 1 as its first entry and zeros everywhere else. Utilizing this, we can make the top row into the form of the standard basis vector. This allows us to once again eliminate the top row and first column. The matrix is now in the form

$$\begin{bmatrix} f_3 & & & -f_1 \\ & \ddots & & \vdots \\ & & f_n & -f_1 \\ -f_2 f_3+f_2 & \cdots & -f_2 f_n+f_2 & (n+h-2)(f_1 f_2)+f_1+f_2 \end{bmatrix}.$$

Notice that this is an $(n-1)\times(n-1)$ matrix. We can add the first $n-2$ rows multiplied by f_2 to the bottom row, which results in

$$\begin{bmatrix} f_3 & & & -f_1 \\ & \ddots & & \vdots \\ & & f_n & -f_1 \\ f_2 & \cdots & f_2 & f_1+f_2+h(f_1 f_2) \end{bmatrix}.$$

□

Theorem 4.2 is useful in computing the sandpile group of specific graphs, as it dramatically reduces the size of the Laplacian used to compute the group. This theorem can also be used to classify the sandpile group of a family of graphs, as the following corollaries show.

Corollary 4.3. *The sandpile group of the parallel composition of n paths*, *where* $n \geq 3$, *with length* a, *where* $a \geq 2$, *is of the form*

$$\left(\bigoplus_{i=1}^{n-2} \mathbb{Z}_a\right) \oplus \mathbb{Z}_{an}.$$

Proof. Since all paths are of length a and $a \geq 2$ we know by Theorem 4.2 that the reduced Laplacian is $\mathbb{Z}$-equivalent to the $(n-1)\times(n-1)$ matrix

$$\begin{bmatrix} a & & & -a \\ & \ddots & & \vdots \\ & & a & -a \\ a & \cdots & a & 2a \end{bmatrix}.$$

Adding the first $n-2$ columns to the rightmost column, and then adding -1 times the first $n-2$ rows to the bottom row produces

$$\begin{bmatrix} a & & & \\ & \ddots & & \\ & & a & \\ & & & an \end{bmatrix}.$$

This presents the group

$$\left(\bigoplus_{i=1}^{n-2} \mathbb{Z}_a\right) \oplus \mathbb{Z}_{an}.$$

□

Corollary 4.4. *The sandpile group of the parallel composition of* $n-1$ *paths, where* $n \geq 3$, *with length* a *and one path of length* b, *where* $a, b \geq 2$, *is of the form*

$$\left(\bigoplus_{i=1}^{n-3} \mathbb{Z}_a\right) \oplus \mathbb{Z}_{\gcd(a,b)} \oplus \mathbb{Z}_{(a^2+(n-1)ab)/\gcd(a,b)}.$$

Proof. Since $a, b \geq 2$ we know by Theorem 4.2 that the reduced Laplacian is $\mathbb{Z}$-equivalent to the $(n-1) \times (n-1)$ matrix

$$\begin{bmatrix} a & & & & -a \\ & \ddots & & & \vdots \\ & & a & & -a \\ & & & b & -a \\ a & \cdots & a & a & 2a \end{bmatrix}.$$

Adding the first $n-3$ columns to the rightmost column, and then adding -1 times the first $n-3$ rows to the bottom row produces

$$\begin{bmatrix} a & & & & \\ & \ddots & & & \\ & & a & & \\ & & & b & -a \\ & & & a & a(n-1) \end{bmatrix}.$$

We can use row operations to perform the extended Euclidean algorithm on the b and a entries of the last two rows, replacing them with $\gcd(a, b)$ and zero. To see how this works, consider the case where $a < b$. The first step of the Euclidean algorithm asks us to divide b by a,

$$b = aq_1 + r_1,$$

and replace b with the remainder r_1. We achieve this in the matrix by multiplying the last row by $-q_1$ and adding to the penultimate row. The second step of the Euclidean algorithm asks us to divide a by the remainder r_1,

$$a = r_1 q_2 + r_2,$$

and replace a with the remainder r_2. We multiply the penultimate row by $-q_2$ and add to the last row. This process continues until one remainder evenly divides the previous. At this point the last nonzero remainder is $\gcd(a, b)$, and the other entry is zero. If needed we can switch the last two rows to ensure $\gcd(a, b)$ is on the diagonal. The entries in the last column started as multiples of a, and are now linear combinations of multiples of a. We write them as $a \cdot j$ and $a \cdot k$, where j and k are some integers. This yields a matrix of the form

$$\begin{bmatrix} a & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \gcd(a,b) & a\cdot j \\ & & & 0 & a\cdot k \end{bmatrix}.$$

Of course $\gcd(a,b)$ divides any multiple of a; therefore we can simplify this expression to

$$\begin{bmatrix} a & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \gcd(a,b) & \\ & & & & a\cdot k \end{bmatrix}.$$

This presents the group

$$\left(\bigoplus_{i=1}^{n-3} \mathbb{Z}_a\right) \oplus \mathbb{Z}_{\gcd(a,b)} \oplus \mathbb{Z}_{ak}.$$

Recall from Section 1 that the order of a sandpile group is the number of spanning trees on its graph [Dhar et al. 1995]. The number of spanning trees in this family of graphs can be found by counting the number of ways to remove an edge from all paths except one. If we leave the path of length b intact, we would need to remove an edge from the $n-1$ paths of length a. There are a^{n-1} ways of doing this. Similarly, we can choose a path of length a to keep intact and remove an edge from the rest of the paths. There are $a^{n-2}b(n-1)$ ways to do this, making the total number of spanning trees $a^{n-1}+a^{n-2}b(n-1)$. Another way of finding the order of the group is to take the product of the components. For this group this is $a^{n-3}\times\gcd(a,b)\times ak$. Setting these values equal to each other and solving for k yields

$$k = \frac{a+b(n-1)}{\gcd(a,b)}.$$

Therefore, the group is

$$\left(\bigoplus_{i=1}^{n-3} \mathbb{Z}_a\right) \oplus \mathbb{Z}_{\gcd(a,b)} \oplus \mathbb{Z}_{(a^2+ab(n-1))/\gcd(a,b)}. \qquad \square$$

We now offer the sandpile groups of undirected graphs made of the parallel composition of h paths of length 1 and n paths, where $n = 0, 1, 2$, of the varying lengths $f_1, \dots, f_n$, where $f_i \geq 2$.

Theorem 4.5. *The sandpile group of the parallel composition of h paths with length 1 is of the form*

$$\mathbb{Z}_h.$$

Proof. The Laplacian of this family of graphs is of the form

$$\begin{bmatrix} h & -h \\ -h & h \end{bmatrix}.$$

The reduced Laplacian is a 1×1 matrix with h as the only entry, representing the group

$$\mathbb{Z}_h. \qquad \square$$

Theorem 4.6. *The sandpile group of the parallel composition of h paths with length 1 and one path with length f_1, where $f_1 \geq 2$, is of the form*

$$\mathbb{Z}_{hf_1+1}.$$

Proof. We will proceed by using a method similar to the one used to prove Lemma 4.1. The Laplacian of this family of graphs has the form

$$\left[\begin{array}{c|c|c} h+1 & D_1 & -h \\ \hline D_1^\intercal & B_1 & C_1^\intercal \\ \hline -h & C_1 & h+1 \end{array}\right].$$

Since the graphs in this family are Eulerian, the choice of sink does not affect the sandpile group. For simplicity's sake, we let the first row and column correspond to the sink. The resulting reduced Laplacian will be of the form

$$\left[\begin{array}{c|c} B_1 & C_1^\intercal \\ \hline C_1 & h+1 \end{array}\right].$$

Lemma 4.1 does not require adding the last column (or row) to any other column (or row). Therefore, we can utilize Lemma 4.1 to show the reduced Laplacian is $\mathbb{Z}$-equivalent to the matrix

$$\begin{bmatrix} f_1 & -1 \\ -(f_1-1) & h+1 \end{bmatrix}.$$

We now add $(f_1 - 1)$ times the right column to the left column. Then we add the left column to the right column. As a result, the top row will have 1 as the first entry and zeros everywhere else. We use the top row to eliminate all other entries in the first column. In matrix form, these steps are

$$\begin{bmatrix} 1 & -1 \\ h(f_1-1) & h+1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ h(f_1-1) & hf_1+1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & hf_1+1 \end{bmatrix}.$$

Consequently, the group presented is

$$\mathbb{Z}_{hf_1+1}. \qquad \square$$

Theorem 4.7. *The sandpile group of the parallel composition of h paths with length* 1 *and two paths with respective lengths f_1 and f_2, where f_1, $f_2 \geq 2$, is of the form*

$$\mathbb{Z}_{f_1+f_2+h(f_1 f_2)}.$$

Proof. The Laplacian of this family of graphs is of the form

$$\left[\begin{array}{c|c|c|c} h+2 & D_1 & D_2 & -h \\ \hline D_1^{\mathsf{T}} & B_1 & & C_1^{\mathsf{T}} \\ \hline D_2^{\mathsf{T}} & & B_2 & C_2^{\mathsf{T}} \\ \hline -h & C_1 & C_2 & h+2 \end{array}\right].$$

Graphs in this family are Eulerian; thus we can let the first row and column represent the sink. The resulting reduced Laplacian will be of the form

$$\left[\begin{array}{c|c|c} B_1 & & C_1^{\mathsf{T}} \\ \hline & B_2 & C_2^{\mathsf{T}} \\ \hline C_1 & C_2 & h+2 \end{array}\right].$$

We utilize Lemma 4.1 and additional steps seen in the proof of Theorem 4.2 to condense B_1 and B_2, producing the matrix

$$\begin{bmatrix} f_1 & 0 & -1 \\ 0 & f_2 & -1 \\ -(f_1-1) & -(f_2-1) & h+2 \end{bmatrix}.$$

We add $(f_1 - 1)$ times the right column to the left column. Then we add the left column to the right column and use the top row to eliminate all other entries in the first column:

$$\begin{bmatrix} 1 & 0 & -1 \\ -(f_1-1) & f_2 & -1 \\ h(f_1-1)+(f_1-1) & -(f_2-1) & h+2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & f_2 & -f_1 \\ 0 & -(f_2-1) & hf_1+f_1+1 \end{bmatrix}.$$

Observe, the top row and left column form the standard basis vector and can therefore be eliminated using the fourth elementary operation:

$$\begin{bmatrix} f_2 & -f_1 \\ -(f_2-1) & hf_1+f_1+1 \end{bmatrix}.$$

Adding the bottom row to the top will produce the matrix

$$\begin{bmatrix} 1 & hf_1+1 \\ -(f_2-1) & hf_1+f_1+1 \end{bmatrix}.$$

We now add $(f_2 - 1)$ times the top row to the bottom row. As a result, the left column has 1 as the first entry and 0 everywhere else. We use this to put the top row and column into the form of the standard basis vector:

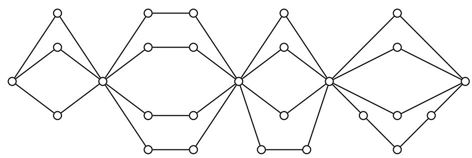
$$\begin{bmatrix} 1 & 0 \\ 0 & f_1+f_2+h(f_1 f_2) \end{bmatrix}$$

Therefore, the group is

$$\mathbb{Z}_{f_1+f_2+h(f_1 f_2)}. \qquad \square$$

5. Conclusion

The techniques of the preceding sections allow us to easily find the sandpile group of a wide range of graphs. For example, consider the following graph:



This graph is the series composition of four parallel graphs. We apply Corollary 4.3 to the first parallel graph from the left. This graph is the parallel composition of three paths of length 2, so it has a sandpile group of

$$\left(\bigoplus_{i=1}^{3-2} \mathbb{Z}_2\right) \oplus \mathbb{Z}_{2\cdot 3} = \mathbb{Z}_2 \oplus \mathbb{Z}_6.$$

Observe that the second graph is the parallel composition of four paths of length 3. Therefore, applying Corollary 4.3 yields

$$\left(\bigoplus_{i=1}^{4-2} \mathbb{Z}_3\right) \oplus \mathbb{Z}_{3\cdot 4} = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{12}.$$

The third graph is the parallel composition of four paths, three of length 2 and a single path of length 3. Therefore, applying Corollary 4.4 yields

$$\left(\bigoplus_{i=1}^{4-3} \mathbb{Z}_2\right) \oplus \mathbb{Z}_{\gcd(2,3)} \oplus \mathbb{Z}_{(2^2+2\cdot 3(4-1))/\gcd(2,3)} = \mathbb{Z}_2 \oplus \mathbb{Z}_{22}.$$

The fourth graph is the parallel composition of four paths, three of length 2 and a single path of length 4. Therefore, applying Corollary 4.4 yields

$$\left(\bigoplus_{i=1}^{4-3} \mathbb{Z}_2\right) \oplus \mathbb{Z}_{\gcd(2,4)} \oplus \mathbb{Z}_{(2^2+2\cdot 4(4-1))/\gcd(2,4)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{14}.$$

The entire graph is formed by using series operations to combine all four parallel graphs. By Theorem 3.1 the resulting sandpile group will be the direct sum of the sandpile groups of all four parallel graphs. Consequently, the sandpile group is

$$\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{22} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{14}.$$

This example shows how the tools presented in this paper allow us to easily compute the sandpile groups of rather complicated graphs.

We have presented the completely general result for the series composition of graphs, namely that the sandpile group is the direct sum of the component sandpile groups. We initially sought a similar result for parallel composition of graphs. While a completely general result about parallel composition eluded us, we were able to prove results about the parallel composition of paths, including a significant reduction in the size of the matrix needed to compute the sandpile groups for such graphs. This work has been fascinating, and we hope to see more general results involving the sandpile groups of the parallel composition of graphs in the future.

References

[Bak et al. 1988] P. Bak, C. Tang, and K. Wiesenfeld, "Self-organized criticality", *Phys. Rev. A* (3) **38**:1 (1988), 364–374. MR Zbl

[Cori and Rossin 2000] R. Cori and D. Rossin, "On the sandpile group of dual graphs", *European J. Combin.* **21**:4 (2000), 447–459. MR Zbl

[Corry and Perkinson 2018] S. Corry and D. Perkinson, *Divisors and sandpiles: an introduction to chip-firing*, American Mathematical Society, Providence, RI, 2018. MR Zbl

[Dhar et al. 1995] D. Dhar, P. Ruelle, S. Sen, and D.-N. Verma, "Algebraic aspects of abelian sandpile models", *J. Phys. A* **28**:4 (1995), 805–831. MR Zbl

[Dummit and Foote 2004] D. S. Dummit and R. M. Foote, *Abstract algebra*, 3rd ed., John Wiley & Sons, Hoboken, NJ, 2004. MR Zbl

[Hou et al. 2006] Y. Hou, C. Woo, and P. Chen, "On the sandpile group of the square cycle C_n^2", *Linear Algebra Appl.* **418**:2-3 (2006), 457–467. MR Zbl

[Hou et al. 2008] Y. Hou, T. Lei, and C. Woo, "On the sandpile group of the graph $K_3 \times C_n$", *Linear Algebra Appl.* **428**:8-9 (2008), 1886–1898. MR Zbl

[Jacobson et al. 2003] B. Jacobson, A. Niedermaier, and V. Reiner, "Critical groups for complete multipartite graphs and Cartesian products of complete graphs", *J. Graph Theory* **44**:3 (2003), 231–250. MR Zbl

[Levine 2009] L. Levine, "The sandpile group of a tree", *European J. Combin.* **30**:4 (2009), 1026–1035. MR Zbl

[Liang et al. 2008] H. Liang, Y.-L. Pan, and J. Wang, "The critical group of $K_m \times P_n$", *Linear Algebra Appl.* **428**:11-12 (2008), 2723–2729. MR Zbl

[Shen and Hou 2008] J. Shen and Y. Hou, "On the sandpile group of $3 \times n$ twisted bracelets", *Linear Algebra Appl.* **429**:8-9 (2008), 1894–1904. MR Zbl

[Toumpakari 2007] E. Toumpakari, "On the sandpile group of regular trees", *European J. Combin.* **28**:3 (2007), 822–842. MR Zbl

Received: 2019-04-29 Revised: 2020-04-27 Accepted: 2020-04-28

kweishaar@regis.edu Department of Mathematics, Regis University, Denver, CO, United States

jseibert@regis.edu Department of Mathematics, Regis University, Denver, CO, United States

involve

msp.org/involve

INVOLVE YOUR STUDENTS IN RESEARCH

Involve showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2020 is US $205/year for the electronic version, and $275/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2020 vol. 13 no. 3