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A Cheeger inequality for graphs based on a reflection principle

Edward Gelernt, Diana Halikias, Charles Kenney and Nicholas F. Marshall

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Given a graph with a designated set of boundary vertices, we define a new notion of a Neumann Laplace operator on a graph using a reflection principle. We show that the first eigenvalue of this Neumann graph Laplacian satisfies a Cheeger inequality.

1. Introduction and main result

1A. ***Introduction.*** Suppose that $G = (V, E)$ is a graph with vertices V and edges E. Let $\partial V \subseteq V$ be a designated set of boundary vertices, and $\mathring{V} := V \setminus \partial V$. We define the doubled graph G' as follows. Let $\mathring{G} = (U, F)$ be an isomorphic copy of the induced subgraph $G[\mathring{V}]$, and let f be an isomorphism from $\mathring{V}$ to U. Set

$$F' := \big\{ \{u, v\} : u \in U,\ v \in \partial V,\ \{f^{-1}(u), v\} \in E \big\}.$$

Then, we define $G' := (V', E')$, where $V' := V \cup U$ and $E' := E \cup F \cup F'$. That is to say, G' is defined by making an isomorphic copy of the interior of G and attaching it to the boundary vertices ∂V as in the original graph; see Figure 1.

Definition 1.1. Let $G' = (V', E')$ be a doubled graph, and let $f : \mathring{V} \to U$ be an isomorphism as above, so that for all $w \in \partial V$ and $v \in \mathring{V}$, $\{v, w\} \in E'$ if and only if $\{f(v), w\} \in E'$. We say that a function $\varphi : V' \to \mathbb{R}$ is even with respect to ∂V if

$$\varphi(v) = \varphi(f(v)) \quad \text{for } v \in \mathring{V},$$

and we say that φ is odd with respect to ∂V if

$$\varphi(v) = -\varphi(f(v)) \quad \text{for } v \in \mathring{V} \qquad \text{and} \qquad \varphi(v) = 0 \quad \text{for } v \in \partial V.$$

Let $L' := D - A$ denote the graph Laplacian of G', where D is the degree matrix of G', and A is the adjacency matrix of G'. The following proposition characterizes the eigenvectors of L' as either even or odd.

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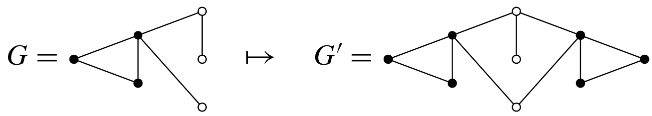

Figure 1. A graph G, and its doubled graph G', where the black and white dots denote interior and boundary vertices, respectively.

Proposition 1.2. *The graph Laplacian L' has $|V|$ eigenvectors that are even with respect to ∂V, and $|\mathring{V}|$ eigenvectors that are odd with respect to ∂V; this accounts for all eigenvectors of L'.*

1B. ***Motivation.*** We are motivated by the observation that the restrictions of the odd and even eigenvectors of L' to the graph G seem like natural Dirichlet and Neumann Laplacian eigenvectors for the graph G, given the respective odd and even behavior of Dirichlet and Neumann Laplacian eigenfunctions on manifolds. In fact, the restriction of the odd eigenvectors of L' to the graph G are eigenvectors of the Dirichlet graph Laplacian defined in [Chung 1997], and inequalities involving the eigenvalues of this operator have been investigated [Chung and Oden 2000]. However, an operator corresponding to the restriction of the even eigenvectors of L' to G has not, to our knowledge been investigated. Chung [1997] defined the Neumann graph Laplacian by enforcing a condition that a discrete derivative vanishes on the boundary nodes of the graph, which results in different eigenvectors than those arising from the even eigenvectors of L'. We note that a Cheeger inequality for Chung's definition of the Neumann graph Laplacian has recently been established in [Hua and Huang 2018].

1C. ***Odd and even eigenvectors.*** The proof of Proposition 1.2 gives some initial insight into the odd and even eigenvectors the graph Laplacian L' on the doubled graph G'.

Proof of Proposition 1.2. The proof is immediate from the block structure of the graph Laplacian L'. Indeed, let $L'(U, W)$ denote the submatrix of L' whose rows and columns are indexed by $U \subseteq V$ and $W \subseteq V$, respectively. We can write

$$L' = \begin{pmatrix} X & Y & 0 \\ Y^\top & Z & Y^\top \\ 0 & Y & X \end{pmatrix},$$

where X is the submatrix $L'(\mathring{V}, \mathring{V})$, Y is the submatrix $L'(\mathring{V}, \partial V)$, and Z is the submatrix $L'(\partial V, \partial V)$. With this notation, the eigenvectors of L' that are even with respect to ∂V are solutions to the equation

$$\begin{pmatrix} X & Y & 0 \\ Y^\top & Z & Y^\top \\ 0 & Y & X \end{pmatrix} \begin{pmatrix} u \\ v \\ u \end{pmatrix} = \mu \begin{pmatrix} u \\ v \\ u \end{pmatrix}.$$

That is to say, the vectors u and v satisfy $Xu + Yv = \mu u$ and $2Y^\top u + Zv = \mu v$. Put differently, when concatenated, u and v form an eigenvector of the matrix

$$L_R := \begin{pmatrix} X & Y \\ 2Y^\top & Z \end{pmatrix}. \tag{1}$$

Observe that L_R is similar to a symmetric matrix,

$$L_R = \begin{pmatrix} I & 0 \\ 0 & \sqrt{2}I \end{pmatrix} \begin{pmatrix} X & \sqrt{2}Y \\ \sqrt{2}Y^\top & Z \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \sqrt{2}I \end{pmatrix}^{-1},$$

and thus by the spectral theorem, L_R has $|V|$ real eigenvectors, which give rise to $|V|$ even eigenvectors of L'. The eigenvectors of L' that are odd with respect to ∂V are solutions to the equation

$$\begin{pmatrix} X & Y & 0 \\ Y^\top & Z & Y^\top \\ 0 & Y & X \end{pmatrix} \begin{pmatrix} u \\ 0 \\ -u \end{pmatrix} = \lambda \begin{pmatrix} u \\ 0 \\ -u \end{pmatrix}.$$

Thus, each vector u such that $Xu = \lambda u$ gives rise to an odd eigenvector of L'. Let

$$L_D := X.$$

Since L_D is symmetric, it follows from the spectral theorem that it has $|\mathring{V}|$ real eigenvectors, and we conclude that L' has $|\mathring{V}|$ odd eigenvectors. □

1D. ***Contribution.*** In this paper, we study the operator L_R defined in (1), which we call the reflected Neumann graph Laplacian. This operator seems to be particularly natural on graphs approximating manifolds. For example, in Remark 1.3, we show that on the path graph, the eigenvectors of the Dirichlet graph Laplacian L_D and the reflected Neumann graph Laplacian L_R are the familiar discrete sine and cosine functions. We remark that the definition of the reflected Neumann graph Laplacian L_R has some similarities to the normalization used in the diffusion maps manifold learning method of [Coifman and Lafon 2006].

Our main result Theorem 1.4 shows that the first eigenvalue of the normalized reflected Neumann graph Laplacian $\mathcal{L}_R$ defined in (2) satisfies a Cheeger inequality. The graph cuts arising from $\mathcal{L}_R$ can differ significantly from graph cuts arising from the standard normalized graph Laplacian $\mathcal{L}$ defined in [Chung 1997]. In Figure 3, we illustrate Theorem 1.4 with an example where the first eigenvector of the Neumann graph Laplacian L_R suggests a drastically different cut than the first eigenvector of the standard graph Laplacian, and describe how the graph cut suggested by L_R is consistent with the Cheeger inequality established in Theorem 1.4. It may be interesting to investigate the analog of other classical eigenvalue inequalities involving these definitions of L_D and L_R for graphs with boundary.

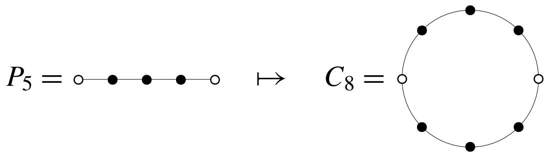

Figure 2. A path graph and its doubled graph.

Remark 1.3. The operators L_D and L_R are particularly natural on the path graph. Let $P_n = (V, E)$ denote the path graph on n vertices, where $V = \{1, \dots, n\}$ and $\{u, v\} \in E$ if and only if $|u - v| = 1$. If $\partial V := \{1, n\}$, then the doubled graph $P_n' = C_{2n-2}$ is the cycle graph on $2n - 2$ vertices; see Figure 2.

Consider L_D and L_R of the path graph P_n. The Dirichlet eigenvectors φ_k and eigenvalues λ_k, which satisfy $L_D \varphi_k = \lambda_k \varphi_k$ for $k = 1, \dots, n-2$, are of the form

$$\lambda_k = 2\left(1 - \cos\left(\frac{\pi k}{n-1}\right)\right) \quad \text{and} \quad \varphi_k(j) = \sin\left(\frac{\pi j k}{n-1}\right)$$

for $j = 1, \dots, n-2$, while the Neumann eigenvectors, ψ_k and μ_k, which satisfy $L_R \psi_k = \mu_k \psi_k$ for $k = 0, \dots, n-1$, are of the form

$$\mu_k = 2\left(1 - \cos\left(\frac{\pi k}{n-1}\right)\right) \quad \text{and} \quad \psi_k(j) = \cos\left(\frac{\pi j k}{n-1}\right)$$

for $j = 0, \dots, n-1$. Thus, the path graph doubling procedure defined in Section 1A gives the familiar sine and cosine functions, which are the Dirichlet and Neumann eigenfunctions of the Laplace operator of the unit interval.

1E. ***Notation and definitions.*** Suppose that $G = (V, E)$ is a graph with vertices V and edges E. Let $\partial V \subseteq V$ be a designated set of boundary vertices, and set $\mathring{V} = V \setminus \partial V$. We can write the adjacency matrix A of the graph G as the block matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{pmatrix},$$

where $A_{11} = A(\mathring{V}, \mathring{V})$, $A_{12} = A(\mathring{V}, \partial V)$, and $A_{22} = A(\partial V, \partial V)$. Motivated by Proposition 1.2 we define the reflected adjacency matrix R by

$$R := \begin{pmatrix} A_{11} & A_{12} \\ 2A_{12}^\top & A_{22} \end{pmatrix}.$$

With this notation, the reflected Neumann Laplacian L_R can be defined by

$$L_R = D - R,$$

where $D = \operatorname{diag}(R\vec{1})$, where $\vec{1}$ denotes a vector whose entries are all 1, and whose dimensions are such that the matrix-vector multiplication is well-defined. We define the normalized reflected Neumann graph Laplacian $\mathcal{L}_R$ by

$$\mathcal{L}_R := D^{-1/2} L_R D^{-1/2}. \tag{2}$$

1F. *Main result.* In this section, we present our main result Theorem 1.4. While the matrix $\mathcal{L}_R$ is not in general symmetric, it is similar to a symmetric matrix; indeed, if

$$Q := \begin{pmatrix} I_{|\mathring{V}|} & 0 \\ 0 & \frac{1}{2} I_{|\partial V|} \end{pmatrix},$$

then the matrix $Q^{1/2} \mathcal{L}_R Q^{-1/2}$ is symmetric, positive-definite, and has the eigenvector $D^{1/2} Q^{1/2} \vec{1}$ of eigenvalue 0. It follows that the first nontrivial eigenvalue λ_R of $\mathcal{L}_R$ satisfies

$$\lambda_R := \inf_{x^\top D^{1/2} Q^{1/2} \vec{1} = 0} \frac{x^\top Q^{1/2} \mathcal{L}_R Q^{-1/2} x}{x^\top x}.$$

Let $E(U, W) := \{\{u, w\} \in E : u \in U,\ w \in W\}$; that is, $E(U, W)$ is the set of edges between U and W. We define a measure $m(U, W)$ on this set of edges by

$$m(U, W) = |E(U, W)| - \tfrac{1}{2} |E(U \cap \partial V, W \cap \partial V)|,$$

and we define the volume $\mathrm{vol}(U)$ of $U \subseteq V$ by

$$\mathrm{vol}(U) := \sum_{u \in U} m(\{u\}, V).$$

The following theorem is our main result.

Theorem 1.4. *Suppose that $G = (V, E)$ is a graph with a designated set of boundary vertices $\partial V \subseteq V$, and define the Cheeger constant h_R by*

$$h_R := \min_{S \subseteq V} \frac{m(S, V \setminus S)}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}}. \tag{3}$$

Then,

$$\sqrt{2\lambda_R} \geq h_R \geq \tfrac{1}{2} \lambda_R,$$

where λ_R is the first nontrivial eigenvalue of $\mathcal{L}_R$.

Recall that the standard Cheeger inequality is constructive in the sense that a cut that achieves the upper bound on the Cheeger constant can be determined from the eigenfunction corresponding to the first eigenvalue of the normalized graph Laplacian $\mathcal{L}$; see [Alon 1986; Cheeger 1970]. Specifically, a partition that achieves the upper bound can be determined by dividing the vertices into two groups based on if the value of the first eigenvector is more or less than some threshold; for a detailed exposition see for example [Chung 1997; 2007]. Similarly, the result of Theorem 1.4 is constructive in the sense that a cut which achieves the upper bound on h_R can be determined from the eigenvector ψ_R of $\mathcal{L}_R$ that corresponds to λ_R.

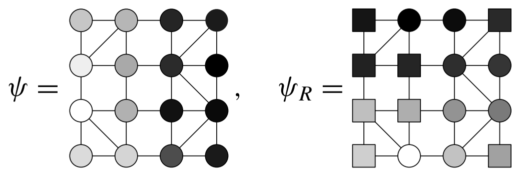

Figure 3. The same graph with vertices colored proportional to ψ (left) and colored proportional to ψ_R (right), where the squares in the right graph denote boundary vertices.

In the following remark, we present an example where the cut arising from ψ_R differs significantly from the cut arising from the first eigenvector ψ of the standard normalized graph Laplacian $\mathcal{L}$.

Remark 1.5. Graph cuts arising from ψ_R can differ significantly from graph cuts arising from ψ. Indeed, on the left of Figure 3 we illustrate a graph whose vertices are colored by greyscale values proportional to ψ. On the right of Figure 3 we illustrate the same graph except several vertices have been designated as boundary vertices (indicated by squares) and the color of the vertices is proportional to ψ_R. Observe that ψ suggests cutting the graph by a vertical line into two equal parts, while ψ_R suggests cutting the graph by a horizontal line into two equal parts.

That ψ_R suggests a horizontal cut of the graph is illustrative of Theorem 1.4. Indeed, it is straightforward to check that the horizontal cut suggested by ψ_R minimizes the cut measure $m(S, V \setminus S)/(\text{vol}(S), \text{vol}(V \setminus S))$ from (3). In contrast, the vertical cut suggested by ψ minimizes the standard cut measure, which is equivalent to the measure $m(S, V \setminus S)/(\text{vol}(S), \text{vol}(V \setminus S))$ in the case that all vertices are interior vertices. Of course, Theorem 1.4 only guarantees that the measure of the cut arising from the eigenvector ψ_R is an upper bound for h_R with value at most $\sqrt{2\lambda_R}$; however, in this simple example the cut arising from ψ_R actually obtains this minimum.

Remark 1.6. Here we visualize the first eigenfunction ψ_R of the reflected Neumann graph Laplacian $\mathcal{L}_R$ on a classic barbell shaped graph; see Figure 4. Observe that in Figure 4 the maximum and minimum value of the eigenvector occur at an interior vertex. This feature of the eigenvectors is interesting in the context of spectral

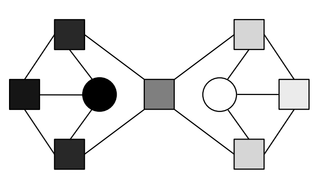

Figure 4. A barbell shaped graph whose vertices are colored proportional to ψ_R, where squares in the graph denote boundary vertices.

clustering, where extreme values of the eigenvectors often correspond to the center of clusters.

1G. *Future directions.* One future direction for this work is the problem of selecting boundary vertices in a principled way. How the boundary is selected may depend on the application at hand. In a social network graph, boundary vertices could correspond to individuals with many connections outside the network. In the context of manifold learning, where the vertices of the graph are points in $\mathbb{R}^n$, boundary vertices could be selected based on the number of points within some ε-neighborhood of each vertex. On the other hand, when a graph is given by sampling from a predefined manifold with boundary, vertices selected from some collar neighborhood of the boundary could be designated as boundary vertices.

Another future direction arises from generalizing the setup under which our work was done. Our graph doubling procedure inputs a graph with boundary and outputs a larger graph, containing the original graph as an induced subgraph, which has a special Z_2 symmetry. Could similar Cheeger results be proven for other reflection procedures? For example, what if $n-1$ copies of the interior vertices were attached, instead of only 1?

Finally, we note a connection between the doubled graph (defined in Section 1A) and numerical analysis that may motivate a direction for future study. Recall that for a path graph P_n the eigenfunctions of the reflected Neumann Laplacian L_R are of the form $\psi_k(j) = \cos(\pi jk/(n-1))$; see Remark 1.3. These Neumann eigenvectors are precisely the basis vectors for the discrete cosine transform (DCT) type I, as classified in [Strang 1999]. The DCT type II, which has basis vectors $\psi_k(j) = \cos\bigl(\pi\bigl(j+\tfrac{1}{2}\bigr)k/n\bigr)$ is also important in numerical analysis; it could be interesting to develop a graph doubling procedure whose Neumann eigenvectors on the path graph are these vectors.

2. Proof of main result

2A. *Summary.* The proof of Theorem 1.4 is divided into two lemmas: first, in Lemma 2.1 we show that $\lambda_R \le 2h_R$, and second, in Lemma 2.2 we show that $h_R^2/2 \le \lambda_R$. The structure of our argument is similar to classical Cheeger inequality proofs; see [Chung 1996; 1997].

2B. *Proof of Theorem 1.4.*

Lemma 2.1 (trivial direction). *We have*

$$\lambda_R \le 2h_R.$$

Proof of Lemma 2.1. Recall that

$$\mathcal{L}_R := D^{-1/2}Q^{1/2}L_R Q^{-1/2}D^{-1/2}.$$

First, we observe that QL_R can be written as

$$QL_R = L - \tfrac{1}{2}L_\partial,$$

where

$$L = \begin{pmatrix} \operatorname{diag}(A_{11}\vec{1}+A_{12}\vec{1})-A_{11} & -A_{12} \\ -A_{12}^\top & \operatorname{diag}(A_{12}^\top\vec{1}+A_{22}\vec{1})-A_{22} \end{pmatrix},$$

and

$$L_\partial := \begin{pmatrix} 0 & 0 \\ 0 & \operatorname{diag}(A_{22}\vec{1})-A_{22} \end{pmatrix}.$$

Observe that L is the standard graph Laplacian of G, while L_∂ is the graph Laplacian of the vertex induced subgraph $G[\partial V]$. Fix a subset $S \subseteq V$, and let χ_S be the indicator function for S. Define

$$x := Q^{1/2}D^{1/2}\chi_S - \frac{\chi_S^\top DQ\vec{1}}{\vec{1}^\top DQ\vec{1}} D^{1/2}Q^{1/2}\vec{1}.$$

By construction, we have $x^\top D^{1/2}Q^{1/2}\vec{1} = 0$, and it follows that

$$\begin{aligned}
\lambda_N &\le \frac{x^\top D^{-1/2}Q^{1/2}L_R Q^{-1/2}D^{-1/2}x}{x^\top x} \\
&= \frac{\chi_S^\top QL_R\chi_S}{\chi_S^\top DQ\chi_S(\vec{1} - \chi_S^\top DQ\chi_S/(\vec{1}^\top DQ1))} = \frac{\chi_S^\top\big(L - \frac{1}{2}L_\partial\big)\chi_S(\vec{1}^\top DQ\vec{1})}{(\chi_S^\top DQ\chi_S^\top)(\chi_{V\setminus S}^\top DQ\chi_{V\setminus S})} \\
&\le \frac{2\cdot\chi_S^\top\big(L - \frac{1}{2}L_\partial\big)\chi_S}{\min\{(\chi_S^\top DQ\chi_S^\top), (\chi_{V\setminus S}^\top DQ\chi_{V\setminus S})\}} = \frac{2\cdot m(S, V\setminus S)}{\min\{\operatorname{vol}(S), \operatorname{vol}(V\setminus S)\}}.
\end{aligned}$$

Since this inequality holds for all subsets $S \subseteq V$, we conclude that $\lambda_R \le 2h_R$. □

Lemma 2.2 (nontrivial direction). *We have*

$$\lambda_R \ge \tfrac{1}{2}h_R^2.$$

Proof of Lemma 2.2. Recall that

$$\lambda_R = \inf_{x^\top D^{1/2}Q^{1/2}\vec{1}=0} \frac{x^\top \mathcal{L}_R x}{x^\top x} = \inf_{y^\top DQ\vec{1}=0} \frac{y^\top QL_R y}{y^\top QDy}.$$

Let g be a vector satisfying

$$\lambda_R = \frac{g^\top QL_R g}{g^\top DQg} \quad \text{and} \quad g^\top QD\vec{1} = 0.$$

Let $\{v_1, \dots, v_n\}$ be an enumeration of the vertices V so that $g_{v_1} \le \dots \le g_{v_n}$ and set $S_j := \{v_1, \dots, v_j\}$, for $j = 1, \dots, n$. Let p be the largest integer such that

$\operatorname{vol}(S_p) \le \frac{1}{2}\operatorname{vol}(V)$; that is,

$$p := \max\bigl\{ j \in \{1, \dots, n\} : \operatorname{vol}(S_j) \le \tfrac{1}{2}\operatorname{vol}(V) \bigr\}.$$

Let g^+ and g^- denote the positive and negative parts of $g - g_{v_p}$, respectively. That is, $g_v^+ := \max\{g_v - g_{v_p}, 0\}$ and $g_v^- := \max\{g_{v_p} - g_v, 0\}$. Let $u \sim v$ denote $\{u, v\} \in E$ and $q = \operatorname{diag}(Q)$. Then

$$\begin{aligned}\lambda_R = \frac{g^\top (L - \frac{1}{2}L_\partial) g}{g^\top D Q g} &= \frac{\sum_{u\sim v}(g_u - g_v)^2 - \frac{1}{2}\sum_{\substack{u\sim v\\ u,v\in\partial V}}(g_u - g_v)^2}{\sum_v g_v^2 d_v q_v} \\ &\ge \frac{\sum_{u\sim v}(g_u - g_v)^2 - \frac{1}{2}\sum_{\substack{u\sim v\\ u,v\in\partial V}}(g_u - g_v)^2}{\sum_v (g(v) - g(v_p))^2 d_v q_v},\end{aligned}$$

where the last inequality holds because we have increased the denominator. From here,

$$\lambda_R \ge \frac{\sum_{u\sim v}((g_u^+ - g_v^+)^2 + (g_u^- - g_v^-)^2) - \frac{1}{2}\sum_{\substack{u\sim v\\ u,v\in\partial V}}((g_u^+ - g_v^+)^2 + (g_u^- - g_v^-)^2)}{\sum_v ((g_v^+)^2 + (g_v^-)^2) d_v q_v}. \quad (4)$$

Recall that

$$\frac{a+b}{c+d} \ge \min\left\{\frac{a}{c}, \frac{b}{d}\right\} \quad (5)$$

for any $a, b \ge 0$ and $c, d > 0$. From (4), we can set

$$a = \sum_{u\sim v}(g_u^+ - g_v^+)^2 - \sum_{\substack{u\sim v\\ u,v\in\partial V}}(g_u^+ - g_v^+)^2, \quad c = \sum_v (g_v^+)^2 d_v q_v,$$

$$b = \sum_{u\sim v}(g_u^- - g_v^-)^2 - \sum_{\substack{u\sim v\\ u,v\in\partial V}}(g_u^- - g_v^-)^2, \quad d = \sum_v (g_v^-)^2 d_v q_v.$$

Observe that a and b are nonnegative. Indeed,

$$a = \sum_{\substack{u\sim v\\ u\notin\partial V \text{ or } v\notin\partial V}} (g_u^+ - g_v^+)^2,$$

which has nonnegative summands, and a similar statement holds for b.

Without loss of generality, (5) implies

$$\lambda_R \ge \frac{\sum_{u\sim v}(g_u^+ - g_v^+)^2 - \frac{1}{2}\sum_{\substack{u\sim v\\ u,v\in\partial V}}(g_u^+ - g_v^+)^2}{\sum_v (g_v^+)^2 d_v q_v}.$$

To simplify notation in the following, let $f = g^+$. We begin by setting

$$\lambda := \frac{\sum_{u\sim v}(f_u - f_v)^2 - \frac{1}{2}\sum_{\substack{u\sim v\\ u,v\in\partial V}}(f_u - f_v)^2}{\sum_v f_v^2 d_v q_v}.$$

Multiplying the numerator and denominator by the same term gives

$$\lambda=\frac{\left(\sum_{u\sim v}(f_u-f_v)^2-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}(f_u-f_v)^2\right)\left(\sum_{u\sim v}(f_u+f_v)^2-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}(f_u+f_v)^2\right)}{\left(\sum_v f_v^2 d_v q_v\right)\left(\sum_{u\sim v}(f_u+f_v)^2-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}(f_u+f_v)^2\right)}.$$

Applying the Cauchy–Schwarz inequality in the numerator gives

$$\lambda\geq\frac{\left(\sum_{u\sim v}|f_u^2-f_v^2|-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}|f_u^2-f_v^2|\right)^2}{\left(\sum_v f_v^2 d_v q_v\right)\left(\sum_{u\sim v}(f_u+f_v)^2-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}(f_u+f_v)^2\right)}.$$

Next, we observe that

$$\sum_{u\sim v}(f_u+f_v)^2-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}(f_u+f_v)^2=\sum_v f_v^2 d_v q_v-\left(\sum_{u\sim v}(f_u-f_v)^2-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}(f_u-f_v)^2\right),$$

and thus it follows that

$$\lambda\geq\frac{\left(\sum_{u\sim v}|f_u^2-f_v^2|-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}|f_u^2-f_v^2|\right)^2}{\left(\sum_v f_v^2 d_v q_v\right)^2(2-\lambda)}.$$

We want to show

$$\sum_{u\sim v}|f_u^2-f_v^2|-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}|f_u^2-f_v^2|\geq\sum_{i=1}^n|f_{v_i}^2-f_{v_{i+1}}^2|m(S_i,V\setminus S_i).$$

We can write

$$\sum_{u\sim v}|f_u^2-f_v^2|-\frac{1}{2}\sum_{\substack{u\sim v\\u,v\in\partial V}}|f_u^2-f_v^2|=\sum_{i=2}^n\sum_{j=1}^{i-1}\left(\chi_{E_{i,j}}-\frac{\chi_{\partial_i}\chi_{\partial_j}}{2}\right)(f_{v_i}^2-f_{v_j}^2),$$

where

$$\chi_{E_{i,j}}=\begin{cases}1 & \text{if } \{v_i,v_j\}\in E,\\ 0 & \text{otherwise}\end{cases}$$

is the indicator function for $\{v_i,v_j\}\in E$, and

$$\chi_{\partial_i}=\begin{cases}1 & \text{if } i\in\partial V,\\ 0 & \text{otherwise}\end{cases}$$

is the indicator function for $v_i\in\partial V$. Note that we are justified in dropping the absolute value signs because $f_{v_i}^2$ is an increasing function of i. Next we write $f_{v_i}^2-f_{v_j}^2$ as a telescoping series

$$f_{v_i}^2-f_{v_j}^2=(f_{v_i}^2-f_{v_{i-1}}^2)+(f_{v_{i-1}}^2-f_{v_{i-2}}^2)+\cdots+(f_{v_{j+1}}^2-f_{v_j}^2),$$

and rearrange terms in the summation to conclude that

$$\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(\chi_{E_{i,j}}-\frac{\chi_{\partial_i}\chi_{\partial_j}}{2}\right)(f_{v_i}^2-f_{v_j}^2)\\ =\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\left(\chi_{E_{j,k+l}}-\frac{\chi_{\partial_j}\chi_{\partial_{k+l}}}{2}\right)\chi_{j\le l}\right)(f_{v_{l+1}}^2-f_{v_l}^2),$$

where

$$\chi_{j\le l}=\begin{cases}1 & \text{if } j\le l,\\ 0 & \text{otherwise.}\end{cases}$$

Then, to complete this step, we note that

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\left(\chi_{E_{j,k+l}}-\frac{\chi_{\partial_j}\chi_{\partial_{k+l}}}{2}\right)\chi_{j\le l}\right)=m(S_l,V\setminus S_l).$$

Returning to our main sequence of inequalities for λ, we have

$$\lambda\ge\frac{\left(\sum_i|f_{v_i}^2-f_{v_{i+1}}^2|m(S_i,V\setminus S_i)\right)^2}{2\left(\sum_v f_v^2 d_v q_v\right)^2}\ge\frac{\left(\alpha\sum_{i=1}^{n}|f_{v_i}^2-f_{v_{i+1}}^2|\min\{\mathrm{vol}(S_i),\mathrm{vol}(V\setminus S_i)\}\right)^2}{2\left(\sum_u f(u)^2 d_u q_v\right)^2},$$

where

$$\alpha:=\min_{1\le i\le n}\frac{m(S_i,V\setminus S_i)}{\min\{\mathrm{vol}(S_i),\mathrm{vol}(V\setminus S_i)\}}.$$

Since $f_{v_i}^2$ is nondecreasing, a rearrangement of the numerator of the previous expression gives

$$\lambda\ge\frac{\alpha^2}{2}\frac{\left(\sum_i(f_{v_i}^2|\min\{\mathrm{vol}(S_i),\mathrm{vol}(V\setminus S_i)\}-\min\{\mathrm{vol}(S_{i+1}),\mathrm{vol}(V\setminus S_{i+1})\}|)\right)^2}{\left(\sum_u f(u)^2 d_u q_u\right)^2}.$$

It follows that

$$\lambda_R\ge\lambda\ge\frac{\alpha^2}{2}\frac{\left(\sum_i f_{v_i}^2 d_{v_i} q_{v_i}\right)^2}{\left(\sum_u f_u^2 d_u q_u\right)^2}=\frac{\alpha^2}{2}\ge\frac{h_R^2}{2},$$

which completes the proof. □

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